

Solution of the vibrating string fixed at both ends using SOV

Diagram of a vibrating string fixed at both ends. The string is represented by a horizontal line of length L , with fixed ends at $x=0$ and $x=L$. The displacement is $u(x,t)$.

Boundary conditions: $u(x=0,t)=0$ and $u(x=L,t)=0$.

Wave equation: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

Initial conditions: $u(x,t=0)=f(x)$ and $\frac{\partial u}{\partial t}(x,t=0)=g(x)$.

Assume $u(x,t)=F(x)G(t)$. Put into the PDE and bring terms that involve x to one side and terms involving t to the other side. c^2 is brought to the x side so that the t side produces a simple equation

Assumed solution: $u(x,t)=F(x)G(t)$

Wave equation: $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$

Initial conditions: $u(x,t=0)=f(x)$ and $\frac{\partial u}{\partial t}(x,t=0)=g(x)$

Separation of variables leads to the equation:

$$c^2 \frac{F''}{F} = \frac{G''}{G} = -\omega^2$$

$$= \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} \quad u(x,t) = u = F(x)G(t)$$

$$u(x,t=0) = f(x) \quad \ddot{G} + \omega^2 G = 0$$

$$\frac{\partial u}{\partial t}(x,t=0) = g(x) \quad G(t) = A \sin \omega t + B \cos \omega t$$

$$\frac{F'' \cdot G}{FG} = \frac{F \ddot{G}}{FG} = -\omega^2$$

$$G(t) = A \sin \omega t + B \cos \omega t$$

$$\frac{\ddot{G}}{G} = -\omega^2$$

$$F'' + \frac{\omega^2}{c^2} F = 0$$

$$F(x) = C \sin \frac{\omega}{c} x + D \cos \frac{\omega}{c} x$$

From the BCs we find:

$$0 = u(x=0, t) = F(x=0)G(t)$$

$$F(x=0) = 0$$

$$0 = u(x=L, t) = F(x=L)G(t)$$

$$F(x=L) = 0$$

Applying these conditions to $F(x)$ at $x=L$, leads to either $C=0$ or $\sin() = 0$. $C=0$ leads to a trivial solution. We are looking for nontrivial solutions.

$$F(x=L) = 0$$

$$F(x=0) = 0 = C \sin \frac{\omega}{c} \cdot 0 + D \cos \frac{\omega}{c} \cdot 0 = D$$

$$F(x) = C \sin \frac{\omega}{c} x$$

$$F(x=L) = 0 = C \sin \frac{\omega}{c} \cdot L$$

$$F(x=0)=0 = C \sin \frac{\omega}{c} \cdot 0 + D \cos \frac{\omega}{c} \cdot 0 = D$$

$$F(x) = C \sin \frac{\omega}{c} x$$

$$F(x=L)=0 = C \sin \frac{\omega}{c} \cdot L \Rightarrow \sin \frac{\omega L}{c} = 0$$

$$\Rightarrow \frac{\omega L}{c} = n\pi$$

Now solve for omega and put back into the equation

$$F(x) = C \sin \frac{n\pi x}{L}$$

$$u_n(x,t) = F \cdot G = \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi c t}{L} + \bar{B}_n \cos \frac{n\pi c t}{L} \right)$$

$$u(x,t) = \sum u_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi c t}{L} + \bar{B}_n \cos \frac{n\pi c t}{L} \right)$$

The question is "What is the starting value of the index n?"

$$u_n(x,t) = F \cdot G = \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi t}{L} + \bar{B}_n \cos \frac{n\pi t}{L} \right)$$

$$u(x,t) = \sum u_n(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi t}{L} + \bar{B}_n \cos \frac{n\pi t}{L} \right)$$

Is $n=0$ a valid index?

$$n=0 \Rightarrow \omega=0 \Rightarrow c^2 \frac{F''}{F} = 0 \Rightarrow F''(x)=0$$

$$F'(x) = C$$

$$F(x) = Cx + D$$

$n=1$

Is $n=0$ a valid index?

$$n=0 \Rightarrow \omega=0 \Rightarrow c^2 \frac{F''}{F} = 0 \Rightarrow F''(x)=0$$

$$F'(x) = C$$

$$F(x) = Cx + D$$

$$F(x=0)=0 = C \cdot 0 + D \Rightarrow F(x) = Cx$$

$$F(x=L) = C \cdot L = 0 \Rightarrow C=0$$

$$\Rightarrow n \neq 0$$

So we see that $n=0$ leads to the solution $F(x)=0$, which is a trivial solution. Now we begin applying the initial conditions to the solution $u(x,t)$ that we found.

Diagram of a string of length L fixed at both ends:

$$u(x=0,t)=0 \quad u(x=L,t)=0$$

General solution:

$$u(x,t) = u = F(x)G(t)$$

Initial condition:

$$u(x,t=0) = f(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi c}{L} t + \bar{B}_n \cos \frac{n\pi c}{L} t \right) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

Equation being pointed to:

$$f(x) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

By multiplying both sides by the $\sin(m\pi x/L)$ and integrating, we will find that the results depends on m and n

Equation being derived:

$$f(x) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

Integral of the product of sines:

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} \frac{L}{2} \\ 0 \end{cases}$$

Integral of $f(x)$ multiplied by $\sin(m\pi x/L)$:

$$\int_0^L f(x) \sin \frac{m\pi x}{L} dx = \int_0^L \left(\sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} dx$$

Trigonometric identity used:

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

The only non zero term that is left is when $m=n$

$$u(x,t) = u = F(x)G(t)$$

$$= \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi c}{L} t + \bar{B}_n \cos \frac{n\pi c}{L} t \right) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

$$\int_0^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx = \begin{cases} \frac{L}{2} & m=n \\ 0 & m \neq n \end{cases}$$

$$\sin(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\int_0^L \sin \frac{m\pi x}{L} dx = \int_0^L \left(\sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L} \right) \sin \frac{m\pi x}{L} dx = \sum_{n=1}^{\infty} \bar{B}_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L} dx = \bar{B}_n \cdot \frac{L}{2}$$

$$0 = u(x=L,t) =$$

$$F(x=L) = 0$$

$$F(x=0) = 0 = C \sin \frac{n\pi x}{L}$$

$$F(x) = C \sin \frac{n\pi x}{L}$$

$$F(x=L) = 0 = C \sin \frac{n\pi L}{L} = C \sin n\pi = 0$$

$$\Rightarrow$$

By using the second initial condition you can get a similar result to find the \bar{A}_n

$$\bar{B}_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\frac{\partial u}{\partial t}(x,t=0) = g(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \frac{n\pi c}{L} \cos \frac{n\pi c}{L} t - \bar{B}_n \frac{n\pi c}{L} \sin \frac{n\pi c}{L} t \right)$$

$$= \sum_{n=1}^{\infty} \bar{A}_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$$

$$\bar{B}_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\frac{\partial u}{\partial t}(x, t=0) = g(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \frac{n\pi c}{L} \cos \frac{n\pi c t}{L} - \bar{B}_n \frac{n\pi c}{L} \sin \frac{n\pi c t}{L} \right)$$

$$g(x) = \sum_{n=1}^{\infty} \bar{A}_n \frac{n\pi c}{L} \sin \frac{n\pi x}{L}$$

$$\bar{A}_n \frac{n\pi c}{L} = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$$\bar{A}_n = \frac{2}{n\pi c} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

So our final solution is as shown below with the definitions for A_n and B_n shown above.

$$\bar{A}_n \frac{n\pi c}{L} \cos \frac{n\pi c t}{L} - \bar{B}_n \frac{n\pi c}{L} \sin \frac{n\pi c t}{L}$$

$$u(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[\bar{A}_n \sin \frac{n\pi c t}{L} + \bar{B}_n \cos \frac{n\pi c t}{L} \right]$$

Thus the overall solution is an infinite series which is a function of n .

$$u(x,t) = u = F(x)G(t)$$

$$= 0 = f(x) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left(\bar{A}_n \sin \frac{n\pi c}{L} t + \bar{B}_n \cos \frac{n\pi c}{L} t \right) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

$$f(x) = \sum_{n=1}^{\infty} \bar{B}_n \sin \frac{n\pi x}{L}$$

$$\int_0^L \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

You can use the trigonometric identities to show that the integral of the $\sin * \cos$ will be equal to zero irrespective of what m and n are.