$$\frac{d}{dx}\left[Sy'\right] + \left[Q+\lambda^2 P\right]y = 0$$

subjectes to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{matrix}\lambda = \lambda_m \\ y = y_m\end{matrix}\right\}$ & $\left.\begin{matrix}\lambda = \lambda_n \\ y = y_n\end{matrix}\right\}$ both ODE & BC are satisfied $y_m \neq y_n$ $\lambda_m \neq \lambda_n$

$$\Rightarrow \quad [Sy_m']' + [Q + \lambda_m^2 P]y_m = 0 \qquad (1)$$

$$[Sy_n']' + [Q + \lambda_n^2 P]y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{ [Sy_m']' \cancel{y_n} + [\underline{\underline{Q}} + \lambda_m^2 P]y_m \right\} y_n - \left\{ [Sy_n']' + [\underline{\underline{Q}} + \lambda_n^2 P]y_n \right\} y_m \right\rangle dx$$

$$\int_a^b \left\{ [Sy_m']' y_n - [Sy_n']' y_m \right\} dx + (\lambda_m^2 - \lambda_n^2) \int_a^b P y_m y_n dx = 0$$

$$\int_a^b [Sy_m']' y_n \, dx = Sy_m' y_n \Big|_a^b - \int_a^b Sy_m' y_n' dx$$ integration by parts

$$\int_a^b [Sy_n']' y_m \, dx = Sy_n' y_m \Big|_a^b - \int_a^b Sy_n' y_m' dx$$

and $\left.\begin{matrix}\text{at } x = a \\ \text{at } x = b\end{matrix}\right\}$ $\begin{matrix}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{matrix}$ either $\alpha = 0$ & $\beta = 0$ or $\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \;\; \text{at both } x = a, b \qquad S[y_m' y_n - y_n' y_m]\Big|_a^b + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_n y_m dx =$$

$$\Rightarrow \text{ if } \lambda_m \neq \lambda_n \quad \boxed{\int_a^b P y_n y_m dx = 0}$$ orthogonality condition but wrt weight fn $P(x)$

example $T'' + \omega^2 T = 0 \quad \Rightarrow \quad S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m dx = 0 \quad \Rightarrow \quad \int_a^b P \sin\omega_n x \cdot \sin\omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $T(x) = 0$ @ $x = L$ $\Rightarrow T_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{l} dx = 0$$

William Reynolds - Soln's of PDEs notes

$$\int_a^b P y_n y_m \, dx = 0 \qquad n \neq m \qquad \text{for sturm-Liouville} \tag{4.2.14}$$

Eqn. (4.2.14) is the <u>orthogonality property</u> of the eigenfunctions. The eigenfunctions are said to be orthogonal <u>with respect to the weight function $P(x)$</u> .

Now, suppose that, in the course of trying to construct the solution to a PDE as a linear combination of eigensolutions of the linear, homogeneous partial problem, we are led to the point where we wish to determine the coefficients in an eigenfunction expansion,

$$f(x) = \sum_{n=1}^{\infty} A_n y_n(x) \tag{4.2.15}$$

where the y_n are eigensolutions of a Sturm-Liouville problem. Multiplying (4.2.15) by Py_m , and integrating over the problem domain,

$$\int_a^b f P y_m \, dx = \sum_{n=1}^{\infty} A_n \int_a^b P y_n y_m \, dx \tag{4.2.16}$$

But, because of the orthogonality property (4.2.14), all of the integrals on the right will drop out, except the one where $n = m$. Hence, we can immediately solve for A_m ,

$$A_m = \frac{\int_a^b f P y_m \, dx}{\int_a^b P y_m^2 \, dx} \tag{4.2.17}$$

The infinite series (4.2.15) will be useless if it fails to converge to $f(x)$. In specific problems where one calculates the A_n it is easy to perform the standard tests for series convergence. It is somewhat more difficult to prove convergence in general. However, if f is square-integrable, i.e., if

$$\int_a^b Pf^2 dx \quad \text{is finite}$$

then the series converges in the sense that[*]

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} A_n y_n(x) \right|^2 dx \to 0$$

This means that, if f is continuous over the interval $a \leq x \leq b$, the series converges uniformly (at all x). However, if f is discontinuous at some point, then the series will give a value at that point that is the average of the values of f at points infinitesimally above and below the point of discontinuity.

There are many problems of interest involving higher order system of linear homogeneous equations. In these cases, there are no theorems or general proofs of convergence of the eigenfunction expansions. One has to proceed by examining each case separately. However, problems arising from well-thought through physical formulations rarely, if ever, give rise to non-convergent expansions, so the analyst is usually safe in going ahead, assuming convergence, and then verifying it after the fact by ratio tests, numerical calculations, or other appropriate means.

↑ read

↓ don't need

4.3 Example - Vibrating String

For the vibrating string problem discussed in §4.1, the solution is given by (4.1.6). The coefficients A_n must be chosen such that (4.1.9) is satisfied. The eigenfunctions X_n are eigensolutions of

$$X_n'' + \lambda_n^2 X_n = 0 \tag{4.3.1}$$

and hence, from Sturm-Liouville theory, have the orthogonality property

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[*] See, for example, Ince, Ordinary Differential Equations, Dover, New York, 1956.

now suppose we want to write $f(x)$ as a fn of the eigenfunctions $y_n($

now suppose $f(x) = \sum A_n y_n(x)$

if we mult. by $P y_m$ & integrate

$$\int_a^b P(x) f(x) y_m(x)\,dx = \int_a^b P \sum A_n y_n(x) y_m(x) dx$$

$$= \sum A_n \int_a^b P y_n y_m \,dx = \begin{cases} A_m \int_a^b P y_m^2 dx & n=m \\ 0 & n \neq m \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m \,dx}{\int_a^b P y_m^2 \,dx}}$$

same as pg 4.7 Reynolds' notes ↑

<u>FOR</u> $T'' + \omega^2 T = 0 \qquad y_m = \sin\frac{m\pi x}{L} \qquad P = 1 \qquad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin\frac{m\pi x}{L} dx \Big/ \int_0^L 1 \cdot \sin^2\frac{m\pi x}{L} dx$$

$$\int_0^L \sin^2\frac{m\pi x}{L} dx = 1/2 \quad \Rightarrow \quad \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2m\pi x}{L}\right) dx = 1/2$$

$A_m = \frac{2}{L} \int_0^L f(x) \sin\frac{m\pi x}{L} dx$ normal fourier coefficient

- Returning to our problem if $f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$
  for the bessel fn. $P(r) = +r$ $\qquad E_i = \dfrac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r\, J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$

since $E_i = \tilde{B}_i$

also $g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$ $\qquad L_i = \dfrac{\int_0^a r\, g(r) J_0(\,) dr}{\int_0^a r\, J_0^2(\,) dr} = \tilde{A}_i \omega_i$

since $L_i = \tilde{A}_i \omega_i \quad \Rightarrow \quad \tilde{A}_i = \frac{L_i}{\omega_i}$

$$\int_0^a r\, J_0^2\left(r\frac{\omega_i}{c}\right) dr$$ can be evaluated as follows

using the ODE : if you remember $J_0\left(\frac{\omega_i}{c} r\right)$ satisfies

$$r R'' + R' + \left(\overbrace{\left[\frac{\omega_i}{c}\right]^2 r}^{\lambda_i^2 r} - 0^2\right) R = \overset{(*)}{0} \qquad R_i = J_0(\lambda_i r)$$

$\equiv$ to $(rR')' + \lambda^2 r R = 0$

let $R(r,\lambda)$ be a solution to $r R'' + R' + (\lambda^2 r) R = 0$
and satisfy that $R(r=0)$ is not ∞
$R(r) = J_0(\lambda r)$

now take $\frac{\partial}{\partial\lambda}(*)$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial\lambda} + \underbrace{R_i \frac{\partial^2 R}{\partial\lambda\partial r}} + 2\lambda r\, R R_i + \lambda^2 r \frac{\partial R}{\partial\lambda} R_i = 0$$

integrate over $0 \le r \le a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r \partial\lambda}\right) + \lambda^2 r \frac{\partial R}{\partial\lambda} + 2\lambda r R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du$, let $u = R_i$ $v = r\frac{\partial^2 R}{\partial r\partial\lambda}$

$$r R_i \frac{\partial^2 R}{\partial r \partial\lambda}\Big|_0^a - \int_0^a r \frac{\partial^2 R}{\partial r\partial\lambda} R_i'\, dr + \int_0^a \left(\lambda^2 r \frac{\partial R}{\partial\lambda} R_i + 2\lambda r R R_i\right) dr = 0$$

R_i by definition is $=0$ @a: $(J_0(\lambda_i a) = 0)$ at $r=0$ at lower limit $\overset{r}{}=0$

integrate by parts 2nd term $$-\int_0^a r R_i' \frac{\partial^2 R}{\partial r\partial\lambda} dr = -r R_i' \frac{\partial R}{\partial\lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial\lambda} (r R_i')' dr,$$ let $u = r R_i'$ $v = \frac{\partial R}{\partial\lambda}$

$$\Rightarrow \underbrace{-r R_i' \frac{\partial R}{\partial\lambda}\Big|_0^a}_{-a R_i' \frac{\partial R}{\partial\lambda}} + \int_0^a \frac{\partial R}{\partial\lambda} \left\{ \underbrace{(r R_i')' + \lambda^2 r R_i}_{=0 \text{ since } R_i \text{ solves } (rR_i')' + \lambda^2 r R_i = 0 \text{ when } \lambda = \lambda_i} \right\} dr + \int_0^a 2\lambda r R R_i\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$; thus the middle term is zero

and $$\int_0^a r R R_i\, dr \Rightarrow \int_0^a r R_i^2\, dr = \frac{a R_i'}{2\lambda_i}\frac{\partial R}{\partial\lambda}\Big|_{r=a}$$

THUS $\int_0^a r\, J_0^2\left(\frac{\omega_i}{c} r\right) dr = \frac{a}{2\left(\frac{\omega_i}{c}\right)}\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2 \cdot a \cdot \frac{\omega_i}{c} = \frac{a^2}{2}\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2$

since $R_i'(r) = \frac{d}{dr} J_0\left(\frac{\omega_i}{c} r\right) = \frac{\omega_i}{c} J_0'\left(\frac{\omega_i}{c} r\right)$ and $R_i\left(\frac{\omega_i}{c} r\right)$ solves D.E. & BC's & equals $J_0\left(\frac{\omega_i}{a} r\right)$

$\frac{\partial R}{\partial \lambda} = \frac{d}{d\lambda} J_0(\lambda r) = r J_0'(\lambda r)$

now $a R_i' \frac{\partial R}{\partial \lambda}\Big|_{\substack{r=a \\ \lambda=\lambda_i}} = a\left[\frac{\omega_i}{c} J_0'\left(\frac{\omega_i}{c} r\right)\right]\left[r J_0'(\lambda r)\right]\Big|_{\substack{r=a \\ \lambda=\lambda_i}} = a^2 \frac{\omega_i}{c}\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2$

THUS $$\tilde{B}_i = \frac{2}{a^2} \frac{\int_0^a r f(r)\, J_0\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2}$$

$$\tilde{A}_i = \frac{2}{a^2 \omega_i} \frac{\int_0^a r g(r)\, J_0\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2}$$

Bessel Relations: $J_0'(\lambda x) = -\lambda J_1(\lambda x) \quad \therefore \quad J_0'\left(\frac{\omega_i}{c} a\right) = -\frac{\omega_i}{c} J_1\left(\frac{\omega_i}{c} a\right)$

- note Problem in Reynolds pg 4.10 involves – SPHERICAL BESSEL FNS.

produces a bessel eq. that solves

$$r^2 R'' + \underline{\underline{2}} r R' + \lambda^2 r^2 R = 0$$ and this has an

$$[r^2 R']' + \lambda^2 r^2 R = 0$$

orthogonality condition $\int_0^r \underline{\underline{r^2}} R_n R_m \, dr = 0$

- in general if $[S y_n']' + [Q + \lambda_n^2 P] y_n = 0$ Sturm-Liouville

under the conditions $\alpha y_n + \beta y_n' = 0$ at $x=a$, $x=b$

y_n is an eigenfunction λ_n eigenvalue

- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x) P(x) y_n(x) dx}{\int^b P(x)\, y_n^2(x) dx}$$

Reynolds' notes

$$R(r,\lambda) = \frac{\sin(\lambda r)}{\lambda r} \tag{4.4.26}$$

$$\left.\frac{\partial R}{\partial \lambda}\right)_{\substack{\lambda = \lambda_n \\ r = r_o}} = \frac{1}{\lambda_n} \cos(\lambda_n r_o) \tag{4.4.27}$$

Noting that $\cos(\lambda_n r_o) = (-1)^n$, (4.2.23) gives

$$A_n = 2(-1)^{n+1}\, T_o \tag{4.4.28}$$

So, our final solution is, from (4.4.12),

$$T(r,t) = 2T_o \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/r_o)}{n\pi r/r_o}\, e^{-n^2\pi^2\alpha t/r_o^2} \tag{4.4.29}$$

Note that the series converges for all t . The series for $\partial T/\partial r$, developed from (4.4.29) by differentiation, will converge for all $t > 0$ because of the exponential, but does not converge at $t = 0$. But this is not a serious limitation. As t increases the series converges more rapidly, and at large t the solution is given (approximately) by just the first term,

$$T \simeq 2T_o \frac{\sin(\pi r/r_o)}{\pi r/r_o}\, e^{-\pi^2\alpha t/r_o^2} \tag{4.4.30}$$

↑ don't need

↓ read

4.5 Sturm-Liouville Denominator Integral for general S-L eqn

In analyses, leading to the Sturm-Liouville problems, the orthogonality property will produce (4.2.17). The denominator integral may be expressed in terms of quantities evaluated at the boundary using a generalization of the trick employed in the previous example. Let $y(x,\lambda)$ be a solution to (4.2.1) not necessarily satisfying the boundary conditions (4.2.2). Then, $y(x,\lambda_n)$

4.2.1 is $\frac{d}{dx}\left[S\frac{dy}{dx}\right] + \left[Q+\lambda^2 P\right]y=0$

4.2.2 is $\alpha y+\beta y'=0$ @ $x=a$ & $x=b$

will be an eigensolution satisfying the Diff. Eq & the boundary conditions. We differentiate (4.2.1) with respect to λ, obtaining

$$\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x\partial\lambda}\right)+\left[Q+\lambda^2 P\right]\frac{\partial y}{\partial\lambda}+2\lambda Py = 0 \qquad (4.5.1)$$

Next, we multiply (4.5.1) by y_n and integrate over the problem range,

$$\int_a^b y_n \left\{\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x\partial\lambda}\right)+\left[Q+\lambda^2 P\right]\frac{\partial y}{\partial\lambda}+2\lambda Py\right\}dx = 0 \qquad (4.5.2)$$

The first integral is integrated twice by parts, and (4.5.2) becomes

$$y_n\, S\,\frac{\partial^2 y}{\partial x\partial\lambda}\bigg|_a^b - \frac{\partial y}{\partial\lambda}\,y_n' S\bigg|_a^b$$

$$+\int_a^b \frac{\partial y}{\partial\lambda}\left\{(Sy_n')' + \left[Q+\lambda^2 P\right]y_n\right\}dx + 2\lambda\int_a^b Pyy_n dx = 0 \qquad (4.5.3)$$

Now, if we set $\lambda = \lambda_n$, the first integral drops out (because the integrand contains the y_n equation), and hence

IF y_n is a soln to Bessel eqn $\Rightarrow \lambda=\lambda_n$. last term of 4.5.3 becomes

$$\int_a^b Py_n^2 dx = \frac{1}{2\lambda_n}\left\{y_n' S\,\frac{\partial y}{\partial\lambda}\bigg|_a^b - y_n S\,\frac{\partial^2 y}{\partial x\partial\lambda}\bigg|_a^b\right\}_{\lambda = \lambda_n} \qquad (4.5.4)$$

Thus, the denominator in A_n can be evaluated without recourse to integration.

4.6 Removal of Inhomogeneities in the PDE and BCs

In the previous problem, the PDE and BCs were homogeneous, and therefore eigensolutions of this homogeneous problem could be found. By taking a