suppose $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}$

$U(0,t) = \phi_0(t)$
$U(L,t) = \phi_1(t)$ } want to remove time dependence in bc's
$U(x,0) = f(x)$ how?
$\frac{\partial U(x,0)}{\partial t} = g(x)$

let $U(x,t) = V(x,t) + \Psi(x,t)$ → Ψ will be picked in simplest manner

<u>choose</u> $V(x,t)$ to have <u>homog bc</u>
$V(0,t) = \phi_0(t) - \Psi(0,t) = 0 \stackrel{SO}{\Rightarrow} \Psi(0,t) = \phi_0(t)$
$V(L,t) = \phi_1(t) - \Psi(L,t) = 0 \stackrel{SO}{\Rightarrow} \Psi(L,t) = \phi_1(t)$
The V is the original problem, viz. I.C. problem with homo BC.
<u>choose in simplest form</u> $\Psi(x,t) = \phi_0(t) + \frac{x}{L}\left[\phi_1(t) - \phi_0(t)\right]$ satisfies BC.

now since

$\frac{1}{c^2} U_{tt} = U_{xx}$

$\frac{1}{c^2}\left[V_{tt} + \Psi_{tt}\right] = \left[V_{xx} + \Psi_{xx}\right]$

$\therefore \frac{1}{c^2}\left[V_{tt}\right] - V_{xx} = \Psi_{xx} - \frac{1}{c^2}\Psi_{tt} = 0 - \frac{1}{c^2}\left[\phi_0''(t) + \frac{x}{L}\left(\phi_1''(t) - \phi_0''(t)\right)\right]$ ← <u>known fn of t & x</u>

$\frac{1}{c^2} V_{tt} - V_{xx} = \underline{G(x,t)}$

also BC's convert to
$U(x,0) = V(x,0) + \Psi(x,0) = V(x,0) + \underbrace{\phi_0(0) + \frac{x}{L}\left(\phi_1(0) - \phi_0(0)\right)} = f(x)$

$\therefore V(x,0) = f(x) - \Psi(x,0) = h(x)$ known fn of x ✓

$\frac{\partial U}{\partial t}(x,0) = \frac{\partial V}{\partial t}(x,0) + \frac{\partial \Psi}{\partial t}(x,0) = \frac{\partial V}{\partial t}(x,0) + \underbrace{\phi_0'(0) + \frac{x}{L}\left(\phi_1'(0) - \phi_0'(0)\right)} = g(x)$

$\frac{\partial V}{\partial t}(x,0) = g(x) - \frac{\partial \Psi}{\partial t}(x,0) = l(x)$ known fn of x ✓

we have removed the <u>time dependence in the BC</u> & put in in the inhomogeneous term of PDE

for $V(x,t)$: $\frac{1}{c^2} V_{tt} - V_{xx} = G(x,t)$ Inhomog PDE time dependent ✓

$V(0,t) = 0$ (1) ✓
$V(L,t) = 0$ (2)

$V(x,0) = h(x)$ ✓

$\frac{\partial V}{\partial t}(x,0) = l(x)$

use the homog PDE to find the spatial function then use variation of parameters

$\frac{1}{c^2} V_{tt} - V_{xx} = 0 \Rightarrow \frac{1}{c^2} F T'' - F'' T = 0$

or $\frac{c^2 F''}{F} = \frac{T''}{T} = -\omega^2$

$T = A\cos\omega t + B\sin\omega t$
$F = C\cos\frac{\omega}{c}x + D\sin\frac{\omega}{c}x$
(1) & (2) $\Rightarrow C = 0$ & $\frac{\omega}{c}L = n\pi$
or $\omega = \frac{n\pi c}{L}$

thus $F = D \sin \frac{n\pi x}{L}$

$T = A\cos \frac{n\pi c}{L} t + B \cos \frac{n\pi c}{L} t$ $\Big\}$ $V = \sum_{n=0}^{\infty} \left\{ A \cos \frac{n\pi c}{L} t + B \cos \frac{n\pi c}{L} t \right\} \sin \frac{n\pi x}{L}$

using Var. of Param

$\therefore$ $V = \sum \left\{ \tilde{A}_n(t) \cos \frac{n\pi c}{L} t + \tilde{B}_n(t) \sin \frac{n\pi c}{L} t \right\} \sin \frac{n\pi x}{L} = \sum E_n(t) . \sin \frac{n\pi x}{L}$

$V_t = \sum E_n' \sin \frac{\pi n x}{L}$ $\qquad$ $V_x = \sum E_n \frac{n\pi x}{L} \cos \frac{n\pi x}{L}$

$V_{tt} = \sum E_n'' \sin \frac{n\pi x}{L}$ $\qquad$ $V_{xx} = -\sum E_n \left(\frac{n\pi \cancel{x}}{L}\right)^2 \sin \frac{n\pi x}{L}$

$\therefore$ $\frac{1}{c^2} V_{tt} - V_{xx} = G(x,t)$

$$\sum \left\{ \frac{1}{c^2} E_n'' + E_n \left(\frac{n\pi \cancel{x}}{L}\right)^2 \right\} \sin \frac{n\pi x}{L} = \underbrace{- \frac{1}{c^2} \left[\phi_0''(t) + \frac{x}{L} \left(\phi_1''(t) - \phi_0''(t) \right) \right]}^{G(x,t)} = \sum_{n=1}^{\infty} H_n(t) \sin \frac{n\pi x}{L}$$

$G(x,t)$ can be expanded as a sine series

$$\therefore \quad H_n(t) = \frac{2}{L} \int_0^L - \frac{1}{c^2} \left[\phi_0'' + \frac{x}{L} \left(\phi_1''(t) - \phi_0'' \right) \right] \sin \frac{n\pi x}{L} dx$$

$$= \frac{2}{L} \left\{ -\frac{1}{c^2} \phi_0'' \left(-\frac{L}{n\pi} \cos \frac{n\pi x}{L} \right) \Big|_0^L - \frac{1}{c^2 L} (\phi_1'' - \phi_0'') \quad (-x \cos \frac{n\pi}{L} \right.$$

thus term by term

$\frac{1}{c^2} E_n'' + \left(\frac{n\pi}{L}\right)^2 E_n = H_n(t)$. Solve by variation of parameters after get homogeneous solutions

from homog eqn. $E_n'' + \left(\frac{n\pi c}{L}\right)^2 E_n = 0$

and to find $H_n(t)$

when must let $E_{n_p} = A_n(t) \cos \frac{n\pi c}{L} t + B_n(t) \sin \frac{n\pi c}{L} t$

and use variation of parameters.

remember if $y_h = C_1 y_1 + C_2 y_2$ satisfies $y'' + b(x) y' + c(x) y = g(x)$

we have $C_1' y_1 + C_2' y_2 = 0$

$C_1' y_1' + C_2' y_2' = g(x)$

and $C_1 = \int \frac{-g y_2}{\text{wronskian}} dx \qquad C_2 = \frac{g y_1}{\text{wronskian}} dx$

wronskian is $y_1 y_2' - y_2 y_1'$

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here $y_1 = \cos \frac{n\pi c}{L} t \qquad y_2 = \sin \frac{n\pi c}{L} t$

wronskian is $\frac{n\pi c}{L}$ $\qquad$ thus $A_n = \int^t \frac{-H_n(\bar{t}) c^2 \sin \frac{n\pi c}{L} \bar{t} \, d\bar{t}}{\frac{n\pi c}{L}}$

$B_n = \int^t \frac{H_n(\bar{t}) c^2 \cos \frac{n\pi c}{L} \bar{t}}{\frac{n\pi c}{L}} d\bar{t}$