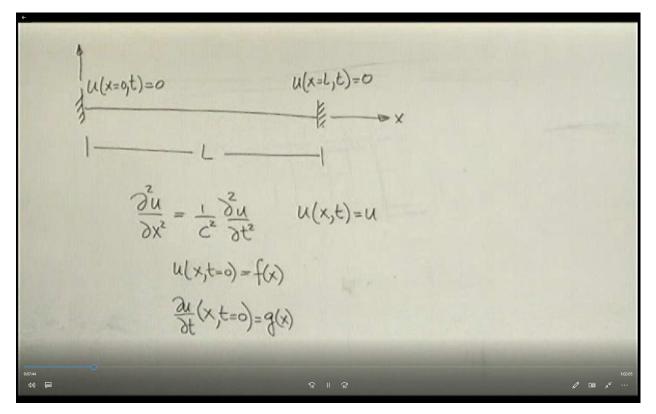
Solution of the vibrating string fixed at both ends using SOV



Assume u(x,t)=F(x)G(t). Put into the PDE and bring terms that involve x to one side and terms involving t to the other side. C² is brought to the x side so that the t side produces a simple equation

 $\frac{\partial u}{\partial x^2} = \frac{1}{C^2} \frac{\partial u}{\partial t^2} \qquad u(x,t) = u = F(x)G(t)$ u(x,t=0) = f(x) $\frac{\partial u}{\partial t}(x,t=0) = g(x)$ $-\sum_{c^2} \frac{F'' \cdot G}{FG} = \frac{F' \cdot G}{FG} = -\omega^2$

 $= \frac{1}{c^2} \frac{\partial u}{\partial t^2} \qquad u(x,t) = u = F(x)G(t)$ u(x,t=0) = f(x) $\frac{\partial u}{\partial t}(x,t=0) = g(x)$ $G + \omega^2 G = 0$ G(t) = A sinut + B cos wt $\frac{F'' \cdot G}{FG} = \frac{FG}{FG} = -\omega^2$

G(t) = A sinut + B cos wt $F'' + \frac{\omega^2}{C^2}F = 0$ G $F(x) = C \sin \frac{\omega}{c} x + D \cos \frac{\omega}{c} x$

From the BCs we find:

o = u(x=o,t) = F(x=o)G(t)F(x=0)=0 O = U(x=L,t) = F(x=L)G(t)F(x=L)=0

Applying these conditions to F(x) at x=L, leads to either C=0 or sin() =0. C=0 leads to a trivial solution. We are looking for nontrivial solutions.

F(x=L)=0 $F(x=0)=0=C\sin \frac{10}{2}.0+D\cos \frac{10}{2}.0=D$ $F(x) = C \sin \frac{\omega}{2} x$ F(x=L)=0=CsinwL

sin 2.0 $F(x) = C \sin \frac{\omega}{C} x$ $F(x=L) = 0 = C \sin \frac{\omega}{C} L \implies \sin \frac{\omega L}{C} = 0$ ×⊞ P

Now solve for omega and put back into the equation

 $F(x) = C \sin \frac{n\pi x}{L}$ $f(x,t) = F \cdot G = \sin \frac{n\pi x}{L} \left(\overline{A} \sin \frac{n\pi x}{L} + \overline{B} \cos \frac{n\pi x}{L} \right)$ $u(x,t) = \overline{Z} U_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\overline{A}_n \sin \frac{n\pi x}{L} + \overline{B} \cos \frac{n\pi x}{L} \right)$

The question is "What is the starting value of the index n?"

$$(x,t) = F \cdot G = \sin \frac{n\pi x}{L} \left(A \sin \frac{n\pi t}{L} + B \cos \frac{n\pi t}{L} \right)$$

$$u(x,t) = \sum U_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(A_n \sin \frac{n\pi t}{L} + B \cos \frac{n\pi t}{L} \right)$$

$$IS \quad n=0 \quad as \text{ usual d index}?$$

$$n=0 \Rightarrow \omega = 0 \Rightarrow c^2 F''_n = 0 \Rightarrow F''_n(x) = 0$$

$$F'(x) = C$$

$$F(x) = C$$

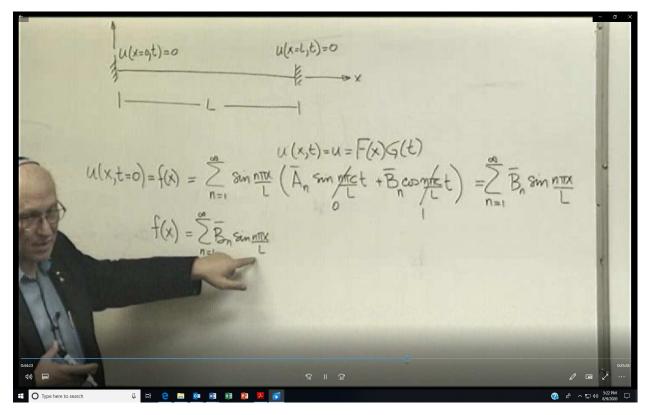
$$F(x) = C \times D$$

$$N=1$$

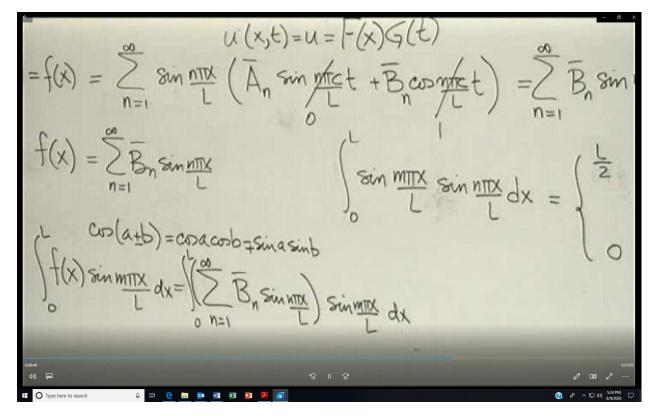
$$IS \quad n=0 \quad as \text{ usual d index}?$$

15 n=0 as valid index?	Constant I
$n=0 \Rightarrow \omega=0 \Rightarrow e^{2}E''=0 \Rightarrow F'(x)=0$	
F'(x) = C	
F(x) = Cx + D	A
$F(x=0)=0=C(0+D) \Rightarrow F(A)=Cx$	1
$F(x=L) = C \cdot L = 0 => C = 0$	13
$\Rightarrow n \neq 0$	-
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	(3) 유 ⁴ 스 뒤 43) <mark>520 PM</mark> 다

So we see that n=0 leads to the solution F(x)=0, which is a trivial solution. Now we begin applying the initial conditions to the solution u(x,t) that we found.



By multiplying both sides by the sin (m*pi*x/L) and integrating, we will find that the results depends on m and n



The only non zero term that is left is when m=n

O = U(x=L,t) = $= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\overline{A}_n \sin \frac{n\pi x}{L} + \overline{B}_n \cos \frac{n\pi x}{L} \right) = \sum_{n=1}^{\infty} \overline{B}_n \sin \frac{n\pi x}{L}.$ F(x=L)=0 F(x=0)=0=Cs $\sum_{n=1}^{\infty} \overline{B_n} \sin \frac{n\pi x}{L} \qquad \int_0^L \sin \frac{n\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} \frac{L}{2} & m=n \\ 0 & L & \frac{1}{2} \end{cases}$ F(x) = C sin s F(x=L)=0=Cor (atb) = coacob = sina sinb $\lim_{L} \max_{dx} = \left(\sum_{n=1}^{\infty} \overline{B}_n \sin \frac{\pi n}{2}\right) \sin \frac{\pi n}{2} dx = \sum_{n=1}^{\infty} \overline{B}_n \int \sin \frac{\pi n}{2} \sin \frac{\pi n}{2} dx = \overline{B}_n$ Type here to searc 0 2 w ×⊞ PE 4

By using the second initial condition you can get a similar result to find the An

 $\overline{B}_{n} = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} dx$ $\frac{\partial u}{\partial t}(x, t=0) = q(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\overline{A}_{n} \cdot \frac{n\pi x}{L} \cos \frac{n\pi x}{L} - \overline{B}_{n} \cdot \frac{n\pi x}{L} \operatorname{Ain} \frac{n\pi x}{L} \right)$ = ZAn ITZ Ain MITX

 $\overline{B}_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$ $\frac{\partial u}{\partial t}(x,t=0) = q(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\overline{A_n} \frac{n\pi x}{L} \cos \frac{n\pi x}{L} - \overline{B_n} \frac{n\pi x}{L} \sin \frac{n\pi x}{L} \right)$ $g(x) = \sum_{n=1}^{\infty} \overline{A}_n \underbrace{n\pi }_{L} Aun \underbrace{n\pi }_{L}$ $\overline{A}_n \underbrace{\operatorname{MTC}}_{L} = \underbrace{\operatorname{R}}_{-}^{L} g(x) \operatorname{Sin} \operatorname{MTX}_{-} dx$ =n $\overline{A}_{n} = \frac{2}{n\pi c} \left(g(x) \sin n\pi x \, dx \right)$ a N 0 w 8

So our final solution is as shown below with the definitions for An and Bn shown above.

Anne contet - Bn ne sin net $u(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[\widehat{A}_n \sin \frac{n\pi c}{L} t + \widehat{B}_n \cos \frac{n\pi c}{L} t \right]$ Ain nII dx dx O Type here to searc

Thus the overall solution is an infinite series which is a function of n.

sin (a+b) = sin a cob ± coa sinb ×∄ PE 0

You can use the trigonometric identities to show that the integral of the sin *cos will be equal to zero irrespective of what m and n are.