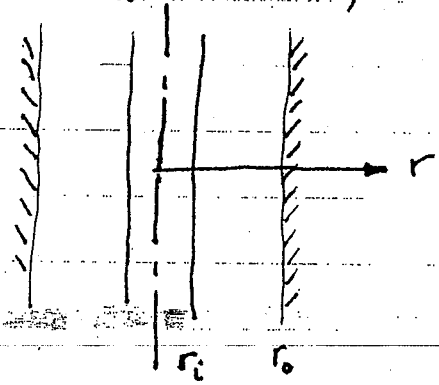
The denominator $\int_a^b P(x)\, y_n^2(x)\,dx = \frac{1}{2\lambda_n}\left\{ y_n' \, S \frac{\partial y}{\partial \lambda}\Big|_a^b - y_n \, S \frac{\partial^2 y}{\partial x \partial \lambda}\Big|_a^b \right\}_{\lambda=\lambda_n}$

$y_n'(z) = \frac{d}{dz} y_n$

$y = y(x;\lambda)$ satisfies ODE & is bounded

$y = y(x;\lambda_n) = y_n$ satisfies ODE & B.C.

- Removing inhomogeneities in the PDE & BC's
  - IN PDE & IN BC.
  - Time history of diffusion of a contaminant $c(r,t)$ in an annular region in which the contaminant in continually produced. (source exists)



$$\frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rS \qquad (3)$$

α diffusivity

S source term

- assume at $r=r_o$ barrier blocks outer diffusion $\ni$ $\frac{\partial c}{\partial r} = 0$ (1)

- Also assume contaminant is convectively removed at $r=r_i$

$$h(c - c_\infty) = D\frac{\partial c}{\partial r} \qquad (2)$$

h convective transport coeff ; c_∞ is the fixed concentration in fluid passing through annular hole

D - diffusion coeff for the contaminant in the solid

- initially $c = c_o$ at $t=0$ (only one IC) two boundaries require 2 BCs (at r_i & r_0)

Where is inhomogeneity? replace c by nc $n=$ const.

(1) homogeneous $\frac{\partial c}{\partial r}$ $(r=r_0)$

(2) no due to hc_∞ term $h(c-c_\infty)=D\frac{\partial c}{\partial r}$ @ $r=r_i$

(3) not PDE due to rs term $\frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right)=\frac{r}{\alpha}\frac{\partial c}{\partial t}-rs$

- Try to remove inhogeneity in PDE & BC together
- since the inhomogeneous term in PDE is a fn of r only choose a solution which is fn of r only : $\psi(r)=c_{ss}$
- since ψ not fn of t $\Rightarrow$ steady state solution ie. let ψ satisfy

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right)=-rs \qquad (3')$$

$$\frac{\partial \psi}{\partial r}=0 \quad \text{at } r=r_0 \qquad (1)$$

$$h(\psi-c_\infty)=D\frac{\partial \psi}{\partial r} \quad \text{at } r=r_i \qquad (2)$$

$$r\psi'=-\frac{r^2 s}{2}+C_1 \qquad \text{from } (3')$$

$$\psi'=-\frac{rs}{2}+\frac{C_1}{r} \Rightarrow \psi=-\frac{r^2 s}{4}+C_1\ln r+C_2$$

$$h(\psi-c_\infty)=D\psi' \Rightarrow +h\left(-\frac{r_i^2 s}{4}+C_1\ln r_i+C_2 - c_\infty\right)=D\left(-\frac{r_i s}{2}+\frac{C_1}{r_i}\right)$$

$$(1)\Rightarrow \psi'=0\Big|_{r=r_0}=-\frac{r_0 s}{2}+\frac{C_1}{r_0}=0 \Rightarrow C_1=\frac{r_0^2 s}{2}$$

$$\Rightarrow C_2=\frac{D}{h}\left(-\frac{r_i s}{2}+\frac{r_0^2 s}{2r_i}\right)+\frac{r_i^2 s}{4}-\frac{r_0^2 s}{2}\ln r_i+c_\infty$$

$$\therefore \psi(r)=(-r^2+r_i^2)\frac{s}{4}+\frac{r_0^2 s}{2}\ln(r/r_i)+\frac{D}{hr_i}\left(\frac{r_0^2 s}{2}-\frac{r_i^2 s}{2}\right)+c_\infty$$

next let's find a $C(r,t) = C_{trans.}$

$$C_{TOT} = C_{ss} + C_{trans.} = \psi(r) + C_{trans}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial C_{TOT}}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial C_{TOT}}{\partial t} - rS \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial C_{tr}}{\partial r} + r\frac{\partial C_{ss}}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial C_{tr}}{\partial t} - rS$$

$$\frac{\partial C_{TOT}}{\partial r} = 0 \quad \text{at } r = r_0 \qquad \frac{\partial C_{tr}}{\partial r} + \frac{\partial C_{ss}}{\partial r} = 0$$

$$h\left(C_{TOT} - C_\infty\right) = D\frac{\partial C_{TOT}}{\partial r} \quad \text{at } r = r_i \qquad h\left(C_{ss} - C_\infty + C_{tr}\right) = D\left[\frac{\partial C_{tr}}{\partial r} + \frac{\partial C_{ss}}{\partial r}\right]$$

$$C(r,t=0) = C_0 = C_{ss} + C_{tr}(r,t=0) = \psi(r) + C_{tr}(r,t=0)$$

transient problem satisfies $\frac{\partial}{\partial r}\left(r\frac{\partial C_{tr}}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial C_{tr}}{\partial t}$

$$\frac{\partial C_{tr}}{\partial r} = 0 \quad @\ r = r_0$$

$$h\,C_{tr} = D\frac{\partial C_{tr}}{\partial r} \quad @\ r = r_i$$

$$C_{tr}(r,t=0) = C_0 - \psi(r)$$

this is homogeneous PDE & BC $\Rightarrow C_{tr}(r,t) = R(r)T(t)$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(RT)\right) = \frac{r}{\alpha}\frac{\partial}{\partial t}(RT)$$

$$rR''T + R'T = \frac{r}{\alpha_0}R\dot{T} \qquad \text{DIVIDE BY } rRT$$

$$\frac{rR''}{rR} + \frac{R'}{rR} = \frac{1}{\alpha}\frac{\dot{T}}{T} = -\lambda^2$$

$$\dot{T} + \lambda^2\alpha T = 0 \qquad T(t) = Ae^{-\lambda^2\alpha t}$$

$$rR'' + R' + \lambda^2 rR = 0 \qquad R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

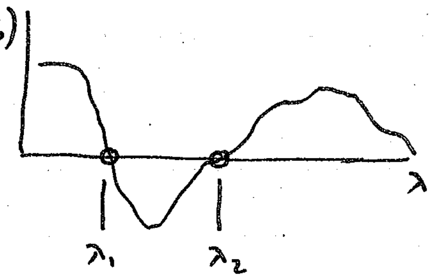
FIRST BC. $\left.\frac{\partial C_{tr}}{\partial r}\right|_{r=r_0} = R'(r_0)T(t) = 0 \Rightarrow R'(r_0) = [C_1 J_0'(\lambda r_0) + C_2 Y_0'(\lambda r_0)]\lambda$

2nd BC $\quad hC_{tr} - D\frac{\partial C_{tr}}{\partial r} = h(C_1 J_0 + C_2 Y_0) - D\lambda(C_1 J_0' + C_2 Y_0') = 0 \quad @\ r = r_i$

put into matrix form
$$\begin{bmatrix} \lambda J_0'(\lambda r_o) & \lambda Y_0'(\lambda r_o) \\ h J_0(\lambda r_i) - D\lambda J_0'(\lambda r_i) & h Y_0(\lambda r_i) - D\lambda Y_0'(\lambda r_i) \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if C_1 & $C_2 \neq 0$ (otherwise $R \equiv 0, \Rightarrow C_{tr} \equiv 0$) $\Rightarrow$ $h\lambda J_0'(\lambda r_o) Y_0(\lambda r_i) - D\lambda^2 J_0'(\lambda r_o) Y_0'(\lambda r_i) - \lambda h J_0(\lambda r_i) Y_0'(\lambda r_o)$
$+ D\lambda^2 Y_0'(\lambda r_o) J_0'(\lambda r_i) = 0 = D(\lambda, r_i, r_o)$

This determinant gives the value of λ



- since $D(\lambda, r_i, r_o) = 0 \Rightarrow$ rows of the matrix are multiples of each other (columns)

$$\begin{vmatrix} 1 & 1 \\ 5 & 5 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} = 0$$

- $\Rightarrow C_1$ is a multiple of C_2
- since $\det(\lambda, r_i, r_o) = 0$ for each $\lambda_n \Rightarrow C_{1n}$ & C_{2n} ie C_i's are fns of choice of λ_n
- only need one of the rows (rows lin.): $C_{1n} J_0'(\lambda_n r_o) + C_{2n} Y_0'(\lambda_n r_o) = 0$

$$\Rightarrow C_{1n} = -C_{2n} \frac{Y_0'(\lambda_n r_o)}{J_0'(\lambda_n r_o)}$$

$$\therefore \quad R_n(r) = C_{2n}\left[-Y_0'(\lambda_n r_o) \frac{J_0(\lambda_n r)}{J_0'(\lambda_n r_o)} + Y_0(\lambda_n r) \right]$$

& $$C_{tr} = \sum_{n=1}^{\infty} \tilde{A}_n \left[-\frac{Y_0'(\lambda_n r_o)}{J_0'(\lambda_n r_o)} J_0(\lambda_n r) + Y_0(\lambda_n r) \right] e^{-\lambda_n^2 \alpha t}$$

$$\tilde{A}_n = A C_{2n}$$

$$\tilde{A}_n = \frac{\int_{r_i}^{r_o} r\, C_{tr}(r, t=0)\, R_n(r)\, dr}{\int_{r_i}^{r_o} r\, R_n^2(r)\, dr} = \frac{\int_{r_i}^{r_o} r\,[c_o - \psi]\, R_n(r)\, dr}{\int_{r_i}^{r_o} r\, R_n^2(r)\, dr}$$