FLORIDA INTERNATIONAL UNIVERSITY
Mechanical and Materials Engineering Department

Fall 2018 Intermediate Analysis of Mechanical Systems EGM 5315

COURSE CONTENT

1. Review of Ordinary Differential Equations (ODE)
   - o Review of First Order ODE (constant and non constant coeff)
   - o Review of Second Order ODE (constant and non constant coeff)

2. Partial Differential Equations Generated by Fluid/Thermal and Solid Mechanics Problems
   - o Recognition of elliptic, parabolic and hyperbolic PDE's
   - o Solution methodology used
     - o Method of Characteristics
     - o Canonical Forms
     - o Reduced canonical forms

   - o Derivation of the governing equation of motion for a continuous system
     - o complete description--boundary conditions

   - o Solution Methodology continued--Separation of Variables
     - o The Eigenvalue Problem
     - o Fourier Series revisited

   - o Sturm-Liouville problem and non-homogeneous boundary conditions

   - o Method of Characteristics Revisited

   - o Transform methods--Laplace and Fourier Transforms

   - o Self-Similar problems

Reference Texts for PDE portion –These can be bought from the department at a total cost of $45
Applied Partial Differential Equations by Donald Trim, PWS-Kent

Certain sections of Solution of Partial Differential Equations by W.C. Reynolds notes from Stanford University will be provided as well.

Reference Texts for the ODE portion:
- o Advanced Calculus for Applications by Hildebrand Prentice-Hall Publishers
- o Elementary Differential Equations by Boyce and DePrima John Wiley Publishers

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Grade will be determined on the basis of	2 Exams	30 % each
	HW/Project	10 %
	Final Exam	30 %

Tentative Grading Scheme:

95 and above	A	80 – 84.99	B	67 – 72.99	C
90 – 94.99	A-	77 – 79.99	B-	60 – 66.99	D
85 – 89.99	B+	73 – 76.99	C+	Below 60	F

Florida International University is a community of faculty, staff and students dedicated to generating and imparting knowledge through 1) excellent teaching and research, 2) the rigorous and respectful exchange of ideas, and 3) community service. All students should respect the right of others to have an equitable opportunity to learn and honestly demonstrate the quality of their learning. Therefore, all students are expected to adhere to a standard of academic conduct, which demonstrates respect for themselves, their fellow students, and the educational mission of the University. All students are deemed by the University to understand that if they are found responsible for academic misconduct, they will be subject to the Academic Misconduct procedures and sanctions, as outlined in the Student Handbook.

Please be on time to class and keep up with the work. There is a lot of work to cover and it will be difficult for you if you do not do the homework assignments. My office hours will be announced at the end of the first week. Please come to see me if you are having problems or have suggestions on how to improve this rather compact course.

At present we will meet in EC3327 on W 10-1150am and also F 11-1150am. However, that is subject to change. The website on which materials related to this class will be http://web.eng.fiu.edu/levy. Videotapes related to this class will be available so that the class can be handled as a distance learning class..

This is a <u>preliminary</u> syllabus subject to change. All changes will be announced on the class website.

O.D.E. REVIEW

- ORDER OF DIFF. EQ. = HIGHEST DERIV.

$$y^{IV} + 2y'' = 0 \qquad\qquad y^{IV} = \frac{d^4y}{dx^4}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^2 w}{\partial x^2} = 0 \qquad\qquad w = w(x,y)$$

- IF COEFFS OF DERIVATIVES ARE CONST. ⇒ CONSTANT COEFF DIFF. EQ.

$$u''(t) + cu'(t) + m\,u(t) = 0 \qquad m, c \text{ CONST.}$$

- IF FN IS FN OF ONE INDEP. VARIABLE ie $y = y(x)$

⇒ ORDINARY DIFF. EQ.

$$a(x)y'' + b(x)y' + c(x)y = 0 \qquad y = y(x)$$

- LINEAR IF ~~[illegible]~~ Power OF y or its derivatives are $y^1, (y')^1$ etc only

$$a(x)\; y'' + b(x)y' + c(x)y = 0$$

- NON LINEAR IF products of y & derivs or powers of y

$$a(x)y'' + b(x)yy' = 0$$

$$a(x)y'' + b(x)(y')^2 + c(x) = 0$$

- HOMOGENEOUS IF TERM THAT DOESN'T INVOLVE y or derivs $= 0$

$$a(x)y'' + b(x)\,y' + c(x)y + d(x) = 0 \qquad \text{HOMOG IF } d(x) = 0$$

- FIRST ORDER ODE : FIND IN ANY O.D.E. BOOK.

$$a(x)y' + b(x)y = c(x) \qquad \div \text{ by } a(x)$$

$$\Rightarrow \quad y' + p(x)y = h(x)$$

- SOLUTION use integrating factor $\mu(x) = e^{\int^x p(t)dt}$

$$y(x) = \frac{1}{\mu(x)}\int^x \mu(t)h(t)\,dt + \frac{\text{const}}{\mu(x)}$$

- need one condition $y = y_0$ at $x = x_0$ to define const

- WHAT IF IT IS HARD TO FIND SOLUTION BY CLOSED FORM

- PICARD'S METHOD Good $_\wedge$ for $|x|<a$ if y & $\partial f/\partial y$ are continuous in $|x| \le a$ $|y| \le b$

WRITE $y' = f(x,y)$ with $y(x=x_0) = y_0$

$$\Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(\bar{x}, y)\, d\bar{x} \quad \text{when we integrate}$$

Picard says: define sequence $y_0, y_1, y_2, \dots y_n \rightarrow y$ in limit

$$y_1 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_0)\, d\bar{x}$$

$$y_2 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_1)\, d\bar{x}$$

$$\vdots$$

$$y_n = y_0 + \int_{x_0}^{x} f(\bar{x}, y_{n-1})\, d\bar{x}$$

- unsatisfactory on practical reasons it is hard to integrate

- example $y' = x - y$ $\quad y(x=0) = 1$ $\quad y_0 = 1$ $\quad x_0 = 0$

$f(x,y) = x - y$

$$y_1 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_0)\, d\bar{x} = 1 + \int_0^x (\bar{x} - 1)\, d\bar{x}$$

$$= 1 + \frac{(\bar{x}-1)^2}{2}\Big|_0^x = 1 + \frac{(x-1)^2}{2} - \frac{(-1)^2}{2} = 1 + \frac{x^2}{2} - x = 1 - x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_1)\, d\bar{x} = 1 + \int_0^x \left[\bar{x} - \left(1 - \bar{x} + \frac{\bar{x}^2}{2}\right)\right] d\bar{x}$$

$$= 1 + \left[-\bar{x} + \frac{\bar{x}^2}{\text{[illegible]}} - \frac{\bar{x}^3}{6}\right]_0^x = 1 - x + \frac{x^2}{\text{[illegible]}} - \frac{x^3}{6}$$

- actual: $\mu(x) = e^{\int^x 1\, dt} = e^x$ $\qquad y' + y = x \quad p(x) = 1 \quad h(x)=$

$$y(x) = \frac{1}{e^x} \int^x e^t\, t\, dt + \frac{c}{e^x}$$

$$= \frac{1}{e^x}\left[te^t - e^t\right]^x + c/e^x = x - 1 + ce^{-x},$$

when $x=0$ $y=1$
$\Rightarrow c = 2$

AT $x=0$ $y=1$ $\Rightarrow$ $C=2$

$$y(x) = x-1+2e^{-x}$$
$$= x-1+2\left[1-x+\frac{x}{2}^{2}-\frac{x}{3!}^{3}+\dots\right]$$
$$= 1-x+x^{2}-\frac{x}{3}^{3}+\dots$$

y_1 error in last term

y_2 error in last term

but sequence tends to y

HW Use Picard's method to find the solution to $y'-x^2y=x$ with $y(x=0)=1$ at $x=.2$ (ie $y(x=.2)=?$) Solution must be accurate to 5 decimal places

- Picard's method is good but unwieldy

~~$y_1 = y(x_0+h) = y_0 + ah + b\frac{h^2}{2} = y_0 + f(x_0,y_0)\,h + \frac{h}{2}\left[f(x_0,y_0) - f(x_0-h, y(x_0-\right.$~~

- ~~requires knowledge of (x_0,y_0) pt & $(x_0-h,\ y(x_0-h))$ pt (x_{-1}, y_{-1})~~
- ~~best to use method like Picard to start up this routine, then adams.~~

- Another method of determining the first pt is Taylor series expansion

Taylor series expansion

$$y_1 = y(x_1) = y(x_0+h) = \overset{y_0}{y(x_0)} + y'(x_0)\,h + y''(x_0)\frac{h^2}{2} + \cdots$$

$$y'(x_0) = f(x_0,y_0)$$

$$y''(x_0) \neq \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = \frac{\partial f}{\partial x} + f\cdot\frac{\partial f}{\partial y} = \frac{d}{dx}(f\ ) = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{dy}{dx}\right)f$$

$$y''' = \frac{\partial^2 f}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\cdot\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right)\cdot\frac{dy}{dx} = f_{xx} + f_x f_y + 2f\,f_{xy} + f^2 f_{yy} + f f_y'$$

for the case $y' = x - y = f$ $\qquad \frac{\partial f}{\partial x} = 1 \qquad \frac{\partial f}{\partial y} = -1 \qquad y'' = 1 - f = 1-(x-y)$

$x_0 = 0 \quad y_0 = 1 \qquad f(x_0,y_0) = x_0 - y_0 = 0 - 1 = -1 \qquad f_{xx}=0 \quad f_x f_y = -1 \quad f_{xy} = 0 \quad f_{yy} = 0$

$$\therefore \quad y(h) = 1 + (-1)h + \frac{h^2}{2}\left(1-(0-1)\right) = 1 - h + h^2 = y_1$$

$$y = y(x) = y(0+\Delta x) = 1 - 1\cdot(\overset{x-0}{[illegible]}) + \frac{(x-0)^2}{2}\cdot 2 = 1 + (0-x) + (0-x)^2 = \quad 1 - x + x^2$$

THUS $\quad x_1 = x_0 + h = 0 + h \qquad y(x_1 = h) = 1 - h + h^2$

and $\quad f(x_1,y_1) = x_1 - y_1 = h - (1 - h + h^2) = -1 + 2h - h^2$

$\therefore \quad x_0 = 0 \quad y_0 = 1 \qquad f(x_0,y_0) = x_0 - y_0 = -1$

$\qquad x_1 = h \quad y_1 = 1 - h + h^2 \qquad f(x_1,y_1) = -1 + 2h + h^2$

By adam's method

$$y_2 = y(x_2) = y_1 + f(x_1,y_1)\,h + \frac{h}{2}\left[f(x_1,y_1) - f(x_0,y_0)\right]$$

and

$$y_n = y_{n-1} + f(x_{n-1},y_{n-1})\,h + \frac{h}{2}\left[f(x_{n-1},y_{n-1}) - f(x_{n-2},y_{n-2})\right]$$

- Thus $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$
  - u_1' & u_2' can be found via Cramer's Rule

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g\,y_2}{y_1y_2' - y_1'y_2} \Rightarrow u_1(x) = \int^x \frac{-g(t)y_2(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} + \cancel{C_1}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1\,g}{y_1y_2' - y_1'y_2} \Rightarrow u_2(x) = \int^x \frac{g(t)y_1(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} + \cancel{C_2}$$

- Note $y_1y_2' - y_1'y_2 = W(y_1, y_2)$: Wronskian
  - if $W(y_1,y_2) \neq 0$ then y_1 & y_2 are linearly independent & $y_p = u_1y_1 + u_2y$
  - if at some pt the Wronskian is zero then it must be zero everywhere and y_1 & y_2 are not linearly independent.

- Example $x^2y'' - 2xy' + 2y = 4x^2$ for $x>0$ and $y_1(x) = x$

$y_1'(x) = 1$

$y_1''(x) = 0$

1) Check if y_1 solves $x^2y_1'' - 2xy_1' + 2y_1 = 0$

$x^2 \cdot 0 - 2x \cdot 1 + 2x \equiv 0$ ✓

2) Find $v(x) \Rightarrow$ $y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0 \Longrightarrow p = -2/x \quad q = 2/x^2$

$$v(x) = \int^x \frac{1}{s^2}\, e^{-\int^s -\frac{2}{t}dt}\, ds = \int^x \frac{1}{s^2}\, e^{2\ln s}\, ds = \int^x \frac{1}{s^2}\, s^2 ds = \underline{x}$$

3) Find $y_h(x)$

$$y_2(x) = v(x)\,y_1(x) = x^2$$

$$\therefore\ y_h(x) = C_1y_1(x) + C_2y_2(x) = C_1x + C_2 \cdot x^2$$

4) Now find total solution

$$y_p(x) = u_1(x) \cdot x + u_2(x) \cdot x^2$$

5) find $u_1(x)$ & $u_2(x)$ $\qquad x^2y''-2xy'+2y=4x^2 \Rightarrow y''-\frac{2}{x}y'+\frac{2}{x^2}y=\underset{g(x)}{\underset{\downarrow}{4}}$

$$u_1(x)=\int^x \frac{-4\cdot t^2}{t\cdot 2t-1\cdot t^2}\,dt+c_1=\int^x -4\,dt+\cancel{c_1}$$

$$=-4x+\cancel{c_1}$$

$$W(y_1,y_2)=2x^2-x^2=x^2$$

$$u_2(x)=\int^x \frac{4\cdot t}{t^2}\,dt+\cancel{c_2}=\int^x \frac{4}{t}\,dt+\cancel{c_2}$$

$$=4\ln x+\cancel{c_2}$$

6) $y_p=u_1y_1+u_2y_2=(-4x+\cancel{c_1})x+(4\ln x+\cancel{c_2})x^2$

$$=\underbrace{-4x^2+4x^2\ln x}_{\text{particular}}+\cancel{\underbrace{c_1x+c_2x^2}_{\text{homog.}}}$$

$$y_T=y_p+y_h=-4x^2+4x^2\ln x+c_1x+c_2x^2$$

HW Find y_2, u_1, u_2, y given $x^2y''+7xy'+5y=x \quad x>0 \quad y_1(x)=\frac{1}{x}$

$x^2y''+xy'+(x^2-\frac{1}{4})y=3x^{3/2}\sin x \quad x>0 \quad y_1(x)=\frac{\sin x}{\sqrt{x}}$

[illegible] 2nd [illegible]

- must use another method to start
- Try Picard

$$y(x_0+h) = y(x_0) + \int_{x_0}^{x_0+h} p \, d\bar{x}$$

$$p(x_0+h) = p(x_0) + \int_{x_0} f(\bar{x}, y, p) \, d\bar{x}$$

given $y(x_0) = y_0$, $p(x_0) = y' = p_0$, $y'' = f(x, y, p)$

Example $y'' - x^2 y' - 2xy = 1$ w/ $y(0) = 1$ $y'(0) = 0$

$x_0 = 0$ $y_0 = 1$ $y'(0) = p_0 = 0$

$y'' = 1 + x^2 p + 2xy = 1 + x^2 y' + 2xy = f(x, y, p)$

$f(\bar{x}_0, y_0, p_0) = 1$

$$y_1 = y_0 + \int p_0 \, d\bar{x}$$

$$p_1 = p_0 + \int_0 f(\bar{x}, y_0, p_0) \, d\bar{x}$$

$$\therefore \quad p_1 = 0 + \int_0^x 1 \, d\bar{x} = x$$

$$\therefore \quad y_1 = 1 + \int_0^x 0 \cdot d\bar{x} = 1$$

$$y_2 = y_0 + \int_0^x p_1 \, d\bar{x} = 1 + \int_0^x \bar{x} \, d\bar{x} = 1 + x^2/2$$

$$p_2 = p_0 + \int_0 f(\bar{x}, y_1, p_1) \, d\bar{x} = 0 + \int_0^x [1 + \bar{x}^2 \cdot \bar{x} + 2\bar{x} \cdot 1] \, d\bar{x} = x + x^2 + \frac{x^4}{4}$$

here $f(\bar{x}, y_1, p_1) = 1 + x^2 \cdot \underset{p_1}{x} + 2x \cdot \underset{y_1}{1}$

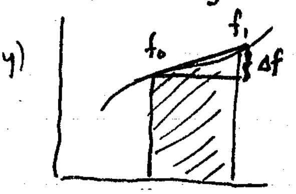
in general $y_{n+1}^{(x)} = y_0 + \int_0^x p_n(\bar{x}) \, d\bar{x}$

$$p_{n+1}^{(x)} = p_0 + \int_0^x f(\bar{x}, y_n, p_n) \, d\bar{x}$$

- This gives a good approx to $y(x=h)$ & $y'(x=h)$ [y_1, p_1 for $x = x_1$
- Then use Adams method to find y_2, p_2 knowing y_1, p_1 & y_0, p_0

- If we think of $y' = f(x,y)$ as y' dependent var.
  x, y independent var.

To find y $\int f(\bar{x}, y) \, d\bar{x}$ find the area under the curve



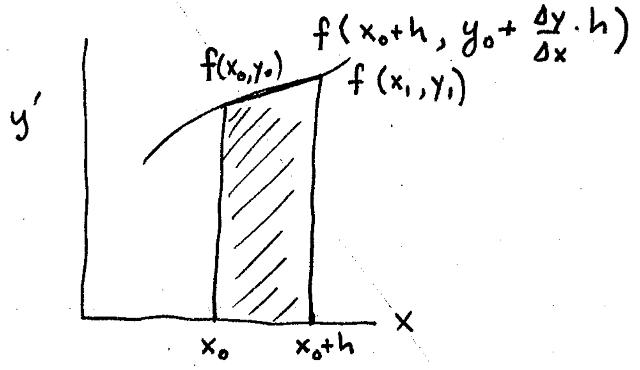
- Adams assumes that $f_1 = f_0 + \frac{\Delta f}{\Delta x} \Delta x$

THUS $\int f \, d\bar{x} = f_0 \Delta x + \frac{\Delta f}{\Delta} \cdot \Delta x^2$

so that for $\Delta x = h$ & $\Delta f = f_0 - f_{-1}$

$$\int f d\bar{x} = f_0 h + \frac{f_0 - f_{-1}}{h} \cdot \frac{h^2}{2} = y_1 - y_0$$

Similarly The 2nd order Runge – Kutta method finds the area under the graph by assuming it to be a trapezoid Error = $O(h^3)$



$$\text{area} = \int f dx = \frac{f(x_1,y_1) + f(x_0,y_0)}{2} h$$

$$\tilde{y}_1 = y(x_1) = y(x_0) + \left.\frac{dy}{dx}\right|_{x=x_0} (x_1 - x_0)$$
$$= y_0 + f(x_0,y_0) h$$

$$\therefore \quad y_1 = y_0 + \frac{1}{2} h \left[f(x_0,y_0) + f\{x_0+h, \underbrace{y_0 + h f(x_0,y_0)}_{\tilde{y}_1}\} \right]$$

HW. for $y' + xy^2 = x$ at $y(0) = 1$

use 2nd order Runge - Kutta to find y at $x = 0.1$ & $x = 0.2$

for $y'' - x^2 y' - 2xy = 1$ and $y(0) = 1$ $y'(0) = 0$

use 2nd order Runge - Kutta to find y at $x = 0.1$ & $x = 0.2$

Note: you will need to do this in two steps & derive the eqns for y & p

1) $y' = p \Rightarrow y_1 = y_0 + \frac{1}{2} h [\tilde{p}_1 + p_0]$

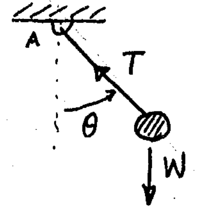
$$p' = f(x,y,p) \Rightarrow p_1 = p_0 + \frac{1}{2} h \left[f\{x_0+h, y_0 + h p_0, \underbrace{p_0 + h f(x_0,y_0,p_0)}_{\tilde{p}_1}\} + f(x_0,y_0,p_0) \right]$$

$$\tilde{p}_1 = p_0 + \left.\frac{\partial p}{\partial x}\right|_{x=x_0} \Delta x = p_0 + f(x_0,y_0,p_0) h$$

$$\tilde{y}_1 = y_0 + \left.\frac{dy}{dx}\right|_{x=x_0} \Delta x = y_0 + p_0 h$$

Self- Starting technique

- Suppose you are given a 2^{nd} order equation with conditions at more than one point



a pendulum

$$\Sigma F_r = T - mg\cos\theta = 0$$

$$\Sigma F_\theta = -mg\sin\theta = ma_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$$

$$\Sigma T_A = -mgl\sin\theta = +I_o\ddot{\theta}$$

for small angular motion $I_o\ddot{\theta} + mgl\theta = 0$

- if you were told that at $t=0$ $\Theta = \theta_o$ and $t=1$ $\Theta = \theta_1$ you would need to know what was $\dot{\theta}$ at $t=0$ to produce $\theta = \theta_1$ @ $t=1$

$$\ddot{\Theta} = -\text{const}\,\Theta \qquad \Theta(t=0) = \bar{\theta}_o \qquad \Theta(t=1) = \bar{\theta}_1$$

BOUNDARY
VALUE
PROBLEM

- Methods of Solution
- Method of finite differences – use taylor expansion about some known value

$$\theta(t+\Delta t) = \theta(t) + \theta'(t)\,\Delta t + \theta''(t)\,\frac{\Delta t^2}{2} + \dots$$

$$\Theta(t-\Delta t) = \theta(t) - \theta'(t)\,\Delta t + \theta''(t)\,\frac{\Delta t^2}{2} + \dots$$

$$\theta''(t) = \frac{\theta(t+\Delta t) - 2\theta(t) + \theta(t-\Delta t)}{\Delta t^2} + \text{error } O(\Delta t^2)$$

CENTERED
DIFFERENCE

$$\theta''(t) + \text{const}\,\theta = 0 \Rightarrow \theta(t+\Delta t) + \theta(t)[-2 + \text{const}\cdot\Delta t^2] + \theta(t-\Delta t) = 0$$

or $\theta(t+\Delta t) = \theta[2 - \text{const}\,\Delta t^2] - \theta(t-\Delta t)$

- take time difference [1−0] and divide by discretization Δt to define n
- θ at next time level depends on θ of the previous 2 time levels
- OK for every time level but cannot start
- since $\theta(t=0)$ and $\theta(t=1)$ are known this is implicit

$$\begin{bmatrix} -1 & (2-\text{const}\,\Delta t^2) & -1 & & \\ & -1 & (&) & -1 \\ & & & & \\ & & & & \end{bmatrix} \begin{pmatrix} \theta_o \\ \theta_1 \\ \theta_2 \\ \theta_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \Rightarrow A\underset{\sim}{\Theta} = b$$

- IF $\dot{\theta}(t=0)$ was given along with $\theta(t=0)$: explicit scheme

$$\frac{\dot{\theta}^2}{2} + \frac{mgl\cos\theta}{I_0} = \mathcal{C}$$

$$\dot{\theta} =$$

$t = 0 \quad \theta = 0$

$\dot{\theta} = ?\alpha$

$$\frac{\dot{\theta}^2}{2} - \frac{mgl}{I_0} 1 = \mathcal{C}.$$

$$\mathcal{C} = \frac{\alpha^2}{2} - \frac{mgl}{I_0}$$

$t = 1 \quad \theta = \pi/2$

$\dot{\theta}^2 =$

$\theta(t=0) = 0 \qquad \theta(t=1) = \pi/2 \qquad \frac{mgl}{I_0} = 1$

$$\frac{\dot{\theta}^2}{2} - 1\cos\theta = \mathcal{C}$$

$$\frac{\dot{\theta}_0^2}{2} = \mathcal{C}$$

$$\frac{\dot{\theta}^2}{2} - \cos\theta = \frac{\dot{\theta}_0^2}{2}$$

$$\frac{\dot{\theta}^2}{2}\Big|_1 = \frac{\dot{\theta}_0^2}{2}$$

$t_n = t_0 + n\Delta t \qquad t_0 = 0 \qquad n = 1, 2, \ldots, N-1$

$t_N = 1$

- This gives a linear system of N-1 equs w/ N-1 unknowns $\theta(t_n) = \theta_n$ $\quad n = 1, 2, \ldots, N-1$

- This gives a system of equs $A\underset{\sim}{\theta} = \underset{\sim}{b}$
  or $\underset{\sim}{\theta} = A^{-1}\underset{\sim}{b}$

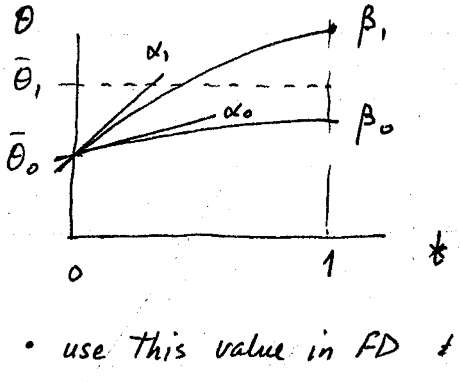
- <u>Shooting Method</u> - used basically w/ Nonlinear type ODE's
  - Difference Methods can be used but require guessing at solution and iterating to improve on the guess
  - $\ddot{\theta} + \dfrac{mgl\sin\theta}{I_0} = 0$ w/ $\theta(t=0) = \bar{\theta}_0$ & $\theta(t=1) = \bar{\theta}_1$

if θ is small

- $\ddot{\theta} + \text{const}\,\theta = 0$ to solve we need $\theta(t=0)$ & $\dot{\theta}(t=0)$
  - only have $\theta(t=0)$ & $\theta(t=1)$
  - assume $\dot{\theta}(t=0) = \alpha_0$

    use the Runge-Kutta Technique and find $\theta(t=1; \alpha_0) = \beta_0$
  - assume $\dot{\theta}(t=0) = \alpha_1$

    use Runge-Kutta technique & find $\theta(t=1; \alpha_1) = \beta_1$



New guess for $\alpha_2 = \dot{\theta}(t=0)$

$$\alpha_2 = \alpha_0 + (\alpha_1 - \alpha_0)\,\frac{\bar{\theta}_1 - \beta_0}{\beta_1 - \beta_0}$$

- use this value in FD & find $\theta(t=1; \alpha_2) = \beta_2$

- FOR SUCCEEDING GUESSES

$$\alpha_{k+1} = \alpha_{k-1} + (\alpha_k - \alpha_{k-1})\,\frac{\bar{\theta}_1 - \beta_{k-1}}{\beta_k - \beta_{k-1}}$$

CHECK IF $\dot{\theta}(t=1; \alpha_{k+1}) = \bar{\theta}_1$

$$\overbrace{\sum_{n=1} A_n n (x-x_0)^{n-1}}^{y'} - x + \sum_{n=0} A_n (x-x_0)^n$$

let $n-1=m$
$n=m+1$

$$\sum_{m=0} A_{m+1}(m+1)(x-x_0)^m + \sum_{n=0} A_n (x-x_0)^n - x$$

$m=0$ $\quad A_{m+1}(m+1) + A_m = 0 \Rightarrow A_1(1) + A_0 = 0$

$m=1$ $\quad A_2 \cdot 2 + A_1 - 1 = 0$

$m=2$ $\quad A_3 \cdot 3 + A_2 = 0$

$$y' = \frac{2x}{1-x} - \frac{1}{1-x} y$$

$$y' + \underset{P}{\left(\frac{1}{1-x}\right)} y = \frac{2x}{1-x}$$

$$\mu = e^{\int p\,dt} = e^{-\ln(1-x)} = \frac{1}{1-x}$$

$$y^* = \frac{1}{\mu}\int \mu h\,dt + \frac{C}{\mu} = \frac{1-x}{1}\int_0^x \frac{1}{1-\bar{x}} \cdot \frac{2\bar{x}}{1-\bar{x}}\,d\bar{x} + C(1-x)$$

$$\int \frac{2\bar{x}}{(1-\bar{x})^2} = \frac{2x}{1-x} + 2L(1-x)$$

$$y = 2x + 2(1-x)\ln(1-x) + C(1-x)$$

let $t=x+1 \implies ty''+2(t-1)y'-3y=0$

let $y=\sum_{n=0}^{\infty} A_n t^{n+r} = A_0t^r+A_1t^{r+1}+A_2t^{r+2}+\dots$

$$y'=\sum_{n=0}^{\infty} A_n t^{n+r-1}(n+r) = rA_0t^{r-1}+(r+1)A_1t^r+(r+2)A_2t^{r+1}+\dots$$

$$y''=\sum_{n=0}^{\infty} A_n t^{n+r-2}(n+r-1)(n+r) = r(r-1)A_0t^{r-2}+(r+1)rA_1t^{r-1}+(r+2)(r+1)A_2t^{r}+\dots$$

$$ty''+2ty'-2y'-3y=0 \implies \sum_{n=0}^{\infty} A_n t^{n+r-1}(n+r)(n+r-1)+2\sum_{n=0}^{\infty} A_n t^{n+r}(n+r)-2\sum_{n=0} A_n t^{n+r-1}(n+r)-3\sum_{n=0} A_n t^{n+r}=0$$

$$\left\{\underline{r(r-1)A_0t^{r-1}}+\underline{\underline{(r+1)rA_1t^r}}+\underline{\underline{(r+2)(r+1)A_2}}t^{r+1}+\dots\right\}+2\left\{\underline{\underline{rA_0t^r}}+\underline{\underline{(r+1)A_1t}}^{r+1}+(r+2)A_2t^{r+2}+\dots\right\}-2\left\{\underline{rA_0t^{r-1}}+\underline{\underline{(r+1)A_1t^r}}+\underline{\underline{(r+2)A_2t^{r+1}}}+\dots\right\}$$

$$-3\left\{\underline{A_0t^r}+\underline{\underline{A_1t}}^{r+1}+A_2t^{r+2}+\dots\right\}=0$$

t^{r-1}: $[r(r-1)A_0-2rA_0]=0$ $\quad A_0[r^2-r-2r]=0 \quad A_0[r^2-3r]=0 \quad r=0, r=3 \implies A_0\neq 0$

t^r: $[(r+1)rA_1+2rA_0-2(r+1)A_1-3A_0]=0 \quad (r+1)(r-2)A_1+(2r-3)A_0=0$ or $A_1=\dfrac{-(2r-3)A_0}{(r+1)(r-2)}$

t^{r+1}: $[(r+2)(r+1)A_2+2(r+1)A_1-2(r+2)A_2-3A_1]=0 \quad (r+2)(r-1)A_2+(2r-1)A_1=0$ or $A_2=-\dfrac{(2r-1)A_1}{(r+2)(r-1)}=\dfrac{(2r-3)(2r-1)A_0}{(r^2-4)(r^2-1)}$

t^{r+2}: $[(r+3)(r+2)A_3+2A_2(r+2)-2(r+3)A_3-3A_2]=0 \quad (r+3)\cdot rA_3+(2r+1)A_2=0$ or $A_3=-(2r+1)A_2/(r+3)r$

t^{r+i}: $[(r+i+1)(r+i)A_{i+1}+2(r+i)A_i-2(r+i+1)A_{i+1}-3A_i]=0 \quad A_{i+1}=-\dfrac{(2r+2i-3)A_i}{(r+i+1)(r+i-2)}$

$$\therefore A_3=\frac{-(2r-3)(2r-1)(2r+1)}{(r+2)(r+1)(r+3)(r-2)(r-1)r}A_0$$

$\cancel{A_{i+1}=(-1)^{i+1}}$

$$y_2=Ay_1\ln x+\bar{y}_2(x)$$

where $\bar{y}_2(x)=\cancel{x^0\sum A_n x^n}$ $\quad B_n$ depends on $r=0$

$$x^0\left[1+\sum_{n=1} B_n x^n\right]$$

For nontrivial solution, the corresponding determinant must be zero, i.e.

$$\det(\Omega,k)=0 \tag{13}$$

Equations (10) and (13) are two nonlinear complex equations for unknowns Ω and k. A modified Muller's method was developed to find k and Ω, thus yielding the resonance frequency and loss factor.

3 Numerical Results

The input parameters, unless stated otherwise, were:

$h_3/L=0.02$, $h_1/h_3=h_2/h_3=1.0$, $h_4/h_3=0.1$, $E_1/E_3=1.0$, $E_{Aa}/E_3=0.368$, $E_{Ma}/E_3=0.117$, $M_f=5.0\ ^0C$, $M_s=23.0\ ^0C$, $A_s=29.0\ ^0C$, $A_f=51.0\ ^0C$, $T_0=25.0\ ^0C$, $G/E_3=0.05$, $\rho_1=\rho_3=7800$ Kg/m^3, $\rho_2=3140$ Kg/m^3, $E_1=E_3=20.6$ N/m^2, $\xi_1=100$ J/(hr.m^2.^{0}C), $\alpha_{T1}=\alpha_{T3}=0.042$ m^2/hr, $\alpha_{T2}=0.35$ m$_2$/hr, L=0.5 m.

Figure 3 shows the effects of SMA layer temperature to system frequency factor for both the first and second modes. If the temperature of the SMA layer is less than the martensite start temperature, the increase of the temperature will decrease the system frequency factor. Beginning at the austenite start temperature, the temperature increase of SMA layer will increase the frequency factor. For the temperature above the austenite finish temperature, the increase of SMA temperature will decrease the frequency factor. This is because temperature increase will generally decrease the frequency factor. But beginning at the austenite start temperature, the phase transformation due to temperature increase will increase the Young's modulus of SMA material, thus causing the increase of the

Series solutions

First Order Equations

- Given $y' = f(x,y)$ A solution exists if $f(x,y)$ is continuous &
  - single valued over the region of interest
  - $\partial f/\partial y$ exists & is continuous

- if so we can assume $y = \sum_{n=0}^{\infty} A_n x^n$ and all the A_n's can be determined in terms of A_0 & A_0 can be found if an initial value is given. if $y = y_0$ at $x = x_0$ use $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ & $A_0 = y_0$

series
$S_i = \sum_{j=0}^{i} a_j$ converges to S

$\lim_{i\to\infty} S_i = S$ if $\sum |a_i|$ converges
& if $\left|\frac{a_{i+1}}{a_i}\right| < 1$

here $a_i = A_i (x-x_0)^i$

- Radius of convergence $\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| |x-x_0| < 1$
- example $y' = x - y$ $y(x=0) = 1$ $f(x,y) = x - y$ $\frac{\partial f}{\partial y} = -1$

let $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ $x_0 = 0$ $y_0 = 1$

@ $x = x_0 = 0$ $y = y_0 = 1 = A_0$

$$y' = \sum_{n=1}^{\infty} A_n n (x-x_0)^{n-1}$$

$$y' - x + y = A_1 + 2A_2 x + 3A_3 x^2 + \cdots - x + [A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \cdots] = 0$$

Collect terms in powers of x $= (A_1 + A_0) + (2A_2 - 1 + A_1)x + (A_2 + 3A_3)x^2 + \cdots + (nA_n + A_{n-1})x^n + \cdots = 0$

$= 0 + 0.x + 0.x^2 +$

since RHS = 0

$$A_1 = -A_0$$

$$A_2 = \frac{1-A_1}{2} = \frac{1}{2} + \frac{A_0}{2} = \frac{1}{2}(1-A_1)$$

$$A_3 = -\frac{A_2}{3} = -\frac{1}{3}\cdot\frac{1}{2}(1-A_1)$$

$$A_4 = -\frac{A_3}{4} = +\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}(1-A_1)$$

$$\vdots$$

$$A_n = -\frac{A_{n-1}}{n} = (-1)^n \cdot \frac{1}{n!}(1-A_1)$$

$$y = \left[+\frac{x^2}{2} - \frac{x^3}{3!} + \cdots\right](1-A_1) + A_0 - A_0 x \qquad 1 - A_1 = 1 + A_0 = 2$$

$$= 1 - x + 2\left[\frac{x^2}{2} - \frac{x^3}{3!} + \cdots\right] = -(1-x) + 2\left[1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right]$$

$$y = 2e^{-x} - 1 + x$$

radius of convergences $\left|\frac{A_{n+1}}{A_n}\right| |x-x_0| < 1 \Rightarrow \frac{1}{(n+1)!}/\frac{1}{n!} \; |x| \to 0 \Rightarrow |x| < \infty$

HW $(1-x)y' = 2x - y$ $y = y_0$ when $x = 0$

$$y = A_0 x^m + A_1 x^{m+1} + A_2 x^{m+2} + \dots + A_n x^{m+n}$$

$$y' = mA_0 x^{m-1} + (m+1)A_1 x^m + (m+2)A_2 x^{m+1} + \dots + (m+n)A_n x^{m+n-1}$$

$$y'' = m(m-1)A_0 x^{m-2} + (m+1)m A_1 x^{m-1} + (m+2)(m+1)A_2 x^m + \dots + (m+n)(m+n-1)A_n x^{m+n-2}$$

$$y'' + xy'' + 2xy' - 3y \quad = (1+x)y'' + 2xy' - 3y = 0 \qquad \text{if } x \neq -1$$

$$m(m-1)A_0 x^{m-2} + (m+1)mA_1 x^{m-1} + (m+2)(m+1)A_2 x^m + \dots (m+n)(m+n-1)A_n x^{m+n-2}$$

$$m(m-1)A_0 x^{m-1} + (m+1)mA_1 x^m + \dots$$

$$2mA_0 x^m + 2(m+1)A_1 x^{m+1} + 2(m+n)A_n x^{m+n}$$

$$-3A_0 x^m + \dots$$

$3m^2A_1$ ou.

$$m(m-1)A_0 x^{m-2} + \left[\cancel{(m+1)mA_1 + m(m-1)A_0}\right]x^{m-1} +$$

either $A_0 = 0$ or $m=0$ $m=1$ $\left[(2m-3)A_0 + (m+1)mA_1 + (m+2)(m+1)A_2\right]x^m$

$$A_1 = \frac{-(m+1)m A_0}{m(m+1)} \qquad A_2 = -\frac{(2m-3)A_0}{(m+2)(m+1)} - \frac{(m+1)mA_1}{(m+2)(m+1)}$$

$m=0$ $\qquad -3A_0 + 2A_2$

$A_0 \neq 0$ $m=1$ $\qquad A_1 = 0$ $\qquad -A_0 + 2A_1 + 6A_2$

$$3y'' + 5y' + 8y = 0 \qquad y = \sum A_n(x^n) \quad \text{near } x=0$$

$$3(n)(n-1)\sum_{n=2} A_n x^{n-2} + 5n\sum_{n=1} A_n x^{n-1} + 8\sum_{n=0} A_n x^n$$

let $m = n-2$
$v = n-1$

$$3(m+2)(m+1)\sum_{m=0} A_{m+2}x^m + 5(v+1)\sum_{v=0} A_{v+1}x^v + 8\sum_{n=0} A_n x^n = 0$$

$n=0$ $\qquad 3(2)(1)\,A_2 + 5\cdot 1 A_1 + 8A_0 = 0 \qquad A_2 = -\frac{5}{6}A_1 - \frac{8}{6}A_0$

$n=1$ $\qquad 3(3)(2)\,A_3 + 5\cdot 2A_2 + 8A_1 = 0 \qquad A_3 = -\frac{10A_2}{3\cdot 6} - \frac{8A_1}{3\cdot 6}$

$$= -\frac{10}{18}\left[-\frac{5}{6}A_1 - \frac{8}{6}A_0\right] - \frac{8}{18}A_1$$

so we see that the A_i's can be written in terms of A_0 & A_1

$$\therefore \quad y = A_0 + A_1 x + \left[-\frac{5}{6}A_1 - \frac{8}{6}A_0\right]x^2 + \left[\frac{2}{18\cdot 6}A_1 + \frac{80}{18\cdot 6}A_0\right]x^3 + \dots$$

Linear Equations of order 2

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

if $P_0(a) \neq 0$ at $x=a$ $\Rightarrow$ $x=a$ is an ordinary point then $y(x) = \sum_{n=0}^{\infty} a_n (x-$

and $y = c_1 y_1(x) + c_2 y_2(x)$ $\qquad y_1(x) = \sum_{n=0}^{\infty} A_n (x-a)^n$

$y_2(x) = \sum_{n=0}^{\infty} B_n (x-a)^n$

$y_1(x)$ and $y_2(x)$ are linearly independent & analytic at $x=a$

IF $P_0(a) = 0$ $\Rightarrow$ $x=a$ is a singular point

- a is a regular singular point if

  $\frac{P_1(x)}{P_0(x)}(x-a)$ and $\frac{P_2(x)}{P_0(x)}(x-a)^2$ can be expanded in Power series about $x=a$

  $\sum_{n=0}^{\infty} C_n (x-a)^n \qquad \sum_{n=0}^{\infty} D_n (x-a)^n$

- a is an irregular singular point if cannot be expanded in series in terms of $x-a$

radius of Convergence is $|x| = a$

- Example — Regular singular point

$$(1+x)y'' + 2xy' - 3y = 0 \quad \text{at } x=-1$$

$$\frac{P_1}{P_0}(x+1) = 2x = 2(x+1) - 2 \qquad C_0 = -2,\ C_1 = 2,\ C_2 \dots, C_\infty =$$

$$\frac{P_2}{P_0}(x+1)^2 = -3(x+1) \qquad D_0 = 0 \quad D_1 = -3 \quad D_2 \cdots D_\infty = 0$$

$$t = \frac{1}{x} \qquad \frac{d}{dx} = \frac{d}{dt} \cdot \frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt}$$

$$\frac{d^2}{dx^2} = \frac{d}{dx}\left(\frac{d}{dx}\right) = +2\frac{1}{x^3}\frac{d}{dt} - \frac{1}{x^2}\frac{d}{dx}\left(\frac{d}{dt}\right)$$
$$+\frac{1}{x^4}\frac{d^2}{dt^2}$$

$$xy'' + y' - y = 0 \implies x\left[\frac{2}{x^3}\frac{dy}{dt} + \frac{1}{x^4}\frac{d^2y}{dt^2}\right] + -\frac{1}{x^2}\frac{dy}{dt} - y = 0$$
$$2t^2\frac{dy}{dt} + t^3\frac{d^2y}{dt^2} - t^2\frac{dy}{dt} - y$$

$$t^3 y'' + 2t^2 y' - y = 0 \qquad t \to 0$$

$$\frac{P_1}{P_0}\cdot t = \frac{+2t^2}{t^3}(t) = +1 \quad \text{can be expressed as } \Sigma C_n(t-0)^n$$

$$\frac{P_2}{P_0}\cdot t^2 = \frac{-1}{t^3}(t^2) = \frac{1}{t} \quad \text{cannot be expressed as } \Sigma D_n(t-0)^n$$

$$(xy')' - y = 0$$

HW.

$xy''+y'-y=0$ [Show that $x=0$ is a regular singular pt

Show that $x=\infty$ ~~is not a~~ ~~does~~ may not have a series solution

x

$$y = \sum_{n=0} A_n x^{m+n} \qquad y' = \Sigma A_n(m+n)\,x^{m+n-1} \qquad y'' = \Sigma A_n(m+n)(m+n-1)x^{m+n-2}$$

$$xy''+y'-y = \sum_{n=0} A_n(m+n)(m+n-1)x^{m+n-1} + \sum_{n=0} A_n(m+n)x^{m+n-1} - \sum_{n=0} A_n x^{m+n}$$

~~let $n-1=v$ $n=v+1$ $\sum_{v=1} A_{v+1}(m+v+1)(m+v)x^{m+v} + \sum_{v=0} A_{v+1}(m+v+1)x^{m+v} - \sum_{n=0} A_n x^{m+n}$~~

~~$[A_1(m+1) + A_0]x^{m+0} + \sum_{n=1}[A_{n+1}(m+n+1)(m+n) + A_{n+1}(m+n+1) - A_n]x^{m+n}$~~

let $n-1=v$ $\quad n=v+1$

$$\sum_{v=-1}\left[A_{v+1}(m+v+1)(m+v) + A_{v+1}(m+v+1)\right]x^{m+v} - \sum_{n=0} A_n x^{m+n} = 0$$

$$\sum_{v=-1}\left[A_{v+1}(m+v+1)^2\right]x^{m+v} - \sum_{n=0} A_n x^{m+n}$$

$$\underbrace{A_0\,(m)^2}x^{m-1} + \sum_{n=0}\left[A_{n+1}(m+n+1)^2 - A_n\right]x^{m+n} = 0$$

if $n=v-1$, $n=0$ $v=1$

$$\sum_{n=0}\left[A_n(m+n)(m+n-1) + A_n(m+n)\right]x^{m+n-1} - \sum_{v=1} A_{v-1}x^{m+v-1} = 0$$

$n=0$ $$\left[A_0 m(m-1) + A_0\cdot m\right]x^{m-1} + \sum_{n=1}\left[A_n(m+n)^2 - A_{n-1}\right]x^{m+n-1} = 0$$

$$A_0 m^2 x^{m-1} + \Sigma[\qquad]x^{m+n-1} = 0 = 0\cdot x^{m-1} + 0x^m + 0x^{m+1} + \cdots$$
$$A_0 m^2 = 0 \quad A_0 \neq 0 \quad m = 0$$

- FOR $x=a$ BEING AN IRREGULAR SINGULAR PT – A SOLUTION IN Power series form may or may not exist.

- FOR REGULAR SINGULAR PTS USE METHOD OF FROBENIUS (ABOUT $x=0$)
  - COEFF OF FIRST TERM IN POWER SERIES EXPANSION = INDICIAL EQN
  - INDICIAL EQUATION GIVES m (FOR 2nd order m_1 & m_2)
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ is not integer

    $\Rightarrow$ 2 DISTINCT SOLNS EACH OF FORM $x^{m_1}\left[1+\sum_{n=0}^{\infty} A_n x^n\right]$ & $x^{m_2}\left[1+\sum_{n=0}^{\infty} B_n x^n\right]$

    where A_n depends on m_1 & B_n depends on m_2
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ IS an integer
    - $\Rightarrow$ LARGER ROOT ALWAYS GIVES SOLN $m = m_1$ $\quad y_1 = x^{m_1}\left[1+\sum_{n=0}^{\infty} A_n x^n\right]$
    - $$y_2(x) = A\, y_1(x) \ln x + \bar{y}_2(x) \qquad \bar{y}_2 = x^{m_2}\left[1+\sum_{n=0}^{\infty} B_n x^n\right]$$
      $$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$m=m_2$ (2nd) can yield a soln no solution or a general soln

  - IF $m_1 = m_2$

$$y_2(x) = y_1(x) \ln x + \bar{y}(x) \qquad \bar{y} = x^{m_1}\sum_{n=0}^{\infty} B_n x^n$$

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

  - AN EASIER FORM IS IF you know $y_1(x)$

$$y_2(x) = \left.\frac{\partial y_1}{\partial m}\right|_{m=m_1}$$

  - WHAT IF $a \neq 0$ let $t = x-a$ $\Rightarrow$ $x=a$ $t=0$

$$\frac{d}{dx}(\ ) = \frac{d}{dt}(\ )\cdot\frac{dt}{dx} = \frac{d}{dt}(\ )$$

  - WHAT IF $a=\infty$ let $t=\frac{1}{x}$ $\Rightarrow$ $x \to \infty$ $t \to 0$

Regular Singular

Example $\quad xy'' + y' - y = 0 \qquad \frac{P_1}{P_0}x = \frac{1}{x}\cdot x = 1 \qquad \frac{P_2}{P_0}\cdot x^2 = \frac{-1}{x}\cdot x^2 = -x$

let $y = x^m \sum_{n=0}^{\infty} A_n x^n$

$$y' = m x^{m-1}\sum_{n=0}^{\infty} A_n x^n + x^m \sum_{n=1}^{\infty} A_n n x^{n-1}$$

$$y'' = m(m-1)x^{m-2}\sum A_n x^n + 2m x^{m-1}\sum A_n n x^{n-1} + x^m \sum_{n=2} A_n n(n-1) x^{n-2}$$

$$\frac{\partial y_1}{\partial m} = \frac{\partial x^m}{\partial m}\bar{y} + x^m\frac{\partial \bar{y}}{\partial m}$$

$$= \frac{\partial e^{m\ln x}}{\partial m}\bar{y} + "$$

$$= (e^{m\ln x}\ln x)\bar{y} + "$$

$$= (x^m \ln x)\bar{y} + x^m\frac{\partial \bar{y}}{\partial m}$$

$y = x^2$ $\quad$ $y = x$

$y' = 2x$ $\quad$ $y' = 1$ $\qquad$ $ay'' + by' + cy$

$y'' = 2$ $\quad$ $y'' = 0$ $\qquad$ $a\cdot 2 + b\cdot 2x + Cx^2 = 0$ $\quad$ or $\quad$ $2 + p\cdot 2x + q\cdot x^2 = 0$

$a\cdot 0 + b\cdot 1 + C\cdot x = 0$ $\qquad$ $0 + p\cdot 1 + q\cdot x = 0$

$$\begin{pmatrix} 2x & x^2 \\ 1 & x \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$$

$$p = \frac{\begin{pmatrix} -2 & x^2 \\ 0 & x \end{pmatrix}}{x^2} = \frac{-2x}{x^2} = \frac{-2}{x}$$

$$q = \frac{\begin{pmatrix} 2x & -2 \\ 1 & 0 \end{pmatrix}}{x^2} = \frac{2}{x^2}$$

$$y'' - \frac{2}{x}y' + \frac{2}{x^2}y = 0$$

$$x^2y'' - 2xy' + 2y = 0$$

$$xy'' + y' - y = m^2 A_0 x^{m-1} + \left[(m+1)^2 A_1 - A_0\right]x^m + \left[(m+2)^2 A_2 - A_1\right]x^{m+1}$$

$$+ \dots + \left[(m+n)^2 A_n - A_{n-1}\right]x^{m+n-1} + \dots = 0$$

$$\Rightarrow \quad A_n = \frac{A_{n-1}}{(m+n)^2} \qquad n = 1 \dots\dots \infty$$

$$\Rightarrow \quad A_0 = 0 \quad \text{or} \quad m = 0 \quad \text{equal roots}$$

$$A_n = \frac{A_{n-1}}{(m+n)^2} = \frac{A_{n-2}}{(m+n)^2(m+n-1)^2} = \dots\dots \frac{A_0}{\left[(m+n)(m+n-1)\cdots(m+1)\right]^2}$$

$$\therefore \quad y_1 = x^m A_0 \sum \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \underset{\frac{x^3}{(m+1)^2(m+2)^2(m+3)^2}}{\dots\dots}\right] = x^m \bar{y}_1$$

$$\text{for } m = 0 \qquad y_1 = x^m A_0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = x^m \bar{y}_1$$

$$y_2 = \frac{\partial y_1}{\partial m} = x^m \ln x \, \bar{y}_1 + x^m A_0 \sum \left[\frac{-2x}{(m+1)^3} - \left\{\frac{2}{(m+1)^3(m+2)^2} + \frac{2}{(m+1)^2(m+2}\right.\right. x^2$$

$$@\ m = 0 \qquad y_2 = \bar{y}_1 \ln x + A_0 \sum \left[\frac{-2x}{1} - \left[\frac{2}{4} + \frac{2}{8}\right]x^2 - \left[\frac{2}{6} + \overset{\frac{2}{12}}{\cancel{\quad}} + \overset{2/18}{\cancel{\quad}}\right]x^3 + \dots\right.$$

$$= \bar{y}_1 \ln x - 2A_0 \left[x + \frac{1}{2!}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{3!}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \dots\right]$$

$$y = C_1 y_1(x) + C_2 y_2(x)$$

- Find a soln to $y'' = xy$ about $x = 1$ ordinary
- if x & x^2 are solns to a differential eqn ~~what~~ how can we define the d.e. about $x = 0$ what about $x = \infty$

  what is the radius of convergence
- How can you classify $2(x-2)^2 x y'' + 3xy' + (x-2)y = 0$

(c) find the series solution to $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)\,y = 0$ near $x = \infty$

HW Due 31 January

1. Find the solution to $y'' = xy$ about $x = 1$

2. If x & x^2 are solutions to a 2nd order ODE, 1) determine the governing diff. equation. 2) What is the classification of the equation near $x=0$ 3) what about at $x = \infty$. What is the radius of convergence

3. How do you classify

$$2(x-2)^2 x y'' + 3xy' + (x-2)y = 0$$

about $x = 2$, $x = 0$, $x = 5$, $x = \infty$

4. Find the series solution to $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$

near $x = \infty$

- PHYSICAL PROCESSES NORMALLY VARY WITH TIME AND LOCATION
- TO UNDERSTAND THESE PROCESSES
  - IT WOULD BE NICE TO KNOW HOW THEY VARY IN TIME & SPACE
  - WHAT DRIVES THESE PROCESSES (HOW PROCESSES DEPEND ON SYSTEM PARAM
  - WHERE THESE PROCESSES WILL BE AT SOME FUTURE TIME OR WHAT WILL HAPPEN AT SOME FUTURE LOCATION
- IT TURNS OUT THAT EQNS THAT DESCRIBE THESE PROCESSES ARE GENERALLY DIFFERENTIAL EQUATIONS
- WHEN THESE PROCESSES DEPEND ON THE VARIATION OF TWO OR MORE QUANTITIES, THEN THESE PROCESSES ARE GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS
- MANY PROCESSES IN NATURE ARE DESCRIBED BY 2nd ORDER P.D.E.
- EXAMPLES VIBRATIONS OF A ROD (LONGITUDINAL VIBRATIONS)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$c = \sqrt{E/\rho}$ BAR VELOCITY

u - LONGITUDINAL DISPLACEMENT

HEAT TRANSFER

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$\frac{k}{c\rho} = \alpha$ THERMAL DIFFUSIVITY

T TEMPERATURE

POTENTIAL FLOW

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0$$

LAPLACE'S EQN

ϕ VELOCITY POTENTIAL $\qquad \underset{\sim}{V} = \nabla \phi$

$v = \frac{\partial \phi}{\partial y} \qquad u = \frac{\partial \phi}{\partial x}$

STEADY STATE INCOMPRESSIBLE CONTINUITY EQN IS $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

IN THESE CASES u, T, ϕ ARE FIELD VARIABLE OR DEPENDENT VARIABLE,

WHILE x, y, t SPACE COORDINATES OR TIME ARE INDEPENDENT VARIABLES

PARTIAL DERIV IS DEFINED BY

$$\frac{\partial}{\partial x} u(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

- THESE THREE EQNS CAN DESCRIBE MANY SYSTEMS IN MECH. ENG.
  - WAVE EQUATION ARISES FROM ACOUSTICS, VIBRATIONS, SHALLOW-WATER WAVE THEORY
  - HEAT EQN ARISES IN HEAT TRANSFER & ONE-DIMENSIONAL DIFFUSION PROBLEMS
  - LAPLACE'S EQN ARISES IN POTENTIAL FLOW THEORY, STEADY STATE HEAT TRANSFER, TORSION OF A BAR, STEADY STATE VIB. OF A MEMBRANE

- DO ALL THREE HAVE ANY COMMON IDEAS?
- HOW CAN THEY BE SOLVED? WHAT METHODS EXIST TO SOLVE THE EQNS
- HOW CAN I DERIVE THE MATHEMATICAL EQN?
- WHAT IS A WELL POSED PROBLEM - CAN I FIND A UNIQUE SOLUTION

- CHARACTERIZATION & CLASSIFICATION
  - MOST GENERAL 2nd ORDER PDE OF A FN $u(x,y)$

$$\varphi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \quad \text{i.e., } u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

- φ: LINEAR WITH RESPECT TO HIGHEST DERIVATIVE WHEN

$$a u_{xx} + b u_{xy} + c u_{yy} + F(x, y, u, u_x, u_y) = 0$$

  - HERE a, b, c are fns of x, y only.
  - IF $F(x, y, u, u_x, u_y)$ quasilinear

  IF $F(x, y, u, u_x, u_y) = d u_x + e u_y + f u + g$ LINEAR

  IF $g = 0$ THEN IT IS HOMOGENEOUS

- WE WILL USE THE METHOD OF CHARACTERISTICS TO FIND SOLUTIONS TO PDE

  - IDEA : TO TRANSFORM EQN SO THAT ALONG CERTAIN LINES - ONLY DERIV ALONG THESE LINES EXIST & CAN BE INTEGRATED AS IF THE EQN WERE ODE. LINES ARE CHARACTERISTICS

- IN LINEAR PROBLEMS CHARACTERISTICS DEPEND ON COEFF a, b, c
- NON LINEAR CAN ALSO DEPEND ON SOLUTION, u, ITSELF

- LOOK AT SIMPLE FIRST ORDER PROBLEM

$$A u_x + B u_t + C = 0 \qquad u=u(x,t) \qquad A, B, C \text{ are fn of } x, t, u$$

- FIND TRANSFORMATION $\xi = \xi(x,t) \qquad \eta = \eta(x,t) \Rightarrow x = x(\eta,\xi) \quad t = t(\eta,\xi)$

  SO THAT EQN ~~ONLY~~ HAS DERIVATIVES OF EITHER ξ (OR η) ONLY

- e.g. $u_x = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x$

- $u_t = u_\xi \xi_t + u_\eta \eta_t$

  PUT INTO $A u_x + B u_t + C = 0$ & COLLECT TERMS IN u_ξ & u_η

- $\Rightarrow \left(A \xi_x + B \xi_t\right) u_\xi + \left(A \eta_x + B \eta_t\right) u_\eta + C = 0$

- WE WANT COEFF OF u_ξ (OR u_η) $= 0$ along const ξ (or η)

- $\Rightarrow$ e.g., $A \xi_x + B \xi_t = 0$ along constant ξ line ①

- ALONG ANY ξ line $d\xi = \xi_x \, dx + \xi_t \, dt$
- ALONG CONSTANT ξ line $d\xi = 0 = \xi_x \, dx + \xi_t \, dt$ ②

THUS ① & ② YIELD $\begin{bmatrix} A & B \\ dx & dt \end{bmatrix} \begin{pmatrix} \xi_x \\ \xi_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- NON TRIVIAL SOLUTION IS $A\,dt - B\,dx = 0$ or $\frac{dx}{dt} = \frac{A}{B}$

- THIS GIVES SLOPE OF CHARACTERISTICS & $x = \int^{t} \frac{A}{B}\,dt' + \text{constant}$

$$\therefore \quad x - \int^{t} \frac{A}{B}\,dt' = \underset{\text{LET IT BE } \xi}{\text{constant}} \Longrightarrow x - \int^{t} \frac{A}{B}\,dt' = \xi$$

THIS IS ONE OF
CHARACTERISTI

- TO FIND η PICK ANY LINE THAT <u>INTERSECTS</u> ξ FOR EXAMPLE $\eta = t$

- THUS $\eta_t = 1$ $\eta_x = 0$ & $(A\eta_x + B\eta_t)\,u_\eta + C = 0 \Longrightarrow Bu_\eta + C = 0$

- WE CAN THEN INTEGRATE THIS ALONG THE CHARACTERISTIC ξ

$$u_\eta = -\frac{C}{B} \qquad u = \int^{\eta} -\frac{C}{B}\,d\bar{\eta} + f(\xi)$$

Exercise 7.1

- EXAMPLE $\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x} = \bar{A}\sin(\pi x/L)$

in our $Au_x + Bu_t + C = 0$
$A = V$ $B = 1$ $C = -\bar{A}\sin($
$u = T$

$$\frac{dx}{dt} = \frac{A}{B} = V \quad \therefore \quad x - Vt = \xi \text{; choose } \eta = t \text{;}$$

$$x = \xi + Vt = \xi + V\eta$$

$$Bu_\eta + C = 1 \cdot T_\eta - \bar{A}\sin\left(\frac{\pi}{L}(\xi + V\eta)\right) = 0 \text{;} \quad \text{INTEGRATE WRT } \eta$$

$$\therefore \quad T = \int^{\eta} \bar{A}\sin\frac{\pi}{L}(\xi + V\eta')\,d\eta' + f(\xi)$$

$$T = -\frac{\bar{A}}{V}\frac{L}{\pi}\cos\frac{\pi}{L}(\xi + V\eta) + f(\xi)$$

$$\therefore \quad T(x,t) = -\frac{\bar{A}}{V}\frac{L}{\pi}\cos\frac{\pi}{L}(x - vt + Vt) + f(x - vt)$$

$$= -\frac{\bar{A}L}{V\pi}\cos\frac{\pi x}{L} + f(x - vt)$$

- FOR 2nd order quasilinear PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0$

A, B, C, D are fns of x, y, u, u_x, u_y

$$C_2 = \frac{C\eta_y}{\eta_x} C_1 \qquad \text{but } \frac{\eta_x}{\eta_y} = -y'$$

$$C_2 = -\frac{C}{y'} C_1$$

$$V_\xi \left[(A\xi_x + B\xi_y) C_1 + \xi_y \left(\frac{-C}{y'}\right) C_1 \right] + W_\xi \left[(C\xi_y) C_1 + (-\xi_x) - \frac{C}{y'} C_1 \right] + C_1 D = 0$$

$$\Rightarrow \quad C_1 \neq 0$$

$$V_\xi \left[(A\xi_x + B\xi_y) + \xi_y \left(\frac{-C}{y'}\right) \right] + W_\xi \left[C\xi_y + \frac{C\xi_x}{y'} \right] + D = 0$$

now $$y - \int \frac{B+\sqrt{}}{2A}\, dx = C_1 = \eta$$

$$y - \int \frac{B-\sqrt{}}{2A}\, dx = C_2 = \xi$$

$$\xi_y = 1 - \int \frac{\partial}{\partial y} [\ -\ ]\, dx$$

$$\xi_x = -[\ -\ ]$$

$$y' = \frac{B-\sqrt{}}{2A} = \frac{2C}{B+\sqrt{}}$$

$$V_\xi \left[A\xi_x + B\xi_y - \xi_y \left[\frac{B+\sqrt{}}{2} \right] \right] + \frac{CW_\xi}{y'} [\, y'\xi_y + \xi_x \,]$$

$$A V_\xi \left[\xi_x + \frac{B-\sqrt{}}{2A} \xi_y \right] + \frac{CW_\xi}{y'}\, d\xi/dx$$

$$\xi_x + y'\xi_y$$

$$AV_\xi \frac{d\xi}{dx} + C\, W_\xi \frac{d\xi}{dy} + D = 0$$

$$A \frac{\partial V}{\partial x} + C \frac{\partial W}{\partial y} + D = 0 \quad \Rightarrow \quad Au_{xx} + Cu_{yy} + D = 0$$

$$\eta_y = 1 + \int \frac{\partial}{\partial y} [\ \ ]\, dx$$

$$\eta_x = [\ +\ ]$$

$$\xi_y = 1 + \int \frac{\partial}{\partial y} [\ \ ]\, dx$$

$$\xi_x = [\ -\ ]$$

$$y_+ = \int \left[\frac{B+\sqrt{}}{2A} \right] dx + C_1 \qquad C_1 = \eta_+ \quad \sim \eta$$

$$y_- = \int \left[\frac{B-\sqrt{}}{2A} \right] dx + C_2 \qquad C_2 = \eta_- \quad \xi$$

For 2nd order quasilinear PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0$

- let $V = u_x \qquad W = u_y$

$\Rightarrow \quad A V_x + B V_y + C W_y + D = 0 \quad (*) \qquad$ note $u_{xy} = V_y$

also $\quad V_y - W_x = 0 \quad (**) \quad \Rightarrow \quad u_{xy} = u_{xy}$

- 2 eqns for V & W

- NEXT TRANSFORM EQNS USING $\xi = \xi(x,y) \qquad \eta = \eta(x,y)$

$$\Rightarrow \quad V_x = V_\xi \xi_x + V_\eta \eta_x \qquad W_x = W_\xi \xi_x + W_\eta \eta_x$$
$$V_y = V_\xi \xi_y + V_\eta \eta_y \qquad W_y = W_\xi \xi_y + W_\eta \eta_y$$

- PUT INTO (*) & (**). TAKE $C_1 \cdot$ TRANSFORMED 1 $+ C_2 \cdot$ TRANSFORMED 2 $= 0$
- Collect terms involving V_ξ, V_η, W_ξ, W_η

$$\cdot \; V_\xi\left[(A\xi_x + B\xi_y)C_1 + (\xi_y)C_2\right] + W_\xi\left[(C\xi_y)C_1 + (-\xi_x)C_2\right]$$
$$+ V_\eta\left[(A\eta_x + B\eta_y)C_1 + (\eta_y)C_2\right] + W_\eta\left[(C\eta_y)C_1 + (-\eta_x)C_2\right] + C_1 D = 0$$

- AS BEFORE WANT COEFF OF DERIVS WRT EITHER ξ (OR η) TO VANISH ALONG CONSTANT ξ (OR η)

- LOOK ALONG CONSTANT η $\Rightarrow$
  V_η & W_η TERMS MUST VANISH

$$(A\eta_x + B\eta_y)C_1 + (\eta_y)C_2 = 0$$
$$(C\eta_y) \quad C_1 + (-\eta_x) C_2 = 0$$

- for non zero solutions $\Rightarrow$ determinant of coeffs of C_1 & $C_2 = 0$

$$\begin{vmatrix} A\eta_x + B\eta_y & \eta_y \\ C\eta_y & -\eta_x \end{vmatrix} = 0$$

or $\quad -A\eta_x^2 - B\eta_x\eta_y - C\eta_y^2 = 0 \qquad$ and $\quad C_2 = C\dfrac{\eta_y}{\eta_x} C_1$

- ALONG constant η lines $\quad d\eta = \eta_x dx + \eta_y dy = 0 \quad$ or $\quad \eta_x = -\eta_y y'$

$$\Rightarrow -\eta_y^2\left[A(y')^2 - By' + C\right] = 0 \qquad \text{if } \eta_y \neq 0$$

- $$\frac{dy}{dx} = y' = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \qquad \text{slope of characteristics}$$

$\Rightarrow \quad f_1(x,y) = \text{const} = \eta + f_2(x,y) = \text{const} = \eta \quad$ or $\quad \xi, \eta$

$u_\xi \xi_{xx} + u_\eta \eta_{xx}$ $\qquad$ $u_\xi \xi_{xy} + u_\eta \eta_{xy}$

$$A\left(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2\right) + B\left(u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y\right)$$
$$+ C\left(u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2\right) + \cdots$$

$u_\xi \xi_{yy} + u_\eta \eta_{yy}$

now $\quad Au_{xx} + Bu_{xy} + Cu_{yy} + D(u_x, u_y, u, x, y) = 0$

transforms to $\quad a_{11}u_{\xi\xi} + \overset{\text{no}}{\cancel{2}}a_{12}u_{\xi\eta} + a_{22}u_{\eta\eta} + \bar{D}(u_\xi, u_\eta, u, \xi, \eta) = 0$

$$a_{11} = A\xi_x^2 + \overset{\text{no x}}{\cancel{2}}B\xi_x\xi_y + C\xi_y^2$$
$$a_{12} = \underset{\uparrow\checkmark}{2}A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + \underset{\uparrow\checkmark}{2}C\xi_y\eta_y$$
$$a_{22} = A\eta_x^2 + \underset{\uparrow\text{no x}}{\cancel{2}}B\eta_x\eta_y + C\eta_y^2$$

if ξ, η are real characteristics then $a_{11}, a_{22} = 0 \qquad a_{12} \neq 0 \qquad (B^2 - 4AC > 0)$

$\therefore$ for hyperbolic gives $u_{\xi\eta} + \dfrac{\bar{D}}{\cancel{2}a_{12}} = 0$

now if $\alpha = \dfrac{\xi+\eta}{2} \quad \beta = \dfrac{\xi-\eta}{2}$ then another form is $u_{\alpha\alpha} - u_{\beta\beta} + 2\overset{\bar{\Phi}}{\cancel{\dfrac{\bar{D}}{a_{12}}}} = 0$

$y - \int^x \frac{B}{2A}d\bar{x}$

if $B^2 - 4AC = 0$ then ξ is a real characteristic $= \cancel{\dfrac{B}{2A}}$ & η is any line that intersects it

$\Rightarrow a_{11} = 0$ & $a_{22} = 0 \quad \therefore$

for parabolic $\quad u_{\eta\eta} + \dfrac{\bar{D}}{a_{22}} = 0$

if $B^2 - 4AC < 0 \qquad \xi$ & η are complex conj imaginary let $\alpha = \dfrac{\xi+\eta}{2} \quad \beta = \dfrac{\xi-\eta}{2i}$ $\qquad \xi = \alpha + i\beta$ $\quad \eta = \alpha - i\beta$

& $a_{11} = a_{22}$ & $a_{12} = 0 \quad \therefore$ for elliptic $\quad u_{\alpha\alpha} + u_{\beta\beta} + \dfrac{\bar{D}}{a_{22}} = 0$

- IF $B^2 - 4AC > 0$ 2 distinct real characteristics : HYPERBOLIC PROBLEM
- IF $B^2 - 4AC = 0$ 1 real characteristic : parabolic problem
- IF $B^2 - 4AC < 0$ no real characteristics : elliptic problem.

$$\left.\begin{aligned} u_{xx} - u_{tt} &= 0 \\ u_{xx} - u_t &= 0 \\ u_{xx} + u_{tt} &= 0 \end{aligned}\right\} \text{examples}$$

- WE THEN HAVE $V_\xi [\qquad\qquad] + W_\xi [\qquad\qquad] + C, D = 0$

- AS BEFORE IF WE SUBSTITUTE FOR $\xi_x, \xi_y, \eta_x, \eta_y$ WE CAN INTEGRATE TO FIND V AS A FN OF W

- SOLUTIONS CAN BE OBTAINED FOR LINEAR PROBLEMS – CONSTANT COEFFICIENT
  - WITH THE TRANSFORMATIONS $\xi = \xi(x,y)$ $\eta = \eta(x,y)$

we can transform the general equation

$$A u_{xx} + B u_{xy} + C u_{yy} + b_1 u_x + b_2 u_y + cu + f(x,y) = 0$$

using the characteristics

$$y - \left[\frac{B + \sqrt{B^2 - 4AC}}{2A}\right] x = \xi \qquad y - \left[\frac{B - \sqrt{B^2 - 4AC}}{2A}\right] x = \eta$$

into the following basic forms.

$$u_{\xi\xi} + u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \qquad \text{elliptic type}$$

$$\left.\begin{aligned} u_{\xi\eta} - \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} &= 0 \\ u_{\xi\xi} - u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} &= 0 \end{aligned}\right\} \qquad \text{hyperbolic type}$$

$$u_{\xi\xi} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \qquad \text{parabolic}$$

Sp06 class

- $xu_{xx} + u_{yy} = 0$ find regions where eq is hyperbolic, elliptic & parabolic

  transform the region in which it is elliptic into canonical form.

  transform to canonical
- $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$
- $\sin^2 y\, u_{xx} - e^{2x}u_{yy} + 3u_x - 5u = 0$

p. 67 15 in Trim

Su12 do 9, 13, 15, 20

p. 67 9, 15, 14, 18

- note that the basic forms still hold first derivatives
- basic forms only convert eqns containing xx, yy, xy derivs to xx, yy or xy
- WANT TO GET RID OF FIRST DERIVATIVES IF POSSIBLE
- to further simplify define

$$u = e^{\lambda\xi+\mu\eta}v \qquad \underline{\text{if}}\ \tilde{b}_1, \tilde{b}_2 \text{ are constant}$$

$$u_\xi = \lambda e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\xi = e^{\lambda\xi+\mu\eta}(v_\xi+\lambda v)$$

$$u_\eta = e^{\lambda\xi+\mu\eta}(v_\eta+\mu v)$$

$$u_{\xi\xi} = \lambda e^{\lambda\xi+\mu\eta}(v_\xi+\lambda v) + e^{\lambda\xi+\mu\eta}(v_{\xi\xi}+\lambda v_\xi) = e^{\lambda\xi+\mu\eta}(v_{\xi\xi}+2\lambda v_\xi+\lambda^2 v)$$

$$u_{\xi\eta} = e^{\lambda\xi+\mu\eta}(v_{\xi\eta}+\lambda v_\eta+\mu v_\xi+\lambda\mu v)$$

$$u_{\eta\eta} = e^{\lambda\xi+\mu\eta}(v_{\eta\eta}+2\mu v_\eta+\mu^2 v)$$

for example put into elliptic type.

$$e^{\lambda\xi+\mu\eta}\left[v_{\xi\xi}+v_{\eta\eta}+(\tilde{b}_1+2\lambda)v_\xi+(\tilde{b}_2+2\mu)v_\eta\cancel{\xi}+(\lambda^2+\mu^2+\tilde{b}_1\lambda+\tilde{b}_2\mu+\tilde{c})v+\tilde{f}_1\right]=$$

$$f_1 = \tilde{f}e^{-(\lambda\xi+\mu\eta)}$$

Pick $\lambda = -\frac{\tilde{b}_1}{2}$ $\quad\mu = -\frac{\tilde{b}_2}{2}$

$$\Rightarrow \quad v_{\xi\xi}+v_{\eta\eta}+\overset{\gamma}{\left[-\frac{\tilde{b}_1^2}{4}-\frac{\tilde{b}_2^2}{4}+\tilde{c}\right]}v+f_1=0$$

we can reduce our basic forms to

elliptic $\quad v_{\xi\xi}+v_{\eta\eta}+\gamma v+f_1=0$

hyperbolic $\quad \begin{cases} v_{\xi\eta}+\gamma v+f_1=0 \\ v_{\xi\xi}-v_{\eta\eta}+\gamma v+f_1=0 \end{cases}$

parabolic $\quad v_{\xi\xi}+b_2 v_\eta+f_1=0$

if $f_1=0$ eigenvalue problem
if $f_1\neq 0$ inhomogeneous eqn.

$\boxed{\cdot\ \text{HW}}$ Choose λ & μ to simplify ie find $u = ve^{-(\lambda x+\mu y)}$ for

① $u_{xy} = \alpha u_x + \beta u_y$

② $u_{xx} - \frac{1}{a^2}u_{yy} = \alpha u_x + \beta u_y + \gamma u$

③ Do p. 67 (Trim) #9, 15, 14,

Example

- look at $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$
- let's reduce this : choose $u = v e^{\lambda x + \mu y}$ $\quad u = u(x,y) \quad v = v(x,y)$

$$u_x = v_x e^{\lambda x+\mu y} + \lambda v e^{\lambda x+\mu y}$$

$$u_{xx} = v_{xx} e^{\lambda x+\mu y} + 2 v_x \lambda e^{\lambda x+\mu y} + \lambda^2 \cancel{v} e^{\lambda x+\mu y} = (v_{xx} + 2\lambda v_x + \lambda^2 v) e^{\lambda x}$$

$$u_{xy} = v_{xy} e^{\lambda x+\mu y} + \mu v_x e^{\lambda x+\mu y} + \lambda v_y e^{\lambda x+\mu y} + \lambda\mu v e^{\lambda x+\mu y}$$

$$= (v_{xy} + \mu v_x + \lambda v_y + \lambda\mu v) e^{\lambda x+\mu y}$$

$$\Rightarrow \quad e^{\lambda x+\mu y} \left[v_{xx} + v_{yy} + v_x(2\lambda+\alpha) + v_y(2\mu+\beta) + v(\lambda^2+\mu^2+\alpha\lambda+\beta\mu+ \right.$$

$$= 0$$

choose $\lambda = -\alpha/2 \quad \frac{\beta}{2} = -\mu \quad \Rightarrow \lambda^2+\mu^2+\alpha\lambda+\beta\mu+\gamma = \gamma - \frac{\alpha^2+\beta^2}{4} =$

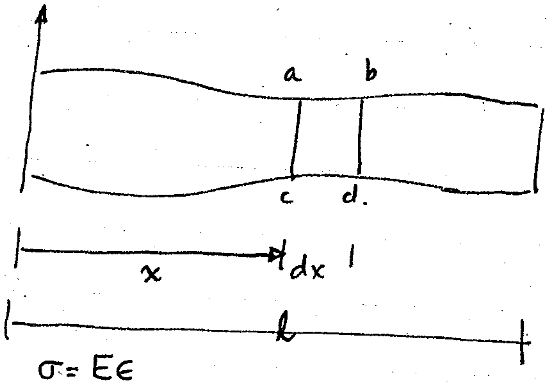
$$\Rightarrow \quad v_{xx} + v_{yy} + \gamma_1 v = 0$$

easier to solve that the previous

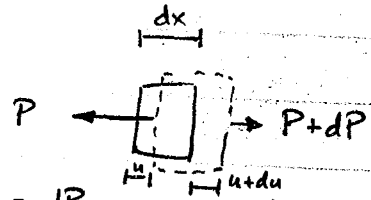
Read Chapter 1 and Chapter 3 in Trim/Reynolds

Derivation of Governing Equations

- Look at longitudinal vibration of a thin rod : $u(x,t)$ - displ of any point of rod.



$\sigma = E\epsilon$



$$\Sigma F = dP = m \cdot \text{accel}$$

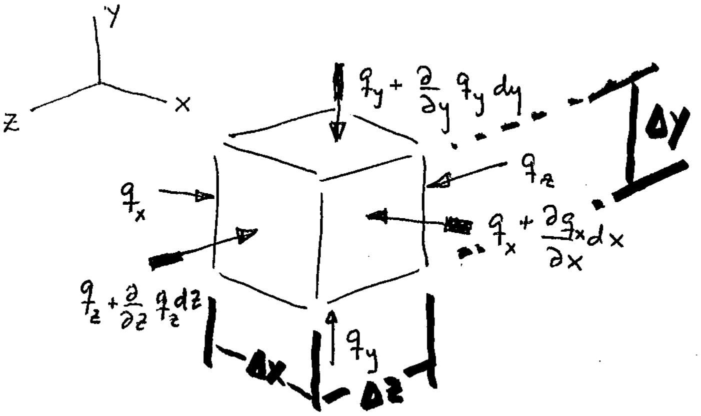
$$\underset{\text{stress}}{P = \sigma A} = \underset{\text{strain}}{E\epsilon A} = EA\frac{\partial u}{\partial x}$$

$$\therefore \quad dP = \frac{\partial \sigma A}{\partial x} dx = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) dx$$

$$m \cdot \text{accel} = \rho \, \overbrace{dx \cdot A}^{\text{Volume}} \cdot \frac{\partial^2 u}{\partial t^2} \qquad m = \rho \cdot \text{volume}$$

$$\therefore \quad \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) + f(x,t) \cdot A = \rho A \frac{\partial^2 u}{\partial t^2}$$

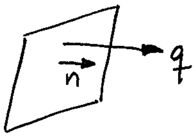
where $f(x,t)$ external force/ ~~unit length~~ per unit volume



Fourier Heat Conduction law

$$q = -kA\frac{\partial T}{\partial n}$$

heat conduction coeff



A = area ⊥ to heat flow

n = normal to area

$$-\left(\frac{\partial q_x}{\partial x}dx + \frac{\partial q_y}{\partial y}dy + \frac{\partial q_z}{\partial z}dz\right) = mc_v\frac{\partial T}{\partial t}$$

$$+\frac{\partial}{\partial x}k_xA_x\frac{\partial T}{\partial x}dx + \frac{\partial}{\partial y}k_yA_y\frac{\partial T}{\partial y}dy + \frac{\partial}{\partial z}k_zA_zdz \; \rho \, d\forall c_v\frac{\partial T}{\partial t}$$; IF $A_x = A_y = A_z = A$ & $A_x dx = A_y dy = A_z dz = d\forall$

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)d\forall + \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)d\forall + \frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)d\forall = \rho c_v d\forall\frac{\partial T}{\partial t}$$

thus $\nabla\cdot(k\nabla T) = \rho c_v\frac{\partial T}{\partial t}$ when $\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$

if k = const $\Rightarrow$ $k\nabla^2T = \rho c_v\frac{\partial T}{\partial t}$ c_v - specific heat of matter at const. volume

or $\nabla^2T = \frac{\rho c_v}{k}\frac{\partial T}{\partial t}$

$\nabla^2T = \alpha\frac{\partial T}{\partial t}$ $\alpha = \frac{\rho c_v}{k}$ = thermal diffusivity

---

now if $T \neq$ fn of t $\nabla^2T = 0$ steady state (Laplace Eq)

$T \neq$ fn of ~~x~~,y $\frac{\partial^2 ~~u~~}{\partial x^2} = \alpha\frac{\partial T}{\partial t}$ one - D heat eqn.

$$\frac{\partial^2T}{\partial x^2} - \alpha\frac{\partial T}{\partial t} = au_{xx} + bu_{xt} + cu_{tt} + du_x + eu_t + fu + g = 0$$

here $a = 1$ $b = c = d = 0$ $e = -\alpha$ $f = g = 0$

$\therefore$ $b^2 - 4ac = 0$ parabolic when $\frac{\partial^2T}{\partial x^2} = \alpha\frac{\partial T}{\partial t}$

Florida International University

MEMORANDUM

TO: **Tenure and Promotion Committee Members**
(Ebadian, Leonard, Levy, Schoephoerster, Wu, Yih)

FROM: **Ibrahim Tansel, Chairperson** I.N.T

DATE: **3 October 1996**

RE: **Tenure and Promotion Folders**

All candidates have handed in their folders for tenure and promotion; please contact me when you are ready to inspect them.

Department of Mechanical Engineering • College of Engineering and Design
University Park, Miami, Florida 33199 • (305) 348-2569 • FAX (305) 348-1932

Example

- look at $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$
- let's reduce this : choose $u = v e^{\lambda x + \mu y}$ $\quad u = u(x,y) \quad v = v(x,y)$

$$u_x = v_x e^{\lambda x+\mu y} + \lambda v e^{\lambda x + \mu y}$$

$$u_{xx} = v_{xx} e^{\lambda x+\mu y} + 2 v_x \lambda e^{\lambda x+\mu y} + \lambda^2 v^{\lambda x+\mu y} = (v_{xx} + 2\lambda v_x + \lambda^2 v) e^{\lambda x+\mu}$$

$$u_{xy} = v_{xy} e^{\lambda x+\mu y} + \mu v_x e^{\lambda x+\mu y} + \lambda v_y e^{\lambda x+\mu y} + \lambda\mu v e^{\lambda x+\mu y}$$

$$= (v_{xy} + \mu v_x + \lambda v_y + \lambda\mu v) e^{\lambda x+\mu y}$$

$$\Rightarrow \quad e^{\lambda x+\mu y} \left[v_{xx} + v_{yy} + v_x (2\lambda + \alpha) + v_y (2\mu + \beta) + v(\lambda^2 + \mu^2 + \alpha\lambda + \beta\mu + \gamma) \right. = 0$$

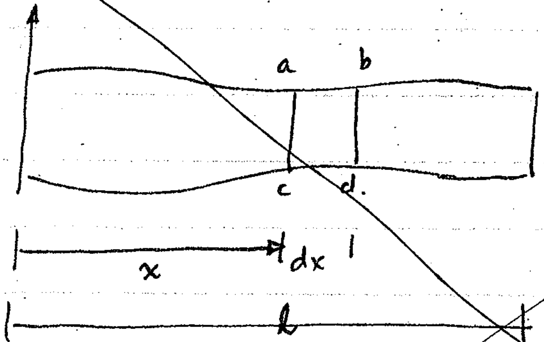
choose $\lambda = -\alpha/2 \quad \frac{\beta}{2} = -\mu \quad \Longrightarrow \lambda^2 + \mu^2 + \alpha\lambda + \beta\mu + \gamma = \gamma - \frac{\alpha^2 + \beta^2}{4} = \gamma$

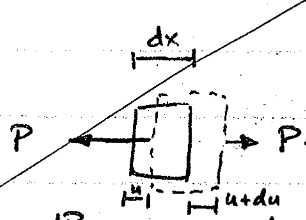
$$\Rightarrow \quad v_{xx} + v_{yy} + \gamma_1 v = 0$$ easier to solve that the previous

Read Chapter 1 and Chapter 3 in Trim/~~Reynolds~~

Derivation of Governing Equations

- Look at longitudinal vibration of a thin rod : $u(x,t)$ - displ of any point of rod.





$\sigma = E\epsilon$

$$\Sigma F = dP = m \cdot \text{accel}$$

$$\underset{\text{stress}}{P = \sigma A} = \underset{\text{strain}}{E\epsilon A} = EA\frac{\partial u}{\partial x}$$

$$\therefore \quad dP = \frac{\partial^{\sigma A}}{\partial x} dx = \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) dx$$

$$m \cdot \text{accel} = \rho \overbrace{dx \cdot A}^{\text{Volume}} \cdot \frac{\partial^2 u}{\partial t^2} \qquad m = \rho \cdot \text{volume}$$

$$\therefore \quad \frac{\partial}{\partial x}\left(EA \frac{\partial u}{\partial x}\right) + f(x,t) \cdot A = \rho A \frac{\partial^2 u}{\partial t^2}$$ where $f(x,t)$ external force/ ~~unit length area~~ per unit volume

- $xu_{xx} + u_{yy} = 0$ find regions where eq is hyperbolic, elliptic & parabolic
  transform the region in which it is elliptic into canonical form.

transform to canonical

- $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$
- $\sin^2 y\, u_{xx} - e^{2x}u_{yy} + 3u_x - 5u = 0$

p. 67 15

do 9, 13, 15, 20

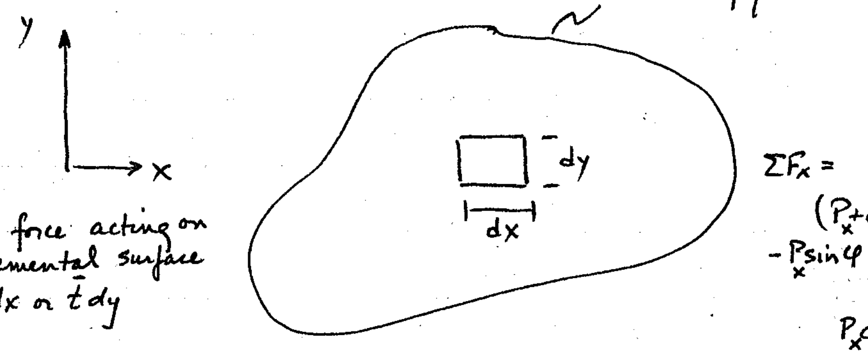
p. 67 9, 15, 14, 18

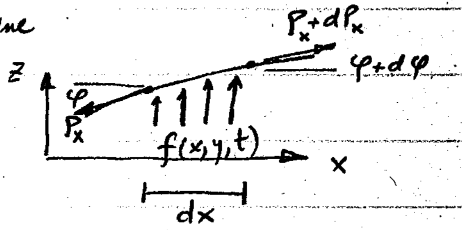
if E and A are not functions of x and $f(x,t) = 0$

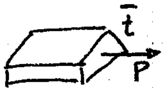
$u_{xx} - \frac{1}{c^2} u_{tt} = 0$ $\quad c = \sqrt{E/\rho}$ is the bar velocity

if $f(x,t) \neq 0$ we have an inhomogeneous problem: example vertical rod where weight cannot be neglected

- vibration of membranes







P force acting on elemental surface $\bar{t}dx$ or $\bar{t}dy$

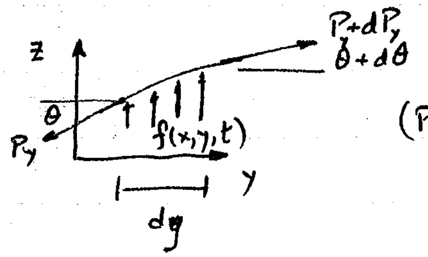
$\Sigma F_x =$

$(P_x + dP_x)\cos(\varphi + d\varphi) - P_x\cos\varphi$

$- P_x \sin\varphi\, d\varphi + dP_x\cos\varphi = 0$ TO

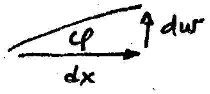
$d(P_x\cos\varphi) = 0$

$P_x\cos\varphi =$ const. or fn of y

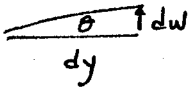


$(P_y + dP_y)\cos(\theta + d\theta) - P_y\cos\theta = 0 \quad = \Sigma F_y$

$d(P_y\cos\theta) = 0 \quad \Rightarrow \quad P_y\cos\theta =$ const or fn of x.



$\cos\varphi = \frac{dx}{\sqrt{d\psi^2 + dw^2}} \cong 1$ if $\left|\frac{\partial w}{\partial x}\right| << 1$



$\cos\theta = \frac{dy}{\sqrt{dy^2 + dw^2}} \cong 1$ if $\left|\frac{\partial w}{\partial y}\right| << 1$

$\Rightarrow$ P_x is essentially constant everywhere
P_y " " " " $\Big\}$ $P_x = P_y = P$

$\Sigma F_z = m \cdot \text{accel} = \rho \frac{\partial^2 w}{\partial t^2} dy dx \cdot \bar{t}$

$$f(x,y,t)dxdy + (P_y + dP_y)\sin(\theta + d\theta) - P_y\sin\theta + (P_x + dP_x)\sin(\varphi + d\varphi) - P_x\sin\varphi = \Sigma F_z$$

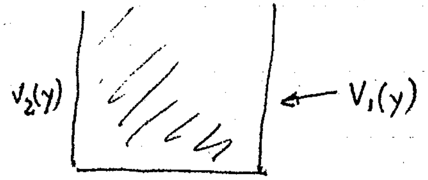
$$d(P_y\sin\theta) + d(P_x\sin\varphi) + f(x,y,t)dxdy = \Sigma F_z$$

$$\frac{d}{dy}\cancel{dy}\underbrace{(P_y\sin\theta)}_{\bar{P}\cdot\bar{t}dx}dy + \frac{d}{dx}\underbrace{(P_x\sin\varphi)}_{\bar{P}\cdot\bar{t}dy}dx + f(x,y,t)dxdy = \Sigma F_z$$

$\sin\theta \sim \frac{\partial w}{\partial y} \qquad \sin\varphi \sim \frac{\partial w}{\partial x}$

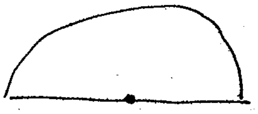
at ∞ V must be bounded.

p. 9 # 2



$V(y=0,x) = V_0(x)$

4



$V(r_0,\theta) = V_1(\theta)$

$V(r,0) = V_2(r)$

$V(r,\pi) = V_3(r)$

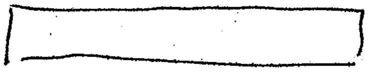
$\frac{\partial V(r_0,\theta)}{\partial r} =$

$\frac{\partial V}{\partial \theta} =$

$\frac{\partial V}{\partial \theta}$

Trim

p. 17 #2



$U(x,t=0) = f(x)$

$U(0,t) = 100$ $U(l,t) = 100$

#3 $\frac{\partial U}{\partial x}(0,t) = 0$

4

$U(L,t) = \frac{100}{T}t$ $t \leq T$

100 $t > T$

$$\frac{\partial}{\partial y}\left(\bar{P}\bar{t}\,\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\bar{P}\bar{t}\,\frac{\partial w}{\partial x}\right) + f(x,y,t) = \rho\frac{\partial^2 w}{\partial t^2}\,\bar{t}$$

IF $\bar{t}$ = const $\quad \frac{\partial}{\partial y}\left(\bar{P}\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial x}\left(\bar{P}\frac{\partial w}{\partial x}\right) + f/\bar{t} = \rho\, w_{tt}$

if $\bar{P}$ is constant ⟹ $\quad w_{xx} + w_{yy} + f/\bar{t}\bar{P} = \rho/\bar{P}\; w_{tt}$

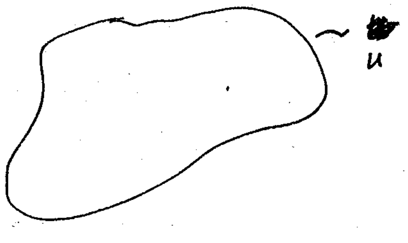
if $f = 0$ ⟹ $\quad w_{xx} + w_{yy} = \frac{1}{c^2} w_{tt}$ and $\nabla^2 w = \frac{1}{c^2} w_{tt} \quad c = \sqrt{\frac{\bar{P}}{\rho}}$

- IF steady state $\frac{\partial}{\partial t} = 0$ ⟹ $\nabla^2 w = 0$ Elliptic eqn.

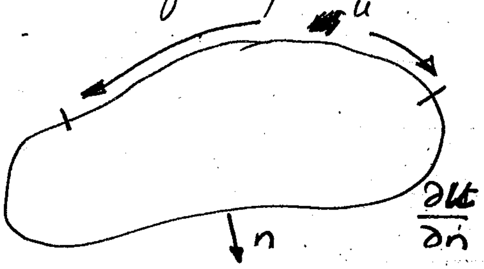
- Boundary Conditions + initial conditions IN GENERAL
  - for each time derivative you need 1 initial condition
  - for each space derivative $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ you need 1 boundary condi

- FOR ELLIPTIC TYPE EQNS need to specify value on bdry, or deriv on bdr or value and normal derivative on different parts of bdry

must also satisfy consistency of Neumann
$\int \frac{\partial u}{\partial n} ds = \iint \nabla^2 u\, dA$
using Divergence theorem.



or



- FOR PARABOLIC TYPE - heat eqn $\quad u_{xx} = a u_t$
  - need 1 initial condition $\quad u(x, t=t_0) = u_0(x)$
  - need value or value + normal deriv on boundary : $\alpha u + \beta u_n$

- FOR HYPERBOLIC - wave eqn
  - need 2 initial conditions $\quad u(x, t=t_0) = u_0(x)$, $\frac{\partial}{\partial t}u(x, t=t_0) = u_1(x)$
  - need value or value + normal deriv on boundary

p.31 #7

- IF NOT ENOUGH INFO IS GIVEN - incompletely posed problem
- IF TOO MUCH INFO OR WRONG TYPE - ill posed problem

- IF INFO GIVEN IS ONLY normal DERIVATIVE ON BOUNDARY: PROBLEM is not unique - can add constant to a solution to get another solution

NAMES of problem type

- Dirichlet - problems with B.C. on the function itself
- Von Neumann - " " " " " derivative of function
- Mixed, Robin, Churchill problems with B.C. on combo of func. & derivative

- Linear homogeneous PDE's can usually be solved by means of separation of variables. (SOV) in simple domains

- Sep. of Variables solutions represent the eigenfunctions ~~modes of vibration~~
- each eigenfunction ~~mode~~ is independent of ~~each~~ other eigenfunctions ~~modes~~, and, solutions are linear combos of these eigenfunctions ~~modes~~.

- Linear problems: where dependent variable & its deriv are to first power; no products ~~like~~ like $u \frac{\partial u}{\partial x}$ see pg 3.2 of Reynolds

- Homogeneous problems: if you multiply dependent variable by a constant, the constant drops out

$$(cu)_{xx} + (cu)_{yy} = 0 \Rightarrow u_{xx} + u_{yy} = 0$$

$$u_{xx} + u_{yy} + f = 0 \quad \text{not homogeneous}$$

- eigenvalue problems: Linear homogeneous PDE's with linear, homogeneous b.c.
- solutions not equal to zero are eigensolutions

- each solution is dependent on a parameter (eigenvalue)
- Vibration problems (eigenvalues are natural frequencies of vibration problem)

- SOV method : assume solution is a product of fns of independe variables $u(x,y,t) = X(x)\,Y(y)\,T(t)$

- in method when put into PDE ; leads to terms in x that are independent of terms in y. Must be true for all values of x, y $\Rightarrow$ must each be equal to same constant.
- SOV reduces PDE's to ODE's
- constant is the eigenvalues

- example vibrating membrane - steady state $w_{xx} \oplus w_{yy} = 0$

$$w(x,y) = X(x)\,Y(y)$$

$$\frac{\partial w}{\partial x} = X'Y \oplus X \cancel{\frac{dY}{dx}}^{0} = X'Y \qquad \frac{\partial^2 w}{\partial x^2} = X''Y$$

$$\frac{\partial^2 w}{\partial y^2} = XY''$$

$$\therefore \quad X''Y \oplus XY'' = 0 \quad \Longrightarrow \quad \frac{X''Y}{XY} = -\frac{XY''}{XY}$$

$$\Longrightarrow \quad \underset{\text{fn of } x}{\frac{X''}{X}} = \ominus \underset{\text{fn of } y}{\frac{Y''}{Y}} = \text{const.} = \lambda^2$$

$$\left.\begin{aligned} X'' - \lambda^2 X &= 0 \\ Y'' + \lambda^2 Y &= 0 \end{aligned}\right\} \text{2 ODES instead of 1 PDE}$$

- solution to $Y'' + \lambda^2 Y = 0$ $^{\text{if } \lambda > 0}$ is $Y_\lambda = A\cos\lambda y + B\sin\lambda y$

  $X'' - \lambda^2 X = 0$ is $X_\lambda = Ce^{\lambda x} + De^{-\lambda x}$

- to solve for A, B, C, D must be given boundary information vibration of circular membrane clamped at edge

$$\nabla^2 W - \frac{1}{c^2} W_{tt} = 0 \qquad c = \sqrt{P/\rho}$$

let $W(r,\theta,t) = w(r,\theta)T(t)$

$$\Rightarrow \quad T \cdot \nabla^2 w - \frac{1}{c^2} w \cdot T'' = 0 \quad \text{or}$$

$$c^2 \frac{\nabla^2 w}{w} - \frac{T''}{T} = 0$$

$$\Rightarrow \quad c^2 \frac{\nabla^2 w}{w} = \frac{T''}{T} = -\omega^2$$ frequency of vibration

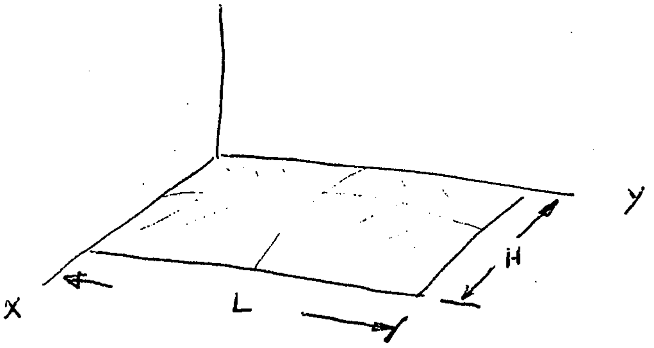
$$\Rightarrow \quad T'' + \omega^2 T = 0$$ ~ temporal mode of vib

$$\nabla^2 w + \frac{\omega^2}{c^2} w = 0 \quad \Rightarrow \quad \nabla^2 w + \lambda^2 w = 0$$ — spatial mode of vibration

$\lambda = \frac{\omega}{c}$ eigenvalue

$w(r,\theta)$ eigenmode

LET'S LOOK AT VIBS OF RECTANGULAR MEMBRANE



$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0$$

with $W(x,y=0,t) = W(x,y=L,t) = 0$
$W(x=0,y,t) = W(x=H,y,t) = 0$
Homogeneous 1st Kind (Neumann)

AGAIN LET $W(x,y,t) = w(x,y)T(t)$ $\quad \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\therefore\ T\nabla^2 w - \frac{1}{c^2} w \ddot{T} = 0$ $\quad \div$ BY wT

AND $\dfrac{c^2 \nabla^2 w}{w} = \dfrac{\ddot{T}}{T} = -\omega^2$

$\Rightarrow\ \ddot{T} + \omega^2 T = 0 \quad \Rightarrow\ T = \mathcal{C}_1 \sin \omega t + \mathcal{C}_2 \cos \omega t$

$\Rightarrow\ \nabla^2 w + \left(\frac{\omega}{c}\right)^2 w = 0$ $\quad$ LET $\lambda = \frac{\omega}{c}$

NOW LET $w(x,y) = \mathbb{X}(x)Y(y)$

$\therefore\ \nabla^2 w + \lambda^2 w = \mathbb{X}''Y + \mathbb{X}Y'' + \lambda^2 \mathbb{X}Y = 0$ $\quad \div$ by $\mathbb{X}Y$

$\Rightarrow\ \left(\frac{\mathbb{X}''}{\mathbb{X}} + \lambda^2\right) = -\left(Y''/Y\right) = \alpha^2$

$\therefore\ Y'' + \alpha^2 Y = 0 \quad \Rightarrow\ Y = A \sin \alpha y + B \cos \alpha y$

$\mathbb{X}'' + (\lambda^2 - \alpha^2)\mathbb{X} = 0 \Rightarrow\ \mathbb{X} = C \sin \beta x + D \cos \beta x \quad \beta = \sqrt{\lambda^2 - \alpha^2}$

$\Rightarrow$ Now $W(x,y=0,t) = w(x,y=0)T(t) = \mathbb{X}(x)Y(0)T(t) = 0$ $\quad$ FOR ALL $x,t \Rightarrow Y(0)=0$

$W(x,y=L,t) = w(x,y=L)T(t) = \mathbb{X}(x)Y(L)T(t) = 0$ $\quad$ " " " $\Rightarrow Y(L)=0$

$\Rightarrow$ FROM $Y(0)=0$ & $Y(L)=0$ $\Rightarrow B=0$ & $Y(L) = A \sin \alpha L = 0 \Rightarrow \alpha L = n\pi$

$\therefore\ Y(y) = A \sin \frac{n\pi}{L} y$ $\quad$ A WILL DEPEND ON n & SO WILL $Y(y)$

FROM $W(x=0,y,t) = w(x=0,y)T(t) = \mathbb{X}(0)Y(y)T(t) = 0$ FOR ALL y & $t \Rightarrow \mathbb{X}(0)=0$

$W(x=H,y,t) = w(x=H,y)T(t) = \mathbb{X}(H)Y(y)T(t)=0$ FOR ALL y & $t \Rightarrow \mathbb{X}(H)=0$

$\Rightarrow \mathbb{X}(0)$ & $\mathbb{X}(H)=0 \Rightarrow D=0$ & $\beta H = m\pi$ $\quad (C\sin\beta H = 0)$

$\therefore \mathbb{X}(x) = C\sin\frac{m\pi}{H}x$ (C WILL DEPEND ON m & SO WILL $\mathbb{X}$)

$$\therefore w(x,y) = \overline{A_n C_m}\sin\frac{n\pi}{L}y\sin\frac{m\pi}{H}x = E_{nm}\sin\frac{n\pi}{L}y\sin\frac{m\pi x}{H}$$

since $\alpha = \frac{n\pi}{L}$ & $\beta = \sqrt{\lambda^2-\alpha^2} = \frac{m\pi}{H}$ $\Rightarrow \lambda = \sqrt{\beta^2+\alpha^2}$

$$= \sqrt{\frac{n^2\pi^2}{L^2}+\frac{m^2\pi^2}{H^2}}$$

- note that n,m start at 1, WHY?
- IF $n=0 \Rightarrow \alpha=0 \Rightarrow -Y''/Y = 0$ or $Y = Ay+B$

since $Y(0)=Y(L)=0$ FROM BC's $\quad \left.\begin{matrix} Y(0)=0 \Rightarrow B=0 \\ Y(L)=0 \Rightarrow A=0 \end{matrix}\right\} \Rightarrow Y(y)=0$

TRIVIAL SOLUTION

$\Rightarrow$ NOTE MUST ALWAYS CHECK $n=0$ CASE BY GOING BACK TO DIFF. EQ.

& USING BCS

$\Rightarrow W(x=0,y,t)=0 \Rightarrow \mathbb{X}(0)=0$ ONLY WORKS FOR <u>ZERO</u> BC's

$W(x=0,y,t)=5 \not\Rightarrow \mathbb{X}(0)=5$

LOWEST FREQUENCY OF VIBRATION $\omega_{mn} = \lambda_{mn}\cdot c = c\pi\sqrt{\frac{n^2}{L^2}+\frac{m^2}{H^2}}\Bigg|_{m=n=1} = c\pi\sqrt{\frac{1}{L^2}+\frac{1}{H^2}}$

$$\text{SO } W(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}\sin\omega_{nm}t + B_{nm}\cos\omega_{nm}t\right]\sin\frac{n\pi y}{L}\sin\frac{m\pi x}{H}$$

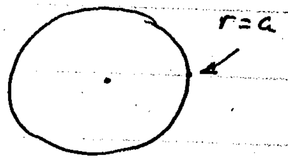
~~and~~ where $\omega_{nm} = c\pi\left[\frac{n^2}{L^2}+\frac{m^2}{H^2}\right]^{1/2}$; $C_{11} = c\pi\left[\frac{1}{L^2}+\frac{1}{H^2}\right.$

EIGENVALUE PROBLEM (POISSON'S EQN IN A CIRCULAR REGION)

- OBTAINED BY SEPARATING VARIABLES – ONLY LOOKS AT SPATIAL MODE SHAPES
- $\nabla^2 W - \frac{1}{c^2} W_{tt} = 0 \Rightarrow W = w(r,\theta)T(t) \Rightarrow \nabla^2 w + \lambda^2 w = 0$ & $T'' + \omega^2 T = 0 \quad \lambda^2 = \omega^2/c^2$
- let us solve $\nabla^2 w + \lambda^2 w = 0$ in a circular region under the condition that $w(x,\theta) = 0$ when $r = a$

$$\nabla^2 w = \frac{\partial^2}{\partial r^2} w + \frac{1}{r}\frac{\partial}{\partial r} w + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} w \qquad \text{in cylindrical coordinates}$$

$$\therefore \quad \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} w + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \lambda^2 w = 0$$



method of S.O.V. yields $w(r,\theta) = R(r)\Theta(\theta)$

$$\therefore \quad \nabla^2 w = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta''$$

and

$$\nabla^2 w + \lambda^2 w = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' + \lambda^2 R\Theta = 0$$

- divide by $R\Theta$ if $w(r,\theta) \neq 0$

$$\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \lambda^2 = 0$$

- $$\Rightarrow \underbrace{\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \lambda^2\right)r^2}_{\text{fn of } r} = \underbrace{-\frac{\Theta''}{\Theta}}_{\text{fn of } \theta} = \text{constant} = k^2 > 0$$

$$\Rightarrow \quad \Theta'' + k^2\Theta = 0 \quad \text{or} \quad \Theta = A\cos k\theta + B\sin k\theta$$

$$\Rightarrow \quad r^2R'' + rR' + (\lambda^2r^2 - k^2)R = 0$$

$$(\lambda r)^2 \frac{d^2}{d(\lambda r)^2}R(\lambda r) + (\lambda r)\frac{d}{d(\lambda r)}R(\lambda r) + (\lambda^2r^2 - k^2)R = 0$$

Look at the Bessel's equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$

IF $x = \lambda r$, $p = k$ then the solutions to $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - p^2)y = 0$

are the solutions to $r^2R'' + rR' + (\lambda^2r^2 - k^2)R = 0$

for example

IF p is not zero or a positive integer

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

Bessel fn of 1st kind $$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!}$$ for $J_{-p}(x)$ replace p by $-p$

if p is zero or a positive integer $p=n$

$$y = C_1 J_n(x) + C_2 Y_n(x)$$ ~ Bessel fn of 2nd Kind

J_n is same as J_p but replace p by n

- $$Y_n(x) = \frac{2}{\pi}\left[\left(\log \tfrac{x}{2} + \gamma\right) J_n(x) - \frac{1}{2} \sum_{k=0}^{n-1} \frac{(n-k-1)!\left(\frac{x}{2}\right)^{2k-n}}{k!} + \frac{1}{2}\sum_{k=0}^{\infty} (-1)^{k+1}\left[\varphi(k) + \varphi(k+n)\right] \frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,(k+n)!}\right]$$
- $$\varphi(k) = \sum_{m=1}^{k} \frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$
- γ is Euler's constant $= \lim_{k\to\infty} \left[\varphi(k) - \log k\right] = 0.5772157\ldots$

(**)
- note that at $x=0$ $J_n(x) =$ bdd & $Y_n(x)$ is ∞ ; $J_{-p}(x)$ is ∞ at $x=0$
- FOR OUR PROBLEM since ω is unknown we don't know which form to use
- Secondly our problem must satisfy the condition that $w(r,\theta)=0$ at $r=a$

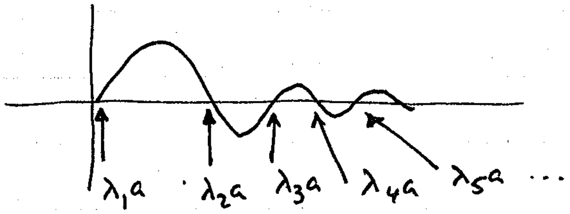
(**)
- IF our problem involves annular membrane - not see $C_2 \neq 0$
- $\Rightarrow$ for circular problems to have unique solutions $\Rightarrow$ $w(r,\Theta) = w(r, \theta + 2n\pi)$

  $\Rightarrow$ k must be integer $\omega = n$
- $\Rightarrow$ $R_n(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$
- also since $w(r=a, \theta) = R_n(\lambda a)\,\Theta(\theta) = 0$ for all Θ $\Rightarrow R_n(\lambda a) = 0$

$$\therefore \quad w(r,\theta) = \sum_{m=1}^{\infty} \sum_{n=0} J_n(\lambda_{nm} r) \left[C_n \cos n\theta + D_n \sin n\theta \right]$$

- since the origin is included and our solution must be bounded at origin $\Rightarrow C_2 \equiv 0$ since $Y_n(0) = \infty$

- $\Rightarrow \quad R_n(\lambda r) = C_1 J_n(\lambda r) \quad \Rightarrow \quad R_n(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$
- THESE POINTS AT WHICH $R_n = 0$ are nodal pts of the mode shape R_n
- Since $J_n(\lambda r)$ is a series made up of terms alternating in sign and each term decreases $\Rightarrow$ there will be a set of values λa for which $R_n(\lambda a) = 0$



thus we can show that $\lambda_1 < \lambda_2 < \dots < \lambda_m$ for each fn J_n and this is true for each n

- The solution then is $w_{n,m}(r,\theta) = J_n(\lambda_{nm} r)\left[C_n \cos n\theta + D_n \sin n\theta\right]$
- see table 3.5.1 Reynolds

$n=0$	$m=1$	$\lambda_{01} \cdot a = 2.40483$
1	1	$\lambda_{11}\, a = 3.83171$
2	1	$\lambda_{21}\, a = 5.13562$
⋮		

we can set in order λ_{nm}

- $w_{nm}(r,\theta)$ represents the mode shapes of vibration
- λ_{mm} represents ~~frequency~~ eigenvalues of vibration

- note: EIGENVALUE PROBLEMS are time independent

DO 3.3 & 3.5 & 3.2 also derive the indicial equation & the series solutions for the bessel fn.

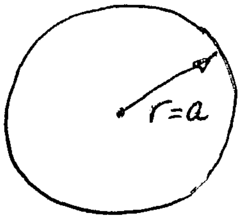
$$x^2 y'' + x y' + (x^2 - p^2) y = 0$$

Pg 44	#4		1/10
38	# 3	nofor	1/10
18	# 11,14		1/10
89	1,5	Fourier	1/10
120	2,6,8		1/10

9/18

ON THURSDAY / WE SOLVED

$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0 \quad \text{IN A CIRCULAR REGION}$$



with $W(r,\theta,t) = W$

AND $W(r=a,\theta,t) = 0$

By writing $W = w(r,\theta)\cdot T(t)$ we can separate spatial & temporal functions

$$\nabla^2 W = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}$$

THUS $$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = T\nabla^2 w - \frac{1}{c^2} w\ddot{T} = 0$$

or $$\frac{c^2\nabla^2 w}{w} = \frac{\ddot{T}}{T} = -\omega^2 \implies T = \mathcal{E}_1\cos\omega t + \mathcal{E}_2 \sin\omega t$$

ω = frequency of vibration

$$\implies \nabla^2 w + \left(\frac{\omega}{c}\right)^2 w = 0 \quad \text{let } \frac{\omega}{c} = \lambda$$

or $\nabla^2 w + \lambda^2 w = 0$ this is an eigenvalue problem
Helmholtz Equation

TO SOLVE EIGENVALUE PROBLEM, LET $w(r,\theta) = R(r)\,\Theta(\theta)$

$$\therefore \nabla^2 w + \lambda^2 w = (R'' + \tfrac{1}{r}R')\Theta + \tfrac{1}{r^2}\Theta'' + \lambda^2 R\Theta = 0$$

$$\implies \frac{r^2(R'' + \frac{1}{r}R') + \lambda^2 r^2}{R} = -\frac{\Theta''}{\Theta} = +k^2$$ SINCE Θ FUNCTION MUST BE PERIODIC

$$\implies \Theta = A\cos k\theta + B\sin k\theta$$

$$\implies r^2R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$$

BESSEL EQN OF FORM $x^2y'' + xy' + (x^2 - p^2)y = 0$ of order $p(k)$

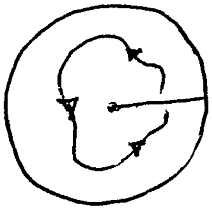
IF $x = \lambda r$
$R = y$
$p = k$

SOLUTIONS ARE

$$y = C_1\, J_p(x) + C_2\, J_{-p}(x) \qquad \text{IF } p \neq 0 \text{ or an integer}$$

$$y = C_1\, J_n(x) + C_2\, Y_n(x) \qquad \text{IF } p \text{ is zero or an integer}$$

- TO DETERMINE IF k IS integer $W(r,\theta,t) = W(r,\theta+2\pi,t) \Rightarrow w(r,\theta) = w(r,\theta+2\pi) \Rightarrow$



$$\Theta \Rightarrow \Theta(\theta+2k\pi) = \Theta(\theta) \Rightarrow \boxed{k=n} = 0,1,2,3,4 \cdots$$

($n=-1,-2$ etc. not linearly independent

$$\therefore\ R(\lambda r) = C_1\, J_n(\lambda r) + C_2\, Y_n(\lambda r)$$

— STOPPED HERE THURSDAY

- NOW $J_n(\lambda r)$ IS BOUNDED AT $r=0$

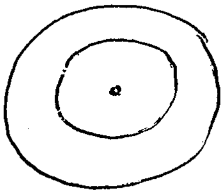
  $Y_n(\lambda r)$ IS <u>NOT</u> BOUNDED AT $r=0$

  PHYSICAL PROBLEM DICTATES THAT $W(r,\theta,t)$ IS BOUNDED AT $r=0$

  $\Longrightarrow$ MUST TAKE $\boxed{C_2 = 0}$ SINCE $Y_n(\lambda r)$ CONTAINS $\log(\lambda r)$ TERM

  $\Longrightarrow$ NOTE $J_{-p}(x)$ IS NOT BOUNDED AT $x=0$ EITHER

  $\Longrightarrow$ FOR AN ANNULAR MEMBRANE ORIGIN NOT INCLUDED THUS $Y_n(\lambda r)$ IS kept



- LET'S LOOK AT HOW TO HANDLE $W(r=a,\theta,t)=0$ IRRESPECTIVE OF t & θ

  $$W(r=a,\theta,t) = w(r=a,\theta)\,T(t) = R(\lambda r=\lambda a)\,\Theta(\theta)\,T(t) = 0$$

  $\Longrightarrow R(\lambda r=\lambda a)=0$ THIS IS THE WAY TO FIND THE λ's

SO FAR $w(r,\theta) = J_n(\lambda r)\left[\bar{A}\cos n\theta + \bar{B}\sin n\theta\right]$

SINCE TRUE FOR ANY n & EQ ($\nabla^2 w + \lambda^2 w = 0$) IS LINEAR

SO $w(r,\theta)$ DEPENDS ON $n \Rightarrow w_n(r,\theta)$

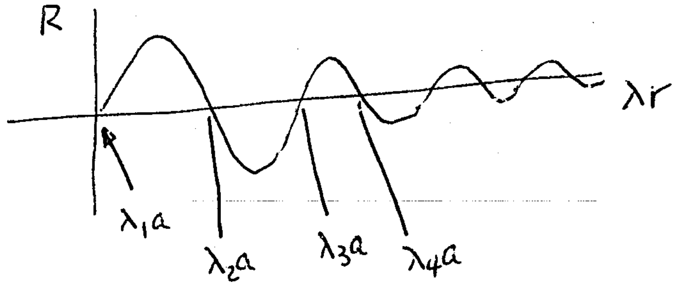
AND

$$w(r,\theta) = \sum_n w_n(r,\theta) = \sum_n J_n(\lambda r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]$$

$\Rightarrow$ SINCE $R(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$

TALK ABOUT TABLE

FOR any n



$\Rightarrow$ THERE ARE AN INFINITE NO. OF VALUES FOR $J_0(\lambda r)$ & $J_1(\lambda r)$, & $J_2(\lambda r)$ ...

$\therefore$ WE MUST NUMBER THE ZEROES OF J_0, J_1, J_2 etc. AND PUT IN ORDER OF INCREASING MAGNITUDE

THUS

$$w(r,\theta) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]$$

AND

$$W(r,\theta,t) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n\cos n\theta + \bar{B}_n\sin n\theta\right]\left[\mathscr{C}_{1_{mn}}\cos\omega_{mn}t + \mathscr{C}_{2_{mn}}\sin\omega_{mn}t\right]$$
$$= \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{C}_n\cos(n\theta + \psi_n)\right]\left[D_{mn}\cos(\omega_{mn}t + \phi_{mn})\right]$$

$\Rightarrow$ ω_{mn} has DOUBLE SUBSCRIPT SINCE $\frac{\omega}{c} = \lambda$ & λ DEPENDS ON m & n

$\Rightarrow$ $\bar{C}_n$, ψ_n, D_{mn} & ϕ_{mn} CANNOT BE FOUND WITHOUT IC's FOR $T(t)$ & BC's ON Θ

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm} J_n(\lambda_{nm} r) \cos(n\theta - \psi) \cos(\omega_{nm} t - \phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm} a \tag{3.5.14}$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for n = 0 the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has <u>two</u> diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02} r) = 0$, i.e. at $\lambda_{02} r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.

TABLE 3.5.1

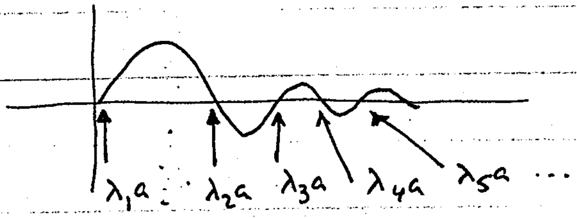
DIMENSIONLESS MEMBRANE FREQUENCIES

n	m	$j_{n,m} = \omega_{nm} r_o / \overset{\approx a}{a} \approx c$
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

- since the origin is included and our solution must be bounded at origin ⇒ $C_2 \equiv 0$ since $Y_n(0) = \infty$

- ⇒ $R_n(\lambda r) = C_1 J_n(\lambda r)$ ⇒ $R_n(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$

- THESE POINTS AT WHICH $R_n = 0$ are nodal pts of the mode shape R_n
- Since $J_n(\lambda r)$ is a series made up of terms alternating in sign and each term decreases ⇒ there will be a set of values λa for which $R_n(\lambda a) = 0$



thus we can show that $\lambda_1 < \lambda_2 < \dots < \lambda_m$ for each fn J_n

and this is true for each n

- The solution then is $w_{n,m}(r,\theta) = J_n(\lambda_{nm} r)\left[C_n \cos n\theta + D_n \sin n\theta\right]$
- see table 3.5.1 Reynolds

$n=0$	$m=1$	$\lambda_{01} \cdot a = 2.40483$
1	1	$\lambda_{11}\ a = 3.83171$
2	1	$\lambda_{21}\ a = 5.13562$
⋮		

we can set in order λ_{nm}

- $W_{nm}(r,\theta)$ represents the mode shapes of vibration
- λ_{nm} represents ~~frequency~~ eigenvalues of vibration

- note: EIGENVALUE PROBLEMS are time independent

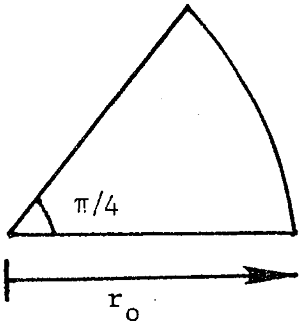
DO 3.3 & 3.5 & 3.2 also derive the indicial equation & the series solutions for the bessel fn.

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

Exercises.

3.1. Find the eigenmodes and frequencies for an annular membrance with inner radius r_i and outer radius r_0. ($u = 0$ at r_i and r_0). For the special case $r_0/r_i = 2.5$, give the lowest four values of $\omega r_0/a$. (HINT: HMF Table 9.7)

3.2. Consider the pie-shaped membrane shown in the sketch. Calculate the eigenmodes and eigenfrequencies, in non-dimensional form.



3.3. Problems 3.3a - 3.3d all deal with acoustic waves in a cylindrical enclosure. In each case the governing PDE for the pressure is

$$c^2\nabla^2 p - p_{tt} = 0$$

where, in polar-cylindrical coordinates

$$\nabla^2 p = p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} + p_{zz}$$

The boundary condition at the solid walls (at $z = 0$, at $z = L$, and at $r = r_0$) is that the derivative of the pressure field normal (perpendicular) to the wall must vanish, i.e.

$$p_z = 0 \quad \text{at} \quad z = 0,L$$

$$p_r = 0 \quad \text{at} \quad r = r_0$$

a. Find the eigenmodes and frequencies for axial modes where $p = p(z,t)$.

b. Find the eigenmodes and frequencies for radial modes where $p = p(r,t)$

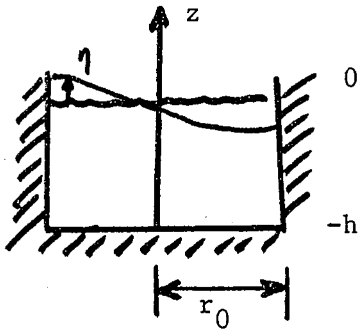
c. Show that there are no modes where

$p = p(\theta, t)$.

d. Find the eigenmodes and frequencies for the general case where

$p = p(r, \theta, z, t)$

3.4 In the analysis of seismic loading on nuclear reactors, oil storage tanks and other large fluid containers, one needs to know the natural frequencies of sloshing motions. This problem will acquaint you with the typical analysis.



Consider a circular geometry, with vertical walls at $r = r_0$, and the bottom at $z = -h$. The equations governing the sloshing are

$$(1) \quad \nabla^2\phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz} = 0$$

$$(2) \quad \frac{\partial\Phi}{\partial t} + g\eta = 0 \quad \text{on} \quad z = 0 \qquad (4) \quad \frac{\partial\Phi}{\partial r} = 0 \quad \text{at} \quad r = r_0$$

$$(3) \quad \frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} = 0 \quad \text{on} \quad z = 0 \qquad (5) \quad \frac{\partial\Phi}{\partial z} = 0 \quad \text{at} \quad z = -h$$

$\Phi(r,\theta,z,t)$ is the velocity potential; the fluid velocity is the gradient of Φ ; $\eta(r,\theta,t)$ is the surface displacement. g is the acceleration of gravity, $g = 9.8\text{m/sec}^2$. Equation (1) is the continuity equation for irrotational flow, (2) is the Bernouli equation applied on the free surface, (3) is a kinematic condition relating surface motion to velocity, and (4) and (5) are boundary conditions that the flow cannot penetrate the wall. Students with expertise in fluid mechanics should derive (1) - (5).

(a) Using the method of separation of variables, derive an expression for the natural frequencies. Express them non-dimensionally as

$$(6) \quad \Omega^2 \equiv \omega^2 r_0/g = f(h/r_0)$$

Express the solution for the surface deflection $\eta(r,\theta,t)$ in the non-dimensional form

$$(7) \quad \frac{\eta}{\eta_a} = F\left(\frac{r}{r_0}\right) G(\omega_{mn}t)\, H(m\theta)$$

where η_a is the maximum deflection at $r = r_0$ (the sloshing amplitude)

(b) For the special case $h/r_0 = \infty$, calculate the values of Ω^2 for the modes having the five lowest natural frequencies, and sketch the node-lines in the surface displacement $\eta(r,\theta,t)$ for each of these modes. Check-point: the fundamental has $\Omega^2 = 1.841$. HINT: See HMF 9.1.1, 9.1.11, Table 9.5.

(c) Consider a large oil tank 30m in diameter, filled to a depth of 10m. Calculate the lowest natural frequency of vibration (hz).

(d) Find a coffee cup, jar, or other circular container. Fill with water to a selected depth, and manually excite the first mode by moving the container sideways. Compare the "measured" frequency (hz) with the value predicted by the analysis. Visualize the radial node-lines of part (d) in your cup by banging it (gently!) on the table.

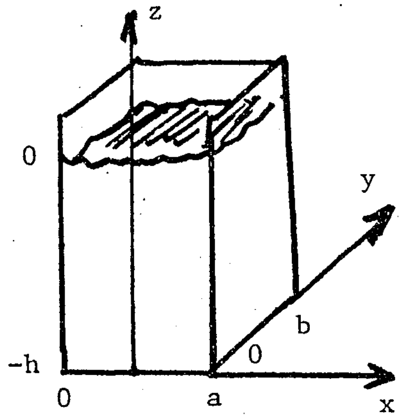
3.5 Consider the sloshing of a fluid in a rectangular tank. The motion is described by the equations of Problem 3.4, except that

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

and (4) is replaced by

$$\phi_x = 0 \quad \text{at} \quad x = 0,a$$

$$\phi_y = 0 \quad \text{at} \quad y = 0,b$$



(a) Calculate the natural frequencies of fluid sloshing in the tank. Show that they are given by

$$\omega^2_{nm} = gk \tanh(kh) \qquad k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

(b) Give the expression for $\eta_{nm}(x,y,t)$, apart from an undetermined phase and amplitude.

(c) Find a bathtub, wash-basin, or kitchen sink, fill with water to a reasonable depth. Manually excite the fundamental sloshing frequency and compare the theoretical value with an "eyeball" experimental measurement (hz).

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm} J_n(\lambda_{nm} r) \cos(n\theta - \psi) \cos(\omega_{nm} t - \phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm} a \tag{3.5.14}$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for $n = 0$ the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has <u>two</u> diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02} r) = 0$, i.e. at $\lambda_{02} r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.

TABLE 3.5.1

DIMENSIONLESS MEMBRANE FREQUENCIES

n	m	$j_{n,m} = \omega_{nm} r_o / a$
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

IF p is not zero or a positive integer

$$y = C_1 J_p(x) + C_2 J_{-p}(x)$$

Bessel fn of 1st kind
$$J_p(x) = \sum_{k=0}^{\infty} \frac{(-1)^k \left(\frac{x}{2}\right)^{2k+p}}{k!\,(k+p)!}$$
for $J_{-p}(x)$ replace p by $-p$

if p is zero or a positive integer $p=n$

$$y = C_1 J_n(x) + C_2 Y_n(x)$$ ~ Bessel fn of 2nd Kind

J_n is same as J_p but replace p by n

- $$Y_n(x) = \frac{2}{\pi}\left[\left(\log\frac{x}{2} + \gamma\right)J_n(x) - \frac{1}{2}\sum_{k=0}^{n-1} \frac{(n-k-1)!\left(\frac{x}{2}\right)^{2k-n}}{k!} + \frac{1}{2}\sum_{k=0}^{\infty}(-1)^{k+1}\left[\varphi(k) + \varphi(k+n)\right]\frac{\left(\frac{x}{2}\right)^{2k+n}}{k!\,(k+n)!}\right]$$
- $$\varphi(k) = \sum_{m=1}^{k}\frac{1}{m} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$
- γ is Euler's constant $= \lim_{k\to\infty}\left[\varphi(k) - \log k\right] = 0.5772157\ldots$

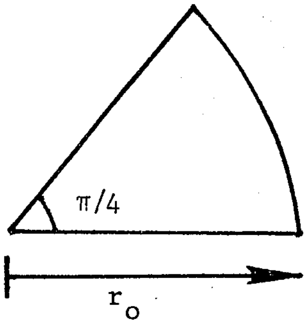
(**)
- note that at $x=0$ $J_n(x) = \text{bdd}$ & $Y_n(x)$ is ∞ ; $J_{-p}(x)$ is ∞ at $x=$
- FOR OUR PROBLEM since ω is unknown we don't know which form to use
- Secondly our problem must satisfy the condition that $w(r,\theta)=0$ at $r=a$

(**)
- IF our problem involves annular membrane – [redacted] $C_2 \neq 0$
- $\Rightarrow$ for circular problems to have unique solutions $\Rightarrow \omega(r,\Theta) = \omega(r,\theta+2n\pi$
  - $\Rightarrow$ <u>k must be integer</u> $\omega = n$
- $\Rightarrow$ $R_n(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$
- also since $w(r=a,\theta) = R_n(\lambda a)\,\Theta(\theta) = 0$ for all Θ $\Rightarrow R_n(\lambda a) = 0$

Exercises.

3.1. Find the eigenmodes and frequencies for an annular membrance with inner radius r_i and outer radius r_0. ($u = 0$ at r_i and r_0). For the special case $r_0/r_i = 2.5$, give the lowest four values of $\omega r_0/a$. (HINT: HMF Table 9.7)

3.2. Consider the pie-shaped membrane shown in the sketch. Calculate the eigenmodes and eigen-frequencies, in non-dimensional form.



3.3. Problems 3.3a - 3.3d all deal with acoustic waves in a cylindrical enclosure. In each case the governing PDE for the pressure is

$$c^2\nabla^2 p - p_{tt} = 0$$

where, in polar-cylindrical coordinates

$$\nabla^2 p = p_{rr} + \frac{1}{r}p_r + \frac{1}{r^2}p_{\theta\theta} + p_{zz}$$

The boundary condition at the solid walls (at $z = 0$, at $z = L$, and at $r = r_0$) is that the derivative of the pressure field normal (perpendicular) to the wall must vanish, i.e.

$$p_z = 0 \quad \text{at} \quad z = 0,L$$

$$p_r = 0 \quad \text{at} \quad r = r_0$$

a. Find the eigenmodes and frequencies for axial modes where $p = p(z,t)$.

b. Find the eigenmodes and frequencies for radial modes where $p = p(r,t)$

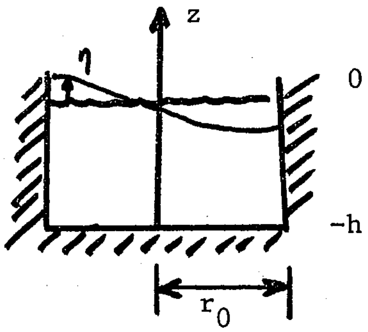
c. Show that there are no modes where

$p = p(\theta,t)$.

d. Find the eigenmodes and frequencies for the general case where

$p = p(r,\theta,z,t)$

3.4 In the analysis of seismic loading on nuclear reactors, oil storage tanks and other large fluid containers, one needs to know the natural frequencies of sloshing motions. This problem will acquaint you with the typical analysis.



Consider a circular geometry, with vertical walls at $r = r_0$, and the bottom at $z = -h$. The equations governing the sloshing are

$$(1) \quad \nabla^2\phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} + \Phi_{zz} = 0$$

$$(2) \quad \frac{\partial\Phi}{\partial t} + g\eta = 0 \qquad \text{on} \quad z = 0$$

$$(3) \quad \frac{\partial\eta}{\partial t} - \frac{\partial\Phi}{\partial z} = 0 \qquad \text{on} \quad z = 0$$

$$(4) \quad \frac{\partial\Phi}{\partial r} = 0 \qquad \text{at} \quad r = r_0$$

$$(5) \quad \frac{\partial\Phi}{\partial z} = 0 \qquad \text{at} \quad z = -h$$

$\Phi(r,\theta,z,t)$ is the velocity potential; the fluid velocity is the gradient of Φ ; $\eta(r,\theta,t)$ is the surface displacement. g is the acceleration of gravity, $g = 9.8\text{m/sec}^2$. Equation (1) is the continuity equation for irrotational flow, (2) is the Bernouli equation applied on the free surface, (3) is a kinematic condition relating surface motion to velocity, and (4) and (5) are boundary conditions that the flow cannot penetrate the wall. Students with expertise in fluid mechanics should derive (1) - (5).

(a) Using the method of separation of variables, derive an expression for the natural frequencies. Express them non-dimensionally as

$$(6) \quad \Omega^2 \equiv \omega^2 r_0/g = f(h/r_0)$$

Express the solution for the surface deflection $\eta(r,\theta,t)$ in the non-dimensional form

$$(7) \quad \frac{\eta}{\eta_a} = F\left(\frac{r}{r_0}\right) G(\omega_{mn}t)\ H(m\theta)$$

where η_a is the maximum deflection at $r = r_0$ (the sloshing amplitude)

(b) For the special case $h/r_0 = \infty$, calculate the values of Ω^2 for the modes having the five lowest natural frequencies, and sketch the node-lines in the surface displacement $\eta(r,\theta,t)$ for each of these modes. Check-point: the fundamental has $\Omega^2 = 1.841$. HINT: See HMF 9.1.1, 9.1.11, Table 9.5.

(c) Consider a large oil tank 30m in diameter, filled to a depth of 10m. Calculate the lowest natural frequency of vibration (hz).

(d) Find a coffee cup, jar, or other circular container. Fill with water to a selected depth, and manually excite the first mode by moving the container sideways. Compare the "measured" frequency (hz) with the value predicted by the analysis. Visualize the radial node-lines of part (d) in your cup by banging it (gently!) on the table.

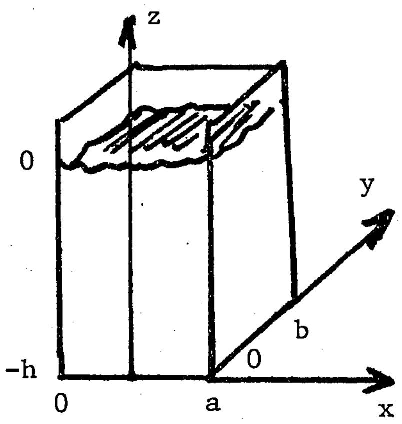
3.5 Consider the sloshing of a fluid in a rectangular tank. The motion is described by the equations of Problem 3.4, except that

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz}$$

and (4) is replaced by

$$\phi_x = 0 \qquad \text{at} \quad x = 0,a$$

$$\phi_y = 0 \qquad \text{at} \quad y = 0,b$$



(a) Calculate the natural frequencies of fluid sloshing in the tank. Show that they are given by

$$\omega^2_{nm} = gk \tanh(kh) \qquad k^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$$

(b) Give the expression for $\eta_{nm}(x,y,t)$, apart from an undetermined phase and amplitude.

(c) Find a bathtub, wash-basin, or kitchen sink, fill with water to a reasonable depth. Manually excite the fundamental sloshing frequency and compare the theoretical value with an "eyeball" experimental measurement (hz).

Table 9.7 BESSEL FUNCTIONS—MISCELLANEOUS ZEROS

s^{th} Zero of $xJ_1(x)-\lambda J_0(x)$

$\lambda\backslash s$	1	2	3	4	5
0.00	0.0000	3.8317	7.0156	10.1735	13.3237
0.02	0.1995	3.8369	7.0184	10.1754	13.3252
0.04	0.2814	3.8421	7.0213	10.1774	13.3267
0.06	0.3438	3.8473	7.0241	10.1794	13.3282
0.08	0.3960	3.8525	7.0270	10.1813	13.3297
0.10	0.4417	3.8577	7.0298	10.1833	13.3312
0.20	0.6170	3.8835	7.0440	10.1931	13.3387
0.40	0.8516	3.9344	7.0723	10.2127	13.3537
0.60	1.0184	3.9841	7.1004	10.2322	13.3686
0.80	1.1490	4.0325	7.1282	10.2516	13.3835
1.00	1.2558	4.0795	7.1558	10.2710	13.3984

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.2558	4.0795	7.1558	10.2710	13.3984	1
0.80	1.3659	4.1361	7.1898	10.2950	13.4169	1
0.60	1.5095	4.2249	7.2453	10.3346	13.4476	2
0.40	1.7060	4.3818	7.3508	10.4118	13.5079	3
0.20	1.9898	4.7131	7.6177	10.6223	13.6786	5
0.10	2.1795	5.0332	7.9569	10.9363	13.9580	10
0.08	2.2218	5.1172	8.0624	11.0477	14.0666	13
0.06	2.2656	5.2085	8.1852	11.1864	14.2100	17
0.04	2.3108	5.3068	8.3262	11.3575	14.3996	25
0.02	2.3572	5.4112	8.4840	11.5621	14.6433	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

s^{th} Zero of $J_1(x)-\lambda xJ_0(x)$

$\lambda\backslash s$	1	2	3	4	5
0.5	0.0000	5.1356	8.4172	11.6198	14.7960
0.6	1.1231	5.2008	8.4569	11.6486	14.8185
0.7	1.4417	5.2476	8.4853	11.6691	14.8346
0.8	1.6275	5.2826	8.5066	11.6845	14.8467
0.9	1.7517	5.3098	8.5231	11.6964	14.8561
1.0	1.8412	5.3314	8.5363	11.7060	14.8636

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
1.00	1.8412	5.3314	8.5363	11.7060	14.8636	1
0.80	1.9844	5.3702	8.5600	11.7232	14.8771	1
0.60	2.1092	5.4085	8.5836	11.7404	14.8906	2
0.40	2.2192	5.4463	8.6072	11.7575	14.9041	3
0.20	2.3171	5.4835	8.6305	11.7745	14.9175	5
0.10	2.3621	5.5019	8.6421	11.7830	14.9242	10
0.08	2.3709	5.5055	8.6445	11.7847	14.9256	13
0.06	2.3795	5.5092	8.6468	11.7864	14.9269	17
0.04	2.3880	5.5128	8.6491	11.7881	14.9282	25
0.02	2.3965	5.5165	8.6514	11.7898	14.9296	50
0.00	2.4048	5.5201	8.6537	11.7915	14.9309	∞

$\langle\lambda\rangle$ = nearest integer to λ.

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BESSEL FUNCTIONS—MISCELLANEOUS ZEROS Table 9.7

s^{th} Zero of $J_0(x)Y_0(\lambda x)-Y_0(x)J_0(\lambda x)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
* 0.80	12.55847 031	25.12877	37.69646	50.26349	62.83026	1
0.60	4.69706 410	9.41690	14.13189	18.84558	23.55876	2
0.40	2.07322 886	4.17730	6.27537	8.37167	10.46723	3
0.20	0.76319 127	1.55710	2.34641	3.13403	3.92084	5
0.10	0.33139 387	0.68576	1.03774	1.38864	1.73896	10
0.08	0.25732 649	0.53485	0.81055	1.08536	1.35969	13 *
0.06	0.18699 458	0.39079	0.59334	0.79522	0.99673	17
0.04	0.12038 637	0.25340	0.38570	0.51759	0.64923	25
0.02	0.05768 450	0.12272	0.18751	0.25214	0.31666	50
0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞

s^{th} Zero of $J_1(x)Y_1(\lambda x)-Y_1(x)J_1(\lambda x)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
* 0.80	12.59004 151	25.14465	37.70706	50.27145	62.83662	1
0.60	4.75805 426	9.44837	14.15300	18.86146	23.57148	2
0.40	2.15647 249	4.22309	6.30658	8.39528	10.48619	3
0.20	0.84714 961	1.61108	2.38532	3.16421	3.94541	5
0.10	0.39409 416	0.73306	1.07483	1.41886	1.76433	10
0.08	0.31223 576	0.57816	0.84552	1.11441	1.38440	13 *
0.06	0.23235 256	0.42843	0.62483	0.82207	1.02001	17
0.04	0.15400 729	0.28296	0.41157	0.54044	0.66961	25
0.02	0.07672 788	0.14062	0.20409	0.26752	0.33097	50
0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞

s^{th} Zero of $J_1(x)Y_0(\lambda x)-Y_1(x)J_0(\lambda x)$

$\lambda^{-1}\backslash s$	1	2	3	4	5	$\langle\lambda\rangle$
* 0.80	6.56973 310	18.94971	31.47626	44.02544	56.58224	1
* 0.60	2.60328 138	7.16213	11.83783	16.53413	21.23751	2
0.40	1.24266 626	3.22655	5.28885	7.36856	9.45462	3
0.20	0.51472 663	1.24657	2.00959	2.78326	3.56157	5
0.10	0.24481 004	0.57258	0.90956	1.25099	1.59489	10
0.08	0.19461 772	0.45251	0.71635	0.98327	1.25203	13 *
0.06	0.14523 798	0.33597	0.53005	0.72594	0.92301	17
0.04	0.09647 602	0.22226	0.34957	0.47768	0.60634	25
0.02	0.04813 209	0.11059	0.17353	0.23666	0.29991	50
0.00	0.00000 000	0.00000	0.00000	0.00000	0.00000	∞

$\langle\lambda\rangle$ = nearest integer to λ.

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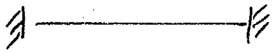
*See page II.

what if we were given

$$u_{xx} = \frac{1}{c^2} u_{tt}$$



BC's are homogeneous

$u(x=0,t)=0 \qquad u(x=L,t)=0$

$u(x,t=0) = f(x)$

$\frac{\partial u}{\partial t}(x,t=0) = g(x)$

let $u(x,t) = F(x)G(t)$

then $F''G = \frac{1}{c^2} FG'' \quad$ or $\quad \frac{c^2F''}{F} = \frac{G''}{G} = -\omega^2$

$G'' + \omega^2 G = 0 \quad$ or $\quad G(t) = B\cos\omega t + A\sin\omega t$

$F'' + \left(\frac{\omega}{c}\right)^2 F = 0 \qquad F(x) = C\cos\lambda x + D\sin\lambda x \qquad \lambda = \frac{\omega}{c}$

- use bc first $\quad u(x=0,t) = F(0)G(t) = 0 \Rightarrow F(0)=0 \Rightarrow C=0$

  $u(x=L,t) = F(L)G(t) = 0 \Rightarrow F(L)=0 \Rightarrow \lambda L = n\pi \quad \lambda = \frac{n\pi}{L}$

  $F_n(x) = D\sin\frac{m\pi x}{L} \qquad$ let $DA = \tilde{A} \quad DB = \tilde{B}$

  $\lambda = \frac{\omega}{c} = \frac{n\pi}{L} \Rightarrow \omega = \frac{n\pi c}{L}$

$$u_n(x,t) = (\tilde{B}_n \cos\omega_n t + \tilde{A}_n \sin\omega_n t)\sin\frac{n\pi x}{L}.$$

$$u = \Sigma u_n = \sum(\tilde{B}_n \cos\omega t + \tilde{A}_n \sin\omega t)\sin\frac{n\pi x}{L}$$

since $\lambda = \frac{\omega}{c}$ & $\lambda = \frac{n\pi}{L}$, $\omega = \frac{n\pi c}{L}$

- now apply IC $\quad$ @ $t=0 \quad u(x,t=0) = f(x) = \Sigma \tilde{B}_n (\cos\omega t)\Big|_{t=0} \sin\frac{n\pi x}{L}$

  $= \Sigma \tilde{B}_n \sin\frac{n\pi x}{L}$

@ $t=0 \quad u_t(x,t=0) = g(x) = \Sigma \tilde{A}\omega \sin\frac{n\pi x}{L}$

$f(x) = \frac{A_0}{2} + A_1\cos\frac{\pi x}{L} + A_2\cos\frac{2\pi x}{L} + \dots + B_1\sin\frac{\pi x}{L} + B_2\sin\frac{2\pi x}{L} + \dots$

where $\quad A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx = 0$

$B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dy = \tilde{B}_n$

$\tilde{A} = \frac{2}{L\omega}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

these were obtained from orthogonality principle that

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = \begin{cases} 0 & n \neq m \\ 1/2 & n = m \end{cases}$$

$$\int_0^L \sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = 0$$

$$\int_0^L \cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} L/2 & n=m \\ 0 & n\neq m \end{cases}$$

---

what if we wanted to find the solution to $\nabla^2 W - \frac{1}{c^2}W_{tt} = 0$ in a circular region including the origin and $W(r=a,\theta,t)=0$

w/ IC $W(r,\theta,t=0) = f(r)$

$W_t(r,\theta,t=0) = g(r)$

if we choose $W(r,\theta,t) = F(r)G(\theta)H(t)$

then $\nabla^2 W - \frac{1}{c^2}W_{tt} = W_{rr} + \frac{1}{r}W_r + \frac{1}{r^2}W_{\theta\theta} - \frac{1}{c^2}W_{tt} = 0$

$$= F''GH + \frac{1}{r}F'GH + \frac{1}{r^2}FG''H - \frac{1}{c^2}FGH'' = 0$$

Divide By FGH & multiply by c^2: $\underbrace{c^2\frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G}}_{\text{spatial}} = \underbrace{\frac{H''}{H}}_{\text{time}} = -\omega^2$ ↖ frequency

$\Rightarrow$ $H'' + \omega^2 H = 0$ or $H(t) = B\cos\omega t + A\sin\omega t$.

and $c^2\frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G} = -\omega^2$ Now multiply both sides by r^2/c^2 & separate

$$r^2\frac{F''}{F} + \frac{r^2}{r}\frac{F'}{F} + \frac{\omega^2 r^2}{c^2} = -\frac{G''}{G} = \alpha^2$$

$\Rightarrow$ $G'' + \alpha^2 G = 0$ or $\underline{G(\theta) = C\cos\alpha\theta + D\sin\alpha\theta}$

$\Rightarrow$ $r^2F'' + rF' + \left(\left[\frac{\omega r}{c}\right]^2 - \alpha^2\right)F = 0$ or $\underline{F(\frac{\omega r}{c}) = M J_\alpha(\frac{\omega r}{c}) + N Y_\alpha(\frac{\omega r}{c})}$

- since we employ a circular region $G(\theta) = G(\theta + 2n\pi)$ $\Rightarrow$ α = integer :
- since we include the origin $\Rightarrow$ $N \equiv 0$

$\therefore$

- Solution is $W(r,\theta,t) = F(\frac{\omega r}{c})\,G(\theta)\,H(t)$ or $J_n(\frac{\omega a}{c}) = 0$
- since $W(r=a,\theta,t) = 0$ $\Rightarrow$ $F(\frac{\omega a}{c}) = 0 = M J_n(\frac{\omega a}{c}) = 0$

- $J_n\left(\frac{\omega a}{c}\right)=0$ defines ω since $\frac{\omega_1 a}{c} = r_1$ so that $J_n(r_1)=0$

  $\frac{\omega_2 a}{c} = r_2$ " " $J_n(r_2)=0$ etc.

- also since initial conditions are independent of θ

  reasonable to assume that $W(r,\theta,t)$ is independent of θ

$\Rightarrow \alpha = 0 \Rightarrow G(\theta) = \text{constant}$ why? $G''+\alpha^2 G = G'' = 0 \Rightarrow G = C_1\theta + C_2$

FOR $G(\theta+2n\pi) = G(\theta) \Rightarrow C_1 = 0 \Rightarrow G(\theta) = \text{constant}$. $\therefore J_n\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$

$$\therefore \quad W(r,\theta,t) \equiv W(r,t) = F\left(\frac{\omega r}{c}\right) H(t) \quad \text{where} \quad F\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$$

and $W_m(r,t) = F\left(\frac{\omega_m r}{c}\right) H(t)$ if $B\cdot C\cdot M = \tilde{B}$, $A\cdot C\cdot M = \tilde{A}$

$$= \left(\tilde{B} \cos\omega_m t + \tilde{A} \sin\omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

$$W(r,t) = \sum_m W_m(r,t) = \sum \left(\tilde{B}_m \cos\omega_m t + \tilde{A}_m \sin\omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

at $t=0$ $\quad W(r,t=0) = f(r) = \sum \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \hat{B}_m R_m(r)$

$t=0$ $\quad W_t(r,t=0) = g(r) = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{A}_m \omega_m R_m(r)$

suppose we can write $\quad f(r) \equiv E_1 J_0\left(\frac{\omega_1 r}{c}\right) + E_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + E_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$$= \sum_{i=1}^{\infty} E_i J_0\left(\frac{\omega_i r}{c}\right) = W(r,t=0) = \sum_{m=1}^{\infty} \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right)$$

$$\Rightarrow E's = \tilde{B}'s \quad \& \quad i=m$$

suppose we can write $\quad g(r) = L_1 J_0\left(\frac{\omega_1 r}{c}\right) + L_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + L_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$$= \sum L_i J_0\left(\frac{\omega_i r}{c}\right) = \left.\frac{\partial W}{\partial t}\right|_{t=0} = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right)$$

$$\Rightarrow L_i = \tilde{A}_i \cdot \omega_i \quad \text{or} \quad \tilde{A}_i = \frac{L_i}{\omega_i} \quad \& \quad i=m$$

- so how do we get the $E's$ & $L's$. See Chapter 8 §8.4

- Sturm-Liouville gives the method for solving for the $\tilde{A}_m$ & $\hat{B}_m$

Given $\quad \frac{d}{dx}\left(S(x)\frac{dy}{dx}\right) + \left[Q(x) + \lambda^2 P(x)\right] y = 0$

$S(x)\,y'' + S'(x)\,y' + \left[Q(x) + \lambda^2 P(x)\right] y = 0 \qquad r^2R'' + rR' + \left[\lambda^2 r^2 - \nu^2\right] R = 0$

if we chose $\Rightarrow$ $S(x) = r$ $\quad Q(x) = -\nu^2/r$

$S'(x) = 1$ $\quad P(x) = r$

if not ie $S(x) = r^2$ $\quad S'(x) = 2r \rightarrow$ doesn't fit

$$\frac{d}{dx}\left[Sy'\right] + \left[Q+\lambda^2 P\right]y = 0$$

subjected to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{matrix}\lambda = \lambda_m \\ y = y_m\end{matrix}\right\}$ & $\left.\begin{matrix}\lambda = \lambda_n \\ y = y_n\end{matrix}\right\}$ both ODE & BC are satisfied $y_m \neq y_n$ $\lambda_m \neq \lambda_n$

$$\Rightarrow \quad [Sy_m']' + [Q+\lambda_m^2 P]y_m = 0 \qquad (1)$$

$$[Sy_n']' + [Q+\lambda_n^2 P]y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{[Sy_m']' + [\underline{\underline{Q}}+\lambda_m^2 P]y_m\right\}y_n - \left\{[Sy_n']' + [\underline{\underline{Q}}+\lambda_n^2 P]y_n\right\}y_m \right\rangle dx$$

$$\int_a^b \left\{[Sy_m']'y_n - [Sy_n']'y_m\right\}dx + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_m y_n \, dx = 0$$

$$\int_a^b [Sy_m']'y_n \, dx = S y_m' y_n \Big|_a^b - \int_a^b S y_m' y_n' \, dx$$ integration by parts

$$\int_a^b [Sy_n']'y_m \, dx = S y_n' y_m \Big|_a^b - \int_a^b S y_n' y_m' \, dx$$

for y_n: at $x = a$ and at $x = b$ $\left.\begin{matrix}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{matrix}\right\}$ either $\alpha = 0$ & $\beta = 0$ or $\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$

also true for y_m:
at $x = a$ $\alpha y_m + \beta y_m' = 0$
$x = b$ $\alpha y_m + \beta y_m' = 0$

det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \quad \text{at both } x=a,b \qquad S[y_m' y_n - y_n' y_m]\Big|_a^b + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_n y_m \, dx = 0$$

$$\Rightarrow \text{ if } \lambda_m \neq \lambda_n \quad \boxed{\int_a^b P y_n y_m \, dx = 0}$$ orthogonality condition but wrt weight fn $P(x)$

example $T'' + \omega^2 T = 0 \Rightarrow S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m \, dx = 0 \Rightarrow \int_a^b P \sin\omega_n x \cdot \sin\omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $T(x) = 0$ @ $x = L$ $\Rightarrow T_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\, dx = 0$$

now suppose we want to write $f(x)$ as a fn of the eigenfunctions $y_n(x)$

now suppose $f(x) = \sum A_n y_n(x)$

if we mult. by Py_m & integrate

$$\int_a^b P(x) f(x) y_m(x)\, dx = \int_a^b P \sum A_n y_n(x) y_m(x) dx$$

$$= \sum A_n \int_a^b P y_n y_m\, dx = \begin{cases} A_m \int_a^b P y_m^2 dx & n=m \\ 0 & n \neq m \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m\, dx}{\int_a^b P y_m^2\, dx}}$$

FOR $T'' + \omega^2 T = 0$ $\quad y_m = \sin \frac{m\pi x}{L}$ $\quad P = 1$ $\quad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin \frac{m\pi x}{L} dx \Big/ \int_0^L \sin^2 \frac{m\pi x}{L} dx$$

$$\int_0^L \sin^2 \frac{m\pi x}{L} dx = L/2 \quad \Rightarrow \quad \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos \frac{2m\pi x}{L}\right) dx = L/2$$

$$A_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$ normal fourier coefficient

- Returning to our problem $f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$

for the bessel fn. $P(r) = +r$

$$E_i = \frac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$$

since $E_i = \tilde{B}_i$

also $g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$

$$L_i = \frac{\int_0^a r g(r) J_0\, dr}{\int_0^a r J_0^2\, dr} = \tilde{A}_i \omega_i$$

since $L_i = \tilde{A}_i \omega_i \Rightarrow \tilde{A}_i = \frac{L_i}{\omega_i}$

$\int_0^a r\ J_0^2(r\frac{\omega_i}{c})dr$ can be evaluated as follows

using the ODE : If you remember $J_0(\frac{\omega_i}{c} r)$ satisfies

$$r R'' + R' + (\overset{\lambda_i^2 r}{[\tfrac{\omega_i}{c}]^2 r} - 0^2)R = 0 \qquad R_i = J_0(\lambda_i r)$$

let $R(r,\lambda)$ be a solution to $rR'' + R' + (\lambda^2 r)R = 0$
and satisfy that $R(r=0)$ is not ∞
$R(r) = J_0(\lambda r)$

now take $\frac{\partial}{\partial\lambda}$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial\lambda} + R_i \frac{\partial^2 R}{\partial\lambda\partial r} + 2\lambda r\ RR_i + \lambda^2 r \frac{\partial R}{\partial\lambda} R_i = 0$$

integrate over $0 \leq r \leq a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r\partial\lambda}\right) + \lambda^2 r \frac{\partial R}{\partial\lambda} + 2\lambda r\, R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du \quad u = R_i \quad v = r\frac{\partial^2 R}{\partial r\partial\lambda}$

$$r R_i \frac{\partial^2 R}{\partial r\partial\lambda}\Big|_0^a - \overset{(*)}{\int_0^a r \frac{\partial^2 R}{\partial r\partial\lambda} R_i'\, dr} + \int_0^a (\overset{(**)}{\lambda^2 r \frac{\partial R}{\partial\lambda} R_i} + \overset{(***)}{2\lambda r\, RR_i}) dr = 0$$

R_i by definition is $=0$ @a:$(J_0(\lambda_i a) = 0)$. For $r=0$ at lower limit $R_i = 0$

integrate by parts

$$(*)\ -\int_0^a r R_i' \frac{\partial^2 R}{\partial r\partial\lambda} dr = -r R_i' \frac{\partial R}{\partial\lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial\lambda}(r R_i')' dr \quad \text{with } u = r R_i' \quad v = \frac{\partial R}{\partial\lambda}$$

$$\Rightarrow \underbrace{-r R_i'}_{-aR_i'\frac{\partial R}{\partial\lambda}} \frac{\partial R}{\partial\lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial\lambda}\Big\{\underbrace{(rR_i')' + \overset{(**)}{\lambda^2 r R_i}}_{=0}\Big\}dr + \int_0^a \overset{(***)}{2\lambda r\, RR_i}\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$; thus the middle term is zero

and $\quad 2\lambda_i \overset{(***)}{\int_0^a r RR_i\, dr} \Rightarrow \int_0^a r R_i^2\, dr = \frac{a\, R_i'}{2\lambda_i}\frac{\partial R}{\partial\lambda}\Big|_{\substack{r=a \\ \lambda=\lambda_i}}$

THUS $$\int_0^a r\,J_0^2\left(\tfrac{\omega_i}{c}r\right)dr = \frac{a}{2\left(\frac{\omega_i}{c}\right)}\left[J_0'\left(\tfrac{\omega_i}{c}a\right)\right]^2 \cdot a\cdot\frac{\omega_i}{c} = \frac{a^2}{2}\left[J_0'\left(\tfrac{\omega_i}{c}a\right)\right]^2$$

since $$R_i'(r) = \frac{d}{dr}\,J_0\left(\tfrac{\omega_i}{c}r\right) = \frac{\omega_i}{c}\,J_0'\left(\tfrac{\omega_i}{c}r\right)$$ with $R_i(\frac{\omega_i}{c}r)$ solves ODE & BCs: $J_0(\frac{\omega_i}{c}r)$

$$\frac{\partial R}{\partial\lambda} = r\,J_0'(\lambda r)$$ with $R(\lambda r)$ solve ODE: $J_0(\lambda r)$

now + $a R_i'\,\frac{\partial R}{\partial\lambda}\Big|_{\substack{r=a\\ \lambda=\lambda_i}} = a\left[\frac{\omega_i}{c}\cdot J_0'\left(\tfrac{\omega_i}{c}r\right)\right]\cdot\left[r J_0'(\lambda r)\right]\Big|_{\substack{r=a\\ \lambda=\lambda_i=\frac{\omega_i}{c}}} = \frac{\omega_i a^2}{c}\left[J_0'\left(\tfrac{\omega_i}{c}a\right)\right]^2$

THUS $$\tilde{B}_i = \frac{2}{a^2}\,\frac{\int_0^a r f(r)\,J_0\left(\frac{\omega_i}{c}r\right)dr}{\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2}$$

$$\tilde{A}_i = \frac{2}{a^2\omega_i}\,\frac{\int_0^a r g(r)\,J_0\left(\frac{\omega_i}{c}r\right)dr}{\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2}$$

$$J_0'(\lambda x) = -\lambda J_1(\lambda x) \quad\therefore\quad J_0'\left(\tfrac{\omega_i}{c}a\right) = -\tfrac{\omega_i}{c}\,J_1\left(\tfrac{\omega_i}{c}a\right)$$

- note the problem on page 4.10 onwards – <u>SPHERICAL</u> BESSEL FNS.

  produces a bessel fn
  $$r^2R'' + \underline{\underline{2}}rR' + \lambda^2r^2R = 0$$ and this has an

  orthogonality condition $$\int_0^r \underline{\underline{r^2}}\,R_n R_m\,dr = 0$$

  $S(x) = r^2$
  $Q(x) = 0$
  $\lambda^2 r^2 = \lambda^2 P$

- in general if $[S y_n']' + [Q + \lambda_n^2 P]\,y_n = 0$ Sturm-Liouville

  under the conditions $\alpha y_n + \beta y_n' = 0$ at $x = a$, $x = b$

  This solution ⇒ y_n is an eigenfunction λ_n eigenvalue

- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x)\,P(x)\,y_n(x)\,dx}{\int_a^b P(x)\,y_n^2(x)\,dx}$$

for $y''+\lambda^2 y=0$ $y(0)=0$ $y(L)=0$ $y_n = \sin\frac{n\pi x}{L}$ $\lambda_n = \frac{n\pi x}{L}$

$y = \sin\lambda x$ solve ODE but not BCs but bdd

$y_n' = \frac{n\pi}{L}\cos\frac{n\pi x}{L}$ $S=1$

$\frac{\partial y}{\partial\lambda} = x\cos\lambda x$ $\frac{\partial^2 y}{\partial x\partial\lambda} = \cos\lambda x - x\lambda\sin\lambda x$ $\int P y_n y_m\, dx = \int_0^L 1\cdot\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}$

$$\frac{1}{2\lambda_n}\left\{\left[\left(\frac{n\pi}{L}\cos\frac{n\pi x}{L}\right)\cdot 1\cdot\left(x\cos\lambda x\right)\right|_0^L - \left(\sin\frac{n\pi x}{L}\right)\cdot 1\cdot\left(\cos\lambda x - x\lambda\sin\lambda x\right)\Big|_0^L\right\}_{\lambda=\lambda_n}$$

$$\frac{1}{2\lambda_n}\left\{\lambda_n x\cos^2\lambda_n x\Big|_0^L - \sin\lambda_n x\cos\lambda_n x + \lambda_n x\sin^2\lambda_n x\Big|_0^L\right\}$$

$$\frac{1}{2\lambda_n}\left\{\lambda_n L\cos^2\lambda_n L - 0 + \lambda_n L\sin^2\lambda_n L\right\} = \frac{\lambda_n L}{2\lambda_n} = \frac{L}{2}$$ since $x=0$ & $\sin\lambda_n x=0$ @ $x=0$

$$\frac{1}{2\lambda_n}\left\{\lambda_n L\left(\cos^2\lambda_n L + \sin^2\lambda_n L\right)\right\} = \frac{L}{2}$$

$$\frac{d}{dx}\left[Sy'\right] + \left[Q+\lambda^2 P\right]y = 0$$

subjectes to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{matrix}\lambda = \lambda_m \\ y = y_m\end{matrix}\right\}$ & $\left.\begin{matrix}\lambda = \lambda_n \\ y = y_n\end{matrix}\right\}$ both ODE & BC are satisfied
$y_m \neq y_n$ $\lambda_m \neq \lambda_n$

$$\Rightarrow \quad [Sy_m']' + [Q + \lambda_m^2 P]y_m = 0 \qquad (1)$$

$$[Sy_n']' + [Q + \lambda_n^2 P]y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{ [Sy_m']' + [\underline{\underline{Q}} + \lambda_m^2 P]y_m \right\} y_n - \left\{ [Sy_n']' + [\underline{\underline{Q}} + \lambda_n^2 P]y_n \right\} y_m \right\rangle dx$$

$$\int_a^b \left\{ [Sy_m']' y_n - [Sy_n']' y_m \right\} dx + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_m y_n \, dx = 0$$

$$\int_a^b [Sy_m']' y_n \, dx = Sy_m' y_n \Big|_a^b - \int_a^b S y_m' y_n' \, dx$$ integration by parts

$$\int_a^b [Sy_n']' y_m \, dx = Sy_n' y_m \Big|_a^b - \int_a^b S y_m' y_m' \, dx$$

and $\left.\begin{matrix}\text{at } x = a \\ \text{at } x = b\end{matrix}\right\}$ $\begin{matrix}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{matrix}$ either $\alpha = 0$ & $\beta = 0$ or
$$\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$$
det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \ \overset{\text{at both}}{x = a, b} \qquad S\left[y_m' y_n - y_n' y_m\right]\Big|_a^b + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_n y_m \, dx = 0$$

$$\Rightarrow \text{if } \lambda_m \neq \lambda_n \quad \boxed{\int_a^b P y_n y_m \, dx = 0}$$ orthogonality condition but wrt weight fn $P(x)$

example $T'' + \omega^2 T = 0 \Rightarrow S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m \, dx = 0 \Rightarrow \int_a^b P \sin\omega_n x \cdot \sin\omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $T(x) = 0$ @ $x = L$ $\Rightarrow T_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} \, dx = 0$$

William Reynolds - Soln's of PDEs notes

$$\int_a^b P y_n y_m dx = 0 \qquad n \neq m \qquad \text{for sturm-Liouville} \tag{4.2.14}$$

Eqn. (4.2.14) is the *orthogonality property* of the eigenfunctions. The eigenfunctions are said to be orthogonal *with respect to the weight function $P(x)$*.

Now, suppose that, in the course of trying to construct the solution to a PDE as a linear combination of eigensolutions of the linear, homogeneous partial problem, we are led to the point where we wish to determine the coefficients in an eigenfunction expansion,

$$f(x) = \sum_{n=1}^{\infty} A_n y_n(x) \tag{4.2.15}$$

where the y_n are eigensolutions of a Sturm-Liouville problem. Multiplying (4.2.15) by Py_m, and integrating over the problem domain,

$$\int_a^b f P y_m dx = \sum_{n=1}^{\infty} A_n \int_a^b P y_n y_m dx \tag{4.2.16}$$

But, because of the orthogonality property (4.2.14), all of the integrals on the right will drop out, except the one where $n = m$. Hence, we can immediately solve for A_m,

$$A_m = \frac{\int_a^b f P y_m dx}{\int_a^b P y_m^2 dx} \tag{4.2.17}$$

The infinite series (4.2.15) will be useless if it fails to converge to $f(x)$. In specific problems where one calculates the A_n it is easy to perform the standard tests for series convergence. It is somewhat more difficult to prove convergence in general. However, if f is square-integrable, i.e., if

$$\int_a^b Pf^2 dx \quad \text{is finite}$$

then the series converges in the sense that[*]

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} A_n y_n(x) \right|^2 dx \rightarrow 0$$

This means that, if f is continuous over the interval $a \leq x \leq b$, the series converges uniformly (at all x). However, if f is discontinuous at some point, then the series will give a value at that point that is the average of the values of f at points infinitesimally above and below the point of discontinuity.

There are many problems of interest involving higher order system of linear homogeneous equations. In these cases, there are no theorems or general proofs of convergence of the eigenfunction expansions. One has to proceed by examining each case separately. However, problems arising from well-thought through physical formulations rarely, if ever, give rise to non-convergent expansions, so the analyst is usually safe in going ahead, assuming convergence, and then verifying it after the fact by ratio tests, numerical calculations, or other appropriate means.

↑ read

↓ don't need

4.3 Example - Vibrating String

For the vibrating string problem discussed in §4.1, the solution is given by (4.1.6). The coefficients A_n must be chosen such that (4.1.9) is satisfied. The eigenfunctions X_n are eigensolutions of

$$X_n'' + \lambda_n^2 X_n = 0 \tag{4.3.1}$$

and hence, from Sturm-Liouville theory, have the orthogonality property

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[*] See, for example, Ince, Ordinary Differential Equations, Dover, New York, 1956.

now suppose we want to write $f(x)$ as a fn of the eigenfunctions fn·

now suppose $f(x) = \sum A_n y_n(x)$

if we mult. by $P y_m$ & integrate

$$\int_a^b P(x) f(x) y_m(x)\,dx = \int_a^b P \sum A_n y_n(x) y_m(x) dx$$

$$= \sum A_n \int_a^b P y_n y_m \, dx = \begin{cases} A_m \int_a^b P y_m^2 dx & n=m \\ 0 & n \neq m \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m \, dx}{\int_a^b P y_m^2 \, dx}}$$

same as pg 4.7
Reynolds' notes
↑

For $T'' + \omega^2 T = 0 \qquad y_m = \sin \frac{m\pi x}{L} \qquad P = 1 \qquad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin \frac{m\pi x}{L} dx \Big/ \int_0^L 1 \cdot \sin^2 \frac{m\pi x}{L} dx$$

$$\int_0^L \sin^2 \frac{m\pi x}{L} dx = L/2 \quad \Rightarrow \quad \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos \frac{2m\pi x}{L}\right) dx = L/2$$

$$A_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$ normal fourier coefficient

- Returning to our problem if $f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$ for the bessel fn. $P(r) = +r$

$$E_i = \frac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r\, J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$$

since $E_i = \tilde{B}_i$

also $g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$

$$L_i = \frac{\int_0^a r\, g(r) J_0(\ ) dr}{\int_0^a r\, J_0^2(\ ) dr} = \tilde{A}_i \omega_i$$

since $L_i = \tilde{A}_i \omega_i \Rightarrow \tilde{A}_i = \frac{L_i}{\omega_i}$

$\int_0^a r\ J_0^2(r\frac{\omega_i}{c})\,dr$ can be evaluated as follows

using the ODE : If you remember $J_0(\frac{\omega_i}{c} r)$ satisfies

$$r R'' + R' + (\overset{\lambda_i^2 r}{[\frac{\omega_i}{c}]^2 r} - 0^2) R = 0 \quad (*) \qquad R_i = J_0(\lambda_i r)$$

$\equiv$ to $(rR')' + \lambda^2 r R = 0$

let $R(r,\lambda)$ be a solution to $r R'' + R' + (\lambda^2 r) R = 0$
and satisfy that $R(r=0)$ is not ∞
$R(r) = J_0(\lambda r)$

now take $\frac{\partial}{\partial \lambda}(*)$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial \lambda} + \underbrace{R_i \frac{\partial^2 R}{\partial \lambda \partial r}} + 2\lambda r\, R R_i + \lambda^2 r \frac{\partial R}{\partial \lambda} R_i = 0$$

integrate over $0 \le r \le a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r \partial \lambda}\right) + \lambda^2 r \frac{\partial R}{\partial \lambda} + 2\lambda r R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du$, let $u = R_i$ $v = r\frac{\partial^2 R}{\partial r \partial \lambda}$

$$r R_i \frac{\partial^2 R}{\partial r \partial \lambda}\Big|_0^a - \int_0^a r \frac{\partial^2 R}{\partial r \partial \lambda} R_i'\, dr + \int_0^a (\lambda^2 r \frac{\partial R}{\partial \lambda} R_i + 2\lambda r R R_i)\, dr = 0$$

R_i by definition is $= 0$ @a:$(J_0(\lambda_i a) = 0)$ at $r = 0$ at lower limit $r = 0$

integrate by parts 2nd term

$$- \int_0^a r R_i' \frac{\partial^2 R}{\partial r \partial \lambda}\, dr = -r R_i' \frac{\partial R}{\partial \lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial \lambda}(r R_i')' dr, \text{ let } u = r R_i' \quad v = \frac{\partial R}{\partial \lambda}$$

$$\Rightarrow \underbrace{-r R_i' \frac{\partial R}{\partial \lambda}}_{-a R_i' \frac{\partial R}{\partial \lambda}}\Big|_0^a + \int_0^a \frac{\partial R}{\partial \lambda}\Big\{ \underbrace{(r R_i')' + \lambda^2 r R_i}_{=0 \text{ since } R_i \text{ solves } (rR_i')' + \lambda^2 r R_i = 0 \text{ when } \lambda = \lambda_i} \Big\} dr + \int_0^a 2\lambda r R R_i\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$; thus the middle term is zero

and $\int_0^a r R R_i\, dr \Rightarrow \int_0^a r R_i^2\, dr = \frac{a R_i'}{2\lambda_i} \frac{\partial R}{\partial \lambda}\Big|_{\substack{r=a \\ \lambda=\lambda_i}}$

THUS $\int_0^a r\,J_0^2\left(\frac{\omega_i}{c}r\right)dr = \frac{a}{2\left(\frac{\omega_i}{c}\right)}\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2 \cdot a \cdot \frac{\omega_i}{c} = \frac{a^2}{2}\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2$

since $R_i'(r) = \frac{d}{dr}J_0\left(\frac{\omega_i}{c}r\right) = \frac{\omega_i}{c}J_0'\left(\frac{\omega_i}{c}r\right)$ and $R_i\left(\frac{\omega_i}{c}r\right)$ solves D.E. & BC's & equals $J_0\left(\frac{\omega_i r}{a}\right)$

$$\frac{\partial R}{\partial\lambda} = \frac{d}{d\lambda}J_0(\lambda r) = r\,J_0'(\lambda r)$$

now $a R_i' \frac{\partial R}{\partial\lambda}\bigg|_{\substack{r=a\\ \lambda=\lambda_i}} = a\left[\frac{\omega_i}{c}J_0'\left(\frac{\omega_i}{c}r\right)\right]\left[r\,J_0'(\lambda r)\right]\bigg|_{\substack{r=a\\ \lambda=\lambda_i}} = a^2\frac{\omega_i}{c}\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2$

THUS $$\tilde{B}_i = \frac{2}{a^2}\,\frac{\int_0^a r f(r)\,J_0\left(\frac{\omega_i}{c}r\right)dr}{\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2}$$

$$\tilde{A}_i = \frac{2}{a^2\omega_i}\,\frac{\int_0^a r g(r)\,J_0\left(\frac{\omega_i}{c}r\right)dr}{\left[J_0'\left(\frac{\omega_i}{c}a\right)\right]^2}$$

Bessel Relations: $J_0'(\lambda x) = -\lambda J_1(\lambda x) \quad \therefore \quad J_0'\left(\frac{\omega_i}{c}a\right) = -\frac{\omega_i}{c}J_1\left(\frac{\omega_i}{c}a\right)$

- note Problem in Reynolds pg 4.10 involves – SPHERICAL BESSEL FNS.

produces a bessel eq. that solves

$$r^2R'' + \underline{\underline{2}}rR' + \lambda^2r^2R = 0 \quad \text{and this has an}$$

$$[r^2R']' + \lambda^2r^2R = 0$$

orthogonality condition $\int_0^r \underline{\underline{r^2}}\,R_n R_m\,dr = 0$

- in general if $[Sy_n']' + [Q + \lambda_n^2 P]\,y_n = 0$ Sturm-Liouville

under the conditions $\alpha y_n + \beta y_n' = 0$ at $x=a$, $x=b$

y_n is an eigenfunction λ_n eigenvalue

- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x)P(x)y_n(x)\,dx}{\int_a^b P(x)\,y_n^2(x)\,dx}$$

Reynolds' notes

$$R(r,\lambda) = \frac{\sin(\lambda r)}{\lambda r} \tag{4.4.26}$$

$$\left.\frac{\partial R}{\partial \lambda}\right)_{\substack{\lambda = \lambda_n \\ r = r_o}} = \frac{1}{\lambda_n} \cos(\lambda_n r_o) \tag{4.4.27}$$

Noting that $\cos(\lambda_n r_o) = (-1)^n$, (4.2.23) gives

$$A_n = 2(-1)^{n+1} T_o \tag{4.4.28}$$

So, our final solution is, from (4.4.12),

$$T(r,t) = 2T_o \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/r_o)}{n\pi r/r_o} e^{-n^2\pi^2\alpha t/r_o^2} \tag{4.4.29}$$

Note that the series converges for all t. The series for $\partial T/\partial r$, developed from (4.4.29) by differentiation, will converge for all $t > 0$ because of the exponential, but does not converge at $t = 0$. But this is not a serious limitation. As t increases the series converges more rapidly, and at large t the solution is given (approximately) by just the first term,

$$T \simeq 2T_o \frac{\sin(\pi r/r_o)}{\pi r/r_o} e^{-\pi^2\alpha t/r_o^2} \tag{4.4.30}$$

↑ don't need

↓ read

4.5 Sturm-Liouville Denominator Integral for general S-L eqn

In analyses, leading to the Sturm-Liouville problems, the orthogonality property will produce (4.2.17). The denominator integral may be expressed in terms of quantities evaluated at the boundary using a generalization of the trick employed in the previous example. Let $y(x,\lambda)$ be a solution to (4.2.1) not necessarily satisfying the boundary conditions (4.2.2). Then, $y(x,\lambda_n)$

4.2.1 is $\frac{d}{dx}\left[S\frac{dy}{dx}\right] + \left[Q+\lambda^2 P\right]y=0$ 4.16

4.2.2 is $\alpha y+\beta y'=0$ @ $x=a$ & $x=b$

will be an eigensolution satisfying the Diff. Eq & the boundary conditions. We differentiate (4.2.1) with respect to λ , obtaining

$$\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x\partial\lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial\lambda} + 2\lambda P y \;=\; 0 \qquad (4.5.1)$$

Next, we multiply (4.5.1) by y_n and integrate over the problem range,

$$\int_a^b y_n \left\{\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x\partial\lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial\lambda} + 2\lambda P y\right\} dx \;=\; 0 \qquad (4.5.2)$$

The first integral is integrated twice by parts, and (4.5.2) becomes

$$y_n\, S\,\frac{\partial^2 y}{\partial x\partial\lambda}\Bigg|_a^b - \frac{\partial y}{\partial\lambda}\, y_n' S\,\Bigg|_a^b$$

$$+\int_a^b \frac{\partial y}{\partial\lambda}\left\{(Sy_n')' + \left[Q + \lambda^2 P\right]y_n\right\} dx + 2\lambda\int_a^b P y y_n dx \;=\; 0 \qquad (4.5.3)$$

Now, if we set $\lambda = \lambda_n$, the first integral drops out (because the integrand contains the y_n equation), and hence

IF y_n is a soln to Bessel eqn $\Rightarrow \lambda = \lambda_n$. last term of 4.5.3 becomes

$$\int_a^b P y_n^2 dx \;=\; \frac{1}{2\lambda_n}\left\{y_n' S\,\frac{\partial y}{\partial\lambda}\Bigg|_a^b - y_n S\,\frac{\partial^2 y}{\partial x\partial\lambda}\Bigg|_a^b\right\}_{\lambda\,=\,\lambda_n} \qquad (4.5.4)$$

Thus, the denominator in A_n can be evaluated without recourse to integration.

4.6 Removal of Inhomogeneities in the PDE and BCs

In the previous problem, the PDE and BCs were homogeneous, and therefore eigensolutions of this homogeneous problem could be found. By taking a

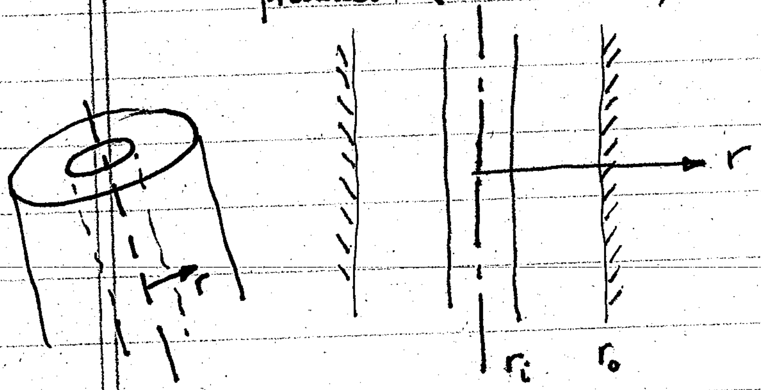
The denominator $$\int_a^b P(x)\, y_n^2(x)\,dx = \frac{1}{2\lambda_n}\left\{ y_n' \, S \frac{\partial y}{\partial \lambda}\Big|_a^b - y_n \, S \frac{\partial^2 y}{\partial x \partial \lambda}\Big|_a^b \right\}_{\lambda=\lambda_n}$$

$$y_n'(z) = \frac{d}{dz} y_n$$

$Y\cancel{y} = Y\cancel{y}(x;\lambda)$ satisfies ODE & is <u>bounded</u>

$y_n = y(x;\lambda_n) = y_n$ satisfies ODE & B.C.

- <u>Removing inhomogeneities in the PDE & BC's</u>
  - IN PDE & IN BC.
  - Time history of diffusion of a contaminant $c(r,t)$ in an annular region in which the contaminant in continually produced. (source exists)



$$\frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rS \qquad (3)$$

α diffusivity

S source term

- assume at $r=r_o$ barrier blocks outer diffusion $\ni$. $\frac{\partial c}{\partial r} = 0$ (1)

- Also assume contaminant is <u>convectively</u> removed at $r=r_i$

$$h(c - c_\infty) = D\frac{\partial c}{\partial r} \qquad (2)$$

h convective transport coeff ; c_∞ is the fixed concentration in fluid passing through annular hole

D - diffusion coeff for the contaminant in the solid

- initially $c = c_0$ at $t=0$ (only one IC.)

two boundaries require 2 BCs (at r_i & r_o)

Where is inhomogeneity? replace c by nc $\quad n = \text{const.}$

(1) homogeneous $\frac{\partial c}{\partial r}$ $(r = r_o)$

(2) no due to hc_∞ term $\quad h(c - c_\infty) = D\frac{\partial c}{\partial r}$ @ $r = r_i$

(3) not PDE due to rS term $\quad \frac{\partial}{\partial r}\left(r\frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rS$

- Try to remove inhogeneity in PDE & BC together
- since the inhomogeneous term in PDE is a fn of r only choose a solution which is fn of r only : $\psi(r) = c_{ss}$

- since ψ not fn of t $\Rightarrow$ steady state solution ie let ψ satisfy

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = -rS \qquad (3')$$

$$\frac{\partial \psi}{\partial r} = 0 \quad \text{at } r = r_o \qquad (1)$$

$$h(\psi - c_\infty) = D\frac{\partial \psi}{\partial r} \quad \text{at } r = r_i \qquad (2)$$

$$r\psi' = -\frac{r^2 S}{2} + C_1 \qquad \text{from } (3')$$

$$\psi' = -\frac{rS}{2} + \frac{C_1}{r} \quad \Rightarrow \quad \psi = -\frac{r^2 S}{4} + C_1 \ln r + C_2$$

$$h(\psi - c_\infty) = D\psi' \Rightarrow +h\left(-\frac{r_i^2 S}{4} + C_1 \ln r_i + C_2 - c_\infty\right) = D\left(-\frac{r_i S}{2} + \frac{C_1}{r_i}\right)$$

$$(1) \Rightarrow \psi' = 0 \Big|_{r = r_o} = -\frac{r_o S}{2} + \frac{C_1}{r_o} = 0 \quad \Rightarrow \quad C_1 = \frac{r_o^2 S}{2}$$

$$\Rightarrow \quad C_2 = \frac{D}{h}\left(-\frac{r_i S}{2} + \frac{r_o^2 S}{2r_i}\right) + \frac{r_i^2 S}{4} - \frac{r_o^2 S}{2}\ln r_i + c_\infty$$

$$\therefore \quad \psi(r) = (-r^2 + r_i^2)\frac{S}{4} + \frac{r_o^2 S}{2}\ln(r/r_i) + \frac{D}{hr_i}\left(\frac{r_o^2 S}{2} - \frac{r_i^2 S}{2}\right) + c_\infty$$

next let's find a $c(r,t) = c_{trans.}$

$$c_{TOT} = c_{ss} + c_{trans.} = \psi(r) + c_{trans}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial c_{TOT}}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c_{TOT}}{\partial t} - rS \Rightarrow \frac{\partial}{\partial r}\left(r\frac{\partial c_{tr}}{\partial r} + \cancel{r\frac{\partial c_{ss}}{\partial r}}\right) = \frac{r}{\alpha}\frac{\partial c_{tr}}{\partial t} \cancel{-rS}$$

$$\frac{\partial c_{TOT}}{\partial r} = 0 \quad \text{at } r=r_0 \qquad \frac{\partial c_{tr}}{\partial r} + \cancel{\frac{\partial c_{ss}}{\partial r}} = 0$$

$$h\left(c_{TOT} - c_\infty\right) = D\frac{\partial c_{TOT}}{\partial r} \quad \text{at } r=r_i \qquad h\left(\cancel{c_{ss} - c_\infty} + c_{tr}\right) = D\left[\frac{\partial c_{tr}}{\partial r} + \cancel{\frac{\partial c_{ss}}{\partial r}}\right]$$

$$c(r,t=0) = c_0 = c_{ss} + c_{tr}(r,t=0) = \psi(r) + c_{tr}(r,t=0)$$

transient problem satisfies $\frac{\partial}{\partial r}\left(r\frac{\partial c_{tr}}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c_{tr}}{\partial t}$

$$\frac{\partial c_{tr}}{\partial r} = 0 \quad @\ r=r_0$$

$$h\,c_{tr} = D\frac{\partial c_{tr}}{\partial r} \quad @\ r=r_i$$

$$c_{tr}(r,t=0) = c_0 - \psi(r)$$

this is homogeneous PDE & BC $\Rightarrow c_{tr}(r,t) = R(r)T(t)$

$$\therefore \quad \frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(RT)\right) = \frac{r}{\alpha}\frac{\partial}{\partial t}(RT)$$

$$rR''T + R'T = \frac{r}{\alpha}R\dot{T} \qquad \text{DIVIDE BY } \underline{r}RT$$

$$\frac{rR''}{rR} + \frac{R'}{rR} = \frac{1}{\alpha}\frac{\dot{T}}{T} = -\lambda^2$$

$$\dot{T} + \lambda^2\alpha T = 0 \qquad T(t) = Ae^{-\lambda^2\alpha t}$$

$$rR'' + R' + \lambda^2 rR = 0 \qquad R(r) = C_1 J_0(\lambda r) + C_2 Y_0(\lambda r)$$

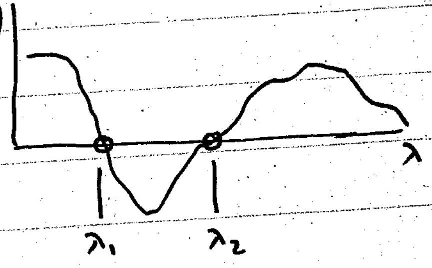
FIRST BC. $\left.\frac{\partial c_{tr}}{\partial r}\right|_{r=r_0} = R'(r_0)T(t) = 0 \Rightarrow R'(r_0) = [C_1 J_0'(\lambda r_0) + C_2 Y_0'(\lambda r_0)]\lambda$

2nd BC $h c_{tr} - D\frac{\partial c_{tr}}{\partial r} = h(C_1 J_0 + C_2 Y_0) - D\lambda(C_1 J_0' + C_2 Y_0') = 0 \quad @\ r=r_i$

$$\begin{bmatrix} \lambda J_o'(\lambda r_o) & \lambda Y_o'(\lambda r_o) \\ h J_o(\lambda r_i) - D\lambda J_o'(\lambda r_i) & h Y_o(\lambda r_i) - D\lambda Y_o'(\lambda r_i) \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if C_1 & $C_2 \neq 0 \Rightarrow$ $h\lambda J_o'(\lambda r_o) Y_o(\lambda r_i) - D\lambda^2 J_o'(\lambda r_o) Y_o'(\lambda r_i) - \lambda h J_o(\lambda r_i) Y_o'(\lambda r_o)$
$+ D\lambda^2 Y_o'(\lambda r_o) J_o'(\lambda r_i) = 0$

This determinant gives the value of λ



- since $D(\lambda, r_i, r_o) = 0 \Rightarrow$ rows of the matrix are multiples of each other
columns

$$\begin{vmatrix} 1 & 1 \\ 5 & 5 \end{vmatrix} = 0 \qquad \begin{vmatrix} 1 & 5 \\ 1 & 5 \end{vmatrix} = 0$$

- $\Rightarrow C_1$ is a multiple of C_2
- since det $(\lambda, r_i, r_o) = 0$ for each $\lambda_n \Rightarrow C_{1n}$ & C_{2n} ie C_is are fns of choice of λ_n
- only need one of the rows: $C_{1n} J_o'(\lambda_n r_o) + C_{2n} Y_o'(\lambda_n r_o) = 0$

$$\Rightarrow C_{1n} = -C_{2n} \frac{Y_o'(\lambda_n r_o)}{J_o'(\lambda_n r_o)}$$

$$\therefore \quad R_n(r) = C_{2n} \left[-Y_o'(\lambda_n r_o) \frac{J_o(\lambda_n r)}{J_o'(\lambda_n r_o)} + Y_o(\lambda_n r) \right]$$

$$\& \quad C_{tr} = \sum_{n=1}^{\infty} \tilde{A}_n \left[-\frac{Y_o'(\lambda_n r_o)}{J_o'(\lambda_n r_o)} J_o(\lambda_n r) + Y_o(\lambda_n r) \right] e^{-\lambda_n^2 \alpha t}$$

$$\tilde{A}_n = A C_{2n}$$

$$\tilde{A}_n = \frac{\int_{r_i}^{r_o} r C_{tr}(r, t=0) R_n(r) \, dr}{\int_{r_i}^{r_o} r R_n^2(r) \, dr}$$

$$\text{Num} = \int_{r_i}^{r_o} r\, c_{t_2}\, R_n\, dr = \int_{r_i}^{r_o} r\Big[\overset{①}{(C_0 - C_\infty)} - \frac{D}{h r_i}\overset{①}{\left(\frac{r_o^2 s}{2} - \frac{r_i^2 s}{2}\right)} - \frac{r_o^2 s}{2}\overset{②①}{\ln(r/r_i)}$$

$$[\ln r - \ln r_i]$$

$$+ \overset{③}{(r^2 - \underset{①}{r_i^2})}\frac{s}{4}\Big] R_n(r)\, dr$$

① involves constant $\cdot \int_{r_i}^{r_o} r\, R_n(r)\, dr \Rightarrow C_0 - C_\infty - \frac{D}{h r_i}\left(\frac{r_o^2 s}{2} - \frac{r_i^2 s}{2}\right) + \frac{r_o^2 s}{2}\ln r_i \; \frac{-r_i^2 s}{4} = \text{const}$

② involves const. $\cdot \int_{r_i}^{r_o} r \ln r \cdot R_n(r)\, dr \Rightarrow \frac{-r_o^2 s}{2} = \text{const}$

③ involves const. $\cdot \int_{r_i}^{r_o} r^3 R_n(r)\, dr \Rightarrow s/4 = \text{const}$

How to get these?

$$r R_n'' + R_n' + \lambda_n^2 r R_n = 0 \qquad (1)$$

$$(r R_n')' + \lambda_n^2 r R_n = 0$$

$$\int (r R_n')'\, dr + \lambda_n^2 \int r R_n\, dr = 0$$

$$r R_n' \Big|_{r_i}^{r_o} + \lambda_n^2 \int r R_n\, dr = 0 \Rightarrow \underline{\overset{①}{\int_{r_i}^{r_o} r R_n(r)\, dr} = -\frac{1}{\lambda_n^2}\, r R_n' \Big|_{r_i}^{r_o}}$$

③: multiply (1) by r^2 and integrate

$$\int r^2 (r R_n')'\, dr + \lambda_n^2 \int r^3 R_n\, dr = 0$$

$u = r^2 \quad du = 2r\,dr$

$dv = (r R_n')' dr = d(r R_n')$

$v = r R_n'$

IBP

Integ by parts

$$r^2 \cdot r R_n' \Big|_{r_i}^{r_o} - \int_{r_i}^{r_o} r R_n' \cdot 2r\, dr + \lambda_n^2 \int_{r_i}^{r_o} r^3 R_n\, dr = 0$$

$$-2 \int r^2 R_n'\, dr$$

$u = r^2 \quad du = 2r\,dr$

$dv = R_n' dr = dR_n \quad V = R_n$

$$r^3 R_n' \Big|_{r_i}^{r_o} - 2\left[r^2 R_n \Big|_{r_i}^{r_o} - \int_{r_i}^{r_o} 2r R_n\, dr\right] + \lambda_n^2 \int r^3 R_n\, dr = 0$$

$$\underline{(r^3 R_n' - 2r^2 R_n)\Big|_{r_i}^{r_o} + 4\left\{-\frac{1}{\lambda_n^2} r R_n'\right\}\Big|_{r_i}^{r_o}} + \lambda_n^2 \int_{r_i}^{r_o} r^3 R_n\, dr = 0$$

②: mult (1) by $\ln r$ and IBP

let $\ln r = u \quad du = \frac{d}{}$

$(r R_n')' dr = dv$

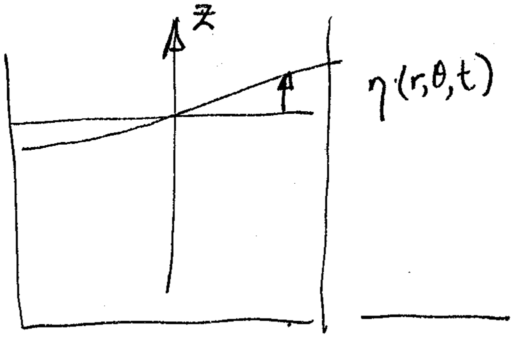
$V = r R$

$$\int \ln r\, (r R_n')'\, dr + \lambda_n^2 \int r \ln r\, R_n\, dr = 0$$

$$\ln r \cdot r R_n' \Big|_{r_i}^{r_o} - \int \frac{1}{r} \cdot r R_n'\, dr + \lambda^2 \int r \ln r\, R_n\, dr = 0$$

$$'' \qquad - \int R_n'\, dr \qquad ''$$

$$'' \qquad - R_n \Big|_{r_i}^{r_o} \qquad ''$$



$\phi(r,\theta,z,t)$ $\quad \nabla^2\phi = 0$ from irrotational flow

$$\nabla\cdot\nabla\phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta} + \phi_{zz} = 0$$

$\eta_t - \phi_z = 0$ @ $z = h$ $\qquad \phi_r = 0$ @ $r = r_0$

$\phi_t + g\eta = 0$ $\quad z = h$ $\qquad \phi_z = 0$ @ $z = 0$

$$\phi = R(r)\,\Theta(\theta)\,Z(z)\,T(t)$$

$$(R'' + \frac{1}{r}R')\Theta Z T + \frac{1}{r^2}\Theta'' R Z T + Z''\Theta R T = 0$$

$$(R'' + \frac{1}{r}R')\Theta Z + \frac{1}{r^2}\Theta'' R Z + R\Theta Z'' = 0 \qquad \div R\Theta Z T$$

$$\frac{(R'' + \frac{1}{r}R')}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = \mp\beta^2 \rightarrow Z = A\sinh\beta z + B\cosh\beta z$$

$$r^2\frac{(R'' + \frac{1}{r}R')}{R} + \beta^2 r^2 = \boxed{-\frac{\Theta''}{\Theta} = +\mu^2} \longrightarrow \Theta = C\sin\mu\theta + D\cos\mu\theta$$

μ = integer

$$r^2R'' + rR' + (\beta^2r^2 - \mu^2)R = 0$$

$$R = E\,J_\mu(\beta r) + F\,\cancel{Y_\mu}(\beta r)$$

0 since origin contained

$$\phi_r\big|_{r=r_0} = \underset{=0}{E\beta J'_\mu(\beta r_0)}$$

$$\phi_{tt} + g\eta_t = \phi_{tt} + g\phi_z = 0 \quad @\ z = h$$

$$\therefore\ T''R\Theta Z + g\left[R\Theta Z'T\right] = 0$$

$$\text{or}\quad \frac{T''}{T} = -g\frac{Z'(h)}{Z(h)} = -\omega^2$$

$$\phi_z\big|_{z=0} \Rightarrow Z'\big|_{z=0} = 0 \Rightarrow \qquad \therefore\ Z = \cancel{A\sinh\beta z}$$

$$\beta A\cosh\beta z + B\beta\sinh\beta z$$

$$\Rightarrow A = 0 \quad\text{or}\quad Z = B\cosh\beta z$$

$$\beta g\cdot\frac{A\,\overset{\sinh}{\cancel{\cosh}}\,\beta}{A\,\cancel{\sinh}\,\beta}$$

$$\frac{\beta h}{h}g\tanh\beta h =$$

$$+\frac{\omega^2 h}{g} = \alpha\tanh\alpha$$

$\simeq 9^{\circ}$++ while the distribution becomes more uniform. The highest values of the SIF are still attained in the $n = 2$ case, and, the $n = 2$ curve exhibits a pattern similar to that for the semi-circular case (see Fig. 16).

From a design point of view, the most critical case is $n = 2$. Figure 17 presents the variation of the SIF along the crack front for $a/t = 0.05$ and various ellipticities for the $n = 2$ case. This figure is typical of the $n = 2$ case for other a/t ratios. From this figure we can see the gradual and continuous transition of the location of K_{max}, the non-normalized value of the maximum SIF, from $\phi = 90°$ for slender, semi-elliptical cracks to $\phi = 0°$ for transverse, semi-elliptical cracks. Furthermore, from the above figure, there appears to be a certain value of ellipticity around $a/c \sim 0.75$ for which the SIF is uniformly distributed along the crack front and has equal K_{max} values at $\phi = 0°$ and $\phi = 90°$.

K_{max} is also found to increase as the relative depth, a/t, increases, as the ellipticity, a/c, decreases, and, is found to reach its highest values for $n = 2$. As the number of cracks in the array increases K_{max} decreases. K_{max}, for $n = 2$, as a function of crack depth for various values of ellipticity are shown in Figure 18. The 2-D case representing $a/c \rightarrow 0$ is also included (Perl and Aroné, 1988).

From the data in Figure 18, showing the normalized maximum SIF, a simple set of expressions for the most critical SIFs may be found. When analyzing these data, distinction is made between sparse and dense radial crack arrays as was done in the 2-D case (see Perl, 1992). In the case of $n = 2$ the array is considered sparse for $a/t < 0.3$ and dense for $a/t \geq 0.3$.

++As the numerical results at $\phi = 0°$ exhibit some uncertainty (see, for example, Kirkhope *et al.*, 1991), and, as the difference in the SIF at $\phi = 0°$ and $\phi = 9°$ is very small, one can still assume that the maximum occurs at $\phi = 0°$.

suppose $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2}$

$$\left.\begin{aligned} U(0,t) &= \phi_0(t) \\ U(L,t) &= \phi_1(t) \end{aligned}\right\} \text{want to remove time dependence in bc's}$$

$U(x,0) = f(x)$

how ?

$\frac{\partial U}{\partial t}(x,0) = g(x)$

let $U(x,t) = V(x,t) + \psi(x,t)$ → ψ will be picked in simplest manner

<u>choose</u> $V(x,t)$ to have <u>homog bc</u>

$V(0,t) = \phi_0(t) - \psi(0,t) = 0 \overset{so}{\Rightarrow} \psi(0,t) = \phi_0(t)$

$V(L,t) = \phi_1(t) - \psi(L,t) = 0 \overset{so}{\Rightarrow} \psi(L,t) = \phi_1(t)$

The V is the original problem, viz. I.C. problem with homo BC.

<u>choose in simplest form</u> $\psi(x,t) = \phi_0(t) + \frac{x}{L}\left[\phi_1(t) - \phi_0(t)\right]$ satisfies BC.

now since

$$\frac{1}{c^2} U_{tt} = U_{xx}$$

$$\frac{1}{c^2}\left[V_{tt} + \psi_{tt}\right] = \left[V_{xx} + \psi_{xx}\right]$$

$$\therefore \frac{1}{c^2}\left[V_{tt}\right] - V_{xx} = \psi_{xx} - \frac{1}{c^2}\psi_{tt} = 0 - \frac{1}{c^2}\left[\phi_0''(t) + \frac{x}{L}\left(\phi_1''(t) - \phi_0''(t)\right)\right] \leftarrow \underline{\underline{\text{known fn of } t \,\&\, x}}$$

$$\frac{1}{c^2} V_{tt} - V_{xx} = \underline{\underline{G(x,t)}}$$

also BC's convert to

$$U(x,0) = V(x,0) + \psi(x,0) = V(x,0) + \underbrace{\phi_0(0) + \frac{x}{L}\left(\phi_1(0) - \phi_0(0)\right)}_{} = f(x)$$

$$\therefore V(x,0) = f(x) - \psi(x,0) = h(x) \quad \text{known fn of } x \checkmark$$

$$\frac{\partial U}{\partial t}(x,0) = \frac{\partial V}{\partial t}(x,0) + \frac{\partial \psi}{\partial t}(x,0) = \frac{\partial V}{\partial t}(x,0) + \underbrace{\phi_0'(0) + \frac{x}{L}\left(\phi_1'(0) - \phi_0'(0)\right)}_{} = g(x)$$

$$\frac{\partial V}{\partial t}(x,0) = g(x) - \frac{\partial \psi}{\partial t}(x,0) = l(x) \quad \text{known fn of } x \checkmark$$

we have removed the <u>time dependence in the BC</u> & put in in the inhomogeneous term of PDE

for $V(x,t)$:

$\frac{1}{c^2} V_{tt} - V_{xx} = G(x,t)$ Inhomog PDE time dependent ✓

$V(0,t) = 0$ (1) ✓

$V(L,t) = 0$ (2)

$V(x,0) = h(x)$ ✓

$\frac{\partial V}{\partial t}(x,0) = l(x)$

use the homog PDE to find the spatial function then use variation of parameters

$\frac{1}{c^2} V_{tt} - V_{xx} = 0 \Rightarrow \frac{1}{c^2} F\cancel{T}'' - F''T = 0$

or $\frac{c^2 F''}{F} = \frac{T''}{T} = -\omega^2$

$T = A\cos\omega t + B\sin\omega t$

$F = \mathscr{C}\cos\frac{\omega}{c}x + \mathscr{D}\sin\frac{\omega}{c}x$

(1) & (2) $\Rightarrow \mathscr{C} = 0$ & $\frac{\omega}{c}L = n\pi$

or $\omega_n = \frac{n\pi c}{L}$

$$T = A\cos\frac{n\pi c}{L}t + B\cos\frac{n\pi c}{L}t \qquad V = \sum_{n=0}\left\{A\cos\frac{n\pi c}{L}t + B\cos\frac{n\pi c}{L}t\right\}\sin\frac{n\pi x}{L}$$

using Var. of Param

$$\therefore\; V = \sum\left\{\tilde{A}_n(t)\cos\frac{n\pi c}{t}t + \tilde{B}_n(t)\sin\frac{n\pi c}{L}t\right\}\sin\frac{n\pi x}{L} = \sum E_n(t)\cdot\sin\frac{n\pi x}{L}$$

$$V_t = \sum E_n'\sin\frac{\pi n x}{L} \qquad V_x = \sum E_n\frac{n\pi x}{L}\cos\frac{n\pi x}{L}$$

$$V_{tt} = \sum E_n''\sin\frac{n\pi x}{L} \qquad V_{xx} = -\sum E_n\left(\frac{n\pi x}{L}\right)^2\sin\frac{n\pi x}{L}$$

$$\therefore\; \frac{1}{c^2}V_{tt} - V_{xx} = G(x,t)$$

$$\sum\left\{\frac{1}{c^2}E_n'' + E_n\left(\frac{n\pi}{L}\right)^2\right\}\sin\frac{n\pi x}{L} = \overbrace{-\frac{1}{c^2}\left[\phi_0''(t) + \frac{x}{l}\left(\phi_1''(t) - \phi_0''(t)\right)\right]}^{G(x,t)} = \sum_{n=1}^{\infty} H_n(t)\sin\frac{n\pi x}{L}$$

G(x,t) can be expanded as a sine series

$$\therefore\; H_n(t) = \frac{2}{L}\int_0^L -\frac{1}{c^2}\left[\phi_0'' + \frac{x}{L}\left(\phi_1''(t) - \phi_0''\right)\right]\sin\frac{n\pi x}{L}dx$$

$$= \frac{2}{L}\left\{-\frac{1}{c^2}\phi_0''\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)\Big|_0^L - \frac{1}{c^2L}\left(\phi_1'' - \phi_0''\right) \quad \left(-x\cos\frac{n\pi x}{L}\right)\right.$$

thus term by term

$$\frac{1}{c^2}E_n'' + \left(\frac{n\pi}{L}\right)^2 E_n = H_n(t).$$

Solve by variation of parameters after get homogeneous solutions

from homog eqn. $E_n'' + \left(\frac{n\pi c}{L}\right)^2 E_n = 0$

and to find $H_n(t)$

when must let $E_{n_p} = A_n(t)\cos\frac{n\pi c}{L}t + B_n(t)\sin\frac{n\pi c}{t}t$

and use variation of parameters.

remember if $y_p = C_1 y_1 + C_2 y_2$ satisfies $y'' + b(x)y' + c(x)y = g(x)$

we have $C_1' y_1 + C_2' y_2 = 0$

$C_1' y_1' + C_2' y_2' = g(x)$

and $C_1 = \int\frac{-g y_2}{\text{wronskian}}dx \qquad C_2 = \int\frac{g y_1}{\text{wronskian}}dx$

wronskian is $y_1 y_2' - y_2 y_1'$

---

here $y_1 = \cos\frac{n\pi c}{t}t \qquad y_2 = \sin\frac{n\pi c}{L}t$

wronskian is $\frac{n\pi c}{L}$

thus $$C_{1n} = \int^t\frac{-H_n(\bar{t})c^2\sin\frac{n\pi c}{L}\bar{t}\,d\bar{t}}{\frac{n\pi c}{L}}$$

$$C_{2n} = \int^t\frac{H_n(\bar{t})c^2\cos\frac{n\pi c}{L}\bar{t}}{\frac{n\pi c}{L}}d\bar{t}$$

Exercises

4.1 The temperature field in a slab, initially at uniform temperature, subjected to a sudden increase in the temperature of one face, is described by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_o$$

$$T(0,t) = T_o \qquad T(L,t) = T_1$$

Develop the solution to this problem, giving expressions for any integrals involved in the solution. Does your (series) solution converge?

4.2 The temperature field in a slab, initially at uniform temperature, subjected to a step input in heat flux at one surface, is described by

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_o$$

$$T(0,t) = T_o \qquad k\left.\frac{\partial T}{\partial x}\right|_{x=L} = q''$$

Solve this problem, giving expressions for any integrals involved in the solution. Does your (series) solution converge?

4.3 The azimuthal velocity field in a cylinder of radius a filled with fluid initially at rest, subject to a sudden rotation of the cylinder is described by

$$\nu\left[\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) - \frac{u}{r}\right] = r\frac{\partial u}{\partial t} \qquad u(r,0) = 0$$

$$u(a,t) = u_o$$

Solve this problem, giving expressions for any integrals involved in the solution. Hint: The steady-state solution is solid body rotation.

4.4 The motion of the fluid in an annular cylinder, set into motion by the sudden rotation of the outer surface, is described by the PDE and initial condition of exercise 4.3, and the boundary conditions

$$u(r_i,t) = 0 \qquad u(r_o,t) = u_o$$

where r_i and r_o are the inner and outer radii, respectively. Solve this problem. Express any integrals involved in terms of functions evaluated at r_i and r_o.

4.5 The concentration of a contaminant in a hollow sphere, initially "clean", subjected to a step jump in the concentration at the inner radius r_i, is described by

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) = \frac{r^2}{\alpha}\frac{\partial c}{\partial t} \qquad c(r,0) = 0$$

$$c(r_i,t) = c_o \qquad c(r_o,t) = 0$$

Solve this problem, developing expressions for any integrals involved in terms of functions evaluated at r_i and r_o. This problem has application in the geological diffusion of nuclear wastes.

4.6 The transient temperature of a circular fin is described by

$$k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \beta^2 r(T-T_o) = \frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$$\frac{\partial T}{\partial r} = 0 \quad \text{at} \quad r = r_o \qquad T(r_i,t) = 0$$

$$T(r,0) = T_o$$

Solve this problem, developing expressions for any integrals in terms of functions evaluated at r_i and r_o.

4.7 The steady potential field in a circular object, with potential specified around the perimeter ($r = a$), is described by

$$\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 \phi}{\partial \theta^2} = 0$$

$$\phi(a,\theta) = f(\theta)$$

Develop the solution to this problem, expressing the result in terms of appropriate integrals.

$$F_o'' = a_o \tag{4.7.27a}$$

$$F_n'' - \left(\frac{2n\pi}{a}\right)^2 F_n = a_n \tag{4.7.27b}$$

$$G_n'' - \left(\frac{2n\pi}{a}\right)^2 G_n = b_n \tag{4.7.27c}$$

Particular solutions to these three ODEs can be obtained by standard methods (e.g., the method of separation of variables).

In this problem, the corners would be singular points. The series solutions would converge everywhere, except at the corners, where the solutions $\phi^{(1)} - \phi^{(4)}$ would all be zero because of the method of solution.

4.8 Some Generalizations

While some problems fall into the Sturm-Liouville form, others do not. However, the same general ideas can be used with the help of a new concept, adjoint operations.

Suppose that the SOV process in a linear, homogeneous PDE problem produces the ODE

$$Lu = Mu + \lambda Nu = 0 \tag{4.8.1}$$

where L , M , and N are linear operators. Suppose that the linear, homogeneous boundary conditions are a set of equations of the form

$$\{B_i u = 0\} \quad \text{at} \quad x = a \text{ or } b \tag{4.8.2}$$

where the B_i are also linear operators. The eigenvalues λ are those values for which non-trivial solutions to (4.8.1), and (4.8.2) exist. The adjoint operators L^* , M^* , N^* , and B_i^* are defined by the requirement that

$$\int_a^b vLudx = \int_a^b uL^*vdx \tag{4.8.3}$$

and hence I_3 can also be evaluated in terms of boundary quantities. Thus, the A_n can be found without recourse to any numerical integration! This is often the case; the key is always integration by parts.

4.7 Splitting

We have seen that problems with linear PDEs and BCs can be solved by constructing linear combinations of the eigensolutions for appropriate homogeneous partial problems. We also saw that in transient problems the inhomogeneities can be "removed" by "splitting" the solution into steady-state and transient parts. The concept of problem splitting can also be used to "remove inhomogeneities" in other problems.

To illustrate the idea, consider the problem shown in Fig. 4.7.1. The PDE is the inhomogeneous Laplace equation,

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = h(x,y) \tag{4.7.1}$$

The domain is the rectangle shown, and the boundary conditions specify ϕ around the boundary, in terms of the functions shown. Note that all of these boundary conditions are inhomogeneous.

To use the methods developed in this chapter, we can "split" the problem into the five problems shown in Fig. 4.7.1. Problem (p) will take care of the inhomogeneity in the PDE. The solution $\phi^{(p)}$ is any particular solution of the PDE, without regard for boundary conditions. It will yield the values of $\phi^{(p)}$ on the boundaries denoted by the functions $g_1 - g_4$. We shall discuss means for finding the particular solution shortly. The four problems $\phi^{(1)} - \phi^{(4)}$ involve homogeneous PDEs and nearly completely homogeneous boundary conditions. Therefore, for each the eigensolutions of the homogeneous partial problem can be found, and then a linear combination of these eigenfunctions taken to construct a solution satisfying the remaining inhomogeneous boundary condition. Note that the sum

$$\phi = \phi^{(p)} + \sum_{k=1}^{4} \phi^{(k)} \tag{4.7.2}$$

satisfies the inhomogeneous PDE and inhomogeneous boundary conditions. This type of splitting can, of course, only be done in linear problems.

Let's presume that we have the particular solution $\phi^{(p)}$, and are ready to solve problems $\phi^{(1)} - \phi^{(4)}$. We will do the $\phi^{(1)}$ problem; the other three are done in the same way.

The $\phi^{(1)}$ PDE is, dropping the superscript (1),

$$\phi_{xx} + \phi_{yy} = 0 \tag{4.7.3}$$

and the boundary conditions are

$$\phi = 0 \quad \text{on} \quad y = 0 \tag{4.7.4}$$

$$\phi = 0 \quad \text{on} \quad x = 0 \tag{4.7.5}$$

$$\phi = 0 \quad \text{on} \quad x = a \tag{4.7.6}$$

$$\phi = f_1(x) - g_1(x) = q(x) \quad \text{on} \quad y = b \tag{4.7.7}$$

We look for eigensolutions to the homogeneous partial problem (4.7.3) - (4.7.6) in the form

$$\phi = X(x)\,Y(y) \tag{4.7.8}$$

and, from (4.7.3), find

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \tag{4.7.9}$$

Hence,

$$X'' + \lambda^2 X = 0 \tag{4.7.10}$$

$$Y'' - \lambda^2 Y = 0 \tag{4.7.11}$$

The decision to name the separation constant $-\lambda^2$ was dictated by the recognition that the X-solutions must oscillate in X in order to match the boundary conditions. The X solution is

$$X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) \tag{4.7.12}$$

The BC (4.7.5) gives $C_2 = 0$. Then, the BC (4.7.6) requires $\sin(\lambda a) = 0$. Hence,

$$\lambda_n a = n\pi \tag{4.7.13}$$

The Y equation solution is

$$Y = C_3 \sinh(\lambda y) + C_4 \cosh(\lambda y) \tag{4.7.14}$$

The BC (4.7.4) requires $C_4 = 0$. Hence, the eigensolutions are (apart from a scaling constant)

$$\phi_n(x,y) = \sin(n\pi x/a) \sinh(n\pi y/a) \tag{4.7.15}$$

Finally, we seek the solution satisfying the inhomogeneous condition (4.7.7) as an expansion in the eigenfunctions,

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n \tag{4.7.16}$$

Thus, at $y = b$,

$$\phi(b,x) = q(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(n\pi b/a) \tag{4.7.17}$$

The orthogonality property for the X_n eigenfunctions is*

$$\int_0^a X_n X_m dx = 0 \qquad n \neq m \tag{4.7.18}$$

---

*Developed in the usual way.

So, multiplying (4.7.17) by $\sin(m\pi x/a)$, and integrating

$$A_m = \frac{\int_0^a q(x) \sin(m\pi x/a)}{\sinh(m\pi b/a) \int_0^a \sin^2(m\pi x/a) dx} \tag{4.7.19}$$

Given $q(x)$, we could compute the A_n . Hence, the $\phi^{(1)}$ solution is completely known.

The $\phi^{(2)}$, $\phi^{(3)}$, and $\phi^{(4)}$ problems could be handled in much the same way. In the $\phi^{(3)}$ problem, the Y equations would again be (4.7.11), and $Y(b) = 0$. Hence, rather than (4.7.14), a "more artistic" form of the Y solution is

$$Y = C_5 \sinh[\lambda(y-b)] + C_6 \cosh[\lambda(y-b)] \tag{4.7.20}$$

because C_6 will have to be zero for $Y(b) = 0$.

Let's now discuss the particular solution. If h depends upon only one of the independent variables, say x , the particular solution may be developed by assuming

$$\phi^{(p)} = F(x) \tag{4.7.21}$$

The inhomogeneous PDE is then

$$F'' = h(x) \tag{4.7.22}$$

which has the solution (by double integration)

$$F = \int_0^x \int_0^\xi h(\sigma)\, d\sigma\, d\xi \tag{4.7.23}$$

If $h = h(x,y)$, the particular solution can be obtained by expanding h in a Fourier series in either x or y. If we choose to do it in x, we would write

$$h(x,y) = \sum_{n=o}^{\infty} a_n(y) \cos(2n\pi x/a) + \sum_{n=1}^{\infty} b_n(y) \sin(2n\pi x/a) \tag{4.7.24}$$

The coefficients a_n and b_n are determined using the orthogonality property of the sine and cosine functions;

$$a_o = \frac{1}{a}\int_0^a h\, dx \tag{4.7.25a}$$

$$a_m = \frac{2}{a}\int_0^a h \cos(2m\pi x/a)dx \tag{4.7.25b}$$

$$b_m = \frac{2}{a}\int_0^a h \sin(2m\pi x/a)dx \tag{4.7.25c}$$

Next, one would look for a particular solution in the form

$$\phi^{(p)} = \sum_{n=o}^{\infty} F_n(y) \cos(2n\pi x/a) + \sum_{n=1}^{\infty} G_n(y) \sin(2n\pi x/a) \tag{4.7.26}$$

Substituting into the PDE, and equating coefficients of the sines and cosines, one finds

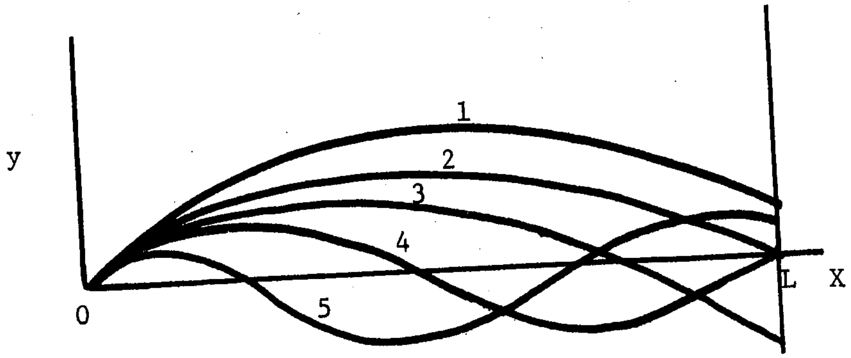


FIGURE 4.2.1

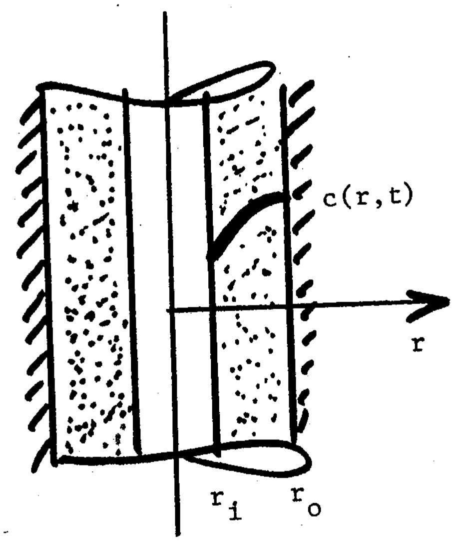


Fig. 4.6.1

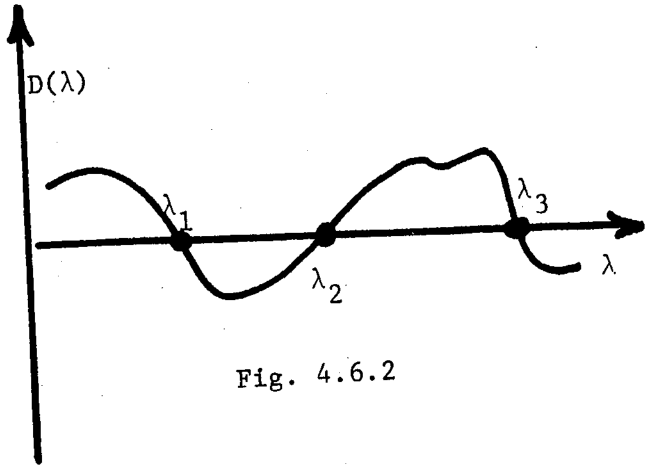


Fig. 4.6.2

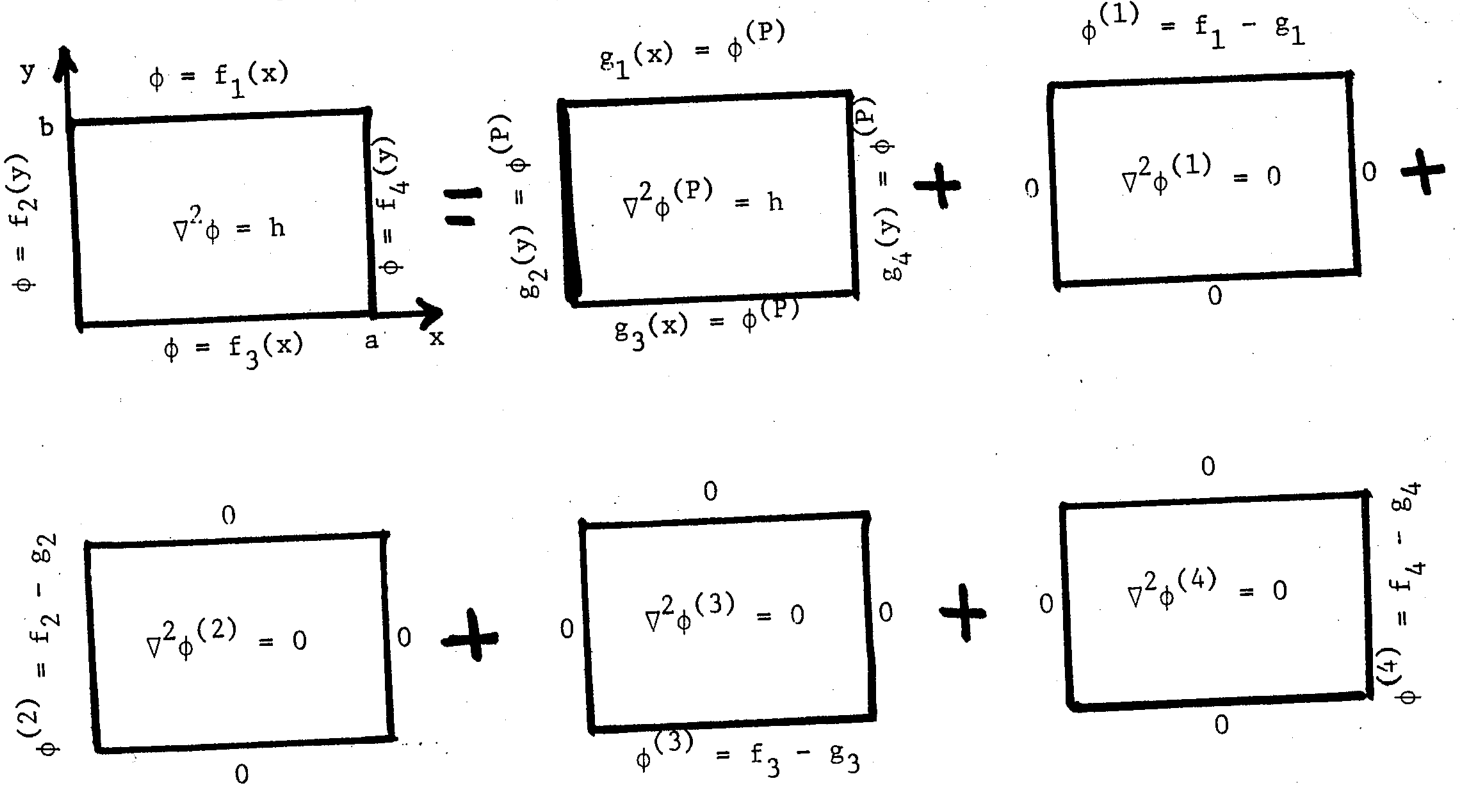


Fig. 4.7.1. Splitting

~~let $z = \sigma/\sqrt{2}$ $\quad z^2 = \sigma^2/2$ $\quad dz = \frac{d\sigma}{\sqrt{2}}$~~

~~$$\eta\sqrt{2}\int_{\infty}^{\eta/\sqrt{2}} e^{-z^2}dz$$~~

~~$$-\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}\,\frac{\eta}{\sqrt{2}}$$~~

~~$$f(\eta) = e^{-\eta^2/2} - \eta\sqrt{\pi/2}\,\mathrm{erfc}\left(\eta/\sqrt{2}\right)$$~~

LESSON # [illegible]

LAPLACE TRANSFORMS - **HANDOUT TABLES**

CAN BE USED TO FIND SOLUTIONS OF PDE's WHEN ONE OR MORE INDEPENDENT VARIABLES CAN RANGE FROM 0 TO ∞

ie. TIME $\quad t \geq 0$

$\quad x \quad\quad x \geq 0$

used to solve : $\quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

we can define $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad s = \text{const.}$

note that $F(s) = \mathcal{L}(f(t))$

for every $f(t) \Longleftrightarrow F(s)$

does $F(s)$ exist for any $f(t)$? NO.

$F(s)$ exists if
- $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \leq t \leq T$ where $t_1 > 0$
- $t^n |f(t)|$ is bounded near $t = 0$ for some n $\quad n < 1$
- $e^{-s_0 t} |f(t)|$ is bounded for large t, for some value s_0

$|f| \leq \frac{M}{t^n}$ $\Leftarrow$

$\left|\int f e^{-st} dt\right| < \int |f| e^{-st} dt$

$< \int \frac{M}{t^n} e^{-st} dt = M\left[\frac{t^{1-n} e^{-st}}{1-n} -\right.$

LAPLACE TRANSFORMS HAVE THE FOLLOWING PROPERTIES § 2.3.

$$\mathcal{L}\{a f_1(t) + b f_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\} = a \cdot F_1(s) + b \cdot F_2(s$$

$$\mathcal{L}\{\frac{d}{dt} f(t)\} = s \cdot F(s) - f(t=0+)$$

$$\mathcal{L}\{d^2/dt^2 f(t)\} = s^2 F(s) - s f(0+) - \frac{d}{dt} f(0+)$$

onvolution $$\mathcal{L}\{\int_0^t f(t-a)\, g(a)\, da\} = F(s)G(s) \quad \text{where } G(s) = \int_0^\infty g(t) e^{-st} dt$$

nit step fn.
eaviside step fn

$$H(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \end{cases} \qquad \mathcal{L}\{H(t)\} = \int_0^\infty H(t) e^{-st} dt = \int_{0+}^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_{0+}^\infty = -\frac{1}{s}(-e^0) = \frac{1}{s}$$

$$\mathcal{L}\{\delta(t)\} = \int_0^\infty \delta(t) e^{-st} dt = e^0 = 1$$

Suppose $\frac{d^2y}{dt^2} = k^2 y$ where $y(0) = 0$ $\frac{dy}{dt} = 1$ at $t = 0$ let $Y(s) = \int_0^\infty y(t) e^{-st} dt$

$$\int \frac{d^2y}{dt^2} e^{-st} dt = \int k^2 y e^{-st} dt \Rightarrow s^2 Y(s) - s\underbrace{y(0+)}_{0} - \underbrace{y'(0+)}_{1} = k^2 Y(s).$$

$$(s^2-k^2)Y(s) = 1 \quad \therefore \quad Y(s) = \frac{1}{s^2-k^2} = \frac{A}{(s-k)} \; \frac{+B}{(s+k)}$$

$A = \frac{1}{2k}$

$B = -\frac{1}{2k}$

$$\therefore \quad Y(s) = \frac{1}{2k}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$

$(A+B)s = 0 \cdot s$ $A = -B$

$k(A-B) = 1$ $A - B = \frac{1}{k}$

look at 29.3.8 or 29.3.17

$$y(t) = \frac{1}{2k}\left[e^{+kt} - e^{-kt}\right] = \frac{1}{k}\sinh kt;\ y(0) = \frac{1}{2k}[0].$$

$$y'(t) = +\frac{k}{2k}\left[e^{-kt} + e^{kt}\right] = \cosh kt;\ y'(0) = \frac{1}{2}[2] = 1.$$

Same result
$$y'' - k^2 y = 0 \qquad y = c_1 e^{kt} + c_2 e^{-kt} \qquad @\ t=0\ y = 0 = c_1 + c_2$$

$$y'(0) = 1 = k[c_1 - c_2] \Rightarrow c_2 = -\frac{1}{2k} \quad c_1 = \frac{1}{2k}.$$

NOTE WE INCORPORATE THE B.C. IN THE SOLUTION

JUST AS WE DID FOR O.D.E WE CAN DO FOR PDE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,t).$$

let $\mathcal{J}(x;s) = \int_0^\infty T(x,t) e^{-st} dt$ or $\mathcal{J}(x;s) = \mathcal{L}\{T(x,t)\}$

$$\mathcal{L}\{\frac{\partial}{\partial x} T\} = \frac{\partial}{\partial x}\int_0^\infty T(x,t) e^{-st} dt = \frac{d}{dx}\mathcal{J}(x;s)$$

but $$\mathcal{L}\{\frac{\partial}{\partial t} T\} = s\,\mathcal{J}(x;s) - T(x,t=0)$$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}T\right\} = \frac{d^2}{dx^2}\mathcal{J}(x;s)$$

$$\therefore \quad \mathcal{L}\left\{\frac{\partial^2}{\partial x^2}T = \frac{1}{\alpha}\frac{\partial T}{\partial t}\right\} \Longrightarrow \frac{d^2}{dx^2}\mathcal{J}(x;s) = \frac{1}{\alpha}\left[s\mathcal{J}(x;s) - T(x,t=0)\right]$$

$$\frac{d^2}{dx^2}\mathcal{J} - \frac{s}{\alpha}\mathcal{J} = -\frac{1}{\alpha}T(x,t=0)$$

Gain by making PDE into ODE
Lose by making it into an inhomog ODE

look at the problem when @ $T(x,t=0)=T_i$ ①

$T(0,t) = T_s$ ②

$T(x,t) \to T_i$ as $x \to \infty$ ③

$$\mathcal{J}'' - \frac{s}{\alpha}\mathcal{J} = -\overset{①}{\frac{T_i}{\alpha}} \qquad \text{let } \mathcal{J} = \mathcal{J}_H + \mathcal{J}_p \qquad \text{let } \mathcal{J}_p = C$$

$$-\frac{s}{\alpha}C = -\frac{T_i}{\alpha} \quad \Rightarrow \quad C = \frac{T_i}{s} = \mathcal{J}_p$$

$$\mathcal{J}_H'' - \frac{s}{\alpha}\mathcal{J}_H = 0 \qquad \mathcal{J}_H = C_1 e^{-\sqrt{\frac{s}{\alpha}}x} + C_2 e^{\sqrt{\frac{s}{\alpha}}x}$$

② $\displaystyle\int_0^\infty T(x=0,t)e^{-st}dt = \mathcal{J}(0;s) = \int_0^\infty T_s e^{-st}dt = T_s/s \Rightarrow \mathcal{J}(0;s) = T_s/s$

③ $\displaystyle\int_0^\infty T(x,t)e^{-st}dt = \mathcal{J}(x;s) = \int_0^\infty T_i e^{-st}dt = T_i/s \Rightarrow \mathcal{J}(x;s) \to \frac{T_i}{s}$ as $x \to \infty$

note $\mathcal{J} = \mathcal{J}_p + \mathcal{J}_H = \dfrac{T_i}{s} + C_1 e^{-\sqrt{\frac{s}{\alpha}}x} + C_2 e^{+\sqrt{s/\alpha}\,x}$ from ③ $\Rightarrow C_2 = 0$

from ② $\dfrac{T_s}{s} = \dfrac{T_i}{s} + C_1 e^0 \Rightarrow C_1 = \dfrac{T_s - T_i}{s}$

$\therefore \mathcal{J} = \dfrac{T_i}{s} + \dfrac{T_s - T_i}{s} e^{-\sqrt{\frac{s}{\alpha}}x}$ from $\underline{29.3.83}$ $k = \frac{x}{\sqrt{\alpha}}$

we want $T(x,t) = \mathcal{L}^{-1}\{\mathcal{J}(x;s)\} = T_i H(t) + T_s - T_i \operatorname{erfc}\left(\frac{x}{\sqrt{\alpha}} \cdot \frac{1}{2\sqrt{t}}\right)$

$\left(\frac{x}{\sqrt{2}\sqrt{2\alpha t}}\right)$

remember $\eta = \dfrac{x}{\sqrt{2\alpha t}}$ $\therefore T(x,t) = T_i + T_s - T_i \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$

$$\frac{T - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$$

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~~$f(\eta) = e^{-\eta^2/2} - \eta\sqrt{\pi/2}\, \operatorname{erfc}(\eta/\sqrt{2})$~~

LESSON # [redacted]

HANDOUT TABLES

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$$\mathcal{L}\{\tfrac{d}{dt} f(t)\} = s\ F(s) - f(t=0+)$$

$$\mathcal{L}\{d^2/dt^2 f(t)\} = s^2F(s) - sf(0+) - \frac{d}{dt}f(0+)$$

Convolution

$$\mathcal{L}\left\{\int_0^t f(t-a)\,g(a)\,da\right\} = F(s)G(s) \quad \text{where } G(s) = \int_0^\infty g(t)e^{-st}dt$$

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$$(s^2-k^2)Y(s) = 1 \quad \therefore \quad Y(s) = \frac{1}{s^2-k^2} = \frac{A}{(s-k)} \frac{+B}{(s+k)}$$

$A = \frac{1}{2k}$

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$$\therefore \quad Y(s) = \frac{1}{2k}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$

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Same result $\Big\{$ $y'' - k^2y = 0$ $\quad y = c_1e^{kt} + c_2e^{-kt}$ $\quad$ @ $t=0$ $y = 0 = c_1 + c_2$

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but $\mathcal{L}\left\{\frac{\partial}{\partial t}T\right\} = s\,\mathcal{J}(x;s) - T(x,t=0)$

$$\mathcal{L}\left\{ \frac{\partial^2}{\partial x^2} T \right\} = \frac{d^2}{dx^2} \mathcal{J}(x;s)$$

$$\therefore \quad \mathcal{L}\left\{ \frac{\partial^2}{\partial x^2} T = \frac{1}{\alpha} \frac{\partial T}{\partial t} \right\} \implies \frac{d^2}{dx^2} \mathcal{J}(x;s) = \frac{1}{\alpha}\left[s\mathcal{J}(x;s) - T(x,t=0) \right]$$

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③ $$\int_0^\infty T(x,t)e^{-st}dt = \mathcal{J}(x;s) = \int_0^\infty T_i e^{-st}dt = T_i/s \implies \mathcal{J}(x;s) \rightarrow \frac{T_i}{s} \text{ as } x \rightarrow \infty$$

note $$\mathcal{J} = \mathcal{J}_p + \mathcal{J}_H = \frac{T_i}{s} + C_1 e^{-\sqrt{\frac{s}{\alpha}}x} + C_2 e^{+\sqrt{s/\alpha}\,x}$$ from ③ $\implies C_2 = 0$

from ② $$\frac{T_s}{s} = \frac{T_i}{s} + C_1 e^0 \implies C_1 = \frac{T_s - T_i}{s}$$

$$\therefore \quad \mathcal{J} = \frac{T_i}{s} + \frac{T_s - T_i}{s} e^{-\sqrt{\frac{s}{\alpha}}x}$$ from <u>29.3.83</u> $k = \frac{x}{\sqrt{\alpha}}$

we want $$T(x,t) = \mathcal{L}^{-1}\{\mathcal{J}(x;s)\} = T_i H(t) + T_s - T_i \operatorname{erfc}\left(\frac{x}{\sqrt{\alpha}} \cdot \frac{1}{2\sqrt{t}}\right)$$

$$\left(\frac{x}{\sqrt{2}\sqrt{2\alpha t}}\right)$$

remember $$\eta = \frac{x}{\sqrt{2\alpha t}} \quad \therefore \quad T(x,t) = T_i + T_s - T_i \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$$

$$\frac{T - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$$

29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

29.1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in 29.1.1 converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

exists, then it converges for all s with $\mathscr{R}s>s_0$, and the image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2 $$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \frac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

Heaviside step function

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function $F(t)$	*Image Function $f(s)$*
29.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s)-F(+0)$
29.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
29.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

---

[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

	Original Function $F(t)$	Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c}e^{(b/c)t}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t)\sum_{n=0}^{\infty}(-1)^n u(t-na)$	$\frac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\lvert F(t)\rvert$	$f(s)\coth\frac{as}{2}$
	Heaviside Expansion Theorem	
29.2.20	$\sum_{n=1}^{m}\frac{p(a_n)}{q'(a_n)}e^{a_n t}$	$\frac{p(s)}{q(s)}, q(s)=(s-a_1)(s-a_2)\dots(s-a_m)$ $p(s)$ a polynomial of degree$<m$
29.2.21	$e^{at}\sum_{n=1}^{r}\frac{p^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree$<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

---

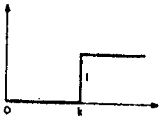
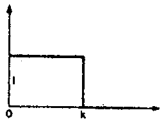
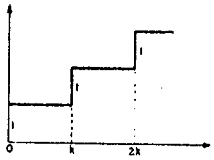
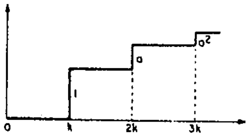
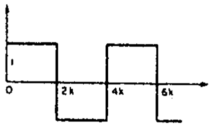
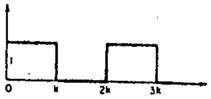
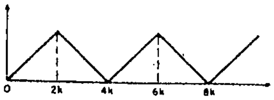
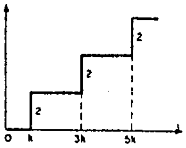
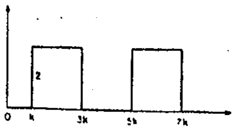
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

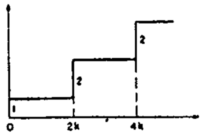
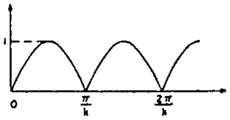
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \quad (n=1,2,3,\ldots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+\frac{1}{2})} \quad (n=1,2,3,\ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot 3\cdot 5\ldots(2n-1)\sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \quad (k>0)$	6	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
29.3.10	$\frac{1}{(s+a)^n} \quad (n=1,2,3,\ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k>0)$	6	$t^{k-1}e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \quad (a\neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \quad (a\neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a}\sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a}\sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at\cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)}\qquad(a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2}=\frac{1}{s^2+2as+(a^2+b^2)}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc}a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf}a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf}a\sqrt{t}$	7

	$f(s)$	$F(t)$	
9.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$	$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
9.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2t}[b-a\ \mathrm{erf}\ a\sqrt{t}]-be^{b^2t}\ \mathrm{erfc}\ b\sqrt{t}$	7
9.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t}\ \mathrm{erfc}\ a\sqrt{t}$	7
9.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$	$\frac{1}{\sqrt{b-a}}e^{-at}\ \mathrm{erf}\ (\sqrt{b-a}\sqrt{t})$	7
9.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}\left[\frac{b}{a}\ \mathrm{erf}\ (a\sqrt{t})-1\right]+e^{b^2t}\ \mathrm{erfc}\ b\sqrt{t}$	7
9.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$	22
9.3.47	$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$	$\frac{n!}{(2n+1)!\sqrt{\pi}}H_{2n+1}(\sqrt{t})$	22
29.3.48	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$	$ae^{-at}[I_1(at)+I_0(at)]$	9
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}\ t\right)$	9
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}\qquad(k>0)$ 6	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}}e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}\ t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$	$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}\ t\right)+I_1\left(\frac{a-b}{2}\ t\right)\right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\frac{1}{t}\ e^{-at}I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}\qquad(k>0)$	$\frac{k}{t}\ e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}\ t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}\qquad(\nu>-1)$	$\frac{1}{a^\nu}\ e^{-\frac{1}{2}at}I_\nu(\frac{1}{2}at)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}\qquad(\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k}\qquad(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$	$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t}J_k(at)$		9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$		9
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}I_{k-\frac{1}{2}}(at)$		6, 10
29.3.61	$\frac{1}{s}e^{-ks}$	$u(t-k)$		
29.3.62	$\frac{1}{s^2}e^{-ks}$	$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu}e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)}u(t-k)$		6
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth\frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty}u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty}a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s}\tanh ks$	$u(t)+2\sum_{n=1}^{\infty}(-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2}\tanh ks$	$tu(t)+2\sum_{n=1}^{\infty}(-1)^n(t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s\sinh ks}$	$2\sum_{n=0}^{\infty}u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s\cosh ks}$	$2\sum_{n=0}^{\infty}(-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
29.3.72	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty} u(t-2nk)$ 	
29.3.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$ 	
29.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
29.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
29.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
29.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
29.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^\mu}e^{-\frac{k}{s}} \quad (\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} J_{\mu-1}(2\sqrt{kt})$	9
29.3.81	$\frac{1}{s^\mu}e^{\frac{k}{s}} \quad (\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{kt})$	9
29.3.82	$e^{-k\sqrt{s}} \quad (k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s}e^{-k\sqrt{s}} \quad (k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}} \quad (k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}} \quad (k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}} \quad (n=0,1,2,\ldots;k\geq 0)$	$(4t)^{\frac{1}{2}n}\,\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}} \quad (n=0,1,2,\ldots;k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}} \quad (k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7

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*See page II.

	$f(s)$		$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}$	$(k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}$	$(k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$	$(k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\frac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	9
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}$	$(k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}}$	$(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks}$	$(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu}$	$(\nu>-1, k\geq 0)$	$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.98	$\frac{1}{s}\ln s$		$-\gamma-\ln t$ ($\gamma=.57721\ 56649\ \ldots$ Euler's constant)	
29.3.99	$\frac{1}{s^k}\ln s$	$(k>0)$	$\frac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	6
29.3.100	$\frac{\ln s}{s-a}$	$(a>0)$	$e^{at}[\ln a+E_1(at)]$	5
29.3.101	$\frac{\ln s}{s^2+1}$		$\cos t\ \mathrm{Si}\ (t)-\sin t\ \mathrm{Ci}\ (t)$	5
29.3.102	$\frac{s\ln s}{s^2+1}$		$-\sin t\ \mathrm{Si}\ (t)-\cos t\ \mathrm{Ci}\ (t)$	5
29.3.103	$\frac{1}{s}\ln(1+ks)$	$(k>0)$	$E_1\left(\frac{t}{k}\right)$	5
29.3.104	$\ln\frac{s+a}{s+b}$		$\frac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\frac{1}{s}\ln(1+k^2s^2)$	$(k>0)$	$-2\ \mathrm{Ci}\left(\frac{t}{k}\right)$	5
29.3.106	$\frac{1}{s}\ln(s^2+a^2)$	$(a>0)$	$2\ln a-2\ \mathrm{Ci}\ (at)$	5

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2) \qquad (a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}(at)]$	**5**
29.3.108	$\ln\frac{s^2+a^2}{s^2}$		$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$		$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$		$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$		$\mathrm{Si}(kt)$	**5**
29.3.112	$e^{k^2s^2}\,\mathrm{erfc}\,ks \qquad (k>0)$	**7**	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\,\mathrm{erfc}\,ks \qquad (k>0)$	**7**	$\mathrm{erf}\,\frac{t}{2k}$	**7**
29.3.114	$e^{ks}\,\mathrm{erfc}\,\sqrt{ks} \qquad (k>0)$	**7**	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\,\mathrm{erfc}\,\sqrt{ks} \qquad (k\geq 0)$	**7**	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\,\mathrm{erfc}\,\sqrt{ks} \qquad (k\geq 0)$	**7**	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\mathrm{erf}\,\frac{k}{\sqrt{s}}$	**7**	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\,\mathrm{erfc}\,\frac{k}{\sqrt{s}}$	**7**	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \qquad (k>0)$	**9**	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \qquad (k>0)$	**9**	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks) \qquad (k>0)$	**9**	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	**9**	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right) \qquad (k>0)$	**9**	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	**9**
29.3.124	$\pi e^{-ks}I_0(ks) \qquad (k>0)$	**9**	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks) \qquad (k>0)$	**9**	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

$\Leftrightarrow \frac{1}{\sqrt{s}}[\mathrm{erfc}\,0 - \mathrm{erfc}\sqrt{ks}]$

$\frac{1-1+\mathrm{erf}\sqrt{ks}}{\sqrt{s}} - \frac{\mathrm{erf}\sqrt{ks}}{\sqrt{s}}$

	$f(s)$			$F(t)$
29.3.126	$e^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as)$	$(a>0; n=0, 1, 2, \ldots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$		5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms[4]

	$\phi(s)$		$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$		$\Phi(t)$
29.4.2	e^{-ks}	$(k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks$	$(k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)}$	$(k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)}$	$(k>0, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)}$	$(0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^b u[t+h-(2n+1)k]$ $-e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s}$	$(0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x} \delta(t)dt=\begin{cases} 0 & (x<0) \\ 1 & (x>0). \end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

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[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

29. Laplace Transforms

9.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a
nplex variable. $F(t)$ is called the original func-
n and $f(s)$ is called the image function. If the
tegral in 29.1.1 converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

xists, then it converges for all s with $\mathscr{R}s>s_0$, and
he image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2 $$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \frac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

Heaviside
step function

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function $F(t)$	*Image Function* $f(s)$
29.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s)-F(+0)$
29.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
29.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

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[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

	Original Function $F(t)$	Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c}e^{(b/c)t}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\frac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t)\sum_{n=0}^{\infty}(-1)^n u(t-na)$	$\frac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\lvert F(t)\rvert$	$f(s)\coth\frac{as}{2}$
	Heaviside Expansion Theorem	
29.2.20	$\sum_{n=1}^{m}\frac{p(a_n)}{q'(a_n)}e^{a_n t}$	$\frac{p(s)}{q(s)}, \ q(s)=(s-a_1)(s-a_2)\ldots(s-a_m)$ $p(s)$ a polynomial of degree $<m$
29.2.21	$e^{at}\sum_{n=1}^{r}\frac{p^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree $<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

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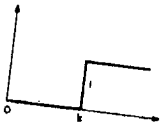
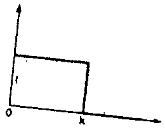
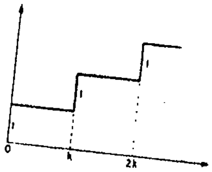
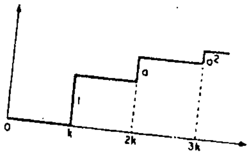
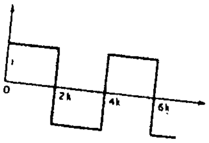
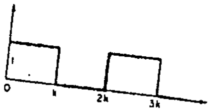
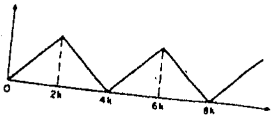
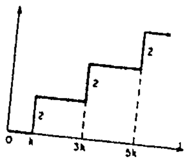
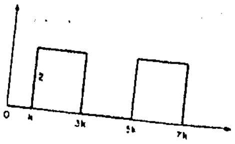
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

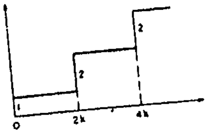
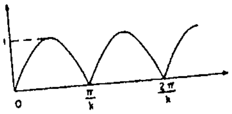
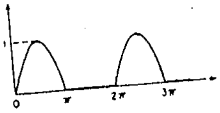
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
	$\frac{1}{s^n}$ $(n=1, 2, 3, \ldots)$		$\frac{t^{n-1}}{(n-1)!}$
	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
	$s^{-3/2}$		$2\sqrt{t/\pi}$
	$s^{-(n+\frac{1}{2})}$ $(n=1, 2, 3, \ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \ldots (2n-1)\sqrt{\pi}}$
7	$\frac{\Gamma(k)}{s^k}$ $(k>0)$	6	t^{k-1}
8	$\frac{1}{s+a}$		e^{-at}
.9	$\frac{1}{(s+a)^2}$		te^{-at}
3.10	$\frac{1}{(s+a)^n}$ $(n=1, 2, 3, \ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
.3.11	$\frac{\Gamma(k)}{(s+a)^k}$ $(k>0)$	6	$t^{k-1}e^{-at}$
.3.12	$\frac{1}{(s+a)(s+b)}$ $(a \neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
9.3.13	$\frac{s}{(s+a)(s+b)}$ $(a \neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
9.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ $(a, b, c$ distinct constants)		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a}\sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a}\sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at\cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2} = \frac{1}{s^2+2as+(a^2+b^2)}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf} a\sqrt{t}$	7

	$f(s)$		$F(t)$	
	$\frac{1}{\sqrt{s}(s+a^2)}$		$\frac{2}{a\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$		$e^{a^2t}[b-a\operatorname{erf}a\sqrt{t}]-be^{b^2t}\operatorname{erfc}b\sqrt{t}$	7
	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$		$e^{a^2t}\operatorname{erfc}a\sqrt{t}$	7
	$\frac{1}{(s+a)\sqrt{s+b}}$		$\frac{1}{\sqrt{b-a}}e^{-at}\operatorname{erf}(\sqrt{b-a}\sqrt{t})$	7
5	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$		$e^{a^2t}\left[\frac{b}{a}\operatorname{erf}(a\sqrt{t})-1\right]+e^{b^2t}\operatorname{erfc}b\sqrt{t}$	7
46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$		$\frac{n!}{(2n)!\sqrt{\pi t}}H_{2n}(\sqrt{t})$	22
.47	$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$		$\frac{n!}{(2n+1)!\sqrt{\pi}}H_{2n+1}(\sqrt{t})$	22
3.48	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$		$ae^{-at}[I_1(at)+I_0(at)]$	9
.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$		$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}t\right)$	9
.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}$	$(k>0)$ 6	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}}e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$		$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}t\right)+I_1\left(\frac{a-b}{2}t\right)\right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$		$\frac{1}{t}e^{-at}I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$	$(k>0)$	$\frac{k}{t}e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}$	$(\nu>-1)$	$\frac{1}{a^\nu}e^{-\frac{1}{2}at}I_\nu(\frac{1}{2}at)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$		$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}$	$(\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k}$	$(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$		$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k$	$(k>0)$	$\frac{ka^k}{t}J_k(at)$		9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}}$	$(\nu>-1)$	$a^\nu I_\nu(at)$		9
29.3.60	$\frac{1}{(s^2-a^2)^k}$	$(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}I_{k-\frac{1}{2}}(at)$		6, 10
29.3.61	$\frac{1}{s}e^{-ks}$		$u(t-k)$		
29.3.62	$\frac{1}{s^2}e^{-ks}$		$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu}e^{-ks}$	$(\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)}u(t-k)$		6
29.3.64	$\frac{1-e^{-ks}}{s}$		$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth\frac{1}{2}ks}{2s}$		$\sum_{n=0}^{\infty}u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$		$\sum_{n=1}^{\infty}a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s}\tanh ks$		$u(t)+2\sum_{n=1}^{\infty}(-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$		$\sum_{n=0}^{\infty}(-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2}\tanh ks$		$tu(t)+2\sum_{n=1}^{\infty}(-1)^n(t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s\sinh ks}$		$2\sum_{n=0}^{\infty}u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s\cosh ks}$		$2\sum_{n=0}^{\infty}(-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
72	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty} u(t-2nk)$ 	
.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$ 	
3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
9.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
9.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^{\mu}}e^{-\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}J_{\mu-1}(2\sqrt{kt})$	9
29.3.81	$\frac{1}{s^{\mu}}e^{\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}I_{\mu-1}(2\sqrt{kt})$	9
29.3.82	$e^{-k\sqrt{s}}\quad(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k\geq 0)$	$(4t)^{\frac{1}{2}n}\,\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7

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*See page II.

	$f(s)$		$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}$	$(k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\frac{k}{2\sqrt{t}}$	
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}$	$(k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$	$(k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\frac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	9
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}$	$(k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}}$	$(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks}$	$(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu}$	$(\nu>-1, k\geq 0)$	$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.98	$\frac{1}{s}\ln s$		$-\gamma-\ln t$ ($\gamma=.57721\ 56649\ldots$ Euler's constant)	
29.3.99	$\frac{1}{s^k}\ln s$	$(k>0)$	$\frac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	6
29.3.100	$\frac{\ln s}{s-a}$	$(a>0)$	$e^{at}[\ln a+E_1(at)]$	5
29.3.101	$\frac{\ln s}{s^2+1}$		$\cos t\ \mathrm{Si}\ (t)-\sin t\ \mathrm{Ci}\ (t)$	5
29.3.102	$\frac{s\ln s}{s^2+1}$		$-\sin t\ \mathrm{Si}\ (t)-\cos t\ \mathrm{Ci}\ (t)$	5
29.3.103	$\frac{1}{s}\ln(1+ks)$	$(k>0)$	$E_1\left(\frac{t}{k}\right)$	5
29.3.104	$\ln\frac{s+a}{s+b}$		$\frac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\frac{1}{s}\ln(1+k^2s^2)$	$(k>0)$	$-2\ \mathrm{Ci}\left(\frac{t}{k}\right)$	5
29.3.106	$\frac{1}{s}\ln(s^2+a^2)$	$(a>0)$	$2\ln a-2\ \mathrm{Ci}\ (at)$	5

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2) \qquad (a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}\,(at)]$	5
29.3.108	$\ln\frac{s^2+a^2}{s^2}$		$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$		$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$		$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$		$\mathrm{Si}\,(kt)$	5
29.3.112	$e^{k^2s^2}\,\mathrm{erfc}\,ks \qquad (k>0)$	7	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\,\mathrm{erfc}\,ks \qquad (k>0)$	7	$\mathrm{erf}\,\frac{t}{2k}$	7
29.3.114	$e^{ks}\,\mathrm{erfc}\sqrt{ks} \qquad (k>0)$	7	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\,\mathrm{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\,\mathrm{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\mathrm{erf}\,\frac{k}{\sqrt{s}}$	7	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\,\mathrm{erfc}\,\frac{k}{\sqrt{s}}$	7	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks) \qquad (k>0)$	9	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right) \qquad (k>0)$	9	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks}I_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks) \qquad (k>0)$	9	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

$\Leftrightarrow \frac{1}{\sqrt{s}}[\mathrm{erfc}\,0-\mathrm{erfc}\sqrt{ks}]$

$\frac{1-1+\mathrm{erf}\sqrt{ks}}{\sqrt{s}}-\frac{\mathrm{erf}\sqrt{ks}}{\sqrt{s}}$

	$f(s)$			$F(t)$
29.3.126	$e^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as)$	$(a>0; n=0, 1, 2, \ldots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$		5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms [4]

	$\phi(s)$		$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$		$\Phi(t)$
29.4.2	e^{-ks}	$(k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks$	$(k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)}$	$(k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)}$	$(k>0, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)}$	$(0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^b u[t+h-(2n+1)k]$ $-e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s}$	$(0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x} \delta(t)dt=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

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[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

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$$\left.\begin{array}{c}\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \\ u(x,t=0) = f(x) \\ \frac{\partial u}{\partial t}(x,t=0) = g(x) \\ u(x=0,t)=0 \\ u(x=L,t)=0\end{array}\right\}$$

Let $U(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$\therefore \frac{d^2}{dx^2}U(x;s) = \frac{1}{a^2}\left[s^2U - su(x,t=0) - \frac{\partial u}{\partial t}(x,t=0)\right]$$
$$= \frac{1}{a^2}\left[s^2U - sf(x) - g(x)\right]$$

or $$\frac{d^2U}{dx^2} - \frac{s^2}{a^2}U = -[sf+g] = -G(x;s)$$

$$U_h = \mathscr{C}_1 \sinh\left(\tfrac{s}{a}x\right) + \mathscr{C}_2 \cosh\left(\tfrac{s}{a}x\right) = \mathscr{C}_1 U_1 + \mathscr{C}_2 U_2$$

using variation of parameters $$U_p = U_1\int_0^x \frac{G(\bar{x};s)\cosh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a} + U_2\int_0^x \frac{G(\bar{x};s)\sinh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a}$$
$$= \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

now $$U_{TOT} = U_h + U_p = \mathscr{C}_1\sinh\left(s/a\, x\right) + \mathscr{C}_2\cosh\left(s/a\, x\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

putting in the B.C.'s $u(x=0,t)=0 \Rightarrow \mathscr{C}_2 = 0$

$$u(x=L,t)=0 \Rightarrow \mathscr{C}_1 = -\frac{\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[s/a(L-\bar{x})\right]d\bar{x}}{\sinh \frac{L}{a}s}$$

$$\therefore U(x;s) = -\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[s/a(L-\bar{x})\right]d\bar{x}\left(\frac{\sinh s/a\, x}{\sinh s/a\, L}\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

Residue Theorem: we can use $u(x,t) = \lim_{s\to a}(s-a)e^{st}U(x;s)$ if $U(x;s) = \frac{Q(x;s)}{s-a}$

the first & second integrals appear to have $s=0$ & $s=\frac{a}{L}n\pi i$ as poles

since $\sinh is = i\sin s$ & $\sin s = 0$ if $s = n\pi$

@ $s\to 0$ $$U(x;s) \to -\int_0^L \frac{a}{s}[sf+g]\,\frac{s}{a}(L-\bar{x})\,d\bar{x}\left(\frac{\frac{s}{a}x}{s/a\, L}\right) + \int_0^x \frac{a}{s}[sf+g]\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

since $\sinh x = \frac{e^x + e^{-x}}{2} = x + x^3/3! + \cdots$

- when $s=0$ $U(x;s)$ tends to a constant not $\frac{Q(x;s)}{s}$ $\therefore$ $s=0$ is not a pole and so the residue $=0$

- when $s = \frac{a}{L}n\pi i$ $\sinh \frac{Ls}{a} = i\sin n\pi$ and

$$U(x;s) \to -\int_0^L \frac{a}{a/L\, n\pi i}\left[\tfrac{a}{L}n\pi i\, f + g\right]\sinh \tfrac{n\pi i}{L}(L-\bar{x})\,d\bar{x}\left(\frac{\sinh \frac{n\pi i}{L}x}{\sinh i n\pi}\right)$$
$$+ \int_0^x \frac{a}{a/L\, n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right]\sinh i\tfrac{n\pi}{L}(x-\bar{x})\,d\bar{x}$$

the 2nd integral is constant & gives no residue the first integral will give a residue at $s = \frac{a}{L}n\pi i$ for each & every n except $n=0$ (this is same as $g=0$)

$$\text{Thus}\underset{s\to\frac{a}{L}n\pi i}{\sum}\left(s-\tfrac{a}{L}n\pi i\right)e^{\overset{\overset{s}{\downarrow}}{\frac{a}{L}n\pi i}t}\left\{-\int_0^L\frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sinh\frac{n\pi i}{L}(L-\bar{x})d\bar{x}\;\frac{\sinh\frac{n\pi i}{L}x}{\sinh i n\pi}\right\}$$

$$\sum e^{\frac{a}{L}n\pi i t}\left\{-\int_0^L\frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]i\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\cdot\frac{i\sin\frac{n\pi x}{L}}{\;}\;\underset{s\to\frac{a}{L}n\pi i}{\mathcal{L}}\frac{\left(s-\frac{a}{L}n\pi i\right)}{\sinh\frac{L}{a}s}\right\}$$

using L'Hôpital's Rule

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{n=\infty}e^{\frac{a}{L}n\pi i t}\left\{+\int_0^L\frac{\cancel{L}}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\;\sin\frac{n\pi x}{L}\cdot\frac{1}{\frac{\cancel{L}}{a}\underbrace{\boxed{\cosh n\pi i}}_{=\cos n\pi=(-1)^n}}\right\}$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}e^{\frac{a}{L}n\pi i t}\left\{\int_0^L\frac{a}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\;\sin\frac{n\pi x}{L}\;(-1)^n\right\}$$

$$\text{but}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}=\sum_{n=-\infty}^{-1}+\sum_{n=1}^{\infty}$$

$$\text{now if } n=-m\qquad\sum_{n=-\infty}^{-1}=\sum_{m=1}^{\infty}e^{\frac{a}{L}(-m)\pi i t}\left\{\int_0^L\frac{a}{(-m)\pi i}\left[\tfrac{a}{L}(-m)\pi i f+g\right]\left(-\sin\frac{m\pi}{L}(L-\bar{x})\right)d\bar{x}\left(-\sin\frac{m\pi x}{L}\right)(-1)^{-m}\right\}$$

since m is dummy index

$$=\sum_{m=1}^{\infty}e^{-\frac{a}{L}n\pi i t}\left\{-\int_0^L\frac{a}{n\pi i}\left[-\tfrac{a}{L}(n\pi i)f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\;\left(\sin\frac{n\pi x}{L}\right)(-1)^n\right\}$$

now by adding the two sums

$$\sum_{n=1}^{\infty}\int_0^L\frac{a}{\cancel{n\pi i}}\left[e^{\frac{a}{L}n\pi i t}+e^{-\frac{a}{L}n\pi i t}\right]\cdot\frac{a}{L}\cancel{(n\pi i)}f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\left(\sin\frac{n\pi x}{L}\right)(-1)^n$$

$$+\int_0^L\frac{a}{n\pi i}\left[e^{a/L\,n\pi i t}-e^{-a/L\,n\pi i t}\right]g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\left(\sin\frac{n\pi x}{L}\right)(-1)^n$$

$$e^{a/L\,n\pi i t}+e^{-a/L\,n\pi i t}=2\cos\tfrac{a}{L}n\pi t$$

$$e^{a/L\,n\pi i t}-e^{-a/L\,n\pi i t}=2i\sin a/L\,n\pi t$$

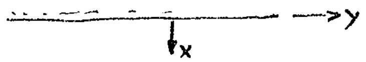
thus

$$u(x,t)=\sum_{n=1}^{\infty}\int_0^L\frac{2a^2}{L}\cos\tfrac{a}{L}n\pi t\;f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\cdot\sin\frac{n\pi x}{L}(-1)^n$$

$$+\sum_{n=1}^{\infty}\int_0^L\frac{2a}{n\pi}\sin\tfrac{a}{L}n\pi t\;g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\cdot\sin\frac{n\pi x}{L}(-1)^n$$

$$\boxed{\begin{array}{l}\text{or}\quad\displaystyle\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left\{A_n\cos\tfrac{a}{L}n\pi t+B_n\sin\tfrac{a}{L}n\pi t\right\}\quad\text{where}\\[2ex]\qquad A_n=\displaystyle\frac{2a^2}{L}\int_0^L f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\cdot(-1)^n\\[2ex]\qquad B_n=\displaystyle\frac{2a}{n\pi}\int_0^L g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})d\bar{x}\;(-1)^n\end{array}}$$

Diffusion of contaminant deposited at $t=0$ on surface of semi infinite slab $\alpha C_{xx} = C_t$



initially $c(x,t=0)=0 \quad x>0$
for $x \to \infty \quad c(x,t) \to 0$
$\int_0^\infty c(x,t)\,dx = Q$ constant total amount

Take L.T. of both sides of integ condition
$$\int_0^\infty c(x,t)\,dx = Q$$

$$\int_0^\infty e^{-st}\int_0^\infty c(x,t)\,dx\,dt = \int_0^\infty Q e^{-st}\,dt = -\frac{Q}{s}e^{-st}\Big|_0^\infty = \frac{Q}{s}.$$

Change order of integration
$$\int_0^\infty\int_0^\infty e^{-st}c(x,t)\,d\cancel{t}\,d\cancel{x} = \int_0^\infty C(x;s)\,dx = \frac{Q}{s}$$

$$\alpha\frac{d^2}{dx^2}C = sC - c(x,t=0) = sC$$

$$C'' - \frac{s}{\alpha}C = 0 \qquad \Longrightarrow C = c_1 e^{\sqrt{\frac{s}{\alpha}}x} + c_2 e^{-\sqrt{\frac{s}{\alpha}}x}$$

as $x \to \infty \quad c \to 0 \qquad x \to \infty \quad C \to 0 \qquad \Rightarrow \quad c_1 = 0$

$$c_2\int_0^\infty e^{-\sqrt{\frac{s}{\alpha}}x}\,dx = \frac{Q}{s}$$

$$c_2\left(-\sqrt{\frac{\alpha}{s}}e^{-\sqrt{\frac{s}{\alpha}}x}\right)\Big|_0^\infty = c_2\sqrt{\frac{\alpha}{s}} = \frac{Q}{s} \qquad \therefore \quad c_2 = \frac{Q}{\sqrt{\alpha s}}$$

$$\therefore \qquad C = \frac{Q}{\sqrt{\alpha}}\underbrace{\frac{1}{\sqrt{s}}e^{-\frac{x}{\sqrt{\alpha}}\cdot\sqrt{s}}} \qquad \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}}e^{-\frac{x}{\sqrt{\alpha}}\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha t}}$$

29.3.84

$$c = \frac{Q}{\sqrt{\alpha}}\frac{1}{\sqrt{\pi t}}e^{-\frac{x^2}{4\alpha t}} = \frac{Q}{\sqrt{\pi\alpha t}}e^{-x^2/4\alpha t}$$

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$$\left.\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \\ u(x,t=0) &= f(x) \\ \frac{\partial u}{\partial t}(x,t=0) &= g(x) \\ u(x=0,t) &= 0 \\ u(x=L,t) &= 0 \end{aligned}\right\}$$

Let $U(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$\therefore \frac{d^2}{dx^2}U(x;s) = \frac{1}{a^2}\left[s^2U - su(x,t=0) - \frac{\partial u}{\partial t}(x,t=0)\right]$$
$$= \frac{1}{a^2}\left[s^2U - sf(x) - g(x)\right]$$

or $$\frac{d^2U}{dx^2} - \frac{s^2}{a^2}U = -[sf+g] = -G(x;s)$$

$$U_h = \mathcal{C}_1 \sinh\left(\tfrac{s}{a}x\right) + \mathcal{C}_2 \cosh\left(\tfrac{s}{a}x\right) = \mathcal{C}_1 U_1 + \mathcal{C}_2 U_2$$

using variation of parameters $$U_p = U_1 \int_0^x \frac{G(\bar{x};s)\cosh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a} - U_2\int_0^x \frac{G(\bar{x};s)\sinh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a}$$
$$= \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

now $$U_{TOT} = U_h + U_p = \mathcal{C}_1 \sinh\left(\tfrac{s}{a}x\right) + \mathcal{C}_2\cosh\left(\tfrac{s}{a}x\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

putting in the B.C.'s $u(x=0,t)=0 \Rightarrow \mathcal{C}_2 = 0$

$$u(x=L,t)=0 \Rightarrow \mathcal{C}_1 = -\frac{\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[\frac{s}{a}(L-\bar{x})\right]d\bar{x}}{\sinh\frac{L}{a}s}$$

$$\therefore \quad U(x;s) = -\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(L-\bar{x})\right]d\bar{x}\left(\frac{\sinh\frac{s}{a}x}{\sinh\frac{s}{a}L}\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

Residue Theorem: we can use $u(x,t) = \lim_{s\to a}(s-a)e^{st}U(x;s)$ if $U(x;s) = \dfrac{Q(x;s)}{s-a}$

the first & second integrals appear to have $s=0$ & $s=\frac{a}{L}n\pi i$ as poles

since $\sinh is = i\sin s$ & $\sin s = 0$ if $s = n\pi$

@ $s\to 0$ $$U(x;s) \to -\int_0^L \frac{a}{s}[sf+g]\,\frac{s}{a}(L-\bar{x})\,dx\left(\frac{\frac{s}{a}x}{\frac{s}{a}L}\right) + \int_0^x \frac{a}{s}[sf+g]\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

since $\sinh x = \frac{e^x - e^{-x}}{2} = x + x^3/3! + \cdots$

- when $s=0$ $U(x;s)$ tends to a constant not $\dfrac{Q(x;s)}{s}$ $\therefore$ $s=0$ is <u>not</u> a pole

  and so the residue $=0$

- when $s = \frac{a}{L}n\pi i$ $\sinh\frac{ds}{a} = i\sin n\pi$ and

$$U(x;s) \to -\int_0^L \frac{a}{\frac{a}{L}n\pi i}\left[\tfrac{a}{L}n\pi i\, f + g\right]\sinh\frac{n\pi i}{L}(L-\bar{x})\,d\bar{x}\left(\frac{\sinh\frac{n\pi i}{L}x}{\sinh i n\pi}\right)$$
$$+ \int_0^x \frac{a}{\frac{a}{L}n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right]\sinh i\frac{n\pi}{L}(x-\bar{x})\,d\bar{x}$$

the 2^{nd} integral is constant & gives no residue the first integral will give a residue at $s=\frac{a}{L}n\pi i$ for each & every n except $n=0$ (this is same as $s=0$)

$$\text{Thus}\sum_{s\to\frac{a}{L}n\pi i}\left(s-\tfrac{a}{L}n\pi i\right)e^{\frac{\overset{s}{\overset{\downarrow}{a}}}{L}n\pi i t}\left\{-\int_0^L\frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sinh\frac{n\pi i}{L}(L-\bar{x})\,d\bar{x}\quad\frac{\sinh\frac{n\pi i}{L}x}{\sinh i n\pi}\right\}$$

$$\sum e^{\frac{a}{L}n\pi i t}\left\{-\int_0^L\frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]i\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\cdot\frac{i\sin\frac{n\pi x}{L}}{}\ \underset{s\to\frac{a}{L}n\pi i}{L}\ \frac{\left(s-\frac{a}{L}n\pi i\right)}{\sinh\frac{L}{a}s}\right\}$$

using L'Hôpital's Rule

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{n=\infty}e^{\frac{a}{L}n\pi i t}\left\{+\int_0^L\frac{\cancel{L}}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\ \sin\frac{n\pi x}{L}\cdot\frac{1}{\frac{\cancel{L}}{a}\underbrace{\boxed{\cosh n\pi i}}_{=\cos n\pi=(-1)^n}}\right\}$$

$$=\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}e^{\frac{a}{L}n\pi i t}\left\{\int_0^L\frac{a}{n\pi i}\left[\tfrac{a}{L}n\pi i f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\ \sin\frac{n\pi x}{L}\ (-1)^n\right\}$$

but $\displaystyle\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}=\sum_{n=-\infty}^{-1}+\sum_{n=1}^{\infty}$

now if $n=-m$ $\displaystyle\sum_{n=-\infty}^{-1}=\sum_{m=1}^{\infty}e^{\frac{a}{L}(-m)\pi i t}\left\{\int_0^L\frac{a}{(-m)\pi i}\left[\tfrac{a}{L}(-m)\pi i f+g\right]\left(-\sin\frac{m\pi}{L}(L-\bar{x})\right)d\bar{x}\left(-\sin\frac{m\pi x}{L}\right)(-1)^{-1}\right.$

since m is dummy index

$$=\sum_{m=1}^{\infty}e^{-\frac{a}{L}n\pi i t}\left\{-\int_0^L\frac{a}{n\pi i}\left[-\tfrac{a}{L}(n\pi i)f+g\right]\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\ \left(\sin\frac{n\pi x}{L}\right)(-1)^n\right\}$$

now by adding the two sums

$$\sum_{n=1}^{\infty}\int_0^L\frac{a}{\cancel{n\pi i}}\left[e^{\frac{a}{L}n\pi i t}+e^{-\frac{a}{L}n\pi i t}\right]\cdot\frac{a}{L}\cancel{(n\pi i)}f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\left(\sin\frac{n\pi x}{L}\right)(-1)^n$$

$$+\int_0^L\frac{a}{n\pi i}\left[e^{a/L\,n\pi i t}-e^{-a/L\,n\pi i t}\right]g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\left(\sin\frac{n\pi x}{L}\right)(-1)^n$$

$$e^{a/L\,n\pi i t}+e^{-a/L\,n\pi i t}=2\cos\tfrac{a}{L}n\pi t$$

$$e^{a/L\,n\pi i t}+e^{-a/L\,n\pi i t}=2i\sin a/L\,n\pi t$$

thus

$$u(x,t)=\sum_{n=1}^{\infty}\int_0^L\frac{2a^2}{L}\cos\tfrac{a}{L}n\pi t\ f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\cdot\sin\frac{n\pi x}{L}\ (-1)^n$$

$$+\sum_{n=1}^{\infty}\int_0^L\frac{2a}{n\pi}\sin\tfrac{a}{L}n\pi t\ g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\cdot\sin\frac{n\pi x}{L}\ (-1)^n$$

$$\boxed{\begin{array}{l}\text{or}\quad\displaystyle\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left\{A_n\cos\tfrac{a}{L}n\pi t+B_n\sin\tfrac{a}{L}n\pi t\right\}\quad\text{where}\\[2ex] \qquad A_n=\dfrac{2a^2}{L}\displaystyle\int_0^L f(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\cdot(-1)^n\\[2ex] \qquad B_n=\dfrac{2a}{n\pi}\displaystyle\int_0^L g(\bar{x})\sin\frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\ (-1)^n\end{array}}$$

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$$\alpha \frac{d^2}{dx^2} J = sJ - T(x,t=0) = sJ - T_i$$

$$J'' - \frac{s}{\alpha} J = -\frac{T_i}{\alpha}$$

$$J = C_1 e^{-\sqrt{s/\alpha}\,x} + C_2 e^{\sqrt{s/\alpha}\,x} + \frac{T_i}{s}$$

$$J \rightarrow \frac{T_i}{s} \text{ as } x \rightarrow \infty \quad \Rightarrow C_2 = 0$$

$$-k \frac{dJ}{dx} = q/s \quad \text{at } x=0$$

$$-k C_1 \cancel{e^{-\sqrt{s/\alpha}\,x}}^{1} \cdot -\sqrt{\frac{s}{\alpha}} = q/s$$

$$C_1 = \frac{q}{sk}\sqrt{\frac{\alpha}{s}} = \frac{q\sqrt{\alpha}}{k} \cdot \frac{1}{s^{3/2}}$$

$$\therefore \quad J = \frac{q}{k}\sqrt{\alpha}\,\frac{1}{s^{3/2}} e^{-\sqrt{s}\,\frac{x}{\sqrt{\alpha}}} + \frac{T_i}{s} \quad \text{for } k = \frac{x}{\sqrt{\alpha}} \geq 0$$

$$J = \frac{q}{k}\sqrt{\alpha}\left[2\sqrt{t/\pi}\, e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{\alpha}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] + T_i$$

$$J = \frac{2q}{k}\sqrt{\frac{\alpha t}{\pi}}\, e^{-\frac{x^2}{4\alpha t}} - \frac{qx}{k} \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)\Big] + T_i$$

$$= \frac{q}{k}\sqrt{\alpha} \cdot 2\sqrt{t/\pi}\left[e^{-\frac{x^2}{4\alpha t}} - \frac{x\sqrt{\pi}}{2\sqrt{\alpha t}} \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] + T_i$$

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \qquad \begin{matrix} x > 0 \\ t > 0 \end{matrix}$$

$$T(x,0) = T_i$$

$$T(x,t) \rightarrow T_i \text{ as } x \rightarrow \infty$$

$$-k \frac{\partial T}{\partial x} = q \quad @ \; x=0 \quad t>0$$

$$\text{let } J(x;s) = \int_0^\infty T(x,t) e^{-st} dt$$

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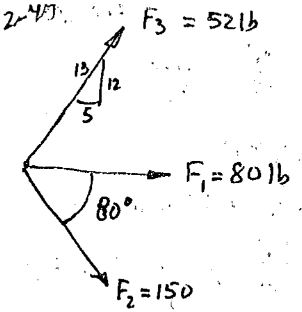


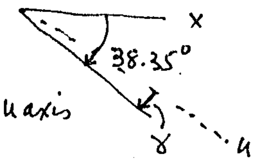
$$\Sigma F_x = 52 \cdot \frac{5}{13} + 80 + 150 \cos 80 = 126.05 \text{ lb}$$

$$\Sigma F_y = 52 \cdot \frac{12}{13} - 150 \sin 80 = -99.72 \text{ lb}$$

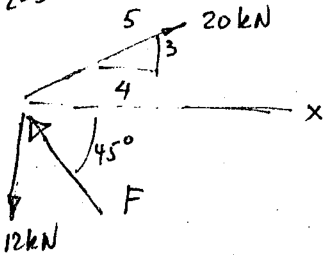
$$R = \sqrt{(126.05)^2 + (99.72)^2} = \underline{160.73 \text{ lb}}$$

$$\tan\theta = \frac{\Sigma F_y}{\Sigma F_x} = -.7911 \qquad \theta = -38.35° \text{ from horizontal}$$

$$\gamma = 38.35 - 25 = \underline{13.35°} \text{ below u axis}$$



2-51



$$\Sigma F_x = 20 \cdot 4/5 - F\cos 45 + 0 = 16 - F\cos 45°$$

$$\Sigma F_y = 20 \cdot 3/5 - 12 + F\sin 45° = F\sin 45°$$

$$R = \sqrt{(\Sigma F_x)^2 + (\Sigma F_y)^2} = \sqrt{256 - 32F\cos 45 + F^2\cos^2 45 + F^2\sin^2 45}$$

$$R = \sqrt{256 - 32F\sin 45 + F^2} \qquad \text{since } \sin 45 = \cos 45$$

for smallest resultant $$\frac{dR}{dF} = 0 = \frac{1}{2\sqrt{256 - 32F\sin 45 + F^2}}(-32\sin 45 + 2F) \Rightarrow F = 16\sin 45 = \underline{11.31 \text{ kN}}$$

$$\therefore \left.\begin{aligned} \Sigma F_x &= 8.00 \\ \Sigma F_y &= 8.00 \end{aligned}\right\} R = 8.00\sqrt{2} = {}^{11.31} \text{ kN}$$

let $U(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{d^2U}{dx^2} = \frac{1}{a^2}\left[s^2U - su(\overset{x,t=0^+}{\cancel{}}) - \frac{\partial u}{\partial t}(x,t=0^+)\right]$$

$$= \frac{1}{a^2}s^2U - \frac{s}{a^2}f(x) - \frac{1}{a^2}g(x)$$

$u(x,t=0) = f(x)$

$\frac{\partial u}{\partial t}(x,t=0) = g(x)$

$u(x=0,t) = 0$

$u(x=L,t) = 0$

$$\therefore \quad \frac{d^2U}{dx^2} - \frac{s^2}{a}U = -\frac{s}{a^2}f(x) - \frac{1}{a^2}g(x) = G(x;s)$$

$$\mathcal{L}\{u(x=0,t)=0\} \Rightarrow U(x=0;s)=0$$

$$\mathcal{L}\{u(x=L,t)=0\} \Rightarrow U(x=L;s)=0$$

Solution of ODE

$U_{TOT} = U_H + U_P$

U_H : $\frac{d^2U}{dx^2} - \frac{s^2}{a^2}U = 0 \quad \Rightarrow \quad U_H = Ae^{\frac{s}{a}x} + Be^{-\frac{s}{a}x} = \bar{A}\sinh\frac{s}{a}x + \bar{B}\cosh\frac{s}{a}x$

U_P : $\frac{d^2U}{dx^2} - \frac{s^2}{a^2}U = G(x;s)$ use variation of parameters $U_P = v_1(x)\sinh\frac{s}{a}x + v_2(x)\cosh\frac{s}{a}x$

$$\Rightarrow v_1'\sinh\frac{s}{a}x + v_2'\cosh\frac{s}{a}x = 0$$

$$\Rightarrow \left(v_1'\cosh\frac{s}{a}x + v_2'\sinh\frac{s}{a}x\right)\frac{s}{a} = G(x;s)$$

$$v_1' = \frac{\begin{pmatrix} 0 & \cosh\frac{s}{a}x \\ G & \frac{s}{a}\sinh\frac{s}{a}x \end{pmatrix}}{\begin{pmatrix} \sinh\frac{s}{a}x & \cosh\frac{s}{a}x \\ \frac{s}{a}\cosh\frac{s}{a}x & \frac{s}{a}\sinh\frac{s}{a}x \end{pmatrix}} = \frac{-G\cosh\frac{s}{a}x}{-\frac{s}{a}} \quad \Bigg\| \quad v_2' = \frac{\begin{pmatrix} \sinh\frac{s}{a}x & 0 \\ \frac{s}{a}\cosh\frac{s}{a}x & G \end{pmatrix}}{-s/a} = \frac{G\sinh\frac{s}{a}x}{-s/a}$$

Now $U_P = \sinh\frac{s}{a}x\int^x \frac{G(\bar{x};s)\cosh\frac{s}{a}\bar{x}}{+s/a}d\bar{x} + \cosh\frac{s}{a}x\int^x\frac{G(\bar{x};s)\sinh\frac{s}{a}\bar{x}}{-s/a}d\bar{x}$

$U_P = \int^x \frac{G(\bar{x};s)}{s/a}\sinh\frac{s}{a}(x-\bar{x})\,d\bar{x}$ using $\sinh(\alpha\pm\beta) = \sinh\alpha\cosh\beta \pm \cosh\alpha\sinh$

$$\therefore \quad U_{TOT} = \bar{A}\sinh\frac{s}{a}x + \bar{B}\cosh\frac{s}{a}x + \frac{a}{s}\int^x G(\bar{x};s)\sinh\frac{s}{a}(x-\bar{x})\,d\bar{x}$$

$$U(x=0) = \bar{B}\cdot 1 + \frac{a}{s}\int^{0} G(\bar{x};s)\sinh\frac{s}{a}(\bar{x})\,d\bar{x} = 0$$

$$\bar{B} = \frac{a}{s}\int^{0} G(\bar{x};s)\sinh\frac{s}{a}\bar{x}\,d\bar{x}$$

$$U(x=L) = \bar{A}\sinh\frac{s}{a}L + \bar{B}\cosh\frac{s}{a}L + \frac{a}{s}\int^{L} G(\bar{x};s)\sinh\frac{s}{a}(L-\bar{x})d\bar{x} = 0$$

$$\bar{A}\sinh\frac{s}{a}L + \frac{a}{s}\left(\int^{0} G(\bar{x};s)\sinh\frac{s}{a}\bar{x}\,d\bar{x}\right)\cosh\frac{s}{a}L + \frac{a}{s}\int^{L} G(\bar{x};s)\sinh\frac{s}{a}(L-\bar{x})d\bar{x} = 0$$

$$\bar{A} = \frac{-\frac{a}{s}\int^{0} G(\bar{x};s)\sinh\frac{s}{a}\bar{x}\,d\bar{x}\,\cosh\frac{s}{a}L + \frac{a}{s}\int^{L} G(\bar{x};s)\sinh\frac{s}{a}(L-\bar{x})d\bar{x}}{\sinh\frac{s}{a}L}$$

$$U_{TOT} = \left\{\frac{-\frac{a}{s}\int^{0}\overset{①}{G\sinh\frac{s}{a}\bar{x}d\bar{x}}\cosh\frac{s}{a}L - \frac{a}{s}\int^{L}\overset{③}{G\sinh\frac{s}{a}(L-\bar{x})d\bar{x}}}{\sinh\frac{s}{a}L}\right\}\sinh\frac{s}{a}x + \frac{a}{s}\int^{0}\overset{②}{G\sinh\frac{s}{a}\bar{x}d\bar{x}}\cdot\cosh\frac{s}{a}x$$

$$+ \frac{a}{s}\int^{x}\overset{④}{G\sinh\frac{s}{a}(x-\bar{x})d\bar{x}}$$

use $\cosh(\alpha+\beta) - \cosh(\alpha-\beta) = 2\sinh\alpha\sinh\beta$

$$①+② = -\frac{a}{s}\int^{0}\frac{G\sinh\frac{s}{a}\bar{x}d\bar{x}}{\sinh\frac{s}{a}L}\left(\cosh\frac{s}{a}L\sinh\frac{s}{a}x - \cosh\frac{s}{a}x\sinh\frac{s}{a}L\right) = -\frac{a}{s}\int^{0}\frac{G\sinh\frac{s}{a}\bar{x}d\bar{x}}{\sinh\frac{s}{a}L}\sinh\frac{s}{a}(x-$$

$$③+④ = -\frac{a}{s}\int^{L} G\sinh\frac{s}{a}(L-\bar{x})d\bar{x}\frac{\sinh\frac{s}{a}x}{\sinh\frac{s}{a}(\cancel{L})} + \frac{a}{s}\int^{x} G\sinh\frac{s}{a}(x-\bar{x})d\bar{x}\cdot\frac{\sinh\frac{s}{a}L}{\sinh\frac{s}{a}L}$$

$$= -\frac{a}{2s}\int^{L}\frac{G\left\{\overset{Ⓔ}{\cosh\frac{s}{a}(L-\bar{x}+x)} - \overset{Ⓓ}{\cosh\frac{s}{a}(L-\bar{x}-x)}\right\}d\bar{x}}{\sinh\frac{s}{a}L} + \frac{a}{2s}\int^{x} G\frac{\left\{\overset{Ⓕ}{\cosh\frac{s}{a}(x-\bar{x}+L)} - \overset{Ⓑ}{\cosh\frac{s}{a}(x-\bar{x}-L)}\right.}{\sinh\frac{s}{a}L}$$

$$①+② = -\frac{a}{2s}\int^{0}\frac{G\left\{\overset{Ⓒ}{\cosh\frac{s}{a}(x-L+\bar{x})} - \overset{Ⓐ}{\cosh\frac{s}{a}(\bar{x}-x+L)}\right\}d\bar{x}}{\sinh\frac{s}{a}L}$$

$$A+B = -\frac{a}{2s}\int_{0}^{x}\frac{G\cosh\frac{s}{a}(\bar{x}-x+L)\,d\bar{x}}{\sinh\frac{s}{a}L} \qquad C+D = \frac{a}{2s}\int_{0}^{L}\frac{G\cosh\frac{s}{a}(\bar{x}+x-L)\,d\bar{x}}{\sinh\frac{s}{a}L}$$

$$E+F = -\frac{a}{2s}\int_x^L \frac{G\cosh\frac{s}{a}(\bar{x}-L-x)\,d\bar{x}}{\sinh\frac{s}{a}L}$$

$$U_{TOT} = \frac{a}{2s}\int_0^L \frac{G\cosh\frac{s}{a}(\bar{x}+x-L)\,d\bar{x}}{\sinh\frac{s}{a}L} + -\frac{a}{2s}\int_0^x G\frac{\cosh\frac{s}{a}(\bar{x}-x+L)\,d\bar{x}}{\sinh\frac{s}{a}L}$$

$$-\frac{a}{2s}\int_x^L \frac{G\cosh\frac{s}{a}(\bar{x}-L-x)\,d\bar{x}}{\sinh\frac{s}{a}L} \quad -\frac{a}{2s}\int_0^x \frac{G\cosh\frac{s}{a}(\bar{x}-L-x)\,d\bar{x}}{\sinh\frac{s}{a}L} \quad +\frac{a}{2s}\int_0^x \frac{G\cosh\frac{s}{a}(\bar{x}-L-x)\,d\bar{x}}{\sinh\frac{s}{a}L}$$

using $\cosh(\alpha+\beta)-\cosh(\alpha-\beta) = 2\sinh\alpha\sinh\beta$ $\qquad \alpha = \bar{x}-x \quad \beta = L$

$$\overset{H}{=\frac{a}{2s}\int_0^L G\frac{\cosh\frac{s}{a}(\bar{x}+x-L)\,d\bar{x}}{\sinh\frac{s}{a}L}} \quad \overset{I}{-\frac{a}{2s}\int_0^L G\frac{\cosh\frac{s}{a}(\bar{x}-x-L)}{\sinh\frac{s}{a}L}\,d\bar{x}} \quad +\frac{a}{2s}\int_0^x 2G\sinh\frac{s}{a}(\bar{x}-x)\,d\bar{x}$$

$$U_{TOT} = \frac{a}{2s}\int_0^L -2G\frac{\sinh\frac{s}{a}(\bar{x}-L)\,d\bar{x}}{\sinh\frac{s}{a}L}\sinh\frac{s}{a}x + \frac{a}{2s}\int_0^x 2G\sinh\frac{s}{a}(\bar{x}-x)\,d\bar{x}$$

H+I combine using $\underbrace{\cosh(\alpha+\beta)-\cosh(\alpha-\beta)}_{=2\sinh\alpha\sinh\beta}$ with $\alpha = \bar{x}-L \quad \beta = x$

$$\mathcal{L}\{u(x,t)\} = \int_0^\infty e^{-st}u(x,t)\,dt = U(x;s)$$

$$\mathcal{L}^{-1}\{U(x;s)\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} U(x;s)e^{st}\,ds = u(x,t)$$

~~Since $\oint f(z)\,dz = 0$~~

~~let $f(z)$ have a pole at a~~

~~$\frac{g(z)}{z-a} = f(z)$ then $g(a)$~~

if $U(x;s) = \frac{Q(x;s)}{s-a}$ then $u(x,t) = \lim_{s\to a}(s-a)e^{st}U(x;s)$

$$= e^{at}Q(x;a)$$

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}\,dz$$

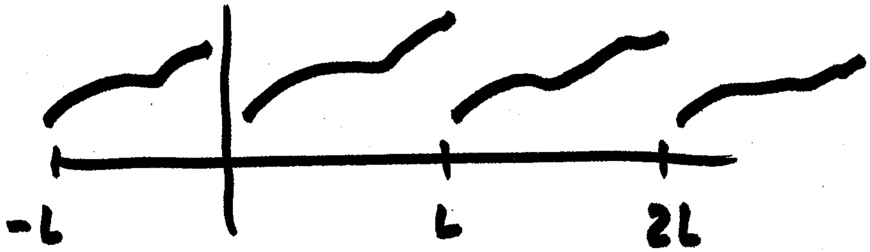
if f is analytic everywhere within & on a closed contour C & z_0 is an interior pt to C then

FOURIER SERIES

$$f(x) = \frac{a_0}{2} + a_1 \cos\frac{\pi x}{L} + a_2 \cos\frac{2\pi x}{L} + \dots$$
$$+ b_1 \sin\frac{\pi x}{L} + b_2 \sin\frac{2\pi x}{L} + \dots$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx \qquad n=0,1,2,\dots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \qquad n=0,1,2..$$



FOURIER SINE TRANSFORM

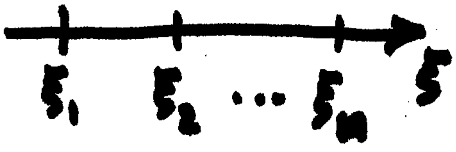
$$-f(x) = f(-x) \quad \text{for} \quad x \in (-L,L)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(\bar{x})\sin\frac{n\pi\bar{x}}{L}\,d\bar{x}$$

LOOK AT WHAT HAPPENS AS $L \to \infty$

$$f(x) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{L}\,\Delta n \qquad \Delta n = n+1-n$$

LET $\xi_n = \frac{n\pi}{L}$; $\Delta\xi_n = \frac{\pi}{L}\Delta n$



FOURIER SERIES

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{\pi x}{L} + a_2\cos\frac{2\pi x}{L} + \dots$$

$$+ b_1\sin\frac{\pi x}{L} + b_2\sin\frac{2\pi x}{L} + \dots$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad n=0,1,2,\dots$$

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad n=0,1,2..$$

FOURIER SINE TRANSFORM

$$-f(x) = f(-x) \quad \text{for } x = (-L, L)$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

LOOK AT WHAT HAPPENS AS L→∞

$$f(x) = \sum_{n=0}^{\infty} b_n\sin\frac{n\pi x}{L}\Delta N \qquad \Delta N = N+1-N$$

LET $\xi_n = \frac{n\pi}{L}$; $\Delta\xi_n = \frac{\pi}{L}\Delta n$

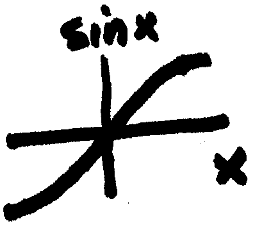
$$f(x) = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}\,\Delta n = \frac{L}{\pi}\sum_{n=0}^{\infty} b_n \sin \overset{\xi_n x}{\boxed{\frac{n\pi x}{L}}}\,\Delta\xi_n$$

$$b_n = \frac{2}{L}\int_0^L f(s)\sin \xi_n s\,ds$$

$$f(x) = \frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=0}^{\infty}\left\{\int_0^L f(s)\sin\xi_n s\,ds\right\}\sin\xi_n x\,\Delta\xi_n$$

L's CANCEL & TAKE limit as L→∞

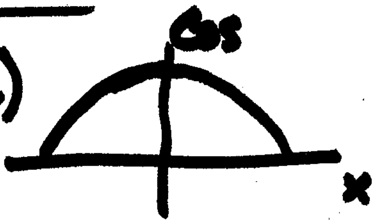
$$f(x) = \frac{2}{\pi}\int_0^\infty d\xi\left\{\int_0^\infty f(s)\sin\xi s\,ds\right\}\sin\xi x$$



$$\begin{cases} \text{LET } F(\xi) = \int_0^\infty f(s)\sin\xi s\,ds \\ f(x) = \frac{2}{\pi}\int_0^\infty F(\xi)\sin\xi x\,d\xi \end{cases}$$

FOR ODD f(x)

IF $f(x) = f(-x)$ FOR $x \in (-L, L)$



$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

as L→∞

$$\begin{cases} F(\xi) = \int_0^\infty f(s)\cos\xi s\,ds \\ f(x) = \frac{2}{\pi}\int_0^\infty F(\xi)\cos\xi x\,d\xi \end{cases}$$

FOR EVEN f(x)

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} = \frac{l}{\pi}\sum_{n=1}^{\infty} b_n \sin\Big(\overset{\xi_n}{\frac{n\pi}{l}}x\Big)\Delta\xi_n$$

$$b_n = \frac{2}{l}\int_0^l f(s)\sin\xi_n s\,ds$$

$$f(x) = \frac{2}{l}\cdot\frac{l}{\pi}\sum_{n=1}^{\infty}\left\{\int_0^l f(s)\sin\xi_n s\,ds\right\}\sin\xi_n x\,\Delta\xi_n$$

l's CANCEL & TAKE limit as $l\to\infty$

$$f(x) = \frac{2}{\pi}\int_0^\infty d\xi\left\{\int_0^\infty f(s)\sin\xi s\,ds\right\}\sin\xi x$$

TR. $F(\xi) = \int_0^\infty f(s)\sin\xi s\,ds$

for odd $f(x)$ $\quad f(x) = \frac{2}{\pi}\int_0^\infty F(\xi)\sin\xi x\,d\xi$

If: $f(x) = f(-x)$ for x in $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

as $l\to\infty$

$F(\xi) = \int_0^\infty f(s)\cos\xi s\,ds$

for even $f(x)$ $\quad f(x) = \frac{2}{\pi}\int_0^\infty F(\xi)\cos\xi x\,d\xi$

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\substack{E(x)\\ \text{even fn}}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\substack{\text{ODD fn}\\ O(x)}}$$

$$\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx = R(\xi)$$

$$(\cos\xi x - i\sin\xi x)\,[E(x) + O(x)]$$

$$\int_{-\infty}^{\infty} E(x)\cos\xi x\,dx - i\int_{-\infty}^{\infty} O(x)\sin\xi x\,dx + \int_{-\infty}^{\infty} O(x)\cos\xi x\,dx - i\int_{-\infty}^{\infty} E(x)\sin\xi x\,dx$$

$$\int_0^{\infty} O(x)\cos\xi x\,dx + \int_{-\infty}^{0} O(x)\cos\xi x\,dx$$

let $z = -x$ $\quad x < 0$, $z > 0$

$$\int_{\infty}^{0} O(-z)\cos(-\xi z)(-dz)$$

$$\int_0^{\infty} O(-z)\cos(-\xi z)\,dz$$

$$\int_0^{\infty} -O(z)\cos\xi z\,dz$$

$\int_{-\infty}^{\infty} O\cos\,dx = 0$ same for $\int_{-\infty}^{\infty} E\sin\,dx = 0$

$$2\int_0^{\infty} E\cos\,dx \;-2i\int_0^{\infty} O\sin\xi\,dx = 2\,\mathcal{F}_c\{E\} + 2\,\mathcal{F}_s\{O\}$$

$$\int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx = F(k)$$

$$f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\substack{E(x)\\ \text{even fn}}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\substack{\text{odd fn}\\ O(x)}}$$

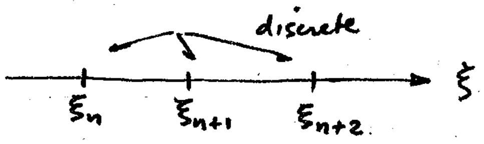
1. Consider $f(x)$ defined $(-L, L)$ $f(x) = -f(-x)$ w/period $2L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} \qquad b_n = \frac{2}{L}\int_0^L f(s) \sin \frac{n\pi s}{L}\, ds$$

as $L \to \infty$
$$\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L} = \sum_{n=0}^{\infty} b_n \sin \frac{n\pi x}{L}$$

$$f(x) = \sum_{n=0}^{\infty} b_n \left(\sin \frac{n\pi x}{L}\right) \Delta n \qquad (n+1) - n = 1 = \Delta n$$

let $\xi_n = \frac{n\pi}{L}$



$$f(x) = \frac{L}{\pi} \sum_{n=0} b_n (\sin \xi_n x)\, \Delta \xi_n$$ ~~as $L \to \infty$ become~~

$$b_n = \frac{2}{L}\int_0^L f(s) \sin \xi_n s\, ds.$$

$f(x)$
$$= \frac{2}{L}\cdot\frac{L}{\pi} \sum_{n=0} \left\{\int_0^L f(s) \sin \xi_n s\, ds\right\} \sin \xi_n x \quad \Delta \xi_n$$

as $L \to \infty$ $\Delta \xi_n \to d\xi$ $\xi_n \to \xi$

$$\sum (\quad) \Delta\xi_n \longrightarrow \int_0^{\infty} (\quad) d\xi$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} d\xi \underbrace{\left\{\int_0^{\infty} ds\, f(s) \sin \xi s\right\}}_{\text{define } F(\xi)} \sin \xi x$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\xi) \sin \xi x\, dx$$
inverse transform

$F(\xi)$ is the fourier sine transform of $f(x)$

look at the cosine transform $f(x) = f(-x)$ even fn w/period $2L$ $(-L, L$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} \quad ; \quad a_n = \frac{2}{L}\int_0^L f(s) \cos \frac{n\pi s}{L}\, ds$$

we can write $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} = \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n' \cos\frac{n\pi x}{L}$

where $a_{-n}' = a_n' = a_n$ & $a_0' = a_0$

as before let $\xi_n = \frac{n\pi}{L}$ with $\Delta n = 1$

$$\Delta n \cdot \frac{\pi}{L}$$

$$\downarrow$$

$$f(x) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \left\{\frac{2}{L}\int_0^L f(s)\cos\xi_n s\, ds\right\} \cos\xi_n x \qquad \Delta\xi_n \cdot \frac{L}{\pi}$$

$$= \frac{1}{\pi}\sum\left\{\int \qquad\qquad \right\}$$

as $L\to\infty$ $\quad \xi_n \to \xi \quad \sum_{-\infty}^{\infty} \to \int_{-\infty}^{\infty}$

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi \cos\xi x \left(\int_0^{\infty} f(s)\cos\xi s\, d\xi\right)$$

since cos & f are symmetric $f(x) = \frac{2}{\pi}\int_0^{\infty} d\xi \cos\xi x \underbrace{\left(\int \qquad\qquad\right)}_{F(\xi)}$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F(\xi)\cos\xi x\, dx$$

$F(\xi)$ is the fourier cosine transform

any function can be written as a sum of even and odd fn.

$$f(x) = f_e(x) + f_o(x) = \int_0^{\infty}\left[F_o(\xi)\sin\xi x + F_e(\xi)\cos\xi x\right]d\xi$$

$$\int_0^{\infty} f_e(s)\cos\xi s\, ds = \int_0^{\infty} f \cos\xi s$$

$$= \frac{2}{\pi}\int_0^{\infty}\int_0^{\infty} f(\xi)\left[\cos\xi(x-s)\right]ds \quad d\xi$$

$$= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\left[f(s)\cos\xi(s-x)\,ds\right] \quad d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[f(s)\, e^{-i\xi(s-x)}\, ds\right] d\xi$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}\int_{-\infty}^{\infty} f(s)\, e^{-i\xi s}\, d\xi\,\Big] d\xi$$

To minimize the confusion w/ the book we will define

Fourier Transform

$$R(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx \qquad R \text{ is } \mathcal{F}\{f\}$$

$$f \text{ is } \mathcal{F}^{-1}\{R\}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} R(\xi)\,d\xi$$

Fourier Cosine transform if f is even

$$R(\xi) = \int_0^{\infty} f(x)\cos\xi x\,dx \qquad R \text{ is } \mathcal{F}_c\{f\}$$

$$f \text{ is } \mathcal{F}_c^{-1}\{R\}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} R(\xi)\cos\xi x\,d\xi$$

Fourier Sine transform if f is odd

$$R(\xi) = \int_0^{\infty} f(x)\sin\xi x\,dx \qquad R \text{ is } \mathcal{F}_s\{f\}$$

$$f \text{ is } \mathcal{F}_s^{-1}\{R\}$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} R(\xi)\sin\xi x\,d\xi$$

What is $\mathcal{F}\{f'\}$? IBP

$$\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx = f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} - (-i\xi)\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

$$\mathcal{F}\{f'\} = i\xi\,\mathcal{F}\{f\} = i\xi R$$

$$\mathcal{F}^{-1}\{i\xi R\} = f'$$

what about $\mathcal{F}\{f''\}$?

$$\int_{-\infty}^{\infty} e^{-i\xi x} f''(x)\,dx = f'(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} - (-i\xi)\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx$$

$$= i\xi\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx = i\xi\,[i\xi R] = -\xi^2 R$$

$$\therefore \quad \mathcal{F}\{f''\} = -\xi^2 R = -\xi^2\mathcal{F}\{f\}$$

$$\therefore \quad \mathcal{F}^{-1}\{-\xi^2 R\} = f''$$

For the sine & cosine transforms.

$$\mathcal{F}_c\{f'\} = \xi\,\mathcal{F}_s\{f\} - f(0+)$$

$$\mathcal{F}_c\{f''\} = -\xi^2\,\mathcal{F}_c\{f\} - f'(0+)$$

$$\mathcal{F}_s\{f'\} = -\xi\,\mathcal{F}_c\{f\}$$

$$\mathcal{F}_s\{f''\} = -\xi^2\,\mathcal{F}_s\{f\} + \xi f(0+).$$

CONVOLUTION INTEGRAL

also $$\mathcal{F}^{-1}\{R_1(\xi)R_2(\xi)\} = \int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi$$

where $$R_1(\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-i\xi x}\,dx$$

$$R_2(\xi) = \int_{-\infty}^{\infty} g(x)\,e^{-i\xi x}\,dx$$ — R_1 is $\mathcal{F}_c\{f\}$ $R_2 = \mathcal{F}_c\{g\}$

$$\mathcal{F}_c^{-1}\{R_1(\xi)R_2(\xi)\} = \tfrac{1}{2}\left\{\int_0^{\infty} f(u)\left[g(x-u) + g(x+u)\right]du\right\}$$ — R_1 is $\mathcal{F}_c\{f\}$ $R_2 = \mathcal{F}_s\{g\}$

$$\mathcal{F}_s^{-1}\{R_1(\xi)R_2(\xi)\} = \tfrac{1}{2}\left\{\int_0^{\infty} f(u)\left[g(x-u) - g(x+u)\right]du\right\}$$

Example:

4.3
Reynolds Notes

$$k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)-\beta^2 r(T-T_0)=\frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$\frac{\partial T}{\partial r}=0$ @ $r=r_0$ & $T(r_i,t)=0$ & $T(r,0)=T_0$

let $T(r,t)=J(r,t)+\psi(r)$

$$k\frac{\partial T}{\partial r}+rk\frac{\partial^2 T}{\partial r^2}-\beta^2 r(T-T_0)=\frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$$k\left(\frac{\partial J}{\partial r}+\psi'\right)+rk\left(\frac{\partial^2 J}{\partial r^2}+\psi''\right)-\beta^2 r(J+\psi-T_0)=\frac{r}{\alpha}\frac{\partial J}{\partial t}$$

choose ψ so that $k\psi'+rk\psi''-\beta^2 r(\psi-T_0)=0$

& $k\frac{\partial J}{\partial r}+rk\frac{\partial^2 J}{\partial r^2}-\beta^2 rJ=\frac{r}{\alpha}\frac{\partial J}{\partial t}$

also $\frac{\partial T}{\partial r}=\frac{\partial J}{\partial r}+\psi'=0$ choose $\left.\frac{\partial J}{\partial r}\right|_{r=r_0}=0$ & & $\psi(r_0)=0$

and $T(r_i=0,t)=J(r_i,t)+\psi(r_i)=0$ choose $J(r_i,t)=0$ & $\psi(r_i)=0$

and $T(r,t=0)=J(r,t=0)+\psi(r)=T_0$ $\Rightarrow$ $J(r,t=0)=T_0-\psi(r)$

$k\frac{\partial}{\partial r}(r\psi')-\beta^2 r(\psi-T_0)=0$ let $\psi=\bar{\psi}+T_0$

Now $k\frac{\partial}{\partial r}(r\psi')-\beta^2 r(\psi-T_0)=0$ Modified Bessel Fn

or $k\frac{\partial}{\partial r}(r\bar{\psi}')-\beta^2 r(\bar{\psi})=0$ $\bar{\psi}(r_i)=-T_0$

& $\psi'(r_0)=0=\bar{\psi}'(r_0)$ & $\psi(r_i)=0=\bar{\psi}(r_i)+T_0$ $\Rightarrow$

$$\bar{\psi}(r)=C_1J_0\left(+\frac{i\beta}{\sqrt{k}}r\right)+C_2Y_0\left(\frac{i\beta}{\sqrt{k}}r\right)$$

$(\cdot r)\equiv\left(\frac{i\beta}{\sqrt{k}}r\right)$

BC1 $\bar{\psi}'(r_0)=\frac{i\beta}{\sqrt{k}}\left[C_1J_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)+C_2Y_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)\right]=0$

BC2 $\bar{\psi}(r_i)=\left[C_1J_0\left(\frac{i\beta}{\sqrt{k}}r_i\right)+C_2Y_0\left(\frac{i\beta}{\sqrt{k}}r_i\right)\right]=-T_0$

$$\text{so}\quad C_1=\frac{\begin{bmatrix}0 & \frac{i\beta}{\sqrt{k}}Y_0'(\cdot r_0)\\ -T_0 & Y_0(\cdot r_i)\end{bmatrix}}{i\beta/\sqrt{k}\left[J_0'(\cdot r_0)Y_0(\cdot r_i)-J_0(\cdot r_i)Y_0'(\cdot r_0)\right]}=\frac{i\beta}{\sqrt{k}}T_0Y_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)/\text{denom}$$

$$C_2=-\frac{i\beta}{\sqrt{k}}T_0J_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)/\text{denom}$$

J satisfies $\frac{k}{r}\frac{\partial J}{\partial r}+\cdot k\frac{\partial^2 J}{\partial r^2}-\beta^2 J=\frac{1}{\alpha}\frac{\partial J}{\partial t}$

let $J(r,t)=R(r)T(t)$

$$\frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)T-\beta^2RT=\frac{1}{\alpha}RT'$$

$$\frac{k}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)-\beta^2=\frac{1}{\alpha}\frac{T'}{T}=-\lambda^2$$

$\Rightarrow$ $T'/T=-\alpha\lambda^2$ $\Rightarrow$ $T'+\alpha\lambda^2T=0$ $\Rightarrow$ $T(t)=Ce^{-\alpha\lambda^2t}$

$$\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)-\left[\frac{\beta^2-\lambda^2}{k}\right]rR=0$$

or $r^2R''+r\cdot R'-\left[\frac{\beta^2-\lambda^2}{k}\right]r^2R=0$ whose soln is $R(r)=C_1J_0\left(\frac{i\mu}{\sqrt{k}}r\right)+C_2Y_0\left(\frac{i\mu}{\sqrt{k}}r\right)$

where $\mu=\sqrt{\beta^2-\lambda^2}$

Now the application of the BCs on

$$\left.\frac{\partial J}{\partial r}\right|_{r=r_0}=0 \quad \forall t \Rightarrow \frac{i\mu}{\sqrt{k}}C_1 J_0'\left(\frac{i\mu}{\sqrt{k}}r_0\right)+\frac{i\mu}{\sqrt{k}}C_2 Y_0'\left(\frac{i\mu}{\sqrt{k}}r_0\right)=0 \qquad (+r)\equiv\frac{i\mu}{\sqrt{k}}r$$

$$J\big|_{r=r_i}=0 \quad \forall t \qquad C_1 J_0\left(\frac{i\mu}{\sqrt{k}}r_i\right)+C_2 Y_0\left(\frac{i\mu}{\sqrt{k}}r_i\right)=0$$

So for C_1 & $C_2 \neq 0 \Rightarrow \frac{i\mu}{\sqrt{k}}\left[J_0'(+r_0)Y_0(+r_i)-Y_0'(+r_0)J_0(+r_i)\right]=0 \Rightarrow$ gives the μ_n values

and $\quad C_{2n}=-C_{1n}J_0\left(\frac{i\mu_n}{\sqrt{k}}r_i\right)/Y_0\left(\frac{i\mu_n}{\sqrt{k}}r_i\right) \qquad \mu_n=\sqrt{\beta^2-\lambda_n^2}$

$$\therefore \quad J=\sum C_{1n}\left[J_0(+r)-Y_0(+r)\cdot J_0(+r_i)/Y_0(+r_i)\right]e^{-\alpha\lambda_n^2 t}$$

$$@\ t=0 \quad J(r,t=0)=T_0-\psi(r)=\sum C_{1n}\left[J_0(+r)-Y_0(+r)\cdot J_0(+r_i)/Y_0(+r_i)\right]$$

$$=-\bar{\psi}(r)$$

$$=-\frac{T_0\left[Y_0'(\cdot r_0)J_0(\cdot r)-J_0'(\cdot r_0)Y_0(\cdot r)\right]}{\left[J_0'(\cdot r_0)Y_0(\cdot r_i)-Y_0'(\cdot r_0)J_0(\cdot r_i)\right]}=\sum C_{1n}\left[\frac{J_0(+r)Y_0(+r_i)-Y_0(+r)J_0(+r_i)}{Y_0(+r_i)}\right]$$

$$=\sum C_{1n}R_n(+r)$$

from what we known $\quad C_{1n}=\dfrac{\int_{r_0}^{r_i} r\left[-\bar{\psi}(r)\right]R_n(+r)\,dr}{\int_{r_i}^{r_0} r R_n^2(+r)\,dr}$

To minimize the confusion w/ the book we will define

Fourier Transform

$$R(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx \qquad R \text{ is } \mathcal{F}\{f\}$$

$$\qquad\qquad f \text{ is } \mathcal{F}^{-1}\{R\}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x} R(\xi)\,d\xi$$

Fourier Cosine transform if f is even

$$R(\xi) = \int_0^\infty f(x)\cos\xi x\,dx \qquad R \text{ is } \mathcal{F}_c\{f\}$$

$$\qquad\qquad f \text{ is } \mathcal{F}_c^{-1}\{R\}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty R(\xi)\cos\xi x\,d\xi$$

Fourier Sine transform if f is odd

$$R(\xi) = \int_0^\infty f(x)\sin\xi x\,dx \qquad R \text{ is } \mathcal{F}_s\{f\}$$

$$\qquad\qquad f \text{ is } \mathcal{F}_s^{-1}\{R\}$$

$$f(x) = \frac{2}{\pi}\int_0^\infty R(\xi)\sin\xi x\,d\xi$$

What is $\mathcal{F}\{f'\}$? IBP

$$\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx = f(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} - (-i\xi)\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$$

$$\mathcal{F}\{f'\} = i\xi\,\mathcal{F}\{f\} = i\xi R$$

$$\mathcal{F}^{-1}\{i\xi R\} = f'$$

what about $\mathcal{F}\{f''\}$?

$$\int_{-\infty}^{\infty} e^{-i\xi x} f''(x)\,dx = f'(x)e^{-i\xi x}\Big|_{-\infty}^{\infty} - (-i\xi)\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx$$

$$= i\xi\int_{-\infty}^{\infty} e^{-i\xi x} f'(x)\,dx = i\xi\,[i\xi R] = -\xi^2 R$$

$$\therefore\quad \mathcal{F}\{f''\} = -\xi^2 R = -\xi^2\mathcal{F}\{f\}$$

$$\therefore \quad \mathcal{F}^{-1}\{-\xi^2 R\} = f''$$

For the sine & cosine transforms.

$$\mathcal{F}_c\{f'\} = \xi\,\mathcal{F}_s\{f\} - f(0+)$$
$$\mathcal{F}_c\{f''\} = -\xi^2\,\mathcal{F}_c\{f\} - f'(0+)$$
$$\mathcal{F}_s\{f'\} = -\xi\,\mathcal{F}_c\{f\}$$
$$\mathcal{F}_s\{f''\} = -\xi^2\,\mathcal{F}_s\{f\} + \xi f(0+).$$

CONVOLUTION INTEGRAL

also $\mathcal{F}^{-1}\{R_1(\xi)R_2(\xi)\} = \int_{-\infty}^{\infty} f(\xi)\,g(x-\xi)\,d\xi$

where $R_1(\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-i\xi x}\,dx$

$R_2(\xi) = \int_{-\infty}^{\infty} g(x)\,e^{-i\xi x}\,dx$ — R_1 is $\mathcal{F}_c\{f\}$ $R_2 = \mathcal{F}_c\{g\}$

$\mathcal{F}_c^{-1}\{R_1(\xi)R_2(\xi)\} = \frac{1}{2}\left\{\int_0^{\infty} R_1(u)\left[R_2(x-u) + R_2(x+u)\right]du\right\}$ — R_1 is $\mathcal{F}_c\{f\}$ $R_2 = \mathcal{F}_s\{g\}$

$\mathcal{F}_s^{-1}\{R_1(\xi)R_2(\xi)] = \frac{1}{2}\left\{\int_0^{\infty} R_1(u)\left[R_2(x-u) - R_2(x+u)\right]du\right\}$

<u>Example</u>:

as long as you have information

look at pg 284 (2) $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}$
$U(0,t) = \bar{U}$
$U(x,0) = 0$

use sine trans form $\mathcal{F}_s\{f'\} = -\omega \mathcal{F}_c\{f\}$

$\mathcal{F}_s\{f''\} = -\omega^2 \mathcal{F}_s\{f\} + \omega f(0+)$

$\Rightarrow$ define $\tilde{U}(\omega;t) = \int_0^\infty u(x,t)\sin\omega x\,dx$ sine transform on x

$$\Rightarrow \quad \frac{d\tilde{U}}{dt} = k\left[-\omega^2\tilde{U} + \omega\bar{U}\right]$$

also $\mathcal{F}_s\{U(x,0)=0\} \Rightarrow \tilde{u}(\omega;0) = 0$

$$\Rightarrow \quad \frac{d\tilde{U}}{dt} + k\omega^2\tilde{U} = k\omega\bar{U} \qquad \tilde{U}_h = C_1 e^{-k\omega^2 t} +$$

$$\tilde{U}_p = \text{const} \Rightarrow \tilde{U}_p = \frac{\cancel{k}}{\cancel{k}\omega}\bar{U}$$

$$\therefore \quad \tilde{U}_{TOT} = C_1 e^{-k\omega^2 t} + \frac{\cancel{k}}{\cancel{k}\omega}\bar{U}$$

$$\tilde{U}_{TOT}(\omega;0) = 0 \Rightarrow C_1 = -\frac{\cancel{k}}{\cancel{k}\omega}\bar{U}$$

$$\therefore \quad \tilde{U}_{TOT} = \frac{\cancel{k}\bar{U}}{\cancel{k}\omega}\left[1 - e^{-k\omega^2 t}\right]$$

pg 274 21(a) Now $\mathcal{F}_s\{\text{erfc}(ax)\} = \frac{(1 - e^{-\omega^2/4a^2})}{\omega}$ let $kt = \frac{1}{4a^2}$

$\Rightarrow a = \frac{1}{2\sqrt{kt}}$

then $u(x,t) = \frac{\bar{U}\cancel{k}}{\cancel{k}}\text{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$

$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{K}g(x,t)$
$U_x(0,t) = \frac{-f_1(t)}{K}$
$U(x,0) = f(x)$

case where $f_1(t) = Q_0$
$f(x) = 0$

look at pg ~~385~~ 284 #5c use cosine transform $\mathcal{F}_c\{f'\} = \omega\mathcal{F}_s\{f\} - f(0+)$

$\mathcal{F}_c\{f''\} = -\omega^2\mathcal{F}_c\{f\} - f'(0+)$

$\Rightarrow \tilde{U}(\omega;t) = \int_0^\infty u(x,t)\cos\omega x\,dx$ cos transform on x

$$\Rightarrow \quad \frac{d\tilde{U}}{dt} = k\left[-\omega^2\tilde{U} + \frac{Q_0}{K}\right]$$

also $\mathcal{F}_c\{U(x,0)=0\} \Rightarrow \tilde{U}(x,0) = 0$

passive and combined vibration control, the combined application of passive damping treatment and active control (SMA actuator) will reduce the vibration time greatly, especially in the case of small damping and slender beam.

4. Conclusions

This part of the paper investigated the vibration control of a simply supported flexible beam with a constrained viscoelastic layer and shape memory layer when the temperature effects of SMA layer on the system were included. The effects of damping material loss factor as well as the slenderness of the beam on the control results were discussed. The following conclusions may be obtained: 1) increase of the damping layer material loss factor will reduce the vibration time and also improve the control results in the case of the combined passive and active vibration control; 2) more time will be spent to control a slender beam than to control a thicker beam or a short beam; and, 3) combined application of passive control (constrained viscoelastic damping treatment) and active vibration control (SMA actuator) will greatly reduce the vibration time.

Acknowledgements

The authors wish to acknowledge the partial support of this work under NASA grant NAG-1-1787.

$$\Rightarrow \quad \frac{du}{dt} + k\omega^2 \tilde{u} = \frac{kQ_0}{\kappa}$$

$$\tilde{u}_h = C_1 e^{-k\omega^2 t}$$

$$\tilde{u}_p = \frac{\cancel{k}Q_0}{\kappa} \cdot \frac{1}{\cancel{k}\omega^2}$$

Now $$\tilde{u}_{TOT} = C_1 e^{-k\omega^2 t} + \frac{\cancel{k}Q_0}{\cancel{\kappa}} \frac{1}{\cancel{k}\omega^2}$$

$$\tilde{u}_{TOT}(x,0) = 0 \Rightarrow C_1 = -\frac{Q_0\cancel{k}}{\cancel{\kappa}} \frac{1}{\cancel{k}\omega^2}$$

$$\therefore \quad \tilde{u}_{TOT}(\omega;t) = -\frac{Q_0\cancel{k}}{\kappa\cancel{k}} \cdot \left(\frac{-1 + e^{-k\omega^2 t}}{\omega^2}\right)$$

Now $$\mathcal{F}_c\left\{ax\,\mathrm{erfc}(ax) - \frac{1}{\sqrt{\pi}} e^{-a^2x^2}\right\} = \frac{a}{\omega^2}\left(-1 + e^{-\omega^2/4a^2}\right)$$ Pg 274 21(b)

let $kt = \frac{1}{4a^2}$ $\quad a = \frac{1}{2\sqrt{kt}}$

$$\tilde{u}(\omega;t) = \underbrace{\frac{-Q_0\cancel{k}}{\kappa\cancel{k}}}_{\text{constant}} \cdot 2\sqrt{kt} \cdot \frac{1}{2\sqrt{kt}} \left(\frac{-1 + e^{-k\omega^2 t}}{\omega^2}\right)$$

$$\Rightarrow \quad u(x,t) = -\frac{Q_0\cancel{k}}{\kappa\cancel{k}} \cdot 2\sqrt{kt} \left\{\frac{1}{2\sqrt{kt}} x\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) - \frac{1}{\sqrt{\pi}} e^{-\frac{x^2}{4kt}}\right\}$$

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Do probl #4, #1 Pg 284

[PUT FIGS. 1-4 HERE]

1) Passive Vibration Control

Figures 1 and 2 show the results of transverse response amplitude versus nondimensional time with $\eta=0.1$, $h_3/L=0.003$ and $\eta=1.0$, $h_3/L=0.003$, respectively. Figures 3 and 4 are the results for $\eta=0.1$, $h_3/L=0.005$ and $\eta=1.0$, $h_3/L=0.005$, respectively. All these results are for the passive vibration control cases (constrained damping treatment only). From the figures we note that a decrease in the ratio h_3/L will increase the response time. This is because reducing the ratio means that we have a more slender beam, thus increasing the vibration time. If we increase the value of the damping layer loss factor, then the vibration time of the beam will decrease. Higher loss factor means high damping ability of the material. More vibration energy will be absorbed, thus reducing the vibration time.

[PUT FIGS. 5-8 HERE]

2) Combined Active and Passive Vibration Control

Active vibration control results are shown in Figures 5-8. Figures 5 and 6 show the results of transverse response amplitude versus nondimensional time with $\eta=0.1$, $h_3/L=0.003$ and $\eta=1.0$, $h_3/L=0.003$, respectively. Figures 7 and 8 are the results for $\eta=0.1$, $h_3/L=0.005$ and $\eta=1.0$, $h_3/L=0.005$, respectively. The same results as passive vibration control cases will be observed: a) increase of damping layer loss factor will reduce the vibration response time, which means that increasing damping may improve active vibration control results; and, b) decrease of the ratio h_3/L will increase the response time. Also, if we compare the results for

$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_0

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = \pm$

Assume for this problem $\sigma_{zz} = \frac{P}{A_0}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = \frac{-\nu}{E}\frac{P}{A_0} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = \frac{-\nu P}{EA_0} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{EA_0} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_0 E}x + f(y,z)$$

$$u_y = -\frac{\nu P}{EA_0}y + g(x,z)$$

$$u_z = \frac{P}{EA_0}z + h(y,x)$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

$$\Rightarrow \frac{\partial f}{\partial z} = -\frac{\partial h}{\partial x} = k(y)$$

$$\therefore f = k_1(y)z + k_2(y)$$

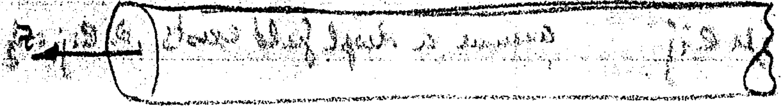
$$h = -k_1(y)x + k_3(y)$$

→ HW #1 complete and solve showing solution is/includes a rigid body rotation/translat

2-D elastostatic problems (isotropic materials)

Plain strain

Elastic solid very long in 1 direction



$$e_{zz} = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = 0$$

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad \mu = \frac{E}{2(1+\nu)}$$

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$$

$$e_{ij} = \frac{-\lambda}{2\mu} \delta_{ij} \sigma_{kk} + \frac{1}{2\mu} \sigma_{ij}$$

$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$u_x = u_x(x,y)$; $u_y = u_y(x,y)$; $u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$; $\neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other cross section. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of z $\Longrightarrow$ $\frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

Solution by Airy Stress fn.

Define a fn ϕ ∋

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) ∴ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists ∋ $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

now $\sigma_{ij,i} = \lambda \delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ gradient relationship $= \lambda (u_{k,kj}) + \mu (u_{j,ii} + u_{i,ji})$

$$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii}$$

now differentiate once

$$(\lambda+\mu)(u_{k,kjj}) + \mu (u_{j,iij}) = 0$$

$$\text{or} \quad (\lambda+2\mu)(u_{k,kjj}) = 0$$

$$\text{or} \quad (\lambda+2\mu) \nabla^2 e_{kk} = 0 \implies \nabla^2 \sigma_{kk} = 0 \implies \nabla^4 \phi = c$$

now $\sigma_{ii} = \overset{3}{\overline{\lambda}} e_{kk} + 2\mu e_{ii} \Rightarrow \sigma_{kk} = (3\lambda + 2\mu) e_{kk}$ or $\nabla^2 \sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove
no summation

1/10/79

Plain Strain From last time

$$\left.\begin{array}{l} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{xy}}{\partial y} = 0 \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} = 0 \end{array}\right\} \begin{array}{l} \sigma_{xx} = \phi_{,yy} \\ \sigma_{xy} = -\phi_{,xy} \\ \sigma_{yy} = \phi_{,xx} \end{array} \quad \text{where } \phi \text{ is the airy stress fn}$$

using $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$$

now take $\frac{\partial}{\partial x_j}$ $\quad (\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0$ since dummy indices rep $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$$\therefore \quad \nabla^2 e_{ii} = 0 \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\quad)_{,ii}$$

now $\sigma_{ii} = \lambda \overset{3}{\overline{\delta_{ii}}} e_{kk} + 2\mu e_{ii} = (3\lambda + 2\mu) e_{ii}$

$$\therefore \quad \nabla^2 e_{ii} \implies \nabla^2 \sigma_{ii} = 0$$

Now in plane strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $e_{zz} = 0$

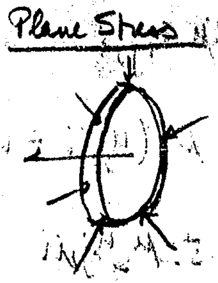
$$\nabla^2 \sigma_{ii} = \nabla^2 (1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu) \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

using Airy stress fns

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2 \phi) = \nabla^4 \phi = 0$$

Plane Stress Plane $\Rightarrow \sigma_{zx} = \sigma_{zy} = 0$; Plane Stress also $\Rightarrow \sigma_{zz} = 0$

If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere



$$0 = \begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{zy} \end{pmatrix}$$

on flat faces

$$\sigma_{zz} = 0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\sigma_{zx} = 0 \Rightarrow e_{zx} = 0$$

$$\sigma_{zy} = 0 \Rightarrow e_{zy} = 0$$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$
$u_y = u_y(x,y)$

Timoshenko & Goodier
Rd 274-277

$\boxed{\text{HW \#1b} \quad \text{prove that this assumption is inconsistent}}$

however we can see that u_x, u_y have z^2 component & that for $z << 1$ then we can assume the above w/o loss in accuracy

hence we can define generalized displ for disc w/ thickness h

$$\bar{U}_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

Plane strain

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \overset{\nu(\sigma_{xx}+\sigma_{yy})}{\sigma_{zz}})\right)$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \dfrac{E}{2(1+\nu)}$ $\quad \therefore \; e_{xx} = \dfrac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

Plane stress

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}})\right) = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy}) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

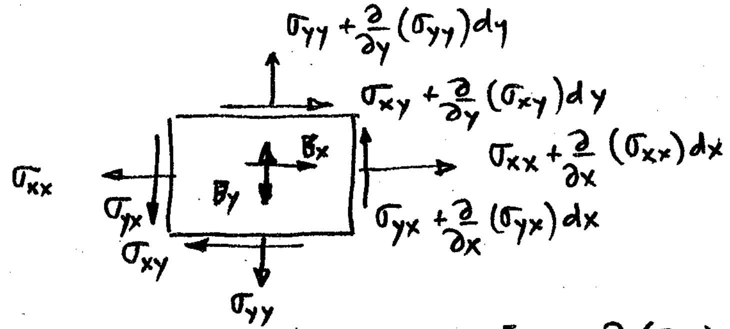
$$e_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

Thus if $\int_{-\infty}^{\infty} f(s) e^{-i\xi s} ds = F(\xi)$ Fourier transfo

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi) e^{i\xi x} d\xi = f(x)$$

Existence if $\int_{-\infty}^{\infty} |f(x)| dx$ exists

Now let's look at a box having stresses as shown, assuming 2-D



$$\Sigma F_x = \left[\sigma_{xx}+\frac{\partial}{\partial x}(\sigma_{xx})dx\right] dy\, dz - \sigma_{xx}\cdot dy\, dz + \left[\sigma_{xy}+\frac{\partial}{\partial y}(\sigma_{xy})dy\right]\cdot dx\cdot dz - \sigma_{xy}\cdot dy\cdot dz$$

$$+ B_x\cdot dxdydz = \left\{\frac{\partial}{\partial x}(\sigma_{xx}) + \frac{\partial}{\partial y}(\sigma_{xy})\right\} dxdydz + B_x\, dxdydz$$

Similarly $\Sigma F_y = \left\{\frac{\partial}{\partial x}(\sigma_{yx}) + \frac{\partial}{\partial y}(\sigma_{yy}) + B_y\right\} dxdydz$

Here B_x, B_y is the body force / unit volume

if we are talking about statics $\Sigma F_x = \Sigma F_y = 0$; $\left(\text{dynamics} = \overbrace{\rho\, dxdydz}^{m} \frac{\partial^2 u_i}{\partial t^2}\right\}$

$$\Rightarrow \frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} + B_x = 0 \qquad \left(= \rho\frac{\partial^2 u_x}{\partial t^2}\right)$$

$$\Rightarrow \frac{\partial}{\partial x}\sigma_{yx} + \frac{\partial}{\partial y}\sigma_{yy} + B_y = 0 \qquad \left(= \rho\frac{\partial^2 u_y}{\partial t^2}\right)$$

- From the moment equation $= \left[\sigma_{xy}+\frac{\partial}{\partial y}(\sigma_{xy})dy\right] dxdz\cdot\frac{dy}{2} + \sigma_{xy}\, dxdz\cdot\frac{dy}{2} - \left[\sigma_{yx}+\frac{\partial}{\partial x}(\sigma_{yx})dx\right.$ $dydz\cdot\frac{dx}{2} - \sigma_{yx}\cdot dydz\cdot\frac{dx}{2} = 0$ for static equilib $\Rightarrow (\sigma_{xy}-\sigma_{yx})\cdot \text{Vol} + \text{h.o.t.} =$

  $\Rightarrow \sigma_{xy} = \sigma_{yx}$

- It the body forces are negligible
$$\frac{\partial}{\partial x}\sigma_{xx} + \frac{\partial}{\partial y}\sigma_{xy} = 0$$
$$\frac{\partial}{\partial x}\sigma_{yx} + \frac{\partial}{\partial y}\sigma_{yy} = 0$$

if we let $\sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2}$ $\sigma_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y} = \sigma_{yx}$ & $\sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2}$

then we satisfy equations of equilib. But what does φ satisfy?

This is obtained from the constitutive equations relating the stresses & strains (remember Hooke's Law?)

- if we let let x axis be the "1" axis , y axis be the "2" axis
- we can write that
- $\sigma_{ij} = \lambda\,\delta_{ij}(e_{11}+e_{22}) + 2\mu\,(e_{ij})$ $\delta_{ij} = 1$ if $i=j$, $= 0$ if $i \neq j$
- λ & μ are Lamé constants related to E & ν
- e_{ij} are the strains; when $i \neq j$ e_{ij} are the shear strains
- $e_{ij} = \frac{\partial u_i}{\partial x_j}$ thus $e_{11} = \frac{\partial u_1}{\partial x_1}$ is the strain in the x direction

  $e_{12} = \frac{\partial u_1}{\partial x_2} + \partial u_2/\partial x_1$ is the shear strain γ_{xy} etc.

- also $\frac{\partial^2 e_{11}}{\partial y^2} + \frac{\partial^2 e_{22}}{\partial x^2} = \frac{\partial^2 e_{12}}{\partial x\partial y}$ this is compatibility of strains
- now $\sigma_{11} + \sigma_{22} = (\lambda+2\mu)(e_{11}+e_{22})$
- and $\nabla^2(\sigma_{11}+\sigma_{22}) = [\nabla^2(e_{11}+e_{22})](\lambda+2\mu) = 0$ if we use the above definitions for σ_{ij} in the equilibrium equations & the definitions from compatibility
- Since $\sigma_{11} = \sigma_{xx} = \frac{\partial^2\varphi}{\partial y^2}$ & $\sigma_{22} = \sigma_{yy} = \frac{\partial^2\varphi}{\partial x^2}$ $\sigma_{11}+\sigma_{22} = \nabla^2\varphi$

$$\therefore \quad \nabla^2(\sigma_{11}+\sigma_{22}) = \nabla^2(\nabla^2\varphi) = \nabla^4\varphi = \frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0$$

in x, y coordinates.

~~If we let $\varphi(x,y) = X(x)Y(y) \Rightarrow X^{IV}Y + 2X''Y'' + Y^{IV}X = 0$~~

~~then $X^{IV}/X + 2\frac{X''}{X}\frac{Y''}{Y} + \frac{Y^{IV}}{Y} = 0$ if we divide by XY~~

let's chose a value for $\varphi = e^{mx+\lambda y} \Rightarrow m^4 + 2m^2\lambda^2 + \lambda^4 = 0$

$(m^2+\lambda^2)^2 = 0$ $m = \pm i\lambda$ or $\lambda = \pm im$

$\varphi = e^{\pm(ix+y)\lambda}$ are solutions of multiplicity (2)

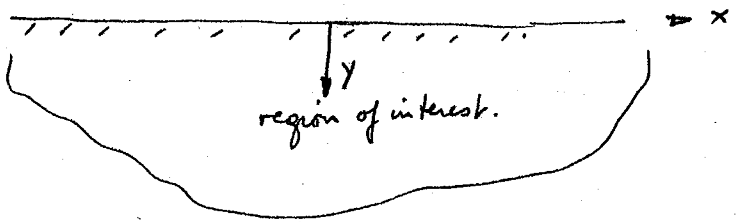
actually $\varphi = \begin{matrix}\cos\lambda x\\ \sin\lambda x\end{matrix}\left[A_1 e^{\lambda y} + A_2 e^{-\lambda y} + A_3 y e^{\lambda y} + A_4 y e^{-\lambda y}\right]$

for the discrete case

also note that A_1, A_2, A_3, A_4 will depend on λ

$$\varphi = \sum_{\lambda} e^{-i\lambda x}\left[A_1(\lambda)\, e^{\lambda y} + A_2(\lambda) e^{-\lambda y} \dots\right]$$

If we look at an semi-∞ region for the continuous λ case



we find that $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left[A e^{-|\lambda| y} + B y e^{-|\lambda| y}\right] d\lambda$

this will account for both + & - values of λ so that stress $\to 0$ as $y \to$

what if $\sigma_{yy}(y=0) = f(x)$ & $\sigma_{xy}(y=0) = g(x)$

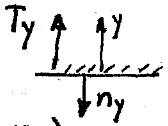
$$\sigma_{yy} = \frac{\partial^2 \varphi}{\partial x^2} = \int_{-\infty}^{\infty} \frac{\partial^2}{\partial x^2}\left\{e^{-i\lambda x}\left[A e^{-|\lambda| y} + B y e^{-|\lambda| y}\right]\right\} d\lambda = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x}\left[\quad\right] d\lambda$$

$$\sigma_{yy}\Big|_{y=0} = f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} \cdot A\, d\lambda \qquad \Rightarrow -2\pi\lambda^2 A \text{ is the fourier transform of } f(x)$$

$$\sigma_{xy}\Big| = -\frac{\partial^2\varphi}{\partial x \partial y} = -\int_{-\infty}^{\infty} \frac{\partial^2}{\partial x \partial y}\left\{e^{-i\lambda x}\left[A e^{-|\lambda| y} + B y e^{-|\lambda| y}\right]\right\} d\lambda = -\int -i\lambda e^{-i\lambda x}\left[-|\lambda| A e^{-|\lambda| y} + B e^{-|\lambda| y} - |\lambda| B y e^{-|\lambda| y}\right] d\lambda$$

$$\sigma_{xy}\Big|_{y=0} = \int_{-\infty}^{\infty} i\lambda e^{-i\lambda x}\left[-|\lambda| A + B\right] d\lambda$$

we define the traction $T_y = \sigma_{yj} \cdot n_j = (\sigma_{yy} \cdot n_y + \sigma_{yx} \cdot n_x)$



$\Rightarrow \sigma_{yy}$ is −

n_x, n_y are unit vectors in x, y

for example: suppose T_y is a delta fn $\delta(x)$ acting in + y direction

then $\quad T_y = \sigma_{yy} \cdot n_y \qquad n_y = -1 \quad \Rightarrow \quad \sigma_{yy} = -\delta(x) \quad$ if $\sigma_{xy} = 0$

$T_x = \sigma_{xj} \cdot n_j \qquad n_x = 0 \quad \Rightarrow \quad \sigma_{xy} = T_x = 0$

$\delta(x-x_0) = \infty$ for $x = x_0$

Aside: Delta fns are defined by

(1) $\delta(x-x_0) = 0$ for $x \neq x_0$

(2) $\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$

(3) $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$

$$\therefore \int_{-\infty}^{\infty} \sigma_{yy}(y=0)\,dx = \int_{-\infty}^{\infty} -\delta(x)\,dx = -1 = -\int_{-\infty}^{\infty} T_y\,dx$$

since $f(x) = -\delta(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A\,d\lambda$ we have to represent $\delta(x) = \int_{-\infty} e^{-i\lambda x} R(\lambda)\,d\lambda$

thus using the fourier transforms $R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\,\delta(x)\,dX = \frac{1}{2\pi}\,e^{i\lambda \cdot 0} = \frac{1}{2\pi}$

thus $R(\lambda) = \frac{1}{2\pi}$ if $f(x) = \delta(x)$ thus $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$

and $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$ is the Fourier Integral Representation of the δ Function

$$-\delta(x) = f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda \Rightarrow \lambda^2 A = \frac{1}{2\pi};$$ and from $g(x) = 0$

$-|\lambda| A + B = 0 \qquad \therefore \quad A(\lambda) = \frac{1}{2\pi\lambda^2} \qquad B(\lambda) = |\lambda| A(\lambda) = \frac{1}{2\pi|\lambda|}$. Substituting into $\sigma_{yy}(x,y)$

$$\therefore \sigma_{yy}(x,y) = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty} d\lambda\, \cancel{\lambda^2} e^{-i\lambda x}\left[\frac{1}{2\pi\cancel{\lambda^2}} e^{-|\lambda| y} + y\,\overset{|\lambda|}{\frac{1}{2\pi\cancel{|\lambda|}}}\, e^{-|\lambda| y}\right]$$

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{\infty}^{\infty} d\lambda\, \underset{\cos + i\sin}{e^{-i\lambda x}}\, \underset{\text{even fn of } \lambda}{e^{-|\lambda| y}}\, \underset{\text{even as a fn of } \lambda}{\{1 + |\lambda| y\}}$$. Since only result exists if $\int_{-\infty}^{\infty}$ even fn

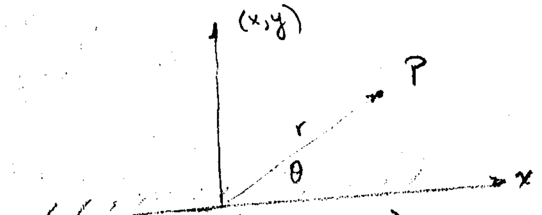
only non zero term is
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \cos\lambda x\, e^{-|\lambda| y}\{1 + |\lambda| y\} = 2 \cdot \frac{-1}{2\pi}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y}\{1 + \lambda y\}$$

Now (1) : $\int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y} = \frac{y}{x^2+y^2}$ ← this is Laplace transf of $\cos\lambda x$ with $y > 0$

(2): $y\int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y}\cdot\lambda$ since integration is wrt λ can takes y outside integral. Notice

that $y\frac{\partial}{\partial y}(1) = -(2) = y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$ $\quad\therefore\quad (2) = -y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\sigma_{yy} = -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} - y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$



$\sigma_{xy} = 0$ ↑ $\sigma_{yy} = -\delta(x)$

$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} = -\frac{2}{\pi}\frac{\sin^3\theta}{r}$$

For $\sigma_{yy}(x,y=0) = -f(x) \overset{}{=} f(x)[-\delta(x)]$ $\quad \sigma_{xy}(x,y=0) = 0$ we can get the answer based on our delta fn result. We know:

1. for a point force applied at at point $x=\xi$ on $y=0$. Then by shift of origin
$$\sigma_{yy}(x,y;\xi) = -\frac{2}{\pi}\frac{y^3}{[(x-\xi)^2+y^2]^2}$$

2. then by the principle of linear superposition with a distrubuted load $f(x)$
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

$\frac{y^3}{[(x-\xi)^2+y^2]^2}$ is the Green's fn for a half space

$$\sigma_{yy} = f(x) = \int_{-\infty}^{\infty} f(\xi)\delta(\xi-x)\,d\xi$$

for these two to be same then the two conditions needed are the $\overset{\text{stress}}{\epsilon_{xx} = A\sigma_{xx} + B\sigma_{yy}} = \overset{\text{strain}}{\epsilon_{xx} = C\sigma_{xx} + D\sigma_{yy}}$

$$\frac{1}{1+\nu_\sigma} = 1-\nu_\epsilon \quad \text{and} \quad \nu_\epsilon = \frac{\nu_\sigma}{1+\nu_\sigma} \quad \Rightarrow \quad \nu_\sigma = \frac{\nu_\epsilon}{1-\nu_\epsilon}$$

(i) Given a complete plain strain soln, get the plane stress solution by leaving μ fixed and replace ν_ϵ by $\frac{\nu}{1+\nu}$

(ii) Given plane stress and want plane strain, replace ν_σ by $\frac{\nu}{1-\nu}$

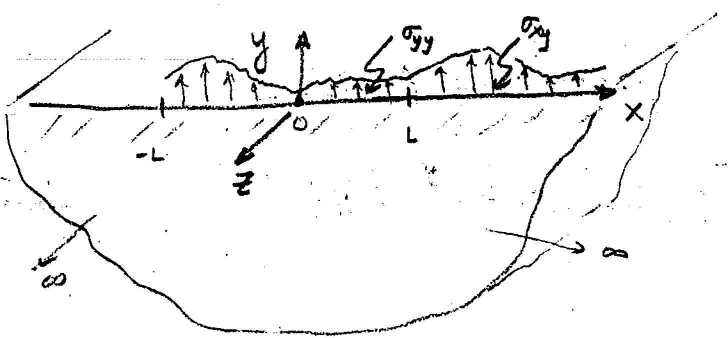
We now look at

2-D problems in rectangular coordinates using Fourier Series (Timo&Good Pg 53 ff)

Rect, strips, half space

$T_i = \sigma_{ij} n_j$

Problem * 1



Look at plane strain problem

Traction boundary value problem

on $y=0$ $\quad \underset{\sim}{n} = \underline{e}_y$

$T_x = \sigma_{xy}$ $\qquad \sigma_{xx} i + \sigma_{xy} j + \sigma_{xz} k$

$T_y = \sigma_{yy}$ $\qquad \sigma_{yx} i + \sigma_{yy} j + \sigma_{yz} k$

$T_z = \sigma_{zy} = 0$ since plain strain $\Leftarrow \epsilon_{yz} = 0$

$\sigma_{zx} i + \sigma_{zy} j + \sigma_{zz} k$

@ ∞ $\sigma \to 0$ $(y \to -\infty)$

We will assume no shear loading $\sigma_{xy} = 0$. Assume $\sigma_{yy} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$

$$f(x) = \sigma_{yy}(x,0) = \sum A_n \sin \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

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Continuing this problem again

For $|x| < \infty$, $0 > y > -\infty$ w/ <u>Periodic</u> loading on $y=0 \Rightarrow \sigma_{yy}, \sigma_{xy}, \sigma_{zy}$ only loading

we look at plane strain problem $\therefore \sigma_{zy} = 0$

We now look at problem $\sigma_{yy} = \sum A_n \sin \frac{\pi x}{L}$ $\qquad \sigma_{xy} = 0$

Note: next problem we will look at $\sigma_{xy} = \sum B \sin \frac{n\pi x}{L}$, $\sigma_{yy} = 0$

finally look at $T_i = \eta \sigma_{yy} + \xi \sigma_{xy}$ where η, ξ are direction cosines

We also impress $\sigma_{ij} \to 0$ as $y \to \pm\infty$

**** Problem given: $\sigma_{yy}(x;0) = f(x)$ $\quad \sigma_{xy} = 0$

$$\sigma_{yy}(x, y=0) = f(x) = \sum A_n \sin\frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

PDE $\nabla^4\phi = 0$ pick $\phi = g(y)\sin\frac{n\pi x}{\ell} = g(y)\sin\gamma_n x$ since it satisfies $\overset{\text{the form of}}{\wedge}$ b/c

$$\therefore \quad \nabla^4\phi = \left[(\gamma_n)^4 g - 2(\gamma_n)^2 g'' + g^{IV}\right]\sin\gamma_n x = 0 \Rightarrow \gamma_n^4 g - 2\gamma_n^2 g'' + g^{IV} = 0 \quad (1)$$

Take $g(y) = e^{Sy} \Rightarrow (1) \Rightarrow (S^2 - \gamma_n^2)^2 = 0$

$\therefore \quad S = \pm\gamma_n \,,\ \pm\gamma_n$

$$\phi_n(x,y) = \sin\frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + C_n y e^{\gamma_n y} + D_n y e^{-\gamma_n y}\right\}$$

$$\phi = \sum_{n=1}^{\infty}\phi_n \qquad \phi_0 = 0$$

using b.c. that $\sigma_{ij} \to 0$ as $y \to -\infty$ pick $\beta_n = D_n = 0$

$$\therefore \phi = \sum_{n=1}^{\infty}\phi_n = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + C_n y e^{\gamma_n y}\right\} \qquad y<0$$

this satisfy compat & equil (since $\nabla^4\phi = 0$ came from there) and b.c. at $-\infty$

now look at b.c. at $y=0$

$$\text{since}\ \left.\begin{matrix}\sigma_{xy}=0\\ @\ y=0\end{matrix}\right\} \xRightarrow{} -\phi_{,xy} = -\sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell}\left(\frac{n\pi}{\ell}\right)\left\{\alpha_n\gamma_n e^{\gamma_n y} + C_n e^{\gamma_n y} + \gamma_n C_n y e^{\gamma_n y}\right\}$$

$$= -\sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell}\left(\frac{n\pi}{\ell}\right)\left\{\alpha_n\gamma_n + C_n\right\} = 0$$

$$\Rightarrow \boxed{C_n = -\gamma_n\alpha_n}$$

$$\therefore \quad \phi = \sum_{n=1}^{\infty}\phi_n = \sum_{n=1}^{\infty}\alpha_n\sin\gamma_n x\,(1-\gamma_n y)\,e^{\gamma_n y}$$

also $\sigma_{yy}(y=0) = \sum_1^{\infty} A_n\sin\frac{n\pi x}{\ell}$; $\phi_{,xx} = -\sum_1^{\infty}\alpha_n\gamma_n^2\sin\gamma_n x\,(1-\gamma_n y)\,e^{\gamma_n y}$

at $y=0$ $\quad \phi_{,xx} = -\sum^{\infty}\alpha_n\gamma_n^2\sin\gamma_n x$

$\therefore \quad A_n = \alpha_n \gamma_n^2 \quad$ or $\quad \boxed{\alpha_n = -\frac{A_n}{\gamma_n^2}}$

$$\therefore \phi(x,y) = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2} \sin\gamma_n x \, (1-\gamma_n y) e^{\gamma_n y} \qquad \text{where } A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \; ; \; \gamma_n = \frac{n\pi}{L}$$

to get displ $\quad \epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1}{E}\left\{\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right\} = -\frac{\nu^{(1+\nu)}}{E}\phi_{,xx} + \frac{1-\nu^2}{E}\phi_{,yy}$

$\epsilon_{zz} = 0 = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$ or $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$

thus $u_x = \int \frac{\partial u_x}{\partial x}\,dx + f(y,\cancel{z})$ $\quad$ now since we only want $u_x = u(x,y)$ plane strain

<u>Problem #2</u> $\quad$ on $y=0 \quad \sigma_{yy} = \frac{a_0}{2} + \sum_1^{\infty} F_n \cos\frac{n\pi x}{L}$

$\sigma_{xy} = 0$

$$\sigma_{yy}(x, y=0) = f(x) = \sum_1^{\infty} F_n \cos\gamma_n x + \frac{a_0}{2} \qquad \text{where } F_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

(2a) if $F_n = 0 \quad$ look at $\phi_0 = \alpha x^2 \quad \nabla^4\phi_0 = 0 \quad$ and since $\sigma_{yy} = \frac{a_0}{2}$

$\sigma_{yy} = \frac{\partial^2\phi_0}{\partial x^2} = 2\alpha \Rightarrow 2\alpha = \frac{a_0}{2} \quad \alpha = \frac{a_0}{4}$

$\sigma_{xx} = 0 \quad \sigma_{xy} = 0 \quad$ since $\phi_0 \neq f(y)$

if load is periodic in direction 1 and $\nabla^4\phi = 0$ is governing pde in a half-space then $\phi \to 0$ in direction 2, as $|x_2| \to \infty$

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<u>Continuing problem #2</u>

$\sigma_{xy} = T_x = 0 \; ; \; \sigma_{yy} = T_y = \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n \cos\gamma_n x \; ; \; \sigma_{yz} = T_z = 0 \;$ on $y = 0 \qquad F_n = \frac{2}{L}\int_0^L f(x)\cos\gamma_n x\,dx$

if $\phi_0 = \alpha x^2$ then $\frac{\partial^2\phi_0}{\partial x^2} = 2\alpha \Rightarrow \alpha = \frac{a_0}{4}$

Try $\phi_n(x,y) = g_n(y)\cos\frac{n\pi x}{L} = g_n(y)\cos\gamma_n x$ (why? because satisfies b.c. $\frac{\partial^2\phi}{\partial x^2} \sim \cos\gamma_n x$)

$$\nabla^4\phi = \cos\gamma_n x\left(\gamma_n^4 g_n - 2\gamma_n^2 g_n'' + g^{iv}\right) = 0 \Rightarrow g_n(y) = \alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + E_n y e^{\gamma_n y} + H_n y e^{-\gamma_n y}$$

as $y \to -\infty \quad \sigma \to 0 \Rightarrow \beta_n, H_n \to 0$

$$\phi = \sum_{n=1}^{\infty} \{\alpha_n + E_n y\} e^{\gamma_n y} \cos \gamma_n x$$

$$\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -\sum_{1}^{\infty} \{-\gamma_n \sin \gamma_n x\} \{(\alpha_n + E_n y)\gamma_n e^{\gamma_n y} + E_n e^{\gamma_n y}\}$$

$$\sigma_{xy}\big|_{y=0} = \sum_{1}^{\infty} \gamma_n \sin \gamma_n x \{\alpha_n \gamma_n + E_n\} = 0 \quad \forall x \Rightarrow \boxed{E_n = -\alpha_n \gamma_n}$$

$$\therefore \quad \phi = \sum_{n=1}^{\infty} \{1 - \gamma_n y\} \alpha_n e^{\gamma_n y} \cos \gamma_n x$$

$$\sigma_{yy} = 0 = \frac{\partial^2 \phi}{\partial x^2} = \sum_{1}^{\infty} (-\gamma_n^2) \{1 - \gamma_n y\} \alpha_n e^{\gamma_n y} \cos \gamma_n x = \sum F_n \cos \gamma_n x$$

$$\sigma_{yy}\big|_{y=0} \Rightarrow \quad \sum_{1}^{\infty} (-\gamma_n^2) \alpha_n \cos \gamma_n x = \sum F_n \cos \gamma_n x$$

take $\boxed{\alpha_n = \frac{-F_n}{\gamma_n^2}}$

$$\phi_0 = \frac{a_0 x^2}{4} \qquad \phi \sim \cos \frac{n\pi x}{L} \qquad \phi \sim \sin \frac{n\pi x}{L}$$

Problem 3: if $f(x) = \frac{a_0}{2} + \sum A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} = \sigma_{yy}$

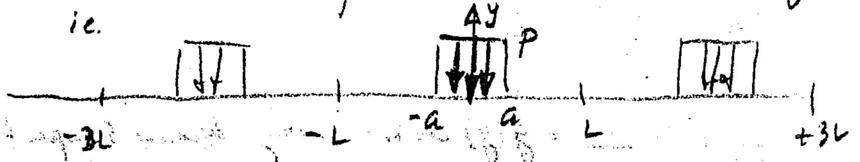
then use superposition of proble #1 & 2

if $f(x) = \frac{a_0}{2} + \sum A_n \cos \frac{n\pi x}{L} + B_n \sin \frac{n\pi x}{L} = \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$

take $\phi_0 = -\frac{a_0 x y}{2}$ $\qquad \phi \sim \sin \frac{n\pi x}{L}$ $\qquad \phi \sim \cos \frac{n\pi x}{L}$ $\qquad$ since $\sigma_{xy} \sim \frac{\partial \phi}{\partial x}$ only

Consider a preleminary proble to the pericl fn.

i.e.



$$\sigma_{yy} = -P \quad (y=0); \quad |x| < a$$
$$0 \qquad a < |x| < L$$

since this an even problem (symmetric) need only cos series

$\sigma_{yy}(y=0) = \frac{a_0}{2} + \sum_1^\infty B_n \cos\frac{n\pi x}{L} = -P$ since $T_y = \sigma_{yj} n_j = \sigma_{yy}(+1) = -P$

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-a}^{a}(-P)\,dx \Rightarrow \frac{2}{2L}\int_0^a -P\,dx = \frac{-Pa}{L}$$

$$B_n = \frac{1}{L}\int_{-a}^{a}(-P)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^a(-P)\cos\frac{n\pi x}{L}\,dx = \frac{-2P}{n\pi}\sin\frac{n\pi a}{L}$$

$\therefore$ $\quad\therefore$ $\quad -P = -\left\{\frac{Pa}{L} + \sum_1^\infty \frac{2P}{n\pi}\sin\frac{n\pi a}{L}\cos\frac{n\pi x}{L}\right\}$

$$\sigma_{yy}(y=0) = -\left\{\frac{2Pa}{2L} + \sum_1^\infty \frac{2Pa}{L}\,\frac{\sin\frac{n\pi a}{L}}{\frac{n\pi a}{L}}\cos\frac{n\pi x}{L}\right\}$$

let $2Pa \to 1$ let $P \to \infty$ $a \to 0$

$$\therefore\ \sigma_{yy} = -\left\{\frac{1}{2L} + \sum_{n=1}^\infty \frac{1}{L}\cos\frac{n\pi x}{L}\right\} = -\delta(x-2mL) \qquad m=0,1,2,\ldots$$

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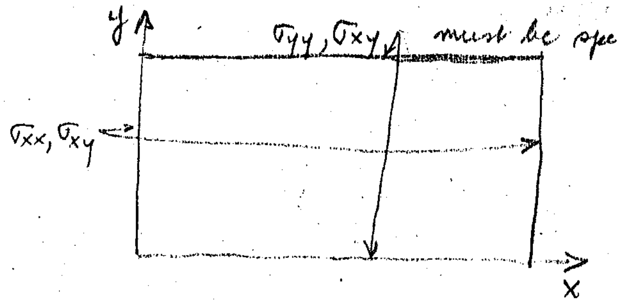
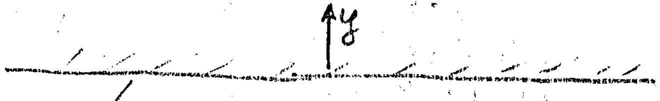
Next order of complexity in problems



P 61 & 62 in Timoshenko
Solution requires eigenfunction expansions
solutions will be of the form
$\sin\beta_n x$ where β_n are complex nos.

Aside :
Why not superposition? It can be done. No reason why not.

Fourier Analysis - Fourier Integrals.
Problem



$\sigma_{xy} = g(x)$ on $y=0$ where $g(x)$ is not periodic
$\sigma_{yy} = f(x)$ on $y=0$

complex imbedding of cosine/sine fn

basic solutions ∋ $|\sigma_{ij}| \to 0$ as $y \to \infty$

$$\text{try } \underset{\lambda}{\phi}(x,y) = \overbrace{e^{-i\lambda x}}\left\{\overbrace{Ae^{-\lambda y} + Bye^{-\lambda y}}\right\} \quad \text{w/ } \lambda > 0 \qquad y > 0$$

then

$\nabla^4\phi = 0$ $\quad \sigma_{ij}$ are bounded as $y \to +\infty$

since this is ^not^ true ^for^ neg values of λ then to remove the restriction of $\lambda > 0$ take $|\lambda|$

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{A(\lambda)e^{-|\lambda|y} + B(\lambda)ye^{-|\lambda|y}\right\}d\lambda$$

$$\sigma_{yy}\Big|_{y=0} = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty} e^{-i\lambda x}\left\{Ae^{-|\lambda|y} + Be^{-|\lambda|y}\cdot y\right\}d\lambda\Big|_{y=0}\cdot\lambda^2$$

$$= -\int_{-\infty}^{\infty} e^{-i\lambda x}\lambda^2\cdot A(\lambda)\,d\lambda = f(x)$$

$$\left[\sigma_{xy}\right]\Big|_{y=0} = -\frac{\partial^2\phi}{\partial x\,\partial y}\Big|_{y=0} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x}\left\{-A|\lambda|e^{-|\lambda|y} + Be^{-|\lambda|y} - B|\lambda|ye^{-|\lambda|y}\right\}d\lambda\Big|_{y=0}$$

$$= \int_{-\infty}^{\infty}\{-i\lambda\}e^{-i\lambda x}\{-|\lambda|A + B\})\,d\lambda = g(x)$$

Before continuing let's look at Fourier Transforms

1. Fourier Sine Transforms

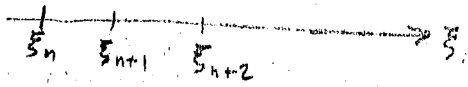
consider $f(x)$ defined on $(-L, L)$ $\quad f(x) \overset{odd}{=} -f(-x)$ w/ period of $2L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad ; \quad b_n = \frac{2}{L}\int_0^L f(s)\sin\frac{n\pi s}{L}\,ds$$

look at what happens as $L \to \infty$ since $\sin 0 = 0$ $\quad \sum_{n=0}^{\infty} = \sum_{n=1}^{\infty}$

$$f(x) = \sum_{n=0}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\Delta n \qquad (n+1) - n = 1 = \Delta n$$

Let $\xi_n = \frac{n\pi}{L}$; $\quad \Delta\xi_n = \frac{\pi\,\Delta n}{L}$



then we can write that

$$f(x) = \frac{L}{\pi}\sum_{n=0}^{\infty} b_n\left(\sin\xi_n x\right)\Delta\xi_n \qquad \text{w/}$$

$$b_n = \frac{2}{L}\int_0^L f(s)\sin\xi_n s\;ds$$

, Put this back into $f(x)$

$$f(x) = \frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=0}^{\infty}\left\{\int_0^L f(s)\sin\xi_n s\,ds\right\}\sin\xi_n x\;\Delta\xi_n$$

as $L \to \infty$ $\quad \xi_n \to \xi$ $\qquad \xi_n = \frac{n\pi}{L}$; as $L\to\infty$ $\xi_n \to 0$ $d\ \Delta\xi_n = \frac{\pi}{L}\xrightarrow{\Delta n} 0$

$$\sum_{n=0}^{\infty}\Delta\xi_n\left(\quad\right) \longrightarrow \int_0^{\infty}\left(\quad\right)d\xi$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty}d\xi\left(\int_0^{\infty}ds\ f(s)\sin\xi s\right)\sin\xi x$$

Let $F(\xi) = \cancel{\frac{2}{\pi}}\int_0^{\infty}ds\,f(s)\sin\xi s$ then

$f(x) = \frac{2}{\pi}\int_0^{\infty}d\xi\ F(\xi)\sin\xi x$ $\qquad$ fourier sine transform pairs

$F(\xi)$ is the fourier sine transform of $f(x)$

2. Fourier Cosine transform

Let $f(x) = f(-x)$ even fn. W/ period $2L$ on $(-L, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}a_n\cos\frac{n\pi x}{L}\quad;\quad a_n = \frac{2}{L}\int_0^L f(s)\cos\frac{n\pi s}{L}\,ds$$

we now write this as a $\sum_{-\infty}^{\infty}$ since fn is even.

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}a_n'\cos\frac{n\pi x}{L}\quad;\quad a_0' = a_0\ ;\ a_n' = a_n\ ,\quad n \geq 1$$

with $a_{-n}' = a_n' = a_n$

let $\xi_n = \frac{n\pi}{L}$ $\quad \Delta n = 1$ $\quad \therefore$ putting a_n back into the sum

sum $\xi_{-\infty}^{\infty}$ $\quad a_n$ $\quad$ redef ξ_n

$$f(x) = \frac{1}{2}\cdot\frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\;\Delta\xi_n = \frac{1}{\pi}\sum_{-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\;\Delta\xi_n$$

as $L \to \infty$ $\quad \xi_n \to \xi$ $\quad \sum_{-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$

$$\therefore\ f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}d\xi\cos\xi x\int_0^{\infty}ds\cos\xi s\ f(s) = \frac{2}{\pi}\int_0^{\infty}d\xi\cos\xi x\int_0^{\infty}ds\cos\xi s\,f(s)$$

since they are symmetric

$\therefore$ if $F(\xi) = \frac{2}{\pi}\int_0^\infty ds \cos\xi s\, f(s)$

$$f(x) \overset{\checkmark}{=} \int_0^\infty d\xi \cos\xi x\, F(\xi)$$ for even fn!

every thing works so long as ① $\int_0^\infty |f(s)|\, ds \leq M$, ② if x_0 is a pt of discontinuity then $f(x_0) = \frac{1}{2}f(x_0^-) + \frac{1}{2}f(x_0^+)$, ③ continuous on intervals & ④ $f'(x^-), f'(x^+)$ exist

3. General Fourier Transform

Let $f(x)$ be defined on $(-\infty, \infty)$; $f(x)$ neither even nor odd – then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = E(x) + O(x)$$

$$E(-x) = E(x) \qquad O(-x) = -O(-x)$$

Call $R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\cos\xi x - i\sin\xi x)(E(x) + O(x))\, dx$

since even product $\int_{-\infty}^{\infty} = 2\int_0^\infty = \pi\cdot\frac{2}{\pi}\int_0^\infty$ odd · odd = even f.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\cos\xi x\, E(x)\, dx - \frac{i}{2\pi}\int_{-\infty}^{\infty}\sin\xi x\, O(x)\, dx + \frac{i}{2\pi}\int_{-\infty}^{\infty}\underbrace{\begin{matrix}(\sin\xi x)(E(x)) \\ (O(x))(\cos\xi x) \\ \text{odd}\cdot\text{even}\end{matrix}}_{\text{odd}} = \text{even}\Big|_{-\infty}^{\infty} = 0$$

thus:

$$R(\xi) = \frac{1}{2}\left[U(\xi) + iV(\xi)\right] = \frac{1}{2}\left[U(\xi) - iV(\xi)\right]$$

with

$$U(\xi) = \frac{2}{\pi}\int_0^\infty \cos\xi x\, E(x)\, dx \quad \& \quad V(\xi) = \frac{2}{\pi}\int_0^\infty \sin\xi x\, O(x)\, dx$$

we can define

$$E(x) = \frac{2}{\pi}\int_0^\infty \cos\xi x\, U(\xi)\, d\xi \quad \& \quad O(x) = \frac{2}{\pi}\int_0^\infty \sin\xi x\, V(\xi)\, d\xi$$

Next time we will look at the proof of $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\, d\xi = f(x)$

$$\frac{\partial \phi}{\partial y} = \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x} \left[-|\lambda| A(\lambda) e^{-|\lambda| y} + B(y) e^{-|\lambda| y} - |\lambda| y B(y) e^{-|\lambda| y} \right]$$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} = \int_{-\infty}^{\infty} dx\, e^{-i\lambda x} \left[\lambda^2 A(\lambda) e^{-|\lambda| y} - 2|\lambda| B(y) e^{-|\lambda| y} + \lambda^2 B(y) e^{-|\lambda| y} \right]$$

$$\frac{\partial \phi}{\partial x} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x} d\lambda \left[A e^{-|\lambda| y} + B y e^{-|\lambda| y} \right]$$

$$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} d\lambda \left[A e^{-|\lambda| y} + B y e^{-|\lambda| y} \right] \; ; \; @ \; y = 0 \qquad \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} A \, d\lambda = -\delta(x)$$

$$\sigma_{xy} = -\frac{\partial \phi}{\partial x \partial y} = -\int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x} d\lambda \left[-|\lambda| (A + By) e^{-|\lambda| y} + B(y) e^{-|\lambda| y} \right] \quad @ \; y = 0 \quad -\int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x} \left[-|\lambda| A + B \right]$$

1/19/79

Recap:

$$R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx \quad ; \quad f(x) = E(x) + O(x) \qquad E(x) = E(-x),\ \ O(x) = -O(-x)$$

$$= \frac{\cancel{1}^{2}}{2\pi}\left\{ \int_0^{\infty} \cos\xi x\, E(x)\,dx - i\int_0^{\infty} \sin\xi x\, O(x)\,dx \right\}$$

$$R(\xi) = \frac{1}{2}\left[U(\xi) - i V(\xi)\right] \quad \text{with } U(\xi) = \frac{2}{\pi}\int_0^{\infty} \cos\xi x\, E(x)\,dx, \quad V(\xi) = \frac{2}{\pi}\int_0^{\infty} \sin\xi x\, O(x)\,dx$$

$U(\xi) = U(-\xi)$ since $\cos\xi x$ is even in ξ $\qquad V(\xi) = +V(-\xi)$ since $\sin\xi x$ is odd in ξ

$$E(x) = \frac{2}{\pi}\int_0^{\infty} \cos\xi x\, U(\xi)\,d\xi \quad ; \quad O(x) = \frac{2}{\pi}\int_0^{\infty} \sin\xi x\, V(\xi)\,d\xi$$

Consider: $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\,d\xi = \frac{\cancel{1}^{2}}{2}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\cos\xi x + i\sin\xi x\right]\left[U(\xi) - iV(\xi)\right]d\xi$

$$= \frac{1}{2}\frac{2}{2\pi}\int_{-\infty}^{\infty} \overset{\text{even} \times \text{even} = \text{even}}{\cos\xi x\, U(\xi)}\,d\xi + \frac{1}{2\pi}\frac{\cancel{1}^{2}}{2}\int_{-\infty}^{\infty} \overset{\text{odd} \times \text{odd} = \text{even}}{\sin\xi x\, V(\xi)}\,d\xi + i\int_{-\infty}^{\infty} \overset{V(\xi)\cdot\cos\xi x - \sin\xi x\cdot U(\xi)}{\text{odd} \cdot \text{even} = \text{odd}} = 0$$

$$= \frac{2}{2}\frac{2\cdot 2}{2\pi}\int_0^{\infty}\left[\cos\xi x\, U(\xi)\,d\xi + \sin\xi x\, V(\xi)\,d\xi\right]$$

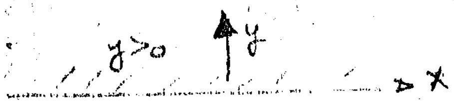
$$= \frac{2}{2}\left\{E(x) + O(x)\right\} = \frac{2}{2}\left\{f(x)\right\} = f(x)$$

$\therefore$ define $R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$ w/ $f(x) = E(x) + O(x)$

then $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\,d\xi$

$R(\xi)$ = Fourier Transform of $f(x)$; $f(x)$ is fourier transform of $R(\xi)$

Return to Half Space problem of 1/17/79

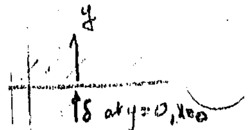


$\sigma_{yy}(y=0) = f(x)$

$\sigma_{xy}(y=0) = g(x)$

$$\phi(x,y) = \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\left[A(\lambda)\, e^{-|\lambda| y} + y B(\lambda)\, e^{-|\lambda| y}\right]$$

Require $f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda$; $g(x) = -i\int_{-\infty}^{\infty} \lambda e^{-i\lambda x}\left\{-|\lambda| A(\lambda) + B(\lambda)\right\} d\lambda$



look at special case where $\sigma_{xy}=0$ and σ_{yy} is the now famous Dirac Delta
take $g(x)=0$ and $\sigma_{yy}(y=0) = f(x) = -\delta(x)$ since $T_y = \sigma_{yj}n_j$ $n_y=-1$ thus
since $\delta(x) = T_y = -\sigma_{yy} \Rightarrow \sigma_{yy} = -\delta(x)$

(note to myself: that T_y the traction in +y direction is $\lim\limits_{\Delta A\to 0}\frac{\Delta F_y}{\Delta A}$ hence direction of traction in this case is in same direction as ΔF_y).

Aside: Delta fns are defined by

$$\begin{cases} (1) & \delta(x-x_0) = 0 \quad \text{for } x\neq x_0 \qquad \delta(x-x_0)=\infty \quad\text{for } x=x_0 \\ (2) & \int_{-\infty}^{\infty}\delta(x-x_0)\,dx = 1 \\ (3) & \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0) \end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\sigma_{yy}(y=0)\,dx = \int_{-\infty}^{\infty}-\delta(x)\,dx = -1 = -\int_{-\infty}^{\infty}T_y\,dx$$

since $f(x) = -\delta(x) = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}A\,d\lambda$ we have to represent $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x}R(\lambda)\,d\lambda$

thus using the fourier transforms $R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda x}\delta(x)\,dX = \frac{1}{2\pi}e^{i\lambda\cdot 0} = \frac{1}{2\pi}$

thus $R(\lambda) = \frac{1}{2\pi}$ if $f(x)=\delta(x)$ thus $\delta(x) = \int_{-\infty}^{\infty}e^{-i\lambda x}R(\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\,d\lambda$

and $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\,d\lambda$ is the Fourier Integral Representation of the δ Function

$$-\delta(x) = f(x) = -\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}A(\lambda)\,d\lambda = \frac{-1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}\,d\lambda \Rightarrow \lambda^2 A = \frac{1}{2\pi}$$ and from $g(x)=0$

$-|\lambda|A + B = 0 \quad \therefore \quad A(\lambda) = \frac{1}{2\pi\lambda^2} \quad B(\lambda) = |\lambda|A(\lambda) = \frac{1}{2\pi|\lambda|}$. Substituting into $\sigma_{yy}(x,y)$

$$\therefore \sigma_{yy}(x,y) = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty}d\lambda\,\cancel{\lambda^2}e^{-i\lambda x}\left[\frac{1}{2\pi\cancel{\lambda^2}}e^{-|\lambda|y} + y\frac{\cancel{|\lambda|}}{2\pi\cancel{|\lambda|}}e^{-|\lambda|y}\right]$$

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{\infty}^{\infty}d\lambda\,\underbrace{e^{-i\lambda x}}_{\cos+i\sin}\underbrace{e^{-|\lambda|y}}_{\text{even fn of }\lambda}\underbrace{\{1+|\lambda|y\}}_{\text{even as a fn of }\lambda}$$. Since only result exists if $\int_{-\infty}^{\infty}$ even fn

only non zero term is

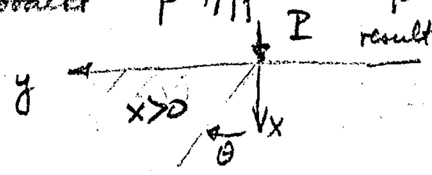
$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty}d\lambda\,\cos\lambda x\,e^{-|\lambda|y}\{1+|\lambda|y\} = 2\cdot\frac{-1}{2\pi}\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}\{1+\lambda y\}\,d\lambda$$

now (1) : $\int_0^{\infty}d\lambda\cos\lambda x\,e^{-\lambda y} = \frac{y}{x^2+y^2}$ this is Laplace transf of $\cos\lambda x$ with $y>0$

(2): $y\int_0^\infty d\lambda \cos\lambda x\, e^{-\lambda y}\cdot\lambda$ since integration is wrt λ can take y outside integral. Notice that $y\frac{\partial}{\partial y}(1) = -(2) = y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$ ∴ $(2) = -y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\sigma_{yy} = -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} - y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

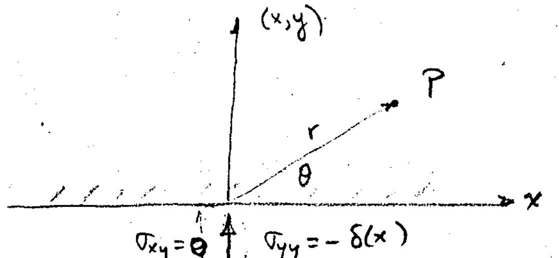
Look @ Timonshenko & Goodier p 97ff



His system looks like this results are same as ours except w/change in x, y and mult by P
Checked out it is correct.

HW find σ_{xy}, σ_{xx} and ϕ for this problem.

1/22/79



$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} = -\frac{2}{\pi}\frac{\sin^3\theta}{r}$$

For $\sigma_{yy}(x, y=0) = -f(x) \overset{= f(x)\,[-\delta(x)]}{}$ $\sigma_{xy}(x, y=0) = 0$ we can get the answer based on our delta fn result. We know:

1. for a point force applied at at point $x = \xi$ on $y = 0$. Then by shift of origin
$$\sigma_{yy}(x, y; \xi) = -\frac{2}{\pi}\frac{y^3}{[(x-\xi)^2+y^2]^2}$$

2. then by the principle of linear superposition with a distributed load f(x)
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

y $f(\xi) = \text{const} = p_0$ $|x| \le l$

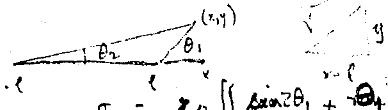
$$= -\frac{2}{\pi}p_0\int_{-l}^{l}\frac{y^3\,d\xi}{((x-\xi)^2+y^2)^2}$$

$$\sigma_{yy} = \cdot\frac{2}{\pi}p_0\int_{x+l}^{x-l}\frac{-y^3\,du}{(u^2+y^2)^2}$$

$$= \frac{2}{\pi}p_0 y^3\left[\frac{u}{2y^2(u^2+y^2)} + \frac{1}{2y^3}\tan^{-1}\frac{u}{y}\right]^{x}_{x+}$$

$$= \frac{2}{\pi}p_0\left\{\frac{y(x-l)}{2((x-l)^2+y^2)} + \frac{1}{2}\tan^{-1}\frac{(x-l)}{y} - \frac{y(x+l)}{2((x+l)^2+y^2)} - \frac{1}{2}\tan^{-1}\left(\frac{x+l}{y}\right)\right\}$$

$\frac{y^3}{[(x-\xi)^2+y^2]^2}$ is the Green's fn for a half space



$\sin\theta_1 \cos\theta_1$

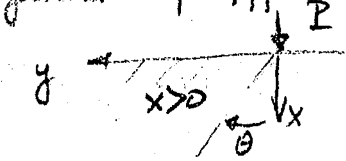
$$\sigma_{yy} = \frac{2}{\pi}p_0\left[\left\{\frac{\sin 2\theta_1}{2} + \theta_1\right\} - \left\{\frac{\sin 2\theta_2}{2} + \theta_2\right\}\right]$$

(2): $y\int_0^\infty d\lambda \cos\lambda x\, e^{-\lambda y}\cdot\lambda$ since integration is wrt λ can take y outside integral. Notice

that $y\frac{\partial}{\partial y}(1) = -(2) = y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \quad \therefore \quad (2) = -y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\sigma_{yy} = -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} - y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

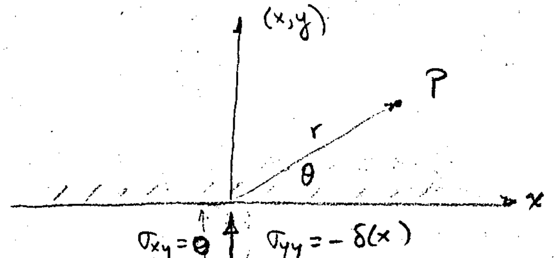
Look @ Timoshenko & Goodier P 97ff



His system looks like this results are same as ours except w/change in x, y and mult by P checked out it is correct.

<u>HW</u> find σ_{xy}, σ_{xx} and ϕ for this problem.

1/22/79



$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} = -\frac{2}{\pi}\frac{\sin^3\theta}{r}$$

For $\sigma_{yy}(x,y=0) = -f(x) \overset{= f(x)\,[-\delta(x)]}{} \quad \sigma_{xy}(x,y=0) = 0$ we can get the answer based on our delta fn result. We know:

1. for a point force applied at at point $x=\xi$ on $y=0$. Then by shift of origin
$$\sigma_{yy}(x,y;\xi) = -\frac{2}{\pi}\frac{y^3}{[(x-\xi)^2+y^2]^2}$$

2. then by the principle of linear superposition with a destributed load $f(x)$
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

if $f(\xi) = \text{const} = p_0 \quad |x|\le l$

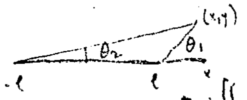
$$= -\frac{2}{\pi}p_0\int_{-l}^{l}\frac{y^3\,d\xi}{((x-\xi)^2+y^2)^2}$$

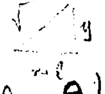
$$\sigma_{yy} = -\frac{2}{\pi}p_0\int_{x+l}^{x-l}\frac{-y^3\,du}{(u^2+y^2)^2}$$

$$= \frac{2}{\pi}p_0 y^3\left[\frac{u}{2y^2(u^2+y^2)} + \frac{1}{2y^3}\tan^{-1}\frac{u}{y}\right]_{x+l}^{x-}$$

$$= \frac{2}{\pi}p_0\left\{\frac{y\cdot(x-l)}{2((x-l)^2+y^2)} + \frac{1}{2}\tan^{-1}\frac{(x-l)}{y} - \frac{y(x+l)}{2((x+l)^2+y^2)} - \frac{1}{2}\tan^{-1}\left(\frac{x+l}{y}\right)\right\}$$

$$\frac{y^3}{[(x-\xi)^2+y^2]^2}$$ is the Green's fn for a half space





$\sin\theta_1, \cos\theta_1$

$$\sigma_{yy} = \frac{2}{\pi}p_0\left[\left\{\frac{\sin 2\theta_1}{2} + \theta_1\right\} - \left\{\frac{\sin 2\theta_2}{2} + \theta_2\right\}\right]$$

8. If $v(x,t)$ and $w(y,t)$ satisfy the heat equation for one-dimensional flow, $v_t = kv_{xx}$, $w_t = kw_{yy}$, show by differentiation that their product $u = vw$ satisfies the heat equation $u_t = k(u_{xx} + u_{yy})$. Use this fact to arrive at the solution of Problem 7.

65. An Application of Fourier Integrals. The face $x = 0$ of a semi-infinite solid $x \geqq 0$ is kept at temperature zero (Fig. 19). Let us find the temperatures $u(x,t)$ in the solid when the initial temperature distribution is $f(x)$, assuming at present that f and f' are sectionally continuous on each finite interval and that f is bounded and absolutely integrable over the positive x axis.

If the solid is considered as a limiting case of a slab $0 \leqq x \leqq c$ as c increases, some condition corresponding to a thermal condition on the face $x = c$ seems to be needed; otherwise the temperature of that face may be increased in any manner as c increases. We require that our function u be bounded, a condition that also implies that there is no instantaneous source of heat on the face $x = 0$ at the instant $t = 0$. Then

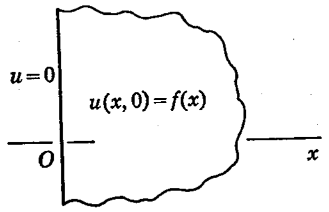


FIG. 19

$$(1)\qquad u_t(x,t) = ku_{xx}(x,t) \qquad (x > 0, t > 0),$$

$$(2)\qquad \begin{aligned} u(0,t) &= 0 && (t > 0),\\ |u(x,t)| &< M && (x > 0, t > 0), \end{aligned}$$

where M is some constant, and

$$(3)\qquad u(x,0) = f(x) \qquad (x > 0).$$

Linear combinations of functions XT will not ordinarily be bounded unless X and T themselves are bounded. Upon separating variables we then have the conditions

$$(4)\qquad \begin{aligned} &X''(x) + \lambda X(x) = 0 && (x > 0),\\ &X(0) = 0, \quad |X(x)| < M_1,\\ &T'(t) + \lambda kT(t) = 0, \quad |T(t)| < M_2 && (t > 0), \end{aligned}$$

where M_1 and M_2 are constants. As pointed out in Sec. 52, the singular eigenvalue problem (4) has continuous eigenvalues $\lambda = \alpha^2$, where α represents *all real positive numbers;* $\sin \alpha x$ are the eigenfunctions. In this case the corresponding functions $T = \exp(-\alpha^2 kt)$ are bounded. The generalized linear combination of the functions XT for all positive α,

$$(5)\qquad u(x,t) = \frac{2}{\pi}\int_0^\infty g(\alpha) \exp(-\alpha^2 kt) \sin \alpha x \, d\alpha,$$

may satisfy all conditions of the boundary value problem if the function g can be determined so that

$$(6)\qquad f(x) = \frac{2}{\pi}\int_0^\infty g(\alpha) \sin \alpha x \, d\alpha \qquad (x > 0).$$

The representation (6) is the Fourier sine integral formula (2), Sec. 52, for our function f if

$$(7)\qquad g(\alpha) = \frac{2}{\pi}\int_0^\infty f(\xi) \sin \alpha\xi \, d\xi \qquad (\alpha > 0).$$

Our formal solution is therefore

$$(8)\qquad u(x,t) = \frac{2}{\pi}\int_0^\infty \exp(-\alpha^2 kt) \sin \alpha x \int_0^\infty f(\xi) \sin \alpha\xi \, d\xi \, d\alpha.$$

We can simplify this result by formally interchanging the order of integration, replacing $2 \sin \alpha x \sin \alpha\xi$ by $\cos \alpha(x - \xi) - \cos \alpha(x + \xi)$, and then applying the integration formula (Problem 12, Sec. 66)

$$(9)\qquad \int_0^\infty \exp(-\alpha^2 b) \cos \alpha r \, d\alpha = \frac{1}{2}\sqrt{\frac{\pi}{b}} \exp\left(-\frac{r^2}{4b}\right) \quad (b > 0).$$

Formula (8) then becomes

$$(10)\qquad u(x,t) = \frac{1}{2\sqrt{\pi kt}}\int_0^\infty f(\xi) \left\{\exp\left[-\frac{(x-\xi)^2}{4kt}\right] - \exp\left[-\frac{(x+\xi)^2}{4kt}\right]\right\} d\xi$$

when $t > 0$. An alternate form of equation (10), obtained by

introducing new variables of integration, is

$$(11)\quad u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-x/(2\sqrt{kt})}^{\infty} \exp(-\eta^2) f(x + 2\eta\sqrt{kt})\, d\eta - \frac{1}{\sqrt{\pi}} \int_{x/(2\sqrt{kt})}^{\infty} \exp(-\eta^2) f(-x + 2\eta\sqrt{kt})\, d\eta.$$

The function u defined by formulas (10) and (11) can be established as a solution of our problem under more relaxed conditions on the function f. Assume only that f is sectionally continuous on some interval $(0,x_0)$ and continuous and bounded when $x \geqq x_0$. Then it can be shown from formula (10) that u satisfies the heat equation (1) because the functions

$$t^{-\frac{1}{2}} \exp[-(x \pm \xi)^2/(4kt)]$$

satisfy that equation and from formula (11) that u satisfies conditions (2). From the two forms (10) and (11) it can be found that $u(x,t) - f(x) \to 0$ as $t \to 0$ at each point x where f is continuous. Details of the proof are tedious.[1]

When $f(x) = 1$, it follows from formula (11) that

$$(12)\quad u(x,t) = \frac{1}{\sqrt{\pi}} \left[\int_{-x/(2\sqrt{kt})}^{\infty} \exp(-\eta^2)\, d\eta - \int_{x/(2\sqrt{kt})}^{\infty} \exp(-\eta^2)\, d\eta \right].$$

In terms of the *error function*

$$(13)\quad \operatorname{erf}(r) = \frac{2}{\sqrt{\pi}} \int_0^r \exp(-\eta^2)\, d\eta,$$

where $\operatorname{erf}(\infty) = 1$, formula (12) can be written

$$(14)\quad u(x,t) = \operatorname{erf}\left(\frac{x}{2\sqrt{kt}}\right).$$

The full verification of this result is not difficult.

66. Temperatures $u(x,t)$ in an Unlimited Medium. As an application of the general Fourier integral formula we shall derive formulas for the temperatures $u(x,t)$ in a medium that occupies all space, when the initial temperature distribution is

---

[1] A similar verification is carried out on pp. 35ff. of Carslaw and Jaeger's book "Conduction of Heat in Solids," 1947.

$f(x)$. We assume that f is bounded and, for the present, that it satisfies conditions under which it is represented by its Fourier integral formula. The boundary value problem consists of a boundedness condition $|u(x,t)| < M$ and the conditions

$$(1)\quad u_t(x,t) = ku_{xx}(x,t) \qquad (-\infty < x < \infty, t > 0),$$

$$(2)\quad u(x,0) = f(x) \qquad (-\infty < x < \infty).$$

Separation of variables leads to the singular eigenvalue problem

$$X''(x) + \lambda X(x) = 0, \qquad |X(x)| < M_1 \quad (-\infty < x < \infty),$$

whose eigenvalues are $\lambda = \alpha^2$, where α is real, and to two linearly independent eigenfunctions $\cos \alpha x$ and $\sin \alpha x$ corresponding to each nonzero value of α. Negative values of α produce no additional eigenfunctions, so we use only the values $\alpha \geqq 0$.

Our generalized linear combination of functions XT becomes

$$(3)\quad u(x,t) = \int_0^{\infty} \exp(-\alpha^2 kt)[A(\alpha)\cos \alpha x + B(\alpha) \sin \alpha x]\, d\alpha.$$

The coefficients A and B are to be determined so that, when $t = 0$, the integral here represents $f(x)$ $(-\infty < x < \infty)$. According to equations (7) and (8) of Sec. 50 and the Fourier integral theorem (Sec. 51), the representation is valid if

$$A(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \cos \alpha\xi\, d\xi, \qquad B(\alpha) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\xi) \sin \alpha\xi\, d\xi.$$

Therefore formula (3) becomes

$$(4)\quad u(x,t) = \frac{1}{\pi} \int_0^{\infty} \exp(-\alpha^2 kt) \int_{-\infty}^{\infty} f(\xi) \cos \alpha(x - \xi)\, d\xi\, d\alpha.$$

If we formally invert the order of integration here, the integration formula (9) of Sec. 65 can be used to write equation (4) in the form

$$(5)\quad u(x,t) = \frac{1}{2\sqrt{\pi kt}} \int_{-\infty}^{\infty} f(\xi) \exp\left[-\frac{(x-\xi)^2}{4kt}\right] d\xi \quad (t > 0).$$

An alternate form of this formula is

$$(6)\quad u(x,t) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(x + 2\eta\sqrt{kt}) \exp(-\eta^2)\, d\eta.$$

Forms (5) and (6) can be established by assuming only that f is sectionally continuous over some bounded interval, $|x| < c$,

[redacted]

Problem Set #2

1) $y'' = xy$ @ $x = 1$

$y'' - xy = 0$ $P_0(x) = 1$ $P_1(x) = 0$ $P_2(x) = -x$ → $x = 1$ is an ordinary point.

$$y(x) = \sum_{n=0}^{\infty} A_n(x-1)^n \qquad y'(x) = \sum_{n=1}^{\infty} nA_n(x-1)^{n-1} \qquad y''(x) = \sum_{n=2}^{\infty} n(n-1)A_n(x-1)^{n-2} \longrightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)A_n(x-1)^{n-2} - x\sum_{n=0}^{\infty} A_n(x-1)^n = 0 \longrightarrow \sum_{n=2}^{\infty} n(n-1)A_n(x-1)^{n-2} - (x-1)\sum_{n=0}^{\infty} A_n(x-1)^n - \sum_{n=0}^{\infty} A_n(x-1)^n = 0$$

$$\longrightarrow \sum_{n=2}^{\infty} n(n-1)A_n(x-1)^{n-2} - \sum_{n=0}^{\infty} A_n(x-1)^{n+1} - \sum_{n=0}^{\infty} A_n(x-1)^n = 0 \longrightarrow$$

$$\sum_{n=2}^{\infty} n(n-1)A_n(x-1)^{n-2} - \sum_{n=3}^{\infty} A_{n-3}(x-1)^{n-2} - \sum_{n=2}^{\infty} A_{n-2}(x-1)^{n-2} = 0 \longrightarrow$$

$$2(1)A_2 - A_0 + \sum_{n=3}^{\infty} (x-1)^{n-2}\left[n(n-1)A_n - A_{n-3} - A_{n-2}\right] = 0 \longrightarrow$$

$$2A_2 - A_0 = 0 \longrightarrow A_2 = \frac{A_0}{2}$$

$$6A_3 - A_0 - A_1 = 0 \longrightarrow A_3 = \frac{A_0 + A_1}{6} = \frac{A_0}{6} + \frac{A_1}{6}$$

$$12A_4 - A_1 - A_2 = 0 \longrightarrow A_4 = \frac{A_1 + A_2}{12} = \frac{A_1}{12} + \frac{A_0}{24}$$

$$A_5 = \frac{A_2 + A_3}{5 \times 4} = \frac{A_0}{40} + \frac{A_0}{120} + \frac{A_1}{120} = \frac{A_0}{30} + \frac{A_1}{120}$$

$\vdots$

$$A_n = \frac{A_{n-3} + A_{n-2}}{n(n-1)}$$

✓

$$\longrightarrow y(x) = A_0 + A_1(x-1) + A_2(x-1)^2 + \sum_{n=3}^{\infty} A_n(x-1)^n \longrightarrow$$

$$y(x) = A_0 + A_1(x-1) + A_0\left[\frac{(x-1)^2}{2} + \frac{(x-1)^3}{6} + \frac{(x-1)^4}{24} + \frac{(x-1)^5}{30} + \cdots\right]$$

$$+ A_1\left[\frac{(x-1)^3}{6} + \frac{(x-1)^4}{12} + \frac{(x-1)^5}{120} + \cdots\right]$$

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3) $2(x-2)^2 xy'' + 3xy' + (x-2)y = 0$ $\qquad P_0(x) = 2(x-2)^2 x \quad P_1(x) = 3x \quad P_2(x) = (x-2)$

@ $x = 2$ $\quad \lim\limits_{x \to x_0} \frac{P_1}{P_0}(x - x_0) = \lim\limits_{x \to 2} \frac{3x}{2x(x-2)^2}(x-2) =$ Not defined $\rightarrow$ irregular Singularity

@ $x = 0$ $\quad \lim\limits_{x \to 0} \frac{3x}{2x(x-2)^2} x = 0$, $\lim \frac{(x-2)}{2x(x-2)^2} . x^2 = 0$ $\rightarrow$ regular singularity

@ $x = 5$ $\quad P_0(5) \neq 0$ $\rightarrow$ ordinary point.

@ $x = \infty$ $\quad t = \frac{1}{x} \quad x \to \infty , \; t \to 0$

$$2\left(\frac{1}{t}-2\right)^2 \frac{1}{t}\left[t^4 y'' + 2t^3 y'\right] + \frac{3}{2}\left[-t^2 y'\right] + \left(\frac{1}{t}-2\right)y = 0 \rightarrow$$

$$(8t^3 - 4t^2 + 2t)y'' + (16t - 11t + 4)y' + \left(\frac{1}{t} - 2\right)y = 0$$

$$\lim_{t \to t_0} \frac{P_1(t)}{P_0(t)}(t - t_0) = \lim_{t \to 0} \frac{16t - 11t + 4}{8t^3 - 4t^2 + 2t} , t = 2$$

✓

$\rightarrow$ regular Singularity

$$\lim_{t \to t_0} \frac{P_2(t)}{P_0(t)}(t - t_0)^2 = \lim \frac{\frac{1}{t} - 2}{8t^3 - 4t^2 + 2t} . t^2 = \frac{1}{2}$$

10

4) $(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0$ $\quad$ @ $x = \infty$ $\quad t = 1/x \quad x \to \infty \quad t \to 0$

$$\frac{d}{dx} = -t^2 \frac{d}{dt} \qquad \frac{d^2}{dx^2} = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}$$

$$\left(1 - \frac{1}{t^2}\right)(t^4 y'' + 2t^2 . y') - \frac{2}{t}(-t^2)y' + \alpha(\alpha+1)y = 0 \rightarrow t^2(t^2-1)y'' + 2t^3 y' + \alpha(\alpha+1)y = 0$$

$$\lim_{t \to t_0} \frac{P_1(t)}{P_0(t)}(t - t_0) = \lim_{t \to 0} \frac{2t^3}{t^4 - t^2} . t = 0 \qquad \lim_{t \to t_0} \frac{P_2(t)}{P_0(t)}(t - t_0)^2 = \lim \frac{\alpha(\alpha+1)}{t^4 - t^2} . t^2 = -\alpha(\alpha+1)$$

$\rightarrow$ regular Singular Point.

$$y(t) = \sum_{n=0}^{\infty} A_n t^{n+r}$$

[redacted]

$$\rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1)A_n t^{n+r-2} \cdot t^2(t^2-1) + \sum_{n=0}^{\infty} 2(n+r)A_n t^{n+r-1} \cdot t^3 + \sum_{n=0}^{\infty} A_n \alpha.(\alpha+1)t^{n+r} = 0$$

$$\rightarrow \sum_{n=0}^{\infty} (n+r)(n+r-1)A_n\left(t^{n+r+2} - t^{n+r}\right) + \sum_{n=0}^{\infty} 2(n+r)A_n t^{n+r+2} + \sum_{n=0}^{\infty} A_n \alpha(\alpha+1)t^{n+r} = 0$$

$$\rightarrow \sum_{n=0}^{\infty} A_n t^{n+r+2}\left((n+r)(n+r-1) + 2(n+r)\right) + \sum_{n=0}^{\infty} t^{n+r}A_n\left(\alpha(\alpha+1) - (n+r)(n+r-1)\right) = 0$$

$$\rightarrow A_0(\alpha(\alpha+1) - r(r-1))t^r + A_1(\alpha(\alpha+1) - r(r+1))t^{r+1} + \sum_{n=2}^{\infty} t^{n+r}A_n\left(\alpha(\alpha+1) - (n+r)(n+r-1)\right) +$$

$$\sum_{n=2}^{\infty} A_{n-2}t^{n+r}(n+r-2)(n+r-1) = 0$$

All coeft's for t^{n+r} should be zero, so we will have:

(indicial eq.) $A_0[\alpha(\alpha+1) - r(r-1)] = 0$ (1)

$A_1[\alpha(\alpha+1) - r(r+1)] = 0$ (2)

(Recurrence Formula) $A_n = \dfrac{(n+r-2)(n+r-1)}{(n+r)(n+r-1) - \alpha(\alpha+1)} A_{n-2}$ (3)

(1) $\rightarrow \alpha(\alpha+1) - r(r-1) = 0 \rightarrow r^2 - r - \alpha(\alpha+1) = 0 \rightarrow r_{1,2} = \dfrac{1 \pm \sqrt{1+4\alpha(\alpha+1)}}{2}$

Based on values of α we could have any of 3 Conditions discussed in class, whic we will investigate later. But considering the values r_1 and r_2 it could be seen that the only for eq (2) to be true is $A_1 = 0$

Case I Consider $r_1 \neq r_2$ and $|r_1 - r_2| = \sqrt{1+4\alpha(\alpha+1)}$ is not an integer:

$y_1(t) = t^{r_1}\left[1 + \sum_{n=1}^{\infty} A_n t^n\right]$ $\quad A_n = \dfrac{(n+r_1-2)(n+r_1-1)}{(n+r_1)(n+r_1-1) - \alpha(\alpha+1)} A_{n-2}$ and $A_1 = 0$

$y_2(t) = t^{r_2}\left[1 + \sum_{n=1}^{\infty} B_n t^n\right]$ $\quad B_n = \dfrac{(n+r_2-2)(n+r_2-1)}{(n+r_2)(n+r_2-1) - \alpha(\alpha+1)} B_{n-2}$ and $B_1 = 0$

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Case II Consider $r_1 \neq r_2$ and $|r_1 - r_2| = \sqrt{1+4\alpha(\alpha+1)}$ is an integer

$$y_1(t) = t^{r_1}\left[1 + \sum_{n=1}^{\infty} A_n t^n\right] \qquad A_n = \frac{(n+r_1-2)(n+r_1-1)}{(n+r_1)(n+r_1-1)-\alpha(\alpha+1)} A_{n-2}, \quad A_1 = 0$$

$$y_2(t) = t^{r_2}\left[1 + \sum_{n=1}^{\infty} B_n t^n\right] + A y_1(t) \ln(t) \qquad B_n = \frac{(n+r_2-2)(n+r_2-1)}{(n+r_2)(n+r_2-1)-\alpha(\alpha+1)}, \quad B_1 = 0$$

Case III Consider $r_1 = r_2$ (it means $\alpha = -\frac{1}{2}$)

$$y_1(t) = t^{r_1}\left[1 + \sum_{n=1}^{\infty} A_n t^n\right] \qquad A_n = \text{Same as above}$$

$$y_2(t) = y_1(t) \ln(t) + t^{r_2} \sum_{n=1}^{\infty} B_n t^n \qquad B_n = \text{Same as above}$$

in all cases the final answer would be $y_{TOT} = c_1 y_1(t) + c_2 y_2(t)$ and all "t"s shall be changed to "$1/x$".

✓

Problem set #3

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3.5) $\nabla^2\phi = \phi_{xx} + \phi_{yy} + \phi_{zz} = 0$ (1)

(2) $\frac{\partial\phi}{\partial t} + g\eta = 0$ @ $z = 0$ (4) $\frac{\partial\phi}{\partial x} = 0$ @ $x = 0, a$ (6) $\frac{\partial\phi}{\partial z} = 0$ @ $z = -h$

(3) $\frac{\partial\eta}{\partial t} - \frac{\partial\phi}{\partial z} = 0$ @ $z = 0$ (5) $\frac{\partial\phi}{\partial y} = 0$ @ $y = 0, b$

$\phi = \phi(X, y, z, T) = X(x)\,Y(y)\,Z(z)\,T(t) \longrightarrow X''YZT + Y''XZT + Z''XYT = 0 \xrightarrow{\times 1/\phi}$

$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$ Sum of three independent functions are equal to zero, so we can say all of them are constants. $\longrightarrow$

$\frac{X''}{X} = -K_x^2$ $\frac{Y''}{Y} = -K_y^2$ $\frac{Z''}{Z} = K_z^2$ $K_z^2 = K_x^2 + K_y^2$ $\longrightarrow$

$X = A\cos(K_x x) + B\sin(K_x x)$ $\phi = XYZT$

$Y = C\cos(K_y y) + D\sin(K_y y)$

$Z = E\sinh(K_z z) + F\cosh(K_z z)$

$\frac{\partial\phi}{\partial x} = YZT\left[-A\sin(K_x x) + B\cos(K_x x)\right]K_x \Big|_{x=0} = 0 \longrightarrow B = 0$ /

$\frac{\partial\phi}{\partial x}\Big|_{x=a} = 0 \longrightarrow K_x = n\pi/a$ ✓

$\longrightarrow K_z^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$

$\frac{\partial\phi}{\partial y}\Big|_{y=0} = 0 \longrightarrow D = 0$ / $\frac{\partial\phi}{\partial y}\Big|_{y=b} = 0 \longrightarrow K_y = \frac{m\pi}{b}$ ✓

$\phi = \cos\left(\frac{n\pi x}{a}\right)\cos\left(\frac{m\pi y}{b}\right)\cdot\left[E\sinh(K_z z) + F\cosh(K_z z)\right]$

~~$\frac{\partial}{\partial t}(2) - g\times(3) \longrightarrow \frac{\partial^2\phi}{\partial t^2} + g\frac{\partial\phi}{\partial z} = 0 \longrightarrow XYZ\ddot{T} + gK_z XYZT = 0 \longrightarrow \frac{\ddot{T}}{T} = -gK_z$~~

~~$\frac{\partial Z}{\partial z} = K_z Z$~~

[redacted]

~~$\frac{\ddot{T}}{T} - g k_z \rightarrow \omega^2 \, g k_z$~~

(6) $\frac{\partial \phi}{\partial z} = 0$ @ $z = -h$ $\rightarrow$ $XYT \cdot k_z \left[E \cosh(k_z(-h)) + F \sinh(k_z \cdot (-h)) \right] = 0$ $\rightarrow$

$E \cosh(k_z h) = F \sinh(k_z h)$ $\rightarrow$ $\tanh(k_z \cdot h) = \frac{E}{F}$

$\underline{z = 0 \rightarrow E}$

$\frac{\partial}{\partial t}(2) - g \times (3)$ $\rightarrow$ $\frac{\partial^2 \phi}{\partial t^2} + g \frac{\partial \phi}{\partial z} = 0 \Big|_{z=0}$ $\rightarrow$ $XYZ\ddot{T} + gXYTk_z\left[E \cosh(k_z z) + F \sinh(k_z z)\right] = 0$

$\rightarrow z = 0 \rightarrow F$

$\rightarrow XYF\ddot{T} + gXYTk_z E = 0$ $\rightarrow$ $\frac{\ddot{T}}{T} = -\frac{E}{F} \cdot k_z \cdot g$ $\rightarrow$

natural frequency $\omega_{mn}^2 = g k_z \frac{E}{F} = g k_z \tanh(k_z \cdot h)$ or.

$$k_z^2 = \frac{\pi^2 n^2}{b^2} + \frac{\pi^2 m^2}{a^2}$$

$T = P \sin \omega_{mn} t + Q \cos \omega_{mn} t$

$g\eta + \frac{\partial \phi}{\partial t} = 0 \Big|_{z=0}$ $\rightarrow$ $\eta = -\frac{1}{g} \cdot \frac{\partial \phi}{\partial t}$ $\rightarrow$ $\eta(x,y,t) = -\frac{F}{g} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cdot \omega_{mn}\left[P \cos \omega_{mn} t - Q \sin \omega_{mn} t\right]$

$\rightarrow \eta_{mn}(x,y,t) = -\frac{F_{mn}}{g} \cos\left(\frac{n\pi x}{a}\right) \cos\left(\frac{m\pi y}{b}\right) \cdot \omega_{mn} \cdot \left[P_{mn} \cos \omega_{mn} t - Q_{mn} \sin \omega_{mn} t\right]$ ✓

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$\boxed{P120/12}$ $\quad \frac{\partial^2 u}{\partial x^2} = \alpha \frac{\partial u}{\partial t} \qquad u = u(x,t) = X(x).T(t) \qquad u(x,t=0) = f(x)$

$u(x=0,t) = 0$

$u(x=L,t) = 0$

$$\rightarrow X''T = \frac{1}{\alpha}.\dot{T}X \rightarrow \frac{X''}{X} = \frac{1}{\alpha}.\frac{\dot{T}}{T} = -k^2 \rightarrow$$

$$X'' + k^2 X = 0 \rightarrow X = A\sin kx + B\cos kx \quad ✓$$

$$\dot{T} + \alpha k^2 T = 0 \rightarrow T = Ce^{-\alpha k^2 t} \quad ✓$$

$$u(x=0,t) = 0 \rightarrow A\sin k(0) + B\cos k(0) = 0 \rightarrow B = 0 \quad ✓$$

$$u(x=L,t) = 0 \rightarrow A\sin kL = 0 \rightarrow k = \frac{n\pi}{L} \quad ✓$$

$$\rightarrow u_n(x,t) = \bar{A}_n \sin(\frac{n\pi}{L}x).e^{-\alpha(\frac{n\pi}{L})^2 t} \rightarrow u(x,t) = \sum_{n=1}^{\infty} \bar{A}_n \sin(\frac{n\pi x}{L})\, e^{-\alpha(\frac{n\pi}{L})^2 t}$$

$$u(x,t=0) = \sum_{n=1}^{\infty} \bar{A}_n \sin(\frac{n\pi x}{L}) = f(x) \rightarrow \bar{A}_n = \frac{2}{L}\int_0^L f(x).\sin\frac{n\pi x}{L}.dx \quad ✓$$

$$\rightarrow u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x).\sin\frac{n\pi x}{L}\,dx\right].\sin(\frac{n\pi x}{L}).e^{-\alpha(\frac{n\pi}{L})^2 t} \quad ✓ \quad 10$$

$\boxed{P139/12}$ $\quad \frac{\partial u}{\partial t} = K.\frac{\partial^2 u}{\partial x^2} - hu + \frac{KI^2}{K\sigma A^2} \qquad 0<x<L \quad t>0 \qquad u = u(x,t)$

(a) $u(x=0,t) = 0 \qquad u(x=L,t) = 0 \qquad u(x,t=0) = 0$

we assume $u(x,t) = v(x,t) + \psi(x) \rightarrow \frac{\partial v}{\partial t} = K\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}\right] - h\left[v + \psi\right] + \frac{KI^2}{K\sigma A^2}$

$$\rightarrow \left[K\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - hv\right] + \underbrace{\left[K\frac{\partial^2 \psi}{\partial x^2} - h\psi + \frac{KI^2}{K\sigma A^2}\right]} = 0 \xrightarrow{KI^2/K\sigma A^2 = Q}$$

we'll choose ψ in such a way that $K\frac{d^2\psi}{dx^2} - h\psi + Q = 0$ and

@ $x=0$ $\psi(x) = 0$

@ $x=L$ $\psi'(x) = 0$ $\rightarrow$

✓

) $\psi'' - \frac{h}{K}\psi + \frac{Q}{K} = 0$ ✓

) $K\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - hv = 0$ ✓

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since Q = const use $\psi_p = D$ and $\quad -\frac{h}{K}D = -\frac{Q}{K}$

$$D = \frac{Q}{h}$$

$$\psi'' - \frac{h}{K}\psi = -\frac{Q}{K} \quad \xrightarrow{\frac{h}{K} = p^2} \quad \psi'' - p^2\psi = -\frac{Q}{K} \qquad \psi = \psi_h + \psi_p$$

$$\psi_h = c_1 e^{px} + c_2 e^{-px} \qquad W(\psi_1, \psi_2) = \begin{vmatrix} \psi_1 & \psi_1' \\ \psi_2 & \psi_2' \end{vmatrix} = \begin{vmatrix} e^{px} & pe^{px} \\ e^{-px} & -pe^{-px} \end{vmatrix} = -2p$$

$$\psi_p = u_1(x)\psi_1(x) + u_2(x)\psi_2(x) \quad \longrightarrow \quad \psi_p = \frac{Q}{Kp^2}$$

$$u_1(x) = \int^x \frac{Q/K \cdot e^{-px}}{-2p} = \frac{Q}{2K}\cdot\frac{1}{p^2}\cdot e^{-px} \qquad \psi(x) = c_1 e^{px} + c_2 e^{-px} + \frac{Q}{2Kp^2}$$

$$u_2(x) = \int^x \frac{-Q/K \cdot e^{px}}{-2p} = \frac{Q}{2K}\cdot\frac{1}{p^2}\cdot e^{px}$$

@ $x = 0$ $\quad \psi(x) = 0$

@ $x = L$ $\quad \psi(L) = 0$ $\quad \longrightarrow$

$$\begin{cases} c_1 + c_2 = -\frac{Q}{Kp^2} \\ c_1 e^{pL} + c_2 e^{-pL} = -\frac{Q}{Kp^2} \end{cases} \longrightarrow \begin{cases} -c_1 e^{pL} - c_2 e^{pL} = \frac{Q}{Kp^2}\cdot e^{pL} \\ c_1 e^{pL} + c_2 e^{-pL} = -\frac{Q}{Kp^2} \end{cases} \longrightarrow \begin{matrix} c_2 = \frac{Q}{Kp^2}(e^{pL} - 1)\cdot\frac{1}{e^{-pL} - e^{pL}} \\ c_1 = -\frac{Q}{Kp^2}(e^{-pL} - 1)\cdot\frac{1}{e^{-pL} - e^{pL}} \end{matrix}$$

$$\longrightarrow \psi(x) = \frac{Q}{Kp^2}\left[1 - \frac{e^{px}(e^{-pL} - 1)}{e^{-pL} - e^{pL}} + \frac{e^{-px}(e^{pL} - 1)}{e^{-pL} - e^{pL}}\right] =$$

$$= \frac{Q}{Kp^2}\left[1 - \frac{e^{px} - e^{-p(L-x)} - e^{-px} + e^{+p(L-x)}}{e^{pL} - e^{-pL}}\right] = \frac{Q}{Kp^2}\left[1 - \frac{\sinh(px) + \sinh(p(L-x))}{\sinh(pL)}\right]$$

$$\longrightarrow \boxed{\psi(x) = \frac{KI^2}{Kh\sigma A^2}\left[1 - \frac{\sinh(\sqrt{\frac{h}{K}}x) + \sinh(\sqrt{\frac{h}{K}}(L-x))}{\sinh(\sqrt{\frac{h}{K}}L)}\right]} \quad ✓$$

$$K\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - hv = 0$$

$v(x,t)$ @ $x = 0$ $= 0$

$v(x,t)$ @ $x = L$ $= 0$

$v(x, t = 0) = -\psi(x)$

$v(x,t) = X(x)T(t) \longrightarrow$

$$K\cdot X''T - X\dot{T} - hXT = 0 \longrightarrow K\cdot\frac{X''}{X} - \frac{\dot{T}}{T} - h = 0 \longrightarrow K\frac{X''}{X} = \frac{\dot{T}}{T} + h = -\alpha \longrightarrow$$

$$\begin{cases} \dot{T} + (\alpha + h)T = 0 & \longrightarrow T(t) = c_1 e^{-(\alpha+h)t} \\ X'' + \frac{\alpha}{K}X = 0 & \longrightarrow X(x) = A\sin\sqrt{\frac{\alpha}{K}}x + B\cos\sqrt{\frac{\alpha}{K}}x \end{cases}$$

Cont'd $\longrightarrow$

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$v(x,t)$ @ $x=0$ $=0$ $\longrightarrow$ $X(0) = A \sin\sqrt{\frac{\alpha}{k}}(0) + B \cos\sqrt{\frac{\alpha}{k}}(0) = 0$ $\longrightarrow B=0$ ✓

$v(x,t)$ @ $x=L$ $=0$ $\longrightarrow$ $X(L) = A \sin\sqrt{\frac{\alpha}{k}}L = 0 \longrightarrow \sqrt{\frac{\alpha}{k}}L = n\pi \longrightarrow \alpha = \frac{n^2\pi^2}{L^2 K}$ ✓

$$\longrightarrow v_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-(\frac{n^2\pi^2}{KL^2}+h)t} \cdot A_n \cdot \sin(\frac{n\pi x}{L}) \longrightarrow v_n(x,t) = \sum_{n=1}^{\infty} \bar{A}_n e^{-(\frac{n^2\pi^2}{KL^2}+h)t} \sin\frac{n\pi x}{L}$$

$v(x,t)$ @ $t=0$ $= -\psi(x)$ $\longrightarrow$ $-\psi(x) = \sum_{n=1}^{\infty} \bar{A}_n \sin\frac{n\pi x}{L} \longrightarrow \bar{A}_n = -\frac{2}{L}\int_0^L \psi(x) \sin\frac{n\pi x}{L} \cdot dx$

$$\psi(x) = \frac{KI^2}{kh\sigma A^2}\left[1 - \frac{\sinh(\sqrt{\frac{h}{k}}x) + \sinh(\sqrt{\frac{h}{k}}(L-x))}{\sinh(\sqrt{\frac{h}{k}}L)}\right]$$

$$\longrightarrow \quad \bar{A}_n = -\frac{2}{L}\int_0^L \psi(x) \cdot \sin\frac{n\pi x}{L}\, dx$$ ✓

$$v_n(x,t) = \sum_{n=1}^{\infty} \bar{A}_n \cdot e^{-(\frac{n^2\pi^2}{KL^2}+h)t} \sin\frac{n\pi x}{L}$$

$$u(x,t) = v_n(x,t) + \psi(x)$$ ✓ 20

$\boxed{P139/15}$ $\quad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad 0<x<L \quad t>0$

$y(0,t) = 0$

$E \cdot \frac{\partial y(L,t)}{\partial x} = F$

$y(x,0) = y_t(x,0) = 0$

$y(x,t) = v(x,t) + \psi(x)$

$$\longrightarrow \frac{\partial^2 v}{\partial t^2} = c^2\left(\frac{\partial^2 v}{\partial x^2} + \frac{d^2\psi}{dx^2}\right) \longrightarrow c^2\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} + \underbrace{c^2\frac{d^2\psi}{dx^2}}_{} = 0$$

we will choose $\psi(x)$ in such a way that

$\frac{d^2\psi}{dx^2} = 0$ $\longleftarrow$

$\psi(x)$ @ $x=0$ $=0$

$\frac{d\psi}{dx}$ @ $x=L$ $= \frac{F}{E}$

$\psi(x) = Ax + B$

$\psi(0) = 0 \longrightarrow B = 0$ ✓

$\left.\frac{d\psi}{dx}\right|_{x=L} \longrightarrow A = \frac{F}{E}$ ✓

$\longrightarrow \psi(x) = \frac{F}{E}x$ ✓

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$$c^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = 0$$

$v(x,t)$ @ $x=0$ $=0$

$E \cdot \frac{\partial v(x,t)}{\partial x}$ @ $x=L$ $=0$

$v(x,t) = X(x) \cdot T(t)$

$v(x,0) = -\Psi(x)$ $\quad v_t(x,0) = 0$

↓

$$c^2 X'' T - \ddot{T} X = 0 \rightarrow c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} = -\omega^2 \rightarrow$$

$$\ddot{T} + \omega^2 T = 0 \rightarrow T = A \sin\omega t + B \cos\omega t$$ ✓

$$X'' + \frac{\omega^2}{c^2} X = 0 \rightarrow X = \bar{C} \sin\frac{\omega}{c}x + D\cos\frac{\omega}{c}x$$ ✓

$v(x,t)$ @ $x=0$ $=0 \rightarrow D = 0$ ✓

$\frac{\partial v}{\partial x}$ @ $x=L$ $=0 \rightarrow \frac{\omega L}{c} = (2n+1) \cdot \frac{\pi}{2} \rightarrow \omega_n = \frac{(2n+1)\pi c}{2L}$ ✓

$v_t(x,t)$ @ $t=0$ $=0 \rightarrow A = 0$ ✓

$v(x,t)$ @ $t=0$ $= -\Psi(x)$ $\therefore$ $v_n(x,t) = \sum_{n=0}^{\infty} \bar{B}_n \cos\frac{(2n+1)\pi c t}{2L} \cdot \sin\frac{(2n+1)\pi x}{2L}$

$$v_n(x,0) = \sum_{n=0}^{\infty} \bar{B}_n \cdot \sin\frac{(2n+1)\pi x}{2L} \rightarrow$$

$$\bar{B}_n = \frac{2}{L}\int_0^L -\Psi(x) \cdot \sin\frac{(2n+1)\pi x}{2L} = \frac{2}{L}\int_0^L -\frac{F}{E} x \sin\frac{(2n+1)\pi x}{2L}\, dx \rightarrow$$

$$\bar{B}_n = -\frac{2F}{EL}\int_0^L \underset{\downarrow\, u}{x} \underbrace{\sin\frac{(2n+1)\pi x}{2L}\, dx}_{dv} = -\frac{2F}{EL}\left[x \cdot \frac{-2L}{(2n+1)\pi} \cos\frac{(2n+1)\pi x}{2L} \Bigg|_0^L + \int_0^L \frac{2L}{(2n+1)\pi} \cdot \cos\frac{(2n+1)\pi x}{2L}\, dx \right]$$

$$= -\frac{2F}{EL} \cdot \frac{2L}{(2n+1)\pi} \cdot \frac{2L}{(2n+1)\pi} \sin\frac{(2n+1)\pi x}{2L} \Bigg|_0^L = -\frac{8FL^2(-1)^n}{E\pi^2(2n+1)^2} \rightarrow$$

$$y_n(x,t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 8FL}{\pi^2 E(2n+1)^2} \cdot \sin\frac{(2n+1)\pi x}{2L} \cdot \cos\frac{(2n+1)\pi c t}{2L} + \frac{F}{E}x$$

✓ 19/20

[redacted] [redacted]

:283/1

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{K}{R}\, g(x,t) \qquad -\infty < x < +\infty \qquad t > 0$$

$$u(x,0) = f(x) \qquad -\infty < x < +\infty$$

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\omega x}\, dx = \tilde{u}(\omega;t)$$

$$\frac{d\tilde{u}}{dt} = K\left[-\omega^2 \tilde{u}(\omega;t)\right] + \frac{K}{R}\tilde{g}(\omega;t) \longrightarrow \frac{d\tilde{u}}{dt} + K\omega^2\tilde{u} = \frac{K}{R}\tilde{g}(\omega,t) \longrightarrow$$

$$\tilde{u}(\omega;t) = c\, e^{-k\omega^2 t} + \int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})\, e^{-k\omega^2(t-\bar{t})}\, d\bar{t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{u}(\omega,t)\, e^{i\omega x}\, d\omega$$

$$\mathcal{F}\{i.c.\} \longrightarrow \tilde{u}(\omega,0) = \tilde{f}(\omega) \longrightarrow \tilde{u}(\omega;t) = \tilde{f}(\omega)\, e^{-k\omega^2 t} + \int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})\, e^{-k\omega^2(t-\bar{t})}\, d$$

$$\longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{k\pi t}}\, e^{-(x-u)^2/4kt}\cdot du + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})\, e^{-k\omega^2(t-\bar{t})}\, d\bar{t}\right\} e^{+i\omega x}\, d\omega$$ ✓

$$\text{if } g(x,t) \equiv 0 \longrightarrow \tilde{g}(\omega,t) \equiv 0 \longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{k\pi t}}\cdot e^{\frac{-(x-u)^2}{4kt}}\cdot du$$ ✓

$$(i)\ f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} \longrightarrow u(x,t) = \int_{-a}^{+a} \frac{1}{2\sqrt{k\pi t}}\, e^{\frac{-(x-u)^2}{4kt}}\cdot du$$ ✓

$$\text{let } \frac{x-u}{2\sqrt{kt}} = v \longrightarrow du = -2\sqrt{kt}\cdot dv \longrightarrow u(x,t) = -\int_{(x-a)/2\sqrt{kt}}^{(x+a)/2\sqrt{kt}} e^{-v^2}\, dv \left(-\frac{1}{\sqrt{\pi}}\right)$$

$$\longrightarrow u(x,t) = +\frac{1}{\sqrt{\pi}}\int_0^{(x+a)/2\sqrt{kt}} e^{-v^2}\, dv - \frac{1}{\sqrt{\pi}}\int_0^{(x-a)/2\sqrt{kt}} e^{-v^2}\, dv \longrightarrow$$

$$u(x,t) = -\frac{1}{2}\,\mathrm{erf}\left(\frac{x-a}{2\sqrt{kt}}\right) + \frac{1}{2}\,\mathrm{erf}\left(\frac{x+a}{2\sqrt{kt}}\right)$$ ✓

Ⓑ

[redacted]

(ii) $f(x) = \begin{cases} 0 & |x|<a \\ 1 & |x|>a \end{cases}$ [redacted] [redacted] $\longrightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}} \int_{-\infty}^{-a} e^{-(x-u)^2/4\kappa t} du + \frac{1}{2\sqrt{\kappa\pi t}} \int_{+a}^{\infty} e^{-(x-u)^2/4\kappa t} du$

$$\longrightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}} \left[\int_{-\infty}^{+\infty} e^{-(x-u)^2/4\kappa t} du - \int_{-a}^{a} e^{-(x-u)^2/4\kappa t} du \right] \quad \xrightarrow{\text{like Part (i)}} \quad v = \frac{x-u}{2\sqrt{\kappa t}}$$

$$u(x,t) = +\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2} dv - \boxed{\frac{1}{2\sqrt{\kappa\pi t}} \int_{-a}^{a} e^{-(x-u)^2/4\kappa t} du} \longrightarrow$$

we know the answer from Part (i)

$$u(x,t) = +1 + \frac{1}{2} \operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right) \quad ✓$$

20/20

P.284 /4 $\quad \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{\kappa}{K} g(x,t) \qquad x>0 \quad t>0$

$u(0,t) = f_1(t) \qquad u(x,0) = f(x)$

$$\tilde{F}_S\{u(x,t)\} = \int_0^\infty u(x,t) \sin \omega x \, dx = \tilde{u}(\omega,t) \longrightarrow \frac{d\tilde{u}}{dt} = \kappa\left[-\omega^2 \tilde{u} + \omega u(0,t)\right] + \frac{\kappa}{K}\tilde{g}$$

$$\longrightarrow \frac{d\tilde{u}}{dt} + \kappa\omega^2 \tilde{u} = \kappa \omega u(0,t) + \frac{\kappa}{K} \tilde{g}(\omega,t) \longrightarrow \quad ✓$$

$$\tilde{u}(\omega,t) = c e^{-\kappa\omega^2 t} + \int_0^t \left[\frac{\kappa}{K} \tilde{g}(\omega,\bar{t}) + \kappa\omega f_1(\bar{t})\right] e^{-\kappa\omega^2 (t-\bar{t})} d\bar{t} \quad ✓$$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \tilde{u}(\omega,t) \sin \omega x \, d\omega$$

IC conversion $\tilde{u}(\omega,0) = c = \tilde{f}(\omega)$

(b) $g(x,t) \equiv 0 \;,\; f_1(t) \equiv 0 \;,\; f(x) = u_0 \longrightarrow \tilde{u}(\omega,t) = \tilde{f}(\omega) e^{-\kappa\omega^2 t}$ ✓

using Convolution integral $\longrightarrow u(x,t) = \frac{1}{2} \int_0^\infty u_0 \frac{1}{\sqrt{\kappa\pi t}} \left[e^{\frac{-(x-u)^2}{4\kappa t}} - e^{\frac{-(x+u)^2}{4\kappa t}} \right] du$ ✓

let $\frac{x-u}{2\sqrt{\kappa t}} = v \longrightarrow du = -2\sqrt{\kappa t}\, dv$ & let $z = \frac{x+u}{2\sqrt{\kappa t}} \longrightarrow du = 2\sqrt{\kappa t}\, dz$

$$u(x,t) = -\frac{u_0}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{-\infty} e^{-v^2} dv - \frac{u_0}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{+\infty} e^{-z^2} dz$$

Cont $\longrightarrow$ ②

[redacted]

ii) $f(x) = \begin{cases} 0 & |x|<a \\ 1 & |x|>a \end{cases} \rightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}}\int_{-\infty}^{-a} e^{-(x-u)^2/4\kappa t}\,du + \frac{1}{2\sqrt{\kappa\pi t}}\int_{+a}^{\infty} e^{-(x-u)^2/4\kappa t}\,du$

$\rightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}}\left[\int_{-\infty}^{+\infty} e^{-(x-u)^2/4\kappa t}\,du - \int_{-a}^{a} e^{-(x-u)^2/4\kappa t}\,du\right]$ $\xrightarrow{\text{like part (i)}}$ $v = \frac{x-u}{2\sqrt{\kappa t}}$

$u(x,t) = +\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-v^2}\,dv - \boxed{\frac{1}{2\sqrt{\kappa\pi t}}\int_{-a}^{a} e^{-(x-u)^2/4\kappa t}\,du} \rightarrow$ we know the answer from part (i)

$u(x,t) = +1 + \frac{1}{2}\operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right)$ ✓ 20/20

P.284 / 4 $\quad \frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + \frac{\kappa}{K}g(x,t) \qquad x>0 \quad t>0$

$u(0,t) = f_1(t) \qquad u(x,0) = f(x)$

$\mathcal{F}_s\{u(x,t)\} = \int_0^\infty u(x,t)\sin wx\,dx = \tilde{u}(w,t) \rightarrow \frac{d\tilde{u}}{dt} = \kappa\left[-w^2\tilde{u} + w u(0,t)\right] + \frac{\kappa}{K}\tilde{g}$

✓

$\rightarrow \frac{d\tilde{u}}{dt} + \kappa w^2\tilde{u} = \kappa w\, u(0,t) + \frac{\kappa}{K}\tilde{g}(w,t) \rightarrow$

$\tilde{u}(w,t) = Ce^{-\kappa w^2 t} + \int_0^t \left[\frac{\kappa}{K}\tilde{g}(w,\bar{t}) + \kappa w\, f_1(\bar{t})\right] e^{-\kappa w^2(t-\bar{t})}\,d\bar{t}$ ✓

IC conversion $\tilde{u}(w,0) = C = \tilde{f}(w)$

$u(x,t) = \frac{2}{\pi}\int_0^\infty \tilde{u}(w,t)\sin wx\,dw$

(b) $g(x,t) \equiv 0, \quad f_1(t) \equiv 0, \quad f(x) = u_0 \rightarrow \tilde{u}(w,t) = \tilde{f}(w)\,e^{-\kappa w^2 t}$ ✓

using Convolution integral $\rightarrow u(x,t) = \frac{1}{2}\int_0^\infty u_0 \frac{1}{\sqrt{\kappa\pi t}}\left[e^{\frac{-(x-u)^2}{4\kappa t}} - e^{\frac{-(x+u)^2}{4\kappa t}}\right]du$ ✓

let $\frac{x-u}{2\sqrt{\kappa t}} = v \rightarrow du = -2\sqrt{\kappa t}\,dv$ & let $z = \frac{x+u}{2\sqrt{\kappa t}} \rightarrow du = 2\sqrt{\kappa t}\,dz$

$u(x,t) = -\frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{\kappa t}}^{-\infty} e^{-v^2}\,dv - \frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{\kappa t}}^{+\infty} e^{-z^2}\,dz$

Cont → ②

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we know $\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}\,du = \text{erfc}(x)$

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_{-x/2\sqrt{\kappa t}}^{\infty} e^{-v^2}\,dv - \frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{\kappa t}}^{\infty} e^{-z^2}\,dz \longrightarrow$$

$$\boxed{u(x,t) = \frac{u_0}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]}$$ ✓

(c) $g(x,t) \equiv 0 \longrightarrow \tilde{g}(w,t) \equiv 0$ ✓

$f(x) \equiv 0 \longrightarrow \tilde{f}(w) \equiv 0$

$f_1(t) = \bar{u} \longrightarrow \tilde{u}(w,t) = \int_0^t \kappa w \bar{u}\, e^{-\kappa w^2(t-\bar{t})}\,d\bar{t}$

$$\longrightarrow \tilde{u}(w,t) = \kappa w \bar{u} \cdot \frac{1}{\kappa w^2} \cdot e^{-\kappa w^2(t-\bar{t})} = \frac{\bar{u}}{w} \cdot e^{-\kappa w^2(t-\bar{t})}\Big|_0^t$$

$$\longrightarrow \tilde{u}(w,t) = \frac{\bar{u}}{w}\left(1 - e^{-\kappa w^2 t}\right) \longrightarrow \tilde{u}(w,t) = \frac{\bar{u}}{w} - \bar{u}\,\frac{e^{-\kappa w^2 t}}{w}$$ ✓

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \tilde{u}(w,t) \cdot \sin wx\,dw$$ ✓

$$\mathcal{F}_s^{-1}\left\{\frac{\bar{u}}{w}\right\} = \frac{2}{\pi}\int_0^{\infty} \frac{\bar{u}\sin wx}{w}\,dw = \frac{2}{\pi} \cdot \bar{u} \cdot \frac{\pi}{2} = \bar{u}$$ we know $\int_0^{\infty} \frac{\sin x}{x}\,dx = \pi/2$

applying Convolution on $\bar{u}\,\frac{e^{-\kappa w^2 t}}{w} \longrightarrow$

$$\mathcal{F}^{-1}\left\{\frac{\bar{u}}{w} \cdot e^{-\kappa w^2 t}\right\} = \frac{1}{2}\int_0^{\infty} \bar{u} \cdot \frac{1}{\sqrt{\kappa \pi t}}\left[e^{\frac{-(x-u)^2}{4\kappa t}} - e^{\frac{-(x+u)^2}{4\kappa t}}\right] du$$ $\underline{\text{from Part (b)}} \longrightarrow$

$$= \frac{\bar{u}}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]$$ ✓ $\Longrightarrow$

$$\boxed{u(x,t) = \bar{u} - \frac{\bar{u}}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right]}$$ ✓

20/20

[redacted]

[redacted] [redacted]

!363 / 2

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{K}{k} g(x,t) \qquad u(x,0) = f(x)$$

Applying Laplace operator to both Side of the equation : [redacted]

let $\tilde{u}(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$S\tilde{u}(x;S) - u(x,0) = K\frac{\partial^2}{\partial x^2}\cdot\tilde{u}(x;S) + \frac{K}{k}\tilde{g}(x;S) \longrightarrow$$

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;S) - \frac{S}{K}\tilde{u}(x;S) + \frac{1}{k}\tilde{g}(x;S) + \underline{\underline{f(x)}} = 0 \longrightarrow$$

s/B $f(x)/K$

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;S) - \frac{S}{K}\tilde{u}(x;S) = -\left(\frac{1}{k}\tilde{g}(x;S) + \frac{1}{K}f(x)\right) \qquad \tilde{u}(x;S) = \tilde{u}_h + \tilde{u}_p$$

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;S) - \frac{S}{K}\tilde{u}(x;S) = 0 \longrightarrow \tilde{u}_h = A\underbrace{e^{\sqrt{\frac{S}{K}}x}}_{\tilde{u}_1} + B\underbrace{e^{-\sqrt{\frac{S}{K}}x}}_{\tilde{u}_2}$$

$$\tilde{u}_p = v_1\tilde{u}_1 + v_2\tilde{u}_2 \qquad W(\tilde{u}_1,\tilde{u}_2) = \begin{vmatrix} e^{\sqrt{\frac{S}{K}}x} & \sqrt{\frac{S}{K}}e^{\sqrt{\frac{S}{K}}x} \\ e^{-\sqrt{\frac{S}{K}}x} & -\sqrt{\frac{S}{K}}e^{-\sqrt{\frac{S}{K}}x} \end{vmatrix} = -2\sqrt{\frac{S}{K}}$$

$$v_1 = \int_{\times}^{x} +\left(\frac{1}{K}f(\bar{x}) + \frac{1}{k}\tilde{g}(\bar{x};s)\right)e^{-\sqrt{\frac{S}{K}}\bar{x}}\cdot -\tfrac{1}{2}\sqrt{\frac{K}{S}}\,d\bar{x}$$

$$v_2 = \int_{\times}^{x} -\left(\frac{1}{K}f(\bar{x}) + \frac{1}{k}\tilde{g}(\bar{x};s)\right)e^{\sqrt{\frac{S}{K}}\bar{x}}\cdot -\tfrac{1}{2}\sqrt{\frac{K}{S}}\,d\bar{x}$$

we can multiple $\tilde{u}_1, \tilde{u}_2$ into the functions inside the integral $\longrightarrow$

$$\longrightarrow \tilde{u}_p = \int_0^x -\sqrt{\frac{K}{S}}\left[\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right]\cdot\left[e^{\sqrt{\frac{S}{K}}(x-\bar{x})} - e^{-\sqrt{\frac{S}{K}}(x-\bar{x})}\right]\cdot\tfrac{1}{2}\,d\bar{x}$$

$$\longrightarrow \tilde{u}_p = -\sqrt{\frac{K}{S}}\int_0^x\left(\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right)\cdot\sinh\left(\sqrt{\tfrac{S}{K}}(x-\bar{x})\right)d\bar{x} \qquad \underline{\tilde{u}_{TOT} = \tilde{u}_h + \tilde{u}_p} \longrightarrow$$

$$\tilde{u}(x;S) = Ae^{\sqrt{\frac{S}{K}}x} + Be^{-\sqrt{\frac{S}{K}}x} - \sqrt{\frac{K}{S}}\int_{\times}^{x}\left(\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right)\cdot\sinh\left(\sqrt{\tfrac{S}{K}}(x-\bar{x})\right)d\bar{x}$$

✓

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what about when the BC is changed to $\frac{\partial U}{\partial x}(0,t) = \frac{-f_1(t)}{K} = -\frac{Q_0}{K}$

[redacted] [redacted]

[redacted]

[redacted]

[illegible]

P.364 / 4

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad x>0 \quad t>0$$

$$\frac{\partial u(0,t)}{\partial x} = -\frac{f_1(t)}{K} \qquad t>0$$

$$u(x,0) = u_0 = cte \qquad x>0$$

we have the general solution from previous exercise also we know:

$g(x,t) \equiv 0$

$f(x) = u_0$

and the solution shall be bounded as $x \to \infty$

$$\tilde{u}(x;s) = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x} - \sqrt{\frac{K}{s}}\int_0^x \frac{u_0}{K}\sinh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)d\bar{x}$$

$x \to \infty$, Solution shall be bounded $\to$ $A = 0$ $\to$

$$\tilde{u}(x;s) = Be^{-\sqrt{\frac{s}{k}}x} - \frac{u_0}{\sqrt{sK}}\int_0^x \sinh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)d\bar{x} = Be^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{\sqrt{sK}}\cdot\sqrt{\frac{K}{s}}\left[\cosh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)\right]_0^x \to$$

$$\tilde{u}(x;s) = Be^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{s}\left(1-\cosh\sqrt{\tfrac{s}{k}}\,x\right)$$

$$\mathcal{L}\left\{\frac{\partial}{\partial x}u(0,t)\right\} = \mathcal{L}\left\{-\frac{f_1(t)}{K}\right\} \to \int_0^\infty \frac{\partial}{\partial x}u(0,t)\,e^{-st}\,dt = -\frac{\tilde{f}_1(s)}{K} \to$$

$$\frac{\partial}{\partial x}\tilde{u}(0;s) = -\frac{\tilde{f}_1(s)}{K}$$

$$\frac{\partial}{\partial x}\tilde{u}(x;s) = -B\sqrt{\frac{s}{K}}\,e^{-\sqrt{\frac{s}{k}}x} - \frac{u_0}{s}\cdot\sqrt{\frac{s}{K}}\sinh\sqrt{\frac{s}{K}}x\,\Bigg|_{x=0} = -B\sqrt{\frac{s}{K}} = \frac{\tilde{f}_1(s)}{K} \to$$

$$B = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}$$

$$\Rightarrow \tilde{u}(x;s) = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}\cdot e^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{s} - u_0\frac{\cosh\sqrt{\frac{s}{k}}x}{s}$$

$$= \underbrace{-\frac{1}{\sqrt{K}}\cdot\tilde{f}_1(s)\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{\sqrt{s}}}_{\downarrow\,\mathcal{L}^{-1}} + \underbrace{\frac{u_0}{s}}_{\downarrow\,\mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow\,\mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow\,\mathcal{L}^{-1}}$$

$$-\frac{1}{\sqrt{K}}\int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}}\cdot f_1(a)\cdot da + u_0 - u_0\,\mathrm{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - u_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Kt}}\right)$$

ou

Cont. → ②

[redacted]

$$u(x,t) = -\frac{1}{\sqrt{K}} \int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}} \cdot f_1(a) \cdot da + u_0 \left[1 - \operatorname{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - \operatorname{erfc}\left(\frac{x}{2\sqrt{Kt}}\right) \right]$$ ✓

P350/28 $y'' + 2y' + y = t \quad y(0) = 0 \quad y'(0) = 1$

$$\left[s^2 Y(s) - s\,\overset{\to 0}{y(t=0^+)} - \overset{\to 1}{y'(t=0^+)}\right] + 2\left[sY(s) - \overset{\to 0}{y(t=0^+)}\right] + Y(s) = \frac{1}{s^2} \longrightarrow$$

$$s^2 Y(s) + 2sY(s) + Y(s) - 1 = \frac{1}{s^2} \longrightarrow Y(s)\left(s^2 + 2s + 1\right) = \frac{1+s^2}{s^2} \longrightarrow Y(s) = \frac{1+s^2}{s^2(s+1)^2}$$

$$\longrightarrow Y(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}$$

$$\longrightarrow \boxed{y(t) = -2 + t + 2e^{-t} + 2te^{-t}}$$ ✓ [redacted]

10

$$\text{let } y(s) = \frac{A}{s} + \frac{Bs+C}{s^2} + \frac{D}{s+1} + \frac{Es+\bar{D}}{(s+1)^2}$$

$$= \frac{As(s+1)^2 + (Bs+C)(s+1)^2 + Ds^2(s+1) + (Es+\bar{D})s^2}{s^2(s+1)^2} = \frac{1+s^2}{s^2(s+1)^2}$$

expand and equate powers of s to find $A, B, C, D, E, \bar{D}$

[redacted]

[redacted]

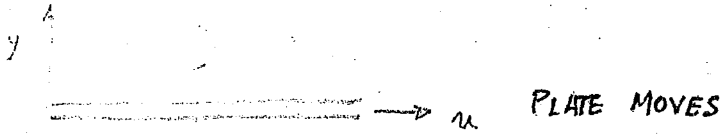
[illegible]

[illegible]

[illegible]

MOTION OF A VISCOUS FLUID OVER AN ∞ PLATE

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$



ν kinematic viscosity

let $u = u(y,t)$ $\qquad u(y=0,t) = at^b \qquad a, b$ fixed (1)

BC $\quad u(y,t) \to 0$ as $y \to \infty$ $\qquad$ (2)

IC $\quad u(y,0) = 0$ $\qquad$ (3)

let $\quad \eta = B \frac{y}{t^n} \qquad u = A f(\eta) \qquad A, B, n$ constant

- for behavior of $u \to 0$ for y large & for t small $\Rightarrow \eta \to \infty$ for y large, t small
  note that (2) & (3) are collapsed into 1 condition
- Check $u(y=0,t) = A\, f(0) = at^b \qquad y=0 \Rightarrow \eta = 0$
  impossible

Must add additional degree of freedom : let $u = A t^m f(\eta)$
pick t^m since $u(y=0,t)$ involves t^b

$\therefore \quad u(y=0,t) = At^m f(0) = at^b$ must pick $m = b$

$A f(0) = a \qquad$ may pick $A = a \Rightarrow f(0) = 1$

Guidance : try to get fns $f(0)$, $f(\infty)$ etc to be either $1, 0, \infty$ but not 2.735, 15.2 etc.

$\therefore \quad u = at^b f(\eta) \qquad \eta = By/t^n$

$\frac{\partial u}{\partial y} = at^b f'(\eta) \frac{d\eta}{dy} = at^b f' \cdot \frac{B}{t^n}$ $\quad\Bigg\|\quad$ $\frac{\partial u}{\partial t} = abt^{b-1} f + at^b f' \cdot \frac{d\eta}{dt}$

$\partial^2 u/\partial y^2 = at^b f'' \; B^2/t^{2n}$ $\quad\Bigg\|\quad$ $= abt^{b-1} f + at^b f' (-nBy/t^{n+1})$

$$\frac{T-T_i}{T_s-T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = 1-\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

1. ALWAYS REMEMBER THAT IF THE PROBLEM IS INDEPENDENT OF LENGTH OR TIME SCALES — WE HAVE SELF SIMILAR SOLUTION

2. SOLUTION WILL REDUCE # OF INDEPENDENT VARIABLES BY 1 & $\underbrace{\text{PDE, BECOMES ODE}}_{\text{OF 2 INDEP. VARIABLES}}$

3. ASSUME GENERAL FORM OF TRANSFORMATION BASED ON B.C's & I.C.

4. IN SIMILARITY PARAMETER η, MOST DIFFERENTIATED VARIABLE SHOULD APPEAR IN NUMERATOR. ie $A\frac{x}{t^n}$

5. TRANSFORM BC & IC USING SIMILARITY PARAMETER AND INSURE THEY ARE SATISFIED

   IF NOT ADD ADDITIONAL DEGREES OF FREEDOM

6. CONVERT DE INTO ONE THAT CONTAINS f & ITS DERIV, η and only one of the independent variables.

   Determine parameters of η to reduce PDE order by one

7. Express BC & IC for reduced problem & solve.

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$(T_s - T_i)\ \frac{A^2}{t^{2n}}\ f''(\eta) = \frac{1}{\alpha}\ (T_s - T_i)\left(\frac{-\eta\, n}{t}\right) f'$$

$$(T_s - T_i)\frac{A}{t^{2n}}\left[f'' + \frac{n\eta}{\alpha A^2} t^{2n-1} f' \right] = 0$$

FOR ODE ; η, f, f', f'' *only* $\Rightarrow$ $t^{2n-1} = 1$ $\Rightarrow n = 1/2$

$\therefore$ $f'' + \frac{1}{2\alpha A^2}\eta f' = 0$ pick $A \Rightarrow f'' + \eta f' = 0$

$\therefore$ choose $A = \frac{1}{\sqrt{2\alpha}}$

choose A to simplify form of equation

$\therefore$ $\eta = \frac{x}{\sqrt{2\alpha t}}$

---

at $x = 0$ $T = T_s$ $\Rightarrow$ $\eta = 0$ $\frac{T_s - T_i}{T_s - T_i} = 1 = f(\eta = 0)$

collapse of 2 conditions to 1:

$t = 0$ $T = T_i$ $\Rightarrow$ $\eta = \infty$ $T = T_i \Rightarrow 0 = f(\eta = \infty)$

$x \to \infty$ $T \to T_i$ $\Rightarrow$ $\eta \to \infty$ $T \to T_i$ $0 \leftarrow f(\eta \to \infty)$

note $f'' + \eta f' = 0$ (2nd order ODE)

$\frac{df'}{d\eta} + \eta f' = 0$ $\Rightarrow$ $\frac{df'}{d\eta} = -\eta f'$ or $\frac{df'}{f'} = -\eta\, d\eta$

$\ln f' = -\eta^2/2 + \ln C_1$

$f' = C_1 e^{-\eta^2/2}$

$df = C_1 e^{-\eta^2/2} d\eta$ or $f = C_1 \int_0^{\eta} e^{-\sigma^2/2} d\sigma + C_2$

when $\eta = 0$ $f(\eta = 0) = 1 = C_2$

$f \to 0$ as $\eta \to \infty$ NOW $\int_0^{\infty} e^{-\sigma^2/2} d\sigma = \sqrt{2}\int_0^{\infty} e^{-z^2} dz = \sqrt{\frac{\pi}{2}}$

$0 = C_1 \cdot \sqrt{\frac{\pi}{2}} + 1$ $C_1 = -\sqrt{\frac{2}{\pi}}$ $\therefore$ $f = 1 - \sqrt{\frac{2}{\pi}}\int_0^{\eta} e^{-\sigma^2/2} d\sigma$

$= 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta/\sqrt{2}} e^{-z^2} dz = 1 - \mathrm{erf}\left(\frac{\eta}{\sqrt{2}}\right.$

$= \mathrm{erfc}\left(\frac{\eta}{\sqrt{2}}\right.$

$$\frac{T - T_i}{T_s - T_i} = f(\eta) = \mathrm{erfc}(\eta/\sqrt{2})$$

- CHOOSING a solution involving η
  - REDUCES PDE TO AN ODE WHICH IS A FN OF η ONLY

- NOTE x, t (2 indep var.) now becomes η (1 indep var)

  $\Rightarrow$ SELF SIMILARITY REDUCE # OF INDEPENDENT VAR. BY 1

METHOD OF APPROACH

- $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ w/ $T(0,t) = T_s$

  $T(x,0) = T_i$

  $T(x,t) \to T_i$ as $x \to \infty$

IN GENERAL

Choose $\eta = \dfrac{Ax}{t^n}$ A, n picked to reduce equ. to ODE

let $\dfrac{T - T_i}{T_s - T_i} = f(\eta)$

- NOTE: SINCE $T_s - T_i$ is a basic aspect of problem

  FORM OF η : SINCE $t = 0$ & $x = \infty$ give T_i

CHOOSE FORM OF η : NUMERATOR – INDEP VAR DIFFERENTIATED MOST **OFTEN**

DENOMINATOR – LEAST OFTEN **DIFFERENTIATED**

$$T = T_i + (T_s - T_i) f(\eta)$$

- $\dfrac{\partial T}{\partial x} = \dfrac{\partial T}{\partial \eta}\dfrac{\partial \eta}{\partial x}$; $\dfrac{\partial \eta}{\partial x} = \dfrac{A}{t^n}$; $\dfrac{\partial T}{\partial \eta} = (T_s - T_i)\dfrac{df}{d\eta}$

$$\therefore \quad \frac{\partial T}{\partial x} = (T_s - T_i) f' \cdot \left(\frac{A}{t^n}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial \eta}\left[(T_s - T_i) f'(\eta)\left(\frac{A}{t^n}\right)\right] \cdot \frac{\partial \eta}{\partial x} = (T_s - T_i)\left(\frac{A^2}{t^{2n}}\right) f''(\eta)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = (T_s - T_i) f' \cdot \frac{-nAx}{t^{n+1}} = (T_s - T_i) f' \left[-\frac{n\eta}{t}\right]$$

GIVE PDE HANDOUT

- CERTAIN PROBLEMS DO NOT HAVE NATURAL SCALE FOR INDEPENDENT VARIABLE

FOR EXAMPLE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_i \quad x > 0$$

$$T(0,t) = T_s$$

and $x \to \infty$ $T \to T_i$

IF FOR INDEPENDENT VARIABLES OF THE PROBLEM $\nexists$ CLUE TO EXISTENCE

$$\left.\begin{array}{l} \cdot \text{ NO CHARACTERISTIC LENGTH} \\ \quad '' \qquad '' \qquad \text{TIME} \end{array}\right\} \Rightarrow \text{SELF-SIMILAR SOLUTION.}$$

- SOLUTION OF ALL PHYSICAL PROBLEMS MAY BE EXPRESSED IN DIMENSIONLESS FORM

~~FROM INDEPENDENT VARIABLES~~

- $\Rightarrow$ t, x must form a dimensionless group

FROM PDE : $x^2 = \alpha t$ or $\frac{x^2}{\alpha t}$, $\frac{\alpha t}{x^2}$, or $\frac{x}{\sqrt{\alpha t}}$ or $\frac{\sqrt{\alpha t}}{x}$

- IF SOLN MADE DIMENSIONLESS BY COMBO OF INDEPENDENT VARIABLES INSTEAD OF GEOMETRY, BC OR IC. – PROBLEM IS SELF SIMILAR

- THERE IS A CHARACTERISTIC TEMP TO THE PROBLEM $T_s - T_i$

could guess

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \frac{x}{\sqrt{\alpha t}}\, h\left(\frac{x}{\sqrt{\alpha t}}\right)$$

- Could choose $\frac{T}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \dots = h\left(\frac{\alpha t}{x^2}\right)$

- define $\eta = \frac{x}{\sqrt{\alpha t}}$ SIMILARITY VARIABLE

$\Rightarrow$ • ALL TEMPERATURE PROFILES FALL ONTO ONE GRAPH

LESSON # 13

~~$\frac{\partial T}{\partial x} = Q_1$ @ $x=0$ $\quad \frac{\partial T}{\partial x} = Q_2$ @ $x=L$ $\Rightarrow \frac{\partial T}{\partial x} = Ax+B \quad B=Q_1$~~

~~$A = \frac{Q_2-Q_1}{L}$~~

~~$\therefore \frac{\partial T}{\partial x} = Q_1 + \frac{Q_2-Q_1}{L}x$ note q is not a fn of time~~

~~Integrate $T_p = Q_1 x + \frac{Q_2-Q_1}{L}\frac{x^2}{2} + f(t)$~~

~~but T satisfies $\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad \frac{\partial T}{\partial t} = f'(t)$.~~

~~$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \alpha^2 \frac{Q_2-Q_1}{L} \quad \therefore \quad f'(t) = \alpha^2 \frac{Q_2-Q_1}{L}$~~

~~$f(t) = \alpha^2 \frac{Q_2-Q_1}{L} t$ + const. $\quad \therefore \quad T = Q_1 x + \frac{Q_2-Q_1}{2L}(x^2+2\alpha^2 t)$ + con~~

~~$T_p = Q_1 x + \frac{Q_2-Q_1}{2L}(x^2+2\alpha^2 t)$~~

SELF-SIMILAR SOLUTIONS

1. SOMETIMES WE WANT TO FIND SOLUTION IN DIMENSIONLESS FORM

   WHY ?

2. RESULTS ARE INDEPENDENT OF SIZE OF SYSTEM

3. CHOOSE SOME LENGTH OR TIME SCALE THAT CHARACTERIZE PROBLEM IN TERMS OF INDEPENDENT VARIABLES.

4. $\frac{x}{L} = \chi$ LENGTH OF PROBLEM

   WHERE DO WE GET L, time

5. LENGTH OR TIME SCALE CAN COME FROM B.C. OR GEOMETRY

6. PROBLEMS WITH NATURAL CHARACTERISTIC SCALES ARE CALLED SCALE-SIMILAR.

   SCALE-SIMILAR SOLUTIONS FOR SYSTEMS OF DIFFERENT SIZES WILL HAVE SAME NONDIM. SOL. IF THEY HAVE SAME DIMENSIONLESS

EGM 5315

Inter. Anal. of Mech. Systems

4/14/05

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ME 238B
Prof Barnett

100

Problem Set #2

1. From the lecture we had

a. $$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{A e^{-|\lambda| y} + B y e^{-|\lambda| y}\right\} d\lambda \quad \text{with } A = \frac{1}{2\pi\lambda^2} \quad B = \frac{1}{2\pi|\lambda|}$$

$$\therefore\ \phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{1}{2\pi\lambda^2} e^{-|\lambda| y} + \frac{1}{2\pi|\lambda|}\, y e^{-|\lambda| y}\right\} d\lambda$$; By differentiating we obtain

$$\frac{\partial\phi}{\partial y} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} y\, e^{-i\lambda x - |\lambda| y} d\lambda \ ; \quad \frac{\partial^2\phi}{\partial x\partial y} = \frac{i}{2\pi}\int_{-\infty}^{\infty} \lambda y\, e^{-i\lambda x - |\lambda| y} d\lambda \ ;$$

thus $\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{i}{2\pi} y \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda$. The only term that will not be zero in the integration is

$$-\frac{i}{2\pi} y \int_{-\infty}^{\infty} (-i\sin\lambda x)\lambda e^{-|\lambda| y} d\lambda = \frac{-y}{2\pi}\int_{-\infty}^{\infty} \lambda\, \mathrm{sen}\,\lambda x\, e^{-|\lambda| y} d\lambda = \frac{-y}{\pi}\int_0^{\infty} \lambda \sin\lambda x\, e^{-\lambda y} d\lambda$$

Note that $\frac{\partial}{\partial x}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda = -\int_0^{\infty} \lambda\, \mathrm{sen}\,\lambda x\, e^{-\lambda y} d\lambda = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2yx}{(x^2+y^2)^2}$

$$\therefore \quad \frac{y}{\pi}\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \underline{\underline{-\frac{2}{\pi}\frac{y^2 x}{(x^2+y^2)^2} = \sigma_{xy}}}$$

b. Since $\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda$. The only terms that will not be zero in the integration is

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\lambda x\{1-|\lambda| y\} e^{-|\lambda| y} d\lambda = -\frac{1}{\pi}\int_0^{\infty} \cos\lambda x\{1-\lambda y\} e^{-\lambda y} d\lambda$$

$$= -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = \underline{\underline{-\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} = \sigma_{xx}}}$$

here we used $\int_0^{\infty} \cos\lambda x\left(-\lambda y e^{-\lambda y}\right) d\lambda = y\frac{\partial}{\partial y}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y} d\lambda$

c. Since

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} \Rightarrow \frac{\partial\phi}{\partial y} = \frac{x^2}{\pi}\frac{1}{(x^2+y^2)} + \hat{f}_1(x) \Rightarrow \phi = \frac{x}{\pi}\arctan\frac{y}{x} + \hat{f}_1(x) y + \hat{f}_2(x$$

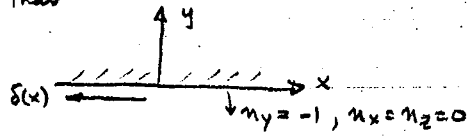
now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{2xy^2}{\pi(x^2+y^2)^2} + \hat{f}_1'(x) = -\sigma_{xy} \Rightarrow \hat{f}_1'(x) = 0$ or $\underline{\hat{f}_1(x) = c_1}$

now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{\pi}\arctan y/x - \frac{1}{\pi}\frac{yx}{x^2+y^2} + \hat{f}_2'\right] = \frac{-2y^3}{\pi(x^2+y^2)^2} + \hat{f}_2'' = \sigma_{yy} \Rightarrow \hat{f}_2''(x) = 0$

or $\hat{f}_2(x) = c_2 x + c_3 \quad \therefore \quad \underline{\underline{\phi(x,y) = \frac{x}{\pi}\arctan\frac{y}{x} + c_1 y + c_2 x + c_3}}$; the last three terms don't play a role in defining the stresses $\Rightarrow$ we can take $c_1 = c_2 = c_3 = 0$ if we wish

2. again let $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}\,d\lambda$

we note that



$\therefore\ T_y = \sigma_{yj}n_j = -\sigma_{yy} = 0 \qquad T_x = -\delta(x) = n_j\sigma_{xj} = -\sigma_{xy}$ thus σ_{xy} must be to the left.

Now $\dfrac{\partial\phi}{\partial x} = \int_{-\infty}^{\infty}(-i\lambda)e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$ and $\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$

since $\sigma_{yy}\Big|_{y=0} = 0 \Rightarrow \quad 0 = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}A\,d\lambda$. It can be shown that $\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}d\lambda \neq 0 \ \therefore \underline{\underline{A \equiv 0}}$

Now $-\dfrac{\partial}{\partial y}\left(\dfrac{\partial\phi}{\partial x}\right) = +\sigma_{xy} = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}\,Be^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

But since $\sigma_{xy} = \delta(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}d\lambda = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}B\,d\lambda \Rightarrow Bi\lambda = \dfrac{1}{2\pi}$ or $\underline{\underline{B = \dfrac{1}{2\pi i\lambda}}}$

hence $\phi(x,y) = \int_{-\infty}^{\infty}e^{-i\lambda x}\dfrac{y}{2\pi i\lambda}e^{-|\lambda|y}d\lambda$; using all this we have

$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2} = \int_{-\infty}^{\infty}\dfrac{-\lambda y}{2\pi i}e^{-i\lambda x}e^{-|\lambda|y}d\lambda$, $\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

Now since $\dfrac{\partial\phi}{\partial y} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{e^{-i\lambda x}}{\lambda}\left[1-|\lambda|y\right]e^{-|\lambda|y}d\lambda$ thus $\dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}-\dfrac{|\lambda|}{\lambda}(2-|\lambda|y)e^{-i\lambda x-|\lambda|y}d\lambda$

using the even/odd argument we then obtain:

a. $\sigma_{yy} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}-\lambda(-i\sin\lambda x)\,y\,e^{-|\lambda|y}d\lambda = \dfrac{y}{\pi}\int_0^{\infty}\lambda\sin\lambda x\,e^{-\lambda y}d\lambda = \dfrac{y}{\pi}\dfrac{-\partial}{\partial x}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right.$

$= \dfrac{-y}{\pi}\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{-y}{\pi}\left[\dfrac{-2yx}{(x^2+y^2)^2}\right] = \underline{\underline{\dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2} = \sigma_{yy}}}$

b. $\sigma_{xy} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\cos\lambda x\,e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}[1-\lambda y]d\lambda = \dfrac{1}{\pi}\left[\dfrac{y}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right.$

$= \dfrac{1}{\pi}\left(\dfrac{y}{x^2+y^2}\right) + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \underline{\underline{\dfrac{2}{\pi}\dfrac{x^2y}{(x^2+y^2)^2} = \sigma_{xy}}}$

c. $\sigma_{xx} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{-i\sin\lambda x}{\lambda}\left\{-|\lambda|e^{-|\lambda|y}\left[2-|\lambda|y\right]\right\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\sin\lambda x\,e^{-\lambda y}(2-\lambda y)\,d\lambda$

$= \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right] = \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\dfrac{y}{x^2+y^2}\right] = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}$

or $\underline{\underline{\sigma_{xx} = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}}}$

d. To obtain ϕ :

$\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2} \Rightarrow \dfrac{\partial\phi}{\partial x} = \dfrac{-y^2}{\pi}\dfrac{1}{x^2+y^2} + \hat{f}_1(y) \Rightarrow \phi = \dfrac{-y}{\pi}\arctan\dfrac{x}{y} + x\hat{f}_1(y) + \hat{f}_2(y)$

$= y/\pi \arctan y/x + x\hat{f}_1(y) + \hat{f}_2(y)$

$$\frac{\partial^2\phi}{\partial x\partial y} = -\sigma_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = -\frac{2yx^2}{\pi(x^2+y^2)^2} + \hat{f}_1' \Rightarrow \hat{f}_1'(y)=0 \text{ or } \hat{f}_1(y) = c_1$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\pi}\text{arc tan}\, y/x + \frac{y}{\pi}\frac{x}{x^2+y^2} + \hat{f}_2' \; ; \; \frac{\partial^2\phi}{\partial y^2} = \frac{2x^3}{\pi(x^2+y^2)^2} + \hat{f}_2''(y) = \sigma_{yy} \Rightarrow \hat{f}_2''(y)=0 \text{ or } \hat{f}_2 = c_2y+c_3$$

$$\therefore \phi(x,y) = \underline{\underline{\frac{y}{\pi}\arctan y/x + c_1x + c_2y + c_3}}$$ same argument on c_1, c_2, c_3 as problem 1.

3a For principal stresses in a plane strain problem $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$: Define λ_i to be the principal stresses $\therefore \underset{\sim}{\sigma}\cdot\underset{\sim}{n} = \lambda\underset{\sim}{n}$ or $\det\begin{pmatrix}\sigma_{xx}-\lambda & \sigma_{xy} & 0\\ \sigma_{xy} & \sigma_{yy}-\lambda & 0\\ 0 & 0 & \sigma_{zz}-\lambda\end{pmatrix} = 0$ $\therefore \lambda_3 = \sigma_{zz}$ and

$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$ after the plug in and the algebra

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \frac{-2\nu y}{(x^2+y^2)} = \lambda_3 \qquad \lambda_1 = 0 \qquad \lambda_2 = \frac{-2y}{\pi(x^2+y^2)}$$

Since $y>0$ and we assume $0<\nu<1$ the stresses are ordered

$(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

3b. again we obtain for plane strain $\lambda_3 = \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$ and

$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \lambda_3 = \frac{2\nu x}{\pi(x^2+y^2)} \qquad \lambda_1 = 0 \; , \; \lambda_2 = \frac{2x}{\pi(x^2+y^2)}$$

for $x>0$ and assuming $0<\nu<1$ the stresses are ordered $(\lambda_2, \lambda_3, \lambda_1)$ in decreasing tension (from left to right). for $x<0$ the stresses are ordered $(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

LESSON # 13

~~$\frac{\partial T}{\partial x} = Q_1$ @ $x=0$ $\frac{\partial T}{\partial x} = Q_2$ @ $x=L$ $\Rightarrow$ $\frac{\partial T}{\partial x} = Ax+B$ $B=Q_1$~~

~~$A = \frac{Q_2-Q_1}{L}$~~

~~$\therefore \frac{\partial T}{\partial x} = Q_1 + \frac{Q_2-Q_1}{L}x$ note q is not a fn of time~~

~~Integrate $T_p = Q_1 x + \frac{Q_2-Q_1}{L}\frac{x^2}{2} + f(t)$~~

~~but T satisfies $\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\frac{\partial T}{\partial t} = f'(t)$.~~

~~$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \alpha^2 \frac{Q_2-Q_1}{L}$ $\therefore$ $f'(t) = \alpha^2 \frac{Q_2-Q_1}{L}$~~

~~$f(t) = \alpha^2 \frac{Q_2-Q_1}{L} t + \text{const.}$ $\therefore$ $T = Q_1 x + \frac{Q_2-Q_1}{2L}(x^2+2\alpha^2 t) + co$~~

~~$T_p = Q_1 x + \frac{Q_2-Q_1}{2L}(x^2+2\alpha^2 t)$~~

SELF-SIMILAR SOLUTIONS

1. SOMETIMES WE WANT TO FIND SOLUTION IN DIMENSIONLESS FORM

WHY ?

2. RESULTS ARE INDEPENDENT OF SIZE OF SYSTEM

3. CHOOSE SOME LENGTH OR TIME SCALE THAT CHARACTERIZE PROBLEM IN TERMS OF INDEPENDENT VARIABLES.

4. $\frac{x}{L} = \chi$ LENGTH OF PROBLEM

WHERE DO WE GET L, time

5. LENGTH OR TIME SCALE CAN COME FROM B.C. OR GEOMETRY

6. PROBLEMS WITH NATURAL CHARACTERISTIC SCALES ARE CALLED SCALE-SIMILAR.

SCALE-SIMILAR SOLUTIONS FOR SYSTEMS OF DIFFERENT SIZES WILL HAVE SAME NONDIM. SOL. IF THEY HAVE SAME DIMENSIONLES PARAM, BC & IC.

GIVE PDE HANDOUT

- CERTAIN PROBLEMS DO NOT HAVE NATURAL SCALE FOR INDEPENDENT VARIABLE.

FOR EXAMPLE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_i \quad x > 0$$
$$T(0,t) = T_s$$

and $x \to \infty$ $T \to T_i$

IF FOR INDEPENDENT VARIABLES OF THE PROBLEM $\nexists$ CLUE TO EXISTENCE

$$\left.\begin{array}{l}\cdot \text{ NO CHARACTERISTIC LENGTH} \\ \quad \text{"} \qquad \text{"} \qquad \text{TIME}\end{array}\right\} \Rightarrow \text{SELF-SIMILAR SOLUTION.}$$

- SOLUTION OF ALL PHYSICAL PROBLEMS MAY BE EXPRESSED IN DIMENSIONLESS FORM

~~FROM INDEPENDENT VARIABLES~~

- $\Rightarrow$ t, x must form a dimensionless group

FROM PDE : $x^2 = \alpha t$ or $\frac{x^2}{\alpha t}$, $\frac{\alpha t}{x^2}$, or $\frac{x}{\sqrt{\alpha t}}$ or $\frac{\sqrt{\alpha t}}{x}$

- IF SOLN MADE DIMENSIONLESS BY COMBO OF INDEPENDENT VARIABLES INSTEAD OF GEOMETRY, BC OR IC. - PROBLEM IS SELF SIMILAR

- THERE IS A CHARACTERISTIC TEMP TO THE PROBLEM $T_S - T_i$

could guess

$$\frac{T - T_i}{T_S - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \frac{x}{\sqrt{\alpha t}}\, h\left(\frac{x}{\sqrt{\alpha t}}\right)$$

- Could choose $\frac{T}{T_S - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \dots = h\left(\frac{\alpha t}{x^2}\right)$

- define $\eta = \frac{x}{\sqrt{\alpha t}}$ SIMILARITY VARIABLE

$\Rightarrow$ • ALL TEMPERATURE PROFILES FALL ONTO ONE GRAPH

- CHOOSING a solution involving η
  - REDUCES PDE TO AN ODE WHICH IS A FN OF η ONLY

- NOTE x, t (2 indep var.) now becomes η (1 indep var)

$\Rightarrow$ SELF SIMILARITY REDUCE # OF INDEPENDENT VAR. BY 1.

METHOD OF APPROACH

- $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$ w/ $T(0,t) = T_s$

$T(x,0) = T_i$

$T(x,t) \rightarrow T_i$ as $x \rightarrow \infty$

IN GENERAL

Choose $\eta = \dfrac{Ax}{t^n}$ A, n picked to reduce eqn. to ODE

let $\dfrac{T - T_i}{T_s - T_i} = f(\eta)$

- NOTE: SINCE $T_s - T_i$ is a basic aspect of problem

FORM OF η : SINCE $t = 0$ & $x = \infty$ give T_i

CHOOSE FORM OF η : NUMERATOR - INDEP VAR DIFFERENTIATED MOST OFTEN

DENOMINATOR - LEAST OFTEN DIFFERENTIATED

$$T = T_i + (T_s - T_i) f(\eta)$$

- $\dfrac{\partial T}{\partial x} = \dfrac{\partial T}{\partial \eta}\dfrac{\partial \eta}{\partial x}$; $\dfrac{\partial \eta}{\partial x} = \dfrac{A}{t^n}$; $\dfrac{\partial T}{\partial \eta} = (T_s - T_i)\dfrac{df}{d\eta}$

$$\therefore \frac{\partial T}{\partial x} = (T_s - T_i) f' \cdot \left(\frac{A}{t^n}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial \eta}\left[(T_s - T_i) f'(\eta)\left(\frac{A}{t^n}\right)\right] \cdot \frac{\partial \eta}{\partial x} = (T_s - T_i)\left(\frac{A^2}{t^{2n}}\right) f''(\eta)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = (T_s - T_i) f' \cdot \frac{-nAx}{t^{n+1}} = (T_s - T_i) f' \left[-\frac{n\eta}{t}\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$(T_s - T_i)\,\frac{A^2}{t^{2n}}\, f''(\eta) = \frac{1}{\alpha}\,(T_s - T_i)\left(\frac{-\eta\, n}{t}\right) f'$$

$$(T_s - T_i)\frac{A}{t^{2n}}\left[f'' + \frac{n\eta}{\alpha A^2} t^{2n-1} f' \right] = 0$$

FOR ODE ; η, f, f', f'' *only* $\Rightarrow$ $t^{2n-1} = 1$ $\Rightarrow n = 1/2$

$\therefore$ $f'' + \frac{1}{2\alpha A^2}\eta f' = 0$ pick $A \Rightarrow f'' + \eta f' = 0$

$\therefore$ choose $A = \frac{1}{\sqrt{2\alpha}}$

$\therefore$ $\eta = \frac{x}{\sqrt{2\alpha t}}$

at $x = 0$ $T = T_s$ $\Rightarrow$ $\eta = 0$ $\frac{T_s - T_i}{T_s - T_i} = 1 = f(\eta = 0)$

collapse of 2 conditions to 1

$t = 0$ $T = T_i$ $\Rightarrow$ $\eta = \infty$ $T = T_i \Rightarrow 0 = f(\eta = \infty)$

$x \to \infty$ $T \to T_i$ $\Rightarrow$ $\eta \to \infty$ $T \to T_i$ $0 \leftarrow f(\eta \to \infty)$

note $f'' + \eta f' = 0$ (2nd order ODE)

$\frac{df'}{d\eta} + \eta f' = 0 \Rightarrow \frac{df'}{d\eta} = -\eta f'$ or $\frac{df'}{f'} = -\eta\, d\eta$

$\ln f' = -\eta^2/2 + \ln C_1$

$f' = C_1 e^{-\eta^2/2}$

$df = C_1 e^{-\eta^2/2} d\eta$ or $f = C_1 \int_0^\eta e^{-\sigma^2/2} d\sigma + C_2$

when $\eta = 0$ $f(\eta = 0) = 1 = C_2$

$f \to 0$ as $\eta \to \infty$ NOW $\int_0^\infty e^{-\sigma^2/2} d\sigma = \sqrt{2}\int_0^\infty e^{-z^2} dz = \sqrt{\frac{\pi}{2}}$

$0 = C_1 \cdot \sqrt{\frac{\pi}{2}} + 1$ $C_1 = -\sqrt{\frac{2}{\pi}}$ $\therefore$ $f = 1 - \sqrt{\frac{2}{\pi}} \int_0^\eta e^{-\sigma^2/2} d\sigma$

$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta/\sqrt{2}} e^{-z^2} dz$

$\frac{T - T_i}{T_s - T_i} = f(\eta) = \text{erfc}(\eta/\sqrt{2})$

$$\frac{T-T_i}{T_s-T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = 1-\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

1. ALWAYS REMEMBER THAT IF THE PROBLEM IS INDEPENDENT OF LENGTH OR TIME SCALES — WE HAVE SELF SIMILAR SOLUTION

2. SOLUTION WILL REDUCE # OF INDEPENDENT VARIABLES BY 1 & $\underbrace{\text{PDE}_{\wedge}\ \text{BECOMES ODE}}_{\text{OF 2 INDEP. VARIABLES}}$

3. ASSUME GENERAL FORM OF TRANSFORMATION BASED ON B.C.'s & I.C.

4. IN SIMILARITY PARAMETER η, MOST DIFFERENTIATED VARIABLE SHOULD APPEAR IN NUMERATOR. ie $\frac{Ax}{t^n}$

5. TRANSFORM BC & IC USING SIMILARITY PARAMETER AND INSURE THEY ARE SATISFIED

   IF NOT ADD ADDITIONAL DEGREES OF FREEDOM

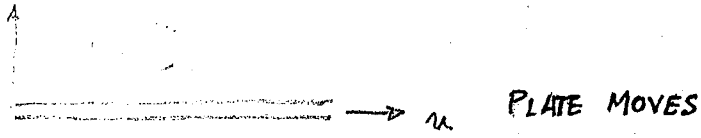
6. CONVERT DE INTO ONE THAT CONTAINS f & ITS DERIV, η and only one of the independent variables t

   Determine parameters of η to reduce PDE order by one

7. Express BC & IC for reduced problem & solve.

MOTION OF A VISCOUS FLUID OVER AN ∞ PLATE

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$



ν kinematic viscosity

let $u = u(y,t)$ $\quad u(y=0,t) = at^b \quad$ a, b fixed (1)

BC $\quad u(y,t) \to 0 \quad$ as $y \to \infty$ (2)

IC $\quad u(y,0) = 0$ (3)

let $\quad \eta = B.\frac{y}{t^n} \qquad u = A\,f(\eta) \qquad A, B, n$ constant

- for behavior of $u \to 0$ for y large & for t small $\Rightarrow \eta \to \infty$ for y large, t small
  note that (2) & (3) are collapsed into 1 condition
- Check $u(y=0,t) = A\,f(0) = at^b \quad y=0 \Rightarrow \eta = 0$
  impossible

Must add additional degree of freedom : let $u = At^m f(\eta)$
pick t^m since $u(y=0,t)$ involves t^b

$\therefore \quad u(y=0,t) = At^m f(0) = at^b$ must pick $m = b$

$A f(0) = a \qquad$ may pick $A = a \Rightarrow f(0) = 1$

Guidance: try to get fns $f(0)$, $f(\infty)$ etc to be either $1, 0, \infty$ but not 2.735, 15.2 etc.

$\therefore \quad u = at^b f(\eta) \qquad \eta = By/t^n$

$\frac{\partial u}{\partial y} = at^b f'(\eta)\,\frac{d\eta}{dy} = at^b f' \cdot \frac{B}{t^n}$ ‖ $\frac{\partial u}{\partial t} = abt^{b-1}f + at^b f' \cdot \frac{d\eta}{dt}$

$\partial^2 u/\partial y^2 = at^b f''\ B^2/t^{2n}$ ‖ $= abt^{b-1}f + at^b f'(-nBy/t^{n+1})$

$$\frac{\partial u}{\partial t} = abt^{b-1}f + at^{b-1}-n\eta f' \; ; \quad \text{put into} \quad \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

$$\therefore B^2 \nu a t^{b-2n} f'' = at^{b-1}\left[bf + f'(-n\eta)\right]$$

if ODE, only f, f', f'' & η appears. So...

let $2n=1 \quad \therefore n=\frac{1}{2}$ $\qquad at^{b-1}$ cancel

$$B^2\nu f'' + \tfrac{1}{2}\eta f' - bf = 0$$

$$2B^2\nu f'' + \eta f' - 2bf = 0 \qquad \text{let } 2B^2\nu = 1 \implies B = \frac{1}{\sqrt{2\nu}}$$

$$\therefore \quad \eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f'' + \eta f' - 2bf = 0$$

from $u(y=0,t) = at^b = at^b f(0) \implies f(0) = 1$

irrespective of t: $u(y,t) \to 0$ as $y \to \infty \implies f(\eta \to \infty) \to 0$

in order to solve $f'' + \eta f' - 2bf = 0$ we need to know b

Suppose $b = \frac{1}{2}$ $\qquad f'' + \eta f' - f = 0 \implies$ 2nd order ODE $\quad f = c_1 f_1 + c_2 f_2$

Use method of reduction in order - if you know a solution f_1

then $f_2 = f_1(\eta)\, g(\eta) \qquad f_2' = f_1'g + f_1 g' \qquad f_2'' = f_1''g + 2f_1'g' + f_1 g''$

PUT INTO ODE

$$\underline{f_1''g} + 2f_1'g_1' + f_1 g'' + \eta[\underline{f_1'g} + f_1 g'] - \underline{f_1 g} = 0$$

$$(f_1'' + \eta f_1' - f_1)^{\checkmark 0} g + f_1 g'' + (\eta f_1 + 2f_1')g' = 0$$

$$\therefore \quad \underset{\frac{d}{d\eta}(\ln g')}{g''/g'} = -\left(\frac{\eta f_1 + 2f_1'}{f_1}\right) = -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\frac{d}{d\eta}\left(\eta^2/2 + 2\ln f_1\right)$$

FIRST ORDER EQN IN g'

NOTICE $f_1 = \eta$ solves , $f_1' = 1 \quad f_1'' = 0$

$$\frac{dg'}{g'} = -\left(\eta + \frac{2}{\eta}\right) d\eta \Rightarrow \ln g' = -\eta^2/2 - 2\ln\eta$$

$$g' = \frac{1}{\eta^2} e^{-\eta^2/2} \qquad g(\eta) = \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$$

Pick ∞ as the lower limit since g' is bounded

$$f = C_1 \eta + C_2 \eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$$

: $f(\eta) \to 0$ as $\eta \to \infty$:

$$\infty > \sigma > \eta \qquad \frac{1}{\sigma^2} < \frac{1}{\eta^2} < \frac{1}{\eta}$$

$$\infty > \sigma^2 > \eta^2 > \eta$$

$$0 < \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma < \int_\infty^\eta \frac{1}{\eta} e^{-\sigma^2/2} d\sigma$$ and $\frac{1}{\eta}\int_\infty^\eta e^{-\sigma^2/2} d\sigma$ is bounded

$$f_2 = \eta\, g(\eta) < \int_\infty^\eta e^{-\sigma^2/2} d\sigma$$ as $\eta \to \infty$ $\int \to 0$

$\therefore$ $f_2 \to 0$ as $\eta \to \infty$ but $f_1 = \eta \to \infty$ as $\eta \to \infty$ $\therefore C_1 = 0$

and $f = C_2 \eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$

→ **INTEGRATE BY PARTS**

let $\frac{1}{\sigma^2} d\sigma = dv \quad w = e^{-\sigma^2/2} \quad v = -\frac{1}{\sigma} \quad dw = e^{-\sigma^2/2} \cdot (-\sigma\, d\sigma)$

: $f(\eta = 0) = 1$

$$C_2 \eta \left[-\frac{1}{\sigma} e^{-\sigma^2/2} \Big|_\infty^\eta - \int_\infty^\eta \left(-\frac{1}{\sigma}\right)(-\sigma) e^{-\sigma^2/2} d\sigma \right]$$

$$\left[-\frac{1}{\eta} e^{-\eta^2/2} - \int_\infty^\eta e^{-\sigma^2/2} d\sigma \right]$$

$$f = -C_2 e^{-\eta^2/2} - C_2 \eta \int_\infty^\eta e^{-\sigma^2/2} d\sigma$$

at $\eta = 0$ $\quad f = -C_2 e^0 - C_2 \cdot 0 \cdot \int_\infty^0 e^{-\sigma^2/2} d\sigma = 1 \qquad C_2 = -1$

$$\therefore \quad f(\eta) = e^{-\eta^2/2} + \eta \int_\infty^\eta e^{-\sigma^2/2} d\sigma$$

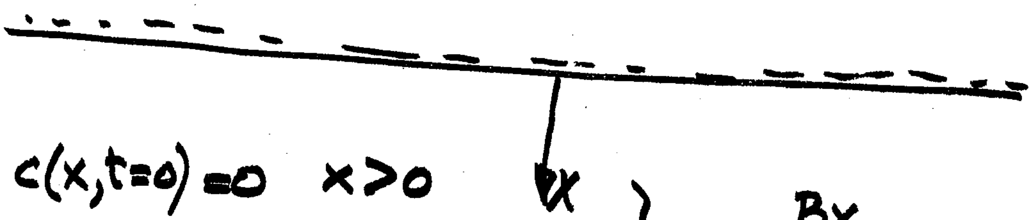
let $z = \sigma/\sqrt{2}$ $\quad z^2 = \sigma^2/2 \quad dz = \frac{d\sigma}{\sqrt{2}}$

$$\eta \cdot \sqrt{2} \int_{\infty}^{\eta/\sqrt{2}} e^{-z^2} dz \qquad \int_0^1 = \int_0^\infty - \int_1^\infty$$

$$- \frac{\sqrt{\pi}}{2} \, \text{erfc} \, \frac{\eta}{\sqrt{2}} \qquad \int_1^\infty = \int_0^\infty - \int_0^1$$

$$f(\eta) = e^{-\eta^2/2} - \eta \sqrt{\pi/2} \, \text{erfc} \left(\eta/\sqrt{2} \right) \qquad \text{erfc} = 1 - \text{erf}$$

$$\alpha \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}$$



initially $c(x,t=0)=0 \quad x>0$

for $x\to\infty \quad c(x,t)\to 0$

$\Big\} \; \eta = \frac{Bx}{t^m}$

$$\int_0^\infty c(x,t)\,dx = Q$$

$$c = A f(\eta)$$

$$\int_0^\infty c(x,t)\,dx = \int_0^\infty A f(\eta)\,d\eta\cdot\frac{t^m}{B} = Q$$ impossible to satisfy

choose

$$c = At^n f(\eta)$$

$$\int_0^\infty c(x,t)\,d\lambda = \int_0^\infty At^n f(\eta)\cdot d\eta\,\frac{t^m}{B} = Q \implies n=-m$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta}\cdot\frac{\partial \eta}{\partial x} = At^n f'\cdot\frac{B}{t^m}\,; \quad \frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial c}{\partial x}\right)\cdot\frac{\partial \eta}{\partial x}$$

$$= At^n f''\cdot\frac{B^2}{t^{m\cdot 2}}$$

$$\frac{\partial c}{\partial t} = \quad = Ant^{n-1}f + At^n f'\cdot\left(-\frac{mBx}{t^{m+1}}\right)$$

$$= Ant^{n-1}f + At^n f'\left(\frac{-m\eta}{t}\right)$$

$$\alpha\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}: \quad \alpha At^n \frac{B^2}{t^{2m}} f'' = Ant^{n-1}f + At^{n-1}f'\cdot(-m\eta)$$

$$\alpha A t^{-3m} B^2 f'' = -Am t^{-m-1}\left[f + \eta f'\right]$$

for "t" to disappear $-3m = -m-1$ $\quad m = \frac{1}{2} = -n$

$$\alpha B^2 f'' + \frac{1}{2}[f + \eta f'] = 0$$

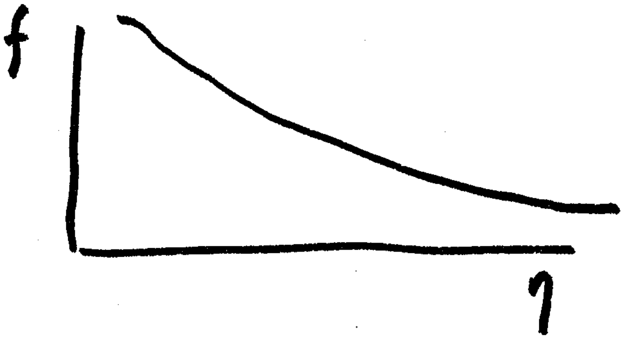
$$2\alpha B^2 f'' + \eta f' + f = 0 \quad \Rightarrow B = \frac{1}{\sqrt{2\alpha}}$$

$$\Rightarrow f'' + \underbrace{\eta f' + f}_{} = 0$$

$$f'' + (\eta f)' = 0 \quad \Rightarrow \quad f' + \eta f = C_1$$

$$\left.\begin{array}{l} c(x,t) \to 0 \text{ as } x \to \infty \\ c(x,t=0) = 0 \end{array}\right\} \eta = \infty \qquad \eta = \frac{Bx}{t^m}$$

$c = At^n f(\eta)$ $\quad$ since $c \to 0$ as $\eta \to \infty$ irrespective of t $\Rightarrow \underline{f(\eta \to \infty) \to 0}$



$\Longrightarrow f' \to 0$ as $\eta \to \infty$

$f' + \eta f = C_1 \quad \Rightarrow \quad C_1 = 0$ since f' & $f \to 0$ as $\eta \to \infty$

$$\Longrightarrow \quad f' + \eta f = 0 \quad \Rightarrow \quad \frac{df}{f} = -\eta\, d\eta$$

$$\ln f = -\eta^2/2 + \ln C_2$$

$$f = C_2 e^{-\eta^2/2}$$

$c = A t^n f(\eta) \qquad n = -m = -1/2$

$\eta = \frac{Bx}{t^m} \cdot = \frac{x}{\sqrt{2\alpha t}}$

$f = C_2 e^{-\eta^2/2} \qquad \int_0^\infty c(x,t)\,dx = \frac{A}{B}\int_0^\infty f(\eta)\,d\eta = Q$

choose $f(\eta=0) = 1 \Longrightarrow C_2 = 1 \Longrightarrow f = e^{-\eta^2/2}$

$$\int_0^\infty c(x,t)\,dx = \frac{A}{B}\int_0^\infty e^{-\eta^2/2}\,d\eta = Q$$

$$= \sqrt{2\alpha}\cdot A \underbrace{\int_0^\infty e^{-z^2}\,dz}_{\frac{\sqrt{\pi}}{2}} \cdot \sqrt{2} = Q$$

$$A = \frac{Q}{\sqrt{\pi\alpha}}$$

$$c = \frac{Q}{\sqrt{\pi\alpha t}}\, e^{-\frac{x^2}{4\alpha t}}$$

- Do 2.1 & 2.2

Exercises:

2.1 The temperature field $T(x,t)$ in a semi-infinite slab with a constant heat flux is described by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad ; \quad T(x,0) = T_i$$

$$T(x,t) \to T_i \quad \text{as} \quad x \to \infty \quad ; \quad -k \frac{\partial T}{\partial x} = q \quad \text{at} \quad x = 0$$

Solve for the temperature field for $x \geq 0$, $t \geq 0$.

2.2 The temperature field in the thermal boundary layer that grows within a hydrodynamic boundary layer at a step in wall temperature is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \to T_\infty \quad \text{as} \quad y \to \infty \quad ; \quad T(x,0) = T_w \; ;$$

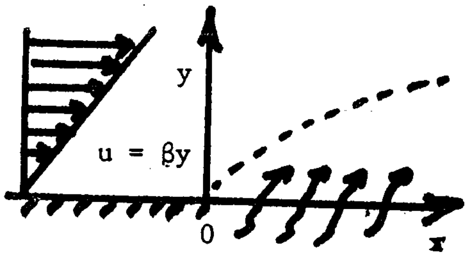
Solve for the temperature field for $x \geq 0$, $y \geq 0$.

2.3 A device for measuring the velocity gradient in flows is shown in the figure. It consists of a heated plate at the wall, over which a thermal boundary layer grows. As long as the thermal boundary layer is confined to the region where the flow velocity u is linear $(u = \beta y)$, the problem is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

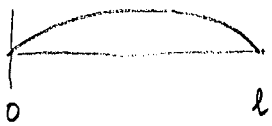
$$T(x,y) \to T_\infty \quad \text{as} \quad y \to \infty \quad ; \quad -k \frac{\partial T}{\partial y} = q \quad \text{at} \quad y = 0$$

Derive an expression relating the local wall temperature, $T_w(x)$, to the flow parameters and x . Evaluate any constants in this expression. Hint: Γ .



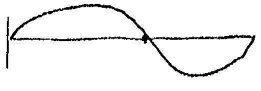
$\sin \frac{n\pi x}{l}$

$n=1$

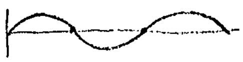


$\underline{k = \frac{\pi c}{l}}$ Fundamental frequency

NEED TO DO



node at $1/2$ $n=2$



nodes at $l/3$, $2l/3$ $n=3$

To show general solution $w_n(x,t) = C_n\left[\sin\frac{n\pi x}{l}\cos\left(\frac{n\pi ct}{l}\right)\right] + D_n\left[\sin\frac{n\pi x}{l}\sin\left(\frac{n\pi ct}{l}\right)\right]$

let $A = \frac{n\pi x}{l}$ $B = \frac{n\pi ct}{l}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ $\cos(A-B) = \cos A\cos B + \sin A\sin B$

$\frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] = \sin A\cos B$ $\frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] = \sin A\sin B$

$$w_n = \left\{\frac{C_n}{2}\sin(A+B) - \frac{D_n}{2}\cos(A+B)\right\} + \left\{\frac{C_n}{2}\sin(A-B) + \frac{D_n}{2}\cos(A-B)\right\}$$

$$= f\left(\frac{n\pi}{l}[x+ct]\right) + g\left(\frac{n\pi}{l}[x-ct]\right) = \mathcal{F}(x+ct) + \mathcal{G}(x-ct)$$

go back to $W_{xx} - \frac{1}{c^2}W_{tt} = 0$ $a=1$ $b=0$ $c=-\frac{1}{c^2}$

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{0 \pm \sqrt{0+4\frac{1}{c^2}}}{2} = \pm\frac{1}{c}$$

or $dt \pm \frac{dx}{c} = 0$ or $t \pm \frac{x}{c} = \text{const.}$ $ct + x = \text{constant}_1 = \varphi_1$

$ct - x = \text{constant}_2 = -\varphi_2$

Characteristics

in general $\underline{w(x,t)} = \underline{f(\varphi_1)} + \underline{g(\varphi_2)}$

LESSON # 2

NEED TO DO $f(\varphi_1)$ represents a wave | a wave moving in +x dir; $g(\varphi_2)$ represents moving in +x
with velocity c

WE CAN USE THIS RESULT TO FIND THE DISPLACEMENT OF AN ∞ LONG STRIN

Using I.C. $W(x,t=0) = W_0(x)$ $-\infty \le x \le \infty$ $\frac{\partial W}{\partial t}(x,t=0) = W_1(x)$

$W(x,t=0) = f(x) + g(x) = W_0(x)$ x is a dummy variable

$\frac{\partial W}{\partial t}(x,t=0) = c f'(x) - c g'(x) = W_1(x) \Rightarrow f'(x) - g'(x) = \frac{1}{c}W_1(x)$

$$f(x) - g(x) = \frac{1}{c}\int_{x_0}^{x} W_1(\xi)\,d\xi$$

$$\left.\begin{array}{l} f+g = W_0(x) \\ f-g = \frac{1}{c}\int_{x_0}^{x} W_1(\tilde{x})d\tilde{x} \end{array}\right\} \qquad f(x) = \frac{1}{2}\left[W_0(x) + \frac{1}{c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma\right]$$

$$g(x) = \frac{1}{2}\left[W_0(x) - \frac{1}{c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma\right]$$

but $f(x+ct) = \frac{1}{2}\left[W_0(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct} W_1(\sigma)\,d\sigma\right]$

$$g(x-ct) = \frac{1}{2}\left[W_0(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct} W_1(\sigma)\,d\sigma\right]$$

$$w(x,t) = f+g = \frac{1}{2}\left[W_0(x+ct) + W_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} W_1(\sigma)\,d\sigma$$

displacement dependent term with $W_1 = 0$ — velocity dependent term, with $W_0 = 0$

• Note that no boundary conditions are needed

Application: let $W_0(x) = \sin \pi x/L \quad 0 \le x \le L$ & $W_1(x) = 0$

$$w(x,t) = \frac{1}{2}\left[\sin\frac{\pi}{L}(x+ct) + \sin\frac{\pi}{L}(x-ct)\right]$$ — 2 wave forms. having ½ height of original, but same form.

Can we use this result for a finite length string? yes, by reflections

TALK ABOUT
LEFT GOING WAVE
RIGHT GOING WAVE

$f(ct) + g(-ct) = 0 : w$

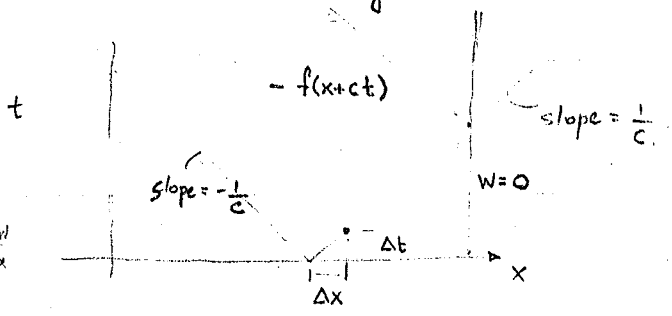
$g(-ct) = -f(ct)$

$f'(ct) + g'(ct) = 0' = \frac{\partial w}{\partial x}$

$f'(ct) = -g'(-ct)$

$f(ct) \quad g(-ct)$

Look at a wave $g(x-ct)$

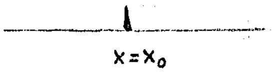


$x - ct = \text{const} \Rightarrow \Delta x - c\Delta t = 0$

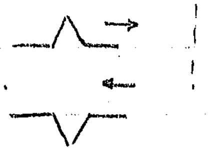
if $\Delta t > 0 \Rightarrow \Delta x > 0$ right moving

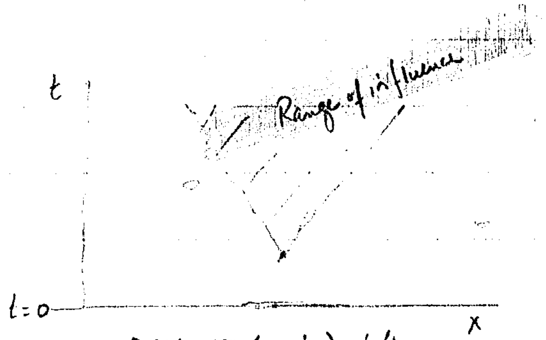
$x + ct = \text{const} \Rightarrow \Delta x + c\Delta t = 0$

if $\Delta t > 0 \Rightarrow \Delta x < 0$

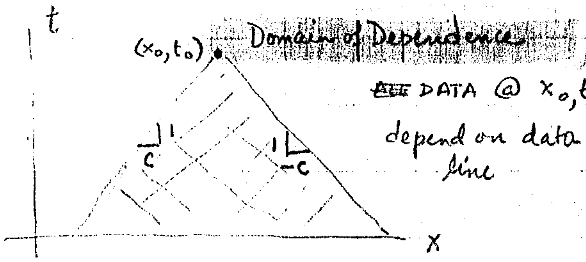


$g(x-ct)$ reflects as $-f(x+ct) \Rightarrow$





DATA AT (x_0,t_0) influence.



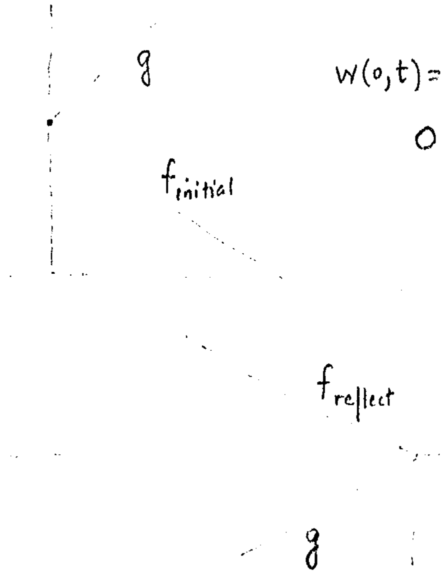
Domain of Dependence

~~ALL~~ DATA @ x_0, t_0 depend on data between line

LET $W(x,t) = f(x+ct) + g(x-ct)$ $\quad W(x,0)=F(x) \quad \frac{\partial W}{\partial t}(x,0)=G(x)$ defined for $0 \le x \le$ only

Assume $W(0,t) = 0$ and $W(l,t)=0$; $0 \le x \le l \quad t \ge 0$

REWRITE THESE TWO LESSONS



$W(0,t)=0 = f(ct) + g(-ct)$ $\quad$ let $ct = u$.

$0 = f(u) + g(-u)$ $\quad$ f is reflected as a g wav

$g(-u) = -f(u)$ $\quad$ g is same form as u but -ive.

- $g(u)$ is defined for $+u$.
- Only then $+g(-u) = -f(u)$ is $g(-u)$ defined
- $f(-u) = -g(u)$

$w(l,t) = 0 = f(l+ct) + g(l-ct)$ $\quad$ let $ct-l=u$

$0 = f(u+2l) + g(-u)$ $\quad ct+l = u+2l$.

$\therefore \quad f(u+2l) = -g(-u) = f(u)$

$-f(-u+2l) = g(u) = -f(-u)$

g is reflected as an f wave

reflected f wave = initial f wave but its argument is increased by $2l$

$f(u) = f(u+2l)$ is a periodicity condition

1) $f(u)=f(u+2l)$ & $f(u)=-g(-u)$ & $g(u)=-f(-u)$.

——— ∘ ——— ∘ ——— ∘ ——— LESSON # 3

Suppose $W(x,0) = F(x) \quad \frac{\partial W}{\partial t}(x,0) = G(x) \qquad 0 \le x \le L$

$W(x,0) = F(x) = f(x) + g(x)$ $\qquad F(u) = f(u) + g(u)$

$\frac{\partial W}{\partial t}(x,0) = G(x) = c[f'(x) - g'(x)]$ $\qquad G(u) = c[f'(u) - g'(u)] = H'(u)$

integrate 1 time $\quad H(u) = c[f(u) - g(u)]$ $\qquad H(u) = \int_0^u G(\bar{u})\,d\bar{u}$

$f(u) = \frac{1}{2}\left[F(u) + \frac{1}{c}H(u)\right]$ $\qquad g(u) = \frac{1}{2}\left[F(u) - \frac{1}{c}H(u)\right]$

also $\quad -g(-u) = f(u) = -\frac{1}{2}\left[F(-u) - \frac{1}{c}H(-u)\right]$

$F(u) = -F(-u)$ $\quad$ or $F(u)$ is odd fn.

$H(u) = H(-u)$ $\quad$ or $H(u)$ is even $\quad$ but $\frac{dH}{du} = G(u) \quad \therefore$ G must b

$$\left.\begin{array}{ll} w_0(x) = \sin x & f(x) = \frac{1}{2}\sin x \\ w_1(x) = 0 & g(x) = \frac{1}{2}\sin x \end{array}\right\} \quad 0 \le x \le \ell$$

if $w(0,t) = 0 \Rightarrow f(x) + g(-x) = 0 \Rightarrow f(x) = -g(-x)$

$$\left.\begin{array}{l} \Rightarrow g(-x) = -f(x) = -\frac{1}{2}\sin x \\ \text{but } f(-x) = -g(x) = -\frac{1}{2}\sin x \end{array}\right\} -\ell \le -x \le 0$$

if $w(\ell,t) = 0 \Rightarrow f(\ell+ct) + g(\ell-ct) = 0$

let $\ell - ct = -u \qquad g(-u) + f(u+2\ell) = 0$

$\Rightarrow -f(u) + f(u+2\ell) = 0 \Rightarrow f(u+2\ell) = f(u)$

@ $t=0$ $\quad \ell - ct = \ell$
$\ell + ct = \ell$

@ $t>0$ $\quad \ell - ct < \ell$
$\ell + ct \geq \ell$

let $\ell - ct = u$
$ct - \ell = -u \qquad ct + \ell = -u + 2\ell$

$\Rightarrow f(u+2\ell) = f(u) \Rightarrow \frac{1}{2}\sin x$

$0 = g(u) + f(-u+2\ell) \Rightarrow g(u) = -f(-u+2\ell)$
$= -f(-u)$

let $ct + \ell = \sigma \qquad \ell - ct = -\sigma + 2\ell$

$\therefore f(\sigma) + g(-\sigma + 2\ell) = 0 \qquad$ for $0 \le \sigma \le \ell$

$f(\sigma)[= -g(-\sigma)] + g(-\sigma+2\ell) = 0 \quad \sigma + 2\ell \qquad \ell$ to 2ℓ.

$\Rightarrow \underline{g(-\sigma) = g(-\sigma + 2\ell)}$

note that if $\quad f(\sigma) = \frac{1}{2}\sin\sigma \qquad \sigma = x + ct$

$g(\sigma) = \frac{1}{2}\sin\sigma \qquad \sigma = x - ct$

$f(-\sigma) = -g(\sigma) = -\frac{1}{2}\sin\sigma \Leftarrow \quad w(0,t) = 0$

$g(-\sigma) = -f(\sigma) = -\frac{1}{2}\sin\sigma \Leftarrow$

$g(-\sigma + 2\ell) = +g(-\sigma) = -\frac{1}{2}\sin\sigma \qquad w(\ell,t) = 0$

$$f(x+ct) = \frac{1}{2}\left[F(x+ct) + \frac{1}{c}\int_0^{x+ct} G(\bar{u})\,d\bar{u}\right]$$

$$g(x-ct) = \frac{1}{2}\left[F(x-ct) - \frac{1}{c}\int_0^{x-ct} G(\bar{u})\,d\bar{u}\right] = \frac{1}{2}\left[F(x-ct) + \frac{1}{c}\int_{x-ct}^{0} G(\bar{u})\,d\bar{u}\right]$$

$$w(x,t) = f+g = \frac{1}{2}\left[F(x+ct) + F(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} G(\bar{u})\,d\bar{u}\right]$$

$f(u) = f(u+2l)$ $\qquad g(-u) = -f(u)$ $\qquad$ B.C.

$\qquad\qquad\qquad\qquad f(-u) = -g(u)$

Use this method for small number of reflections (1 - 5)

Use SOV method for large number of reflections

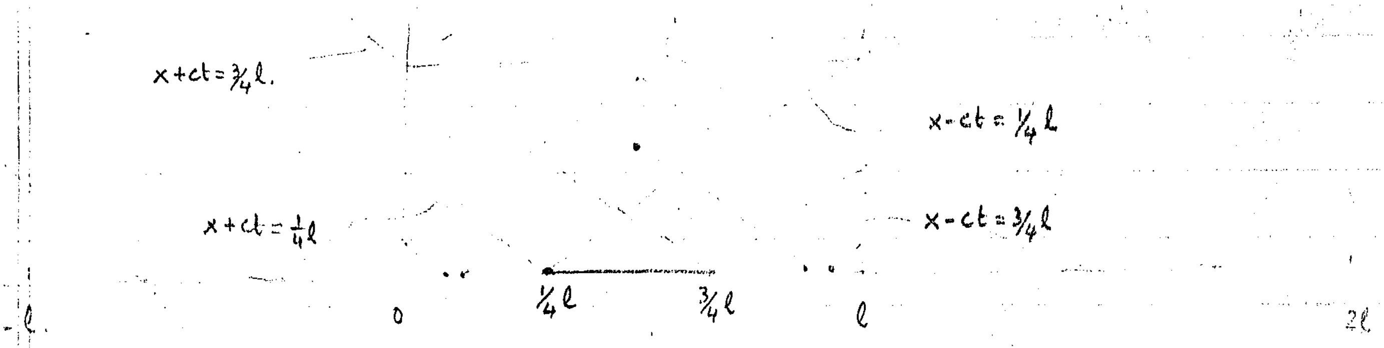
FOR EXAMPLE LET $F(x) = \sin x \quad \frac{1}{4}l \leq x \leq \frac{3}{4}l \qquad G(x) = 0$

$f(x) \cdot \frac{1}{2}F(x) = \frac{1}{2}\sin x \qquad f(x+ct) = \frac{1}{2}\sin(x+ct)$

$g(x) = \frac{1}{2}F(x) = \frac{1}{2}\sin x \qquad g(x-ct) = \frac{1}{2}\sin(x-ct)$

$f(-x) = -g(x) = -\frac{1}{2}\sin x \qquad f(x) = f(x+2l)$

$g(-x) = -f(x) = -\frac{1}{2}\sin x \qquad f(-x) = f(-x+2l) = -\frac{1}{2}\sin(x)$



BY LAPLACE TRANSFORMS $\quad$ let $\mathcal{U}(x;s) = \int_0^t u(x,t)e^{-st}dt$

$u_{xx} - u_{tt} = 0 \qquad \mathcal{L}\{u_{xx} - u_{tt}\} = \mathcal{U}_{xx} - \{s^2\mathcal{U} - s\,u(x,0) - u_t(x,0)\} = 0$

$u(x,t=0) = xe^{-x} \qquad \mathcal{U}_{xx} - s^2\mathcal{U} + s\,xe^{-x} = 0$

$u_t(x,0) = 0 \qquad \mathcal{U}_{xx} - s^2\mathcal{U} = -sxe^{-x}$

$u(0,t) = 0 \qquad \mathcal{L}\{u(0,t)=0\} \Rightarrow \mathcal{U}(0;s) = 0$

Let $\mathcal{U} = \mathcal{U}_H + \mathcal{U}_P \qquad \mathcal{U}_H$ solves: $\mathcal{U}_{xx} - s^2\mathcal{U} = 0$

$\qquad\qquad\qquad\qquad \mathcal{U}_P$ solves: $\mathcal{U}_{xx} - s^2\mathcal{U} = sxe^{-x}$

$$u_{xx} - s^2 u = \frac{s}{x+1}$$

$$\frac{1}{e^{s^2x}} \int \frac{e^{+s^2x}}{x+1}\, dx$$

If you remember, for $u_{xx} - \frac{1}{c^2} u_{tt} = 0$ under $u(x,0) = w_0(x)$
$u_t(x,0) = w_1(x)$

we found $u(x,t) = \mathcal{F}(x+ct) + \mathcal{G}(x-ct)$

and that $\mathcal{F}(x) = \frac{1}{2} w_0(x) + \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)\,d\sigma \qquad (1)$

$\mathcal{G}(x) = \frac{1}{2} w_0(x) - \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)\,d\sigma \qquad (2)$

we said that

$$u(x,t) = \frac{1}{2}\left[w_0(x+ct) + w_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} w_1(\sigma)\,d\sigma$$

This was good for $|x+ct| \leq \infty$ & $|x-ct| \leq \infty$ for the infinite line

- what if now we wanted to impose $u(x=0,t)=0$ $u(x=L,t)=0$
  - That is, that $w_0(x)$ & $w_1(x)$ are only defined for $0 \leq x \leq L$
- we found that from $u(x=0,t)=0$ $\mathcal{F}(\sigma) = -\mathcal{G}(-\sigma)$ $(\sigma = ct)$

  we found that from $u(x=L,t)=0$ $\mathcal{G}(-\sigma) = -\mathcal{F}(+\sigma+2L)$ $(\sigma = \ell - ct)$

  and since $\mathcal{G}(-\sigma) = -\mathcal{F}(\sigma) = -\mathcal{F}(\sigma+2L)$

Then the conditions imposed on $\mathcal{F}$ & $\mathcal{G}$ for arguments $\sigma > L$ & $\sigma < 0$

were $\mathcal{F}(-\sigma) = -\mathcal{G}(\sigma)$ (3)
$\mathcal{G}(-\sigma) = -\mathcal{F}(\sigma)$ (4)
} here $0 \leq \sigma \leq L$

and $\mathcal{F}(\sigma+2L) = \mathcal{F}(\sigma)$ (5)
$\mathcal{G}(\sigma+2L) = \mathcal{G}(\sigma)$ (6)
} here $0 \leq \sigma \leq L$

so if $w_0(x) = \sin x$ $0 \leq x \leq L$ $w_1(x) = 0$

then $\mathcal{F}(x) = \frac{1}{2}\sin x$ $0 \leq x \leq L$ from (1) & (2) (5)

$\mathcal{G}(x) = \frac{1}{2}\sin x$ $0 \leq x \leq L$ (6)

thus

$$\left.\begin{aligned} \mathcal{F}(x+ct) &= \tfrac{1}{2}\sin(x+ct) \quad ① \\ \mathcal{G}(x-ct) &= \tfrac{1}{2}\sin(x-ct) \quad ② \end{aligned}\right\} \text{ for } \begin{aligned} 0 \leq x+ct \leq L \\ 0 \leq x-ct \leq L \end{aligned}$$

for $\mathcal{F}(\sigma) = -\mathcal{G}(-\sigma)$ $-L \leq \sigma \leq 0$ note $0 \leq -\sigma \leq L$
$\mathcal{G}(\sigma) = -\mathcal{F}(-\sigma)$

but $\mathcal{G}(-\sigma) = \frac{1}{2}\sin(-\sigma) = -\mathcal{F}(\sigma)$ remember $-\sigma$ is >0

$\mathcal{F}(-\sigma) = \frac{1}{2}\sin(-\sigma) = -\mathcal{G}(\sigma)$ $\mathcal{G}(\ ) = \frac{1}{2}\sin(\ )$

or $\mathcal{F}(\sigma) = \frac{1}{2}\sin\sigma$ $-L \leq \sigma \leq 0$ (7) from (3) & (6)

$\mathcal{G}(\sigma) = \frac{1}{2}\sin\sigma$ $-L \leq \sigma \leq 0$ (8) from (4) & (5)

thus $\mathcal{F}(x+ct) = \frac{1}{2}\sin(x+ct)$ ③ $-L \leq x+ct \leq 0$ (9) here $\sigma = x+ct$

$\mathcal{G}(x-ct) = \frac{1}{2}\sin(x-ct)$ ④ $-L \leq x-ct \leq 0$ (10) $\sigma = x-ct$

What about in the region $L \leq x \leq 2L$

here we use $\mathcal{F}(\sigma) = \mathcal{F}(\sigma+2L)$; if $-L \leq \sigma \leq 0 \Rightarrow L \leq \sigma+2L \leq 2$

$\mathcal{G}(\sigma) = \mathcal{G}(\sigma+2L)$

$\therefore$ from (7) & (8)

$$\mathcal{F}(\sigma) = \frac{1}{2}\sin\sigma = \mathcal{F}(\sigma+2L)$$
$$\mathcal{G}(\sigma) = \frac{1}{2}\sin\sigma = \mathcal{G}(\sigma+2L)$$

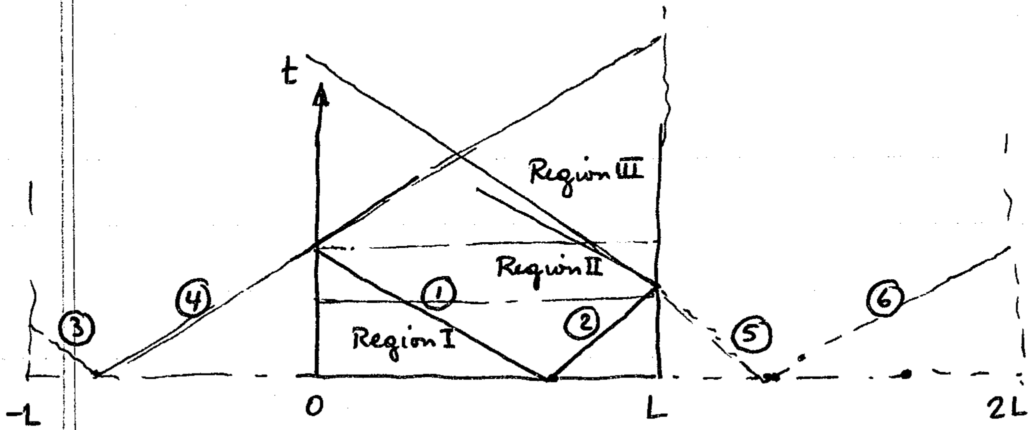
since we want the argument of the function $L \leq \eta \leq 2L$ let $\eta = \sigma+2L$

$\therefore \sigma = \eta - 2L$

$$\therefore \left.\begin{aligned}\mathcal{F}(\eta) &= \mathcal{F}(\eta-2L) = \frac{1}{2}\sin(\eta-2L)\\ \mathcal{G}(\eta) &= \mathcal{G}(\eta-2L) = \frac{1}{2}\sin(\eta-2L)\end{aligned}\right\}\text{ here } L \leq \eta \leq 2L$$

$$\therefore \left.\begin{aligned}\mathcal{F}(x+ct) &= \frac{1}{2}\sin(x+ct-2L) \quad ⑤\\ \mathcal{G}(x-ct) &= \frac{1}{2}\sin(x-ct-2L) \quad ⑥\end{aligned}\right\}\text{ here } \begin{aligned}L \leq x+ct \leq 2L\\ L \leq x-ct \leq 2L\end{aligned}$$

Now let's look at what happens



in region I $\quad u(x,t) =$ ① + ②

II $\quad u(x,t) =$ ① + ⑤

III $\quad u(x,t) =$ ④ + ⑤

etc.

7.4. Given $u_{xx} - u_{tt} = 0$; $\underset{= u_0(x)}{u(x,0) = 0}$ $\underset{= u_1(x)}{u_t(x,0) = e^{-x^2}}$ for $-\infty \le x \le \infty$

this problem just requires the use of the D'Alembert Solution which is good for $-\infty \le x \le \infty$

$$F(x) = \frac{1}{2} u_0(x) + \frac{1}{2c}\int_{x_0}^{x} u_1(\sigma)\,d\sigma = \frac{1}{2}0 + \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}\,d\sigma$$

$$G(x) = \frac{1}{2} u_0(x) - \frac{1}{2c}\int_{x_0}^{x} u_1(\sigma)\,d\sigma = \frac{1}{2}\cdot 0 - \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}\,d\sigma$$

but $u(x,t) = F(x+ct) + G(x-ct)$

$$= \frac{1}{2}\int_{x_0}^{x+ct} e^{-\sigma^2}\,d\sigma - \frac{1}{2}\int_{x_0}^{x-ct} e^{-\sigma^2}\,d\sigma = \frac{1}{2}\int_{x-ct}^{x+ct} e^{-\sigma^2}\,d\sigma$$

or in terms of the erf function

$$u(x,t) = \frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right]$$

Note that these formulas are good for $-\infty \le x+ct \le \infty$ & $-\infty \le x-ct \le \infty$

7.5 Given $u_{xx} - u_{tt} = 0$ with $\underset{= u_0(x)}{u(x,0) = xe^{-x}}$ $0 \le x \le \infty$

and $\underset{= u_1(x)}{u_t(x,0) = 0}$ and $u(0,t) = 0$

Since the initial data is <u>not</u> given over $-\infty \le x \le \infty$ but only over part of it $(0 \le x \le \infty)$, we must find a way to extend the initial data so that it is defined over the entire $-\infty \le x \le \infty$. we shall use the boundary condition $u(0,t) = 0$

$u(x,t) = F(x+t) + G(x-t)$ from the PDE

(1) thus $u(0,t) = F(t) + G(-t) = 0$ or $G(-t) = -F(t)$

This means that if $F(t)$ is defined for $t > 0$ then G is defined

(2) for $-t < 0$. Similarly $-G(t) = F(-t)$ (let $-t = \sigma$ in the above eqn, then let $\sigma = t$).

Therefore we have found out how to define F & G for negative arguments as well

Now $F(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_{x_0}^{x} u_1(\sigma)d\sigma$ is good for $-\infty \leq x \leq \infty$ ie for negative as well as positive arguments. For <u>positive</u> arguments

(3) $F(x) = \frac{1}{2}xe^{-x} + \frac{1}{2}\cdot 0 = \frac{1}{2}xe^{-x} \qquad 0 \leq x \leq +\infty$

Also $G(x) = \frac{1}{2}u_0(x) + \frac{1}{2c}\int_{x_0}^{x} u_1(\sigma)d\sigma$ is good for $-\infty \leq x \leq \infty$ ie for negative as well as positive arguments. For positive arguments

(4) $G(x) = \frac{1}{2}xe^{-x} - \frac{1}{2}\cdot 0 = \frac{1}{2}xe^{-x} \qquad 0 \leq x \leq +\infty$

But $\left.\begin{array}{l} F(-x) = -G(x) \\ G(-x) = -F(x) \end{array}\right\}$ from before; here $x \geq 0$ and $G(x)$ & $F(x)$ are defined

Thus $F(-x) = -\frac{1}{2}xe^{-x} = \frac{1}{2}(-x)e^{(-x)} \qquad x \geq 0$

and $F(\sigma) = \frac{1}{2}\sigma e^{\sigma} \qquad \sigma \leq 0$

Also $G(-x) = -F(x) = -\frac{1}{2}xe^{-x} = \frac{1}{2}(-x)e^{(-x)} \qquad x \geq 0$

$G(\sigma) = \frac{1}{2}\sigma e^{\sigma} \qquad \sigma \leq 0$

(5) Thus for $x \leq 0$ $\qquad F(x) = \frac{1}{2}xe^{x}$

(6) $\qquad G(x) = \frac{1}{2}xe^{x}$

(7) Now for $x+t > 0$ $\quad F(x+t) = \frac{1}{2}(x+t)e^{-(x+t)}$ $\quad$ from (3)

(8) for $x+t < 0$ $\quad F(x+t) = \frac{1}{2}(x+t)e^{(x+t)}$ $\quad$ from (5)

(9) for $x-t > 0$ $\quad G(x-t) = \frac{1}{2}(x-t)e^{-(x-t)}$ $\quad$ from (4)

(10) for $x-t < 0$ $\quad G(x-t) = \frac{1}{2}(x-t)e^{(x-t)}$ $\quad$ from (6)

what is $u(x,t)$ when $t=1$? Remember that the original region we are interested in is $0 \leq x \leq \infty$ & $t \geq 0$

$\therefore$ @ $t=1$ $\quad x+t = x+1$ which is always > 0

$\therefore$ $F(x+t)$ is <u>always</u> given by (7)

@ $t=1$ $x-t = x-1$. This is >0 when $x>1$ & <0 when $x<1$
$\therefore$ for $x>1$ $G(x-t)$ is given by (9) & for $x<1$ $G(x-t)$ is given by (10)

Thus $u(x,t)$ for $x \geqslant 1$ & $t=1$ is given by $\frac{1}{2}(x+t)e^{-(x+t)} + \frac{1}{2}(x-t)e^{-(x-t}$
and for $0 \leq x \leq 1$ & $t=1$ $u(x,t) = \frac{1}{2}(x+t)e^{-(x-t)} + \frac{1}{2}(x-t)e^{(x-t)}$

7.6 Given $u_{xx} - u_{tt} = 0$ for $(0 \leq x \leq 1)$ with $u(x,0)=0$

$$\text{and } u_t(x,0) = \begin{cases} 1 & 0 \leq x \leq \frac{1}{2} \\ 0 & \frac{1}{2} \leq x \leq 1 \end{cases} \quad \text{and } u(0,t)=0, \ u(1,t)=0$$

Again we have a situation where the initial data is <u>not</u> given over $-\infty \leq x \leq \infty$, but only over part of it $(0 \leq x \leq 1)$. We must extend the initial data so that it is defined over the entire $-\infty \leq x \leq \infty$. Having done so we can use the D'Alembert formula. If you remember we did this in class $u(0,t)=0 \implies -F(\sigma) = G(-\sigma)$ and $-G(\sigma) = F(-\sigma)$

$-1 \leq -\sigma \leq 0$ Also $u(1,t)=0 \implies F(\tilde{\sigma}) = F(\tilde{\sigma}+2)$ & $G(\tilde{\sigma}) = G(\tilde{\sigma}+2)$

$1 \leq \tilde{\sigma}+2 \leq 2$ We have thus extended the definition of F & G but we must write the argument of F & G in terms of one variable e.g. $\tilde{\sigma}$ not $\tilde{\sigma}+2$.
Why? remember we must define $F(\sigma)$ & $G(\sigma)$ then let $\sigma = x+t, x-t$
$\therefore$ $F(\tilde{\sigma}) = F(\tilde{\sigma}+2) \implies F(\sigma-2) = F(\sigma)$ & $G(\sigma-2) = G(\sigma)$ where $1 \leq \sigma \leq 2$ and $-1 \leq \sigma - 2 \leq 0$. Now let us find $F(\sigma)$ & $G(\sigma)$ in $0 \leq \sigma \leq 1$ using the D'Alembert Formula with the lower limit as zero

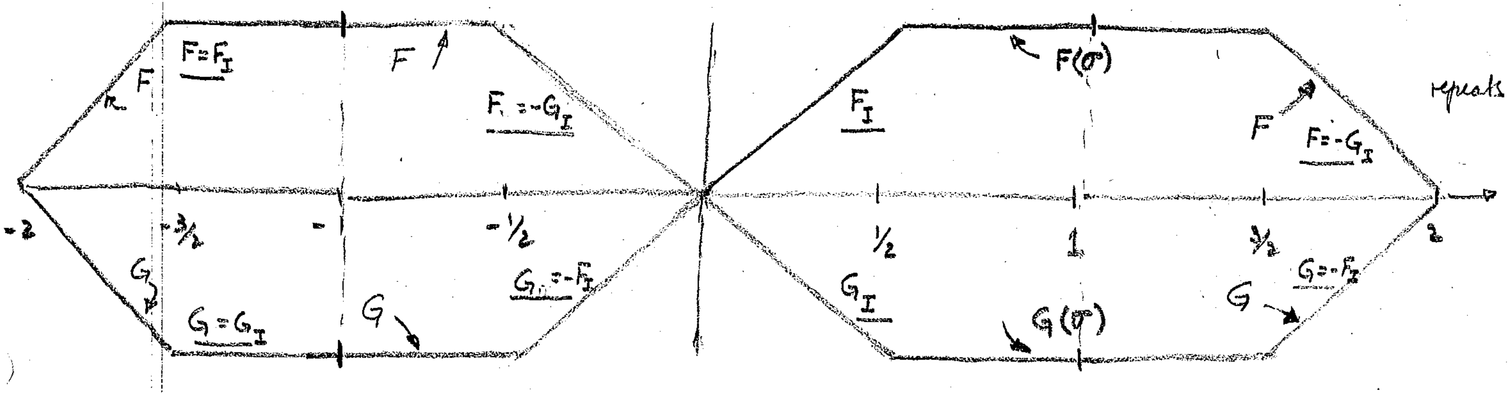
$$F(\sigma) = \frac{1}{2}u_0(\sigma) + \frac{1}{2c}\int_0^{\sigma} u_1(x)dx = \frac{1}{2}\cdot 0 + \frac{1}{2}\int_0^{\sigma} 1\,dx \qquad 0 \leq \sigma \leq \frac{1}{2}$$
$$= \frac{\sigma}{2}$$
$$F(\sigma) = \frac{1}{2}u_0(\sigma) + \frac{1}{2c}\int_0^{\sigma_2} u_1(x)dx = \frac{1}{2}\cdot 0 + \frac{1}{2}\int_0^{1/2} 1\cdot dx \qquad \frac{1}{2} \leq \sigma \leq 1$$
$$= \frac{1}{4}$$

$$G(\sigma) = \frac{1}{2}u_0(\sigma) - \frac{1}{2c}\int_0^{\sigma} u_1(x)dx = \frac{1}{2}\cdot 0 - \frac{1}{2}\int_0^{\sigma} 1\,dx = -\frac{\sigma}{2} \qquad 0 \le \sigma \le \frac{1}{2}$$

$$G(\sigma) = \frac{1}{2}u_0(\sigma) - \frac{1}{2c}\int_0^{\sigma} u_1(x)dx = \frac{1}{2}\cdot 0 - \frac{1}{2}\int_0^{1/2} 1\,dx = -\frac{1}{4} \qquad \frac{1}{2} \le \sigma \le 1$$

Now for $-1 \le -\sigma \le 0$ $\quad F(-\sigma) = -G(\sigma) = \begin{cases} +\sigma/2 \\ +1/4 \end{cases}$ or $F(\sigma) = \begin{cases} -\sigma/2 \\ 1/4 \end{cases}$ $-1 \le \sigma \le$

$$G(-\sigma) = -F(\sigma) = \begin{cases} -\sigma/2 \\ -1/4 \end{cases} \quad \text{or} \quad G(\sigma) = \begin{cases} \sigma/2 \\ -1/4 \end{cases} \quad -1 \le \sigma \le 0$$



Now for the interval $1 \le (\sigma+2) \le 2$ $\qquad F(\sigma+2) = F(\sigma) \quad -1 \le \sigma \le 0$

$$\underset{\uparrow}{F(\sigma+2)} = F(\sigma) = \begin{cases} -\sigma/2 \\ \underset{\uparrow}{1/4} \end{cases}$$ definition

but <u>we want to define</u> $F(\sigma)$ in the interval $1 \le \sigma \le 2$

replace σ by $\sigma - 2$ $\Rightarrow$ $F(\sigma) = F(\sigma-2) = \begin{cases} -\sigma/2 + 1 \\ 1/4 \end{cases}$ where $1 \le \sigma \le 2$

similarly we can define $G(\sigma+2) = G(\sigma) \quad -1 \le \sigma \le 0 \qquad 1 \le \sigma+2 \le 2$

$$\underset{\uparrow}{G(\sigma+2)} = G(\sigma) = \begin{cases} \sigma/2 \\ \underset{\uparrow}{-1/4} \end{cases} \quad \text{or} \quad G(\sigma) = \begin{cases} \sigma/2 - 1 \\ -1/4 \end{cases} \quad 1 \le \sigma \le 2$$

let's look at $u(x,t) = F(x+t) + G(x-t)$

$x+t > 0 \quad F(x+t) = \begin{cases} \frac{x+t}{2} & \text{for } 0 \le x+t \le 1/2 \\ 1/4 & \text{for } \frac{1}{2} \le x+t \le 1 \end{cases}$

$x+t < 0 \quad F(x+t) = \begin{cases} -\frac{(x+t)}{2} & -\frac{1}{2} \le x+t \le 0 \\ 1/4 & -1 \le x+t \le -1/2 \end{cases}$

$x-t>0$ $\quad G(x-t) = \begin{cases} -\frac{x-t}{2} & \text{for } 0 \le x-t \le 1/2 \\ -1/4 & \text{for } \frac{1}{2} \le x-t \le 1 \end{cases}$

$x-t<0$ $\quad G(x-t) = \begin{cases} \frac{x-t}{2} & \text{for } -1/2 \le x-t \le 0 \\ -1/4 & -1 \le x-t \le -1/2 \end{cases}$

Now for $1 \le x+t \le 2$ $\quad F(x+t) = \begin{cases} -\frac{(x+t)}{2}+1 & \text{for } \frac{3}{2} \le x+t \le 2 \\ 1/4 & \text{for } 1 \le x+t \le 3/2 \end{cases}$

$1 \le x-t \le 2$ $\quad G(x-t) = \begin{cases} \frac{(x-t)}{2}-1 & \text{for } 3/2 \le x-t \le 2 \\ -1/4 & \text{for } 1 \le x-t \le 3/2 \end{cases}$

$F(\sigma-2)=F(\sigma)$
$0 \le \sigma \le 1$

Now for $-2 \le x+t \le -1$ $\quad F(x+t) = \begin{cases} +\frac{(x+t)}{2}+1 & \text{for } -2 \le x+t \le -3/2 \\ 1/4 & \text{for. } -3/2 \le x+t \le -1 \end{cases}$

$G(\sigma-2)=G(\sigma)$

$-2 \le x-t \le -1$ $\quad G(x-t) = \begin{cases} -\frac{(x-t)}{2}-1 & \text{for } -2 \le x-t \le -3/2 \\ -1/4 & \text{for } -3/2 \le x-t \le -1 \end{cases}$

and so on to get the rest

Now by Fourier Series – assume $u(x,t) = w(x)\,F(t)$

$u_{xx} - u_{tt} = 0 \implies w''F - wF'' = 0$ or $\frac{w''}{w} - \frac{F''}{F} = 0$ or $\frac{w''}{w} = \frac{F''}{F} = -\lambda^2$

$\therefore \quad w'' + \lambda^2 w = 0 \implies w(x) = A\cos\lambda x + B\sin\lambda x$

since $u(0,t)=0$ & $u(1,t)=0 \implies A=0$ & $\sin\lambda = 0 \Rightarrow \lambda = n\pi$

$n \ne 0.$ If $\lambda = 0$ then $w(x) = Ax+B \implies A=B=0 \Rightarrow$ Trivial Sol

$\therefore \quad w(x) = B\sin n\pi x$ (eigenfunction) with $\lambda = n\pi$ (eigenvalue)

now $F'' + \lambda^2 F = 0 \implies F(t) = \tilde{C}\cos\lambda t + \tilde{D}\sin\lambda t$ and

$$u(x,t) = \sum_{n=1}^{\infty} \sin n\pi x \left[C_n \cos n\pi t + D_n \sin n\pi t \right] \qquad C_n = \tilde{C}B \quad D_n = \tilde{D}B$$

To satisfy the initial conditions that at $t=0$ $\quad u(x,t=0)=0 \Rightarrow C_n = 0$ for all n. To satisfy $u_t(x,t=0) = u_1(x) = \begin{cases} 1 & 0 \le x \le 1/2 \\ 0 & \frac{1}{2} \le x \le 0 \end{cases}$

then $u_1(x) = \sum_{n=1}^{\infty} \sin n\pi x \cdot n\pi D_n = \frac{\partial}{\partial t}\left(\sum \sin n\pi x \cdot D_n \sin n\pi t \right)$ then evaluate at $t=0$

$$\therefore \quad u_1(x) = \sum_{n=1}^{\infty} D_n \cdot n\pi \sin n\pi x = \sum E_n \sin n\pi x \qquad E_n = D_n \cdot n\pi$$

this a fourier sine series where $E_n = \frac{2}{L}\int_0^L \overset{u_1(x)}{} \sin\frac{n\pi x}{L}\,dx$ with $L=1$

$$\therefore \quad D_n \cdot n\pi = 2\int_0^1 u_1(x)\sin n\pi x\,dx = 2\left[\int_0^{1/2} 1 \sin n\pi x\,dx + \int_{1/2}^1 0 \cdot \sin n\pi x\,dx\right]$$

$$= -2 \left.\frac{\cos n\pi x}{n\pi}\right|_0^{1/2} \quad \text{or} \quad D_n = \frac{-2}{n^2\pi^2} \cos n\pi x \Big|_0^{1/2} = \frac{-2}{n^2\pi^2}\left[\cos\frac{n\pi}{2} - 1\right]$$

Not necessary for proof just for your info

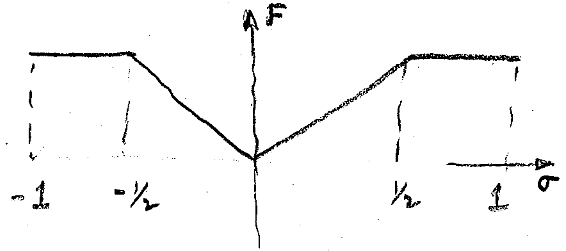
when n is odd $D_n = +\frac{2}{n^2\pi^2}$; when n is even $D_n = \frac{-2}{n^2\pi^2}\left[(-1)^{n/2} - 1\right]$

further when $\frac{n}{2}$ is ODD ie $n = 2, 6, 10, \ldots$ $(-1)^{n/2} - 1 = -2$

$n/2$ is even ie $n = 4, 8, 12, \ldots$ $(-1)^{n/2} - 1 = 0$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} D_n \sin n\pi x \sin n\pi t = \sum_{n=1}^{\infty} \frac{D_n}{2}\left[\cos n\pi(x-t) - \cos n\pi(x+t)\right]$$

Now let's write $F(\sigma)$ & $G(\sigma)$ as general fourier series & compare, e.g. $F(\sigma) = \frac{a_0}{2} + \sum a_n \cos n\pi\sigma + \sum b_n \sin n\pi\sigma$



this pattern repeats every period of $\sigma = 2$.

Now: $\int_{-1}^1$ even fn · odd fn $= \int_{-1}^1$ odd fn

$\int_{-1}^1$ odd fn $=$ even fn $\Big|_{-1}^1 = 0$. Note: $F(\sigma)$ is an even fn of σ also

Remember $a_n \sim \int F(\sigma)\cos n\pi\sigma\,d\sigma$ $\quad b_n \sim \int F(\sigma)\sin n\pi\sigma\,d\sigma$

$\therefore \int_{-1}^1 F(\sigma)\sin n\pi\sigma\,d\sigma = 0$ but for $F(\sigma)$ to be an even fn. $\Rightarrow b_n = 0$

$\therefore$ the only non zero terms will be due to $\int_{-1}^1 F(\sigma)\cos n\pi\sigma\,d\sigma$

since $F(\sigma)$ is even $\Rightarrow \int_{-1}^1 F(\sigma)\cos n\pi\sigma\,d\sigma = 2\int_0^1 F(\sigma)\cos n\pi\sigma\,d\sigma$

this is the definition for a_n

Further $F(\sigma) = \sigma/2 \qquad 0 \le \sigma \le 1/2$

$\qquad\qquad\qquad 1/4 \qquad \frac{1}{2} \le \sigma \le 1$

$$\therefore \quad a_n = 2\int_0^1 F(\sigma)\cos n\pi\sigma\, d\sigma = 2\left[\int_0^{1/2} \frac{\sigma}{2}\cos n\pi\sigma\, d\sigma + \int_{1/2}^1 \frac{1}{4}\cos n\pi\sigma\, d\sigma\right]$$

$$\int_0^{1/2} \frac{\sigma}{2}\cos n\pi\sigma\, d\sigma = \frac{\sigma \sin n\pi\sigma}{2n\pi}\Big|_0^{1/2} - \frac{1}{2n\pi}\int_0^{1/2} \sin n\pi\sigma\, d\sigma = \left(\frac{\sigma \sin n\pi\sigma}{2n\pi} + \frac{1}{2n^2\pi^2}\cos n\pi\sigma\right)_0^{1/2}$$

$$\frac{1}{4}\int_{1/2}^1 \cos n\pi\sigma\, d\sigma = \frac{1}{4n\pi}\sin n\pi\sigma\Big|_{1/2}^1 = -\frac{1}{4n\pi}\sin\frac{n\pi}{2}$$

$$\therefore \quad a_n = 2\left[\frac{1}{4n\pi}\sin\frac{n\pi}{2} + \frac{1}{2n^2\pi^2}\cos\frac{n\pi}{2} - \frac{1}{2n^2\pi^2} - \frac{1}{4n\pi}\sin\frac{n\pi}{2}\right] = \frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)$$

The definition of $a_0 = 2\int_0^1 F(\sigma)\,d\sigma = 2\left[\int_0^{1/2}\frac{\sigma}{2}\,d\sigma + \int_{1/2}^1 \frac{1}{4}\,d\sigma\right] = 2\left\{\frac{\sigma^2}{4}\Big|_0^{1/2} + \frac{\sigma}{4}\Big|_{1/2}^1\right\}$

$$= 2\left[\frac{1}{16} + \frac{1}{8}\right] = 2\cdot\frac{3}{16}$$

$$\therefore \quad F(\sigma) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi\sigma = \frac{3}{16} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos n\pi\sigma$$

where $\sigma = x + t$

$$F(x+t) = \frac{3}{16} + \sum \frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos n\pi(x+t)$$

if we do the same thing for $G(\sigma) = \frac{-\sigma}{2} \qquad 0 \le \sigma \le 1/2$

$\qquad\qquad\qquad\qquad\qquad\qquad -1/4 \qquad 1/2 \le \sigma \le 1$

we will find that $G(\sigma) = \frac{-3}{16} - \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos n\pi\sigma$

(note that $G(\sigma) = -F(\sigma)$) and $G(x-t) = -\frac{3}{16} - \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\cos n\pi(x-$

Thus $u(x,t) = F(x+t) + G(x-t) = \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\left(\cos\frac{n\pi}{2} - 1\right)\left[\cos n\pi(x+t) - \cos n\pi(x-t)\right]$

separation of variables given $u(x,t) = \sum \frac{D_n}{2}\left[\cos n\pi(x-t) - \cos n\pi(x+t)\right]$ $\quad D_n = \frac{-2}{n^2\pi^2}(\cos\frac{n\pi}{2} - 1$

SAME RESULT!