IF p is not zero or a positive integer

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

Bessel fn of 1st kind $$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!}$$ for $J_{-p}(x)$ replace p by $-p$

if p is zero or a positive integer $p=n$

$$y = C_1 J_n(x) + C_2 Y_n(x)$$ ~ Bessel fn of 2nd Kind

J_n is same as J_p but replace p by n

- $$Y_n(x) = \frac{2}{\pi}\left[\left(\log \frac{x}{2} + \gamma\right) J_n(x) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\left(\frac{x}{2}\right)^{2k-n}}{k!} + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1}\left[\varphi(k) + \varphi(k+n)\right] \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,(k+n)!}\right]$$

- $$\varphi(k) = \sum_{m=1}^{k} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

- γ is Euler's constant $= \lim_{k\to\infty} \left[\varphi(k) - \log k\right] = 0.5772157\dots$

(**) • note that at $x=0$ $J_n(x) =$ bdd & $Y_n(x)$ is ∞ ; $J_{-p}(x)$ is ∞ at x:

- FOR OUR PROBLEM since ω is unknown we don't know which form to use
- Secondly our problem must satisfy the condition that $w(r,\theta)=0$ at $r=a$

(**) • IF our problem involves annular membrane – [redacted] $C_2 \neq 0$

- $\Rightarrow$ for circular problems to have unique solutions $\Rightarrow w(r,\Theta) = w(r,\theta+2\pi$

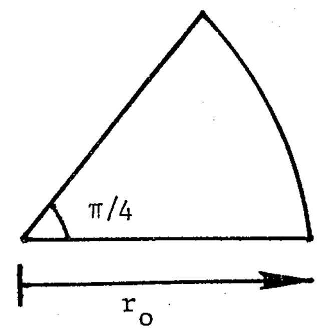
  $\Rightarrow$ <u>k must be integer</u> $\omega = n$

- $\Rightarrow$ $R_n(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$
- also since $w(r=a,\theta) = R_n(\lambda a)\,\Theta(\theta) = 0$ for all Θ $\Rightarrow R_n(\lambda a) = 0$

Exercises.

3.1. Find the eigenmodes and frequencies for an annular membrance with inner radius r_i and outer radius r_0. ($u = 0$ at r_i and r_0). For the special case $r_0/r_i = 2.5$, give the lowest four values of $\omega r_0/a$. (HINT: HMF Table 9.7)

3.2. Consider the pie-shaped membrane shown in the sketch. Calculate the eigenmodes and eigen-frequencies, in non-dimensional form.



3.3. Problems 3.3a - 3.3d all deal with acoustic waves in a cylindrical enclosure. In each case the governing PDE for the pressure is

$$c^2\nabla^2 p - p_{tt} = 0$$

where, in polar-cylindrical coordinates

$$\nabla^2 p = p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} + p_{zz}$$

The boundary condition at the solid walls (at $z = 0$, at $z = L$, and at $r = r_0$) is that the derivative of the pressure field normal (perpendicular) to the wall must vanish, i.e.

$$p_z = 0 \quad \text{at } z = 0,L$$

$$p_r = 0 \quad \text{at } r = r_0$$

a. Find the eigenmodes and frequencies for axial modes where $p = p(z,t)$.

b. Find the eigenmodes and frequencies for radial modes where $p = p(r,t)$

c. Show that there are no modes where

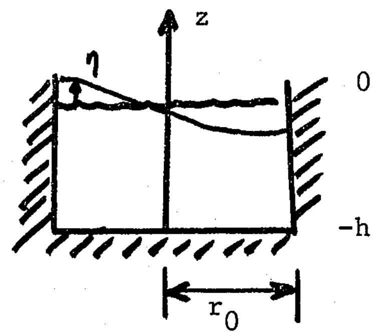
$p = p(\theta,t)$.

d. Find the eigenmodes and frequencies for the general case where

$p = p(r,\theta,z,t)$

3.4 In the analysis of seismic loading on nuclear reactors, oil storage tanks and other large fluid containers, one needs to know the natural frequencies of sloshing motions. This problem will acquaint you with the typical analysis.

Consider a circular geometry, with vertical walls at $r = r_0$, and the bottom at $z = -h$. The equations governing the sloshing are



(1) $\nabla^2\phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz} = 0$

(2) $\frac{\partial\Phi}{\partial t} + g\eta = 0$ on $z = 0$

(3) $\frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} = 0$ on $z = 0$

(4) $\frac{\partial\Phi}{\partial r} = 0$ at $r = r_0$

(5) $\frac{\partial\Phi}{\partial z} = 0$ at $z = -h$

$\Phi(r,\theta,z,t)$ is the velocity potential; the fluid velocity is the gradient of Φ ; $\eta(r,\theta,t)$ is the surface displacement. g is the acceleration of gravity, $g = 9.8\text{m/sec}^2$. Equation (1) is the continuity equation for irrotational flow, (2) is the Bernouli equation applied on the free surface, (3) is a kinematic condition relating surface motion to velocity, and (4) and (5) are boundary conditions that the flow cannot penetrate the wall. Students with expertise in fluid mechanics should derive (1) - (5).

(a) Using the method of separation of variables, derive an expression for the natural frequencies. Express them non-dimensionally as

$$(6) \quad \Omega^2 \equiv \omega^2 r_0/g = f(h/r_0)$$

Express the solution for the surface deflection $\eta(r,\theta,t)$ in the non-dimensional form

$$(7) \quad \frac{\eta}{\eta_a} = F\left(\frac{r}{r_0}\right) G(\omega_{mn}t)\ H(m\theta)$$

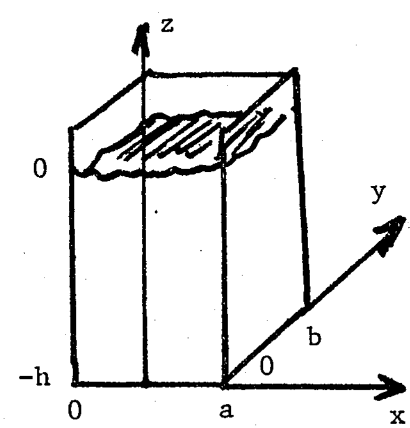
where η_a is the maximum deflection at $r = r_0$ (the sloshing amplitude)

(b) For the special case $h/r_0 = \infty$, calculate the values of Ω^2 for the modes having the five lowest natural frequencies, and sketch the node-lines in the surface displacement $\eta(r,\theta,t)$ for each of these modes. Check-point: the fundamental has $\Omega^2 = 1.841$. HINT: See HMF 9.1.1, 9.1.11, Table 9.5.

(c) Consider a large oil tank 30m in diameter, filled to a depth of 10m. Calculate the lowest natural frequency of vibration (hz).

(d) Find a coffee cup, jar, or other circular container. Fill with water to a selected depth, and manually excite the first mode by moving the container sideways. Compare the "measured" frequency (hz) with the value predicted by the analysis. Visualize the radial node-lines of part (d) in your cup by banging it (gently!) on the table.

3.5 Consider the sloshing of a fluid in a rectangular tank. The motion is described by the equations of Problem 3.4, except that



$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

and (4) is replaced by

$$\phi_x = 0 \quad \text{at} \quad x = 0,a$$

$$\phi_y = 0 \quad \text{at} \quad y = 0,b$$

(a) Calculate the natural frequencies of fluid sloshing in the tank. Show that they are given by

$$\omega^2_{nm} = gk \tanh(kh) \qquad k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

(b) Give the expression for $\eta_{nm}(x,y,t)$, apart from an undetermined phase and amplitude.

(c) Find a bathtub, wash-basin, or kitchen sink, fill with water to a reasonable depth. Manually excite the fundamental sloshing frequency and compare the theoretical value with an "eyeball" experimental measurement (hz).

BESSEL FUNCTIONS—MISCELLANEOUS ZEROS Table 9.7

s^{th} Zero of $J_0(x)Y_0(\lambda x)-Y_0(x)J_0(\lambda x)$

	$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$	
*	0.80	12.55847 031	25.12877	37.69646	50.26349	62.83026	1	
	0.60	4.69706 410	9.41690	14.13189	18.84558	23.55876	2	
	0.40	2.07322 886	4.17730	6.27537	8.37167	10.46723	3	
	0.20	0.76319 127	1.55710	2.34641	3.13403	3.92084	5	
	0.10	0.33139 387	0.68576	1.03774	1.38864	1.73896	10	
	0.08	0.25732 649	0.53485	0.81055	1.08536	1.35969	13	*
	0.06	0.18699 458	0.39079	0.59334	0.79522	0.99673	17	
	0.04	0.12038 637	0.25340	0.38570	0.51759	0.64923	25	
	0.02	0.05768 450	0.12272	0.18751	0.25214	0.31666	50	
	0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞	

s^{th} Zero of $J_1(x)Y_1(\lambda x)-Y_1(x)J_1(\lambda x)$

	$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$	
*	0.80	12.59004 151	25.14465	37.70706	50.27145	62.83662	1	
	0.60	4.75805 426	9.44837	14.15300	18.86146	23.57148	2	
	0.40	2.15647 249	4.22309	6.30658	8.39528	10.48619	3	
	0.20	0.84714 961	1.61108	2.38532	3.16421	3.94541	5	
	0.10	0.39409 416	0.73306	1.07483	1.41886	1.76433	10	
	0.08	0.31223 576	0.57816	0.84552	1.11441	1.38440	13	*
	0.06	0.23235 256	0.42843	0.62483	0.82207	1.02001	17	
	0.04	0.15400 729	0.28296	0.41157	0.54044	0.66961	25	
	0.02	0.07672 788	0.14062	0.20409	0.26752	0.33097	50	
	0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞	

s^{th} Zero of $J_1(x)Y_0(\lambda x)-Y_1(x)J_0(\lambda x)$

	$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$	
*	0.80	6.56973 310	18.94971	31.47626	44.02544	56.58224	1	
*	0.60	2.60328 138	7.16213	11.83783	16.53413	21.23751	2	
	0.40	1.24266 626	3.22655	5.28885	7.36856	9.45462	3	
	0.20	0.51472 663	1.24657	2.00959	2.78326	3.56157	5	
	0.10	0.24481 004	0.57258	0.90956	1.25099	1.59489	10	
	0.08	0.19461 772	0.45251	0.71635	0.98327	1.25203	13	*
	0.06	0.14523 798	0.33597	0.53005	0.72594	0.92301	17	
	0.04	0.09647 602	0.22226	0.34957	0.47768	0.60634	25	
	0.02	0.04813 209	0.11059	0.17353	0.23666	0.29991	50	
	0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞	

$\langle\lambda\rangle$ = nearest integer to λ.

Compiled from British Association for the Advancement of Science, Bessel functions, Part I. Functions of orders zero and unity, Mathematical Tables, vol. VI (Cambridge Univ. Press, Cambridge, England, 1950) (with permission).

*See page II.

Table 9.7 **BESSEL FUNCTIONS—MISCELLANEOUS ZEROS**

s^{th} Zero of $xJ_1(x)-\lambda J_0(x)$

$\lambda \backslash s$	1	2	3	4	5
0.00	0.0000	3.8317	7.0156	10.1735	13.3237
0.02	0.1995	3.8369	7.0184	10.1754	13.3252
0.04	0.2814	3.8421	7.0213	10.1774	13.3267
0.06	0.3438	3.8473	7.0241	10.1794	13.3282
0.08	0.3960	3.8525	7.0270	10.1813	13.3297
0.10	0.4417	3.8577	7.0298	10.1833	13.3312
0.20	0.6170	3.8835	7.0440	10.1931	13.3387
0.40	0.8516	3.9344	7.0723	10.2127	13.3537
0.60	1.0184	3.9841	7.1004	10.2322	13.3686
0.80	1.1490	4.0325	7.1282	10.2516	13.3835
1.00	1.2558	4.0795	7.1558	10.2710	13.3984

$\lambda^{-1} \backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.2558	4.0795	7.1558	10.2710	13.3984	1
0.80	1.3659	4.1361	7.1898	10.2950	13.4169	1
0.60	1.5095	4.2249	7.2453	10.3346	13.4476	2
0.40	1.7060	4.3818	7.3508	10.4118	13.5079	3
0.20	1.9898	4.7131	7.6177	10.6223	13.6786	5
0.10	2.1795	5.0332	7.9569	10.9363	13.9580	10
0.08	2.2218	5.1172	8.0624	11.0477	14.0666	13
0.06	2.2656	5.2085	8.1852	11.1864	14.2100	17
0.04	2.3108	5.3068	8.3262	11.3575	14.3996	25
0.02	2.3572	5.4112	8.4840	11.5621	14.6433	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

s^{th} Zero of $J_1(x)-\lambda xJ_0(x)$

$\lambda \backslash s$	1	2	3	4	5
0.5	0.0000	5.1356	8.4172	11.6198	14.7960
0.6	1.1231	5.2008	8.4569	11.6486	14.8185
0.7	1.4417	5.2476	8.4853	11.6691	14.8346
0.8	1.6275	5.2826	8.5066	11.6845	14.8467
0.9	1.7517	5.3098	8.5231	11.6964	14.8561
1.0	1.8412	5.3314	8.5363	11.7060	14.8636

$\lambda^{-1} \backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.8412	5.3314	8.5363	11.7060	14.8636	1
0.80	1.9844	5.3702	8.5600	11.7232	14.8771	1
0.60	2.1092	5.4085	8.5836	11.7404	14.8906	2
0.40	2.2192	5.4463	8.6072	11.7575	14.9041	3
0.20	2.3171	5.4835	8.6305	11.7745	14.9175	5
0.10	2.3621	5.5019	8.6421	11.7830	14.9242	10
0.08	2.3709	5.5055	8.6445	11.7847	14.9256	13
0.06	2.3795	5.5092	8.6468	11.7864	14.9269	17
0.04	2.3880	5.5128	8.6491	11.7881	14.9282	25
0.02	2.3965	5.5165	8.6514	11.7898	14.9296	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

$\langle\lambda\rangle$ = nearest integer to λ.

Compiled from H. S. Carslaw and J. C. Jaeger, Conduction of heat in solids (Oxford Univ. Press, London, England, 1947) and British Association for the Advancement of Science, Bessel functions, Part I. Functions of orders zero and unity, Mathematical Tables, vol. VI (Cambridge Univ. Press, Cambridge, England, 1950)(with permission).