(b) Assume a particular solution of the form

$$y_p(x) = u(x)\cos x + v(x)\sin x.$$

(Note that the constants or parameters $c_1$ and $c_2$ have been replaced by functions $u(x)$ and $v(x)$. Our objective will be to obtain two equations in $u'(x)$ and $v'(x)$ that can then be solved simultaneously.) Differentiate to obtain

$$y_p'(x) = -u(x)\sin x + v(x)\cos x$$

with

$$u'(x)\cos x + v'(x)\sin x = 0.$$

Observe that this last condition simplifies $y_p'(x)$, $y_p''(x)$ and provides a second equation in $u'(x)$ and $v'(x)$.

(c) Differentiate $y_p'(x)$ in part (b) and substitute into the given differential equation to obtain

$$-u'(x)\sin x + v'(x)\cos x = \tan x.$$

(d) Solve the system

$$\begin{aligned} -u'(x)\sin x + v'(x)\cos x &= \tan x \\ u'(x)\cos x + v'(x)\sin x &= 0 \end{aligned}$$

for $u'(x)$ and $v'(x)$ by Cramer's rule or by elimination.

(e) Integrate $u'(x)$ and $v'(x)$ to find $u(x)$ and $v(x)$.

(f) Find $y_p(x)$ and thus obtain the general solution. Note that success in using the method of variation of parameters is contingent on being able to obtain $u(x)$ and $v(x)$ from $u'(x)$ and $v'(x)$.

**14.** Use the method of Exercise 13 to obtain the general solutions of each of the following equations:

(a) $y'' - y' = \sec^2 x - \tan x$
(b) $y'' - 2y' + y = \exp(x)/(1 - x)^2$
(c) $y'' + y = \sec x \tan x$
(d) $y'' + y = \sec x$
(e) $y'' - 2y' + y = e^x/x^2$
(f) $y'' + 4y = \cot 2x$

**15.** Solve the initial-value problem

$$y'' - 2y' + y = e^x/(1 - x)^2, \qquad y(0) = 2, \qquad y'(0) = 6.$$

**16.** Verify that $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of

$$x^3y'' + x^2y' - xy = 0.$$

Then use this information and the method of variation of parameters to find the general solution of

$$x^3y'' + x^2y' - xy = x/(1 + x).$$

**17.** (a) Solve the equation

$$y'' - y = x\sin x$$

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**18.** Consider

$$y'' + ay' + by = f(x),$$

where $a$ and $b$ are constants with $b \neq 0$ and $f(x)$ is a polynomial of degree $n$. Sho that this equation always has a solution that is a polynomial of degree

**19.** In Exercise 18, show that the solution is a polynomial of degree $n + 1$ in the ca where $b = 0$.

**20.** If $u(x)$, $v(x)$, and $w(x)$ are differentiable functions of $x$, use the formula f differentiating a product,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

to find $d(uvw)/dx$.

**21.** Show that reconciling the coefficients of like terms in the method of undetermine coefficients depends on the forming of a linearly independent set by the functio involved. (*Hint*: Recall that if $f_i(x)$, $i = 1, 2, \ldots, n$, are $n$ linearly independe functions over the reals, then

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

holds only if each $c_i = 0$.)

## 1.3 CAUCHY-EULER EQUATIONS

In the two preceding sections we discussed second-order linear ordina differential equations with *constant* coefficients. While equations of this ty will occur frequently throughout the remaining chapters, we will also ha occasion to solve other linear differential equations, that is, equations of t form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x). \qquad \text{(1.3-}$$

One type of linear differential equation with variable coefficients th can be reduced to a form already considered is the **Cauchy–Euler equatio** which has the normal form

$$x^2y'' + axy' + by = f(x), \qquad x > 0, \qquad \text{(1.3-}$$

where $a$ and $b$ are constants. This equation is also called a *Cauchy** equatic an *Euler*† equation, and an *equidimensional* equation. The last term com from the fact that the physical dimension of $x$ in the left-hand member of E (1.3-2) is immaterial, since replacing $x$ by $cx$, where $c$ is a nonzero constai leaves the dimension of the left-hand member unchanged. We will meet a form of this equation later in our study of boundary-value problems having circular symmetry. We assume throughout this section that $x \neq 0$. For the most part we assume that $x > 0$, although we also deal with the case $x < 0$ later. We begin with an example to illustrate the method of solution.

**EXAMPLE 1.3–1** Find the complementary solution of the equation

$$x^2y'' + 2xy' - 2y = x^2 \exp(-x), \qquad x > 0.$$

**Solution** This is a Cauchy–Euler equation, and we make the following substitutions in the reduced equation

$$y_c(x) = x^m, \qquad y_c' = mx^{m-1}, \qquad y_c'' = m(m-1)x^{m-2},$$

so the homogeneous equation becomes

$$[m(m-1) + 2m - 2]x^m = 0.$$

Because of the restriction $x \neq 0$, we must have

$$m^2 + m - 2 = 0,$$

which has roots $m_1 = -2$ and $m_2 = 1$. Thus

$$y_c(x) = c_1x^{-2} + c_2x. \quad ■$$

We remark that the substitution, $y_c(x) = x^m$, did not come from thin air. It was dictated by the *form* of the left-hand member of the differential equation, which in turn ensured that each term of the equation would contain the common factor $x^m$.

It should be pointed out also that if we were interested in obtaining the *general* solution of the equation in Example 1.3–1, we could use the complementary solution above and the method of **variation of parameters.** (See Exercise 13 in Section 1.2.)

**EXAMPLE 1.3–2** Obtain the complementary solution of

$$x^2y'' + 3xy' + y = x^3, \qquad x > 0.$$

**Solution** This time the substitution $y_c = x^m$ leads to

$$m^2 + 2m + 1 = 0,$$

which has a double root $m = -1$. Hence we have *one* solution of the homogeneous equation, namely,

$$y_1(x) = x^{-1}.$$

One might "guess" that a second linearly independent solution could be obtained by multiplying $y_1(x)$ by $x$. This procedure, however, is limited to the case of equations with *constant* coefficients and is thus not applicable here. It is



easy to check that $y = 1$ is *not* a solution. In this case we can use the methc **reduction of order** to find a second solution.

We set $y_2(x) = u(x)/x$ and compute two derivatives. Thus

$$y_2'(x) = \frac{u'x - u}{x^2},$$

$$y_2''(x) = \frac{x^2(xu'' - 2u') + 2ux}{x^4},$$

and substitution into the homogeneous equation results in

$$xu'' + u' = 0,$$

which can be solved* by setting $u' = v$ and **separating the variables.** The:

$$v = \frac{du}{dx} = \frac{c_2}{x},$$

and†

$$u = c_2 \log x.$$

Hence

$$y_2(x) = \frac{c_2}{x} \log x,$$

and the complementary solution is

$$y_c(x) = \frac{c_1}{x} + \frac{c_2}{x} \log x.$$

We shall see later that the function $\log x$ occurs in the case of repea roots. ■

**EXAMPLE 1.3–3** Find the complementary solution of

$$x^2y'' + xy' + y = \cos x, \qquad x > 0.$$

**Solution** In this example we have, after substituting $y_c(x) = x^m$,

$$m^2 + 1 = 0$$

so that the solutions are $x^i$ and $x^{-i}$. Hence the complementary solution is

$$y_c(x) = C_1x^i + C_2x^{-i}. \qquad (1.3$$

A more useful form can be obtained, however, by replacing $C_1$ and $C_2$ $(c_1 - ic_2)/2$ and $(c_1 + ic_2)/2$, respectively, and noting that

$$x^i = \exp(i \log x) = \cos(\log x) + i \sin(\log x).$$

*An alternative method is to note that $xu'' + u' = d(xu') = 0$, leading to $xu' = c_2$.
†We will consistently use $\log x$ for the *natural logarithm* of $x$.

With these changes the complementary solution (1.3–3) can be written

$$y_c(x) = c_1 \cos(\log x) + c_2 \sin(\log x). \quad \blacksquare$$

We shall summarize the various cases that occur when solving the *homogeneous* Cauchy–Euler equation

$$x^2y'' + axy' + by = 0. \tag{1.3–4}$$

Substitution of $y_c(x) = x^m$ and its derivatives into Eq. (1.3–4) leads to the equation

$$m(m-1) + am + b = 0$$

or

$$m^2 + (a-1)m + b = 0. \tag{1.3–5}$$

This is called the **auxiliary equation** of the homogeneous Cauchy–Euler equation (1.3–4).

**Case I.** $(a-1)^2 - 4b > 0$. The roots of Eq. (1.3–5) are real and unequal, say, $m_1$ and $m_2$. Then

$$y_c(x) = c_1x^{m_1} + c_2x^{m_2}, \tag{1.3–6}$$

and since the Wronskian

$$\begin{vmatrix} x^{m_1} & x^{m_2} \\ m_1x^{m_1-1} & m_2x^{m_2-1} \end{vmatrix} = (m_2 - m_1)x^{m_1+m_2-1} \neq 0,$$

showing that $x^{m_1}$ and $x^{m_2}$ are linearly independent.*

**Case II.** $(a-1)^2 - 4b = 0$. The roots of Eq. (1.3–5) are real and equal, say, $m_1 = m_2 = m$. Then

$$y_1(x) = x^m$$

is one solution of Eq. (1.3–4). A second solution can be found by the method of reduction of order. Let

$$y_2(x) = x^m u(x)$$

be a second solution, differentiate twice, and substitute into Eq. (1.3–4). Then

$$u[m(m-1) + am + b]x^m + u'(2m+a)x^{m+1} + u''x^{m+2} = 0.$$

* ... $x > 0$ is essential here.



But the coefficient of $u$ vanishes because $x^m$ is a solution of Eq. (1.3–4) and $2m + a = 1$ from Eq. (1.3–5). Thus

$$xu'' + u' = 0,$$

which is satisfied by $u = \log x$ and

$$y_2(x) = x^m \log x.$$

In this case the complementary solution is

$$y_c(x) = x^m(c_1 + c_2 \log x). \quad ($$

**Case III.** $(a-1)^2 - 4b < 0$. The roots of Eq. (1.3–5) are complex conju-gates, say, $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$. Then two linearly indep[endent] solutions of the homogeneous equation are

$$y_1(x) = x^{\alpha+\beta i} = x^\alpha x^{\beta i} = x^\alpha \exp(i\beta \log x)$$

and

$$y_2(x) = x^{\alpha-\beta i} = x^\alpha x^{-\beta i} = x^\alpha \exp(-i\beta \log x).$$

Using Euler's formula* to transform the exponential gives us

$$y_1(x) = x^\alpha[\cos(\beta \log x) + i \sin(\beta \log x)]$$

and

$$y_2(x) = x^\alpha[\cos(\beta \log x) - i \sin(\beta \log x)].$$

Hence the complementary solution becomes

$$y_c(x) = x^\alpha[c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)].$$

If the general solution to Eq. (1.3–2) is required, it is necessary t[o] ... particular solution to the appropriate complementary solution. A pa[rticular] solution can be found by the method of variation of parameters, al[though] difficulties may be encountered, as was mentioned in Exercise 13 of S[ection] 1.2. Note that the method of undetermined coefficients is not applicabl[e] since the Cauchy–Euler differential equation does not have c[onstant] coefficients.

There is an alternative method for solving Eq. (1.3–4). Since $x >$ ... can make the substitution

$$u = \log x.$$

This leads to $x = \exp u$ and, using the chain rule, to

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du}.$$

*$\exp(i\theta) = \cos\theta + i \sin\theta$.

We also have

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right)$$
$$= \frac{1}{x}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} - \frac{1}{x^2}\frac{dy}{du}$$
$$= \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right).$$

This method has the advantage that Eq. (1.3–4) is transformed into

$$\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = f(\exp u).$$

Thus the differential equation has constant coefficients, and the methods of Section 1.1 are available for finding the complementary solution of the Cauchy–Euler equation. In fact, if $f(\exp u)$ has the proper form, then the method of undetermined coefficients may lead to the general solution of the nonhomogeneous equation quite easily.

We have considered exclusively the case where $x > 0$. If solutions are desired for values of $x$ satisfying $x < 0$, then $x$ may be replaced by $-x$ in the differential equation and in the solution.

## Key Words and Phrases

| | |
|---|---|
| Cauchy-Euler equation | separating the variables |
| variation of parameters | auxiliary equation |
| reduction of order | |

## 1.3 Exercises

In the following exercises, assume that the independent variable is positive unless otherwise stated.

1. Use the substitution $u = \log x$ to solve each of the following equations.
   (a) $x^2y'' + 2xy' - 2y = 0$ (Compare Example 1.3–1)
   (b) $x^2y'' + 3xy' + y = 0$ (Compare Example 1.3–2)
   (c) $x^2y'' + xy' + y = 0$ (Compare Example 1.3–3)
2. Obtain the general solution of the equation
$$x^2y'' + 3xy' + y = x^3.$$
(*Hint*: Use the substitution of Exercise 1 and the result of Example 1.3–2.)



3. Use Euler's formula
$$\exp(i\theta) = \cos\theta + i\sin\theta$$
to fill in the details in Case III.

•• 4. Solve the initial-value problem
$$x^2y'' - 2y = 0, \quad y(1) = 6, \quad y'(1) = 3.$$
5. Find the general solution of
$$x^2y'' + 5xy' - 5y = 0.$$
6. Find the general solution of
$$t^2y'' + 5ty' + 5y = 0.$$
7. Find the general solution of
$$r^2u'' + 3ru' + u = 0.$$
8. Obtain the general solution for
$$x^2y'' + xy' - 9y = x^2 - 2x.$$
9. Find the general solution of
$$x^2y'' + xy' + 4y = \log x.$$
10. Solve the initial-value problem
$$x^2y'' - xy' + 2y = 5 - 4x, \quad y(1) = 0, \quad y'(1) = 0.$$
11. Solve each of the following initial-value problems.
   (a) $x^2y'' + xy' = 0, \quad y(1) = 1, \quad y'(1) = 2$
   (b) $x^2y'' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1$

••• 12. Use the method of reduction of order to find a second solution for each c following differential equations, given one solution as shown.
   (a) $x^2y'' - xy' + y = 0, \quad y_1(x) = x$
   (b) $xy'' + 3y' = 0, \quad y_1(x) = 2$
   (c) $x^2y'' + xy' - 4y = 0, \quad y_1(x) = x^2$
   (d) $x^2y'' - xy' + y = 0, \quad y_1(x) = x\log x^2$
13. Show that the products $xy'$ and $x^2y''$ remain unchanged if $x$ is replaced b where $c$ is a nonzero constant.
14. Show that the substitution $x = \exp u$ transforms the equation
$$x^2y'' + axy' + by = 0,$$
where $a$ and $b$ are constants, into
$$\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = 0.$$
15. Solve each of the following initial-value problems.
   (a) $4x^2y'' - 4xy' + 3y = 0, \quad y(1) = 0, \quad y'(1) = 1$
   (b) $x^2y'' + 5xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3$

16. Obtain the general solution of the equation

$$x^2y'' + axy' = 0,$$

where $a$ is a constant.

## 1.4 INFINITE SERIES

Since we will need to solve second-order linear equations with variable coefficients that are not of Cauchy–Euler type (Eq. 1.3–2), we must explore other methods of solution. One powerful method is the power series method. In using this method we assume that the solution to a given differential equation can be expressed as a power series. Inasmuch as we will need certain facts about infinite series and their convergence, we digress in this section to review some aspects of these.

Each of the following is an example of a **series of constants:**

$$1 + 2 + 3 + 4 + \cdots + n + \cdots, \qquad \textbf{(1.4–1)}$$

$$1 - 1 + 1 - 1 + 1 - + \cdots + (-1)^{n+1} + \cdots, \qquad \textbf{(1.4–2)}$$

$$0 + 0 + 0 + \cdots + 0 + \cdots, \qquad \textbf{(1.4–3)}$$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \qquad \textbf{(1.4–4)}$$

$$a(1 + r + r^2 + \cdots + r^n + \cdots), \qquad \textbf{(1.4–5)}$$

$$1 - \frac{1}{3} + \frac{1}{5} - + \cdots + \frac{(-1)^{n+1}}{2n-1} + \cdots. \qquad \textbf{(1.4–6)}$$

The series in (1.4–1) is **divergent** because the **partial sums**

$$S_1 = 1, \quad S_2 = 1 + 2, \quad S_3 = 1 + 2 + 3, \quad S_4 = 1 + 2 + 3 + 4, \quad \ldots$$

form a **sequence**

$$\{S_1, S_2, S_3, \ldots\} = \{1, 3, 6, 10 \ldots\},$$

which has *no* **limit point.*** On the other hand, the series in (1.4–2) is divergent because its sequence of partial sums has *two* limit points, +1 and 0. The series in (1.4–3) is a trivial example of a **convergent** series, since its sequence of partial sums has a unique limit point, namely zero.

*A point is called a *limit point* of a sequence if an infinite number of terms in the sequence are within a distance of $\epsilon$ of the point, where $\epsilon$ is an arbitrarily small positive number. A limit point need not be unique and need not be an element of the sequence. For example, 1 is the unique limit point of the sequence

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots\right\}$$



By a more sophisticated test (the **integral test**) it can be shown that t series of (1.4–4) is convergent for $p > 1$ and divergent for $p \leq 1$. Wh $p = 1$, the series is called a **harmonic series.** The series of (1.4–5) is **geometric series** with first term $a$ and **common ratio** $r$. It can be shown (by t ratio test) that (1.4–5) converges if $|r| < 1$ and diverges if $|r| \geq 1$ a $a \neq 0$. The sum of (1.4–5) can be written in **closed form** as

$$a\sum_{n=0}^{\infty} r^n = \frac{a}{1-r}, \qquad |r| < 1.$$

Finally, the series of (1.4–6) is an example of an **alternating series** that can proved (using a theorem of Leibniz*) to be convergent because the followi two conditions hold:

1. The absolute value of each term is less than or equal to the absolute va of its predecessor.
2. The limiting value of the $n$th term is zero as $n \to \infty$.

We remark that it is one thing to determine whether a given series c verges, but quite another to determine what it converges to. It is not obvio for example, that the sum of the series in (1.4–6) is $\pi/4$, although we will tain this result and some others in the exercises for Section 4.3. (See Exerci 23 and 25 of that section.)

Of greater interest to us than a series of constants will be **power series** the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots. \qquad \textbf{(1.4}$$

Such a series is called a power series in $x - x_0$. A power series *always* c verges. For example, (1.4–7) converges for $x = x_0$, but we will be interested convergence on an *interval* such as $(x_0 - R, x_0 + R)$. We call $R$ ($R > 0$) **radius of convergence** of the power series and determine its value by using **ratio test**† as shown in the next example.

**EXAMPLE 1.4–1** Find the radius of convergence of the series

$$(x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

*Gottfried Wilhelm Leibniz (1646–1716), the co-inventor (with Sir Isaac Newton) of calculus, who proved the theorem in 1705.

†Also called D'Alembert's ratio test after Jean-le-Rond D'Alembert (1717–1783), a Fre mathematician who made important contributions in analytical mechanics.

**Solution** It will be convenient to write the series using summation notation,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

According to the ratio test, a series converges whenever

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1,$$

where $u_n$ represents the $n$th term of the series. In the present case,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}(x-1)^n} \right|$$
$$= |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1| < 1.$$

Hence $-1 < x - 1 < 1$ or $0 < x < 2$, showing that the *radius* of convergence is 1. In many problems it is necessary to examine the endpoints of the **interval of convergence** as well, and this must be done separately. It can be shown (Exercise 3) that the interval of convergence here is $0 < x \le 2$. ■

Obtaining a power series representation of a function is an important mathematical technique that has many applications. Recall that the **Maclaurin*** **series** for a function $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots. \quad (1.4\text{–}8)$$

Following are some familiar Maclaurin series expansions:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \quad (1.4\text{–}9)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \cdots, \quad (1.4\text{–}10)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \cdots, \quad (1.4\text{–}11)$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots. \quad (1.4\text{–}12)$$

*After Colin Maclaurin (1698–1746), a Scottish mathematician.



The Maclaurin series for exp $x$, sin $x$, and cos $x$ converge for all fin values of $x$, while the series (1.4–12) has an interval of convergence given $-1 < x \le 1$. All four of the series are particular cases of Eq. (1.4–8). Usi summation notation, we can write*

$$\exp x = \sum_{n=0} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=1} \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

$$\cos x = \sum_{n=1} \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!}$$

$$\log(1+x) = \sum_{n=1} \frac{(-1)^{n+1}x^n}{n}$$

where we have used the convention $0! = 1$. Observe that $n$ is a **dummy ind** and may be replaced by something else if this is desirable. For example, repla ing $n$ by $m - 1$ in Eq. (1.4–12) produces

$$\log(1+x) = \sum_{m=0} \frac{(-1)^m x^{m-1}}{m-1}.$$

We shall make use of this flexibility of the dummy index in the next sectio

A Maclaurin series representation of a function can be thought of as approximation of the function in the neighborhood of $x = 0$ as shown in F 1.4–1. If a series expansion about some point other than $x = 0$ is require then we can use a **Taylor† series**

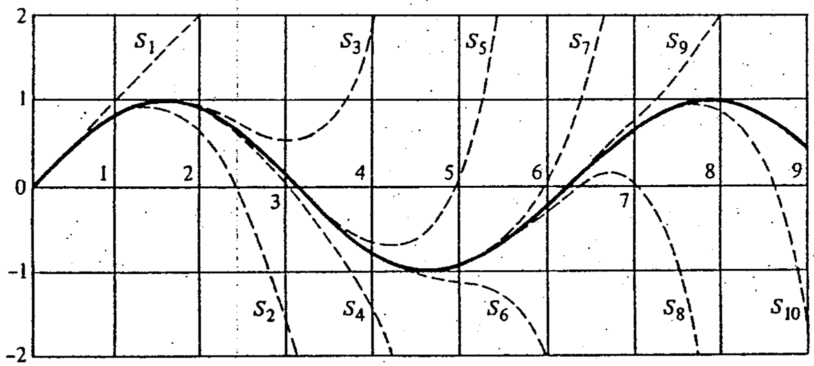
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots. \quad (1.4\text{–}$$

Note that Maclaurin's series is a special case of Taylor's series, the case wh $a = 0$.

It would appear from Eqs. (1.4–8) and (1.4–13) that any function tl has an infinite number of derivatives that are defined at $x = a$ can be rep sented by a Taylor series in the neighborhood of $x = a$. This is not entir

*We shall omit the upper values of $n$ on summations henceforth if they are $n = \infty$.

†After Brook Taylor (1685–1731), a British mathematician who discovered it in 17 Historically, Taylor's series predated Maclaurin's series.




$$S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}, \quad S_n(x) \to \sin x \text{ as } n \to \infty$$

**Figure 1.4-1**
**Approximations of partial sums to sin *x*. (From H. M. Kammerer, "Sine and Cosine Approximation Curves," *MAA Monthly* 43, p. 293.)**

true. The conjecture in the last statement represents an oversimplification of the facts.* We will discuss this and related topics further in the next section.

## Key Words and Phrases

| | |
|---|---|
| series of constants | closed form |
| divergent | alternating series |
| partial sums | power series |
| sequence | radius of convergence |
| limit point | ratio test |
| convergent | interval of convergence |
| integral test | Maclaurin series |
| harmonic series | dummy index |
| geometric series | Taylor series |
| common ratio | |

## 1.4 Exercises

• 1. Write the sequence of partial sums for each of the following series.
(a) $1 - 1 + 1 - 1 + - \cdots + (-1)^{n+1} + \cdots$
(b) $0 + 0 + 0 + 0 + \cdots$

*A small caveat is necessary here, since there are some (pathological) functions that have



(c) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$
(d) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

2. What is the sum of the series in Exercise 1(d)?

3. Show that the series of Example 1.4-1 converges when $x = 2$ and diverges wh $x = 0$.

4. Use the ratio test to show that the Maclaurin series for exp $x$, sin $x$, and cos converge for all $x$.

5. Show that the interval of convergence of the Maclaurin series for log $(1 + x)$ $-1 < x \leq 1$. See Eq. (1.4-12).

6. Verify that each of the following summations is correct.

(a) $\sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

(b) $\cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$

(c) $\exp(-x) = \sum_{n=1} \frac{(-1)^{n-1} x^{n-1}}{(n-1)!}$

(d) $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0} \frac{x^{2n}}{(2n)!}$

(e) $\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=1} \frac{x^{2n-1}}{(2n-1)!}$

*7. Use Maclaurin's series to compute $\sin \frac{\pi}{4}$ and $\cos \frac{\pi}{4}$ to four decimals. (*Hint*: U the fact that the error has the same sign as the first neglected term but has a small absolute value.)

•• 8. Consider the sequence

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \ldots\right\}.$$

(a) Write the $n$th term of the sequence.
(b) Show that 1 is the limit point of the sequence.

9. Apply the ratio test to the series (1.4-5) to show that the series converges f $|r| < 1$.

10. (a) Identify the series

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots.$$

(b) Find the sum of the series.

*Calculator problem.

11. Find the radius of convergence of each of the following power series.

(a) $(x - 1) + \frac{(x-1)^3}{3} + \frac{(x-1)^5}{5} + \cdots$

(b) $1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots$

(c) $1 + \frac{(x+3)}{2} + \frac{(x+3)^2}{3} + \frac{(x+3)^3}{4} + \cdots$

(d) $x + \frac{2!x^2}{2^2} + \frac{3!x^3}{3^3} + \frac{4!x^4}{4^4} + \cdots$ (*Hint*: Use the limit definition of $e$.)

(e) $1 + \frac{(x+2)}{3} + \frac{(x+2)^2}{2 \cdot 3^2} + \frac{(x+2)^3}{3 \cdot 3^3} + \cdots$

(f) $1 + \frac{(x-1)^2}{2!} + \frac{(x-1)^4}{4!} + \frac{(x-1)^6}{6!} + \cdots$

12. Find the interval of convergence of each of the following power series. If the interval is finite, investigate the convergence of the series at the endpoints of the interval.

(a) $\sum_{n=0} \frac{x^n}{2^n}$

(b) $\sum_{n=0} \frac{x^{2n+1}}{n!}$

(c) $\sum_{n=1} \frac{(x-1)^n}{2n}$

(d) $\sum_{n=1} \frac{x^n}{n^2}$

(e) $\sum_{n=1} \frac{x^n}{n^n}$

13. Verify that each of the following series is convergent.

(a) $\sum_{n=1} \frac{n}{(n+1)^3}$

(b) $\sum_{n=1} \frac{1}{n!}$

(c) $\sum_{n=1} n\left(\frac{1}{2}\right)^n$

(d) $\sum \frac{\sqrt{n+3}}{(n-1)^2}$



14. Verify that each of the following series is divergent.

(a) $\sum_{n=2} \frac{1}{\sqrt{n}\log n}$

(b) $\sum_{n=1} \frac{2^n}{n^2}$

(c) $\sum_{n=1} \frac{(n-1)^n}{n!}$

(d) $\sum_{n=1} \frac{n}{(n+1)^2}$

15. Determine the values of $p$ for which the following series converges and div

$$\sum_{n=2} \frac{1}{n^p \log n},$$

where $p$ is a positive integer.

••• 16. Use the integral test to determine for what values of $p$ ($p$ is a real numbe series

$$\sum_{n=1} \frac{1}{n^p}$$

converges.

17. Consider the series

$$\frac{1}{1 \cdot 3} + \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} + \cdots.$$

(a) Write $S_n$, the sum of the first $n$ terms in closed form. (*Hint*: Decompo $n$th term of the series into partial fractions.)
(b) Obtain the sum of the series.

18. Generalize Exercise 17 for

$$\sum_{n=1} \frac{1}{n(n+p)}$$

where $p$ is a positive integer.

## 1.5 SERIES SOLUTIONS

Before we give an example of how a power series solution of a linear diff tial equation can be obtained, we need the results of two theorems. Thes presented without proofs in order to preserve the continuity

**THEOREM 1.5-1**
*A power series $\sum_{n=0} a_n(x - x_0)^n$ and its derivative $\sum_{n=1} na_n(x - x_0)^{n-1}$ have the same radius of convergence.*

**THEOREM 1.5-2**
*Let a function $f(x)$ be represented by a power series $\sum_{n=0} a_n(x - x_0)^n$ in the interior of its interval of convergence. Then the function is differentiable there, and its derivative is given by*

$$f'(x) = \sum_{n=1} na_n(x - x_0)^{n-1}.$$

We are now ready to look at a simple differential equation with a view to solving it by using series. To begin with, we will take $x_0$ to be zero. Later we will indicate why this is not always possible.

**EXAMPLE 1.5-1** Find a solution of the equation $y'' - xy = 0$.

**Solution** We *assume* that there is a solution of the form

$$y = \sum_{n=0} a_n x^n.$$

Then

$$y' = \sum_{n=1} na_n x^{n-1}$$

and

$$y'' = \sum_{n=2} n(n-1)a_n x^{n-2}.$$

Substituting these values into the given equation produces

$$\sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=0} a_n x^{n+1} = 0.$$

In order to collect terms it would be convenient to have $x^n$ in both summations. This can be accomplished by realizing that $n$ is a *dummy index* of summation and can be replaced by any other letter just as we change variables in definite integrals. Accordingly, we replace *each* $n$ in the first sum by $n + 2$ and each $n$ in the second sum by $n - 1$. Then

$$\sum_{n=0} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1} a_{n-1}x^n = 0.$$



Next we combine the two sums into one with $n$ going from 1 to $\infty$, adding a terms that are left out of this sum. Thus

$$\sum_{n=1} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n + 2a_2 = 0,$$

which is a linear combination of $1, x, x^2, \ldots$. Since the set of functions

$$\{1, x, x^2, x^3, \ldots\}$$

is a linearly independent set, a linear combination of these functions can zero if and only if *each* coefficient is zero. Hence

$$2a_2 = 0,$$

and, in general,

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0.$$

From the first of these, $a_2 = 0$, and from the second we obtain the **recursi formula**

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \qquad n = 1, 2, \ldots.$$

- For $n = 1$ we have $a_3 = \dfrac{a_0}{3\cdot 2}$, so that $a_0$ can be arbitrary.
- For $n = 2$ we have $a_4 = \dfrac{a_1}{4\cdot 3}$, so that $a_1$ can be arbitrary.
- For $n = 3$ we have $a_5 = \dfrac{a_2}{5\cdot 4} = 0$; consequently, $a_2, a_5, a_8, \ldots$ are zero.
- For $n = 4$ we have $a_6 = \dfrac{a_0}{3\cdot 2\cdot 6\cdot 5}$.
- For $n = 5$ we have $a_7 = \dfrac{a_4}{7\cdot 6} = \dfrac{a_1}{4\cdot 3\cdot 7\cdot 6}$, and so on.

The solution to the given differential equation is

$$y = a_0 + a_1 x + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + \frac{a_0}{180}x^6 + \frac{a_1}{504}x^7 + \cdots$$

This last equation can also be written as

$$y = a_0\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right),$$

which shows the two arbitrary constants we expect to find in the solution second-order differential equation. It can be shown (Exercise 1) that b series converge for $-\infty < x < \infty$. ■

While many series can be written in **closed form**, for example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{1.5-1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots, \tag{1.5-2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots, \tag{1.5-3}$$

$$\log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \tag{1.5-4}$$

this is not always possible. If a function can be represented in an open interval containing $x_0$ by a *convergent* series of the form $\sum_{n=0} a_n(x - x_0)^n$, then the function is said to be **analytic** at $x = x_0$. The functions in Eqs. (1.5-1) through (1.5-4) are all analytic at $x = 0$. If a function is analytic at every point where it is defined, it is called an **analytic function.** All polynomials are analytic, and so are rational functions except where their denominators vanish.

Now let us look at another example of a series solution of a differential equation.

**EXAMPLE 1.5-2** Solve the equation

$$(x - 1)y'' - xy' + y = 0.$$

**Solution** As before, assume

$$y = \sum_0 a_n x^n, \qquad y' = \sum_1 n a_n x^{n-1}, \qquad y'' = \sum_2 n(n-1)a_n x^{n-2},$$

and substitute into the given differential equation. Then

$$\sum_2 n(n-1)a_n x^{n-1} - \sum_2 n(n-1)a_n x^{n-2} - \sum_1 n a_n x^n + \sum_0 a_n x^n = 0.$$

Replace $n$ by $n + 1$ in the first sum and replace $n$ by $n + 2$ in the second sum so that we have

$$\sum_1 (n+1)n a_{n+1} x^n - \sum_0 (n+2)(n+1)a_{n+2} x^n - \sum_1 n a_n x^n + \sum_0 a_n x^n = 0$$

or

$$\sum [n(n+1)a_{n+1} - (n+1)(n+2)a_{n+2} - n a_n + a_n]x^n - 2a_2 + a_0 = 0.$$



Equating the coefficients of various powers of $x$ to zero gives us the follo

$a_0$ is arbitrary,
$a_1$ is arbitrary,

$$a_2 = \frac{1}{2}a_0,$$

$$a_{n+2} = \frac{n(n+1)a_{n+1} + (1-n)a_n}{(n+1)(n+2)}, \qquad n = 1, 2, \ldots,$$

$$a_3 = \frac{2a_2}{2\cdot 3} = \frac{a_2}{3} = \frac{a_0}{3\cdot 2},$$

$$a_4 = \frac{6a_3 - a_2}{3\cdot 4} = \frac{a_3}{2} - \frac{a_2}{12} = \frac{a_0}{12} - \frac{a_0}{24} = \frac{a_0}{4!},$$

etc. Hence

$$y = a_1 x + a_0\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right),$$

and it can be shown (Exercise 2) that $x$ and $e^x$ are two linearly indepe solutions of the given equation. Here the solution can be written in c form in contrast to the solution of Example 1.5-1 (Exercise 3). ■

Unfortunately, the series method of solving ordinary differential e tions is not as simple as the last two examples seem to indicate. Conside equation

$$2x^2y'' + 5xy' + y = 0.$$

We leave it as an exercise (Exercise 4) to show that the series method with 0 will produce only the trivial solution $y = 0$. Yet the given equation Cauchy–Euler equation, and both $x^{-1/2}$ and $1/x$ are solutions (Exercise 5). answer to the apparent mystery lies in the fact that the individual solutions Cauchy–Euler equation are not linearly independent on any interval th cludes the origin. Recall that in Section 1.3 we solved Cauchy–Euler equa assuming that $x > 0$ *or* $x < 0$.

Consider the most general second-order, linear, homogeneous ordi differential equation,

$$y'' + P(x)y' + Q(x)y = 0. \qquad (1.$$

Those values of $x$, call them $x_0$, at which *both* $P(x)$ and $Q(x)$ are analyti called **ordinary points** of Eq. (1.5-5). If either $P(x_0)$ or $Q(x_0)$ is not ana then $x_0$ is a **singular point** of Eq. (1.5-5). If, however, $x_0$ is a singular poi Eq. (1.5-5) but *both* $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic at $x$ = then $x_0$ is a **regular singular point** of Eq. (1.5-5). All other singular point called **irregular singular points.**

**EXAMPLE 1.5–3** Classify the singular points of the equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,$$

where $n = 0, 1, 2, \ldots$.

The only singular points are $x_0 = \pm 1$. If $x_0 = -1$, then

$$\frac{(x + 1)(-2x)}{1 - x^2} = \frac{2x}{x - 1} \quad \text{and} \quad \frac{(x + 1)^2 n(n + 1)}{1 - x^2} = \frac{n(n + 1)(x + 1)}{x - 1}.$$

Since both of these rational functions are analytic at $x = -1$, the latter is a regular singular point. Similarly, for $x_0 = 1$ we have

$$\frac{(x - 1)(-2x)}{1 - x^2} = \frac{2x}{x + 1} \quad \text{and} \quad \frac{(x - 1)^2 n(n + 1)}{1 - x^2} = \frac{n(n + 1)(1 - x)}{1 + x},$$

so that $x_0 = 1$ is also a regular singular point. ■

The point of all this is contained in an 1865 theorem due to Fuchs.* Fuchs' theorem states that it is always possible to obtain *at least one* power series solution to a linear differential equation provided that the assumed series solution is about an ordinary point or, at worst, a regular singular point.

The work of Fuchs was extended by Frobenius,† who in 1874 suggested that instead of assuming a series solution of the form $\sum_0 a_n x^n$, one should use the form $\sum_0 a_n x^{n+r}$. The use of this form to solve linear, ordinary differential equations is known today as the **method of Frobenius**. We illustrate with an example using the Cauchy–Euler equation referred to above.

**EXAMPLE 1.5–4** Solve the equation

$$2x^2y'' + 5xy' + y = 0$$

by the method of Frobenius.

**Solution** We have

$$y = \sum_0 a_n x^{n+r},$$

$$y' = \sum_0 (n + r)a_n x^{n+r-1},$$

$$y'' = \sum_0 (n + r)(n + r - 1)a_n x^{n+r-2},$$

*Lazarus Fuchs (1833–1902), a German mathematician.
†Georg Frobenius (1849–1917), a German mathematician.



and on substituting into the given equation we have

$$\sum_0 [2(n + r)(n + r - 1) + 5(n + r) + 1]a_n x^{n+r} = 0.$$

Since the coefficient of $x^{n+r}$ must be zero for $n = 0, 1, 2, \ldots$, we have $n = 0$,

$$(2r^2 + 3r + 1)a_0 = 0.$$

Choosing $a_0$ to be arbitrary, that is, nonzero, produces

$$2r^2 + 3r + 1 = 0,$$

which is called the **indicial equation**. Its roots are $-1$ and $-1/2$. In genei

$$a_n(2n^2 + 4nr + 3n) = 0, \qquad n = 1, 2, \ldots,$$

which can be satisfied only by taking $a_n = 0$, $n = 1, 2, \ldots$. Hence we left with the two possibilities,

$$y_1(x) = a_0 x^{-1} \quad \text{and} \quad y_2(x) = b_0 x^{-1/2}.$$

Note that the two constants are arbitrary, since each root of the indicial eq tion leads to an infinite series. In this example, however, each series cons of a single term. ■

When solving *second-order* linear differential equations by the meth of Frobenius, the indicial equation is a quadratic equation, and th possibilities exist. We list these together with their consequences here.

1. If the roots of the indicial equation are *equal*, then only *one* solution be obtained.
2. If the roots of the indicial equation differ by a number that is not integer, then two linearly independent solutions may be obtained.
3. If the roots of the indicial equation differ by an integer, then the lar integer of the two will yield a solution, whereas the smaller may or n not yield a solution.

It should be mentioned that the theory behind the method of Frobenius is by nc means simple. A good discussion of the various cases that may arise (although the case where the indicial equation has complex roots is omitted) can be found in Albert L. Rabenstein, *Elementary Differential Equations with Linear Algebra*, 3d ed. (New York: Academic Press, 1982), pp. 391 ff.

We conclude this section by solving two important differential eq tions that will appear later in the text in connection with certain types boundary-value problems.

**EXAMPLE 1.5-5** Obtain a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0, \qquad n = 0, 1, 2, \ldots . \tag{1.5-6}$$

This equation is known as **Bessel's differential equation.** It was originally obtained by Friedrich Wilhelm Bessel (1784–1846), a German mathematician, in the course of his studies of planetary motion. Since then, this equation has appeared in problems of heat conduction, electromagnetic theory, and acoustics that are expressed in *cylindrical coordinates*.

**Solution** Since the coefficients are not constant, we seek a series solution. Multiplying Eq. (1.5-6) by $x^2$, we obtain

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{1.5-7}$$

We note that $x = 0$ is a regular singular point; hence we use the method of Frobenius. Assume that

$$y = \sum_{m=0} a_m x^{m+r},$$

$$y' = \sum_{m=0} a_m(m + r)x^{m+r-1},$$

$$y'' = \sum_{m=0} a_m(m + r)(m + r - 1)x^{m+r-2}$$

and substitute into Eq. (1.5-7). Then

$$\sum_{m=0} a_m(m + r)(m + r - 1)x^{m+r} + \sum_{m=0} a_m(m + r)x^{m+r} + \sum_{m=0} a_m x^{m+r+2} - n^2 \sum_{m=0} a_m x^{m+r} = 0.$$

If we replace $m$ by $m - 2$ in the third series, the last equation can be written as

$$\sum_{m=2} [a_m(m + r)(m + r - 1) + a_m(m + r) + a_{m-2} - n^2a_m]x^{m+r} + a_0r(r - 1)x^r + a_0rx^r - n^2a_0x^r + a_1r(r + 1)x^{r+1} + a_1(r + 1)x^{r+1} - n^2a_1x^{r+1} = 0.$$

Simplifying, we get

$$\sum_{m=2} [a_m((m + r)^2 - n^2) + a_{m-2}]x^{m+r} + a_0(r^2 - n^2)x^r + a_1(r^2 + 2r + 1 - n^2)x^{r+1} = 0.$$

The coefficient of $x^r$ must be zero; hence if we assume $a_0 \neq 0$, then we obtain $r = \pm n$. We choose the positive sign, since $n$ was defined as a non-



we may choose $a_1 = 0$. Then the recursion formula is obtained by settir coefficient of $x^{m+r}$ equal to zero. Thus

$$a_m = \frac{-a_{m-2}}{m(m + 2n)}, \qquad m = 2, 3, \ldots .$$

The first few coefficients can be computed from this formula. They a follows:

$$m = 2: \quad a_2 = \frac{-a_0}{2^2(n + 1)},$$

$$m = 4: \quad a_4 = \frac{-a_2}{2^3(n + 2)} = \frac{a_0}{2^4 \cdot 2(n + 1)(n + 2)},$$

$$m = 6: \quad a_6 = \frac{-a_4}{2^2 \cdot 3(n + 3)} = \frac{a_0}{2^6 \cdot 3!(n + 1)(n + 2)(n +}$$

In general we have

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m}m!(n + 1)(n + 2) \cdots (n + m)}, \qquad m = 1, 2, \ldots ,$$

and a solution to Eq. (1.5-6) can be written as

$$y_n(x) = a_0 \sum_{m=0} \frac{(-1)^m x^{2m+n}}{2^{2m}m!(n + 1)(n + 2) \cdots (n + m)}$$

$$= 2^n n! a_0 \sum_{m=0} \frac{(-1)^m}{m!(m + n)!}\left(\frac{x}{2}\right)^{2m+n}. \quad \blacksquare$$

The **Bessel function of the first kind of order $n$** is defined by givi the value $1/2^n n!$. We have

$$J_n(x) = \sum_{m=0} \frac{(-1)^m}{m!(m + n)!}\left(\frac{x}{2}\right)^{2m+n}, \qquad n = 0, 1, 2, \ldots , \quad (1$$

a solution of Bessel's differential equation. We will consider this functi greater detail in Chapter 7.

**EXAMPLE 1.5-6** Obtain a solution of the equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0, \qquad (1$$

where $n$ is a constant. This equation is known as **Legendre's differ equation.***

*After Adrien Marie Legendre (1752–1833), a French mathematician who is known for his work in number theory, elliptic functions, and calculus of variations.

**Solution** Since $x = \pm 1$ are regular singular points (see Example 1.5-3), we may assume a power series about $x = 0$, which is an ordinary point. Accordingly, put

$$y = \sum_{m=0} a_m x^m, \qquad y' = \sum_{m=1} a_m m x^{m-1}, \qquad y'' = \sum_{m=2} a_m m(m-1)x^{m-2}.$$

Substituting these values into Eq. (1.5-9) produces

$$\sum_{m=2} a_m m(m-1)x^{m-2} - \sum_{m=2} a_m m(m-1)x^m - 2\sum_{m=1} a_m m x^m + n(n+1)\sum_{m=0} a_m x^m = 0.$$

Replacing $m$ by $m + 2$ in the first sum, we get

$$\sum_{m=0} a_{m+2}(m+2)(m+1)x^m - \sum_{m=2} a_m(m-1)x^m - 2\sum_{m=1} a_m m x^m + n(n+1)\sum_{m=0} a_m x^m = 0,$$

or

$$\sum_{m=2} [a_{m+2}(m+2)(m+1) - a_m m(m-1) - 2a_m m + a_m n(n+1)]x^m + 2a_2 + 6a_3 x - 2a_1 x + n(n+1)a_0 + n(n+1)a_1 x = 0.$$

Setting the coefficient of each power of $x$ equal to zero in the above, we have

$$2a_2 + n(n+1)a_0 = 0, \qquad a_2 = \frac{-n(n+1)a_0}{2}, \qquad a_0 \text{ arbitrary};$$

$$6a_3 - 2a_1 + n(n+1)a_1 = 0, \qquad a_3 = \frac{[2 - n(n+1)]a_1}{6}, \qquad a_1 \text{ arbitrary}.$$

In general, we can say

$$a_{m+2}(m+2)(m+1) - [m(m-1) + 2m - n(n+1)]a_m = 0;$$

$$a_{m+2} = \frac{m(m+1) - n(n+1)}{(m+2)(m+1)} a_m;$$

$$a_{m+2} = \frac{(m-n)(m+n+1)}{(m+2)(m+1)} a_m, \qquad m = 0, 1, 2, \ldots . \quad \textbf{(1.5-10)}$$

Equation (1.5-10) is the recurrence relation from which the coefficients can be found.



Computing the first few coefficients gives us

$$a_2 = \frac{-n(n+1)}{1\cdot 2} a_0,$$

$$a_4 = \frac{(2-n)(n+3)}{4\cdot 3} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0,$$

$$a_6 = \frac{(4-n)(n+5)}{6\cdot 5} a_4 = \frac{-n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!} a_0,$$

$$a_3 = \frac{(1-n)(n+2)}{3\cdot 2} a_1 = \frac{-(n-1)(n+2)}{3!} a_1,$$

$$a_5 = \frac{(3-n)(n+4)}{5\cdot 4} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1,$$

$$a_7 = \frac{(5-n)(n+6)}{7\cdot 6} a_5 = \frac{-(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} a_1.$$

Hence a solution to Legendre's equation can be written as

$$y_n(x) = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + - \cdots\right] + a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!}x^5 - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + - \cdots\right]. \quad \textbf{(1.5}$$

Both series converge for $-1 < x < 1$.

If $n = 0, 2, 4, \ldots$ and $a_1$ is chosen to be zero, then the solutions, usi Eq. (1.5-11), become

$$y_0(x) = a_0,$$

$$y_2(x) = a_0(1 - 3x^2),$$

$$y_4(x) = a_0\left(1 - 10x^2 + \frac{35}{3}x^4\right), \qquad \text{etc.}$$

If we also impose the condition that $y_n(1) = 1$, then we can evaluate the $a_0$ to obtain

$$P_0(x) = 1,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1), \qquad (1.5\text{–}12)$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \ldots.$$

These *polynomials* are called the **Legendre polynomials of even degree.**

If $n = 1, 3, 5, \ldots$ and $a_0$ is chosen to be zero, then the solutions, using Eq. (1.5–11), become

$$y_1(x) = a_1 x,$$
$$y_3(x) = a_1\left(x - \frac{5}{3}x^3\right),$$
$$y_5(x) = a_1\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right), \qquad \text{etc.}$$

If we again impose the condition that $y_n(1) = 1$, then we can evaluate the $a_1$ to obtain

$$P_1(x) = x, \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \qquad (1.5\text{–}13)$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \ldots.$$

These polynomials are called the **Legendre polynomials of odd degree.** ■

The **Legendre polynomials** will be of use in Section 7.3, since they arise in boundary-value problems expressed in *spherical coordinates*.

## Key Words and Phrases

recursion formula
closed form
analytic
analytic function
ordinary point
singular point
regular and irregular singular point
method of Frobenius
indicial equation
Bessel's differential equation
Bessel function of the first kind of order $n$,
Legendre's differential equation
Legendre polynomials of even degree
Legendre polynomials of odd degree



## 1.5 Exercises

• **1.** (a) Show that one solution of the differential equation $y'' - xy = 0$ in E 1.5–1 can be written as

$$y_1(x) = a_0 \sum_{n=0} \frac{1\cdot 4\cdot 7\cdots(3n-2)}{(3n)!}x^{3n}.$$

(b) Write the second solution in a similar form.
(c) Find the radius of convergence of the series in part (a).
(d) Observe that $x_0 = 0$ is an ordinary point and hence that both solutic analytic.

**2.** Verify that $y_1(x) = x$ and $y_2(x) = e^x$ are linearly independent solutions differential equation $(x - 1)y'' - xy' + y = 0$.

**3.** (a) Show that the solution obtained in Example 1.5–2 is equivalent to

$$y(x) = c_1 x + c_2 e^x.$$

(b) For what values of $x$ is the above solution valid?

**4.** Show that assuming a solution of the form $y = \sum_0 a_n x^n$ for the equation

$$2x^2y'' + 5xy' + y = 0$$

leads to the trivial solution $y = 0$.

**5.** Verify that $x^{-1/2}$ and $1/x$ are linearly independent solutions of the equa Exercise 4 on every interval not containing the origin.

•• **6.** Classify the singular points of each of the following differential equations
(a) $x^2y'' + xy' + (x^2 - n^2)y = 0, \quad n = 0, 1, 2, \ldots$
(b) $x^3y'' - xy' + y = 0$
(c) $x^2y'' + (4x - 1)y' + 2y = 0$
(d) $x^3(x - 1)^2y'' + x^4(x - 1)^3y' + y = 0$

**7.** Use power series to solve each of the following equations.
(a) $y'' + y = 0$
(b) $y'' - y = 0$
(c) $y' - y = x^2$
(Note that the power series method is not limited to *homogeneous* equa
(d) $y' - xy = 0$
(If possible, write the solution in closed form.)
(e) $(1 + x^2)y'' + 2xy' - 2y = 0$

**8.** Solve each of the following differential equations by the method of Frobe
(a) $xy'' + y' + xy = 0$
(b) $4xy'' + 2y' + y = 0$
(c) $x^2y'' + 2xy' - 2y = 0$

**9.** Solve the equation

$$xy'' + 2y' = 0$$

by two methods. (*Hint*: $x$ is an integrating factor.)

10. Solve the equation

$$y'' - xy' - y = 0$$

by assuming a solution that is a power series in $(x - 1)$. In this case the coefficient $x$ must also be written in terms of $x - 1$. This can be done by assuming that $x = A(x - 1) + B$ and determining the constants $A$ and $B$.

11. Obtain a solution of

$$xy'' + y' + 4xy = 0.$$

12. Obtain a solution of

$$xy'' - 2y = 0.$$

••• 13. The differential equation

$$y'' - xy = 0$$

is known as **Airy's equation,*** and its solutions are called **Airy functions** (Fig. 1.5-1), which have applications in the theory of diffraction.
(a) Obtain the solution in terms of a power series in $x$.
(b) Obtain the solution in terms of a power series in $(x - 1)$. (Compare Exercise 10.)

14. Compare the solutions of Exercise 13 with those of $y'' - y = 0$. Comment.

15. Solve the initial-value problem

$$y'' + xy = 2, \qquad y(0) = y'(0) = 1.$$

16. Solve the differential equation

$$4xy'' + 2y' + y = 0.$$

17. Find the interval of convergence of the two series in the solution of Exercise 16.

18. Solve the initial-value problem

$$y'' + y' + xy = 0, \qquad y(0) = y'(0) = 1.$$

19. Obtain the general solution of

$$2x^2y'' - xy' + (1 - x^2)y = 0.$$

20. Obtain the general solution of

$$x^2y'' + x^2y' - 2y = 0.$$

21. Illustrate Theorem 1.5-1 by differentiating the series in Exercise 11 of Section 1.4 and finding the radii of convergence of the differentiated series. (Note that this does *not* constitute a *proof* of the theorem.)

22. Illustrate Theorem 1.5-2 by differentiating the functions and series in Eqs. (1.5-1) through (1.5-4).

23. Find the general solution of

$$y'' + xy' + y = 0.$$

*After Sir George B. Airy (1801-1892), an English mathematician and astronomer.



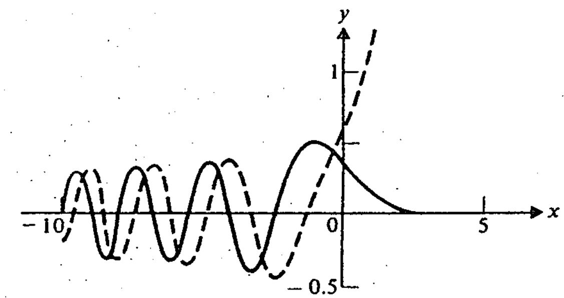



**Figure 1.5-1**
**Airy functions.**

## 1.6 UNIFORM CONVERGENCE

We give a brief discussion of some of the theoretical aspects in the present s tion, not for the sake of the theory per se but to prepare for a wide variety applications. It will be shown, for example, that we will have to broaden o concept of convergence.

### Pointwise Convergence

Recall what is meant by saying that a sequence of functions, defined i $a \le x \le b$,

$$\{f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots\},$$

converges to $f(x)$. When we write

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \text{ in } [a, b],$$

we mean that the difference between $f(x)$ and $f_n(x)$ can be made arbitrar small provided that $n$ is taken large enough.

In more precise mathematical language we would say that given a positive number $\epsilon$ and any point $x_0$ in the interval $[a, b]$, we can satisfy

$$|f_n(x_0) - f(x_0)| < \epsilon \qquad (1.6$$

whenever $n \ge N(\epsilon, x_0)$, an integer. In other words, given $\epsilon > 0$, we can fi an integer $N$ such that the inequality (1.6-1) holds whenever $n \ge N$. The i portant thing to notice here is that $N$ will usually depend on $\epsilon$ *and* $x_0$, so write $N(\epsilon, x_0)$.