24.7. Find a recurrence formula and the indicial equation for an infinite series solution around $x = 0$ for the differential equation

$$2x^2y'' + 7x(x + 1)y' - 3y = 0$$

It follows from Problem 24.2 that $x = 0$ is a regular singular point of the differential equation, so Theorem 24.1 holds. Substituting Eqs. (*24.2*) through (*24.4*) into the left side of the given differential equation and combining coefficients of like powers of x, we obtain

$$x^{\lambda}[2\lambda(\lambda - 1)a_0 + 7\lambda a_0 - 3a_0] + x^{\lambda+1}[2(\lambda + 1)\lambda a_1 + 7\lambda a_0 + 7(\lambda + 1)a_1 - 3a_1] + \cdots$$
$$+ x^{\lambda+n}[2(\lambda + n)(\lambda + n - 1)a_n + 7(\lambda + n - 1)a_{n-1} + 7(\lambda + n)a_n - 3a_n] + \cdots = 0$$

Dividing by x^{λ} and simplifying, we have

$$(2\lambda^2 + 5\lambda - 3)a_0 + x[(2\lambda^2 + 9\lambda + 4)a_1 + 7\lambda a_0] + \cdots$$
$$+ x^n\{[2(\lambda + n)^2 + 5(\lambda + n) - 3]a_n + 7(\lambda + n - 1)a_{n-1}\} + \cdots = 0$$

Factoring the coefficient of a_n and equating each coefficient to zero, we find

$$(2\lambda^2 + 5\lambda - 3)a_0 = 0 \tag{1}$$

and, for $n \geq 1$,

$$[2(\lambda + n) - 1][(\lambda + n) + 3]a_n + 7(\lambda + n - 1)a_{n-1} = 0$$

or,

$$a_n = \frac{-7(\lambda + n - 1)}{[2(\lambda + n) - 1][(\lambda + n) + 3]}a_{n-1} \tag{2}$$

Equation (*2*) is a recurrence formula for this differential equation.

From (*1*), either $a_0 = 0$ or

$$2\lambda^2 + 5\lambda - 3 = 0 \tag{3}$$

It is convenient to keep a_0 arbitrary; therefore, we require λ to satisfy the indicial equation (*3*).

24.8. Find the general solution near $x = 0$ of $2x^2y'' + 7x(x + 1)y' - 3y = 0$.

The roots of the indicial equation given by (*3*) of Problem 24.7 are $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = -3$. Since $\lambda_1 - \lambda_2 = \frac{7}{2}$, the solution is given by Eqs. (*24.5*) and (*24.6*). Substituting $\lambda = \frac{1}{2}$ into (*2*) of Problem 24.7 and simplifying, we obtain

$$a_n = \frac{-7(2n - 1)}{2n(2n + 7)}a_{n-1} \qquad (n \geq 1)$$

Thus,

$$a_1 = -\frac{7}{18}a_0, \qquad a_2 = -\frac{21}{44}a_1 = \frac{147}{792}a_0, \qquad \ldots$$

and

$$y_1(x) = a_0x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right)$$

Substituting $\lambda = -3$ into (*2*) of Problem 24.7 and simplifying, we obtain

$$a_n = \frac{-7(n - 4)}{n(2n - 7)}a_{n-1} \qquad (n \geq 1)$$

Thus,

$$a_1 = -\frac{21}{5}a_0, \qquad a_2 = -\frac{7}{3}a_1 = \frac{49}{5}a_0, \qquad a_3 = -\frac{7}{3}a_2 = -\frac{343}{15}a_0, \qquad a_4 = 0$$

and, since $a_4 = 0$, $a_n = 0$ for $n \geq 4$. Thus,

$$y_2(x) = a_0x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$$

The general solution is

$$y = c_1y_1(x) + c_2y_2(x)$$
$$= k_1x^{1/2}\left(1 - \frac{7}{18}x + \frac{147}{792}x^2 + \cdots\right) + k_2x^{-3}\left(1 - \frac{21}{5}x + \frac{49}{5}x^2 - \frac{343}{15}x^3\right)$$

where $k_1 = c_1a_0$ and $k_2 = c_2a_0$.

$$Y'' = 0 \implies Y = A + By \qquad Y(0)=0 \;\&\; Y(L)=0$$

$$Y(0) = A + 0 = 0 \implies A = 0$$

$$Y(L) = BL = 0 \implies B = 0 \qquad n=0 \implies Y_0(y) = 0$$

$y u_{xx} + u_{yy} = 0$ $\qquad b^2 - 4ac = 0^2 - 4y \cdot 1 = -4y \begin{cases} =0 & y=0 \\ >0 & y<0 \\ <0 & y>0 \end{cases}$

look at $y<0$ hyperbolic PDE $\quad$ now let $z = -y$ $\quad \frac{\partial}{\partial y} = -\frac{\partial}{\partial z} \quad \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\right) = -\frac{\partial}{\partial z}\left(-\frac{\partial}{\partial z}\right) = \partial^2/\partial z^2$

$$-z u_{xx} + u_{zz} = 0$$

now $\quad \frac{dy}{dx} = -\frac{dz}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{0 \pm \sqrt{0^2 + 4z}}{-2z} = \frac{\pm 2\sqrt{z}}{-2z} = \mp \frac{1}{\sqrt{z}} \qquad z>0$

$\therefore \quad \frac{dz}{dx} = \pm \frac{1}{\sqrt{z}} \qquad \pm x = \frac{2}{3} z^{3/2} + C \qquad x - \frac{2}{3} z^{3/2} = \xi \qquad \xi_x = 1 \quad \xi_z = -z^{1/2}$

$-(x + 2/3\, z^{3/2}) = \eta \qquad (-x - 2/3\, z^{3/2} = c)$ let $c = \pm\eta$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi \cdot 1 + u_\eta \cdot 1 \qquad \xi - \eta = 2x \qquad \eta_x = 1 \quad \eta_z = -z^{1/2}$

$+\eta + \xi = -\frac{4}{3} z^{3/2}$

$u_z = -u_y = u_\xi \xi_z + u_\eta \eta_z = u_\xi(-\sqrt{z}) + u_\eta(\sqrt{z})$

$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial \xi}(u_x) \cdot \xi_x + \frac{\partial}{\partial \eta}(u_x)\eta_x = (u_{\xi\xi} + u_{\eta x})1 + (u_{\xi\eta} + u_{\eta\eta})1 = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{zz} = \frac{\partial}{\partial z}(u_z) = \frac{\partial}{\partial \xi}(u_z)\cdot\xi_z + \frac{\partial}{\partial \eta}(u_z)\eta_z + \frac{\partial}{\partial z}(u_z) = [u_{\xi\xi}(-\sqrt{z}) + u_{\xi\eta}(\sqrt{z})](\sqrt{z}) + [u_{\xi\eta}(-\sqrt{z}) + u_{\eta\eta}(\sqrt{z})](\sqrt{z})$

$+ \; u_\xi \frac{-1}{2\sqrt{z}} + u_\eta \frac{-1}{2\sqrt{z}}$

$-z u_{xx} + u_{zz} = -z\cancel{u_{\xi\xi}} + 2z\cancel{u_{\xi\eta}} - z\cancel{u_{\eta\eta}} + \cancel{u_{\xi\xi}} \cdot z + u_{\xi\eta}\cancel{(2z)} + \cancel{u_{\eta\eta}} z + \frac{-1}{2\sqrt{z}}(u_\eta + u_\xi) = 0$

$+4z\, u_{\eta\xi} - \frac{1}{2\sqrt{z}}(u_\eta + u_\xi) = 0$

$u_{\eta\xi} - \frac{1}{8z^{3/2}}(u_\eta + u_\xi) = 0 \qquad 6(\eta + \xi) = 6 \cdot \frac{4}{3} z^{3/2} = -8z^{3/2}$

$u_{\eta\xi} + \frac{1}{6(\xi + \eta)}(u_\eta + u_\xi) = 0 \qquad$ if $\; x - \frac{2}{3}(-y)^{3/2} = \xi$

$-x - 2/3(-y)^{3/2} = \eta$

if $\; x - 2/3\, z^{3/2} = \xi$

$x + 2/3\, z^{3/2} = \eta \qquad u_{\eta\xi} - \frac{1}{6(\xi - \eta)}(u_\eta - u_\xi) = 0$

$x - 2/3(-y)^{3/2} = \xi$

$x + 2/3(-y)^{3/2} = \eta$

$$\frac{\partial U}{\partial t} = \kappa \frac{\partial^2 U}{\partial x^2} \qquad \frac{\partial U}{\partial x}(x=0,t) = q/\kappa \qquad \frac{\partial U}{\partial x}(x=L,t) = 0$$

let $U = V + \psi(x)$

$$\frac{\partial V}{\partial t} + 0 = \kappa\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}\right) \qquad \frac{\partial U}{\partial x} = \frac{\partial V}{\partial x} + \psi'$$

let $\frac{\partial V}{\partial x}(x=0,t) = 0 \quad \psi'_{(0)} = q/\kappa$

$\frac{\partial V}{\partial x}(x=L,t) = 0 \quad \psi'_{(L)} = 0$

let $\psi' = \frac{-q}{\kappa L}(x-L)$

this satisfies bc's but $\psi'' = -\frac{q}{\kappa L}$

$$\therefore \frac{\partial V}{\partial t} - \kappa \frac{\partial^2 V}{\partial x^2} = \frac{-q}{L}$$ which is nonzero

homogeneous if $V(x,t) = e^{-\lambda^2 \kappa t}\left[A\sin\lambda x + B\cos\lambda x\right]$

$$\frac{\partial^2 V}{\partial x^2} = -\lambda^2 [\quad] e$$

$$\frac{\partial V}{\partial t} = -\lambda^2 \kappa\, e^{-\lambda^2 \kappa t}[\quad]$$ for all $\lambda > 0$

that's the homog $\quad \frac{\dot T}{T} - \kappa \frac{V''}{V} = 0 \Rightarrow \frac{\dot T}{\kappa T} = \frac{V''}{V} = -\lambda^2$

Now let $V(x,t) = e^{-\lambda^2 \kappa t}\left[A(t)\sin\lambda x + B(t)\sin\lambda x\right]$

$V_x = A\lambda\cos\lambda x - B\lambda\sin\lambda x$ $\qquad$ if $\lambda = 0 \Rightarrow V = Ax + B \quad V_x = A = 0 \;\therefore$

@ 0 $\quad A\lambda = 0 \quad A = 0, \lambda = 0$

@ $x = L$ $\quad -B\lambda\sin\lambda L = 0 \quad \lambda L = n\pi \quad \lambda = \frac{n\pi}{L}$

$$V_h = e^{-\lambda^2\kappa t}\left[A\sin\frac{n\pi x}{L} + B\cos\frac{n\pi x}{L}\right]$$

use variation of param

$$\dot V_h = -\lambda^2\kappa\, e^{-\lambda^2\kappa t}[\quad] + e^{-\lambda^2\kappa t}\left[\dot A\sin\frac{n\pi x}{L} + \dot B\cos\frac{n\pi x}{L}\right]$$

$$\kappa V_{h_{xx}} = -\lambda^2\kappa e^{-\lambda^2\kappa t}\left[\cancel{+}A\sin + \cancel{+}B\cos\right]$$

$$\dot V_h - \kappa V_h = e^{-\lambda^2\kappa t}\left(\dot A\sin\frac{n\pi x}{L} + \dot B\cos\frac{n\pi x}{L}\right) = -\frac{q}{L}$$

$$\dot B e^{-\lambda^2\kappa t} = \frac{2}{L}\int_0^L -\frac{q}{L}\cos\frac{n\pi x}{L}\,dx$$

$$\therefore \dot B = e^{\lambda^2\kappa t}\cdot\frac{2}{L}\int_0^L -q/L\cos\frac{n\pi x}{L}\,dx$$

$$B = \frac{e^{\lambda^2\kappa t}}{\lambda^2\kappa}\cdot\left(\frac{-2q}{L^2}\right)\frac{\sin\frac{n\pi x}{L}}{\frac{n\pi}{L}}\Bigg|_0^L = 0$$

$$\dot A e^{-\lambda^2\kappa t} = \frac{2}{L}\int_0^L \frac{-q}{L}\sin\frac{n\pi x}{L}\,dx$$

4.6 $$k\left(r\frac{\partial^2 T}{\partial r^2}+\frac{\partial T}{\partial r}\right)-\beta^2(T-T_o)=\frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$\frac{\partial T}{\partial r}=0$ @ $r=r_o$ $\quad T(r=r_i,t)=0$

$T(r,t=0)=T_o$

let $\mathcal{L}\{T(r,t)\}=\int_0^\infty T(r,t)e^{-st}dt=J(r;s)$

let $\mathcal{L}\left\{\frac{\partial T}{\partial t}\right\}=\int_0^\infty \frac{\partial T}{\partial t}e^{-st}dt=$ let $du=\frac{\partial T}{\partial t}dt$ & $v=e^{-st}$, $u=T$ $\quad dv=-se^{-st}$

$Te^{-st}\Big|_0^\infty+s\int Te^{-st}dt$

$-T(r,t=0)+s\mathcal{L}\{T$

$$k\left(r\frac{\partial^2 J}{\partial r^2}+\frac{\partial J}{\partial r}\right)-\beta^2\left(J+\frac{T_o}{s}\right)=\frac{r}{\alpha}\left[sJ-T_o\right]$$

$$k\left(r\frac{\partial^2 J}{\partial r^2}+\frac{\partial J}{\partial r}\right)-\beta^2 J-\frac{rs}{\alpha}J=\beta^2\frac{T_o}{s}-\frac{rT_o}{\alpha}$$

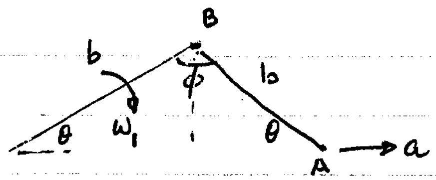
$$(rJ''+J')-\left(\frac{\beta^2}{k}+\frac{rs}{\alpha k}\right)J=\frac{\beta^2 T_o}{sk}-\frac{rT_o}{\alpha k}$$

$$r\cdot 0+A-\left(\frac{\beta^2}{k}+\frac{rs}{\alpha k}\right)(Ar+B)=\frac{\beta^2 T_o}{sk}+\frac{rT_o}{\alpha k}$$ let $J_p=Ar+B$, $J_p'=A$, $J_p''=0$

$\cancel{A-\frac{\beta^2}{k}B=\frac{\beta^2 T_o}{sk}}$

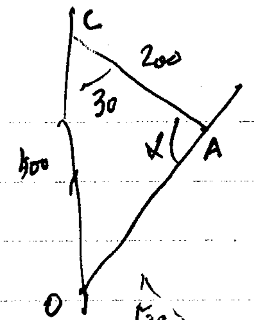
~~let $J=r^m$~~ $\quad \mathcal{L}\left\{\frac{\partial T}{\partial r}\right\}=J'(r=r_o)=0 \qquad \mathcal{L}\{T\}=J(r=r_i)=0$

[illegible]



~~$x^2 = 2b^2 - 2b^2\cos$~~

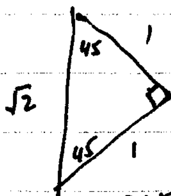
$$x^2 = b^2 + b^2 - 2b^2\cos\theta$$
$$x^2 = 2b^2(1-\cos\theta)$$
$$2x\dot{x} = 2b^2(\sin\theta)\dot{\theta}$$
$$2\dot{x}^2 + 2x\ddot{x} = 2b^2\cos\theta\,\dot{\theta}^2 + \sin\theta\,\ddot{\theta}$$



$$OA = \sqrt{400^2 + 200^2 - 2\cdot 400\cdot 200\cdot \cos 30}$$
$$= 247.863$$

$$\frac{\sin\alpha}{400} = \frac{\sin 30}{OA} \qquad \alpha = 53.794^\circ$$

$$180 - \alpha - 30 = \angle COA = 96.206^\circ$$

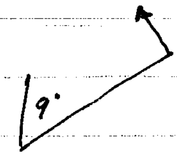


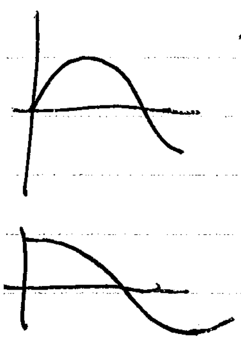
$$-.6\cos 30 i - .6\sin 30 j = V_A$$

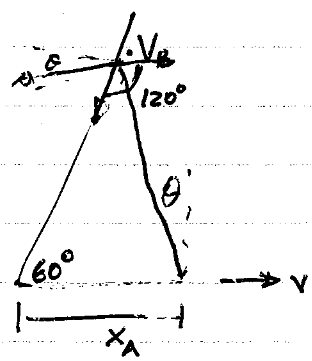
$$V_A = V_o + V_{A/o}$$

~~$2+1 - 2\cdot\sqrt{2}\cdot 1\cos 45$~~ $-.6\cos 30 i - .6\sin 30 j$ 0

~~$\frac{1}{\sqrt{2}}$~~







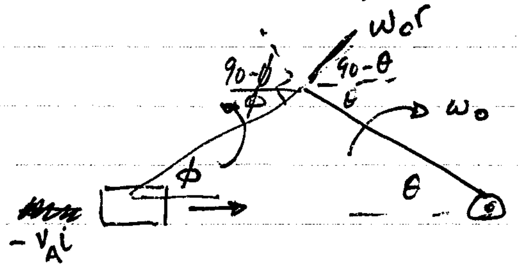
$$\underline{V}_B = \underline{V}_A + \underline{V}_{B/A}$$

$$V_B\cos 240\, i + V_B\sin 240\, j = V\underline{i} + \omega L\cos\theta\, i + \omega L\sin\theta\, \underline{j}$$

$V_B =$ $\qquad V_B\cos 240 = V - \omega L\cos\theta$

$$V_B\sin 240 = -\omega L\sin\theta$$

$$L^2 = x_A^2 + y^2 - 2x_A y\cos 60^\circ$$
$$0 = 2x_A\dot{x}_A + 2y\dot{y} - 2(\dot{x}_A y + x_A\dot{y})\cos 60^\circ$$
$$= 2x_A$$



$$V_B = \omega_0 r[(\cos 90-\theta)i + \sin(90-\theta)j]$$

$$\underline{V}_B = \underline{V}_A + \omega$$

$$V_A\underline{i} + \omega l(\cos 90-\phi)\underline{i} + \omega l\sin(90-\phi)\underline{j}$$

$$V_A - \omega l\sin\phi = \omega_0 r\sin\theta \qquad \frac{V_A - \omega_0 r\sin\theta}{\omega l} = \sin\phi$$

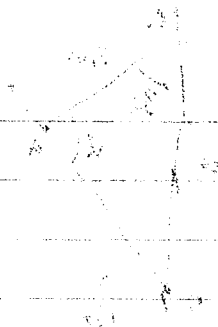
$$\omega l(\cos\phi) = \omega_0 r\cos\theta$$

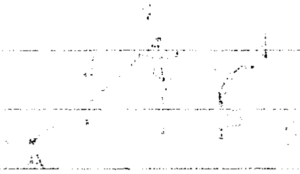
$$= (\omega_0 r)^2 = V_A^2 - 2\omega l V_A\sin\phi + \omega^2 l^2$$

$$V_A^2 - 2V_A[V_A - \omega_0 r\sin\theta] + \omega^2 l^2$$

$$\frac{1}{l}\sqrt{(\omega_0 r)^2 - V_A^2 + 2V_A^2 + 2V_A\omega_0 r\sin\theta} = \omega$$

[illegible]





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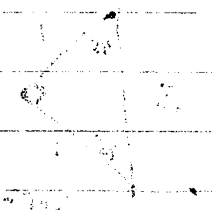

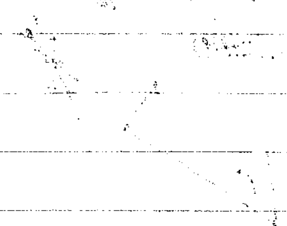

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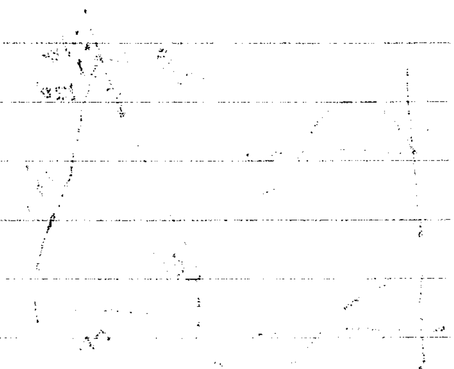

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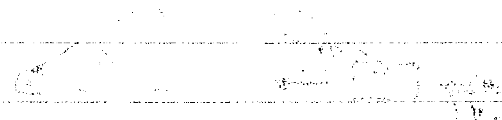

[illegible]

(b) Assume a particular solution of the form

$$y_p(x) = u(x)\cos x + v(x)\sin x.$$

(Note that the constants or parameters c_1 and c_2 have been replaced by functions $u(x)$ and $v(x)$. Our objective will be to obtain two equations in $u'(x)$ and $v'(x)$ that can then be solved simultaneously.) Differentiate to obtain

$$y_p'(x) = -u(x)\sin x + v(x)\cos x$$

with

$$u'(x)\cos x + v'(x)\sin x = 0.$$

Observe that this last condition simplifies $y_p'(x)$, $y_p''(x)$ and provides a second equation in $u'(x)$ and $v'(x)$.

(c) Differentiate $y_p'(x)$ in part (b) and substitute into the given differential equation to obtain

$$-u'(x)\sin x + v'(x)\cos x = \tan x.$$

(d) Solve the system

$$\begin{aligned} -u'(x)\sin x + v'(x)\cos x &= \tan x \\ u'(x)\cos x + v'(x)\sin x &= 0 \end{aligned}$$

for $u'(x)$ and $v'(x)$ by Cramer's rule or by elimination.

(e) Integrate $u'(x)$ and $v'(x)$ to find $u(x)$ and $v(x)$.

(f) Find $y_p(x)$ and thus obtain the general solution. Note that success in using the method of variation of parameters is contingent on being able to obtain $u(x)$ and $v(x)$ from $u'(x)$ and $v'(x)$.

14. Use the method of Exercise 13 to obtain the general solutions of each of the following equations.

(a) $y'' - y' = \sec^2 x - \tan x$

(b) $y'' - 2y' + y = \exp(x)/(1 - x)^2$

(c) $y'' + y = \sec x \tan x$

(d) $y'' + y = \sec x$

(e) $y'' - 2y' + y = e^x/x^2$

(f) $y'' + 4y = \cot 2x$

15. Solve the initial-value problem

$$y'' - 2y' + y = e^x/(1 - x)^2, \qquad y(0) = 2, \qquad y'(0) = 6.$$

16. Verify that $y_1(x) = x$ and $y_2(x) = 1/x$ are solutions of

$$x^3y'' + x^2y' - xy = 0.$$

Then use this information and the method of variation of parameters to find the general solution of

$$x^3y'' + x^2y' - xy = x/(1 + x).$$

17. (a) Solve the equation

$$y'' - y = x \sin x$$

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18. Consider

$$y'' + ay' + by = f(x),$$

where a and b are constants with $b \neq 0$ and $f(x)$ is a polynomial of degree n. Sho that this equation always has a solution that is a polynomial of degree

19. In Exercise 18, show that the solution is a polynomial of degree $n + 1$ in the ca where $b = 0$.

20. If $u(x)$, $v(x)$, and $w(x)$ are differentiable functions of x, use the formula f differentiating a product,

$$\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx},$$

to find $d(uvw)/dx$.

21. Show that reconciling the coefficients of like terms in the method of undetermin coefficients depends on the forming of a linearly independent set by the functio involved. (*Hint*: Recall that if $f_i(x)$, $i = 1, 2, \ldots, n$, are n linearly independe functions over the reals, then

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

holds only if each $c_i = 0$.)

1.3 CAUCHY-EULER EQUATIONS

In the two preceding sections we discussed second-order linear ordina differential equations with *constant* coefficients. While equations of this ty will occur frequently throughout the remaining chapters, we will also ha occasion to solve other linear differential equations, that is, equations of t form

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = f(x). \qquad (1.3\text{-}$$

One type of linear differential equation with variable coefficients th can be reduced to a form already considered is the **Cauchy–Euler equatio** which has the normal form

$$x^2y'' + axy' + by = f(x), \qquad x > 0, \qquad (1.3\text{-}$$

where a and b are constants. This equation is also called a *Cauchy** equatic an *Euler*† equation, and an *equidimensional* equation. The last term com from the fact that the physical dimension of x in the left-hand member of E (1.3-2) is immaterial, since replacing x by cx, where c is a nonzero consta leaves the dimension of the left-hand member unchanged. We will meet a form of this equation later in our study of boundary-value problems having circular symmetry. We assume throughout this section that $x \neq 0$. For the most part we assume that $x > 0$, although we also deal with the case $x < 0$ later. We begin with an example to illustrate the method of solution.

EXAMPLE 1.3–1 Find the complementary solution of the equation

$$x^2y'' + 2xy' - 2y = x^2 \exp(-x), \quad x > 0.$$

Solution This is a Cauchy–Euler equation, and we make the following substitutions in the reduced equation

$$y_c(x) = x^m, \quad y_c' = mx^{m-1}, \quad y_c'' = m(m-1)x^{m-2},$$

so the homogeneous equation becomes

$$[m(m-1) + 2m - 2]x^m = 0.$$

Because of the restriction $x \neq 0$, we must have

$$m^2 + m - 2 = 0,$$

which has roots $m_1 = -2$ and $m_2 = 1$. Thus

$$y_c(x) = c_1x^{-2} + c_2x. \quad ■$$

We remark that the substitution, $y_c(x) = x^m$, did not come from thin air. It was dictated by the *form* of the left-hand member of the differential equation, which in turn ensured that each term of the equation would contain the common factor x^m.

It should be pointed out also that if we were interested in obtaining the *general* solution of the equation in Example 1.3–1, we could use the complementary solution above and the method of **variation of parameters**. (See Exercise 13 in Section 1.2.)

EXAMPLE 1.3–2 Obtain the complementary solution of

$$x^2y'' + 3xy' + y = x^3, \quad x > 0.$$

Solution This time the substitution $y_c = x^m$ leads to

$$m^2 + 2m + 1 = 0,$$

which has a double root $m = -1$. Hence we have *one* solution of the homogeneous equation, namely,

$$y_1(x) = x^{-1}.$$

One might "guess" that a second linearly independent solution could be obtained by multiplying $y_1(x)$ by x. This procedure, however, is limited to the case of equations with *constant* coefficients and is thus not applicable here. It is

easy to check that $y = 1$ is *not* a solution. In this case we can use the me
reduction of order to find a second solution.

We set $y_2(x) = u(x)/x$ and compute two derivatives. Thus

$$y_2'(x) = \frac{u'x - u}{x^2},$$

$$y_2''(x) = \frac{x^2(xu'' - 2u') + 2ux}{x^4},$$

and substitution into the homogeneous equation results in

$$xu'' + u' = 0,$$

which can be solved* by setting $u' = v$ and **separating the variables.**

$$v = \frac{du}{dx} = \frac{c_2}{x},$$

and†

$$u = c_2 \log x.$$

Hence

$$y_2(x) = \frac{c_2}{x} \log x,$$

and the complementary solution is

$$y_c(x) = \frac{c_1}{x} + \frac{c_2}{x} \log x.$$

We shall see later that the function $\log x$ occurs in the case of
roots. ■

EXAMPLE 1.3–3 Find the complementary solution of

$$x^2y'' + xy' + y = \cos x, \quad x > 0.$$

Solution In this example we have, after substituting $y_c(x) = x^m$,

$$m^2 + 1 = 0$$

so that the solutions are x^i and x^{-i}. Hence the complementary solutic

$$y_c(x) = C_1x^i + C_2x^{-i}.$$

A more useful form can be obtained, however, by replacing C_1 an
$(c_1 - ic_2)/2$ and $(c_1 + ic_2)/2$, respectively, and noting that

$$x^i = \exp(i \log x) = \cos(\log x) + i \sin(\log x).$$

*An alternative method is to note that $xu'' + u' = d(xu') = 0$, leading to xu'

†We will consistently use $\log x$ for the *natural logarithm* of x.

With these changes the complementary solution (1.3–3) can be written

$$y_c(x) = c_1 \cos(\log x) + c_2 \sin(\log x). \quad \blacksquare$$

We shall summarize the various cases that occur when solving the *homogeneous* Cauchy–Euler equation

$$x^2y'' + axy' + by = 0. \tag{1.3–4}$$

Substitution of $y_c(x) = x^m$ and its derivatives into Eq. (1.3–4) leads to the equation

$$m(m-1) + am + b = 0$$

or

$$m^2 + (a-1)m + b = 0. \tag{1.3–5}$$

This is called the **auxiliary equation** of the homogeneous Cauchy–Euler equation (1.3–4).

Case I. $(a-1)^2 - 4b > 0$. The roots of Eq. (1.3–5) are real and unequal, say, m_1 and m_2. Then

$$y_c(x) = c_1x^{m_1} + c_2x^{m_2}, \tag{1.3–6}$$

and since the Wronskian

$$\begin{vmatrix} x^{m_1} & x^{m_2} \\ m_1x^{m_1-1} & m_2x^{m_2-1} \end{vmatrix} = (m_2 - m_1)x^{m_1+m_2-1} \neq 0,$$

showing that x^{m_1} and x^{m_2} are linearly independent.*

Case II. $(a-1)^2 - 4b = 0$. The roots of Eq. (1.3–5) are real and equal, say, $m_1 = m_2 = m$. Then

$$y_1(x) = x^m$$

is one solution of Eq. (1.3–4). A second solution can be found by the method of reduction of order. Let

$$y_2(x) = x^m u(x)$$

be a second solution, differentiate twice, and substitute into Eq. (1.3–4). Then

$$u[m(m-1) + am + b]x^m + u'(2m + a)x^{m+1} + u''x^{m+2} = 0.$$

*... > 0 is essential here.

But the coefficient of u vanishes because x^m is a solution of Eq. (1.3–
$2m + a = 1$ from Eq. (1.3–5). Thus

$$xu'' + u' = 0,$$

which is satisfied by $u = \log x$ and

$$y_2(x) = x^m \log x.$$

In this case the complementary solution is

$$y_c(x) = x^m(c_1 + c_2 \log x).$$

Case III. $(a-1)^2 - 4b < 0$. The roots of Eq. (1.3–5) are complex gates, say, $m_1 = \alpha + \beta i$ and $m_2 = \alpha - \beta i$. Then two linearly indep solutions of the homogeneous equation are

$$y_1(x) = x^{\alpha+\beta i} = x^\alpha x^{\beta i} = x^\alpha \exp(i\beta \log x)$$

and

$$y_2(x) = x^{\alpha-\beta i} = x^\alpha x^{-\beta i} = x^\alpha \exp(-i\beta \log x).$$

Using Euler's formula* to transform the exponential gives us

$$y_1(x) = x^\alpha[\cos(\beta \log x) + i \sin(\beta \log x)]$$

and

$$y_2(x) = x^\alpha[\cos(\beta \log x) - i \sin(\beta \log x)].$$

Hence the complementary solution becomes

$$y_c(x) = x^\alpha[c_1 \cos(\beta \log x) + c_2 \sin(\beta \log x)].$$

If the general solution to Eq. (1.3–2) is required, it is necessary t particular solution to the appropriate complementary solution. A pa solution can be found by the method of variation of parameters, al difficulties may be encountered, as was mentioned in Exercise 13 of 1.2. Note that the method of undetermined coefficients is not applicabl since the Cauchy–Euler differential equation does not have c coefficients.

There is an alternative method for solving Eq. (1.3–4). Since x can make the substitution

$$u = \log x.$$

This leads to $x = \exp u$ and, using the chain rule, to

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{x}\frac{dy}{du}.$$

*$\exp(i\theta) = \cos\theta + i \sin\theta$.

We also have

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{du}\right)\\ &= \frac{1}{x}\frac{d}{du}\left(\frac{dy}{du}\right)\frac{du}{dx} - \frac{1}{x^2}\frac{dy}{du}\\ &= \frac{1}{x^2}\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right).\end{aligned}$$

This method has the advantage that Eq. (1.3–4) is transformed into

$$\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = f(\exp u).$$

Thus the differential equation has constant coefficients, and the methods of Section 1.1 are available for finding the complementary solution of the Cauchy–Euler equation. In fact, if $f(\exp u)$ has the proper form, then the method of undetermined coefficients may lead to the general solution of the nonhomogeneous equation quite easily.

We have considered exclusively the case where $x > 0$. If solutions are desired for values of x satisfying $x < 0$, then x may be replaced by $-x$ in the differential equation and in the solution.

Key Words and Phrases

Cauchy-Euler equation
variation of parameters
reduction of order
separating the variables
auxiliary equation

1.3 Exercises

In the following exercises, assume that the independent variable is positive unless otherwise stated.

1. Use the substitution $u = \log x$ to solve each of the following equations.
   (a) $x^2y'' + 2xy' - 2y = 0$ (Compare Example 1.3–1)
   (b) $x^2y'' + 3xy' + y = 0$ (Compare Example 1.3–2)
   (c) $x^2y'' + xy' + y = 0$ (Compare Example 1.3–3)
2. Obtain the general solution of the equation
$$x^2y'' + 3xy' + y = x^3.$$
(*Hint*: Use the substitution of Exercise 1 and the result of Example 1.3–2.)

3. Use Euler's formula
$$\exp(i\theta) = \cos\theta + i\sin\theta$$
to fill in the details in Case III.
4. ●● Solve the initial-value problem
$$x^2y'' - 2y = 0, \quad y(1) = 6, \quad y'(1) = 3.$$
5. Find the general solution of
$$x^2y'' + 5xy' - 5y = 0.$$
6. Find the general solution of
$$t^2y'' + 5ty' + 5y = 0.$$
7. Find the general solution of
$$r^2u'' + 3ru' + u = 0.$$
8. Obtain the general solution for
$$x^2y'' + xy' - 9y = x^2 - 2x.$$
9. Find the general solution of
$$x^2y'' + xy' + 4y = \log x.$$
10. Solve the initial-value problem
$$x^2y'' - xy' + 2y = 5 - 4x, \quad y(1) = 0, \quad y'(1) = 0.$$
11. Solve each of the following initial-value problems.
    (a) $x^2y'' + xy' = 0, \quad y(1) = 1, \quad y'(1) = 2$
    (b) $x^2y'' - 2y = 0, \quad y(1) = 0, \quad y'(1) = 1$
12. ●●● Use the method of reduction of order to find a second solution for e following differential equations, given one solution as shown.
    (a) $x^2y'' - xy' + y = 0, \quad y_1(x) = x$
    (b) $xy'' + 3y' = 0, \quad y_1(x) = 2$
    (c) $x^2y'' + xy' - 4y = 0, \quad y_1(x) = x^2$
    (d) $x^2y'' - xy' + y = 0, \quad y_1(x) = x\log x^2$
13. Show that the products xy' and x^2y'' remain unchanged if x is repla where c is a nonzero constant.
14. Show that the substitution $x = \exp u$ transforms the equation
$$x^2y'' + axy' + by = 0,$$
where a and b are constants, into
$$\frac{d^2y}{du^2} + (a-1)\frac{dy}{du} + by = 0.$$
15. Solve each of the following initial-value problems.
    (a) $4x^2y'' - 4xy' + 3y = 0, \quad y(1) = 0, \quad y'(1) = 1$
    (b) $x^2y'' + 5xy' + 4y = 0, \quad y(1) = 1, \quad y'(1) = 3$

16. Obtain the general solution of the equation

$$x^2y'' + axy' = 0,$$

where a is a constant.

INFINITE SERIES

Since we will need to solve second-order linear equations with variable coefficients that are not of Cauchy–Euler type (Eq. 1.3-2), we must explore other methods of solution. One powerful method is the power series method. In using this method we assume that the solution to a given differential equation can be expressed as a power series. Inasmuch as we will need certain facts about infinite series and their convergence, we digress in this section to review some aspects of these.

Each of the following is an example of a **series of constants:**

$$1 + 2 + 3 + 4 + \cdots + n + \cdots, \tag{1.4-1}$$

$$1 - 1 + 1 - 1 + 1 - + \cdots + (-1)^{n+1} + \cdots, \tag{1.4-2}$$

$$0 + 0 + 0 + \cdots + 0 + \cdots, \tag{1.4-3}$$

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots, \tag{1.4-4}$$

$$a(1 + r + r^2 + \cdots + r^n + \cdots), \tag{1.4-5}$$

$$1 - \frac{1}{3} + \frac{1}{5} - + \cdots + \frac{(-1)^{n+1}}{2n-1} + \cdots. \tag{1.4-6}$$

The series in (1.4-1) is **divergent** because the **partial sums**

$$S_1 = 1, \quad S_2 = 1 + 2, \quad S_3 = 1 + 2 + 3, \quad S_4 = 1 + 2 + 3 + 4, \quad \ldots$$

form a **sequence**

$$\{S_1, S_2, S_3, \ldots\} = \{1, 3, 6, 10 \ldots\},$$

which has *no* **limit point.*** On the other hand, the series in (1.4-2) is divergent because its sequence of partial sums has *two* limit points, $+1$ and 0. The series in (1.4-3) is a trivial example of a **convergent** series, since its sequence of partial sums has a unique limit point, namely zero.

*A point is called a *limit point* of a sequence if an infinite number of terms in the sequence are within a distance of ϵ of the point, where ϵ is an arbitrarily small positive number. A limit point need not be unique and need not be an element of the sequence. For example, 1 is the unique limit point of the sequence

$$\left\{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16} \cdots\right\}$$

By a more sophisticated test (the **integral test)** it can be shown that t series of (1.4-4) is convergent for $p > 1$ and divergent for $p \leq 1$. Wh $p = 1$, the series is called a **harmonic series.** The series of (1.4-5) is **geometric series** with first term a and **common ratio** r. It can be shown (by t ratio test) that (1.4-5) converges if $|r| < 1$ and diverges if $|r| \geq 1$ a $a \neq 0$. The sum of (1.4-5) can be written in **closed form** as

$$a\sum_{n=0}^{\infty} r^n = \frac{a}{1-r}, \qquad |r| < 1.$$

Finally, the series of (1.4-6) is an example of an **alternating series** that can proved (using a theorem of Leibniz*) to be convergent because the followi two conditions hold:

1. The absolute value of each term is less than or equal to the absolute va of its predecessor.
2. The limiting value of the nth term is zero as $n \to \infty$.

We remark that it is one thing to determine whether a given series c verges, but quite another to determine what it converges to. It is not obvio for example, that the sum of the series in (1.4-6) is $\pi/4$, although we will tain this result and some others in the exercises for Section 4.3. (See Exerci 23 and 25 of that section.)

Of greater interest to us than a series of constants will be **power series** the form

$$a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_n(x - x_0)^n + \cdots. \tag{1.4}$$

Such a series is called a power series in $x - x_0$. A power series *always* c verges. For example, (1.4-7) converges for $x = x_0$, but we will be interested convergence on an *interval* such as $(x_0 - R, x_0 + R)$. We call R ($R > 0$) **radius of convergence** of the power series and determine its value by using **ratio test†** as shown in the next example.

EXAMPLE 1.4-1 Find the radius of convergence of the series

$$(x - 1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots.$$

*Gottfried Wilhelm Leibniz (1646–1716), the co-inventor (with Sir Isaac Newton) of calculus, who proved the theorem in 1705.

†Also called D'Alembert's ratio test after Jean-le-Rond D'Alembert (1717–1783), a Fre mathematician who made important contributions in analytical mechanics.

Solution It will be convenient to write the series using summation notation,

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}.$$

According to the ratio test, a series converges whenever

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| < 1,$$

where u_n represents the nth term of the series. In the present case,

$$\lim_{n\to\infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+2}(x-1)^{n+1}}{n+1} \cdot \frac{n}{(-1)^{n+1}(x-1)^n} \right|$$
$$= |x-1| \lim_{n\to\infty} \frac{n}{n+1} = |x-1| < 1.$$

Hence $-1 < x - 1 < 1$ or $0 < x < 2$, showing that the *radius* of convergence is 1. In many problems it is necessary to examine the endpoints of the **interval of convergence** as well, and this must be done separately. It can be shown (Exercise 3) that the interval of convergence here is $0 < x \le 2$. ■

Obtaining a power series representation of a function is an important mathematical technique that has many applications. Recall that the **Maclaurin*** **series** for a function $f(x)$ is given by

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots. \tag{1.4-8}$$

Following are some familiar Maclaurin series expansions:

$$\exp x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots, \tag{1.4-9}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots + \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!} + \cdots, \tag{1.4-10}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots + \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!} + \cdots, \tag{1.4-11}$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - + \cdots + \frac{(-1)^{n+1}x^n}{n} + \cdots. \tag{1.4-12}$$

*After Colin Maclaurin (1698–1746), a Scottish mathematician.

The Maclaurin series for exp x, sin x, and cos x converge for a
values of x, while the series (1.4–12) has an interval of convergence gi
$-1 < x \le 1$. All four of the series are particular cases of Eq. (1.4–8).
summation notation, we can write*

$$\exp x = \sum_{n=0} \frac{x^n}{n!}$$

$$\sin x = \sum_{n=1} \frac{(-1)^{n+1}x^{2n-1}}{(2n-1)!}$$

$$\cos x = \sum_{n=1} \frac{(-1)^{n+1}x^{2n-2}}{(2n-2)!}$$

$$\log(1+x) = \sum_{n=1} \frac{(-1)^{n+1}x^n}{n}$$

where we have used the convention $0! = 1$. Observe that n is a **dumm**
and may be replaced by something else if this is desirable. For example,
ing n by $m - 1$ in Eq. (1.4–12) produces

$$\log(1+x) = \sum_{m=0} \frac{(-1)^m x^{m-1}}{m-1}.$$

We shall make use of this flexibility of the dummy index in the next s

A Maclaurin series representation of a function can be thought o
approximation of the function in the neighborhood of $x = 0$ as shown
1.4-1. If a series expansion about some point other than $x = 0$ is re
then we can use a **Taylor†** **series**

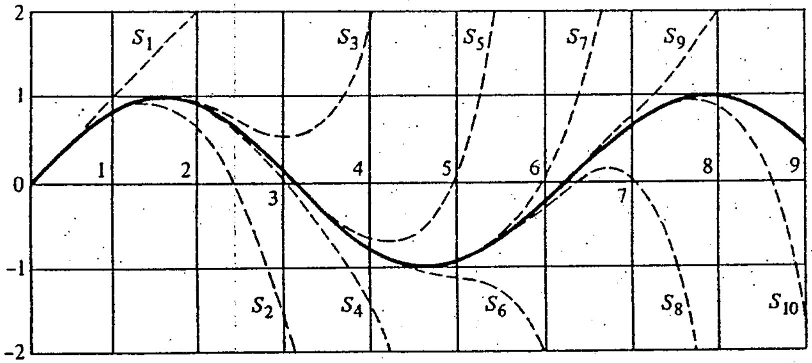
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$
$$+ \frac{f'''(a)}{3!}(x-a)^3 + \cdots. \qquad ($$

Note that Maclaurin's series is a special case of Taylor's series, the cas
$a = 0$.

It would appear from Eqs. (1.4–8) and (1.4–13) that any functi
has an infinite number of derivatives that are defined at $x = a$ can b
sented by a Taylor series in the neighborhood of $x = a$. This is not

*We shall omit the upper values of n on summations henceforth if they are $n =$

†After Brook Taylor (1685–1731), a British mathematician who discovered it
Historically, Taylor's series predated Maclaurin's series.



$$S_n(x) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^{2k-1}}{(2k-1)!}, \quad S_n(x) \to \sin x \text{ as } n \to \infty$$

Figure 1.4-1
Approximations of partial sums to sin *x*. (From H. M. Kammerer, "Sine and Cosine Approximation Curves," *MAA Monthly* 43, p. 293.)

true. The conjecture in the last statement represents an oversimplification of the facts.* We will discuss this and related topics further in the next section.

Key Words and Phrases

series of constants	closed form
divergent	alternating series
partial sums	power series
sequence	radius of convergence
limit point	ratio test
convergent	interval of convergence
integral test	Maclaurin series
harmonic series	dummy index
geometric series	Taylor series
common ratio	

1.4 Exercises

1. Write the sequence of partial sums for each of the following series.
   (a) $1 - 1 + 1 - 1 + - \cdots + (-1)^{n+1} + \cdots$
   (b) $0 + 0 + 0 + 0 + \cdots$

*A small caveat is necessary here, since there are some (pathological) functions that have ... of all orders at a point yet cannot be represented by a Taylor series there.

   (c) $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots$
   (d) $1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$

2. What is the sum of the series in Exercise 1(d)?

3. Show that the series of Example 1.4-1 converges when $x = 2$ and diverges wh $x = 0$.

4. Use the ratio test to show that the Maclaurin series for $\exp x$, $\sin x$, and cos converge for all x.

5. Show that the interval of convergence of the Maclaurin series for $\log(1 + x)$ $-1 < x \le 1$. See Eq. (1.4-12).

6. Verify that each of the following summations is correct.

   (a) $\sin x = \sum_{n=0} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

   (b) $\cos x = \sum_{n=0} \frac{(-1)^n x^{2n}}{(2n)!}$

   (c) $\exp(-x) = \sum_{n=1} \frac{(-1)^{n-1} x^{n-1}}{(n-1)!}$

   (d) $\cosh x = \frac{e^x + e^{-x}}{2} = \sum_{n=0} \frac{x^{2n}}{(2n)!}$

   (e) $\sinh x = \frac{e^x - e^{-x}}{2} = \sum_{n=1} \frac{x^{2n-1}}{(2n-1)!}$

*7. Use Maclaurin's series to compute $\sin \frac{\pi}{4}$ and $\cos \frac{\pi}{4}$ to four decimals. (*Hint:* U the fact that the error has the same sign as the first neglected term but has a small absolute value.)

•• 8. Consider the sequence

$$\left\{ \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \cdots \right\}.$$

   (a) Write the nth term of the sequence.
   (b) Show that 1 is the limit point of the sequence.

9. Apply the ratio test to the series (1.4-5) to show that the series converges f $|r| < 1$.

10. (a) Identify the series

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \cdots.$$

   (b) Find the sum of the series.

*Calculator problem.

11. Find the radius of convergence of each of the following power series.

(a) $(x-1) + \dfrac{(x-1)^3}{3} + \dfrac{(x-1)^5}{5} + \cdots$

(b) $1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$

(c) $1 + \dfrac{(x+3)}{2} + \dfrac{(x+3)^2}{3} + \dfrac{(x+3)^3}{4} + \cdots$

(d) $x + \dfrac{2!x^2}{2^2} + \dfrac{3!x^3}{3^3} + \dfrac{4!x^4}{4^4} + \cdots$ (*Hint*: Use the limit definition of e.)

(e) $1 + \dfrac{(x+2)}{3} + \dfrac{(x+2)^2}{2\cdot 3^2} + \dfrac{(x+2)^3}{3\cdot 3^3} + \cdots$

(f) $1 + \dfrac{(x-1)^2}{2!} + \dfrac{(x-1)^4}{4!} + \dfrac{(x-1)^6}{6!} + \cdots$

12. Find the interval of convergence of each of the following power series. If the interval is finite, investigate the convergence of the series at the endpoints of the interval.

(a) $\displaystyle\sum_{n=0} \frac{x^n}{2^n}$

(b) $\displaystyle\sum_{n=0} \frac{x^{2n+1}}{n!}$

(c) $\displaystyle\sum_{n=1} \frac{(x-1)^n}{2n}$

(d) $\displaystyle\sum_{n=1} \frac{x^n}{n^2}$

(e) $\displaystyle\sum_{n=1} \frac{x^n}{n^n}$

13. Verify that each of the following series is convergent.

(a) $\displaystyle\sum_{n=1} \frac{n}{(n+1)^3}$

(b) $\displaystyle\sum_{n=1} \frac{1}{n!}$

(c) $\displaystyle\sum_{n=1} n\left(\frac{1}{2}\right)^n$

(d) $\displaystyle\sum_{n=1} \frac{\sqrt{n+3}}{(n-1)^2}$

14. Verify that each of the following series is divergent.

(a) $\displaystyle\sum_{n=2} \frac{1}{\sqrt{n}\log n}$

(b) $\displaystyle\sum_{n=1} \frac{2^n}{n^2}$

(c) $\displaystyle\sum_{n=1} \frac{(n-1)^n}{n!}$

(d) $\displaystyle\sum_{n=1} \frac{n}{(n+1)^2}$

15. Determine the values of p for which the following series converges an

$$\sum_{n=2} \frac{1}{n^p \log n},$$

where p is a positive integer.

••• 16. Use the integral test to determine for what values of p (p is a real n series

$$\sum_{n=1} \frac{1}{n^p}$$

converges.

17. Consider the series

$$\frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots.$$

(a) Write S_n, the sum of the first n terms in closed form. (*Hint*: Dec nth term of the series into partial fractions.)
(b) Obtain the sum of the series.

18. Generalize Exercise 17 for

$$\sum_{n=1} \frac{1}{n(n+p)}$$

where p is a positive integer.

1.5 SERIES SOLUTIONS

Before we give an example of how a power series solution of a linea tial equation can be obtained, we need the results of two theorems. presented without proofs in order to preserve the continuity

THEOREM 1.5-1
A power series $\sum_{n=0} a_n(x - x_0)^n$ *and its derivative* $\sum_{n=1} na_n(x - x_0)^{n-1}$ *have the same radius of convergence.*

THEOREM 1.5-2
Let a function $f(x)$ *be represented by a power series* $\sum_{n=0} a_n(x - x_0)^n$ *in the interior of its interval of convergence. Then the function is differentiable there, and its derivative is given by*

$$f'(x) = \sum_{n=1} na_n(x - x_0)^{n-1}.$$

We are now ready to look at a simple differential equation with a view to solving it by using series. To begin with, we will take x_0 to be zero. Later we will indicate why this is not always possible.

EXAMPLE 1.5-1 Find a solution of the equation $y'' - xy = 0$.

Solution We *assume* that there is a solution of the form

$$y = \sum_{n=0} a_n x^n.$$

Then

$$y' = \sum_{n=1} na_n x^{n-1}$$

and

$$y'' = \sum_{n=2} n(n-1)a_n x^{n-2}.$$

Substituting these values into the given equation produces

$$\sum_{n=2} n(n-1)a_n x^{n-2} - \sum_{n=0} a_n x^{n+1} = 0.$$

In order to collect terms it would be convenient to have x^n in both summations. This can be accomplished by realizing that n is a *dummy index* of summation and can be replaced by any other letter just as we change variables in definite integrals. Accordingly, we replace *each* n in the first sum by $n + 2$ and each n in the second sum by $n - 1$. Then

$$\sum_{n=0} (n+2)(n+1)a_{n+2}x^n - \sum_{n=1} a_{n-1}x^n = 0.$$

Next we combine the two sums into one with n going from 1 to ∞, adding a terms that are left out of this sum. Thus

$$\sum_{n=1} [(n+2)(n+1)a_{n+2} - a_{n-1}]x^n + 2a_2 = 0,$$

which is a linear combination of $1, x, x^2, \ldots$. Since the set of functions

$$\{1, x, x^2, x^3, \ldots\}$$

is a linearly independent set, a linear combination of these functions can zero if and only if *each* coefficient is zero. Hence

$$2a_2 = 0,$$

and, in general,

$$(n+2)(n+1)a_{n+2} - a_{n-1} = 0.$$

From the first of these, $a_2 = 0$, and from the second we obtain the **recursi formula**

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)}, \qquad n = 1, 2, \ldots.$$

- For $n = 1$ we have $a_3 = \dfrac{a_0}{3 \cdot 2}$, so that a_0 can be arbitrary.
- For $n = 2$ we have $a_4 = \dfrac{a_1}{4 \cdot 3}$, so that a_1 can be arbitrary.
- For $n = 3$ we have $a_5 = \dfrac{a_2}{5 \cdot 4} = 0$; consequently, $a_2, a_5, a_8, \ldots$ are zero.
- For $n = 4$ we have $a_6 = \dfrac{a_0}{3 \cdot 2 \cdot 6 \cdot 5}$.
- For $n = 5$ we have $a_7 = \dfrac{a_4}{7 \cdot 6} = \dfrac{a_1}{4 \cdot 3 \cdot 7 \cdot 6}$, and so on.

The solution to the given differential equation is

$$y = a_0 + a_1 x + \frac{a_0}{6}x^3 + \frac{a_1}{12}x^4 + \frac{a_0}{180}x^6 + \frac{a_1}{504}x^7 + \cdots.$$

This last equation can also be written as

$$y = a_0\left(1 + \frac{x^3}{6} + \frac{x^6}{180} + \cdots\right) + a_1\left(x + \frac{x^4}{12} + \frac{x^7}{504} + \cdots\right),$$

which shows the two arbitrary constants we expect to find in the solution of a second-order differential equation. It can be shown (Exercise 1) that b series converge for $-\infty < x < \infty$. ■

While many series can be written in **closed form**, for example,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \tag{1.5-1}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \cdots, \tag{1.5-2}$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \cdots, \tag{1.5-3}$$

$$\log (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + - \cdots, \tag{1.5-4}$$

this is not always possible. If a function can be represented in an open interval containing x_0 by a *convergent* series of the form $\sum_{n=0} a_n(x - x_0)^n$, then the function is said to be **analytic** at $x = x_0$. The functions in Eqs. (1.5–1) through (1.5–4) are all analytic at $x = 0$. If a function is analytic at every point where it is defined, it is called an **analytic function.** All polynomials are analytic, and so are rational functions except where their denominators vanish.

Now let us look at another example of a series solution of a differential equation.

EXAMPLE 1.5–2 Solve the equation

$$(x - 1)y'' - xy' + y = 0.$$

Solution As before, assume

$$y = \sum_0 a_n x^n, \quad y' = \sum_1 n a_n x^{n-1}, \quad y'' = \sum_2 n(n-1)a_n x^{n-2},$$

and substitute into the given differential equation. Then

$$\sum_2 n(n-1)a_n x^{n-1} - \sum_2 n(n-1)a_n x^{n-2} - \sum_1 n a_n x^n + \sum_0 a_n x^n = 0.$$

Replace n by $n + 1$ in the first sum and replace n by $n + 2$ in the second sum so that we have

$$\sum_1 (n+1)n a_{n+1} x^n - \sum_0 (n+2)(n+1)a_{n+2} x^n - \sum_1 n a_n x^n + \sum_0 a_n x^n = 0$$

or

$$\sum [n(n+1)a_{n+1} - (n+1)(n+2)a_{n+2} - n a_n + a_n]x^n - 2a_2 + a_0 = 0.$$

Equating the coefficients of various powers of x to zero gives us the

a_0 is arbitrary,
a_1 is arbitrary,

$$a_2 = \frac{1}{2} a_0,$$

$$a_{n+2} = \frac{n(n+1)a_{n+1} + (1-n)a_n}{(n+1)(n+2)}, \qquad n = 1, 2, \ldots,$$

$$a_3 = \frac{2a_2}{2\cdot 3} = \frac{a_2}{3} = \frac{a_0}{3\cdot 2},$$

$$a_4 = \frac{6a_3 - a_2}{3\cdot 4} = \frac{a_3}{2} - \frac{a_2}{12} = \frac{a_0}{12} - \frac{a_0}{24} = \frac{a_0}{4!},$$

etc. Hence

$$y = a_1 x + a_0\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right),$$

and it can be shown (Exercise 2) that x and e^x are two linearly in solutions of the given equation. Here the solution can be written form in contrast to the solution of Example 1.5–1 (Exercise 3).

Unfortunately, the series method of solving ordinary differen tions is not as simple as the last two examples seem to indicate. Co equation

$$2x^2y'' + 5xy' + y = 0.$$

We leave it as an exercise (Exercise 4) to show that the series method 0 will produce only the trivial solution $y = 0$. Yet the given equ Cauchy–Euler equation, and both $x^{-1/2}$ and $1/x$ are solutions (Exerci answer to the apparent mystery lies in the fact that the individual solu Cauchy–Euler equation are not linearly independent on any interva cludes the origin. Recall that in Section 1.3 we solved Cauchy–Euler assuming that $x > 0$ *or* $x < 0$.

Consider the most general second-order, linear, homogeneous differential equation,

$$y'' + P(x)y' + Q(x)y = 0.$$

Those values of x, call them x_0, at which *both* $P(x)$ and $Q(x)$ are an called **ordinary points** of Eq. (1.5–5). If either $P(x_0)$ or $Q(x_0)$ is not then x_0 is a **singular point** of Eq. (1.5–5). If, however, x_0 is a singula Eq. (1.5–5) but *both* $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ are analytic a then x_0 is a **regular singular point** of Eq. (1.5–5). All other singular called **irregular singular points.**

XAMPLE 1.5-3 Classify the singular points of the equation

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,$$

here $n = 0, 1, 2, \ldots$.

The only singular points are $x_0 = \pm 1$. If $x_0 = -1$, then

$$\frac{(x + 1)(-2x)}{1 - x^2} = \frac{2x}{x - 1} \quad \text{and} \quad \frac{(x + 1)^2 n(n + 1)}{1 - x^2} = \frac{n(n + 1)(x + 1)}{x - 1}.$$

ince both of these rational functions are analytic at $x = -1$, the latter is a egular singular point. Similarly, for $x_0 = 1$ we have

$$\frac{(x - 1)(-2x)}{1 - x^2} = \frac{2x}{x + 1} \quad \text{and} \quad \frac{(x - 1)^2 n(n + 1)}{1 - x^2} = \frac{n(n + 1)(1 - x)}{1 + x},$$

o that $x_0 = 1$ is also a regular singular point. ■

The point of all this is contained in an 1865 theorem due to Fuchs.* Fuchs' theorem states that it is always possible to obtain *at least one* power series solution to a linear differential equation provided that the assumed series solution is about an ordinary point or, at worst, a regular singular point.

The work of Fuchs was extended by Frobenius,† who in 1874 suggested that instead of assuming a series solution of the form $\sum_0 a_n x^n$, one should use the form $\sum_0 a_n x^{n+r}$. The use of this form to solve linear, ordinary differential equations is known today as the **method of Frobenius**. We illustrate with an example using the Cauchy–Euler equation referred to above.

EXAMPLE 1.5-4 Solve the equation

$$2x^2y'' + 5xy' + y = 0$$

by the method of Frobenius.

Solution We have

$$y = \sum_0 a_n x^{n+r},$$

$$y' = \sum_0 (n + r)a_n x^{n+r-1},$$

$$y'' = \sum_0 (n + r)(n + r - 1)a_n x^{n+r-2},$$

*Lazarus Fuchs (1833–1902), a German mathematician.
†Georg Frobenius (1849–1917), a German mathematician.

and on substituting into the given equation we have

$$\sum_0 [2(n + r)(n + r - 1) + 5(n + r) + 1]a_n x^{n+r} = 0.$$

Since the coefficient of x^{n+r} must be zero for $n = 0, 1, 2, \ldots$, we have $n = 0$,

$$(2r^2 + 3r + 1)a_0 = 0.$$

Choosing a_0 to be arbitrary, that is, nonzero, produces

$$2r^2 + 3r + 1 = 0,$$

which is called the **indicial equation**. Its roots are -1 and $-1/2$. In gene

$$a_n(2n^2 + 4nr + 3n) = 0, \qquad n = 1, 2, \ldots,$$

which can be satisfied only by taking $a_n = 0$, $n = 1, 2, \ldots$. Hence we left with the two possibilities,

$$y_1(x) = a_0 x^{-1} \quad \text{and} \quad y_2(x) = b_0 x^{-1/2}.$$

Note that the two constants are arbitrary, since each root of the indicial eq tion leads to an infinite series. In this example, however, each series cons of a single term. ■

When solving *second-order* linear differential equations by the meth of Frobenius, the indicial equation is a quadratic equation, and th possibilities exist. We list these together with their consequences here.

1. If the roots of the indicial equation are *equal*, then only *one* solution be obtained.
2. If the roots of the indicial equation differ by a number that is not integer, then two linearly independent solutions may be obtained.
3. If the roots of the indicial equation differ by an integer, then the lar integer of the two will yield a solution, whereas the smaller may or n not yield a solution.

It should be mentioned that the theory behind the method of Frobenius is by no means simple. A good discussion of the various cases that may arise (although the case where the indicial equation has complex roots is omitted) can be found in Albert L. Rabenstein, *Elementary Differential Equations with Linear Algebra*, 3d ed. (New York: Academic Press, 1982), pp. 391 ff.

We conclude this section by solving two important differential eq tions that will appear later in the text in connection with certain types boundary-value problems.

EXAMPLE 1.5-5 Obtain a solution of the differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0, \qquad n = 0, 1, 2, \ldots . \tag{1.5-6}$$

This equation is known as **Bessel's differential equation**. It was originally obtained by Friedrich Wilhelm Bessel (1784–1846), a German mathematician, in the course of his studies of planetary motion. Since then, this equation has appeared in problems of heat conduction, electromagnetic theory, and acoustics that are expressed in *cylindrical coordinates*.

Solution Since the coefficients are not constant, we seek a series solution. Multiplying Eq. (1.5-6) by x^2, we obtain

$$x^2y'' + xy' + (x^2 - n^2)y = 0. \tag{1.5-7}$$

We note that $x = 0$ is a regular singular point; hence we use the method of Frobenius. Assume that

$$y = \sum_{m=0} a_m x^{m+r},$$

$$y' = \sum_{m=0} a_m(m+r)x^{m+r-1},$$

$$y'' = \sum_{m=0} a_m(m+r)(m+r-1)x^{m+r-2}$$

and substitute into Eq. (1.5-7). Then

$$\sum_{m=0} a_m(m+r)(m+r-1)x^{m+r} + \sum_{m=0} a_m(m+r)x^{m+r} + \sum_{m=0} a_m x^{m+r+2} - n^2 \sum_{m=0} a_m x^{m+r} = 0.$$

If we replace m by $m - 2$ in the third series, the last equation can be written as

$$\sum_{m=2} [a_m(m+r)(m+r-1) + a_m(m+r) + a_{m-2} - n^2 a_m]x^{m+r} + a_0 r(r-1)x^r + a_0 r x^r - n^2 a_0 x^r + a_1 r(r+1)x^{r+1} + a_1(r+1)x^{r+1} - n^2 a_1 x^{r+1} = 0.$$

Simplifying, we get

$$\sum_{m=2} [a_m((m+r)^2 - n^2) + a_{m-2}]x^{m+r} + a_0(r^2 - n^2)x^r + a_1(r^2 + 2r + 1 - n^2)x^{r+1} = 0.$$

The coefficient of x^r must be zero; hence if we assume $a_0 \neq 0$, then we obtain $r = \pm n$. We choose the positive sign, since n was defined as a non-

we may choose $a_1 = 0$. Then the recursion formula is obtained b[y] coefficient of x^{m+r} equal to zero. Thus

$$a_m = \frac{-a_{m-2}}{m(m+2n)}, \qquad m = 2, 3, \ldots .$$

The first few coefficients can be computed from this formula. T[...] follows:

$$m = 2: \quad a_2 = \frac{-a_0}{2^2(n+1)},$$

$$m = 4: \quad a_4 = \frac{-a_2}{2^3(n+2)} = \frac{a_0}{2^4 \cdot 2(n+1)(n+2)},$$

$$m = 6: \quad a_6 = \frac{-a_4}{2^2 \cdot 3(n+3)} = \frac{a_0}{2^6 \cdot 3!(n+1)(n+2[\ldots]}$$

In general we have

$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}, \qquad m = 1, 2, \ldots$$

and a solution to Eq. (1.5-6) can be written as

$$y_n(x) = a_0 \sum_{m=0} \frac{(-1)^m x^{2m+n}}{2^{2m} m!(n+1)(n+2)\cdots(n+m)}$$

$$= 2^n n! a_0 \sum_{m=0} \frac{(-1)^m}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n}. \qquad \blacksquare$$

The **Bessel function of the first kind of order n** is defined b[y] the value $1/2^n n!$. We have

$$J_n(x) = \sum_{m=0} \frac{(-1)^m}{m!(m+n)!}\left(\frac{x}{2}\right)^{2m+n}, \qquad n = 0, 1, 2, \ldots$$

a solution of Bessel's differential equation. We will consider this f[...] greater detail in Chapter 7.

EXAMPLE 1.5-6 Obtain a solution of the equation

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

where n is a constant. This equation is known as **Legendre's d[...] equation.***

*After Adrien Marie Legendre (1752–1833), a French mathematician who is kn[...]

Solution Since $x = \pm 1$ are regular singular points (see Example 1.5-3), we may assume a power series about $x = 0$, which is an ordinary point. Accordingly, put

$$y = \sum_{m=0} a_m x^m, \quad y' = \sum_{m=1} a_m m x^{m-1}, \quad y'' = \sum_{m=2} a_m m(m-1)x^{m-2}.$$

Substituting these values into Eq. (1.5-9) produces

$$\sum_{m=2} a_m m(m-1)x^{m-2} - \sum_{m=2} a_m m(m-1)x^m - 2\sum_{m=1} a_m m x^m + n(n+1)\sum_{m=0} a_m x^m = 0.$$

Replacing m by $m + 2$ in the first sum, we get

$$\sum_{m=0} a_{m+2}(m+2)(m+1)x^m - \sum_{m=2} a_m(m-1)x^m - 2\sum_{m=1} a_m m x^m + n(n+1)\sum_{m=0} a_m x^m = 0,$$

or

$$\sum_{m=2} [a_{m+2}(m+2)(m+1) - a_m m(m-1) - 2a_m m + a_m n(n+1)]x^m + 2a_2 + 6a_3 x - 2a_1 x + n(n+1)a_0 + n(n+1)a_1 x = 0.$$

Setting the coefficient of each power of x equal to zero in the above, we have

$$2a_2 + n(n+1)a_0 = 0, \qquad a_2 = \frac{-n(n+1)a_0}{2}, \qquad a_0 \text{ arbitrary;}$$

$$6a_3 - 2a_1 + n(n+1)a_1 = 0, \qquad a_3 = \frac{[2 - n(n+1)]a_1}{6}, \qquad a_1 \text{ arbitrary.}$$

In general, we can say

$$a_{m+2}(m+2)(m+1) - [m(m-1) + 2m - n(n+1)]a_m = 0;$$

$$a_{m+2} = \frac{m(m+1) - n(n+1)}{(m+2)(m+1)} a_m;$$

$$a_{m+2} = \frac{(m-n)(m+n+1)}{(m+2)(m+1)} a_m, \qquad m = 0, 1, 2, \ldots. \quad \text{(1.5-10)}$$

Equation (1.5-10) is the recurrence relation from which the coefficients can be found.

Computing the first few coefficients gives us

$$a_2 = \frac{-n(n+1)}{1\cdot 2} a_0,$$

$$a_4 = \frac{(2-n)(n+3)}{4\cdot 3} a_2 = \frac{n(n-2)(n+1)(n+3)}{4!} a_0,$$

$$a_6 = \frac{(4-n)(n+5)}{6\cdot 5} a_4 = \frac{-n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!} a_0,$$

$$a_3 = \frac{(1-n)(n+2)}{3\cdot 2} a_1 = \frac{-(n-1)(n+2)}{3!} a_1,$$

$$a_5 = \frac{(3-n)(n+4)}{5\cdot 4} a_3 = \frac{(n-1)(n-3)(n+2)(n+4)}{5!} a_1,$$

$$a_7 = \frac{(5-n)(n+6)}{7\cdot 6} a_5 = \frac{-(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!} a_1.$$

Hence a solution to Legendre's equation can be written as

$$y_n(x) = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n-2)(n+1)(n+3)}{4!}x^4 - \frac{n(n-2)(n-4)(n+1)(n+3)(n+5)}{6!}x^6 + - \cdots\right]$$
$$+ a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{5!} - \frac{(n-1)(n-3)(n-5)(n+2)(n+4)(n+6)}{7!}x^7 + - \cdots\right]. \quad \text{(1.}$$

Both series converge for $-1 < x < 1$.

If $n = 0, 2, 4, \ldots$ and a_1 is chosen to be zero, then the solutions, usi Eq. (1.5-11), become

$$y_0(x) = a_0,$$
$$y_2(x) = a_0(1 - 3x^2),$$
$$y_4(x) = a_0\left(1 - 10x^2 + \frac{35}{3}x^4\right), \qquad \text{etc.}$$

If we also impose the condition that $y_n(1) = 1$, then we can evaluate the a_0 to obtain

$$P_0(x) = 1,$$
$$P_2(x) = \frac{1}{2}(3x^2 - 1), \tag{1.5-12}$$
$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \ldots .$$

These *polynomials* are called the **Legendre polynomials of even degree.**

If $n = 1, 3, 5, \ldots$ and a_0 is chosen to be zero, then the solutions, using Eq. (1.5-11), become

$$y_1(x) = a_1 x,$$
$$y_3(x) = a_1\left(x - \frac{5}{3}x^3\right),$$
$$y_5(x) = a_1\left(x - \frac{14}{3}x^3 + \frac{21}{5}x^5\right), \quad \text{etc.}$$

If we again impose the condition that $y_n(1) = 1$, then we can evaluate the a_1 to obtain

$$P_1(x) = x, \quad P_3(x) = \frac{1}{2}(5x^3 - 3x),$$
$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x), \ldots . \tag{1.5-13}$$

These polynomials are called the **Legendre polynomials of odd degree.** ■

The **Legendre polynomials** will be of use in Section 7.3, since they arise in boundary-value problems expressed in *spherical coordinates.*

Key Words and Phrases

recursion formula
closed form
analytic
analytic function
ordinary point
singular point
regular and irregular singular point
method of Frobenius
indicial equation
Bessel's differential equation
Bessel function of the first kind of order n,
Legendre's differential equation
Legendre polynomials of even degree
Legendre polynomials of odd degree

1.5 Exercises

• **1.** **(a)** Show that one solution of the differential equation $y'' - xy = 0$ 1.5-1 can be written as

$$y_1(x) = a_0 \sum_{n=0} \frac{1\cdot 4\cdot 7 \cdots (3n-2)}{(3n)!} x^{3n}.$$

(b) Write the second solution in a similar form.
(c) Find the radius of convergence of the series in part (a).
(d) Observe that $x_0 = 0$ is an ordinary point and hence that both s analytic.

2. Verify that $y_1(x) = x$ and $y_2(x) = e^x$ are linearly independent solu differential equation $(x - 1)y'' - xy' + y = 0$.

3. **(a)** Show that the solution obtained in Example 1.5-2 is equivalent

$$y(x) = c_1 x + c_2 e^x.$$

(b) For what values of x is the above solution valid?

4. Show that assuming a solution of the form $y = \sum_0 a_n x^n$ for the equ

$$2x^2y'' + 5xy' + y = 0$$

leads to the trivial solution $y = 0$.

5. Verify that $x^{-1/2}$ and $1/x$ are linearly independent solutions of the Exercise 4 on every interval not containing the origin.

•• **6.** Classify the singular points of each of the following differential equa
(a) $x^2y'' + xy' + (x^2 - n^2)y = 0, \quad n = 0, 1, 2, \ldots$
(b) $x^3y'' - xy' + y = 0$
(c) $x^2y'' + (4x - 1)y' + 2y = 0$
(d) $x^3(x - 1)^2y'' + x^4(x - 1)^3y' + y = 0$

7. Use power series to solve each of the following equations.
(a) $y'' + y = 0$
(b) $y'' - y = 0$
(c) $y' - y = x^2$
(Note that the power series method is not limited to *homogeneous*
(d) $y' - xy = 0$
(If possible, write the solution in closed form.)
(e) $(1 + x^2)y'' + 2xy' - 2y = 0$

8. Solve each of the following differential equations by the method of F
(a) $xy'' + y' + xy = 0$
(b) $4xy'' + 2y' + y = 0$
(c) $x^2y'' + 2xy' - 2y = 0$

9. Solve the equation

$$xy'' + 2y' = 0$$

by two methods. (*Hint:* x is an integrating factor.)

10. Solve the equation

$$y'' - xy' - y = 0$$

by assuming a solution that is a power series in $(x - 1)$. In this case the coefficient x must also be written in terms of $x - 1$. This can be done by assuming that $x = A(x - 1) + B$ and determining the constants A and B.

11. Obtain a solution of

$$xy'' + y' + 4xy = 0.$$

12. Obtain a solution of

$$xy'' - 2y = 0.$$

13. The differential equation

$$y'' - xy = 0$$

is known as **Airy's equation,*** and its solutions are called **Airy functions** (Fig. 1.5-1), which have applications in the theory of diffraction.

(a) Obtain the solution in terms of a power series in x.

(b) Obtain the solution in terms of a power series in $(x - 1)$. (Compare Exercise 10.)

14. Compare the solutions of Exercise 13 with those of $y'' - y = 0$. Comment.

15. Solve the initial-value problem

$$y'' + xy = 2, \qquad y(0) = y'(0) = 1.$$

16. Solve the differential equation

$$4xy'' + 2y' + y = 0.$$

17. Find the interval of convergence of the two series in the solution of Exercise 16.

18. Solve the initial-value problem

$$y'' + y' + xy = 0, \qquad y(0) = y'(0) = 1.$$

19. Obtain the general solution of

$$2x^2y'' - xy' + (1 - x^2)y = 0.$$

20. Obtain the general solution of

$$x^2y'' + x^2y' - 2y = 0.$$

21. Illustrate Theorem 1.5-1 by differentiating the series in Exercise 11 of Section 1.4 and finding the radii of convergence of the differentiated series. (Note that this does *not* constitute a *proof* of the theorem.)

22. Illustrate Theorem 1.5-2 by differentiating the functions and series in Eqs. (1.5-1) through (1.5-4).

23. Find the general solution of

$$y'' + xy' + y = 0.$$

*After Sir George B. Airy (1801–1892), an English mathematician and astronomer.

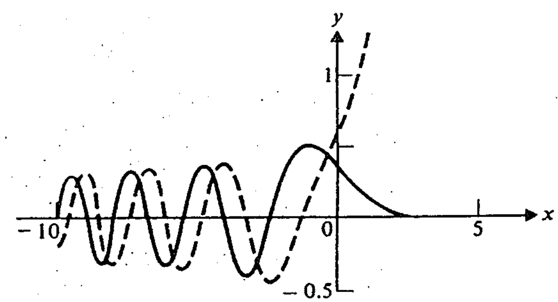


Figure 1.5-1
Airy functions.

1.6 UNIFORM CONVERGENCE

We give a brief discussion of some of the theoretical aspects in the present s tion, not for the sake of the theory per se but to prepare for a wide variety applications. It will be shown, for example, that we will have to broaden o concept of convergence.

Pointwise Convergence

Recall what is meant by saying that a sequence of functions, defined $a \leq x \leq b$,

$$\{f_1(x), f_2(x), f_3(x), \ldots, f_n(x), \ldots\},$$

converges to $f(x)$. When we write

$$f(x) = \lim_{n\to\infty} f_n(x) \qquad \text{for all } x \text{ in } [a, b],$$

we mean that the difference between $f(x)$ and $f_n(x)$ can be made arbitrar small provided that n is taken large enough.

In more precise mathematical language we would say that given a positive number ϵ and any point x_0 in the interval $[a, b]$, we can satisfy

$$|f_n(x_0) - f(x_0)| < \epsilon \qquad (1.6$$

whenever $n \geq N(\epsilon, x_0)$, an integer. In other words, given $\epsilon > 0$, we can fi an integer N such that the inequality (1.6-1) holds whenever $n \geq N$. The i portant thing to notice here is that N will usually depend on ϵ *and* x_0, so write $N(\epsilon, x_0)$.

(a) Show that

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{1}{x^2}\frac{d^2y}{dz^2} - \frac{1}{x^2}\frac{dy}{dz}.$$

(b) Show that the Euler equation becomes

$$\frac{d^2y}{dz^2} + (\alpha - 1)\frac{dy}{dz} + \beta y = 0.$$

Letting r_1 and r_2 denote the roots of $r^2 + (\alpha - 1)r + \beta = 0$, show that

(c) if r_1 and r_2 are real and unequal, then

$$y = c_1e^{r_1z} + c_2e^{r_2z} = c_1x^{r_1} + c_2x^{r_2},$$

(d) if r_1 and r_2 are equal, then

$$y = (c_1 + c_2z)e^{r_1z} = (c_1 + c_2 \ln x)x^{r_1},$$

(e) if r_1 and r_2 are complex conjugates, $r_1 = \lambda + i\mu$, then

$$y = e^{\lambda z}(c_1 \cos \mu z + c_2 \sin \mu z) = x^\lambda[c_1 \cos (\mu \ln x) + c_2 \sin (\mu \ln x)].$$

5. Using the method of Problem 4, solve the following differential equations for $x > 0$.

(a) $x^2y'' - 2y = 0$

(b) $x^2y'' - 3xy' + 4y = \ln x$

(c) $x^2y'' + 7xy' + 5y = x$

(d) $x^2y'' - 2xy' + 2y = 3x^2 + 2 \ln x$

(e) $x^2y'' + xy' + 4y = \sin (\ln x)$

(f) $3x^2y'' + 12xy' + 9y = 0$

6. Show that if $L[y] = x^2y'' + \alpha xy' + \beta y$, then

$$L[(-x)^r] = (-x)^rF(r)$$

for $x < 0$, where $F(r) = r(r-1) + \alpha r + \beta$. Hence conclude that if $r_1 \neq r_2$ are roots of $F(r) = 0$, then linearly independent solutious of $L[y] = 0$ for $x < 0$ are $(-x)^{r_1}$ and $(-x)^{r_2}$.

7. Suppose that x^{r_1} and x^{r_2} are solutions of an Euler equation, where $r_1 \neq r_2$, and r_1 is an integer. According to Eq. (24) the general solution in any interval not containing the origin is $y = c_1|x|^{r_1} + c_2|x|^{r_2}$. Show that the general solution can also be written as $y = k_1x^{r_1} + k_2|x|^{r_2}$.

Hint: Show by a proper choice of constants that the expressions are identical for $x > 0$; and by a different choice of constants that they are identical for $x < 0$.

8. If the constants α and β in the Euler equation $x^2y'' + \alpha xy' + \beta y = 0$ are complex numbers, it is still possible to obtain solutions of the form x^r. However, in general, the solutions will no longer be real-valued. Determine the general solution of each of the following equations for the interval $x > 0$.

(a) $x^2y'' + 2ixy' - iy = 0$

(b) $x^2y'' + (1 - i)xy' + 2y = 0$

(c) $x^2y'' + xy' - 2iy = 0$

Hint: See Problems 13, 14, and 15 of Section 3.5.1.

4.5 SERIES SOLUTIONS NEAR A REGULAR SINGULAR POINT, PART I

We will now consider the question of solving the general linear second order equation

$$P(x)y'' + Q(x)y' + R(x)y = 0 \tag{1}$$

in the neighborhood of a regular singular point $x = x_0$. For convenience we will assume that $x_0 = 0$. If $x_0 \neq 0$ the equation can be transformed into one for which the regular singular point is at the origin by letting $x - x_0$ equal z.

The fact that $x = 0$ is a regular singular point of Eq. (1) means that $xQ(x)/P(x) = xp(x)$ and $x^2R(x)/P(x) = x^2q(x)$ have finite limits as $x \to 0$, and are analytic at $x = 0$ having power series expansions of the form

$$xp(x) = \sum_{n=0}^{\infty} p_nx^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_nx^n, \tag{2}$$

which are convergent for some interval $|x| < \rho$, $\rho > 0$, about the origin. It is convenient for theoretical purposes to divide Eq. (1) by $P(x)$ and then multiply by x^2, obtaining

$$x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{3a}$$

or

$$x^2y'' + x(p_0 + p_1x + \cdots + p_nx^n + \cdots)y' + (q_0 + q_1x + \cdots + q_nx^n + \cdots)y = 0. \tag{3b}$$

If all of the p_n and q_n are zero except

$$p_0 = \lim_{x\to 0} \frac{xQ(x)}{P(x)}, \qquad q_0 = \lim_{x\to 0} \frac{x^2R(x)}{P(x)},$$

then Eq. (3) reduces to the Euler equation

$$x^2y'' + p_0xy' + q_0y = 0,$$

which was discussed in the previous section. In general, of course, some of the p_n and q_n, $n \geq 1$, will not be zero. However, the essential character of solutions of Eq. (3) is identical to that of solutions of the Euler equation. The presence of the variable terms $xp(x)$ and $x^2q(x)$ merely complicates the calculations.

We shall primarily restrict our discussion to the interval $x > 0$. The interval $x < 0$ can be treated, just as for the Euler equation, by making the change of variable $x = -\xi$ and then solving the resulting equation for $\xi > 0$.

Since the coefficients in Eq. (3) are "Euler coefficients" times power series, it is natural to seek solutions of the form of "Euler solutions" times power series:

$$y = x^r(a_0 + a_1x + \cdots + a_nx^n + \cdots) = x^r\sum_{n=0}^{\infty} a_nx^n. \tag{4}$$

As part of our problem we have to determine:

1. The values of r for which Eq. (1) has a solution of the form (4).
2. The recurrence relation for the a_n.
3. The radius of convergence of the series $\sum_{n=0}^{\infty} a_n x^n$.

We shall find that Eq. (1) will always have at least one solution of the form (4) with a_0 arbitrary, and possibly a second solution corresponding to a second value of r. If there is not a second solution of the form (4), the second solution will involve a logarithmic term just as did the second solution of the Euler equation when the roots were equal. In theory, of course, once we obtain the first solution, which will be of the form (4), a second solution can be obtained by the method of reduction of order. Unfortunately, this is not always a convenient method for obtaining a second solution; we will give an alternate method in Sections 4.6 and 4.7.

The general theory is due to the German mathematician Frobenius (1849–1917) and is fairly complicated. Rather than trying to present this theory we shall simply assume in this and the next three sections that there does exist a solution of the stated form. In particular we assume that any power series in an expression for a solution has a nonzero radius of convergence, and shall concentrate on showing how to determine the coefficients in such a series.

Let us first illustrate the method of Frobenius for a specific differential equation which has two solutions of the form (4). Consider the equation

$$2x^2y'' - xy' + (1 + x)y = 0. \tag{5}$$

By comparison with Eq. (3), it is clear that $x = 0$ is a regular singular point of Eq. (5). We will formally try to find a solution of Eq. (5) of the form (4) for $x > 0$. Then y' and y'' are given by

$$y' = \sum_{n=0}^{\infty} a_n(r + n)x^{r+n-1}, \tag{6}$$

and

$$y'' = \sum_{n=0}^{\infty} a_n(r + n)(r + n - 1)x^{r+n-2}. \tag{7}$$

Hence

$$2x^2y'' - xy' + (1 + x)y = \sum_{n=0}^{\infty} 2a_n(r + n)(r + n - 1)x^{r+n} - \sum_{n=0}^{\infty} a_n(r + n)x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+1}.$$

Noting that the last sum can be written in the form $\sum_{n=1}^{\infty} a_{n-1}x^{r+n}$ and combining terms gives

$$2x^2y'' - xy' + (1 + x)y = a_0[2r(r - 1) - r + 1]x^r + \sum_{n=1}^{\infty} \{[2(r + n)(r + n - 1) - (r + n) + 1]a_n + a_{n-1}\}x^{r+n} = 0. \tag{8}$$

If Eq. (8) is to be identically satisfied, the coefficient of each power of x in Eq. (8) must be zero. Corresponding to x^r we obtain, since $a_0 \neq 0$,*

$$2r(r - 1) - r + 1 = 0; \tag{9}$$

and from the coefficient of x^{r+n} we have

$$[2(r + n)(r + n - 1) - (r + n) + 1]a_n + a_{n-1} = 0, \qquad n \geq 1. \tag{10}$$

Equation (9) is called the *indicial equation* for Eq. (5). It is a quadratic equation in r, and its roots determine the two possible values of r for which there may be solutions of Eq. (5) of the form (4). Factoring Eq. (9) gives $(2r - 1)(r - 1) = 0$; hence the roots of the indicial equation are

$$r_1 = 1, \qquad r_2 = \tfrac{1}{2}. \tag{11}$$

In the following, whenever the roots of the indicial equation are real and unequal, the larger one will be denoted by r_1. The roots of the indicial equation are often referred to as the *exponents of the singularity* at the regular singular point.

Equation (10) gives a recurrence relation for the coefficients a_n. Corresponding to $r = r_1 = 1$, we have, on transposing a_{n-1} and dividing by the coefficient of a_n,

$$a_n = \frac{-1}{2(n + 1)n - (n + 1) + 1}a_{n-1}, \qquad n \geq 1$$
$$= \frac{-1}{(2n + 1)n}a_{n-1}.$$

Thus

$$a_n = \frac{-1}{(2n + 1)n} \cdot \frac{-1}{(2n - 1)(n - 1)}a_{n-2}$$

$$= \frac{(-1)^n}{(2n + 1)n(2n - 1)(n - 1)\cdots(5 \cdot 2)(3 \cdot 1)}a_0. \tag{12}$$

* Nothing is gained by taking $a_0 = 0$, as this would simply mean that the series would start with the term a_1x^{r+1} or, if $a_1 = 0$, the term a_2x^{r+2}, and so on. Suppose the first nonzero term is a_mx^{r+m}; this is equivalent to using the series (4) with r replaced by $r + m$ and a_m replaced by a_0. Indeed r is chosen to be the lowest power of x appearing in the series and we take $a_0 \neq 0$ as its coefficient.

Hence one solution of Eq. (5), omitting the constant multiplier a_0, is

$$y_1(x) = x\left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^n}{(2n+1)n(2n-1)(n-1)\cdots(5\cdot 2)(3\cdot 1)}\right], \qquad x > 0. \tag{13}$$

To determine the radius of convergence of the series in Eq. (13) we use the ratio test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim_{n\to\infty}\frac{|x|}{(2n+3)(n+1)} = 0$$

for all x. Thus the series converges for all x.

Corresponding to the second root $r = r_2 = \frac{1}{2}$, we proceed similarly:

$$\begin{aligned} a_n &= \frac{-1}{2(n+\frac{1}{2})(n-\frac{1}{2}) - (n+\frac{1}{2}) + 1}\, a_{n-1} \\ &= \frac{-1}{2n(n-\frac{1}{2})}\, a_{n-1} \\ &= \frac{-1}{2n(n-\frac{1}{2})}\,\frac{-1}{(2n-2)(n-\frac{3}{2})}\, a_{n-2} \\ &\;\;\vdots \\ &= \frac{(-1)^n}{2n(n-\frac{1}{2})(2n-2)(n-\frac{3}{2})\cdots(4\cdot\frac{3}{2})(2\cdot\frac{1}{2})}\, a_0, \qquad n \geq 1. \end{aligned} \tag{14}$$

Again omitting the constant multiplier a_0, we obtain a second solution

$$y_2(x) = x^{1/2}\left[1 + \sum_{n=1}^{\infty}\frac{(-1)^n x^n}{2n(n-\frac{1}{2})(2n-2)(n-\frac{3}{2})\cdots(4\cdot\frac{3}{2})(2\cdot\frac{1}{2})}\right], \qquad x > 0. \tag{15}$$

As before, we can show that the series in Eq. (15) converges for all x. Since the leading terms in the series solutions y_1 and y_2 are x and $x^{1/2}$, respectively, it is clear that the solutions are linearly independent. Hence the general solution of Eq. (5) is

$$y = c_1 y_1(x) + c_2 y_2(x), \qquad x > 0.$$

The preceding example is illuminating in several respects. Let us reconsider our calculations from a more general viewpoint. Let

$$\begin{aligned} F(r) &= 2r(r-1) - r + 1 \\ &= (2r-1)(r-1); \end{aligned} \tag{16}$$

then the indicial equation is $F(r) = 0$, and the recurrence relation (10) becomes

$$F(r+n)a_n + a_{n-1} = 0, \qquad n \geq 1. \tag{17}$$

There are two possible sources of trouble. First, if the two roots r_1 and r_2 of the indicial equation are equal, then we will obtain only one series solution of the form (4). Second, there is the possibility that for a given root r_1 or r_2 we may not be able to solve for one of the a_n from Eq. (17) because $F(r+n)$ may be zero. When can this happen? The only zeros of $F(r)$ are r_1 and r_2, and if $r_1 > r_2$, then clearly $r_1 + n > r_1 > r_2$ for all n and $F(r_1 + n) \neq 0$ for all n. Thus there will never be any difficulty in computing the series solution corresponding to the larger of the roots of the indicial equation. On the other hand consider $F(r_2 + n)$, and suppose that $r_1 - r_2 = N$, a positive integer. Then for $n = N$ we have

$$F(r_2 + N)a_N + a_{N-1} = 0;$$

but $F(r_2 + N) = F(r_1) = 0$ and we cannot solve for a_N, and hence cannot obtain a second series solution. In more general problems, as we shall see in the next section, the recurrence relation will be more complicated and even though $F(r_2 + N)$ may vanish, it may be possible to determine a second solution of the form (4) corresponding to $r = r_2$. In any event the cases $r_1 = r_2$ and $r_1 - r_2 = N$, a positive integer, require special attention.

Finally, we note that if the roots of the indicial equation are complex, then this type of difficulty cannot occur; on the other hand the solutions corresponding to r_1 and r_2 will be complex-valued functions of x. However, just as for the Euler equation, it is possible to obtain real-valued solutions by taking suitable linear combinations of the original solutions.

Before turning to a general discussion and the special cases $r_1 = r_2$ and $r_1 - r_2 = N$, a positive integer, there is a practical point that should be mentioned. If P, Q, and R are polynomials, it is much better to work directly with Eq. (1) than with Eq. (3). This avoids the necessity of expressing $xQ(x)/P(x)$ and $x^2R(x)/P(x)$ as power series. For example, it is more convenient to consider the equation

$$x(1+x)y'' + 2y' + xy = 0$$

than to write it in the form

$$x^2y'' + x\frac{2}{1+x}y' + \frac{x^2}{1+x}y = 0,$$

and then to expand $2/(1+x)$ and $x^2/(1+x)$, which gives

$$x^2y'' + x[2(1 - x + x^2 - \cdots)]y' + (x^2 - x^3 + x^4 - \cdots)y = 0.$$

5. Consider the differential equation

$$x^3y'' + \alpha xy' + \beta y = 0$$

where $\alpha \neq 0$ and β are real constants.

(a) Show that $x = 0$ is an irregular singular point.

(b) Show that if we attempt to determine a solution of the form $x^r \sum_{n=0}^{\infty} a_n x^n$, the indicial equation for r will be linear, and as a consequence there will be only one formal solution of this form.

6. Consider the differential equation

$$y'' + \frac{\alpha}{x^s}y' + \frac{\beta}{x^t}y = 0, \tag{i}$$

where $\alpha \neq 0$ and $\beta \neq 0$ are real numbers, and s and t are positive integers which for the moment are arbitrary.

(a) Show that if $s > 1$ or $t > 2$ the point $x = 0$ is an irregular singular point.

(b) Suppose we try to find a solution of Eq. (i) of the form

$$y = \sum_{n=0}^{\infty} a_n x^{r+n}, \qquad x > 0. \tag{ii}$$

Show that if $s = 2$ and $t = 2$ there is only one possible value of r for which there is a formal solution of Eq. (i) of the form (ii).

(c) Show that if $s = 1$ and $t = 3$ there are no solutions of Eq. (i) of the form (ii).

(d) Show that the maximum values of s and t for which the indicial equation is quadratic in r (and hence we can hope to find two solutions of the form (ii)) are $s = 1$ and $t = 2$. These are precisely the conditions that distinguish a "weak singularity," or a regular singular point from an irregular singular point, as we defined them in Section 4.3.

As a note of caution we should point out that while it is sometimes possible to obtain a formal series solution of the form (ii) at an irregular singular point, the series may not converge.

*4.6 SERIES SOLUTIONS NEAR A REGULAR SINGULAR POINT; $r_1 = r_2$ AND $r_1 - r_2 = N$

We now consider the cases in which the roots r_1 and r_2 of the indicial equation are equal or differ by a positive integer, $r_1 - r_2 = N$. If $r_1 = r_2$, we would expect by analogy with the Euler equation that the second solution will contain a logarithmic term. This may also be the case when the roots differ by an integer. The general situation, including the case in which r_1 and r_2 do not differ by an integer, is summarized in the following theorem.

Theorem 4.4. *Consider the differential equation*

$$L[y] = x^2y'' + x[xp(x)]y' + [x^2q(x)]y = 0, \tag{1}$$

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where $x = 0$ is a regular singular point. Then the functions $xp(x)$ and $x^2q(x)$ are analytic at $x = 0$ with power series representations

$$xp(x) = \sum_{n=0}^{\infty} p_n x^n, \qquad x^2q(x) = \sum_{n=0}^{\infty} q_n x^n, \tag{2}$$

which converge for $|x| < \rho$, $\rho > 0$. Let r_1 and r_2, where $r_1 \geq r_2$ if they are real, be the roots of the indicial equation

$$F(r) = r(r-1) + p_0 r + q_0 = 0. \tag{3}$$

Then in either of the intervals $-\rho < x < 0$ or $0 < x < \rho$, Eq. (1) has two linearly independent solutions y_1 and y_2 of the following form.

1. *If $r_1 - r_2$ is not an integer, then*

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \tag{4a}$$

$$y_2(x) = |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} a_n(r_2)x^n\right]. \tag{4b}$$

2. *If $r_1 = r_2$, then*

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \tag{5a}$$

$$y_2(x) = y_1(x)\ln|x| + |x|^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n. \tag{5b}$$

3. *If $r_1 - r_2 = N$, a positive integer, then*

$$y_1(x) = |x|^{r_1}\left[1 + \sum_{n=1}^{\infty} a_n(r_1)x^n\right], \tag{6a}$$

$$y_2(x) = ay_1(x)\ln|x| + |x|^{r_2}\left[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n\right]. \tag{6b}$$

The coefficients $a_n(r_1)$, $a_n(r_2)$, $b_n(r_1)$, $c_n(r_2)$, and the constant a can be determined by substituting the form of the series solution for y in Eq. (1). The constant a may turn out to be zero. Each of the series in Eqs. (4), (5), and (6) converges for $|x| < \rho$ and defines a function that is analytic at $x = 0$.

The case in which $r_1 - r_2$ is not an integer was discussed in Section 4.5.1. We turn now to the case $r_1 = r_2$, and assume that $x > 0$. The case $x < 0$ can be treated, as before, by letting $x = -\xi$.

The method of finding the second solution is essentially the same as that which we used in finding the second solution of the Euler equation (see Section 4.4) when the roots of the indicial equation were equal. We seek a solution of Eq. (1) of the form

$$y = \phi(r, x) = x^r\sum_{n=0}^{\infty} a_n x^n = x^r\left(a_0 + \sum_{n=1}^{\infty} a_n x^n\right), \tag{7}$$

Bessel Equation of Order Zero. This example illustrates the situation in which the roots of the indicial equation are equal. Setting $\nu = 0$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + x^2y = 0. \tag{2}$$

Substituting

$$y = \phi(r, x) = a_0x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \tag{3}$$

we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} a_n[(n+r)(n+r-1) + (n+r)]x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0[r(r-1) + r]x^r + a_1[(r+1)r + (r+1)]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty} \{a_n[(n+r)(n+r-1) + (n+r)] + a_{n-2}\}x^{r+n} = 0. \end{aligned} \tag{4}$$

The roots of the indicial equation $F(r) = r(r-1) + r = 0$ are $r_1 = 0$ and $r_2 = 0$; hence we have the case of equal roots. The recurrence relation is

$$a_n(r) = \frac{-a_{n-2}(r)}{(n+r)(n+r-1) + (n+r)} = -\frac{a_{n-2}(r)}{(n+r)^2}, \qquad n \geq 2. \tag{5}$$

To determine $y_1(x)$ we set r equal to 0. Then from Eq. (4) it follows that $a_1 = 0$. Hence from Eq. (5), $a_3 = a_5 = a_7 = \cdots = a_{2n+1} = \cdots = 0$. Further

$$a_n(0) = -a_{n-2}(0)/n^2, \qquad n = 2, 4, 6, 8, \ldots,$$

or letting $n = 2m$,

$$\begin{aligned} a_{2m}(0) &= -\frac{a_{2m-2}(0)}{(2m)^2} \\ &= +\frac{a_{2m-4}(0)}{(2m)^2(2m-2)^2} = \frac{(-1)a_{2m-6}(0)}{(2m)^2(2m-2)^2(2m-4)^2} \\ &\;\;\vdots \\ &= \frac{(-1)^m a_0}{(2m)^2(2m-2)^2(2m-4)^2\cdots 2^2} \\ &= \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{6}$$

Hence

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \qquad x > 0. \tag{7}$$

The function in brackets is known as the *Bessel function of the first kind of order zero*, and is denoted by $J_0(x)$. It follows from Theorem 4.4 that the series converges for all x, and that J_0 is analytic at $x = 0$. Some of the important properties of J_0 are discussed in the problems.

In this example we will determine $y_2(x)$ by computing $a_n'(0)$. The alternate procedure in which we simply substitute the form (5b) of Section 4.6 in Eq. (2), and then determine the b_n is discussed in Problem 7. First we note from Eq. (4) that since $(r+1)^2a_1(r) = 0$ it follows that not only does $a_1(0) = 0$, but also $a_1'(0) = 0$. It is easy to deduce from the recurrence relation (5) that $a_3'(0) = a_5'(0) = \cdots = a_{2n+1}'(0) = \cdots = 0$; hence we need only compute $a_{2m}'(0)$, $m = 1, 2, 3, \ldots$. From Eq. (5)

$$\begin{aligned} a_{2m}(r) &= -\frac{a_{2m-2}(r)}{(2m+r)^2} = +\frac{a_{2m-4}(r)}{(2m+r)^2(2m-2+r)^2} \\ &\;\;\vdots \\ &= \frac{(-1)^m a_0}{(2m+r)^2(2m-2+r)^2(2m-4+r)^2\cdots(2+r)^2}, \\ &\qquad\qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{8}$$

The computation of $a_{2m}'(r)$ can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n}$$

then

$$\begin{aligned} f'(x) &= \beta_1(x-\alpha_1)^{\beta_1-1}[(x-\alpha_2)^{\beta_2}\cdots(x-\alpha_n)^{\beta_n}] \\ &\quad + \beta_2(x-\alpha_2)^{\beta_2-1}[(x-\alpha_1)^{\beta_1}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n}] + \cdots; \end{aligned}$$

and hence for x not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Thus, from Eq. (8)

$$\frac{a_{2m}'(r)}{a_{2m}(r)} = -2\left(\frac{1}{2m+r} + \frac{1}{2m-2+r} + \cdots + \frac{1}{2+r}\right),$$

and setting r equal to 0 we obtain

$$a_{2m}'(0) = -2\left[\frac{1}{2m} + \frac{1}{2(m-1)} + \frac{1}{2(m-2)} + \cdots + \frac{1}{2}\right]a_{2m}(0).$$

Substituting for $a_{2m}(0)$ from Eq. (6), and letting

$$H_m = \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1, \tag{9}$$

we obtain finally

$$a_{2m}'(0) = -H_m\frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots. \tag{10}$$

The second solution of the Bessel equation of order zero is obtained by setting $a_0 = 1$, and substituting for $y_1(x)$ and $a_{2m}'(0) = b_{2m}(0)$ in Eq. (5b) of

Bessel Equation of Order Zero. This example illustrates the situation in which the roots of the indicial equation are equal. Setting $\nu = 0$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + x^2y = 0. \tag{2}$$

Substituting

$$y = \phi(r, x) = a_0x^r + \sum_{n=1}^{\infty} a_n x^{r+n}, \tag{3}$$

we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty} a_n[(n + r)(n + r - 1) + (n + r)]x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n+2} \\ &= a_0[r(r-1) + r]x^r + a_1[(r+1)r + (r+1)]x^{r+1} \\ &\quad + \sum_{n=2}^{\infty}\{a_n[(n+r)(n+r-1) + (n+r)] + a_{n-2}\}x^{r+n} = 0. \end{aligned} \tag{4}$$

The roots of the indicial equation $F(r) = r(r-1) + r = 0$ are $r_1 = 0$ and $r_2 = 0$; hence we have the case of equal roots. The recurrence relation is

$$a_n(r) = \frac{-a_{n-2}(r)}{(n+r)(n+r-1) + (n+r)} = -\frac{a_{n-2}(r)}{(n+r)^2}, \qquad n \geq 2. \tag{5}$$

To determine $y_1(x)$ we set r equal to 0. Then from Eq. (4) it follows that $a_1 = 0$. Hence from Eq. (5), $a_3 = a_5 = a_7 = \cdots = a_{2n+1} = \cdots = 0$. Further

$$a_n(0) = -a_{n-2}(0)/n^2, \qquad n = 2, 4, 6, 8, \ldots,$$

or letting $n = 2m$,

$$\begin{aligned} a_{2m}(0) &= -\frac{a_{2m-2}(0)}{(2m)^2} \\ &= +\frac{a_{2m-4}(0)}{(2m)^2(2m-2)^2} = \frac{(-1)a_{2m-6}(0)}{(2m)^2(2m-2)^2(2m-4)^2} \\ &\;\;\vdots \\ &= \frac{(-1)^m a_0}{(2m)^2(2m-2)^2(2m-4)^2\cdots 2^2} \\ &= \frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{6}$$

Hence

$$y_1(x) = a_0\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}\right], \qquad x > 0. \tag{7}$$

The function in brackets is known as the *Bessel function of the first kind of order zero*, and is denoted by $J_0(x)$. It follows from Theorem 4.4 that the series converges for all x, and that J_0 is analytic at $x = 0$. Some of the important properties of J_0 are discussed in the problems.

In this example we will determine $y_2(x)$ by computing $a_n'(0)$. The alternate procedure in which we simply substitute the form (5b) of Section 4.6 in Eq. (2), and then determine the b_n is discussed in Problem 7. First we note from Eq. (4) that since $(r+1)^2a_1(r) = 0$ it follows that not only does $a_1(0) = 0$, but also $a_1'(0) = 0$. It is easy to deduce from the recurrence relation (5) that $a_3'(0) = a_5'(0) = \cdots = a_{2n+1}'(0) = \cdots = 0$; hence we need only compute $a_{2m}'(0)$, $m = 1, 2, 3, \ldots$. From Eq. (5)

$$\begin{aligned} a_{2m}(r) &= -\frac{a_{2m-2}(r)}{(2m+r)^2} = +\frac{a_{2m-4}(r)}{(2m+r)^2(2m-2+r)^2} \\ &\;\;\vdots \\ &= \frac{(-1)^m a_0}{(2m+r)^2(2m-2+r)^2(2m-4+r)^2\cdots(2+r)^2}, \\ &\qquad\qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{8}$$

The computation of $a_{2m}'(r)$ can be carried out most conveniently by noting that if

$$f(x) = (x-\alpha_1)^{\beta_1}(x-\alpha_2)^{\beta_2}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n}$$

then

$$\begin{aligned} f'(x) &= \beta_1(x-\alpha_1)^{\beta_1-1}[(x-\alpha_2)^{\beta_2}\cdots(x-\alpha_n)^{\beta_n}] \\ &\quad + \beta_2(x-\alpha_2)^{\beta_2-1}[(x-\alpha_1)^{\beta_1}(x-\alpha_3)^{\beta_3}\cdots(x-\alpha_n)^{\beta_n}] + \cdots; \end{aligned}$$

and hence for x not equal to $\alpha_1, \alpha_2, \ldots, \alpha_n$

$$\frac{f'(x)}{f(x)} = \frac{\beta_1}{x-\alpha_1} + \frac{\beta_2}{x-\alpha_2} + \cdots + \frac{\beta_n}{x-\alpha_n}.$$

Thus, from Eq. (8)

$$\frac{a_{2m}'(r)}{a_{2m}(r)} = -2\left(\frac{1}{2m+r} + \frac{1}{2m-2+r} + \cdots + \frac{1}{2+r}\right),$$

and setting r equal to 0 we obtain

$$a_{2m}'(0) = -2\left[\frac{1}{2m} + \frac{1}{2(m-1)} + \frac{1}{2(m-2)} + \cdots + \frac{1}{2}\right]a_{2m}(0).$$

Substituting for $a_{2m}(0)$ from Eq. (6), and letting

$$H_m = \frac{1}{m} + \frac{1}{m-1} + \cdots + \frac{1}{2} + 1, \tag{9}$$

we obtain finally

$$a_{2m}'(0) = -H_m\frac{(-1)^m a_0}{2^{2m}(m!)^2}, \qquad m = 1, 2, 3, \ldots. \tag{10}$$

The second solution of the Bessel equation of order zero is obtained by setting $a_0 = 1$, and substituting for $y_1(x)$ and $a_{2m}'(0) = b_{2m}(0)$ in Eq. (5b) of

Section 4.6. We obtain

$$y_2(x) = J_0(x)\ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}, \qquad x > 0. \tag{11}$$

In place of y_2, the second solution is usually taken to be a certain linear combination of J_0 and y_2. It is known as the Bessel function of the second kind of order zero, and is denoted by Y_0. Following Copson [Chapter 12], we define*

$$Y_0(x) = \frac{2}{\pi}[y_2(x) + (\gamma - \ln 2)J_0(x)]. \tag{12}$$

Here γ is a constant, known as the Euler-Máscheroni (1750–1800) constant; it is defined by the equation

$$\gamma = \lim_{n\to\infty}(H_n - \ln n) \cong 0.5772. \tag{13}$$

Substituting for $y_2(x)$ in Eq. (12) we obtain

$$Y_0(x) = \frac{2}{\pi}\left[\left(\gamma + \ln\frac{x}{2}\right)J_0(x) + \sum_{m=1}^{\infty}\frac{(-1)^{m+1}H_m}{2^{2m}(m!)^2}x^{2m}\right], \qquad x > 0. \tag{14}$$

The general solution of the Bessel equation of order zero for $x > 0$ is

$$y = c_1J_0(x) + c_2Y_0(x).$$

Notice that since $J_0(x) \to 1$ as $x \to 0$, $Y_0(x)$ has a logarithmic singularity at $x = 0$; that is, $Y_0(x)$ behaves as $(2/\pi)\ln x$ when $x \to 0$ through positive values. Thus if we are interested in solutions of Bessel's equation of order zero which are finite at the origin, which is often the case, we must discard Y_0. The graphs of the functions J_0 and Y_0 are shown in Figure 4.2.

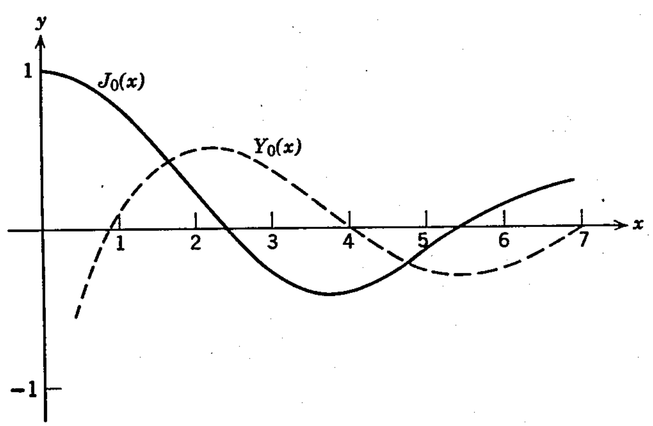

FIGURE 4.2 The Bessel functions of order zero.

* Other authors use other definitions for Y_0. The present choice for Y_0 is also known as the Weber (1842–1913) function.

Bessel Equation of Order One-Half. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer, but there is no logarithmic term in the second solution. Setting $\nu = \frac{1}{2}$ in Eq. (1) gives

$$L[y] = x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0. \tag{15}$$

If we substitute the series (3) for $y = \phi(r, x)$, we obtain

$$\begin{aligned} L[\phi](r, x) &= \sum_{n=0}^{\infty}[(r+n)(r+n-1) + (r+n) - \tfrac{1}{4}]a_nx^{r+n} + \sum_{n=0}^{\infty}a_nx^{r+n+2} \\ &= (r^2 - \tfrac{1}{4})a_0x^r + [(r+1)^2 - \tfrac{1}{4}]a_1x^{r+1} \\ &\qquad + \sum_{n=2}^{\infty}\{[(r+n)^2 - \tfrac{1}{4}]a_n + a_{n-2}\}x^{r+n} = 0. \end{aligned} \tag{16}$$

The roots of the indicial equation are $r_1 = \frac{1}{2}$, $r_2 = -\frac{1}{2}$; hence the roots differ by an integer. The recurrence relation is

$$[(r+n)^2 - \tfrac{1}{4}]a_n = -a_{n-2}, \qquad n \geq 2. \tag{17}$$

Corresponding to the larger root $r_1 = \frac{1}{2}$ we find from Eq. (16) that $a_1 = 0$, and hence from Eq. (17) that $a_3 = a_5 = \cdots = a_{2n+1} = \cdots = 0$. Further, for $r = \frac{1}{2}$

$\frac{1}{4} + n + n^2 - \frac{1}{4} = n(1+n$

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \qquad n = 2, 4, 6, \ldots,$$

or letting $n = 2m$,

$$\begin{aligned} a_{2m} &= -\frac{a_{2m-2}}{(2m+1)2m} = \frac{a_{2m-4}}{(2m+1)(2m)(2m-1)(2m-2)} \\ &\ \ \vdots \\ &= \frac{(-1)^ma_0}{(2m+1)!}, \qquad m = 1, 2, 3, \ldots. \end{aligned} \tag{18}$$

Hence

$$\begin{aligned} y_1(x) &= x^{1/2}\left[1 + \sum_{m=1}^{\infty}\frac{(-1)^mx^{2m}}{(2m+1)!}\right] \\ &= x^{-1/2}\sum_{m=0}^{\infty}\frac{(-1)^mx^{2m+1}}{(2m+1)!}, \qquad x > 0. \end{aligned} \tag{19}$$

The infinite series in Eq. (19) is precisely the Taylor series for $\sin x$; hence one solution of the Bessel equation of order one-half is $x^{-1/2}\sin x$. The Bessel function of the first kind of order one-half, $J_{1/2}$, is defined as $(2/\pi)^{1/2}y_1$. Thus

$$J_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2}\sin x, \qquad x > 0. \tag{20}$$

Corresponding to the root $r_2 = -\frac{1}{2}$ it is possible that we may have difficulty in computing a_1 since $N = r_1 - r_2 = 1$. However, it is clear from Eq. (16) that for $r = -\frac{1}{2}$ the coefficients of a_0 and a_1 are zero, and hence a_0 and a_1 can be chosen arbitrarily. Then corresponding to a_0 we obtain $a_2, a_4, \ldots$ from the recurrence relation (17); and corresponding to a_1 we obtain $a_3, a_5, a_7, \ldots$. Hence the second solution will not involve a logarithmic term. It is left as an exercise for the student to show from Eq. (17) that for $r = -\frac{1}{2}$,

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!}, \qquad n = 1, 2, \ldots,$$

and

$$a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}, \qquad n = 1, 2, \ldots .$$

Hence

$$y_2(x) = x^{-1/2}\left[a_0 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + a_1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}\right]$$

$$= a_0 \frac{\cos x}{x^{1/2}} + a_1 \frac{\sin x}{x^{1/2}}, \qquad x > 0. \tag{21}$$

The constant a_1 simply introduces a multiple of $y_1(x)$. The second linearly independent solution of the Bessel equation of order one-half is usually taken to be the solution generated by a_0 with $a_0 = (2/\pi)^{1/2}$. It is denoted by $J_{-1/2}$. Then

$$J_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos x, \qquad x > 0. \tag{22}$$

The general solution of Eq. (15) is $y = c_1 J_{1/2}(x) + c_2 J_{-1/2}(x)$.

Bessel Equation of Order One. This example illustrates the situation in which the roots of the indicial equation differ by a positive integer and the second solution involves a logarithmic term. Setting $\nu = 1$ in Eq. (1) gives

$$L[y] = x^2 y'' + xy' + (x^2 - 1)y = 0. \tag{23}$$

If we substitute for $y = \phi(r, x)$ the series (3), and collect terms as in the previous examples, we obtain

$$L[\phi](r, x) = a_0(r^2 - 1)x^r + a_1[(r+1)^2 - 1]x^{r+1} + \sum_{n=2}^{\infty} \{[(r+n)^2 - 1]a_n + a_{n-2}\}x^{r+n} = 0. \tag{24}$$

The roots of the indicial equation are $r_1 = 1$ and $r_2 = -1$. The recurrence relation is

$$[(r+n)^2 - 1]a_n(r) = -a_{n-2}(r), \qquad n \geq 2. \tag{25}$$

Corresponding to the larger root $r = 1$ the recurrence relation is

$$a_n = -\frac{a_{n-2}}{(n+2)n}, \qquad n = 2, 4, 6, \ldots .$$

We also find from Eq. (24) that $a_1 = 0$, and hence from the recurrence relation $a_3 = a_5 = \cdots = 0$. For even values of n, let $n = 2m$; then

$$a_{2m} = -\frac{a_{2m-2}}{2^2(m+1)m} = +\frac{a_{2m-4}}{2^4(m+1)m \cdot m(m-1)}$$

$$\vdots$$

$$= \frac{(-1)^m a_0}{2^{2m}(m+1)!\, m!}, \qquad m = 1, 2, 3, \ldots . \tag{26}$$

With $a_0 = 1$ we have

$$y_1(x) = x \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\, m!}. \tag{27}$$

The Bessel function of the first kind of order one is usually taken to be $\frac{1}{2}y_1$ and is denoted by J_1:

$$J_1(x) = \tfrac{1}{2} y_1(x) = \frac{x}{2} \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m+1)!\, m!}. \tag{28}$$

The series converges absolutely for all x; hence the function J_1 is defined for all x.

In determining a second solution of Bessel's equation of order one, we will illustrate the method of direct substitution. According to Theorem 4.4 we assume that

$$y_2(x) = aJ_1(x) \ln x + x^{-1}\left[1 + \sum_{n=1}^{\infty} b_n x^n\right], \qquad x > 0. \tag{29}$$

Computing $y_2'(x)$, $y_2''(x)$, substituting in Eq. (23), and making use of the fact that J_1 is a solution of Eq. (23) gives

$$2axJ_1'(x) + \sum_{n=0}^{\infty} [(n-1)(n-2)b_n + (n-1)b_n - b_n]x^{n-1} + \sum_{n=0}^{\infty} b_n x^{n+1} = 0 \tag{30}$$

where $b_0 = 1$. Substituting for $J_1(x)$ from Eq. (28), shifting the indices of summation in the two series, and carrying out several steps of algebra gives

$$-b_1 + [0 \cdot b_2 + b_0]x + \sum_{n=2}^{\infty} [(n^2 - 1)b_{n+1} + b_{n-1}]x^n = -a\left[x + \sum_{m=1}^{\infty} \frac{(-1)^m (2m+1)x^{2m+1}}{2^{2m}(m+1)!\, m!}\right]. \tag{31}$$

From Eq. (31) we observe first that $b_1 = 0$, and $a = -b_0 = -1$. Next since there are only odd powers of x on the right, the coefficient of each even power of x on the left must be zero. Thus, since $b_1 = 0$, we have $b_3 = b_5 = \cdots = 0$. Corresponding to the odd powers of x we obtain the recurrence relation (let $n = 2m + 1$)

$$[(2m+1)^2 - 1]b_{2m+2} + b_{2m} = \frac{(-1)(-1)^m(2m+1)}{2^{2m}(m+1)!\,m!}, \qquad m = 1, 2, 3, \ldots. \tag{32}$$

When we set $m = 1$ in Eq. (32) we obtain

$$(3^2 - 1)b_4 + b_2 = (-1)3/(2^2 \cdot 2!).$$

Notice that b_2 can be selected *arbitrarily*, and then this equation determines b_4. Also notice that in the equation for the coefficient of x, b_2 appeared multiplied by 0, and that equation was used to determine a. That b_2 is arbitrary is not surprising, since b_2 is the coefficient of x in the expression $x^{-1}\left[1 + \sum_{n=1}^{\infty} b_n x^n\right]$. Consequently, b_2 simply generates a multiple of J_1, and y_2 is only determined up to an additive multiple of J_1. In accord with the usual practice we choose $b_2 = 1/2^2$. Then we obtain

$$\begin{aligned} b_4 &= \frac{-1}{2^4 \cdot 2}\left[\tfrac{3}{2} + 1\right] = \frac{-1}{2^4 2!}\left[\left(1 + \tfrac{1}{2}\right) + 1\right] \\ &= \frac{(-1)}{2^4 \cdot 2!}(H_2 + H_1). \end{aligned}$$

Although it is not an easy task to show, the solution of the recurrence relation (32) is

$$b_{2m} = \frac{(-1)(-1)^m(H_m + H_{m-1})}{2^{2m} m!\,(m-1)!}, \qquad m = 1, 2, \ldots$$

with the understanding that $H_0 = 1$. Thus

$$y_2(x) = -J_1(x)\ln x + \frac{1}{x}\left[1 - \sum_{m=1}^{\infty} \frac{(-1)^m(H_m + H_{m-1})}{2^{2m} m!\,(m-1)!} x^{2m}\right], \qquad x > 0. \tag{33}$$

The calculation of $y_2(x)$ using the alternate procedure [see Eqs. (15) and (16) of Section 4.6] in which we determine the $c_n'(r_2)$ is also fairly complicated. However, the latter procedure does yield the general formula for the b_{2m} without the necessity of solving a recurrence relation of the form (32). In this regard the reader may wish to compare the calculations of the second solution of Bessel's equation of order zero in the text and in Problem 7.

The second solution of Eq. (23), the Bessel function of the second kind of order one, Y_1, is usually taken to be a certain linear combination of J_1 and y_2. Following Copson [Chapter 12], Y_1 is defined as

$$Y_1(x) = \frac{2}{\pi}[-y_2(x) + (\gamma - \ln 2)J_1(x)], \tag{34}$$

where γ, the Euler-Máscheroni constant, is defined in Eq. (13). The general solution of Eq. (23) for $x > 0$ is

$$y = c_1 J_1(x) + c_2 Y_1(x).$$

Notice that while J_1 is analytic at $x = 0$, the second solution Y_1 becomes unbounded in the same manner as $1/x$ as $x \to 0$.

PROBLEMS

1. Show that each of the following differential equations has a regular singular point at $x = 0$, and determine two linearly independent solutions for $x > 0$.

(a) $x^2y'' + 2xy' + xy = 0$ (b) $x^2y'' + 3xy' + (1 + x)y = 0$
(c) $x^2y'' + xy' + 2xy = 0$ (d) $x^2y'' + 4xy' + (2 + x)y = 0$

2. Find two linearly independent solutions of the Bessel equation of order $\frac{3}{2}$,

$$x^2y'' + xy' + (x^2 - \tfrac{9}{4})y = 0,$$

for $x > 0$.

3. Show that the Bessel equation of order one-half,

$$x^2y'' + xy' + (x^2 - \tfrac{1}{4})y = 0, \qquad x > 0,$$

can be reduced to the equation

$$v'' + v = 0$$

by the change of dependent variable $y = x^{-1/2}v(x)$. From this conclude that $y_1(x) = x^{-1/2}\cos x$ and $y_2(x) = x^{-1/2}\sin x$ are solutions of the Bessel equation of order one-half.

4. Show directly that the series for $J_0(x)$, Eq. (7), converges absolutely for all x.

5. Show directly that the series for $J_1(x)$, Eq. (28), converges absolutely for all x and that $J_0'(x) = -J_1(x)$.

6. Consider the Bessel equation of order ν

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0.$$

Take ν real and greater than zero.

(a) Show that $x = 0$ is a regular singular point, and that the roots of the indicial equation are ν and $-\nu$.

(b) Corresponding to the larger root ν, show that one solution is

$$y_1(x) = x^\nu\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(m+\nu)(m+\nu-1)\cdots(2+\nu)(1+\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

(c) If 2ν is not an integer show that a second solution is

$$y_2(x) = x^{-\nu}\left[1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,(m-\nu)(m-\nu-1)\cdots(2-\nu)(1-\nu)}\left(\frac{x}{2}\right)^{2m}\right].$$

Note that $y_1(x)$ is analytic at $x = 0$, and that $y_2(x)$ is unbounded as $x \to 0$.

(d) Verify by direct methods that the power series in the expressions for $y_1(x)$ and $y_2(x)$ converge absolutely for all x. Also verify that y_2 is a solution provided only that ν is not an integer.

7. In this section we showed that one solution of Bessel's equation of order zero,

$$L[y] = x^2y'' + xy' + x^2y = 0$$

is J_0, where $J_0(x)$ is given by Eq. (7) with $a_0 = 1$. According to Theorem 4.4 a second solution will have the form $(x > 0)$

$$y_2(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} b_n x^n.$$

(a) Show that

$$L[y_2](x) = \sum_{n=2}^{\infty} n(n-1)b_n x^n + \sum_{n=1}^{\infty} nb_n x^n + \sum_{n=1}^{\infty} b_n x^{n+2} + 2xJ_0'(x). \qquad \text{(i)}$$

(b) Substituting the series representation for $J_0(x)$ in Eq. (i), show that

$$b_1x + 2^2b_2x^2 + \sum_{n=3}^{\infty}(n^2b_n + b_{n-2})x^n = -2\sum_{n=1}^{\infty}\frac{(-1)^n 2nx^{2n}}{2^{2n}(n!)^2}. \qquad \text{(ii)}$$

(c) Note that only even powers of x appear on the right-hand side of Eq. (ii). Show that $b_1 = b_3 = b_5 = \cdots = 0$, $b_2 = 1/2^2(1!)^2$, and that

$$(2n)^2b_{2n} + b_{2n-2} = -2(-1)^n(2n)/2^{2n}(n!)^2, \qquad n = 2, 3, 4, \ldots.$$

Deduce that

$$b_4 = \frac{-1}{2^24^2}\left(1 + \tfrac{1}{2}\right) \qquad \text{and} \qquad b_6 = \frac{1}{2^24^26^2}\left(1 + \tfrac{1}{2} + \tfrac{1}{3}\right).$$

The general solution of the recurrence relation is $b_{2n} = (-1)^{n+1}H_n/2^{2n}(n!)^2$, and substituting in the expression for $y_2(x)$ we obtain the solution given in Eq. (11).

8. By a suitable change of variables it is often possible to transform a differential equation with variable coefficients into a Bessel equation of a certain order. For example, show that a solution of

$$x^2y'' + (\alpha^2\beta^2x^{2\beta} + \tfrac{1}{4} - \nu^2\beta^2)y = 0, \qquad x > 0$$

is given by $y = x^{1/2}f(\alpha x^\beta)$ where $f(\xi)$ is a solution of the Bessel equation of order ν.

9. Using the result of Problem 8 show that the general solution of the Airy equation

$-1+3=\frac{1}{2}$ ... $\beta = \frac{3}{2}$ $\nu = \frac{1}{3}$ $\alpha = \pm\frac{2}{3}i$

$$y'' - xy = 0, \qquad x > 0$$

is $y = x^{1/2}[c_1f_1(\frac{2}{3}x^{3/2}) + c_2f_2(\frac{2}{3}x^{3/2})]$ where $f_1(\xi)$ and $f_2(\xi)$ are linearly independent solutions of the Bessel equation of order one-third.

10. It can be shown that J_0 has infinitely many zeros for $x > 0$. In particular the first three zeros are approximately 2.405, 5.520, and 8.653 (see Figure 4.2, page 194). Letting λ_j, $j = 1, 2, \ldots$, denote the zeros of J_0, it follows that

$$J_0(\lambda_j x) = \begin{cases} 1, & x = 0, \\ 0, & x = 1. \end{cases}$$

Verify that $y = J_0(\lambda_j x)$ satisfies the differential equation

$$y'' + \frac{1}{x}y' + \lambda_j^2 y = 0, \qquad x > 0.$$

Hence show that

$$\int_0^1 xJ_0(\lambda_i x)J_0(\lambda_j x)\,dx = 0 \qquad \text{if} \qquad \lambda_i \neq \lambda_j.$$

This important property of $J_0(\lambda_i x)$, known as the orthogonality property, is useful in solving boundary value problems. See Sections 11.3 and 11.6.
Hint: Write the differential equation for $J_0(\lambda_i x)$. Multiply it by $xJ_0(\lambda_j x)$ and subtract it from $xJ_0(\lambda_i x)$ times the differential equation for $J_0(\lambda_j x)$. Then integrate from 0 to 1.

REFERENCES

Coddington, E. A., *An Introduction to Ordinary Differential Equations*, Prentice-Hall, Englewood Cliffs, N.J., 1961.

Copson, E. T., *An Introduction to the Theory of Functions of a Complex Variable*, Oxford University, Oxford, 1935.

Proofs of Theorems 4.1, 4.3, and 4.4 can be found in intermediate or advanced books; for example, see Chapters 3 and 4 of Coddington, or Chapters 3 and 4 of

Rainville, E. D., *Intermediate Differential Equations*, 2nd ed., Macmillan, New York, 1964.

Also see these texts for a discussion of the point at infinity, which was mentioned in Problem 10 of Section 4.3. The behavior of solutions near an irregular singular point is an even more advanced topic; a brief discussion can be found in Chapter 5 of

Coddington, E. A. and Levinson, N., *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

More complete discussions of the Bessel equation, the Legendre equation, and many of the other named equations can be found in advanced books on differential equations, methods of applied mathematics, and special functions. A text dealing with special functions such as the Legendre polynomials, the Bessel functions, etc., is

Hochstadt, H., *Special Functions of Mathematical Physics*, Holt, Rinehart & Winston, New York, 1961.

5 Looking at 2^{nd} order systems

- Analytical solution of linear 2^{nd} order ODE

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

$$a(x)y'' + b(x)y' + c(x)y = h(x) \Rightarrow p(x) = \frac{b(x)}{a(x)};\quad q(x) = \frac{c(x)}{a(x)};\quad g(x) = \frac{h(x)}{a(x)}$$

- Will use variation of parameters to define solution

- For homogeneous equation $y'' + p(x)y' + q(x)y = 0$
    - Assume a solution $y_1(x)$ to the equation is known
    - Assume also $y_2(x)$ another linearly independent solution

    - $y_2(x)$ can be written as $y_2(x) = v(x)\,y_1(x)$

    - $v(x)$ satisfies the 1^{st} order OLDE — Reduction in Order

$$v(x)\,\underbrace{\cancel{[y_1'' + p(x)y_1' + q(x)y_1]}}_{0} + y_1 v'' + (2y_1' + p y_1)\,v' = 0 \quad \text{or} \quad \frac{v''}{v'} = -\frac{2y_1'}{y_1} - p$$

$$\frac{dv'}{v'} = -\frac{2dy_1}{y_1} - p\,dx$$

$$\ln v' = -2\ln y_1 - \int p\,dx$$

$$v' = \frac{1}{y_1^2}\cdot e^{-\int p\,dx}$$

and $$v(x) = \int^x \frac{1}{y_1^2(s)}\, e^{-\int^s p(t)\,dt}\, ds$$

- $$y_2(x) = v\,y_1 = y_1(x)\int^x \frac{1}{y_1^2(s)}\, e^{-\int^s p(t)\,dt}\, ds$$

so that $$y_h = C_1 y_1(x) + C_2 y_2(x)$$

satisfies $$y_h'' + p\,y_h' + q\,y_h = 0$$

- To solve and obtain the ~~particular~~ total solution to $y'' + py' + qy = g$

    - Assume $y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$ — Method of Variation of parameters

- If we assume $u_1'y_1 + u_2'y_2 = 0 \Rightarrow y_p' = u_1 y_1' + u_2 y_2'$

- Thus $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$
  - u_1' & u_2' can be found via Cramer's Rule

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g\, y_2}{y_1 y_2' - y_1' y_2} \Rightarrow u_1(x) = \int^x \frac{-g(t) y_2(t)\, dt}{y_1(t) y_2'(t) - y_1'(t) y_2(t)} + \cancel{c}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1\, g}{y_1 y_2' - y_1' y_2} \Rightarrow u_2(x) = \int^x \frac{g(t) y_1(t)\, dt}{y_1(t) y_2'(t) - y_1'(t) y_2(t)} + \cancel{c_2}$$

- Note $y_1 y_2' - y_1' y_2 = W(y_1, y_2)$: Wronskian
  - if $W(y_1, y_2) \neq 0$ then y_1 & y_2 are linearly independent & $y_p = u_1 y_1 + u_2 y_2$
  - if at some pt the Wronskian is zero then it must be zero everywhere and y_1 & y_2 are **not** linearly independent.

- Example $x^2 y'' - 2xy' + 2y = 4x^2$ for $x > 0$ and $y_1(x) = x$
  $y_1'(x) = 1$
  $y_1''(x) = 0$

1) Check if y_1 solves $x^2 y_1'' - 2xy_1' + 2y_1 = 0$
   $x^2 \cdot 0 - 2x \cdot 1 + 2x \equiv 0$ ✓

2) Find $v(x) \Rightarrow$ $y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0 \Longrightarrow p = -2/x \quad q = 2/x^2$

$$v(x) = \int^x \frac{1}{s^2} e^{-\int^s -\frac{2}{t} dt} ds = \int^x \frac{1}{s^2} e^{2 \ln s} ds = \int^x \frac{1}{s^2} s^2 ds = \underline{x}$$

3) Find $y_h(x)$

$$y_2(x) = v(x)\, y_1(x) = x^2$$

$$\therefore y_h(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 x + C_2 \cdot x^2$$

4) Now find total solution

$$y(x) = u_1(x) \cdot x + u_2(x) \cdot x^2$$

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5) find $u_1(x)$ & $u_2(x)$ $\qquad x^2y'' - 2xy' + 2y = 4x^2 \Rightarrow y'' - \frac{2}{x}y' + \frac{2}{x^2}y = \underset{g(x)}{4}$

$$u_1(x) = \int^x \frac{-4 \cdot t^2}{t \cdot 2t - 1 \cdot t^2}\, dt + \cancel{C_1} = \int^x -4\, dt + \cancel{C_1}$$

$$= -4x + \cancel{C_1} \qquad W(y_1, y_2) = 2 \cdot x^2 - x^2 = x^2$$

$$u_2(x) = \int^x \frac{4 \cdot t}{t^2}\, dt + \cancel{C_2} = \int^x \frac{4}{t}\, dt + \cancel{C_2}$$

$$= 4 \ln x + \cancel{C_2}$$

6) $y_p = u_1 y_1 + u_2 y_2 = (-4x + \cancel{C_1})\, x + (4 \ln x + \cancel{C_2}) \cdot x^2$

$$= \underbrace{-4x^2 + 4x^2 \ln x}_{\text{particular}} + \underbrace{\cancel{C_1 x + C_2 x^2}}_{\text{homog.}}$$

$$y_T = y_p + y_h = -4x^2 + 4x^2 \ln x + C_1 x + C_2 x^2$$

HW Find y_2, u_1, u_2, y given $x^2y'' + 7xy' + 5y = x \quad x > 0 \quad y_1(x) = \frac{1}{x}$

$x^2y'' + xy' + (x^2 - \frac{1}{4})y = 3x^{3/2}\sin x \quad x > 0 \quad y_1(x) = \frac{\sin x}{\sqrt{x}}$

$(1-x^2)y'' - 2xy' + 2y = 0$ near $x=\infty$. let $x = \frac{1}{t}$; $\frac{d}{dx} = -t^2\frac{d}{dt}$; $\frac{d^2}{dx^2} = t^4\frac{d^2}{dt^2} + 2t^3\frac{d}{dt}$

then $\left(1-\frac{1}{t^2}\right)\left[t^4 Y'' + 2t^3 Y'\right] - \frac{2}{t}\left[-t^2\frac{dY}{dt}\right] + 2Y = 0$

or

$$\frac{1}{t^2}(t^2-1)\left[t^4Y'' + 2t^3Y'\right] + 2tY' + 2Y = 0$$

$$t^2(t^2-1)Y'' + 2t^3Y' + 2Y = 0$$

$P_0(t) = t^2(t^2-1)$

$P_1(t) = 2t^3$ $\qquad P_0(t=0) = 0 \Rightarrow t=0$ ($x=\infty$) is a singular pt

$P_2(t) = 2$ $\qquad \frac{P_1(t)}{P_0(t)}\cdot t = \frac{2t^3\cdot t}{t^2(t^2-1)} = \frac{2t^2}{t^2-1}$ $\quad \lim_{t\to 0} = \frac{0}{-1} = 0$ ✓

$\frac{P_2(t)}{P_0(t)}\cdot t^2 = \frac{2t^2}{t^2(t^2-1)} = \frac{2}{t^2-1}$ $\quad \lim_{t\to 0} = \frac{2}{-1} = -2$ ✓ regular singular

let $Y(t) = \sum_{n=0}^{\infty} A_n t^{n+r}$ $\quad Y'(t) = \sum_{n=0}^{\infty} A_n (n+r) t^{n+r-1}$ $\quad Y''(t) = \sum_{n=0}^{\infty} A_n(n+r)(n+r-1)t^{n+r-2}$

$$(t^4-t^2)Y'' + 2t^3Y' + 2Y = \overset{①}{\sum_{n=0}^{\infty} A_n(n+r)(n+r-1)t^{n+r+2}} - \overset{③}{\sum_{n=0}^{\infty} A_n(n+r)(n+r-1)t^{n+r}} + \overset{②}{2\sum_{n=0}^{\infty} A_n(n+r)t^{n+r}}$$

$$+ \overset{④}{2\sum_{n=0}^{\infty} A_n t^{n+r}} = 0$$

$$(①+②) + (③+④) = \sum_{n=0}^{\infty} A_n t^{n+r+2}\Big[\underbrace{(n+r)(n+r-1) + 2(n+r)}_{(n+r)(n+r+1)}\Big] - \sum_{n=0}^{\infty} A_n t^{n+r}\left[(n+r)(n+r-1) - 2\right] = 0$$

now let $n+2 = m$ & $n = m-2$

$$\sum_{n=0}^{\infty} A_n t^{n+r+2}(n+r)(n+r+1) = \sum_{m=2} A_{m-2}t^{m+r}(m+r-2)(m+r-1)$$

and since n & m are arbitrary indices

$$\left\langle \sum_{n=2}^{\infty}\left\{A_{n-2}(n+r-2)(n+r-1)\right\} - \left\{\sum_{n=0}^{\infty} A_n\left[(n+r)(n+r-1)-2\right]\right\}\right\rangle t^{n+r} = 0 = \sum_{n=0}^{\infty} 0\cdot t^{n+r}$$

and $-A_0[(0+r)(0+r-1)-2]t^r - A_1[(1+r)(r)-2]t^{r+1} + \sum_{n=2}^{\infty} A_{n-2}(n+r-2)(n+r-1) - A_n[(n+r)(n+r-1)-2]\,t^{n+r} = 0 = \sum_{n=0} 0\, t^{n+r}$

so at $n=0$: $-A_0\left[(0+r)(0+r-1)-2\right] = 0 \Rightarrow$ either $A_0 = 0$ or $-r(r-1)+2 = +r^2-r-2 = 0 \Rightarrow r=2, r=-1$

if $r=2$ & $r=-1$, then at $n=+1$ $-A_1\left[(1+r)(r)-2\right] = 0$ but $r(r+1)-2 \neq 0 \Rightarrow A_1 \equiv 0$

for any n : $A_n\left[(n+r)(n+r-1)-2\right] - \left[(n+r-2)(n+r-1)\right]A_{n-2} = 0$ or $\boxed{A_n = A_{n-2}\frac{(n+r-2)(n+r-1)}{(n+r)(n+r-1)-2}}$

$(1-x^2)y'' - 2xy' + 2y = 0$ near $x = \infty$. Let $x = \frac{1}{\xi}$; $\frac{dy}{dx} = -\xi^2\frac{dy}{d\xi}$; $\frac{d^2y}{dx^2} = \xi^4\frac{d^2y}{d\xi^2} + 2\xi^3\frac{dy}{d\xi}$

then $\left(1-\frac{1}{\xi^2}\right)\left[\xi^4 Y'' + 2\xi^3 Y'\right] - \frac{2}{\xi}\left[-\xi^2\frac{dY}{d\xi}\right] + 2Y = 0$

or $\frac{1}{\xi^2}(\xi^2-1)\left[\xi^4 Y'' + 2\xi^3 Y'\right] + 2\xi Y' + 2Y = 0$

$\xi^2(\xi^2-1)Y'' + 2\xi^3 Y' + 2Y = 0$

$P(\xi) = \xi^2(\xi^2-1)$

$Q(\xi) = 2\xi^3$ $\qquad$ $\xi = 0$ is a singular pt.

$R(\xi) = 2$ $\qquad$ $\xi\frac{Q(\xi)}{P(\xi)} = \frac{2\xi^4}{\xi^2(\xi^2-1)} = \frac{2\xi^2}{\xi^2-1} = \sum_{n=0}^{\infty} p_n\xi^n$

$\xi^2\frac{R(\xi)}{P(\xi)} = \frac{2\xi^2}{\xi^2(\xi^2-1)} = \frac{2}{\xi^2-1} = \sum_{n=0}^{\infty} q_n\xi^n$ $\quad$ regular singular

Let $Y(\xi) = \sum_{n=0}^{\infty} A_n\xi^{n+r}$, $\quad Y'(\xi) = \sum_{n=0}^{\infty} A_n(n+r)\xi^{n+r-1}$, $\quad Y''(\xi) = \sum_{n=0}^{\infty} A_n(n+r)(n+r-1)\xi^{n+r-2}$

$(\xi^4-\xi^2)Y'' + 2\xi^3Y' + 2Y = \sum_{n=0}^{\infty} A_n(n+r)(n+r-1)\xi^{n+r+2} - \sum_{n=0}^{\infty} A_n(n+r)(n+r-1)\xi^{n+r} + 2\sum_{n=0}^{\infty} A_n(n+r)\xi^{n+r+2}$

$+ 2\sum_{n=0}^{\infty} A_n\xi^{n+r} = 0$

[illegible]

Now let $m = n+2$ [illegible]

$\sum_{n=0}^{\infty} A_n\xi^{n+r+2}(n+r)(n+r+1) = \sum_{m=2}^{\infty} A_{m-2}\xi^{m+r}(m+r-2)(m+r-1)$

[illegible]

[illegible]

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2}$$

This series is recognized as $I_0(x)$. Thus

$$J_0(ix) = I_0(x) \tag{2·3·16}$$

2·4 THE GENERAL BESSEL EQUATION

In the previous sections you have been introduced to the family of Bessel equations (ordinary and modified) and the corresponding family of functions. As engineers it would be convenient to have at our disposal a fast method of obtaining solutions to the various forms of Bessel's equation. This section presents a "cookbook" method of obtaining solutions.

The general Bessel equation is given in Refs. [4, 9] as

$$x^2u'' + [(1 - 2A)x - 2Bx^2]u' + [C^2D^2x^{2C} + B^2x^2 - B(1 - 2A)x + A^2 - C^2n^2]u = 0 \tag{2·4·1}$$

This has the solution

$$u = x^A e^{Bx}[C_1 J_n(Dx^C) + C_2 Y_n(Dx^C)] \tag{2·4·2}$$

When you meet an equation you think might be a Bessel equation, first attempt to put it into the form shown in Eq. (2·4·1). The next step is to solve for A and B by comparing the coefficient of u' in your equation to Eq. (2·4·1). Then solve for C, D, and n by comparing the coefficients of u in each equation and using the previously found values of A and B. If you cannot find A, B, C, D, and n, you have either made a mistake or else your equation is not a Bessel equation.

Whenever D is found to be an imaginary number, J_n and Y_n should be replaced by I_n and K_n with the same argument except for omitting the $i = \sqrt{-1}$. This follows from the relationship between the ordinary Bessel functions of complex argument and the modified Bessel functions developed in Sec. 2·3·3.

As an example of how to make use of the general Bessel equation, consider the circular-fin problem discussed in Sec. 2·0. The differential equation (2·0·2) was found to be

$$u'' + \frac{1}{r}u' - M^2u = 0 \tag{2·4·3}$$

The boundary conditions are that $u(a) = 1$ and $u'(1) = 0$.

The first step is to put Eq. (2·4·3) into the standard form comparable to Eq. (2·4·1). Thus multiplying through by r^2 gives

$$r^2u'' + ru' - M^2r^2u = 0 \qquad (2\cdot4\cdot4)$$

By comparison of Eq. (2·4·4) with Eq. (2·4·1),

$$\begin{array}{lll} 1 - 2A = 1 & \text{or} & A = 0 \\ 2B = 0 & \text{or} & B = 0 \\ 2C = 2 & \text{or} & C = 1 \\ C^2D^2 = -M^2 & \text{or} & D = \dfrac{iM}{C} = iM \\ C^2n^2 = 0 & \text{or} & n = 0 \end{array}$$

Therefore the solution to Eq. (2·4·4) is given by Eq. (2·4·2) as

$$u(r) = C_1J_0(iMr) + C_2Y_0(iMr) \qquad (2\cdot4\cdot5)$$

Since J_0 and Y_0 have an imaginary argument i, they can be related to I_0 and K_0. From Appendix B we find that $J_0(ix) = I_0(x)$ and $Y_0(ix) = iI_0(x) - (2/\pi)K_0(x)$. Thus Eq. (2·4·5) can be rewritten as

$$u(r) = C_1I_0(Mr) + C_2\left[iI_0(Mr) - \frac{2}{\pi}K_0(Mr)\right]$$

$$= (C_1 + iC_2)I_0(Mr) + \left(-\frac{2}{\pi}C_2\right)K_0(Mr)$$

or

$$u(r) = C_3I_0(Mr) + C_4K_0(Mr) \qquad (2\cdot4\cdot6)$$

The engineer usually skips over the intermediate steps shown above for going between Eqs. (2·4·5) and (2·4·6). They are shown here to be sure you understand what is behind this change of functions.

The constants C_3 and C_4 are evaluated by making the solution satisfy the boundary conditions. First, since $u(a) = 1$,

$$1 = C_3I_0(Ma) + C_4K_0(Ma) \qquad (2\cdot4\cdot7)$$

Next, we must evaluate the derivative of u to use in the second boundary condition. Differentiating Eq. (2·4·6),

$$\frac{du}{dr} = C_3\frac{d}{dr}I_0(Mr) + C_4\frac{d}{dr}K_0(Mr)$$

$= C_3 \frac{dI_0(M}{d(M}$

Now, from App

$I_0'(x) = \frac{d}{dx}I_0($

and

$K_0'(x) = \frac{d}{dx}K$

Thus

$\frac{dI_0(Mr)}{d(Mr)} = I_1(M$

and

$\frac{dK_0(Mr)}{d(Mr)} = -K$

The expression f

$\frac{du}{dr} = C_3I_1(Mr)M$

Since the secon
yields

$0 = C_3MI_1(M)$

or

$0 = C_3I_1(M) -$

Equations (2·4·7)
for C_3 and C_4. Th
as

$I_0(Ma)C_3 + K_0($

$I_1(M)C_3 - K_1($

and then solving

$C_3 = \dfrac{\begin{vmatrix} 1 & K_0(\\ 0 & -K \end{vmatrix}}{\begin{vmatrix} I_0(Ma) & K \\ I_1(M) & - \end{vmatrix}}$

$$= C_3 \frac{dI_0(Mr)}{d(Mr)} \frac{d(Mr)}{dr} + C_4 \frac{dK_0(Mr)}{d(Mr)} \frac{d(Mr)}{dr}$$

Now, from Appendix B,

$$I_0'(x) = \frac{d}{dx} I_0(x) = I_1(x)$$

and

$$K_0'(x) = \frac{d}{dx} K_0(x) = -K_1(x)$$

Thus

$$\frac{dI_0(Mr)}{d(Mr)} = I_1(Mr)$$

and

$$\frac{dK_0(Mr)}{d(Mr)} = -K_1(Mr)$$

The expression for the derivative then becomes

$$\frac{du}{dr} = C_3 I_1(Mr)M - C_4 K_1(Mr)M$$

Since the second boundary condition is $u'(1) = 0$, the above yields

$$0 = C_3 M I_1(M) - C_4 M K_1(M)$$

or

$$0 = C_3 I_1(M) - C_4 K_1(M) \tag{2·4·8}$$

Equations (2·4·7) and (2·4·8) must now be solved simultaneously for C_3 and C_4. This is most easily done by rewriting the equations as

$$I_0(Ma)C_3 + K_0(Ma)C_4 = 1$$
$$I_1(M)C_3 - K_1(M)C_4 = 0$$

and then solving by determinants to get

$$C_3 = \frac{\begin{vmatrix} 1 & K_0(Ma) \\ 0 & -K_1(M) \end{vmatrix}}{\begin{vmatrix} I_0(Ma) & K_0(Ma) \\ I_1(M) & -K_1(M) \end{vmatrix}} = \frac{-K_1(M)}{-I_0(Ma)K_1(M) - I_1(M)K_0(Ma)}$$

$$C_4 = \frac{\begin{vmatrix} I_0(Ma) & 1 \\ I_1(M) & 0 \end{vmatrix}}{\begin{vmatrix} I_0(Ma) & K_0(Ma) \\ I_1(M) & -K_1(M) \end{vmatrix}} = \frac{-I_1(M)}{-I_0(Ma)K_1(M) - I_1(M)K_0(Ma)}$$

The negative signs can all be canceled in the above and C_3 and C_4 substituted into Eq. (2·4·6) to yield

$$u(r) = \frac{K_1(M)I_0(Mr) + I_1(M)K_0(Mr)}{I_0(Ma)K_1(M) + I_1(M)K_0(Ma)} \tag{2·4·9}$$

This completes the solution of Eq. (2·4·3) using the general Bessel equation procedure. From here on, this will be the way to solve Bessel equations. As engineers we need not be concerned with series solutions each time we encounter a Bessel equation. The notion of a series solution should be valuable, however, in that

1. It should have helped your understanding of Bessel functions, and

2. You may sometime need this method to solve non-Bessel equations.

To complete this section, it might be well to give a numerical example to be sure that the Bessel function tables are being correctly used and that the translation from normalized temperature back to temperature in degrees Fahrenheit is understood.

EXAMPLE For the circular-fin problem discussed in Sec. 2·0 and solved above, determine the tip temperature if $M = 2.0$, $a = r_i/r_o = 0.8$, $t_i = 300°\text{F}$, and $t_\infty = 100°\text{F}$.

At the tip, the normalized variable r is unity. Thus Eq. (2·4·9) gives

$$u(1) = \frac{K_1(M)I_0(M) + I_1(M)K_0(M)}{I_0(Ma)K_1(M) + I_1(M)K_0(Ma)}$$

$$= \frac{K_1(2.0)I_0(2.0) + I_1(2.0)K_0(2.0)}{I_0(1.6)K_1(2.0) + I_1(2.0)K_0(1.6)}$$

Using a table of Bessel functions (Ref. [8], for example),

$$u(1) = \frac{0.1399(2.280) + 1.591(0.1139)}{1.750(0.1399) + 1.591(0.1880)} = 0.9196$$

Since we had defined $u = (t - t_\infty)/(t_i - t_\infty)$, we have

$$\frac{t_{\text{tip}} - t_\infty}{t_i - t_\infty} = 0.9196$$

or

$$\begin{aligned} t_{\text{tip}} &= t_\infty + 0.9196(t_i - t_\infty) \\ &= 100°\text{F} + 0.9196(300°\text{F} - 100°\text{F}) \end{aligned}$$

and finally,

$$t_{\text{tip}} = 283.9°\text{F}$$

2·5 THOMSON FUNCTIONS

Another set of functions, closely related to the Bessel functions, are the Thomson functions. These often appear in the analysis of oscillating heat-transfer systems. They are defined and briefly discussed in this section. For more detailed comments you should refer to any of several books discussing these functions [1, 2, 3].

These functions are best defined by their relation to the modified Bessel functions as follows:

$$I_0(x\sqrt{i}) = \operatorname{ber} x + i \operatorname{bei} x \tag{2·5·1}$$

$$K_0(x\sqrt{i}) = \operatorname{ker} x + i \operatorname{kei} x \tag{2·5·2}$$

Here, ber, bei, ker, and kei are called *Thomson functions*. They are real functions that have been tabulated [8]. Their series representations can be found by substituting $x\sqrt{i}$ into the series for I_0 or K_0 and separating the result into real and imaginary parts. For example, consider the series for $I_0(x)$:

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n}(n!)^2}$$

Replacing x by $x\sqrt{i}$ gives

$$\begin{aligned} I_0(x\sqrt{i}) &= \sum_{n=0}^{\infty} \frac{x^{2n}(\sqrt{i})^{2n}}{2^{2n}(n!)^2} \\ &= \sum_{n=0}^{\infty} \frac{x^{2n} i^n}{2^{2n}(n!)^2} \end{aligned}$$

This series can be rewritten as two series, the first containing the

IV

The ν & η that lead to the canonical form are given by (we want only $P_{\nu\eta}$)

$$a\nu_x^2 + b\nu_x\nu_y + c\nu_y^2 = 0$$
$$a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 = 0$$

If we require that $\nu_x, \nu_y, \eta_x, \eta_y \neq 0$ then

$$a\left(\frac{\nu_x}{\nu_y}\right)^2 + b\left(\frac{\nu_x}{\nu_y}\right) + c = 0$$
$$a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\left(\frac{\eta_x}{\eta_y}\right) + c = 0$$

ie $\frac{\nu_x}{\nu_y}$ and $\frac{\eta_x}{\eta_y}$ satisfy the same eq. ~~are roots of the eq~~

$$\lambda_1 = \frac{\nu_x}{\nu_y} = \frac{-b+\sqrt{b^2-4ac}}{2a} \qquad \lambda_2 = \frac{\eta_x}{\eta_y} = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

remember $b^2 - 4ac > 0$ so these are real

So if we can find $\nu_x = \lambda_1 \nu_y$ $\eta_x = \lambda_2 \eta_y$ then PDE can be reduced to canonical form

For wave Eq

$$\lambda_1 = \frac{0 + \frac{2}{c}}{2} = \frac{1}{c} \qquad \lambda_2 = \frac{-1}{c}$$

$$\nu_x = \frac{\nu_t}{c} \qquad \eta_x = \frac{\eta_t}{c}$$

$$\nu = x + ct$$
$$\eta = x - ct$$

Family of curves: $\nu = x + ct =$ Const.
$\eta = x - ct =$ const.

are characteristic curves

slopes are λ_1 and $-\lambda_2$

Plug back in transformation (from 1)

$$\left(2a\nu_x\eta_x + b(\nu_x\eta_t + \nu_t\eta_x) + 2c\nu_t\eta_t\right)P_{\nu\eta} = 0$$

$a = 1 \quad b = 0 \quad c = -\frac{1}{c^2}$

$\nu_x = 1 = \eta_x$

$\nu_t = c \quad \eta_t = -c$

$$\left(2 + 0 - \frac{2}{c^2} c(-c)\right) p_{\nu\eta} = 0$$

$$4 p_{\nu\eta} = 0 \text{ or}$$

$$\frac{\partial^2 p}{\partial \eta \partial \nu} = 0$$ canonical form of wave eq

Solution:

Integrate wrt η

$\frac{\partial p}{\partial \nu} = f(\nu)$ some fn of ν

Integrate wrt ν

$p = F(\nu) + G(\eta)$ $\quad (F = \int f)$

$= F(x+ct) + G(x-ct)$

Any functions of $x+ct$ and $x-ct$ solve the wave Eq

This is an example of how PDE solutions can often be determined more by BC's and IC's than eq itself

Finding $\eta + \nu$ to help get solution is called

end day — Method of Characteristics

Homework due Oct 9: Trim p.67, Sect 1.8, probs 1-4 → circle characteristics

Method of characteristics

Idea is to transform indep variables ~~so that~~ along curves so that only derivatives along these curves exist in transformed eq. Then, usually can just integrate

For wave equation, transformed $x + t$ to $\nu + \eta$ and ~~got~~ elimination of $\phi_{\nu\nu}$ and $\phi_{\eta\eta}$ gave characteristic curves $\nu = x + ct$ $\eta = x - ct$. This transformed wave eq to $\frac{\partial^2 \phi}{\partial\eta\partial\nu} = 0$ which was easily integrated.

In linear problems, char curves depend on constants a b c in front of 2nd deriv's

Non-linear: Can depend on solution itself

→ Insert

Characteristic curves have physical meaning

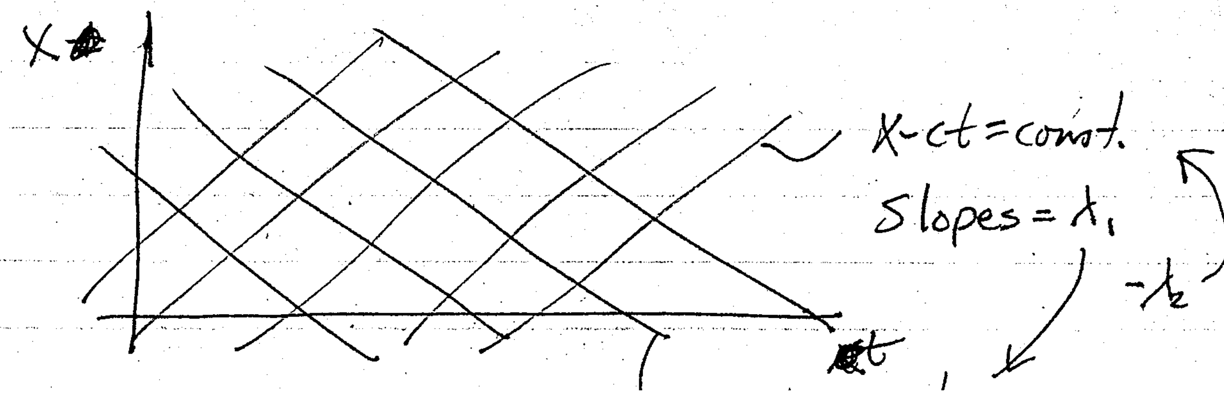
Wave equation:

Characteristic curves were:

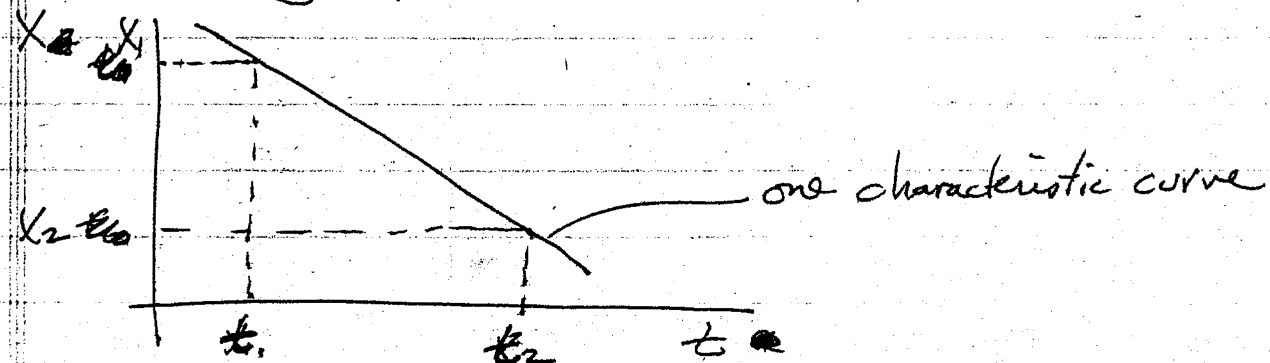
$$\nu = x + ct = \text{const.}$$

$$\eta = x - ct = \text{const.}$$

Choose some values for const's. :



pressure signals propagate along these lines
take only $p' = f(x-ct)$



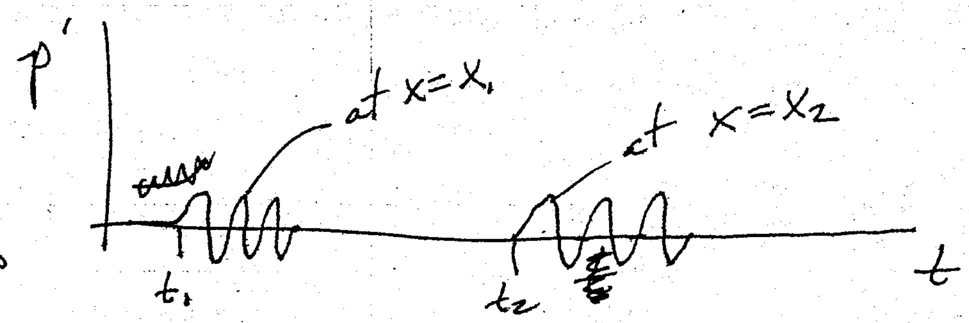
$\cancel{p'(x_1,t_1) = p'}$

$$p'(x_1,t_1) = f(x_1-ct_1)$$
$$p'(x_2,t_2) = f(x_2-ct_2)$$

but ~~x_1, x_2, t_1, t_2 lie~~
(x_1,t_1) (x_2,t_2) lie on line $x-ct = \text{const.}$

$$x_1 - ct_1 = x_2 - ct_2$$

So $p'(x_1,t_1) = p'(x_2,t_2)$



Insert p. 48

slope of characteristic line $= c = \dfrac{x_2-x_1}{t_2-t_1}$
$=$ speed at which p' travels from x_1 to x_2

Hyperbolic PDE's have two <u>real</u> characteristics
$b^2-4ac>0$

insert p. 49

Parabolic PDE's have one real characteristic

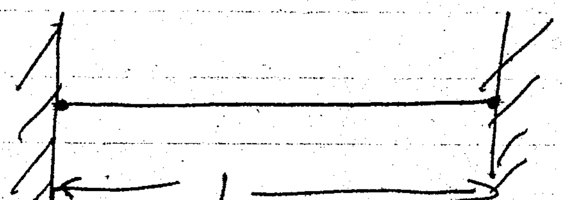
$$b^2 - 4ac = 0 \quad \lambda_1 = \lambda_2 = \frac{-b}{2a}$$

Elliptic PDE's have no real characteristics

$$b^2 - 4ac < 0 \quad \lambda_1 = \alpha + i\beta \quad \lambda_2 = \alpha - i\beta$$

Now, apply above solution to solve an initial boundary value problem

~~E~~ String with fixed ends



Movement is governed by 1 D wave equation

$$\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

$y(x)$ is vertical position of string

$y(0,t) = 0 \qquad t > 0$

$y(L,t) = 0 \qquad t > 0$

$y(x,0) = f(x) \qquad 0 < x < L \qquad f(x) =$ initial position

$y_t(x,0) = g(x) \qquad 0 < x < L \qquad g(x) =$ " velocity

We know $y = F(x+ct) + G(x-ct)$

$F(x) + G(x) = f(x)$

~~$cF(x) \; G$~~

$cF'(x) - cG'(x) = g(x)$

Differentiate 1st: $f'(x) = F'(x) + G'(x)$

~~s~~ mult by c + ~~subtract~~ add to 2nd

$$F'(x) = \frac{1}{2c}\left[c f'(x) + g(x)\right]$$

"chi"

$$F(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int^{x} g(\xi)\,d\xi + D$$

- PHYSICAL PROCESSES NORMALLY VARY WITH TIME AND LOCATION
- TO UNDERSTAND THESE PROCESSES
  - IT WOULD BE NICE TO KNOW HOW THEY VARY IN TIME & SPACE
  - WHAT DRIVES THESE PROCESSES (HOW PROCESSES DEPEND ON SYSTEM PARAM
  - WHERE THESE PROCESSES WILL BE AT SOME FUTURE TIME OR WHAT WILL HAPPEN AT SOME FUTURE LOCATION
- IT TURNS OUT THAT EQNS THAT DESCRIBE THESE PROCESSES ARE GENERALLY DIFFERENTIAL EQUATIONS
- WHEN THESE PROCESSES DEPEND ON THE VARIATION OF TWO OR MORE QUANTITIES, THEN THESE PROCESSES ARE GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS
- MANY PROCESSES IN NATURE ARE DESCRIBED BY 2^{nd} ORDER P.D.E.
- EXAMPLES VIBRATIONS OF A ROD (LONGITUDINAL VIBRATIONS)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

$c = \sqrt{\frac{E}{\rho}}$ BAR VELOCITY

u - LONGITUDINAL DISPLACEMENT

HEAT TRANSFER

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$\frac{k}{c\rho} = \alpha$ THERMAL DIFFUSIVITY

T TEMPERATURE

POTENTIAL FLOW

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0$$

LAPLACE'S EQN

ϕ VELOCITY POTENTIAL $\quad \underset{\sim}{V} = \nabla \phi$

$v = \frac{\partial \phi}{\partial y} \quad u = \frac{\partial \phi}{\partial x}$

STEADY STATE INCOMPRESSIBLE

CONTINUITY EQN IS $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

IN THESE CASES u, T, ϕ ARE FIELD VARIABLE OR DEPENDENT VARIABLE

WHILE x, y, t SPACE COORDINATES OR TIME ARE INDEPENDENT VARIABLES

PARTIAL DERIV IS DEFINED BY

$$\frac{\partial}{\partial x} u(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x, t) - u(x,t)}{\Delta x}$$

- THESE THREE EQNS CAN DESCRIBE MANY SYSTEMS IN MECH. ENG.
  - WAVE EQUATION ARISES FROM ACOUSTICS, VIBRATIONS, SHALLOW-WATER WAVE THEORY
  - HEAT EQN ARISES IN HEAT TRANSFER & ONE-DIMENSIONAL DIFFUSION PROBLEMS
  - LAPLACE'S EQN ARISES IN POTENTIAL FLOW THEORY, STEADY STATE HEAT TRANSFER, TORSION OF A BAR, STEADY STATE VIB. OF A MEMBRANE

- DO ALL THREE HAVE ANY COMMON IDEAS?
- HOW CAN THEY BE SOLVED? WHAT METHODS EXIST TO SOLVE THE EQNS
- HOW CAN I DERIVE THE MATHEMATICAL EQN?
- WHAT IS A WELL POSED PROBLEM - CAN I FIND A UNIQUE SOLUTION

- CHARACTERIZATION & CLASSIFICATION
  - MOST GENERAL 2^{nd} ORDER PDE OF A FN $u(x,y)$

$$\varphi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \quad \text{i.e., } u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

- φ: LINEAR WITH RESPECT TO HIGHEST DERIVATIVE WHEN

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + F(x, y, u, u_x, u_y) = 0$$

  - HERE a, b, c are fns of x, y only.
  - IF $F(x, y, u, u_x, u_y)$ quasilinear

    IF $F(x, y, u, u_x, u_y) = d\,u_x + e\,u_y + fu + g$ LINEAR

    IF $g = 0$ THEN IT IS HOMOGENEOUS

- WE WILL USE THE METHOD OF CHARACTERISTICS TO FIND SOLUTIONS TO PDE

  - IDEA : TO TRANSFORM EQN SO THAT ALONG CERTAIN LINES - ONLY DERIV ALONG THESE LINES EXIST & CAN BE INTEGRATED AS IF THE EQN WERE ODE. LINES ARE CHARACTERISTICS

- IN LINEAR PROBLEMS ~~SOLUTION~~ CHARACTERISTICS DEPEND ON COEFF a, b, c
- NON LINEAR CAN ALSO DEPEND ON SOLUTION, u, ITSELF

- LOOK AT SIMPLE FIRST ORDER PROBLEM

$$A u_x + B u_t + C = 0 \qquad u = u(x,t) \qquad A, B, C \text{ are fn of } x, t, u$$

- FIND TRANSFORMATION $\xi = \xi(x,t)$ $\eta = \eta(x,t)$ $\Rightarrow$ $x = x(\eta,\xi)$ $t = t(\eta,\xi)$

  SO THAT EQN ~~only~~ HAS DERIVATIVES OF EITHER ξ (OR η) ONLY

- e.g. $u_x = \dfrac{\partial u}{\partial \xi}\dfrac{\partial \xi}{\partial x} + \dfrac{\partial u}{\partial \eta}\dfrac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x$

- $u_t = u_\xi \xi_t + u_\eta \eta_t$

  PUT INTO $A u_x + B u_t + C = 0$ & COLLECT TERMS IN u_ξ & u_η

- $\Rightarrow (A\xi_x + B\xi_t) u_\xi + (A\eta_x + B\eta_t) u_\eta + C = 0$

- WE WANT COEFF OF u_ξ (OR u_η) $= 0$ along const ξ (or η)

- $\Rightarrow$ e.g., $A\xi_x + B\xi_t = 0$ along constant ξ line (1)

- ALONG ANY ξ line $d\xi = \xi_x\,dx + \xi_t\,dt$
- ALONG CONSTANT ξ line $d\xi = 0 = \xi_x\,dx + \xi_t\,dt$ (2)

THUS (1) & (2) YIELD $\begin{bmatrix} A & B \\ dx & dt \end{bmatrix}\begin{pmatrix} \xi_x \\ \xi_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

- NON TRIVIAL SOLUTION IS $A\,dt - B\,dx = 0$ or $\frac{dx}{dt} = \frac{A}{B}$

- THIS GIVES SLOPE OF CHARACTERISTICS & $x = \int^t \frac{A}{B}\,dt' + \text{constant}$

$$\therefore \quad x - \int^t \frac{A}{B}\,dt' = \underset{\text{LET IT BE } \xi}{\text{constant}} \implies x - \int^t \frac{A}{B}\,dt' = \xi$$

THIS IS ONE OF CHARACTERIST.

- TO FIND η PICK ANY LINE THAT <u>INTERSECTS</u> ξ FOR EXAMPLE
$\eta = t$

- THUS $\eta_t = 1$ $\eta_x = 0$ & $(A\eta_x + B\eta_t)\,u_\eta + C = 0 \implies B u_\eta + C = 0$

  - WE CAN THEN INTEGRATE THIS ALONG THE CHARACTERISTIC ξ

$$u_\eta = -\frac{C}{B} \qquad u = \int^\eta -\frac{C}{B}\,d\bar{\eta} + f(\xi)$$

Exercise 7.1

in our $Au_x + Bu_t + C = 0$

- EXAMPLE $\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x} = \bar{A}\sin(\pi x/L)$

$A = V$ $B = 1$ $C = -\bar{A}\sin($
$u = T$

$$\frac{dx}{dt} = \frac{A}{B} = V \quad \therefore \quad x - Vt = \xi;\ \text{choose } \eta = t;$$

$$x = \xi + Vt = \xi + V\eta$$

$$B u_\eta + C = 1 \cdot T_\eta - \bar{A}\sin\left(\frac{\pi}{L}(\xi + V\eta)\right) = 0; \quad \text{INTEGRATE WRT } \eta$$

$$\therefore \quad T = \int^\eta \bar{A}\sin\frac{\pi}{L}(\xi + V\eta')\,d\eta' + f(\xi)$$

$$T = -\frac{\bar{A}L}{V\pi}\cos\frac{\pi}{L}(\xi + V\eta) + f(\xi)$$

$$\therefore \quad T(x,t) = -\frac{\bar{A}}{V}\frac{L}{\pi}\cos\frac{\pi}{L}(x - vt + vt) + f(x - vt)$$

$$= -\frac{\bar{A}L}{V\pi}\cos\frac{\pi x}{L} + f(x - vt)$$

- FOR 2nd order quasilinear PDE $A u_{xx} + B u_{xy} + C u_{yy} + D = 0$
A, B, C, D are fns of x, y, u, u_x, u_y

For 2nd order quasilinear PDE $\quad Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0$

- let $\quad V = u_x \qquad W = u_y$

$\Rightarrow \quad A V_x + B V_y + C W_y + D = 0 \quad (*) \qquad$ note $u_{xy} = V_y$

also $\quad V_y - W_x = 0 \quad (**) \quad \Rightarrow \quad u_{xy} = u_{xy}$

- 2 eqns for V & W

- NEXT TRANSFORM EQNS USING $\xi = \xi(x,y) \qquad \eta = \eta(x,y)$

$$\Rightarrow \quad V_x = V_\xi \xi_x + V_\eta \eta_x \qquad W_x = W_\xi \xi_x + W_\eta \eta_x$$
$$V_y = V_\xi \xi_y + V_\eta \eta_y \qquad W_y = W_\xi \xi_y + W_\eta \eta_y$$

- PUT INTO $(*)$ & $(**)$. TAKE $\quad C_1 \cdot$ TRANSFORMED 1 $+ C_2 \cdot$ TRANSFORMED 2 $= 0$
- Collect terms involving $V_\xi, \ V_\eta, \ W_\xi, \ W_\eta$

- $$V_\xi\left[(A\xi_x + B\xi_y)C_1 + (\xi_y)C_2\right] + W_\xi\left[(C\xi_y)C_1 + (-\xi_x)C_2\right]$$
$$+ V_\eta\left[(A\eta_x + B\eta_y)C_1 + (\eta_y)C_2\right] + W_\eta\left[(C\eta_y)C_1 + (-\eta_x)C_2\right] + C_1 D = 0$$

- AS BEFORE WANT COEFF OF DERIVS WRT EITHER ξ (OR η) TO VANISH ALONG CONSTANT ξ (OR η)

- LOOK ALONG CONSTANT η $\Rightarrow$
  V_η & W_η TERMS MUST VANISH

$$(A\eta_x + B\eta_y)C_1 + (\eta_y)C_2 = 0$$
$$(C\eta_y)\ C_1 + (-\eta_x)\ C_2 = 0$$

- for non zero solutions $\Rightarrow$
  determinant of coeffs of C_1 & $C_2 = 0$

$$\begin{vmatrix} A\eta_x + B\eta_y & \eta_y \\ C\eta_y & -\eta_x \end{vmatrix} = 0$$

or $\quad -A\eta_x^2 - B\eta_x\eta_y - C\eta_y^2 = 0 \qquad$ and $\quad C_2 = C\dfrac{\eta_y}{\eta_x}C_1$

- ALONG constant η lines $\quad d\eta = \eta_x dx + \eta_y dy = 0 \quad$ or $\quad \eta_x = -\eta_y y'$

$$\Rightarrow -\eta_y^2\left[A(y')^2 - By' + C\right] = 0 \qquad \text{if } \eta_y \neq 0$$

- $$\frac{dy}{dx} = y' = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} \qquad \text{slope of characteristics}$$

$\Rightarrow \quad f(x,y) = \text{Const} = \eta, \ f_2(x,y) = \text{Const} = \eta \quad$ or $\quad \xi, \eta$

- IF $B^2 - 4AC > 0$ 2 distinct real characteristics : HYPERBOLIC PROBLEM
- IF $B^2 - 4AC = 0$ 1 real characteristic : parabolic problem
- IF $B^2 - 4AC < 0$ no real characteristics : elliptic problem.

$$\left.\begin{aligned} u_{xx} - u_{tt} &= 0 \\ u_{xx} - u_t &= 0 \\ u_{xx} + u_{tt} &= 0 \end{aligned}\right\} \text{examples}$$

- WE THEN HAVE $V_\xi [\qquad] + W_\xi [\qquad] + C, D = 0$

- AS BEFORE IF WE SUBSTITUTE FOR $\xi_x, \xi_y, \eta_x, \eta_y$ WE CAN INTEGRATE TO FIND V AS A FN OF W

- SOLUTIONS CAN BE OBTAINED FOR LINEAR PROBLEMS – CONSTANT COEFFICIENT
  - WITH THE TRANSFORMATIONS $\xi = \xi(x,y)$ $\eta = \eta(x,y)$

  we can transform the general equation

$$A u_{xx} + B u_{xy} + C u_{yy} + b_1 u_x + b_2 u_y + cu + f(x,y) = 0$$

using the characteristics

$$y - \left[\frac{B + \sqrt{B^2 - 4AC}}{2A}\right] x = \xi \qquad y - \left[\frac{B - \sqrt{B^2 - 4AC}}{2A}\right] x = \eta$$

into the following basic forms.

$$u_{\xi\xi} + u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \qquad \text{elliptic type}$$

$$\left.\begin{aligned} &u_{\xi\eta} - \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c}u + \tilde{f} = 0 \\ &u_{\xi\xi} - u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c}u + \tilde{f} = 0 \end{aligned}\right\} \qquad \text{hyperbolic type}$$

$$u_{\xi\xi} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c}u + \tilde{f} = 0 \qquad \text{parabolic}$$

$$A\left(u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2\right) + B\left(u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y\right)$$
$$+ C\left(u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2\right) + \cdots$$

$u_\xi\xi_{xx} + u_\eta\eta_{xx}$ $\quad$ $u_\xi\xi_{xy} + u_\eta\eta_{xy}$ $\quad$ $u_\xi\xi_{yy} + u_\eta\eta_{yy}$

now $\quad Au_{xx} + Bu_{xy} + Cu_{yy} + D(u_x, u_y, u, x, y) = 0$

transforms to $\quad a_{11}u_{\xi\xi} + \cancel{2}a_{12}u_{\xi\eta} + a_{22}u_{\eta\eta} + \bar{D}(u_\xi, u_\eta, u, \xi, \eta) = 0$ (no)

$$a_{11} = A\xi_x^2 + \cancel{2}B\xi_x\xi_y + C\xi_y^2 \quad \text{(no x)}$$
$$a_{12} = 2A\xi_x\eta_x + B(\xi_x\eta_y + \eta_x\xi_y) + 2C\xi_y\eta_y$$
$$a_{22} = A\eta_x^2 + \cancel{2}B\eta_x\eta_y + C\eta_y^2 \quad \text{(no x)}$$

if ξ, η are real characteristics then $a_{11}, a_{22} = 0 \quad a_{12} \neq 0 \quad (B^2 - 4AC > 0)$

$\therefore$ for hyperbolic gives $u_{\xi\eta} + \dfrac{\bar{D}}{\cancel{2}a_{12}} = 0$

now if $\alpha = \dfrac{\xi+\eta}{2} \quad \beta = \dfrac{\xi-\eta}{2}$ then another form is $u_{\alpha\alpha} - u_{\beta\beta} + \cancel{\dfrac{2\bar{D}}{a_{12}}} \; \bar{\Phi} = 0$

if $B^2 - 4AC = 0$ then ξ is a real characteristic $= y - \int^x \dfrac{B}{2A}d\tilde{x}$ & η is any line that intersects it

$\Rightarrow a_{11} = 0$ & $a_{12} = 0 \quad \therefore$

for ~~elliptic~~ parabolic $\quad u_{\eta\eta} + \dfrac{\bar{D}}{a_{22}} = 0$

if $B^2 - 4AC < 0 \quad \xi$ & η are imaginary complex conj let $\alpha = \dfrac{\xi+\eta}{2} \quad \beta = \dfrac{\xi-\eta}{2i}$ $\quad \xi = \alpha + i\beta, \; \eta = \alpha - i\beta$

& $a_{11} = a_{22}$ & $a_{12} = 0 \quad \therefore$ for elliptic $\quad u_{\alpha\alpha} + u_{\beta\beta} + \dfrac{\bar{D}}{a_{22}} = 0$

- note that the basic forms still hold first derivatives
- basic forms only convert eqns containing xx, yy, xy derivs to xx, yy or xy
- WANT TO GET RID OF FIRST DERIVATIVES IF POSSIBLE
- to further simplify define

$$u = e^{\lambda\xi+\mu\eta} v \qquad \text{if } \tilde{b}_1, \tilde{b}_2 \text{ are constant}$$

$$u_\xi = \lambda e^{\lambda\xi+\mu\eta} v + e^{\lambda\xi+\mu\eta} v_\xi = e^{\lambda\xi+\mu\eta}(v_\xi + \lambda v)$$

$$u_\eta = e^{\lambda\xi+\mu\eta}(v_\eta + \mu v)$$

$$u_{\xi\xi} = \lambda e^{\lambda\xi+\mu\eta}(v_\xi + \lambda v) + e^{\lambda\xi+\mu\eta}(v_{\xi\xi} + \lambda v_\xi) = e^{\lambda\xi+\mu\eta}(v_{\xi\xi} + 2\lambda v_\xi + \lambda^2 v)$$

$$u_{\xi\eta} = e^{\lambda\xi+\mu\eta}(v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v)$$

$$u_{\eta\eta} = e^{\lambda\xi+\mu\eta}(v_{\eta\eta} + 2\mu v_\eta + \mu^2 v)$$

for example put into elliptic type.

$$e^{\lambda\xi+\mu\eta}\left[v_{\xi\xi} + v_{\eta\eta} + (\tilde{b}_1 + 2\lambda)v_\xi + (\tilde{b}_2 + 2\mu)v_\eta + (\lambda^2 + \mu^2 + \tilde{b}_1\lambda + \tilde{b}_2\mu + \tilde{c})v + f_1\right]$$

$$f_1 = \tilde{f} e^{-(\lambda\xi+\mu\eta)}$$

Pick $\lambda = \frac{-\tilde{b}_1}{2} \qquad \mu = -\tilde{b}_2/2$

$$\Rightarrow \quad v_{\xi\xi} + v_{\eta\eta} + \overbrace{\left[-\frac{\tilde{b}_1^2}{4} - \frac{\tilde{b}_2^2}{4} + \tilde{c}\right]}^{\gamma} v + f_1 = 0$$

we can reduce our basic forms to

$$\left.\begin{array}{ll} \text{elliptic} & v_{\xi\xi} + v_{\eta\eta} + \gamma v + f_1 = 0 \\ \text{hyperbolic} & \begin{cases} v_{\xi\eta} + \gamma v + f_1 = 0 \\ v_{\xi\xi} - v_{\eta\eta} + \gamma v + f_1 = 0 \end{cases} \end{array}\right\} \begin{array}{ll} \text{if } f_1 = 0 & \text{eigenvalue problem} \\ \text{if } f_1 \neq 0 & \text{inhomogeneous eqn} \end{array}$$

parabolic $\quad v_{\xi\xi} + b_2 v_\eta + f_1 = 0$

$\boxed{\text{HW}}$ Choose λ & μ to simplify ie find $u = v e^{-(\lambda x + \mu y)}$ for

① $u_{xy} = \alpha u_x + \beta u_y$

② $u_{xx} - \frac{1}{a^2} u_{yy} = \alpha u_x + \beta u_y + \gamma u$

③ Do p. 67 (Trim) # 9, 15, 14,

Example

- look at $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$
- let's reduce this : choose $u = v e^{\lambda x + \mu y}$ $u = u(x,y)$ $v = v(x,y)$

$$u_x = v_x e^{\lambda x+\mu y} + \lambda v e^{\lambda x+\mu y}$$

$$u_{xx} = v_{xx} e^{\lambda x+\mu y} + 2 v_x \lambda e^{\lambda x+\mu y} + \lambda^2 v e^{\lambda x+\mu y} = (v_{xx} + 2\lambda v_x + \lambda^2 v) e^{\lambda x+\mu y}$$

$$u_{xy} = v_{xy} e^{\lambda x+\mu y} + \mu v_x e^{\lambda x+\mu y} + \lambda v_y e^{\lambda x+\mu y} + \lambda\mu v e^{\lambda x+\mu y}$$

$$= (v_{xy} + \mu v_x + \lambda v_y + \lambda\mu v) e^{\lambda x+\mu y}$$

$$\Rightarrow \quad e^{\lambda x+\mu y}\left[v_{xx} + v_{yy} + v_x(2\lambda+\alpha) + v_y(2\mu+\beta) + v(\lambda^2+\mu^2+\alpha\lambda+\beta\mu+\gamma)\right] = 0$$

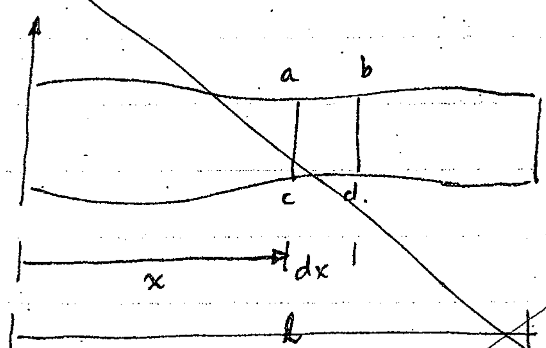
choose $\lambda = -\alpha/2$ $\quad \frac{\beta}{2} = -\mu \quad \Rightarrow \lambda^2+\mu^2+\alpha\lambda+\beta\mu+\gamma = \gamma - \frac{\alpha^2+\beta^2}{4} = \gamma_1$

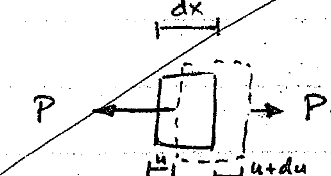
$$\Rightarrow \quad v_{xx} + v_{yy} + \gamma_1 v = 0$$ easier to solve that the previous

Read Chapter 1 and Chapter 3 in Trim/~~Reynolds~~

Derivation of Governing Equations

- Look at longitudinal vibration of a thin rod : $u(x,t)$ - displ of any point of rod.





$\sigma = E\epsilon$

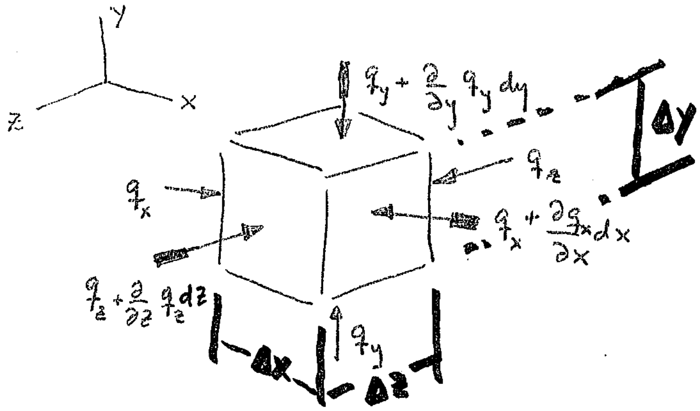
$$\Sigma F = dP = m \cdot \text{accel}$$

$$P = \underset{\text{stress}}{\sigma} A = E\underset{\text{strain}}{\epsilon} A = EA\frac{\partial u}{\partial x}$$

$$\therefore \quad dP = \frac{\partial \sigma A}{\partial x} dx = \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) dx$$

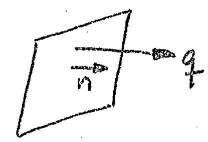
$$m \cdot \text{accel} = \rho \overbrace{dx \cdot A}^{\text{Volume}} \cdot \frac{\partial^2 u}{\partial t^2} \qquad m = \rho \cdot \text{volume}$$

$$\therefore \quad \frac{\partial}{\partial x}\left(EA\frac{\partial u}{\partial x}\right) + f(x,t) \cdot A = \rho A \frac{\partial^2 u}{\partial t^2}$$ where $f(x,t)$ external force/ ~~unit length~~ per unit volume



Fourier Heat Conduction law

$$q = -kA\frac{\partial T}{\partial n}$$



heat conduction coeff

A = area ⊥ to heat flow

n = normal to area

$$-\left(\frac{\partial q_x}{\partial x}dx + \frac{\partial q_y}{\partial y}dy + \frac{\partial q_z}{\partial z}dz\right) = mc_v\frac{\partial T}{\partial t}$$

$$+\frac{\partial}{\partial x}k_xA_x\frac{\partial T}{\partial x}dx + \frac{\partial}{\partial y}k_yA_y\frac{\partial T}{\partial y}dy + \frac{\partial}{\partial z}k_zA_z\frac{\partial T}{\partial z}dz = \rho\, d\forall\, c_v\frac{\partial T}{\partial t}$$

; IF $A_x = A_y = A_z = A$ & $A_x dx = A_y dy = A_z dz = d\forall$

$$\frac{\partial}{\partial x}\left(k_x\frac{\partial T}{\partial x}\right)d\forall + \frac{\partial}{\partial y}\left(k_y\frac{\partial T}{\partial y}\right)d\forall + \frac{\partial}{\partial z}\left(k_z\frac{\partial T}{\partial z}\right)d\forall = \rho c_v\, d\forall\frac{\partial T}{\partial t}$$

thus $\nabla\cdot(k\nabla T) = \rho c_v\frac{\partial T}{\partial t}$ when $\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}$

if k = const $\Rightarrow$ $k\nabla^2 T = \rho c_v\frac{\partial T}{\partial t}$ c_v - specific heat of matter at const. volume

or $\nabla^2 T = \frac{\rho c_v}{k}\frac{\partial T}{\partial t}$

$\nabla^2 T = \alpha\frac{\partial T}{\partial t}$ $\alpha = \frac{\rho c_v}{k}$ = thermal diffusivity

---

now if $T \neq$ fn of t $\nabla^2 T = 0$ steady state (Laplace Eq)

$T \neq$ fn of z, y $\frac{\partial^2 T}{\partial x^2} = \alpha\frac{\partial T}{\partial t}$ one - D heat eqn.

$$\frac{\partial^2 T}{\partial x^2} - \alpha\frac{\partial T}{\partial t} = au_{xx} + bu_{xt} + cu_{tt} + du_x + eu_t + fu + g = 0$$

here $a = 1$ $b = c = d = 0$ $e = -\alpha$ $f = g = 0$

$\therefore$ $b^2 - 4ac = 0$ parabolic when $\frac{\partial^2 T}{\partial x^2} = \alpha\frac{\partial T}{\partial t}$

- $xu_{xx} + u_{yy} = 0$ find regions where eq is hyperbolic, elliptic & parabolic
  transform the region in which it is elliptic into canonical form.

  transform to canonical

- $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$
- $\sin^2 y\, u_{xx} = e^{2x}u_{yy} + 3u_x - 5u = 0$

p. 67 15

do 9, 13, 15, 20

p. 67 9, 15, 14, 18

PDE'S OF MATH. PHYSICS by TYCHONOV & SYMARSKI

1

CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

Many problems of mathematical physics lead to partial differential equations. Differential equations of the second order occur most frequently; in this chapter we shall consider their classification.

1-1. DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH TWO INDEPENDENT VARIABLES

A relation between an unknown function $u(x, y)$ and its partial derivatives[1] up to and including the second order derivatives is designated as a partial differential equation of the second order in the two independent variables x and y:

$$\varphi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 .$$

The equation has an analogous form for more than two independent variables.

A partial differential equation of the second order is called linear with respect to the highest derivative, if it has the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F(x, y, u, u_x, u_y) = 0 , \qquad (1\text{-}1.1)$$

where the coefficients a_{11}, a_{12}, and a_{22} are functions of x and y.

If the coefficients are not only functions of x and y, but also F is a function of x, y, u, u_x, and u_y, then it is called a quasilinear differential equation.

The equation is called linear if it is linear in the higher derivatives u_{xx}, u_{xy}, u_{yy}, also in the function $u(x, y)$, and in the first derivatives u_x, u_y:

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y + cu + f = 0 , \qquad (1\text{-}1.2)$$

where a_{11}, a_{12}, a_{22}, b_1, b_2, c, and f are functions which depend only on x and y. If the coefficients are independent of x and y, then Eq. (1-1.2) is a linear differential equation with constant coefficients. Equation (1-1.2) is called homogeneous if $f(x, y) \equiv 0$.

With the aid of a unique inverse transformation

[1] For the derivatives we use the symbols

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}, \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y}, \quad u_{yy} = \frac{\partial^2 u}{\partial y^2}.$$

$$\xi = \varphi(x, y), \qquad \eta = \psi(x, y)$$

we obtain, under certain assumptions on φ and ψ, a new differential equation which is equivalent to the original equation. There now arises the question of how the variables ξ and η are to be selected so that the transformed differential equation assumes as simple a form as possible. This question will be answered now for a linear equation of the form (1-1.1) with two independent variables x and y.

If we transform the derivatives to the new variables, we obtain

$$\begin{aligned} u_x &= u_\xi \xi_x + u_\eta \eta_x \\ u_y &= u_\xi \xi_y + u_\eta \eta_y \\ u_{xx} &= u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx} \\ u_{xy} &= u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi\xi_{xy} + u_\eta\eta_{xy} \\ u_{yy} &= u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi\xi_{yy} + u_\eta\eta_{yy}\,. \end{aligned} \tag{1-1.3}$$

If these expressions are inserted into (1-1.1) an equation results of the form

$$\bar{a}_{11}u_{\xi\xi} + 2\bar{a}_{12}u_{\xi\eta} + \bar{a}_{22}u_{\eta\eta} + \bar{F} = 0\,, \tag{1-1.4}$$

where

$$\begin{aligned} \bar{a}_{11} &= a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 \\ \bar{a}_{12} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \eta_x\xi_y) + a_{22}\xi_y\eta_y \\ \bar{a}_{22} &= a_{11}\eta_x^2 + 2a_{12}\eta_x\eta_y + a_{22}\eta_y^2 \end{aligned}$$

and $\bar{F}$ is independent of the partial derivatives of the second order of $u(\xi, \eta)$ with respect to ξ and η. If the initial equation is linear, that is,

$$F(x, y, u, u_x, u_y) = b_1 u_x + b_2 u_y + cu + f(x, y)\,,$$

then $\bar{F}$ has the form

$$\bar{F}(\xi, \eta, u, u_\xi, u_\eta) = \beta_1 u_\xi + \beta_2 u_\eta + \gamma u + \delta(\xi, \eta)\,;$$

that is, the transformed differential equation is likewise linear.[2]

We now want to choose the transformation such that the coefficient $\bar{a}_{11}$ vanishes. To this end, we examine a partial differential equation of the first order of the form

$$a_{11}z_x^2 + 2a_{12}z_xz_y + a_{22}z_y^2 = 0\,. \tag{1-1.5}$$

Let $z = \varphi(x, y)$ be an arbitrary particular solution of this equation. If we set $\xi = \varphi(x, y)$, then the coefficient $\bar{a}_{11}$ is obviously equal to zero. In this manner the above-mentioned problem of the selection of the new independent variables ξ and η is linked with the solution of Eq. (1-1.5).

First we shall prove the following lemmas.

Lemma 1. If $z = \varphi(x, y)$ is a particular solution of the equation

[2] If the transformation of the variables is linear, then $\bar{F} = F$, since the second derivatives of ξ and η vanish in formula (1-1.3), and in the expression for $\bar{F}$ none of the transformations of the second derivatives appears in the preceding sums.

$$a_{11}z_x^2 + 2a_{12}z_x z_y + a_{22}z_y^2 = 0\ ,$$

then $\varphi(x, y) = C$ is a general integral of the ordinary differential equation

$$a_{11}dy^2 - 2a_{12}dx\,dy + a_{22}dx^2 = 0\ . \tag{1-1.6}$$

Lemma 2. Conversely, if $\varphi(x, y) = C$ is a general integral of the ordinary differential equation

$$a_{11}\,dy^2 - 2a_{12}\,dx\,dy + a_{22}\,dx^2 = 0\ , \tag{1-1.6}$$

then $z = \varphi(x, y)$ satisfies Eq. (1-1.5).

Proof of the first lemma. Since the function $z = \varphi(x, y)$ satisfies Eq. (1-1.5), the equation

$$a_{11}\left(\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22} = 0 \tag{1-1.7}$$

is valid for all x, y of the region in which $z = \varphi(x, y)$ is defined and $\varphi_y(x, y) \neq 0$. The relation $\varphi(x, y) = C$ is a general integral of Eq. (1-1.6) if the function y defined implicitly by $\varphi(x, y) = C$ satisfies Eq. (1-1.6). If, namely, $y = f(x, C)$ is this function, then it satisfies

$$\frac{dy}{dx} = -\left[\frac{\varphi_x(x, y)}{\varphi_y(x, y)}\right]_{y=f(x,C)} \tag{1-1.8}$$

and hence

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[a_{11}\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22}\right]_{y=f(x,C)} = 0\ ,$$

so that $y = f(x, C)$ satisfies Eq. (1-1.6). The expression in the brackets vanishes, not only for $y = f(x, C)$, but for all values of x, y.

Proof of the second lemma. Let $\varphi(x, y) = C$ be a general integral of Eq. (1-1.6). We can show that for each point (x, y)

$$a_{11}\varphi_x^2 + 2a_{12}\varphi_x\varphi_y + a_{22}\varphi_y^2 = 0 \tag{1-1.7}$$

is valid. Let (x_0, y_0) be any given point. If it can be shown that Eq. (1-1.7) is satisfied at this point, it follows that Eq. (1-1.7) is valid at all points, since (x_0, y_0) is arbitrarily chosen. The function $\varphi(x, y)$ then represents a solution of Eq. (1-1.7). We now construct through (x_0, y_0) an integral curve of Eq. (1-1.6) in which we set $\varphi(x_0, y_0) = C_0$ and consider the curve $y = f(x, C_0)$. Obviously, $y_0 = f(x_0, C_0)$. For all points of this curve the following equation is valid,

$$a_{11}\left(\frac{dy}{dx}\right)^2 - 2a_{12}\frac{dy}{dx} + a_{22} = \left[a_{11}\left(-\frac{\varphi_x}{\varphi_y}\right)^2 - 2a_{12}\left(-\frac{\varphi_x}{\varphi_y}\right) + a_{22}\right]_{y=f(x,C_0)} = 0\ .$$

If we set $x = x_0$ in this equation, we obtain

$$a_{11}\varphi_x^2(x_0, y_0) + 2a_{12}\varphi_x(x_0, y_0)\varphi_y(x_0, y_0) + a_{22}\varphi_y^2(x_0, y_0) = 0 ,$$

which was to be proved.[3]

Equation (1-1.6) is called the characteristic equation of the differential Eq. (1-1.1); its integrals are called characteristics.

If we set $\xi = \varphi(x, y)$, where $\varphi(x, y) = \text{const.}$ is a general integral of Eq. (1-1.6), then we find that the coefficient of $u_{\xi\xi}$ vanishes. Likewise the coefficient of $u_{\eta\eta}$ equals zero if $\psi(x, y) = \text{const.}$ is an additional general integral of (1-1.6) independent of $\varphi(x, y)$ [see footnote 5] and if we set $\eta = \psi(x, y)$.

Equation (1-1.6) yields two equations

$$\frac{dy}{dx} = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \tag{1-1.9}$$

$$\frac{dy}{dx} = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} , \tag{1-1.10}$$

The sign of the expression under the root determines the type of the differential equation

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + F = 0 . \tag{1-1.1}$$

At the point M we shall say that it is[4]

of the hyperbolic type if at this point $a_{12}^2 - a_{11}a_{22} > 0$,
of the elliptic type, if at this point $a_{12}^2 - a_{11}a_{22} < 0$,
of the parabolic type, if at this point $a_{12}^2 - a_{11}a_{22} = 0$.

We can easily show the validity of the expression

$$\bar{a}_{12}^2 - \bar{a}_{11}\bar{a}_{22} = (a_{12}^2 - a_{11}a_{22})\ (\xi_x\eta_y - \xi_y\eta_x)^2 ,$$

from which the invariance of the type of equation follows under a transformation of the variables. At different points of the region of definition, the equation can be of changing type.

For the following considerations we take as basic a region G, at each point of which Eq. (1-1.1) is of one and the same type. Through each point of the region G two characteristics arise which are real and distinct for a differential equation of the hyperbolic type, complex and distinct for a differential equation of the elliptic type, and real and equal for a differential equation of the parabolic type. We shall investigate each of these cases separately.

1. For an equation of the hyperbolic type $a_{12}^2 - a_{11}a_{22} > 0$, the right sides of the differential Eqs. (9) and (10) are real and distinct. The general

[3] This relationship between Eqs. (1-1.5) and (1-1.6) is the equivalent of the well-known relation between a linear partial differential equation of the first order and a system of ordinary differential equations. This can be shown if the left side of Eq. (1-1.5) is represented as the product of two linear differential expressions.

[See V. I. Smirnov: Course in Higher Mathematics, Part II, 2d ed., Berlin, 1958, p. 62, and V. V. Stepanov: Textbook of Differential Equations, Berlin, 1956, p. 328 (Translated from Russian).

[4] This terminology is taken from the theory of curves of the second order.

integrals $\varphi(x, y) = C$ and $\psi(x, y) = C$ of these equations determine a real set of characteristics. We shall set

$$\xi = \varphi(x, y) \qquad \eta = \psi(x, y) \tag{1-1.11}$$

and reduce Eq. (1-1.4) after division by the coefficient of $u_{\xi\eta}$ to the form

$$u_{\xi\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{with} \quad \Phi = -\frac{\bar{F}}{2\bar{a}_{12}} \quad \bar{a}_{12} \neq 0 .$$

This is the so-called canonical form for an equation of the hyperbolic type.[5]

Frequently, a second canonical form is used. If we set

$$\xi = \alpha + \beta \qquad \eta = \alpha - \beta$$

i.e.,

$$\alpha = \frac{\xi + \eta}{2} \qquad \beta = \frac{\xi - \eta}{2}$$

where α and β are the new variables, then

$$u_\xi = \frac{1}{2}(u_\alpha + u_\beta) \qquad u_\eta = \frac{1}{2}(u_\alpha - u_\beta) \qquad u_{\xi\eta} = \frac{1}{4}(u_{\alpha\alpha} - u_{\beta\beta}) ,$$

whereby Eq. (1-1.4) finally assumes the form

$$u_{\alpha\alpha} - u_{\beta\beta} = \Phi_1 \qquad \Phi_1 = 4\Phi$$

2. For an equation of the parabolic type, $a_{12}^2 - a_{11}a_{22} = 0$. Consequently Eqs. (1-1.9) and (1-1.10) coincide, and we obtain only a single general integral of Eq. (1-1.6): $\varphi(x, y) = \text{const}$. In this case, we set

$$\xi = \varphi(x, y) \quad \text{and} \quad \eta = \eta(x, y) ,$$

where $\eta(x, y)$ is an arbitrary function independent of φ. By this choice of the variables we find

$$\bar{a}_{11} = a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 = (\sqrt{a_{11}}\,\xi_x + \sqrt{a_{22}}\,\xi_y)^2 = 0 ,$$

[5] The introduction of the new variables ξ and η through the functions φ and ψ is only possible when these functions are independent of each other. Thus it is sufficient that the corresponding functional determinant obtained from these functions be distinct from zero. This is the case here, since if

$$\begin{vmatrix} \varphi_x & \psi_x \\ \varphi_y & \psi_y \end{vmatrix}$$

at any point M were zero, then for this point the columns of the determinant would be proportional to each other; hence

$$\frac{\varphi_x}{\varphi_y} = \frac{\psi_x}{\psi_y} ,$$

but since

$$\frac{\varphi_x}{\varphi_y} = -\frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \quad \text{and} \quad \frac{\psi_x}{\psi_y} = -\frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} \quad a_{12}^2 - a_{11}a_{22} > 0 ,$$

this is impossible (without loss of generality we assume $a_{11} \neq 0$). Thus, the independence of functions φ and ψ is demonstrated.

since $a_{12} = (a_{11})^{1/2}(a_{22})^{1/2}$; from this it follows that

$$\begin{aligned}\bar{a}_{12} &= a_{11}\xi_x\eta_x + a_{12}(\xi_x\eta_y + \xi_y\eta_x) + a_{22}\xi_y\eta_y \\ &= (\sqrt{a_{11}}\,\xi_x + \sqrt{a_{22}}\,\xi_y)\,(\sqrt{a_{11}}\,\eta_x + \sqrt{a_{22}}\,\eta_y) = 0\,.\end{aligned}$$

After dividing Eq. (1-1.4) by the coefficient of $u_{\eta\eta}$, the canonical form

$$u_{\eta\eta} = \Phi(\xi, \eta, u, u_\xi, u_\eta) \qquad \text{with} \qquad \Phi = -\frac{\bar{F}}{\bar{a}_{22}} \qquad \bar{a}_{22} \neq 0\,.$$

results for an equation of the parabolic type. If, in particular, u_ξ does not appear in this equation, then it is an ordinary differential equation with ξ as a parameter.

3. For an equation of the elliptic type, $a_{12}^2 - a_{11}a_{22} < 0$, and the right sides of Eqs. (1-1.9) and (1-1.10) are complex conjugates of each other. Thus, if

$$\varphi(x, y) = C$$

is a complex integral of the differential Eq. (1-1.9), then

$$\varphi^*(x, y) = C\,,$$

where φ^* is a complex function conjugate to φ, a general integral of Eq. (1-1.10), and a complex conjugate to (1-1.9). We introduce now complex variables by setting

$$\xi = \varphi(x, y) \qquad \eta = \varphi^*(x, y)\,.$$

In this way an equation of the elliptic type, as in the case of the hyperbolic type, is converted to another form.

In order to avoid calculations with complex variables we introduce new real variables α and β, through

$$\alpha = \frac{\varphi + \varphi^*}{2} \qquad \beta = \frac{\varphi - \varphi^*}{2i}\,,$$

such that

$$\xi = \alpha + i\beta \qquad \eta = \alpha - i\beta\,.$$

Thus we obtain

$$\begin{aligned}&a_{11}\xi_x^2 + 2a_{12}\xi_x\xi_y + a_{22}\xi_y^2 \\ &\quad = (a_{11}\alpha_x^2 + 2a_{12}\alpha_x\alpha_y + a_{22}\alpha_y^2) - (a_{11}\beta_x^2 + 2a_{12}\beta_x\beta_y + a_{22}\beta_y^2) \\ &\qquad + 2i(a_{11}\alpha_x\beta_x + a_{12}(\alpha_x\beta_y + \alpha_y\beta_x) + a_{22}\alpha_y\beta_y) = 0\,,\end{aligned}$$

from which it follows that

$$\bar{a}_{11} = \bar{a}_{22} \qquad \text{and} \qquad \bar{a}_{12} = 0\,.$$

After dividing by the coefficient of $u_{\alpha\alpha}$, Eq. (1-1.4) takes the form[6]

[6] Such a transformation is valid only if the coefficients of Eq. (1-1.1) are analytic functions. Namely, if $a_{12}^2 - a_{11}a_{22} < 0$, then Eqs. (1-1.9) and (1-1.10) are complex; consequently, the function y takes on complex values. We can only speak of the solutions of such equations when the coefficients of $a_{ik}(x, y)$ are defined for complex values of y. For the conversion of the differential equation of the elliptic type to canonical form we shall limit ourselves to equations with analytic coefficients.

$$u_{\alpha\alpha} + u_{\beta\beta} = \Phi(\alpha, \beta, u, u_\alpha, u_\beta) \qquad \text{with} \qquad \Phi = -\frac{\bar{F}}{\bar{a}_{22}} \qquad \bar{a}_{22} \neq 0 .$$

Depending on the sign of the discriminant $a_{12}^2 - a_{11}a_{22}$, the following canonical forms of Eq. (1-1.1) result:

hyperbolic type: $\quad u_{xx} - u_{yy} = \Phi \quad$ or $\quad u_{xy} = \Phi$

elliptic type: $\quad u_{xx} + u_{yy} = \Phi$

parabolic type: $\quad u_{xx} = \Phi$

1-2. DIFFERENTIAL EQUATIONS OF THE SECOND ORDER WITH SEVERAL INDEPENDENT VARIABLES

We shall consider now the linear differential equation with real coefficients.

$$\sum_{j=1}^{n} \sum_{i=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0 \qquad a_{ij} = a_{ji} , \tag{1-2.1}$$

where $a, b, c,$ and f are functions of $x_1, x_2, \cdots, x_n$. We introduce a new variable ξ_k by

$$\xi_k = \xi_k(x_1, x_2, \cdots, x_n) \qquad k = 1, \cdots, n .$$

Then

$$u_{x_i} = \sum_{k=1}^{n} u_{\xi_k} \alpha_{ik}$$

$$u_{x_i x_j} = \sum_{k=1}^{n} \sum_{l=1}^{n} u_{\xi_k \xi_l} \alpha_{ik} \alpha_{jl} + \sum_{k=1}^{n} u_{\xi_k} (\xi_k)_{x_i x_j} ,$$

where for brevity $\alpha_{ik} = \partial \xi_k / \partial x_i$ is introduced.

If now we substitute the expressions for the partial derivatives into the initial equation, we obtain

$$\sum_{k=1}^{n} \sum_{l=1}^{n} \bar{a}_{kl} u_{\xi_k \xi_l} + \sum_{k=1}^{n} \bar{b}_k u_{\xi_k} + cu + f = 0$$

with

$$a_{kl} = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \alpha_{ik} \alpha_{jl}$$

$$\bar{b}_k = \sum_{i=1}^{n} b_i \alpha_{jk} + \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} (\xi_k)_{x_i x_j} .$$

We now consider the quadratic form

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^0 y_i y_j , \tag{1-2.2}$$

whose coefficients coincide with the coefficient of a_{ij} of the initial equation at a point $M_0(x_1^0, \cdots, x_n^0)$. Under a linear transformation

$$y_i = \sum_{k=1}^{n} \alpha_{ik} \eta_k$$

If Eq. (1-2.1) at a given point M belongs to a definite type, it can be transformed to the corresponding canonical form.

We now investigate further whether an equation in a definite neighborhood of a point M can be transformed into the corresponding canonical form, if at all points of this neighborhood it belongs to one and the same type. If Eq. (1-2.1) can be transformed to the simplest form in a region in which the elements of the coefficient matrix off the principal diagonal vanish, then the functions

$$\xi_i(x_1, x_2, \cdots, x_n) \qquad i = 1, 2, \cdots, n$$

must satisfy the relation $\bar{a}_{kl} = 0$ for $k \neq 1$. The number of these relations is equal to $n(n-1)/2$, and hence for $n > 3$ it is larger than the number n of the functions ξ_i to be determined. For $n = 3$ the nondiagonal elements of the coefficient matrix $(\bar{a}_{ik})$ usually can be made to vanish; then, however, the elements of the principal diagonal can be distinct from each other.

Consequently, for $n \geqq 3$ it is impossible in a neighborhood of the point M to transform the differential Eq. (1-2.1) to canonical form. For $n = 2$, it can happen that the single nondiagonal coefficient of the second-order matrix vanishes, and the two coefficients on the principal diagonal are equal to each other as outlined earlier in this section.

If the coefficients of Eq. (1-2.1) are constant, then after a transformation of (1-2.1) to canonical form at a point M we obtain an equation which has the same canonical form in the entire region of definition.

1-3. THE CANONICAL FORMS OF LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

In the case of two independent variables, a linear equation of the second order with constant coefficients has the form

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1 u_x + b_2 u_y + cu + f(x, y) = 0 . \tag{1-3.1}$$

A characteristic equation with constant coefficients corresponds to it. Consequently, the characteristics in this case are the straight lines

$$y = \frac{a_{12} + \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} x + C_1 \qquad y = \frac{a_{12} - \sqrt{a_{12}^2 - a_{11}a_{22}}}{a_{11}} x + C_2 .$$

After a corresponding transformation of the variables, (1-3.1) assumes one of the following simple forms:

$$\text{elliptic type:} \qquad u_{\xi\xi} + u_{\eta\eta} + b_1 u_\xi + b_2 u_\eta + cu + f = 0 \tag{1-3.2}$$

$$\text{hyperbolic type:} \qquad \begin{cases} u_{\xi\eta} - b_1 u_\xi + b_2 u_\eta + cu + f = 0 \quad \text{or} \\ u_{\xi\xi} - u_{\eta\eta} + b_1 u_\xi + b_2 u_\eta + cu + f = 0 \end{cases} \tag{1-3.3}$$

$$\text{parabolic type:} \qquad u_{\xi\xi} + b_1 u_\xi + b_2 u_\eta + cu + f = 0 \tag{1-3.4}$$

For further simplification we introduce

$$u = e^{\lambda\xi + \mu\eta} v ,$$

which yields a new function of v where λ and μ are still undetermined constants. Then

$$\begin{aligned}
u_\xi &= e^{\lambda\xi+\mu\eta}(v_\xi + \lambda v) \\
u_\eta &= e^{\lambda\xi+\mu\eta}(v_\eta + \mu v) \\
u_{\xi\xi} &= e^{\lambda\xi+\mu\eta}(v_{\xi\xi} + 2\lambda v_\xi + \lambda^2 v) \\
u_{\xi\eta} &= e^{\lambda\xi+\mu\eta}(v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v) \\
u_{\eta\eta} &= e^{\lambda\xi+\mu\eta}(v_{\eta\eta} + 2\mu v_\eta + \mu^2 v)
\end{aligned}$$

If we substitute these expressions for the derivatives in Eq. (1-3.2), after division by $e^{\lambda\xi+\mu\eta}$ we obtain

$$v_{\xi\xi} + v_{\eta\eta} + (b_1 + 2\lambda)\, v_\xi + (b_2 + 2\mu)\, v_\eta + (\lambda^2 + \mu^2 + b_1\lambda + b_2\mu + c)\, v + f_1 = 0\,.$$

If in this equation the parameters λ and μ are so chosen that there are two coefficients, say in which both of the first derivatives are made to vanish, that is, $\lambda = -(b_1/2)$ and $\mu = -(b_2/2)$, then we obtain

$$v_{\xi\xi} + v_{\eta\eta} + \gamma v + f_1 = 0\,,$$

where γ is a constant defined by c, b_1, and b_2, and $f_1 = f \cdot e^{-(\lambda\xi+\mu\eta)}$. In the same manner we can derive the equations corresponding to (1-3.3) and (1-3.4). Thus, we are led to the following canonical forms for differential equations with constant coefficients:

$$\begin{array}{lll}
\text{elliptic type:} & v_{\xi\xi} + v_{\eta\eta} + \gamma v + f_1 = 0 & \\
\text{hyperbolic type:} & \left\{\begin{array}{r} v_{\xi\eta} + \gamma v + f_1 = 0 \\ v_{\xi\xi} - v_{\eta\eta} + \gamma v + f_1 = 0 \end{array}\right. & \text{or} \\
\text{parabolic type:} & v_{\xi\xi} + b_2 v_\eta + f_1 = 0 &
\end{array}$$

We have already noted (1–2) that a differential equation with constant coefficients in the case of several independent variables,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu + f = 0\,,$$

under a suitable linear transformation of the variables, can be transformed to a canonical form which is the same for all points in the region of definition. If now we set

$$u = e^{\sum_{i=1}^{n} \lambda_i x_i}\, v\,,$$

a new function of v is introduced, and if λ_i is selected appropriately, the transformed equation can be further simplified so that in the case $n = 2$, a corresponding canonical form obtains.

Problems

1. Determine the region in which

$$u_{xx} + y u_{yy} = 0$$

is hyperbolic, elliptic, or parabolic, and transform the differential equation,

in the region in which it is hyperbolic, to canonical form.

2. Transform the following differential equations to canonical form:

a) $u_{xx} + xyu_{yy} = 0$.

b) $yu_{xx} - xu_{yy} + u_x + yu_y = 0$.

c) $e^{2x}u_{xx} + 2e^{x+y}u_{xy} + e^{2y}u_{yy} = 0$.

d) $u_{xx} + (1+y)^2 u_{yy} = 0$.

e) $xu_{xx} + 2\sqrt{xy}\, u_{xy} + yu_{yy} - u_x = 0$.

f) $(x-y)u_{xx} + (xy - y^2 - x + y)u_{xy} = 0$.

g) $y^2 u_{xx} - e^{2x}u_{yy} + u_x = 0$.

h) $\sin^2 y u_{xx} - e^{2x}u_{yy} + 3u_x - 5u = 0$.

i) $u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$.

3. Transform the following differential equation to canonical form and simplify it as much as possible:

$$au_{xx} + 2au_{xy} + au_{yy} + bu_x + cu_y + u = 0 .$$

4. Simplify the following equations with constant coefficients by introducing the function $v = ue^{\lambda x + \mu y}$ and by a suitable selection of the parameters λ and μ:

a) $u_{xx} + u_{yy} + \alpha u_x + \beta u_y + \gamma u = 0$.

b) $u_{xx} = \dfrac{1}{a^2} u_y + \alpha u + \beta u_x$.

c) $u_{xx} - \dfrac{1}{a^2} u_{yy} = \alpha u_x + \beta u_y + \gamma u$.

d) $u_{xy} = \alpha u_x + \beta u_y$.

$$k\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) - \beta^2 r(T-T_0) = \frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$\frac{\partial T}{\partial r} = 0$ @ $r = r_0$, $T(r_i,t) = 0$ & $T(r,0) = T_0$

let $T(r,t) = J(r,t) + \psi(r)$

$$k\frac{\partial T}{\partial r} + rk\frac{\partial^2 T}{\partial r^2} - \beta^2 r(T-T_0) = \frac{r}{\alpha}\frac{\partial T}{\partial t}$$

$$k\left(\frac{\partial J}{\partial r} + \psi'\right) + rk\left(\frac{\partial^2 J}{\partial r^2} + \psi''\right) - \beta^2 r(J+\psi-T_0) = \frac{r}{\alpha}\frac{\partial J}{\partial t}$$

choose ψ so that $k\psi' + rk\psi'' - \beta^2 r(\psi - T_0) = 0$

& $k\frac{\partial J}{\partial r} + rk\frac{\partial^2 J}{\partial r^2} - \beta^2 r J = \frac{r}{\alpha}\frac{\partial J}{\partial t}$

also $\frac{\partial T}{\partial r} = \frac{\partial J}{\partial r} + \psi' = 0$ choose $\left.\frac{\partial J}{\partial r}\right|_{r=r_0} = 0$ & $\psi'(r_0) = 0$

and $T(r_i = 0, t) = J(r_i,t) + \psi(r_i) = 0$ choose $J(r_i,t) = 0$ & $\psi(r_i) = 0$

and $T(r,t=0) = J(r,t=0) + \psi(r) = T_0 \Rightarrow J(r,t=0) = T_0 - \psi(r)$

Now $k\frac{\partial}{\partial r}(r\psi') - \beta^2 r(\psi - T_0) = 0$ let $\psi = \bar\psi + T_0$

or $k\frac{\partial}{\partial r}(r\bar\psi') - \beta^2 r(\bar\psi) = 0$ Modified Bessel Fn

$\psi(r_i) = 0 = \bar\psi(r_i) + T_0 \Rightarrow \bar\psi(r_i) = -T_0$

& $\psi'(r_0) = 0 = \bar\psi'(r_0)$ &

$$\bar\psi(r) = C_1 J_0\left(\frac{+i\beta}{\sqrt{k}}r\right) + C_2 Y_0\left(\frac{i\beta}{\sqrt{k}}r\right) \qquad (\cdot r) \equiv \left(\frac{i\beta}{\sqrt{k}}r\right)$$

BC1 $$\bar\psi'(r_0) = \frac{i\beta}{\sqrt{k}}\left[C_1 J_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right) + C_2 Y_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)\right] = 0$$

BC2 $$\bar\psi(r_i) = \left[C_1 J_0\left(\frac{i\beta}{\sqrt{k}}r_i\right) + C_2 Y_0\left[\frac{i\beta}{\sqrt{k}}r_i\right)\right] = -T_0$$

so $$C_1 = \frac{\begin{bmatrix} 0 & \frac{i\beta}{\sqrt{k}}Y_0'(\cdot r_0) \\ -T_0 & Y_0(\cdot r_i)\end{bmatrix}}{\frac{i\beta}{\sqrt{k}}\left[J_0'(\cdot r_0)Y_0(\cdot r_i) - J_0(\cdot r_i)Y_0'(\cdot r_0)\right]} = \frac{i\beta}{\sqrt{k}}T_0 Y_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)/\text{denom}$$

$$C_2 = -\frac{i\beta}{\sqrt{k}}T_0 J_0'\left(\frac{i\beta}{\sqrt{k}}r_0\right)/\text{denom}$$

J satisfies $\frac{k}{r}\frac{\partial J}{\partial r} + k\frac{\partial^2 J}{\partial r^2} - \beta^2 J = \frac{1}{\alpha}\frac{\partial J}{\partial t}$

let $J(r,t) = R(r)T(t)$

$$\frac{k\partial}{r\partial r}\left(r\frac{\partial R}{\partial r}\right)T - \beta^2 RT = \frac{1}{\alpha}RT'$$

$$\frac{k}{rR}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \beta^2 = \frac{1}{\alpha}\frac{T'}{T} = -\lambda^2$$

$\Rightarrow T'/T = -\alpha\lambda^2 \Rightarrow T' + \alpha\lambda^2 T = 0 \Rightarrow T(t) = Ce^{-\alpha\lambda^2 t}$

$$\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - \left[\frac{\beta^2-\lambda^2}{k}\right]rR = 0$$

or $r^2R'' + r\cdot R' - \left[\frac{\beta^2-\lambda^2}{k}\right]r^2R = 0$ whose soln is $R(r) = C_1 J_0\left(\frac{i\mu}{\sqrt{k}}r\right) + C_2 Y_0\left(\frac{i\mu}{\sqrt{k}}r\right)$

where $\mu = \sqrt{\beta^2-\lambda^2}$

Now the application of the BC's on $\mathcal{J}$ give $\mathcal{J} = \sum \left[C_1 J_0(+r) + C_2 Y_0(+r) \right] e^{-\alpha \lambda^2 t}$

$$\left.\frac{\partial \mathcal{J}}{\partial r}\right|_{r=r_0} = 0 \quad \forall t \Rightarrow \frac{i\mu}{\sqrt{k}} C_1 J_0'\left(\frac{i\mu}{\sqrt{k}} r_0\right) + \frac{i\mu}{\sqrt{k}} C_2 Y_0'\left(\frac{i\mu}{\sqrt{k}} r_0\right) = 0 \qquad (+r) \equiv \frac{i\mu}{\sqrt{k}} r$$

$$\mathcal{J}\big|_{r=r_i} = 0 \quad \forall t \qquad C_1 J_0\left(\frac{i\mu}{\sqrt{k}} r_i\right) + C_2 Y_0\left(\frac{i\mu}{\sqrt{k}} r_i\right) = 0$$

So for C_1 & $C_2 \neq 0 \Rightarrow \frac{i\mu}{\sqrt{k}} \left[J_0'(+r_0) Y_0(+r_i) - Y_0'(+r_0) J_0(+r_i) \right] = 0 \Rightarrow$ gives the μ_n values

and $C_{2n} = -C_{1n} J_0\left(\frac{i\mu_n}{\sqrt{k}} r_i\right) / Y_0\left(\frac{i\mu_n}{\sqrt{k}} r_i\right)$ $\qquad \mu_n = \sqrt{\beta^2 - \lambda_n^2}$

$$\therefore \quad \mathcal{J} = \sum C_{1n} \left[J_0(+r) - Y_0(+r) \cdot J_0(+r_i) / Y_0(+r_i) \right] e^{-\alpha \lambda_n^2 t}$$

@ $t=0$ $\quad \mathcal{J}(r, t=0) = T_0 - \psi(r) = \sum C_{1n} \left[J_0(+r) - Y_0(+r) \cdot J_0(+r_i) / Y_0(+r_i) \right]$

$$= -\bar{\psi}(r)$$

$$= -\frac{T_0 \left[Y_0'(\cdot r_0) J_0(\cdot r) - J_0'(\cdot r_0) Y_0(\cdot r) \right]}{\left[J_0'(\cdot r_0) Y_0(\cdot r_i) - Y_0'(\cdot r_0) J_0(\cdot r_i) \right]} = \sum C_{1n} \left[\frac{J_0(+r) Y_0(+r_i) - Y_0(+r) J_0(+r_i)}{Y_0(+r_i)} \right]$$

$$= \sum C_{1n} R_n(+r)$$

from what we known $\quad C_{1n} = \dfrac{\int_{r_i}^{r_0} r \left[-\bar{\psi}(r) \right] R_n(+r)\, dr}{\int_{r_i}^{r_0} r R_n^2(+r)\, dr}$

 [redacted]

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{\kappa}{K} g(x,t) \qquad -\infty < x < +\infty \quad t > 0$$

$$u(x,0) = f(x) \qquad -\infty < x < +\infty$$

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\omega x}\, dx = \tilde{u}(\omega;t)$$

$$\frac{d\tilde{u}}{dt} = \kappa\left[-\omega^2 \tilde{u}(\omega;t)\right] + \frac{\kappa}{K}\tilde{g}(\omega;t) \longrightarrow \frac{d\tilde{u}}{dt} + \kappa\omega^2 \tilde{u} = \frac{\kappa}{K}\tilde{g}(\omega,t) \longrightarrow$$

$$\tilde{u}(\omega;t) = c\, e^{-\kappa\omega^2 t} + \int_0^t \frac{\kappa}{K}\tilde{g}(\omega,\bar{t})\, e^{-\kappa\omega^2(t-\bar{t})}\, d\bar{t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{u}(\omega,t)\, e^{i\omega x}\, d\omega$$

$$\mathcal{F}\{i.c.\} \longrightarrow \tilde{u}(\omega,0) = \tilde{f}(\omega) \longrightarrow \tilde{u}(\omega;t) = \tilde{f}(\omega)e^{-\kappa\omega^2 t} + \int_0^t \frac{\kappa}{K}\tilde{g}(\omega,\bar{t})e^{-\kappa\omega^2(t-\bar{t})}\, d\bar{t}$$

$$\longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{\kappa\pi t}}\, e^{-(x-u)^2/4\kappa t}\, du + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left\{\int_0^t \frac{\kappa}{K}\tilde{g}(\omega,\bar{t})e^{-\kappa\omega^2(t-\bar{t})}\, d\bar{t}\right\} e^{+i\omega x}\, d\omega$$

$$\text{if } g(x,t) \equiv 0 \longrightarrow \tilde{g}(\omega,t) \equiv 0 \longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{\kappa\pi t}}\cdot e^{\frac{-(x-u)^2}{4\kappa t}}\, du$$

$$(i)\ f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases} \longrightarrow u(x,t) = \int_{-a}^{+a} \frac{1}{2\sqrt{\kappa\pi t}}\, e^{\frac{-(x-u)^2}{4\kappa t}}\, du$$

$$\text{let } \frac{x-u}{2\sqrt{\kappa t}} = v \longrightarrow du = -2\sqrt{\kappa t}\, dv \longrightarrow u(x,t) = -\int_{(x-a)/2\sqrt{\kappa t}}^{(x+a)/2\sqrt{\kappa t}} e^{-v^2}\, dv \left(-\frac{1}{\sqrt{\pi}}\right)$$

$$\longrightarrow u(x,t) = +\frac{1}{\sqrt{\pi}}\int_0^{(x+a)/2\sqrt{\kappa t}} e^{-v^2}\, dv - \frac{1}{\sqrt{\pi}}\int_0^{(x-a)/2\sqrt{\kappa t}} e^{-v^2}\, dv \longrightarrow$$

$$u(x,t) = -\frac{1}{2}\operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right)$$

(ii) $f(x) = \begin{cases} \text{[redacted]} \\ 1 & |x|>a \end{cases} \longrightarrow u(x,t) = \dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_{-\infty}^{-a} e^{-(x-u)^2/4kt}\, du + \dfrac{1}{2\sqrt{k\pi t}}\int_{+a}^{\infty} e^{-(x-u)^2/4kt}\, du$

$$\longrightarrow u(x,t) = \frac{1}{2\sqrt{k\pi t}}\left[\int_{-\infty}^{+\infty} e^{-(x-u)^2/4kt}\,du - \int_{-a}^{a} e^{-(x-u)^2/4kt}\,du\right] \xrightarrow[\ v = \frac{x-u}{2\sqrt{kt}}\ ]{\text{like part (i)}}$$

$$u(x,t) = +\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-v^2}\,dv - \boxed{\frac{1}{2\sqrt{k\pi t}}\int_{-a}^{a} e^{-(x-u)^2/4kt}\,du} \longrightarrow$$

we know the answer from part (i)

$$u(x,t) = +1 + \frac{1}{2}\operatorname{erf}\left(\frac{x-a}{2\sqrt{kt}}\right) - \frac{1}{2}\operatorname{erf}\left(\frac{x+a}{2\sqrt{kt}}\right) \checkmark$$

$\frac{20}{20}$

P.284 / 4 $\quad \dfrac{\partial u}{\partial t} = k\dfrac{\partial^2 u}{\partial x^2} + \dfrac{k}{K} g(x,t) \qquad x>0 \quad t>0$

$$u(0,t) = f_1(t) \qquad u(x,0) = f(x)$$

$$\mathcal{F}_s\{u(x,t)\} = \int_0^{\infty} u(x,t)\sin wx\,dx = \tilde{u}(w,t) \longrightarrow \frac{d\tilde{u}}{dt} = k\left[-w^2\tilde{u} + w\,u(0,t)\right] + \frac{k}{K}\tilde{g}$$

$$\longrightarrow \frac{d\tilde{u}}{dt} + kw^2\tilde{u} = kw\,u(0,t) + \frac{k}{K}\tilde{g}(w,t) \longrightarrow$$

$$\tilde{u}(w,t) = Ce^{-kw^2 t} + \int_0^t \left[\frac{k}{K}\tilde{g}(w,\bar{t}) + kw\,f_1(\bar{t})\right] e^{-kw^2(t-\bar{t})}\,d\bar{t}$$

IC conversion $\tilde{u}(w,0) = C = \tilde{f}(w)$

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty} \tilde{u}(w,t)\sin wx\,dw$$

(b) $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, $f(x) = u_0 \longrightarrow \tilde{u}(w,t) = \tilde{f}(w)\,e^{-kw^2 t}$ ✓

using Convolution integral $\longrightarrow u(x,t) = \dfrac{1}{2}\displaystyle\int_0^{\infty} u_0 \frac{1}{\sqrt{k\pi t}}\left[e^{\frac{-(x-u)^2}{4kt}} - e^{\frac{-(x+u)^2}{4kt}}\right] du$ ✓

let $\dfrac{x-u}{2\sqrt{kt}} = v \longrightarrow du = -2\sqrt{kt}\,dv$ & let $z = \dfrac{x+u}{2\sqrt{kt}} \longrightarrow du = 2\sqrt{kt}\,dz$

$$u(x,t) = -\frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{kt}}^{-\infty} e^{-v^2}\,dv - \frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{kt}}^{+\infty} e^{-z^2}\,dz$$

Cont. → 2

"we know $\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}\,du = \operatorname{erfc}(x)$ [redacted]

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_{-x/2\sqrt{kt}}^{\infty} e^{-v^2}\,dv - \frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{kt}}^{\infty} e^{-z^2}\,dz \longrightarrow$$

$$u(x,t) = \frac{u_0}{2}\left[\operatorname{erfc}\left(\frac{-x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right]$$ ✓

(c) $g(x,t) \equiv 0 \longrightarrow \tilde{g}(\omega,t) \equiv 0$

$f(x) \equiv 0 \longrightarrow \tilde{f}(\omega) \equiv 0$ ✓

$f_1(t) = \bar{u} \longrightarrow \tilde{u}(\omega,t) = \int_0^t k\omega\bar{u}\, e^{-k\omega^2(t-\bar{t})}\,d\bar{t}$

$$\longrightarrow \tilde{u}(\omega,t) = k\omega\bar{u}\cdot\frac{1}{k\omega^2}\cdot e^{-k\omega^2(t-\bar{t})} = \frac{\bar{u}}{\omega}\cdot e^{-k\omega^2(t-\bar{t})}\Big|_0^t$$

$$\longrightarrow \tilde{u}(\omega,t) = \frac{\bar{u}}{\omega}\left(1 - e^{-k\omega^2 t}\right) \longrightarrow \tilde{u}(\omega,t) = \frac{\bar{u}}{\omega} - \bar{u}\,\frac{e^{-k\omega^2 t}}{\omega}$$ ✓

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\tilde{u}(\omega,t)\cdot\sin\omega x\,d\omega$$ ✓

$\mathcal{F}_s^{-1}\left\{\frac{\bar{u}}{\omega}\right\} = \frac{2}{\pi}\int_0^{\infty}\frac{\bar{u}\sin\omega x}{\omega}\,d\omega = \frac{2}{\pi}\cdot\bar{u}\cdot\frac{\pi}{2} = \bar{u}$ we know $\int_0^{\infty}\frac{\sin x}{x}\,dx = \pi/2$

applying convolution on $\bar{u}\,\frac{e^{-k\omega^2 t}}{\omega} \longrightarrow$

$\mathcal{F}^{-1}\left\{\frac{\bar{u}}{\omega}\cdot e^{-k\omega^2 t}\right\} = \frac{1}{2}\int_0^{\infty}\bar{u}\cdot\frac{1}{\sqrt{k\pi t}}\left[e^{\frac{-(x-u)^2}{4kt}} - e^{\frac{-(x+u)^2}{4kt}}\right]du$ from part (b

$$= \frac{\bar{u}}{2}\left[\operatorname{erfc}\left(\frac{-x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right]$$ ✓ $\Longrightarrow$

$$u(x,t) = \bar{u} - \frac{\bar{u}}{2}\left[\operatorname{erfc}\left(\frac{-x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{x}{2\sqrt{kt}}\right)\right]$$ ✓

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:363 / 2 [redacted]

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{\kappa}{K} g(x,t) \qquad u(x,0) = f(x)$$

Applying Laplace operator to both side of the equation : [redacted] let $\tilde{u}(x;s) = \int_0^\infty u(x,t) e^{-st} dt$

$$s\tilde{u}(x;s) - u(x,0) = \kappa \frac{\partial^2}{\partial x^2} \cdot \tilde{u}(x;s) + \frac{\kappa}{K} \tilde{g}(x;s) \longrightarrow$$

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x;s) - \frac{s}{\kappa} \tilde{u}(x;s) + \frac{1}{K} \tilde{g}(x;s) + f(x) = 0 \longrightarrow$$

s/B $f(x)/\kappa$

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x;s) - \frac{s}{\kappa} \tilde{u}(x;s) = -\left(\frac{1}{K} \tilde{g}(x;s) + \frac{1}{\kappa} f(x)\right) \qquad \tilde{u}(x;s) = \tilde{u}_h + \tilde{u}_p$$

$$\frac{\partial^2}{\partial x^2} \tilde{u}(x;s) - \frac{s}{\kappa} \tilde{u}(x;s) = 0 \longrightarrow \tilde{u}_h = A \underbrace{e^{\sqrt{\frac{s}{\kappa}}x}}_{\tilde{u}_1} + B \underbrace{e^{-\sqrt{\frac{s}{\kappa}}x}}_{\tilde{u}_2}$$

$$\tilde{u}_p = v_1 \tilde{u}_1 + v_2 \tilde{u}_2 \qquad W(\tilde{u}_1, \tilde{u}_2) = \begin{vmatrix} e^{\sqrt{\frac{s}{\kappa}}x} & \sqrt{\frac{s}{\kappa}} e^{\sqrt{\frac{s}{\kappa}}x} \\ e^{-\sqrt{\frac{s}{\kappa}}x} & -\sqrt{\frac{s}{\kappa}} e^{-\sqrt{\frac{s}{\kappa}}x} \end{vmatrix} = -2\sqrt{\frac{s}{\kappa}}$$

$$v_1 = \int^x + \left(\frac{1}{\kappa} f(\bar{x}) + \frac{1}{K} \tilde{g}(\bar{x};s)\right) e^{-\sqrt{\frac{s}{\kappa}}\bar{x}} \cdot -\frac{1}{2}\sqrt{\frac{\kappa}{s}}\, d\bar{x}$$

$$v_2 = \int^x - \left(\frac{1}{\kappa} f(\bar{x}) + \frac{1}{K} \tilde{g}(\bar{x};s)\right) e^{\sqrt{\frac{s}{\kappa}}\bar{x}} \cdot -\frac{1}{2}\sqrt{\frac{\kappa}{s}}\, d\bar{x}$$

we can multiple $\tilde{u}_1, \tilde{u}_2$ into th functions inside the integral $\longrightarrow$

$$\longrightarrow \tilde{u}_p = \int_0^x -\sqrt{\frac{\kappa}{s}} \left[\frac{f(\bar{x})}{\kappa} + \frac{\tilde{g}(\bar{x};s)}{K}\right] \cdot \left[e^{\sqrt{\frac{s}{\kappa}}(x-\bar{x})} - e^{-\sqrt{\frac{s}{\kappa}}(x-\bar{x})}\right] \cdot \frac{1}{2}\, d\bar{x}$$

$$\longrightarrow \tilde{u}_p = -\sqrt{\frac{\kappa}{s}} \int_0^x \left(\frac{f(\bar{x})}{\kappa} + \frac{\tilde{g}(\bar{x};s)}{K}\right) \cdot \sinh\left(\sqrt{\frac{s}{\kappa}}(x-\bar{x})\right) d\bar{x} \qquad \xrightarrow{\tilde{u}_{TOT} = \tilde{u}_h + \tilde{u}_p}$$

$$\tilde{u}(x;s) = A e^{\sqrt{\frac{s}{\kappa}}x} + B e^{-\sqrt{\frac{s}{\kappa}}x} - \sqrt{\frac{\kappa}{s}} \int_x^x \left(\frac{f(\bar{x})}{\kappa} + \frac{\tilde{g}(\bar{x};s)}{K}\right) \cdot \sinh\left(\sqrt{\frac{s}{\kappa}}(x-\bar{x})\right) d\bar{x}$$

✓

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what about when the BC is changed to $\frac{\partial U}{\partial x}(0,t) = \frac{-f(t)}{\kappa} = -\frac{Q_0}{\kappa}$

P.364 / 4

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad x>0 \quad t>0$$

$$\frac{\partial u(0,t)}{\partial x} = -\frac{f_1(t)}{K} \qquad t>0$$

$$u(x,0) = u_0 = cte \qquad x>0$$

we have the general solution from previous exercise also we know:

$g(x,t) \equiv 0$

$f(x) = u_0$

and the solution shall be bounded as $x \to \infty$

$$\tilde{u}(x;s) = A e^{\sqrt{\frac{s}{k}}x} + B e^{-\sqrt{\frac{s}{k}}x} - \sqrt{\frac{K}{s}}\int_0^x \frac{u_0}{K}\sinh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)d\bar{x}$$

$x \to \infty$, Solution shall be bounded $\to$ $A = 0$ $\to$

$$\tilde{u}(x;s) = B e^{-\sqrt{\frac{s}{k}}x} - \frac{u_0}{\sqrt{sK}}\int_0^x \sinh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)d\bar{x} = B e^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{\sqrt{sK}}\cdot\sqrt{\frac{K}{s}}\left[\cosh\left(\sqrt{\tfrac{s}{k}}(x-\bar{x})\right)\right]_0^x -$$

$$\tilde{u}(x;s) = B e^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{s}\left(1-\cosh\sqrt{\tfrac{s}{k}}x\right)$$

$$\mathcal{L}\left\{\frac{\partial}{\partial x}u(0,t)\right\} = \mathcal{L}\left\{-\frac{f_1(t)}{K}\right\} \to \int_0^\infty \frac{\partial}{\partial x}u(0,t)e^{-st}dt = -\frac{\tilde{f}_1(s)}{K} \to$$

$$\frac{\partial}{\partial x}\tilde{u}(0;s) = -\frac{\tilde{f}_1(s)}{K}$$

$$\frac{\partial}{\partial x}\tilde{u}(x;s) = -B\sqrt{\frac{s}{k}}e^{-\sqrt{\frac{s}{k}}x} - \frac{u_0}{s}\cdot\sqrt{\frac{s}{k}}\sinh\sqrt{\frac{s}{k}}x\Big|_{x=0} = -B\sqrt{\frac{s}{K}} = \frac{\tilde{f}_1(s)}{K} \to$$

$$B = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}$$

$$\Rightarrow \tilde{u}(x;s) = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}\cdot e^{-\sqrt{\frac{s}{k}}x} + \frac{u_0}{s} - u_0\frac{\cosh\sqrt{\frac{s}{k}}x}{s}$$

$$= \underbrace{-\frac{1}{\sqrt{K}}\cdot\tilde{f}_1(s)\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{\sqrt{s}}}_{\downarrow \mathcal{L}^{-1}} + \underbrace{\frac{u_0}{s}}_{\downarrow \mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow \mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow \mathcal{L}^{-1}}$$

$$-\frac{1}{\sqrt{K}}\int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}}\cdot f_1(a)\cdot da + u_0 - u_0\,\mathrm{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - u_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Kt}}\right)$$

ok

Cont. $\to$

[redacted]

$$u(x,t) = -\frac{1}{\sqrt{K}}\int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}} \cdot f_1(a)\cdot da + u_0\left[1 - \operatorname{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - \operatorname{erfc}\left(\frac{x}{2\sqrt{Kt}}\right)\right]$$ ✓

P350 / 28 $\quad y'' + 2y' + y = t \quad y(0) = 0 \quad y'(0) = 1$

$$\left[s^2 Y(s) - s\,y(t=0^+) - y'(t=0^+)\right] + 2\left[sY(s) - y(t=0^+)\right] + Y(s) = \frac{1}{s^2} \longrightarrow$$

$$s^2 Y(s) + 2sY(s) + Y(s) - 1 = \frac{1}{s^2} \rightarrow Y(s)\left(s^2+2s+1\right) = \frac{1+s^2}{s^2} \rightarrow Y(s) = \frac{1+s^2}{s^2(s+1)^2}$$

$$\longrightarrow Y(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}$$

$$\longrightarrow y(t) = -2 + t + 2e^{-t} + 2te^{-t}$$ ✓ [redacted]

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let $y(s) = \frac{A}{s} + \frac{Bs+C}{s^2} + \frac{D}{s+1} + \frac{Es+\bar{D}}{(s+1)^2}$

$$= \frac{As(s+1)^2 + (Bs+C)(s+1)^2 + Ds^2(s+1) + (Es+\bar{D})s^2}{s^2(s+1)^2} = \frac{1+s^2}{s^2(s+1)^2}$$

expand and equate powers of s to find A, B, C, D, E, $\bar{D}$

EGM 5315 10/2/97

what if we wanted to find the solution to

$\nabla^2 W - \frac{1}{c^2} W_{tt} = 0$ in a circular region including the origin and $W(r=a, \theta, t) = 0$

w/ IC $W(r,\theta,t=0) = f(r)$

$W_t(r,\theta,t=0) = g(r)$

if we choose $W(r,\theta,t) = F(r)G(\theta)H(t)$

then
$$\nabla^2 W - \frac{1}{c^2} W_{tt} = W_{rr} + \frac{1}{r} W_r + \frac{1}{r^2} W_{\theta\theta} - \frac{1}{c^2} W_{tt} = 0$$
$$= F''GH + \frac{1}{r} F'GH + \frac{1}{r^2} FG''H - \frac{1}{c^2} FGH'' = 0$$

Divide By FGH & multiply by c^2: $\underbrace{c^2 \frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G}}_{\text{spatial}} = \underbrace{\frac{H''}{H}}_{\text{time}} = -\omega^2$ ← frequ

$\Rightarrow H'' + \omega^2 H = 0$ or $H(t) = B\cos\omega t + A\sin\omega t$.

and $c^2 \frac{F''}{F} + \frac{c^2}{r}\frac{F'}{F} + \frac{c^2}{r^2}\frac{G''}{G} = -\omega^2$ Now multiply both sides by r^2/c & separate

$$r^2 \frac{F''}{F} + \frac{r^2}{r}\frac{F'}{F} + \frac{\omega^2 r^2}{c^2} = -\frac{G''}{G} = \alpha^2$$

$\Rightarrow G'' + \alpha^2 G = 0$ or $\underline{G(\theta) = C\cos\alpha\theta + D\sin\alpha\theta}$

$\Rightarrow r^2 F'' + rF' + \left(\left[\frac{\omega r}{c}\right]^2 - \alpha^2\right)F = 0$ or $\underline{F(\frac{\omega r}{c}) = M J_\alpha(\frac{\omega r}{c}) + N Y_\alpha(\frac{\omega r}{c}}$

- since we employ a circular region $G(\theta) = G(\theta + 2\alpha\pi) \Rightarrow \alpha = \text{intege}$
- since we include the origin $\Rightarrow N \equiv 0$

$\therefore$

- Solution is $W(r,\theta,t) = F(\frac{\omega r}{c}) G(\theta) H(t)$
- since $W(r=a,\theta,t) = 0 \Rightarrow F(\frac{\omega a}{c}) = 0 = M J_n(\frac{\omega a}{c}) = 0$ or $J_n(\frac{\omega a}{c}) = $

- $J_n\left(\frac{\omega a}{c}\right) = 0$ defines ω since $\frac{\omega_1 a}{c} = r_1$ so that $J_n(r_1) = 0$
$\frac{\omega_2 a}{c} = r_2$ " " $J_n(r_2) = 0$ etc

- also since <u>initial conditions</u> are independent of θ
reasonable to assume that $W(r,\theta,t)$ is independent of θ
$\Rightarrow \alpha = 0 \Rightarrow G(\theta) = \text{constant}$ why? $G'' + \alpha^2 G = G'' = 0 \Rightarrow G = C_1\theta + C_2$
FOR $G(\theta + 2n\pi) = G(\theta) \Rightarrow C_1 = 0 \Rightarrow G(\theta) = \text{constant}$.

$$\therefore \quad W(r,\theta,t) \equiv W(r,t) = F\left(\frac{\omega r}{c}\right) H(t) \quad \text{where} \quad F\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$$

if $B \cdot C \cdot M = \tilde{B}$
$A \cdot C \cdot M = \tilde{A}$

$$\text{and} \quad W_m(r,t) = F\left(\frac{\omega_m r}{c}\right) H(t)$$
$$= \left(\tilde{B}\cos\omega_m t + \hat{A}\sin\omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

$$W(r,t) = \sum_m W_m(r,t) = \sum\left(\tilde{B}_m\cos\omega_m t + \tilde{A}_m \sin\omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

$$\text{at } t=0 \quad W(r,t=0) = f(r) = \sum \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \hat{B}_m R_m(r)$$
$$t=0 \quad W_t(r,t=0) = g(r) = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{A}_m \omega_m R_m(r)$$

$$\text{suppose} \quad f(r) \equiv E_1 J_0\left(\frac{\omega_1 r}{c}\right) + E_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + E_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$$
$$= \sum_{i=1}^{\infty} E_i J_0\left(\frac{\omega_i r}{c}\right)$$
$$\Rightarrow \quad E_i's = \tilde{B}_i's$$

$$g(r) = L_1 J_0\left(\frac{\omega_1 r}{c}\right) + L_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + L_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$$
$$= \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$$
$$\Rightarrow \quad L_i = \tilde{A}_i \omega_i \quad \text{or} \quad \tilde{A}_i = \frac{L_i}{\omega_i}$$

- so how do we get the $E_i's$ & $L_i's$?

- Sturm-Liouville gives the method for solving for the $\tilde{A}_m$ & $\hat{B}_m$

$$\frac{d}{dx}\left(S(x)\frac{dy}{dx}\right) + \left[Q(x) + \lambda^2 P(x)\right] y = 0$$
$$S(x)y'' + S'(x)y' + \left[Q(x) + \lambda^2 P(x)\right] y = 0 \qquad r^2 R'' + rR' + \left[\lambda^2 r^2 - \nu^2\right] R = 0$$
$$\Rightarrow S(x) = r \quad Q(x) = -\nu^2/r$$
$$S'(x) = 1 \quad P(x) = +r$$

$$\frac{d}{dx}\left[Sy'\right] + \left[Q+\lambda^2 P\right]y = 0$$

subjected to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{matrix}\lambda = \lambda_m \\ y = y_m\end{matrix}\right\}$ & $\left.\begin{matrix}\lambda = \lambda_n \\ y = y_n\end{matrix}\right\}$ both ODE & BC are satisfied $y_m \neq y_n \quad \lambda_m \neq \lambda_n$

$$\Rightarrow \quad [Sy_m']' + [Q + \lambda_m^2 P]y_m = 0 \qquad (1)$$

$$[Sy_n']' + [Q + \lambda_n^2 P]y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{[Sy_m']' + [\underline{\underline{Q}} + \lambda_m^2 P]y_m\right\}y_n - \left\{[Sy_n']' + [\underline{\underline{Q}} + \lambda_n^2 P]y_n\right\}y_m \right\rangle dx$$

$$\int_a^b \left\{[Sy_m']'y_n - [Sy_n']'y_m\right\}dx + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_m y_n \, dx = 0$$

$$\int_a^b [Sy_m']'y_n \, dx = Sy_m' y_n \Big|_a^b - \int_a^b Sy_m' y_n' \, dx \qquad \text{integration by parts}$$

$$\int_a^b [Sy_n']'y_m \, dx = Sy_n' y_m \Big|_a^b - \int_a^b Sy_n' y_m' \, dx$$

and $\left.\begin{matrix}\text{at } x = a \\ \text{at } x = b\end{matrix}\right\}$ $\begin{matrix}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{matrix}$ either $\alpha = 0$ & $\beta = 0$ or $\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$

det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \;\; \text{at both } x = a, b \qquad S\left[y_m' y_n - y_n' y_m\right]\Big|_a^b + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_n y_m \, dx = 0$$

$$\Rightarrow \text{if } \lambda_m \neq \lambda_n \quad \boxed{\int_a^b P y_n y_m \, dx = 0}$$

orthogonality condition but wrt weight fn $P(x)$

example $T'' + \omega^2 T = 0 \quad \Rightarrow \quad S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m \, dx = 0 \quad \Rightarrow \quad \int_a^b P \sin\omega_n x \cdot \sin\omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $\quad T(x) = 0$ @ $x = L \quad \Rightarrow T_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}\,dx = 0$$

now suppose we want to write $f(x)$ as a fn of the eigenfunctions $y_n(x)$.

now suppose $f(x) = \sum A_n y_n(x)$

if we mult. by $P y_m$ & integrate

$$\int_a^b P(x) f(x) y_m(x)\,dx = \int_a^b P \sum A_n y_n(x) y_m(x)\,dx$$

$$= \sum A_n \int_a^b P y_n y_m\,dx = \begin{cases} A_m \int_a^b P y_m^2\,dx & n=m \\ 0 & n \neq n \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m\,dx}{\int_a^b P y_m^2\,dx}}$$

For $T'' + \omega^2 T = 0 \qquad y_m = \sin\frac{m\pi x}{L} \qquad P = 1 \qquad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin\frac{m\pi x}{L}\,dx \Big/ \int_0^L \sin^2\frac{m\pi x}{L}\,dx$$

$$\int_0^L \sin^2\frac{m\pi x}{L}\,dx = 1/2 \quad \Rightarrow \quad \int_0 \left(\frac{1}{2} - \frac{1}{2}\cos\frac{2m\pi x}{L}\right)dx = 1/2$$

$$A_m = \frac{2}{L}\int_0^L f(x)\sin\frac{m\pi x}{L}\,dx \qquad \text{normal fourier coefficient}$$

- Returning to our problem $f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$

for the bessel fn. $P(r) = +r$

$$E_i = \frac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$$

since $E_i = \tilde{B}_i$

also $g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right)$

$$L_i = \frac{\int_0^a r g(r) J_0\,dr}{\int_0^a r J_0^2\,dr} = \tilde{A}_i \omega_i$$

since $L_i = \tilde{A}_i \omega_i \quad \Rightarrow \quad \tilde{A}_i = \frac{L_i}{\omega_i}$

$$\int_0^a r\, J_0^2\!\left(r\frac{\omega_i}{c}\right) dr$$ can be evaluated as follows

using the ODE : if you remember $J_0\!\left(\frac{\omega_i}{c} r\right)$ satisfies

$$r R'' + R' + \Big(\underbrace{\left[\tfrac{\omega_i}{c}\right]^2 r}_{\lambda_i^2 r} - 0^2\Big)R = 0 \qquad R_i = J_0(\lambda_i r)$$

let $R(r,\lambda)$ be a solution to $r R'' + R' + (\lambda^2 r)R = 0$
and satisfy that $R(r=0)$ is not ∞
$R(r) = J_0(\lambda r)$

now take $\frac{\partial}{\partial \lambda}$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial \lambda} + R_i \frac{\partial^2 R}{\partial \lambda \partial r} + 2\lambda r\, R R_i + \lambda^2 r \frac{\partial R}{\partial \lambda} R_i$$

integrate over $0 \le r \le a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r \partial \lambda}\right) + \lambda^2 r \frac{\partial R}{\partial \lambda} + 2\lambda r R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du \quad u = R_i \quad v = r\frac{\partial^2 R}{\partial r \partial \lambda}$

$$r R_i \frac{\partial^2 R}{\partial r \partial \lambda}\Big|_0^a - \int_0^a r \frac{\partial^2 R}{\partial r \partial \lambda} R_i'\, dr + \int_0^a \left(\lambda^2 r \frac{\partial R}{\partial \lambda} R_i + 2\lambda r R R_i\right) dr = 0$$

R_i by definition is $=0$ @a: $(J_0(\lambda_i a) = 0)$ $\quad r = 0$ at lower limit $R_i = 0$

integrate by parts

$$-\int_0^a r R_i' \frac{\partial^2 R}{\partial r \partial \lambda} dr = -r R_i' \frac{\partial R}{\partial \lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial \lambda}(r R_i')' dr \quad u = r R_i' \quad v = \frac{\partial R}{\partial \lambda}$$

$$\Rightarrow \underbrace{-r R_i' \frac{\partial R}{\partial \lambda}\Big|_0^a}_{-a R_i' \frac{\partial R}{\partial \lambda}} + \int_0^a \frac{\partial R}{\partial \lambda}\Big\{\underbrace{(r R_i')' + \lambda^2 r R_i}_{=0}\Big\} dr + \int_0^a 2\lambda r R R_i\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$, thus the middle term is zero

and $$\int_0^a r R R_i\, dr \Rightarrow \int_0^a r R_i^2\, dr = \frac{a}{2\lambda_i} R_i' \frac{\partial R}{\partial \lambda}\Big|_{\substack{r=a\\ \lambda=\lambda_i}}$$

THUS $\int_0^a r\, J_0^2\!\left(\frac{\omega_i}{c} r\right) dr = \frac{a}{2\left(\frac{\omega_i}{c}\right)} \left[J_0'\!\left(\frac{\omega_i}{c} a\right)\right]^2 \cdot \frac{a \cdot \omega_i}{c} = \frac{a^2}{2}\left[J_0'\!\left(\frac{\omega_i}{c} a\right)\right]^2$

since $R_i'(r) = \frac{d}{dr} J_0\!\left(\frac{\omega_i}{c} r\right) = \frac{\omega_i}{c} J_0'\!\left(\frac{\omega_i}{c} r\right)$

$$\frac{\partial R}{\partial \lambda} = r\, J_0'(\lambda r)$$

now $R_i' \frac{\partial R}{\partial \lambda}\Big|_{\substack{r=a \\ \lambda=\lambda_i}} = \frac{\omega_i}{c} \cdot r\, J_0'\!\left(\frac{\omega_i}{c} r\right) \cdot J_0'(\lambda r) = \frac{\omega_i a}{c}\left[J_0'\!\left(\frac{\omega_i}{c} a\right)\right]^2$

THUS $$\tilde{B}_i = \frac{2}{a^2} \frac{\int_0^a r f(r)\, J_0\!\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\!\left(\frac{\omega_i}{c} a\right)\right]^2}$$

$$\tilde{A}_i = \frac{2}{a^2 \omega_i} \frac{\int_0^a r g(r)\, J_0\!\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\!\left(\frac{\omega_i}{c} a\right)\right]^2}$$

Bessel Relations: $J_0'(\lambda x) = -\lambda J_1(\lambda x)$ $\therefore$ $J_0'\!\left(\frac{\omega_i}{c} a\right) = -\frac{\omega_i}{c} J_1\!\left(\frac{\omega_i}{c} a\right)$

- note Problem in Reynolds pg 4.10 involves – SPHERICAL BESSEL FNS.

produces a bessel fn that solves

$r^2 R'' + 2 r R' + \lambda^2 r^2 R = 0$ and this has an

$[r^2 R']' + \lambda^2 r^2 R = 0$

orthogonality condition $\int_0^r r^2 R_n R_m \, dr = 0$

- in general if $[S y_n']' + [Q + \lambda_n^2 P] y_n = 0$ Sturm–Liouville

under the conditions $\alpha y_n + \beta y_n' = 0$ at $x = a$, $x = b$

y_n is an eigenfunction λ_n eigenvalue

- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x) P(x) y_n(x)\, dx}{\int_a^b P(x)\, y_n^2(x)\, dx}$$

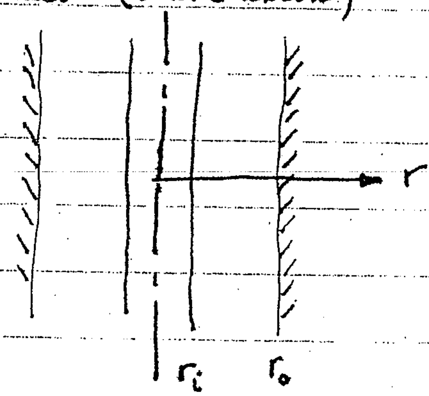
The denominator $\int_a^b P(x)\, y_n^2(x)\,dx = \frac{1}{2\lambda_n}\left\{ y_n' S \frac{\partial y}{\partial \lambda}\Big|_a^b - y_n S \frac{\partial^2 y}{\partial x \partial \lambda}\Big|_a^b \right\}_{\lambda=\lambda}$

$y_n'(z) = \frac{d}{dz} y_n$

$y = y(x;\lambda)$ satisfies ODE & is bounded

$y = y(x;\lambda_n) = y_n$ satisfies ODE & B.C.

- Removing inhomogeneities in the PDE & BC's
  - IN PDE & IN BC.
  - Time history of diffusion of a contaminant $c(r,t)$ in an annular region in which the contaminant in continually produced. (source exists)



$$\frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) = \frac{r}{\alpha}\frac{\partial c}{\partial t} - rs \qquad (3)$$

α diffusivity

s source term

- assume at $r = r_o$ barrier blocks outer diffusion $\therefore \frac{\partial c}{\partial r} = 0$ (1)

- Also assume contaminant is convectively removed at $r = r_i$

$$h(c - c_\infty) = D \frac{\partial c}{\partial r} \qquad (2)$$

h convective transport coeff ; c_∞ is the fixed concentration in fluid passing through annular hole

D - diffusion coeff for the contaminant in the solid

- initially $c = c_0$ at $t = 0$ (only one IC.)

- $J_n\left(\frac{\omega a}{c}\right) = 0$ defines ω since $\frac{\omega_1 a}{c} = r_1$ so that $J_n(r_1) = 0$
  $\frac{\omega_2 a}{c} = r_2$ " " $J_n(r_2) = 0$ etc.

- also since <u>initial conditions</u> are independent of θ
  reasonable to assume that $W(r,\theta,t)$ is independent of θ

$\Rightarrow \quad \alpha = 0 \quad \Rightarrow \quad G(\theta) = \text{constant}$ why? $G'' + \alpha^2 G = G'' = 0 \Rightarrow G = C_1\theta + C_2$
FOR $G(\theta + 2n\pi) = G(\theta) \Rightarrow C_1 = 0 \Rightarrow G(\theta) = \text{constant}$. $\therefore J_n\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$

$$\therefore \quad W(r,\theta,t) \equiv W(r,t) = F\left(\frac{\omega r}{c}\right) H(t) \quad \text{where} \quad F\left(\frac{\omega r}{c}\right) = J_0\left(\frac{\omega r}{c}\right)$$

and $W_m(r,t) = F\left(\frac{\omega_m r}{c}\right) H(t)$ if $B \cdot C \cdot M = \tilde{B}$, $A \cdot C \cdot M = \tilde{A}$

$$= \left(\tilde{B} \cos \omega_m t + \tilde{A} \sin \omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

$$W(r,t) = \sum_m W_m(r,t) = \sum \left(\tilde{B}_m \cos \omega_m t + \tilde{A}_m \sin \omega_m t\right) J_0\left(\frac{\omega_m r}{c}\right)$$

at $t=0$ $\quad W(r,t=0) = f(r) = \sum \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{B}_m R_m(r)$

$t=0$ $\quad W_t(r,t=0) = g(r) = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right) = \sum \tilde{A}_m \omega_m R_m(r)$

suppose we can write $\quad f(r) \equiv E_1 J_0\left(\frac{\omega_1 r}{c}\right) + E_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + E_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$$= \sum_{i=1}^{\infty} E_i J_0\left(\frac{\omega_i r}{c}\right) = W(r,t=0) = \sum_{m=1}^{\infty} \tilde{B}_m J_0\left(\frac{\omega_m r}{c}\right)$$

$$\Rightarrow \quad E's = \tilde{B}'s \quad \& \quad i = m$$

suppose we can write $\quad g(r) = L_1 J_0\left(\frac{\omega_1 r}{c}\right) + L_2 J_0\left(\frac{\omega_2 r}{c}\right) + \dots + L_n J_0\left(\frac{\omega_n r}{c}\right) + \dots$

$$= \sum L_i J_0\left(\frac{\omega_i r}{c}\right) = \left.\frac{\partial W}{\partial t}\right|_{t=0} = \sum \tilde{A}_m \omega_m J_0\left(\frac{\omega_m r}{c}\right)$$

$$\Rightarrow \quad L_i = \tilde{A}_i \omega_i \quad \text{or} \quad \tilde{A}_i = \frac{L_i}{\omega_i} \quad \& \quad i = m$$

- so how do we get the $E's$ & $L's$. See Chapter 8 §8.4

- Sturm-Liouville gives the method for solving for the $\tilde{A}_m$ & $\tilde{B}_m$

Given $\quad \frac{d}{dx}\left(S(x)\frac{dy}{dx}\right) + \left[Q(x) + \lambda^2 P(x)\right] y = 0$

$S(x) y'' + S'(x) y' + \left[Q(x) + \lambda^2 P(x)\right] y = 0 \qquad r^2 R'' + r R' + \left[\lambda^2 r^2 - \nu^2\right] R = 0$

if we chose $\Rightarrow$ $S(x) = r \quad Q(x) = -\nu^2/r$
$S'(x) = 1 \quad P(x) = r$

if not ie $S(x) = r^2 \quad S'(x) = 2r \rightarrow$ doesn't fit

$$\frac{d}{dx}\left[Sy'\right] + \left[Q+\lambda^2 P\right]y = 0$$

subjected to boundary conditions $\alpha y + \beta y' = 0$ at $x = a$

$\alpha y + \beta y' = 0$ at $x = b$

- Homogeneous ODE & B.C.

- assume for $\left.\begin{matrix}\lambda = \lambda_m \\ y = y_m\end{matrix}\right\}$ & $\left.\begin{matrix}\lambda = \lambda_n \\ y = y_n\end{matrix}\right\}$ both ODE & BC are satisfied

$y_m \neq y_n \quad \lambda_m \neq \lambda_n$

$$\Rightarrow \quad [Sy_m']' + [Q+\lambda_m^2 P]y_m = 0 \qquad (1)$$

$$[Sy_n']' + [Q+\lambda_n^2 P]y_n = 0 \qquad (2)$$

$$\Rightarrow \int_a^b \left\langle \left\{[Sy_m']' + [\underline{\underline{Q}}+\lambda_m^2 P]y_m\right\}y_n - \left\{[Sy_n']' + [\underline{\underline{Q}}+\lambda_n^2 P]y_n\right\}y_m \right\rangle dx$$

$$\int_a^b \left\{[Sy_m']'y_n - [Sy_n']'y_m\right\}dx + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_m y_n \, dx = 0$$

$$\int_a^b [Sy_m']'y_n \, dx = Sy_m' y_n \Big|_a^b - \int_a^b S y_m' y_n' \, dx \qquad \text{integration by parts}$$

$$\int_a^b [Sy_n']'y_m \, dx = Sy_n' y_m \Big|_a^b - \int_a^b S y_n' y_m' \, dx$$

and $\left.\begin{matrix}\text{at } x=a \\ \text{at } x=b\end{matrix}\right\} \begin{matrix}\alpha y_n + \beta y_n' = 0 \\ \alpha y_m + \beta y_m' = 0\end{matrix}$ either $\alpha = 0$ & $\beta = 0$ or $\begin{bmatrix} y_n & y_n' \\ y_m & y_m' \end{bmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix}$

det wronskian of $y_n, y_m = 0$.

$$\Rightarrow \quad y_n y_m' - y_m y_n' = 0 \ \text{at both } x = a, b \qquad S\left[y_m' y_n - y_n' y_m\right]\Big|_a^b + (\lambda_m^2 - \lambda_n^2)\int_a^b P y_n y_m \, dx = 0$$

$$\Rightarrow \text{ if } \lambda_m \neq \lambda_n \qquad \boxed{\int_a^b P y_n y_m \, dx = 0}$$

orthogonality condition but wrt weight fn $P(x)$

example $\quad T'' + \omega^2 T = 0 \quad \Rightarrow \quad S(x) = 1 \quad Q(x) = 0 \quad \lambda^2 = \omega^2 \quad P(x) = 1$

$$\Rightarrow \int P y_n y_m \, dx = 0 \quad \Rightarrow \quad \int_a^b P \sin\omega_n x \cdot \sin\omega_m x \, dx = 0$$

bc. suppose $T(x) = 0$ @ $x = 0$ $\quad T(x) = 0$ @ $x = L \quad \Rightarrow T_n(x) = \sin\frac{n\pi x}{L}$

$$\int_0^L \sin\frac{n\pi x}{L} \sin\frac{m\pi x}{L} \, dx = 0$$

now suppose we want to write $f(x)$ as a fn of the eigenfunctions $y_n(x)$

now suppose $f(x) = \sum A_n y_n(x)$

if we mult. by $P y_m$ & integrate

$$\int_a^b P(x) f(x) y_m(x)\,dx = \int_a^b P \sum A_n y_n(x) y_m(x)\,dx$$

$$= \sum A_n \int_a^b P y_n y_m \,dx = \begin{cases} A_m \int_a^b P y_m^2 dx & n=m \\ 0 & n \neq m \end{cases}$$

$$\therefore \quad \boxed{A_m = \frac{\int_a^b P(x) f(x) y_m \,dx}{\int_a^b P y_m^2 \,dx}}$$

FOR $\quad T'' + \omega^2 T = 0 \qquad y_m = \sin \frac{m\pi x}{L} \qquad P = 1 \qquad a=0,\ b=L$

$$A_m = \int_0^L f(x) \sin \frac{m\pi x}{L} dx \Big/ \int_0^L \sin^2 \frac{m\pi x}{L} dx$$

$$\int_0^L \sin^2 \frac{m\pi x}{L} dx = L/2 \quad \Rightarrow \quad \int_0^L \left(\frac{1}{2} - \frac{1}{2}\cos \frac{2m\pi x}{L}\right) dx = L/2$$

$$A_m = \frac{2}{L} \int_0^L f(x) \sin \frac{m\pi x}{L} dx$$ normal fourier coefficient

• Returning to our problem $\quad f(r) = \sum E_i J_0\left(\frac{\omega_i r}{c}\right)$

for the bessel fn. $\quad P(r) = +r \qquad E_i = \dfrac{\int_0^a r f(r) J_0\left(\frac{\omega_i r}{c}\right) dr}{\int_0^a r J_0^2\left(\frac{\omega_i r}{c}\right) dr} = \tilde{B}_i$

since $E_i = \tilde{B}_i$

also $\quad g(r) = \sum L_i J_0\left(\frac{\omega_i r}{c}\right) \qquad L_i = \dfrac{\int_0^a r g(r) J_0 \,dr}{\int_0^a r J_0^2 \,dr} = \tilde{A}_i \omega_i$

since $\quad L_i = \tilde{A}_i \omega_i \quad \Rightarrow \quad \tilde{A}_i = \frac{L_i}{\omega_i}$

$\int_0^a r\, J_0^2(r\frac{\omega_i}{c})\,dr$ can be evaluated as follows

using the ODE : if you remember $J_0(\frac{\omega_i}{c}r)$ satisfies

$$r R'' + R' + \left(\overset{\lambda_i^2 r}{[\tfrac{\omega_i}{c}]^2 r} - 0^2\right)R = 0 \qquad R_i = J_0(\lambda_i r)$$

let $R(r,\lambda)$ be a solution to $r R'' + R' + (\lambda^2 r)R = 0$
and satisfy that $R(r=0)$ is not ∞
$R(r) = J_0(\lambda r)$

now take $\frac{\partial}{\partial\lambda}$ and multiply by R_i

$$R_i\, r \frac{\partial^3 R}{\partial r^2 \partial\lambda} + R_i \frac{\partial^2 R}{\partial\lambda\partial r} + 2\lambda r\, R R_i + \lambda^2 r \frac{\partial R}{\partial\lambda} R_i = 0$$

integrate over $0 \le r \le a$

$$\int_0^a R_i \left\{ \frac{\partial}{\partial r}\left(r \frac{\partial^2 R}{\partial r\partial\lambda}\right) + \lambda^2 r \frac{\partial R}{\partial\lambda} + 2\lambda r R \right\} dr = 0$$

integrate by parts the first term. $\int u\,dv = uv - \int v\,du$ $u = R_i$ $v = r\frac{\partial^2 R}{\partial r\partial\lambda}$

$$r R_i \frac{\partial^2 R}{\partial r\partial\lambda}\Big|_0^a - \overset{*}{\int_0^a r \frac{\partial^2 R}{\partial r\partial\lambda} R_i'\, dr} + \int_0^a \Big(\overset{**}{\lambda^2 r \frac{\partial R}{\partial\lambda} R_i} + \overset{***}{2\lambda r R R_i}\Big) dr = 0$$

R_i by definition is $=0$ @a: $(J_0(\lambda_i a) = 0)$. For $r=0$ at lower limit $R_i = 0$

integrate by parts

$$(*)\quad -\int_0^a r R_i' \frac{\partial^2 R}{\partial r\partial\lambda}\, dr = -r R_i' \frac{\partial R}{\partial\lambda}\Big|_0^a + \int_0^a \frac{\partial R}{\partial\lambda}(r R_i')' \quad \text{with } u = r R_i' \quad v = \frac{\partial R}{\partial\lambda}$$

$$\Rightarrow \underbrace{-r R_i' \frac{\partial R}{\partial\lambda}\Big|_0^a}_{-a R_i' \frac{\partial R}{\partial\lambda}} + \int_0^a \frac{\partial R}{\partial\lambda}\underbrace{\left\{ (r R_i')' + \overset{**}{\lambda^2 r R_i} \right\}}_{=0} dr + \int_0^a \overset{***}{2\lambda r R R_i}\, dr = 0$$

if $R(r,\lambda)$ is $R_i \Rightarrow \lambda = \lambda_i$; thus the middle term is zero

and $$2\lambda_i \int_0^a \overset{***}{r R R_i}\, dr \Rightarrow \int_0^a r R_i^2\, dr = \frac{a R_i' \frac{\partial R}{\partial\lambda}}{2\lambda_i}\Bigg|_{\substack{r=a\\ \lambda=\lambda_i}}$$

THUS $$\int_0^a r\, J_0^2\left(\frac{\omega_i}{c} r\right) dr = \frac{a}{2\left(\frac{\omega_i}{c}\right)} \left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2 \cdot \frac{a \cdot \omega_i}{c} = \frac{a^2}{2}\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2$$

since $$R_i'(r) = \frac{d}{dr} J_0\left(\frac{\omega_i}{c} r\right) = \frac{\omega_i}{c} J_0'\left(\frac{\omega_i}{c} r\right)$$

$$\frac{\partial R}{\partial \lambda} = r\, J_0'(\lambda r)$$

now $$R_i' \frac{\partial R}{\partial \lambda}\bigg|_{\substack{r=a \\ \lambda=\lambda_i}} = \frac{\omega_i}{c} \cdot r\, J_0'\left(\frac{\omega_i}{c} r\right) \cdot J_0'(\lambda r) = \frac{\omega_i a}{c}\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2$$

THUS $$\tilde{B}_i = \frac{2}{a^2} \frac{\int_0^a r f(r)\, J_0\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2}$$

$$\tilde{A}_i = \frac{2}{a^2 \omega_i} \frac{\int_0^a r g(r)\, J_0\left(\frac{\omega_i}{c} r\right) dr}{\left[J_0'\left(\frac{\omega_i}{c} a\right)\right]^2}$$

$$J_0'(\lambda x) = -\lambda J_1(\lambda x) \quad \therefore \quad J_0'\left(\frac{\omega_i}{c} a\right) = -\frac{\omega_i}{c} J_1\left(\frac{\omega_i}{c} a\right)$$

- note the problem on page 4.10 on wards – <u>SPHERICAL</u> BESSEL FNS.
  produces a bessel fn

$$r^2 R'' + \underline{\underline{2}} r R' + \lambda^2 r^2 R = 0$$ and this has an

orthogonality condition $$\int_0^r \underline{\underline{r^2}} R_n R_m\, dr = 0$$

$S(x) = r^2$
$Q(x) = 0$
$\lambda^2 r^2 = \lambda^2 P$

- in general if $[S y_n']' + [Q + \lambda_n^2 P] y_n = 0$ Sturm-Liouville
  under the conditions $\alpha y_n + \beta y_n' = 0$ at $x=a$, $x=b$
  y_n is an eigenfunction λ_n eigenvalue
- then if we want to construct $f(x) = \sum A_n y_n(x)$

$$A_n = \frac{\int_a^b f(x) P(x) y_n(x)\, dx}{\int_a^b P(x)\, y_n^2(x)\, dx}$$

$$\int_a^b P y_n y_m dx = 0 \qquad n \neq m \tag{4.2.14}$$

Eqn. (4.2.14) is the <u>orthogonality property</u> of the eigenfunctions. The eigenfunctions are said to be orthogonal <u>with respect to the weight function $P(x)$</u> .

Now, suppose that, in the course of trying to construct the solution to a PDE as a linear combination of eigensolutions of the linear, homogeneous partial problem, we are led to the point where we wish to determine the coefficients in an eigenfunction expansion,

$$f(x) = \sum_{n=1}^{\infty} A_n y_n(x) \tag{4.2.15}$$

where the y_n are eigensolutions of a Sturm-Liouville problem. Multiplying (4.2.15) by Py_m , and integrating over the problem domain,

$$\int_a^b f P y_m dx = \sum_{n=1}^{\infty} A_n \int_a^b P y_n y_m dx \tag{4.2.16}$$

But, because of the orthogonality property (4.2.14), all of the integrals on the right will drop out, except the one where $n = m$. Hence, we can immediately solve for A_m ,

$$A_m = \frac{\int_a^b f P y_m dx}{\int_a^b P y_m^2 dx} \tag{4.2.17}$$

The infinite series (4.2.15) will be useless if it fails to converge to $f(x)$. In specific problems where one calculates the A_n it is easy to perform the standard tests for series convergence. It is somewhat more difficult to prove convergence in general. However, if f is square-integrable, i.e., if

$$\int_a^b Pf^2 dx \quad \text{is finite}$$

then the series converges in the sense that*

$$\lim_{N\to\infty} \int_a^b \left| f(x) - \sum_{n=1}^{N} A_n y_n(x) \right|^2 dx \to 0$$

This means that, if f is continuous over the interval $a \leq x \leq b$, the series converges uniformly (at all x). However, if f is discontinuous at some point, then the series will give a value at that point that is the average of the values of f at points infinitesimally above and below the point of discontinuity.

There are many problems of interest involving higher order system of linear homogeneous equations. In these cases, there are no theorems or general proofs of convergence of the eigenfunction expansions. One has to proceed by examining each case separately. However, problems arising from well-thought through physical formulations rarely, if ever, give rise to non-convergent expansions, so the analyst is usually safe in going ahead, assuming convergence, and then verifying it after the fact by ratio tests, numerical calculations, or other appropriate means.

4.3 Example - Vibrating String

For the vibrating string problem discussed in §4.1, the solution is given by (4.1.6). The coefficients A_n must be chosen such that (4.1.9) is satisfied. The eigenfunctions X_n are eigensolutions of

$$X_n'' + \lambda_n^2 X_n = 0 \tag{4.3.1}$$

and hence, from Sturm-Liouville theory, have the orthogonality property

*See, for example, Ince, Ordinary Differential Equations, Dover, New York, 1956.

$$R(r,\lambda) = \frac{\sin(\lambda r)}{\lambda r} \qquad (4.4.26)$$

$$\left.\frac{\partial R}{\partial \lambda}\right)_{\substack{\lambda = \lambda_n \\ r = r_o}} = \frac{1}{\lambda_n} \cos(\lambda_n r_o) \qquad (4.4.27)$$

Noting that $\cos(\lambda_n r_o) = (-1)^n$, (4.2.23) gives

$$A_n = 2(-1)^{n+1} T_o \qquad (4.4.28)$$

So, our final solution is, from (4.4.12),

$$T(r,t) = 2T_o \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin(n\pi r/r_o)}{n\pi r/r_o} e^{-n^2\pi^2\alpha t/r_o^2} \qquad (4.4.29)$$

Note that the series converges for all t . The series for $\partial T/\partial r$, developed from (4.4.29) by differentiation, will converge for all $t > 0$ because of the exponential, but does not converge at $t = 0$. But this is not a serious limitation. As t increases the series converges more rapidly, and at large t the solution is given (approximately) by just the first term,

$$T \simeq 2T_o \frac{\sin(\pi r/r_o)}{\pi r/r_o} e^{-\pi^2\alpha t/r_o^2} \qquad (4.4.30)$$

4.5 Sturm-Liouville Denominator Integral

In analyses, leading to the Sturm-Liouville problems, the orthogonality property will produce (4.2.17). The denominator integral may be expressed in terms of quantities evaluated at the boundary using a generalization of the trick employed in the previous example. Let $y(x,\lambda)$ be a solution to (4.2.1) not necessarily satisfying the boundary conditions (4.2.2). Then, $y(x,\lambda_n)$

will be an eigensolution satisfying the boundary conditions. We differentiate (4.2.1) with respect to λ, obtaining

$$\frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x \partial \lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial \lambda} + 2\lambda P y \;=\; 0 \tag{4.5.1}$$

Next, we multiply (4.5.1) by y_n and integrate over the problem range,

$$\int_a^b y_n \left\{ \frac{\partial}{\partial x}\left(S\,\frac{\partial^2 y}{\partial x \partial \lambda}\right) + \left[Q + \lambda^2 P\right]\frac{\partial y}{\partial \lambda} + 2\lambda P y \right\} dx \;=\; 0 \tag{4.5.2}$$

The first integral is integrated twice by parts, and (4.5.2) becomes

$$y_n\, S\,\frac{\partial^2 y}{\partial x \partial \lambda}\Bigg|_a^b - \frac{\partial y}{\partial \lambda}\, y_n' S\Bigg|_a^b$$

$$+\int_a^b \frac{\partial y}{\partial \lambda}\left\{ (S y_n')' + \left[Q + \lambda^2 P\right] y_n \right\} dx + 2\lambda \int_a^b P y y_n\, dx \;=\; 0 \tag{4.5.3}$$

Now, if we set $\lambda = \lambda_n$, the first integral drops out (because the integrand contains the y_n equation), and hence

$$\int_a^b P y_n^2\, dx \;=\; \frac{1}{2\lambda_n}\left\{ y_n' S\,\frac{\partial y}{\partial \lambda}\Bigg|_a^b - y_n S\,\frac{\partial^2 y}{\partial x \partial \lambda}\Bigg|_a^b \right\}_{\lambda = \lambda_n} \tag{4.5.4}$$

Thus, the denominator in A_n can be evaluated without recourse to integration.

4.6 Removal of Inhomogeneities in the PDE and BCs

In the previous problem, the PDE and BCs were homogeneous, and therefore eigensolutions of this homogeneous problem could be found. By taking a

$$u(0,t) = X(0)\,T(t) = 0 \Rightarrow \forall t \quad X(0) = 0 \Rightarrow c_3 = 0$$

$$u(L,t) = X(L)\,T(t) = 0 \Rightarrow \forall t \quad X(L) = 0 \Rightarrow c_4 \sin \lambda L = 0$$

$$\therefore \quad \lambda L = n\pi \quad \text{or} \quad \lambda_n = \frac{n\pi}{L} = \frac{\omega_n}{a}$$

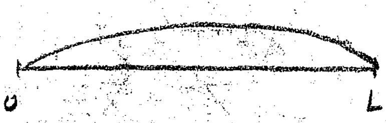
λ_n are the EV's

$$\therefore \quad X(x) = c_{4_n} \sin \frac{n\pi x}{L} \quad \overrightarrow{EV} \qquad \therefore \quad \frac{n\pi a}{L} = \omega_n$$

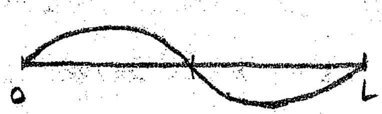
thus $u(x,t) = X(x)T(t) = \sum_{n=0}^{\infty} A_n \sin \frac{n\pi x}{L} \left(\cos\left[\frac{n\pi a t}{L} - \phi_n\right]\right)$

A_n, ϕ_n are not determined since we have not given IC.

$$u_1(x,t) = A_1 \sin \frac{\pi x}{L} \cos\left(\frac{\pi a t}{L} - \phi_1\right)$$

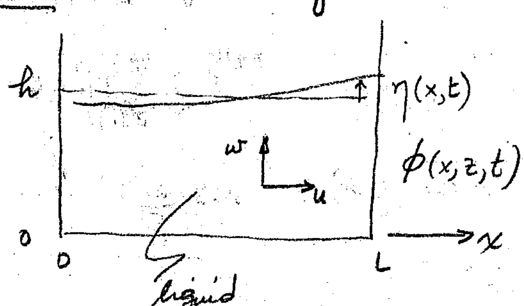


$$u_2(x,t) = A_2 \sin \frac{2\pi x}{L} \cos\left(\frac{2\pi a t}{L} - \phi_2\right)$$



EV problems arise because of the homogeneities of the problem or the b.c.

Proble Sloshing



potential function $\phi \ni \frac{\partial \phi}{\partial x} = u$

$$\frac{\partial \phi}{\partial z} = w$$

DE: Continuity $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = \phi_{xx} + \phi_{zz} = 0$

$$\frac{\partial \phi}{\partial t} + g\eta = 0 \quad \text{at } z = h \qquad (2)$$

g: accel of gravity

Bernoulli Eq (momentum Eq)

$$\frac{\partial \eta}{\partial t} - \frac{\partial \phi}{\partial z} = 0 \quad @ \; z = h \text{ on surface}$$

Kinematics

$\frac{\partial \phi}{\partial x} = u = 0$ @ $x = 0, L$ and $\frac{\partial \phi}{\partial z} = 0 = w$ @ $z = 0$ (4) impermeable wall bc. fluid can't go through wall

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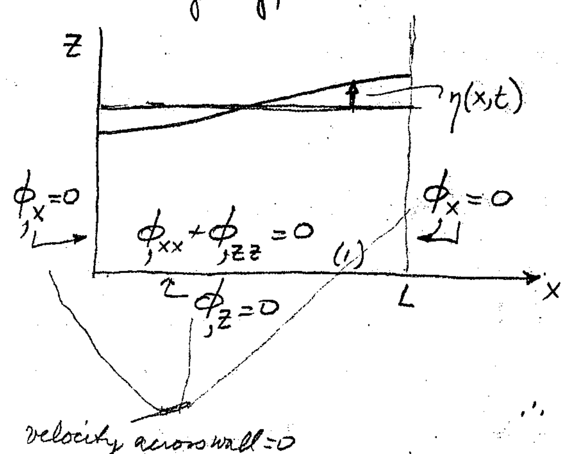
Sloshing Phys/Math



on $z = h$

$\phi_{,t} + g\eta = 0$ (2) $\phi : \phi(x,z,t)$ velocity pot.

$\eta_{,t} - \phi_{,z} = 0$ (3)

Assume $\phi = X(x) Z(z) T(t)$

$\eta = F(x) G(t)$

$\therefore \; X Z T' + g F G = 0$ (2)

$X'' Z T + X Z'' T = 0$ (1) $\Rightarrow \frac{X''}{X} = \frac{-Z''}{Z} = -\lambda^2$

For (1):

$X'' + \lambda^2 X = 0$; $Z'' - \lambda^2 Z = 0 \Rightarrow X = c_1 \sin\lambda x + c_2 \cos\lambda x$; $Z = c_3 \sinh\lambda z + c_4 \cosh\lambda z$

BC $\left.\phi_{,x}\right|_{x=0} = 0 \Rightarrow X'(0) = 0 \Rightarrow c_1 = 0$

$\left.\phi_{,x}\right|_{x=L} = 0 \Rightarrow X'(L) = 0 \Rightarrow \lambda L = n\pi$ $\Big\} \longrightarrow X = c_2 \cos\frac{n\pi x}{L}$

$\left.\phi_{,z}\right|_{z=0} \Rightarrow Z'(0) = 0 \Rightarrow c_3 = 0 \longrightarrow Z = c_4 \cosh\lambda z$

combine (2) & (3) take $\frac{\partial}{\partial t}(2) - g(3) \Rightarrow \phi_{,tt} + g\phi_{,z} = 0$ or

on $z = h$ $\; X Z T'' + g X Z' T = 0$ thus $\frac{T''}{Z} + g \frac{Z'(h)}{Z(h)} T = 0$

now $\frac{Z'(h)}{Z(h)} = \frac{+\lambda c_4 \sinh(\lambda h)}{c_4 \cosh(\lambda h)} = \lambda \tanh(\lambda h)$ let $\underline{\frac{\omega^2}{g} = \lambda \tanh\lambda h}$

$\therefore \; T = c_5 \sin\omega t + c_6 \cos\omega t = a \cos(\omega t - \psi)$

a: amplitude, ψ: phase shift

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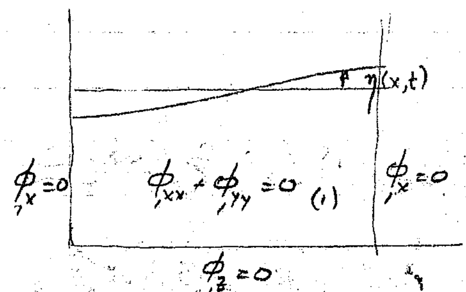


on $z=h$ $\quad \phi_{,t}+g\eta=0$ (2)

$\eta_{,t}-\phi_{,z}=0$ (3)

$\phi = XZT$

$\eta = FG$

$\phi = A\cos\lambda x \cosh\lambda z \cos(\omega t-\psi)$

$\lambda_n = n\pi/L$

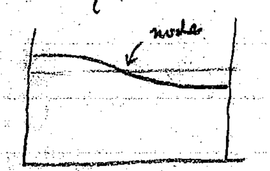
$\omega_n^2 = g\lambda_n \tanh\lambda_n h$

$T = a\cos(\omega t-\psi)$

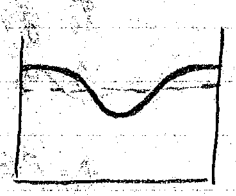
for $h\to\infty$ $\quad \tanh(\lambda_n h)\to 1 \quad \omega_n^2 = g\lambda_n = \frac{n\pi g}{L} \quad \omega=\sqrt{\frac{n\pi g}{L}} \quad f=\frac{\omega}{2\pi}$

Use equation (2) $\quad \eta = -\phi_{,t}/g = \boxed{\frac{A\omega}{g}\cos\lambda x\cosh\lambda h\sin(\omega t-\psi) = \eta(x,t)}$

First mode $n=1$ $\quad \eta(x,t) = $ const $\quad$ const $\times\cos\left(\frac{\pi x}{L}\right)\sin(\omega t-\psi)$



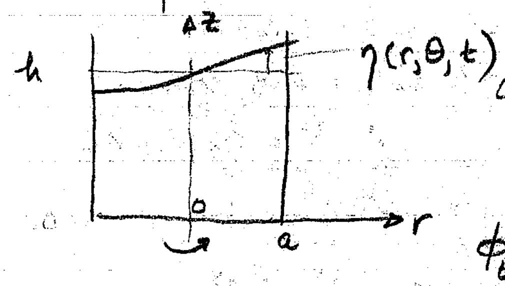
$n=2$



When do we use separation of var

If a linear homogeneous d.e. in a simple geometry w/ zero b.c. give eigensolution

Circular Cylindrical coordinate system.



define a velocity potential ϕ

Continuity $\quad \nabla^2\phi = 0 = \phi_{rr}+\frac{1}{r}\phi_r+\frac{1}{r^2}\phi_{\theta\theta}+\phi_{zz}$

B.C. $\quad \left.\begin{array}{l}\phi_{,z}=0 \text{ @ } z=0\\ \phi_{,r}=0 \text{ @ } r=a\end{array}\right\}$ solid wall bc

$\left.\begin{array}{ll}\phi_t+g\eta=0 \ (2) & \eta_t-\phi_z=0 \ (3)\\ \text{Dynamics} & \text{Kinematics}\end{array}\right\}$ at $z=h$

assume $\quad \phi = R(r)\,\Theta(\theta)\,Z(z)\,T(t)$

$$\underbrace{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}}_{\text{independent of } z} + \underbrace{\frac{Z''}{Z}}_{\text{fn of } z} = 0$$

$$\therefore \quad \frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = -\frac{Z''}{Z} = -\lambda^2 \quad \Rightarrow \quad Z'' - \lambda^2 Z = 0$$

$$r^2\left[\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right] = -r^2\lambda^2 \quad \Rightarrow \quad r^2\frac{R''}{R} + r\frac{R'}{R} + r^2\lambda^2 = -\frac{\Theta''}{\Theta} = \beta^2$$

$$\Rightarrow \quad \Theta'' + \beta^2\Theta = 0$$

and $r^2R'' + rR' + (r^2\lambda^2 - \beta^2)R = 0$

$\Theta = C_1 \sin\beta\theta + C_2\cos\beta\theta \qquad$ solutions must be periodic in $\theta \Rightarrow \beta = \text{integer} = n$

AMF 9.1.1: $\quad z^2\dfrac{d^2w}{dz^2} + z\dfrac{dw}{dz} + (z^2 - \nu^2)w = 0 \qquad$ let $\lambda r = z \quad w = R \quad \nu = n$ defines a bessel

$$R = C_1 J_n(\lambda R) + C_2 Y_n(\lambda R)$$

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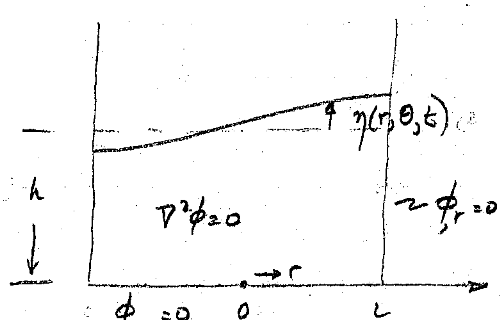


$$\left.\begin{aligned} \phi_t + g\eta &= 0 \quad (2) \\ \eta_t - \phi_z &= 0 \quad (3) \end{aligned}\right\} \text{@ } z = h$$

$\phi = R\Theta ZT$

$\Theta'' + n^2\Theta = 0$

$r^2R'' + rR' + (r^2\lambda^2 - n^2)R = 0$

$Z'' - \lambda^2 Z = 0$

$(2) \,\&\, (3) \Rightarrow \quad \phi_{tt} + g\phi_z = 0 \quad$ @ $z = h$

$T''Z + gZ'T = 0 \quad$ or $\quad \dfrac{T''}{T} = -g\dfrac{Z'(h)}{Z(h)} = -\omega^2 \quad \therefore$ let $g\,Z'(h)/Z(h) = \omega^2$

time: $\quad \therefore \quad T'' + \omega^2 T = 0 \quad \Rightarrow \quad T = A_1\sin\omega t + A_2\cos\omega t$

height: $\quad Z'' - \lambda^2 Z = 0 \quad \Rightarrow \quad Z = B_1\sinh\lambda z + B_2\cosh\lambda z$

$\phi_{,z} = 0 \Rightarrow Z'(0) = 0 \Rightarrow B_1 \equiv 0 \qquad \therefore \boxed{Z = B_2\cosh\lambda z}$

hence $\Rightarrow Z'/Z\big|_{z=h} = \lambda\tanh\lambda h \quad \Rightarrow \quad \boxed{\omega^2 = g\lambda\tanh\lambda h}$

ω^2 determined by $\lambda \quad$: @ $h \to \infty \quad \omega^2 \to g\lambda$

- Radial problem : $\phi_{,r} = 0$ $\forall$ points on wall $\therefore R'(a) = 0$

2nd BC : R is finite everywhere

$$r^2R'' + rR' + (\lambda^2 r^2 - n^2)R = 0$$

Linearly independent soln : pick any two

$J_n(\lambda r)$ $J_{-n}(\lambda r)$ are l.i. if $n \neq$ integer

$Y_n(\lambda r)$, $H_n^{(1)}(\lambda r)$, $H_n^{(2)}(\lambda r)$, $N_N(\lambda r)$

let $R(r) = c_1 J_n(\lambda r) + c_2 Y_n(\lambda r)$ $Y_n(z) \to \infty$ at $z = 0$ $\therefore c_2 = 0$ for finite soln.

$$\therefore R(r) = c_1 J_n(\lambda r)$$

$R'(a) = c_1 \lambda J_n'(\lambda a) = 0$ So on pg 409 of the HMF $j_{0,s} = J_0(x_s) = 0$

$j'_{0,s} = J_0'(x_s) = 0$

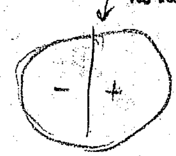
n	s	λa	mode #
0	1	3.831	3
0	2	7.0156	
1	1	1.841	1
1	2	5.331	
2	1	3.054	2

from mode 1 : $R_{1,1} = c_1 J_1(\lambda_{1,1} r)$ $\lambda_{1,1} = \frac{1.841}{a}$

for infinite depth $\omega_{1,1}^2 = g\lambda_{1,1} = g\frac{1.841}{a}$

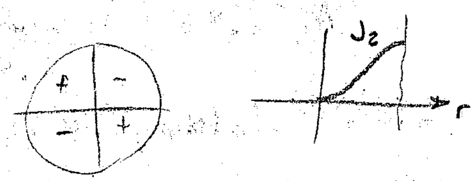
now mode 1 $\Rightarrow$ $\underline{n = 1}$ $\Theta'' + \Theta = 0$ $\Rightarrow$ $\Theta = A\cos\theta + B\sin\theta$

node line

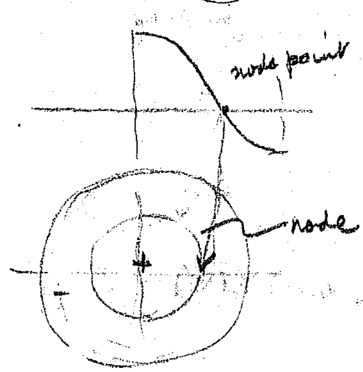


$$\phi_{1,1} = A\cos\theta \cosh\left(\frac{1.841z}{a}\right) J_1\left(\frac{1.841r}{a}\right) \cos\left(\sqrt{\frac{1.841g}{a}}\, t\right)$$

for mode 2 :



for 3 :



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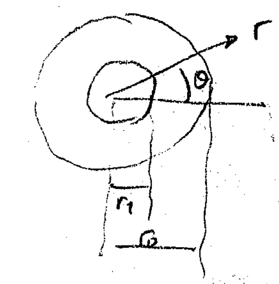
Annular Membrane



$c^2\nabla^2 u - u_{tt} = 0$ u is deflection c^2 = membrane param.

$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$

BC $u = 0$ @ $r = r_i, r_o$

Linear homogeneous w/char length : use separation of variables

Try $u = R\Theta T$

$$\therefore \quad \underbrace{c^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta}\right)}_{\text{fn of } r,\theta} = \underbrace{\frac{T''}{T}}_{\text{fn of } t} = \text{const} = -\omega^2$$

$$T'' + \omega^2 T = 0 \Rightarrow T = c_1\cos\omega t + c_2\sin\omega t = B\cos(\omega t - \phi)$$

$$r^2\left(\frac{R''}{R} + \frac{1}{r}\frac{R'}{R}\right) + \frac{\omega^2 r^2}{c^2} = -\frac{\Theta''}{\Theta} = \beta^2$$

$$\Theta'' + \beta^2\Theta = 0 \Rightarrow \Theta = \hat{c}_1\sin\beta\theta + \hat{c}_2\cos\beta\theta$$ for continuity in θ $\underline{\underline{\beta \equiv \text{integer}}} = n$

$$r^2R'' + rR' - (\beta^2 - \alpha^2r^2)R = 0 \qquad \text{where } \alpha^2 = \frac{\omega^2}{c^2}$$

$$r^2R'' + rR' + (\alpha^2r^2 - n^2)R = 0 \qquad R = A_1J_n(\alpha r) + A_2Y_n(\alpha r)$$

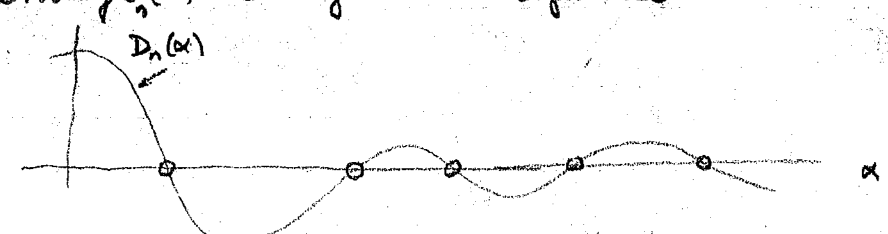
BC. $u = 0$ $r = r_i, r_o$ $R(r_i) = 0$ or $R(r_o) = 0$

$$A_1J_n(\alpha r_i) + A_2Y_n(\alpha r_i) = 0$$
$$A_1J_n(\alpha r_o) + A_2Y_n(\alpha r_o) = 0$$

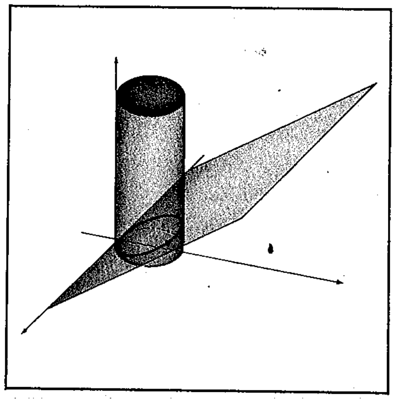
For non trivial A_1, A_2

$$\therefore \quad \det\left[J_n(\alpha r_i)Y_n(\alpha r_o) - J_n(\alpha r_o)Y_n(\alpha r_i)\right] = 0 = D_n(\alpha)$$

thus the roots of $D_n(\alpha) = 0$ give the eigenvalues α



let $x = \alpha r_i$ $\lambda = r_o/r_i$ $\det\left(J_n(x)Y_n(\lambda x) - J_n(\lambda x)\cdot Y_n(x)\right) = 0$



CHAPTER

EIGHT

Special Functions

8.1 Introduction

In Chapters 3–7, discussions have been confined to (initial) boundary value problems expressed in Cartesian coordinates (with the exception of Laplace's equation in polar coordinates in Section 5.3). When separation of variables, finite Fourier transforms, and Laplace transforms are applied to initial boundary value problems in polar, cylindrical, and spherical coordinates, new functions arise, namely, Bessel functions and Legendre functions. In Sections 8.3 and 8.5, we introduce these functions as solutions of ordinary differential equations, as this is how they arise in the context of PDEs. Bessel's differential equation and Legendre's differential equation are homogeneous, second-order, linear differential equations with variable coefficients. The most general form of such an equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{1}$$

A point x_0 is said to be an *ordinary point* of this differential equation when the functions $Q(x)/P(x)$ and $R(x)/P(x)$ have convergent Taylor series about x_0; otherwise, x_0 is called a *singular point*. When x_0 is an ordinary point of (1), there exist two independent solutions $y_1(x)$ and $y_2(x)$, both with Taylor series convergent in some interval $|x - x_0| < \delta$. A general solution of the differential equation valid in this interval is $c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants.

When x_0 is a singular point of (1), independent solutions in the form of power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ about x_0 may not exist. In this case, it is customary to search for solutions in the form

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}, \tag{2}$$

called *Frobenius* solutions. Solutions of this type may or may not exist, depending on the severity of the singularity. A singular point x_0 is said to be *regular* if

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2\frac{R(x)}{P(x)}$$

both have Taylor series expansions about x_0. Otherwise, x_0 is said to be an *irregular* singular point.

When x_0 is a regular singular point of (1), a Frobenius solution (2) always leads to a quadratic equation for the unknown index r. Depending on the nature of the roots of this quadratic, called the *indicial equation*, three situations arise; they are summarized in the following theorem.

Theorem 1

Let r_1 and r_2 be the indicial roots for a Frobenius solution of (1) about a regular singular point x_0. To find linearly independent solutions of (1), it is necessary to consider the cases in which the difference $r_1 - r_2$ is not an integer, is zero, or is a positive integer.

Case 1: $r_1 \neq r_2$ and $r_1 - r_2 \neq$ integer.
In this case, two linearly independent solutions,

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{with } a_0 = 1 \tag{3a}$$

and

$$y_2(x) = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n(x - x_0)^n \quad \text{with } b_0 = 1, \tag{3b}$$

always exist.

Case 2: $r_1 = r_2 = r$.
In this case, one Frobenius solution,

$$y_1(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{with} \quad a_0 = 1, \tag{4a}$$

is obtained. A second (independent) solution exists in the form

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^r \sum_{n=1}^{\infty} A_n(x - x_0)^n, \qquad x > x_0. \tag{4b}$$

29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

29.1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in **29.1.1** converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

exists, then it converges for all s with $\mathscr{R}s>s_0$, and the image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2
$$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \frac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

Heaviside step function

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function $F(t)$	*Image Function $f(s)$*
29.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s)-F(+0)$
29.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
29.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

	Original Function $F(t)$	Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c}e^{(b/c)t}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t)\sum_{n=0}^{\infty}(-1)^n u(t-na)$	$\dfrac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\lvert F(t)\rvert$	$f(s)\coth\frac{as}{2}$
	Heaviside Expansion Theorem	
29.2.20	$\sum_{n=1}^{m}\frac{p(a_n)}{q'(a_n)}e^{a_n t}$	$\frac{p(s)}{q(s)}$, $q(s)=(s-a_1)(s-a_2)\ldots(s-a_m)$ $p(s)$ a polynomial of degree$<m$
29.2.21	$e^{at}\sum_{n=1}^{r}\frac{p^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree$<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

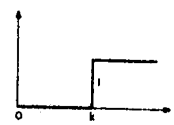
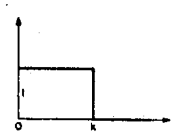
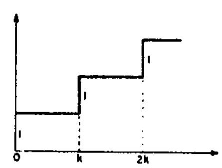
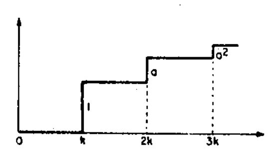
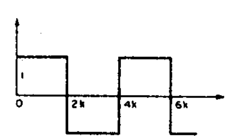
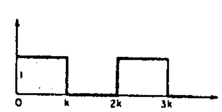
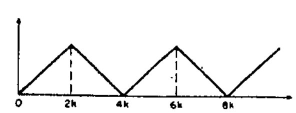
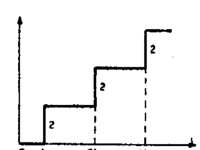
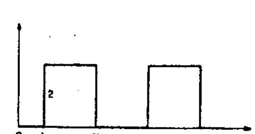
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

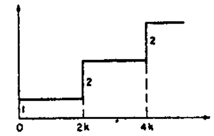
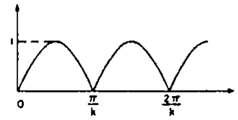
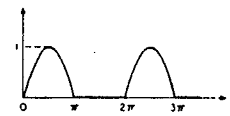
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \qquad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+\frac{1}{2})} \qquad (n=1, 2, 3, \ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot 3\cdot 5 \ldots (2n-1)\sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \qquad (k>0)$	6	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
29.3.10	$\frac{1}{(s+a)^n} \qquad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \qquad (k>0)$	6	$t^{k-1}e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \qquad (a\neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \qquad (a\neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a}\sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a}\sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at\cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2} = \frac{1}{s^2+2as+(a^2+b^2)}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf} a\sqrt{t}$	7

	$f(s)$	$F(t)$	
29.3.41	$\dfrac{1}{\sqrt{s}(s+a^2)}$	$\dfrac{2}{a\sqrt{\pi}}\, e^{-a^2t}\displaystyle\int_0^{a\sqrt{t}} e^{\lambda^2}d\lambda$	7
29.3.42	$\dfrac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$	$e^{a^2t}[b-a \operatorname{erf} a\sqrt{t}]-be^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.43	$\dfrac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2t} \operatorname{erfc} a\sqrt{t}$	7
29.3.44	$\dfrac{1}{(s+a)\sqrt{s+b}}$	$\dfrac{1}{\sqrt{b-a}}\, e^{-at} \operatorname{erf}\, (\sqrt{b-a}\sqrt{t})$	7
29.3.45	$\dfrac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$	$e^{a^2t}\left[\dfrac{b}{a} \operatorname{erf}\, (a\sqrt{t})-1\right]+e^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.46	$\dfrac{(1-s)^n}{s^{n+\frac{1}{2}}}$	$\dfrac{n!}{(2n)!\sqrt{\pi t}}\, H_{2n}(\sqrt{t})$	22
29.3.47	$\dfrac{(1-s)^n}{s^{n+\frac{3}{2}}}$	$\dfrac{n!}{(2n+1)!\sqrt{\pi}}\, H_{2n+1}(\sqrt{t})$	22
29.3.48	$\dfrac{\sqrt{s+2a}}{\sqrt{s}}-1$	$ae^{-at}[I_1(at)+I_0(at)]$	9
29.3.49	$\dfrac{1}{\sqrt{s+a}\sqrt{s+b}}$	$e^{-\frac{1}{2}(a+b)t}I_0\left(\dfrac{a-b}{2}\, t\right)$	9
29.3.50	$\dfrac{\Gamma(k)}{(s+a)^k(s+b)^k}\quad (k>0)\quad 6$	$\sqrt{\pi}\left(\dfrac{t}{a-b}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\dfrac{a-b}{2}\, t\right)$	10
29.3.51	$\dfrac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$	$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\dfrac{a-b}{2}\, t\right)+I_1\left(\dfrac{a-b}{2}\, t\right)\right]$	9
29.3.52	$\dfrac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$	$\dfrac{1}{t}\, e^{-at}I_1(at)$	9
29.3.53	$\dfrac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}\quad (k>0)$	$\dfrac{k}{t}\, e^{-\frac{1}{2}(a+b)t}I_k\left(\dfrac{a-b}{2}\, t\right)$	9
29.3.54	$\dfrac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}\quad (\nu>-1)$	$\dfrac{1}{a^\nu}\, e^{-\frac{1}{2}at}I_\nu(\tfrac{1}{2}at)$	9
29.3.55	$\dfrac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$	9
29.3.56	$\dfrac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}\quad (\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\dfrac{1}{(s^2+a^2)^k}\quad (k>0)$	$\dfrac{\sqrt{\pi}}{\Gamma(k)}\left(\dfrac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$	$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t} J_k(at)$		9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$		9
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$		6, 10
29.3.61	$\frac{1}{s} e^{-ks}$	$u(t-k)$		
29.3.62	$\frac{1}{s^2} e^{-ks}$	$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu} e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)} u(t-k)$		6
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s} \tanh ks$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2} \tanh ks$	$tu(t)+2\sum_{n=1}^{\infty} (-1)^n (t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s \sinh ks}$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s \cosh ks}$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
29.3.72	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty}u(t-2nk)$ 	
29.3.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$ 	
29.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
29.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
29.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
29.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
29.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^{\mu}}e^{-\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}J_{\mu-1}(2\sqrt{kt})$	9
29.3.81	$\frac{1}{s^{\mu}}e^{\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}I_{\mu-1}(2\sqrt{kt})$	9
29.3.82	$e^{-k\sqrt{s}}\quad(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k\geq 0)$	$(4t)^{\frac{1}{2}n}\,\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7

*See page II.

	$f(s)$	$F(t)$	
29.3.89	$\frac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}\quad(k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.90	$\frac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}\quad(k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7
29.3.91	$\frac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}\quad(k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\tfrac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	9
29.3.92	$\frac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}\quad(k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.93	$\frac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}\quad(k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.94	$\frac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}}\quad(k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}}\quad(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks}\quad(k>0)$	$\frac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.97	$\frac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu}\quad(\nu>-1, k\geq 0)$	$\left(\frac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.98	$\frac{1}{s}\ln s$	$-\gamma-\ln t\,(\gamma=.57721\ 56649\ldots$ Euler's constant)	
29.3.99	$\frac{1}{s^k}\ln s\quad(k>0)$	$\frac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	6
29.3.100	$\frac{\ln s}{s-a}\quad(a>0)$	$e^{at}[\ln a+E_1(at)]$	5
29.3.101	$\frac{\ln s}{s^2+1}$	$\cos t\ \mathrm{Si}\ (t)-\sin t\ \mathrm{Ci}\ (t)$	5
29.3.102	$\frac{s\ln s}{s^2+1}$	$-\sin t\ \mathrm{Si}\ (t)-\cos t\ \mathrm{Ci}\ (t)$	5
29.3.103	$\frac{1}{s}\ln(1+ks)\quad(k>0)$	$E_1\left(\frac{t}{k}\right)$	5
29.3.104	$\ln\frac{s+a}{s+b}$	$\frac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\frac{1}{s}\ln(1+k^2s^2)\quad(k>0)$	$-2\ \mathrm{Ci}\left(\frac{t}{k}\right)$	5
29.3.106	$\frac{1}{s}\ln(s^2+a^2)\quad(a>0)$	$2\ln a-2\ \mathrm{Ci}\ (at)$	5

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2) \qquad (a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}(at)]$	5
29.3.108	$\ln\frac{s^2+a^2}{s^2}$		$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$		$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$		$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$		$\mathrm{Si}(kt)$	5
29.3.112	$e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	7	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	7	$\operatorname{erf}\frac{t}{2k}$	7
29.3.114	$e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k>0)$	7	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\operatorname{erf}\frac{k}{\sqrt{s}}$	7	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\operatorname{erfc}\frac{k}{\sqrt{s}}$	7	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks) \qquad (k>0)$	9	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right) \qquad (k>0)$	9	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks}I_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks) \qquad (k>0)$	9	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

$\Rightarrow \frac{1}{\sqrt{s}}[\operatorname{erfc}(0)-\operatorname{erfc}\sqrt{ks}]$

$\frac{1-1+\operatorname{erf}\sqrt{ks}}{\sqrt{s}} - \frac{\operatorname{erf}\sqrt{ks}}{\sqrt{s}}$

	$f(s)$		$F(t)$
29.3.126	$e^{as}E_1(as) \qquad (a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as) \qquad (a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as) \qquad (a>0; n=0,1,2,\ldots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$	5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms [4]

	$\phi(s)$	$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$	$\Phi(t)$
29.4.2	$e^{-ks} \qquad (k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}} \qquad (k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}} \qquad (k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks} \qquad (k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks} \qquad (k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks \qquad (k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)} \qquad (k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)} \qquad (k>0, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)} \qquad (0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^{b}u[t+h-(2n+1)k] - e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s} \qquad (0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)\,dt$, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x}\delta(t)dt=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

10. Solve Exercise 9 for every contour that does not touch the half $x \geqq 0$ of the real axis. *Ans.* $-\pi i$.

11. Note the single-valued function

$$f(z) = z^{\frac{1}{2}} = \sqrt{r}\exp\frac{i\theta}{2} \qquad \left(r > 0, -\frac{\pi}{2} \leqq \theta < \frac{3\pi}{2}\right),$$
$$f(0) = 0$$

is continuous throughout the half plane $0 \leqq \theta \leqq \pi$, $r \geqq 0$. Let C denote the entire boundary of the half disk $r \leqq 1$, $0 \leqq \theta \leqq \pi$, where C is described in the positive direction. Show that

$$\int_C f(z)\,dz = 0$$

by computing the integrals of f over the semicircle and over the two radii on the x axis. Why does the Cauchy-Goursat theorem not apply here?

12. *Nested Intervals.* An infinite sequence of closed intervals $a_n \leqq x \leqq b_n$ $(n = 0, 1, 2, \ldots)$ is determined according to some rule of selecting half intervals, so that the interval (a_1,b_1) is either the left-hand or right-hand half of a given interval (a_0,b_0); then (a_2,b_2) is one of the two halves of (a_1,b_1), and so on. Prove that there is a point x_0 which belongs to every one of the closed intervals (a_n,b_n).

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \leqq a_n \leqq a_{n+1} < b_0$; hence they have a limit A as $n \to \infty$. Show likewise that the end points b_n have a limit B; then that $B = A = x_0$.

13. *Nested Squares.* A square σ_0: $a_0 \leqq x \leqq b_0$, $c_0 \leqq y \leqq d_0$, where $b_0 - a_0 = d_0 - c_0$, is divided into four equal squares by lines parallel to the coordinate axes. One of those four smaller squares σ_1: $a_1 \leqq x \leqq b_1$, $c_1 \leqq y \leqq d_1$, where $b_1 - a_1 = d_1 - c_1$, is selected according to some rule, and it is divided into four equal squares, one of which, σ_2, is selected, etc. (Sec. 47). Prove that there is a point (x_0,y_0) which belongs to every one of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the results of Exercise 12 to each of the sequences $a_n \leqq x \leqq b_n$ and $c_n \leqq y \leqq d_n$ $(n = 0, 1, 2, \ldots)$.

51. The Cauchy Integral Formula.

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere within and on a closed contour C. If z_0 is any point interior to C, then*

analytic means no poles within & on C

$$(1) \qquad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz,$$

important

where the integral is taken in the positive sense around C.

Formula (1) is *Cauchy's integral formula.* It shows that the value of a function that is analytic in a region is determined throughout the region by its values on the boundary. Thus there is no choice of ways in which the function can be defined at points away from the boundary once the function is defined on the boundary. Every alteration of values of the function at interior points must be accompanied by a change of its values on the boundary, if the function is to remain analytic. We shall see further evidence of this *organic* character of analytic functions as we proceed.

According to the Cauchy integral formula, for example, if C is the circle $|z| = 2$ described in the positive sense, then, taking z_0 to be $-i$, we can write

$$\int_C \frac{z\,dz}{(9 - z^2)(z + i)} = 2\pi i\frac{-i}{9 - i^2} = \frac{\pi}{5},$$

since the function $f(z) = z/(9 - z^2)$ is analytic within and on C.

To prove the theorem, let C_0 be a circle about z_0,

$$|z - z_0| = r_0,$$

whose radius r_0 is small enough that C_0 is interior to C (Fig. 38). The function $f(z)/(z - z_0)$ is analytic at all points within and on C except the point z_0.

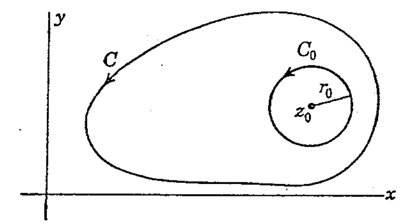


FIG. 38

Hence its integral around the boundary of the ring-shaped region between C and C_0 is zero, according to the Cauchy-Goursat theorem; that is,

$$\int_C \frac{f(z)\,dz}{z - z_0} - \int_{C_0} \frac{f(z)\,dz}{z - z_0} = 0,$$

where both integrals are taken counterclockwise.

Since the integrals around C and C_0 are equal, we can write

$$(2) \qquad \int_C \frac{f(z)\,dz}{z - z_0} = f(z_0)\int_{C_0}\frac{dz}{z - z_0} + \int_{C_0}\frac{f(z) - f(z_0)}{z - z_0}\,dz.$$

But $z - z_0 = r_0e^{i\theta}$ on C_0 and $dz = ir_0e^{i\theta}\,d\theta$, so that

$$(3) \qquad \int_{C_0}\frac{dz}{z - z_0} = i\int_0^{2\pi} d\theta = 2\pi i,$$

for every positive r_0. Also, f is continuous at the point z_0. Hence, if we select any positive number ϵ, then a positive number δ exists such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| \leqq \delta.$$

We take r_0 equal to that number δ. Then $|z - z_0| = \delta$, and

$$\left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0} dz \right| \leqq \int_{C_0} \frac{|f(z) - f(z_0)|}{|z - z_0|} |dz| < \frac{\epsilon}{\delta}(2\pi\delta) = 2\pi\epsilon.$$

The absolute value of the last integral in equation (2) can therefore be made arbitrarily small by taking r_0 sufficiently small. But since the other two integrals in that equation are independent of r_0, in view of equation (3), this one must be independent of r_0 also. Its value must therefore be zero. Equation (2) then reduces to the formula

$$\int_C \frac{f(z)\,dz}{z - z_0} = 2\pi i f(z_0),$$

and the theorem is proved.

52. Derivatives of Analytic Functions. A formula for the derivative $f'(z_0)$ can be written formally by differentiating the integral in Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz \tag{1}$$

with respect to z_0, inside the integral sign. Thus,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} dz. \tag{2}$$

important

As before, we assume that f is analytic within and on the closed contour C and that z_0 is within C. To establish formula (2), we first note that, according to (1),

$$\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i\,\Delta z_0} \int_C \left(\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0} \right) f(z)\,dz$$
$$= \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0 - \Delta z_0)(z - z_0)}.$$

The last integral approaches the integral

$$\int_C \frac{f(z)\,dz}{(z - z_0)^2}$$

as Δz_0 approaches zero; for the difference between that integral and this one reduces to

$$\Delta z_0 \int_C \frac{f(z)\,dz}{(z - z_0)^2(z - z_0 - \Delta z_0)}.$$

Let M be the maximum value of $|f(z)|$ on C and let L be the length of C. Then, if d_0 is the shortest distance from z_0 to C and if $|\Delta z_0| < d_0$, we can write

$$\left| \Delta z_0 \int_C \frac{f(z)\,dz}{(z - z_0)^2(z - z_0 - \Delta z_0)} \right| < \frac{ML|\Delta z_0|}{d_0^2(d_0 - |\Delta z_0|)},$$

and the last fraction approaches zero when Δz_0 approaches zero. Consequently,

$$\lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^2},$$

and formula (2) is established.

If we differentiate both members of equation (2) and assume that the order of differentiation with respect to z_0 and integration with respect to z can be interchanged, we find that

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^3}. \tag{3}$$

This formula can be established by the same method that was used to establish formula (2). For it follows from formula (2) that

$$2\pi i \frac{f'(z_0 + \Delta z_0) - f'(z_0)}{\Delta z_0}$$
$$= \int_C \left[\frac{1}{(z - z_0 - \Delta z_0)^2} - \frac{1}{(z - z_0)^2} \right] \frac{f(z)\,dz}{\Delta z_0}$$
$$= \int_C \frac{2(z - z_0) - \Delta z_0}{(z - z_0 - \Delta z_0)^2(z - z_0)^2} f(z)\,dz.$$

Following the same procedure that was used before, we can show that the limit of the last integral, as Δz_0 approaches zero, is

$$2 \int_C \frac{f(z)\,dz}{(z - z_0)^3},$$

and formula (3) follows at once.

We have now established the existence of the derivative of the

function f' at each point z_0 interior to the region bounded by the curve C.

We recall our definition that a function f is analytic at a point z_1 if and only if there is a neighborhood about z_1 at each point of which $f'(z)$ exists. Hence f is analytic in some neighborhood of the point. If the curve C used above is a circle $|z - z_1| = r_1$ in that neighborhood, then $f''(z)$ exists at each point inside the circle, and therefore f' is analytic at z_1. We can apply the same argument to the function f' to conclude that its derivative f'' is analytic at z_1, etc. Thus the following fundamental result is a consequence of formula (3).

Theorem. *If a function f is analytic at a point, then its derivatives of all orders, $f', f'', \ldots$, are also analytic functions at that point.*

Since f' is analytic and therefore continuous, and since

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

it follows that the partial derivatives of $u(x,y)$ and $v(x,y)$ of the first order are continuous. Since $f''(z)$ is analytic and

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} - i\frac{\partial^2 u}{\partial x\,\partial y},$$

etc., it follows that the partial derivatives of u and v of all orders are continuous functions of x and y at each point where f is analytic. This result was anticipated in Sec. 20, for the partial derivatives of the second order, in the discussion of harmonic functions.

The argument used in establishing formulas (2) and (3) can be applied successively to obtain a formula for the derivative of any given order. But mathematical induction can now be applied to establish the general formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{4}$$

important

That is, if we assume that this formula is true for any particular integer $n = k$, we can show by proceeding as before that it is true if $n = k + 1$. The details of the proof can be left to the reader, with the suggestion that in the algebraic simplifications he retain the difference $(z - z_0)$ throughout as a single term.

The closed contour C here, as well as in Cauchy's integral formula, can be replaced by *the oriented boundary B of a multiply connected closed region R* of the type described in the theorem in Sec. 49, when f is analytic in R and z_0 is any interior point of R. Our derivations of the Cauchy integral formula and its extensions (4) are still valid when C is replaced by B.

53. Morera's Theorem. In Sec. 50 we proved that the derivative of the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

exists at each point of a simply connected domain D, in fact, that

$$F'(z) = f(z).$$

We assumed there that f is analytic in D. But in our proof we used only two properties of the analytic function f, namely, that it is continuous in D and that its integral around every closed contour interior to D vanishes. Thus, when f satisfies those two conditions, the function F is analytic in D.

We proved in Sec. 52 that the derivative of every analytic function is analytic. Since $F'(z) = f(z)$, it follows that f is analytic. The following theorem, due to E. Morera (1856–1909), is therefore established.

Theorem. *If a function f is continuous throughout a simply connected domain D and if, for every closed contour C interior to D,*

$$\oint_C f(z)\,dz = 0,$$

important

then f is analytic throughout D.

Morera's theorem serves as a converse of the Cauchy-Goursat theorem.

54. Maximum Moduli of Functions. Let f be analytic at a point z_0. If C_0 denotes any one of the circles $|z - z_0| = r_0$ within and on which f is analytic, then, according to Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{z - z_0}.$$

It follows that

$$|f(z_0)| \leqq \frac{1}{2\pi r_0}\int_{C_0} |f(z)|\,|dz| = A_0, \tag{1}$$

pg 284 3. $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \qquad x>0,\ t>0$

$\frac{\partial u}{\partial x}(0,t) = -K^{-1}Q_0 = \text{Constant} \quad t>0$

$u(x,0)=0 \qquad x>0$

Using the Fourier Cosine Transform (since bc. is on <u>derivative</u> of u)

$$U(\xi;t) = \int_0^\infty u(x,t)\cos\xi x\,dx$$

$$\mathcal{F}_c\left\{\frac{\partial u}{\partial t}\right\} \to \frac{\partial U}{\partial t}(\xi;t)$$

$$\mathcal{F}_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\xi^2 U(\xi;t) - \frac{\partial}{\partial x}u(x=0+,t) = -\xi^2 U + \frac{Q_0}{K}$$

$$\therefore \frac{dU}{dt} = k\left[-\xi^2 U + \frac{Q_0}{K}\right] \quad \text{or} \quad \frac{dU}{dt} + k\xi^2 U = \frac{kQ_0}{K}$$

$$\left.\begin{array}{l} U_{homog} = Ce^{-k\xi^2 t} \\ U_{particular} = \dfrac{Q_0}{K\xi^2} \end{array}\right\} \quad U(\xi;t) = Ce^{-k\xi^2 t} + \frac{Q_0}{K\xi^2}$$

now $\mathcal{F}_c\{u(x,t=0)=0\} \longrightarrow U(\xi;t=0)=0 \qquad \therefore\ C = -\frac{Q_0}{K\xi^2}$

$$\therefore\quad U(\xi;t) = \frac{Q_0}{K\xi^2}\left[1-e^{-k\xi^2 t}\right]$$

and $u(x,t) = \frac{2}{\pi}\int_0^\infty \frac{Q_0}{K}\left[\frac{1-e^{-k\xi^2 t}}{\xi^2}\right]\cos\xi x\,d\xi = -\frac{2Q_0}{\pi K}\int_0^\infty \left(\frac{-1+e^{-kt\xi^2}}{\xi^2}\right)\cos\xi x\,d\xi$

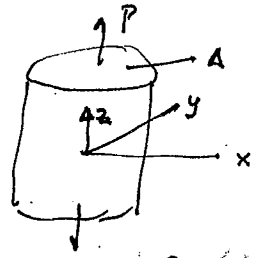
from pg 274 exercise 21 (b)

$$\mathcal{F}_c\left\{ax\,\text{erfc}(ax) - \left(\frac{1}{\sqrt{\pi}}\right)e^{-a^2x^2}\right\} = \frac{a}{\xi^2}\left(-1+e^{-\xi^2/4a^2}\right)$$

if we let $4a^2 = \frac{1}{kt}$ or $a = \frac{1}{2\sqrt{kt}}$

then $-\frac{2Q_0}{\pi K}\int_0^\infty \left(\frac{-1+e^{-kt\xi^2}}{\xi^2}\right)\cos\xi x\,d\xi = -\frac{Q_0}{K}\cdot 2\sqrt{kt}\left\{\frac{x}{2\sqrt{kt}}\,\text{erfc}\left(\frac{x}{2\sqrt{kt}}\right) - \frac{1}{\sqrt{\pi}}e^{-\frac{x^2}{4kt}}\right\}$

and $u(x,t) = -\frac{Q_0}{K}\left\{x\,\text{erfc}\left(\frac{x}{2\sqrt{kt}}\right) - 2\sqrt{\frac{kt}{\pi}}\,e^{-\frac{x^2}{4kt}}\right\}$



$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_0

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = \pm h$

Assume for this problem $\sigma_{zz} = \frac{P}{A_0}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = -\frac{\nu}{E}\frac{P}{A_0} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = -\frac{\nu P}{E A_0} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{EA_0} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_0 E}x + f(y,z)$$

$$u_y = -\frac{\nu P}{EA_0}y + g(x,z)$$

$$u_z = \frac{P}{EA_0}z + h(y,x)$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

$\Rightarrow \frac{\partial f}{\partial z} = -\frac{\partial h}{\partial x} = k(y)$

$\therefore f = k_1(y)z + k_2(y)$

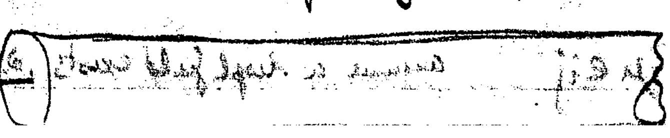
$h = -k_1(y)\cancel{x} + k_3(y)$

→ HW #1 complete and solve showing solution is/includes a rigid body rotation/translation

2-D elastostatic problems (isotropic materials)

<u>Plain strain</u>

~~Elastic solid~~ very long in 1 direction



$$e_{zz} = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = 0$$

$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$\therefore\ u_x = u_x(x,y)\ ;\ u_y = u_y(x,y)\ ;\ u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0\ ;\ \neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other cross section. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \quad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of z $\Rightarrow$ $\frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

Solution by Airy Stress fn

Define a fn ϕ $\ni$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2}\ ;\quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2}\ ;\quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) $\therefore$ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists $\ni e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

now $\sigma_{ij,i} = \lambda \delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ gradient relationship $= \lambda (u_{k,kj}) + \mu (u_{j,ii} + u_{i,ji})$

$$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii} = 0$$

now differentiate once

$$(\lambda+\mu)(u_{k,kjj}) + \mu (u_{j,iij}) = 0$$

$$\text{or} \quad (\lambda + 2\mu)(u_{k,kjj}) = 0$$

$$\text{or} \quad (\lambda+2\mu) \nabla^2 e_{kk} = 0 \Rightarrow \nabla^2 \sigma_{kk} = 0 \Rightarrow \nabla^4 \phi = 0$$

now $\sigma_{ii} = \overset{3}{\delta_{ii}} \lambda e_{kk} + 2\mu e_{ii}$ (no summation) $\Rightarrow \sigma_{kk} = (3\lambda + 2\mu) e_{kk}$ or $\nabla^2 \sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove

1/10/79

<u>Plain Strain</u> From last time

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{xy} &= -\phi_{,xy} \\ \sigma_{yy} &= \phi_{,xx} \end{aligned}$$

where ϕ is the airy stress fn

using $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$$

now take $\frac{\partial}{\partial x_j}$ $\quad (\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0$ $\quad$ since dummy indices rep $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$$\therefore \quad \nabla^2 e_{ii} = 0 \quad \text{where} \quad \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\ )_{,ii}$$

now $\sigma_{ii} = \lambda \overset{3}{\delta_{ii}} e_{kk} + 2\mu e_{ii} = (3\lambda + 2\mu) e_{ii}$

$$\therefore \quad \nabla^2 e_{ii} \Rightarrow \nabla^2 \sigma_{ii} = 0$$

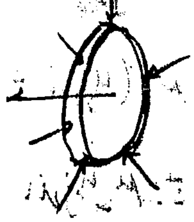
<u>Now in plane strain</u> $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $e_{zz} = 0$

$$\nabla^2 \sigma_{ii} = \nabla^2 (1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu) \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

using Airy stress fns

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2\phi) = \nabla^4\phi = 0$$

Plane Stress Plane $\Rightarrow \sigma_{zx} = \sigma_{zy} = 0$; Plane Stress also $\Rightarrow \sigma_{zz} = 0$



If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere

$$0 = \begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{zy} \end{pmatrix} \text{ on flat faces}$$

$$\sigma_{zz} = 0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$$

$$\sigma_{zx} = 0 \Rightarrow e_{zx} = 0$$

$$\sigma_{zy} = 0 \Rightarrow e_{zy} = 0$$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$, $u_y = u_y(x,y)$

Timoshenko & Goodier Rd 274-277 $\boxed{\text{HW \#16 prove that this assumption is inconsistent}}$

however we can see that u_x, u_y have z^2 component & that for $z \ll 1$ then we can assume the above w/o loss in accuracy

hence we can define generalized displ for disc w/ thickness h

$$U_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

Plane strain

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) \qquad \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \dfrac{E}{2(1+\nu)}$ $\quad \therefore e_{xx} = \dfrac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

Plane stress

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}})\right) = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

$$e_{xy} = \frac{1}{2\mu}\sigma_{xy}$$

for these two to be same then the two conditions needed are the $\overset{\text{stress}}{\epsilon_{xx} = A\sigma_{xx} + B\sigma_{yy}} = \overset{\text{strain}}{\epsilon_{xx} = C\sigma_{xx} + D\sigma_{yy}}$

$$\frac{1}{1+\nu_\sigma} = 1 - \nu_\epsilon \quad \text{and} \quad \nu_\epsilon = \frac{\nu_\sigma}{1+\nu_\sigma} \quad \Rightarrow \quad \nu_\sigma = \frac{\nu_\epsilon}{1-\nu_\epsilon}$$

(i) Given a complete plain strain soln, get the plane stress solution by leaving μ fixed and replace ν_ϵ by $\frac{\nu}{1+\nu}$

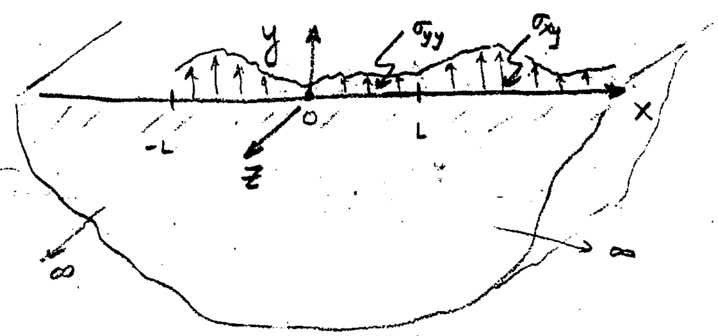
(ii) Given plane stress and want plane strain, replace ν_σ by $\frac{\nu}{1-\nu}$

We now look at

2-D problems in rectangular coordinates using Fourier Series (Timo & Good Pg 53 ff)
Rect, strips, half space

$$T_i = \sigma_{ij} n_j$$

Problem # 1



Look at plane strain problem
Traction boundary value problem
on $y=0$ $\quad \underset{\sim}{n} = \underset{\sim}{e}_y$

$T_x = \sigma_{xy}$ $\quad \sigma_{xx} i + \sigma_{xy} j + \sigma_{xz} k$

$T_y = \sigma_{yy}$ $\quad \sigma_{yx} i + \sigma_{yy} j + \sigma_{yz} k$

$T_z = \sigma_{zy} = 0$ since plain strain $\epsilon_{yz} = 0$ $\quad \sigma_{zx} i + \sigma_{zy} j + \sigma_{zz} k$

@ ∞ $\sigma \to 0$ $(y \to -\infty)$

We will assume no shear loading $\sigma_{xy} = 0$. Assume $\sigma_{yy} = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L}$

$$f(x) = \sigma_{yy}(x,0) = \sum A_n \sin \frac{n\pi x}{L} \qquad A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

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Continuing this problem again

for $|x| < \infty$, $0 > y > -\infty$ w/ Periodic loading on $y = 0 \Rightarrow \sigma_{yy}, \sigma_{xy}, \sigma_{zy}$ only loading

we look at plane strain problem i.e. $\sigma_{zy} = 0$

We now look at problem $\sigma_{yy} = \sum A_n \sin \frac{\pi x}{L}$ $\quad \sigma_{xy} = 0$

Note: next problem we will look at $\sigma_{xy} = \sum B \sin \frac{n\pi x}{L}$, $\sigma_{yy} = 0$

finally look at $T_i = \eta \sigma_{yy} + \xi \sigma_{xy}$ where η, ξ are direction cosines

We also impress $\sigma_{ij} \to 0$ as $y \to -\infty$

**** Problem given: $\sigma_{yy}(x,0) = f(x)$ $\qquad \sigma_{xy} = 0$

$\sigma_{yy}(x, y=0) = f(x) = \sum A_n \sin\frac{n\pi x}{L}$ $\qquad A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$

PDE $\nabla^4\phi = 0$ $\quad$ pick $\phi_n = g(y)\sin\frac{n\pi x}{\ell} = g(y)\sin\gamma_n x$ $\quad$ since it satisfies the form of b/c

$\therefore \quad \nabla^4\phi = \left[(\gamma_n)^4 g - 2(\gamma_n)^2 g'' + g^{iv}\right]\sin\gamma_n x = 0 \Rightarrow \gamma_n^4 g - 2\gamma_n^2 g'' + g^{iv} = 0 \quad (1)$

Take $g(y) = e^{sy} \Rightarrow (1) \Rightarrow (s^2 - \gamma_n^2)^2 = 0$

$\therefore \quad s = \pm\gamma_n, \pm\gamma_n$

$$\phi_n(x,y) = \sin\frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + C_n y e^{\gamma_n y} + D_n y e^{-\gamma_n y}\right\}$$

$\phi = \sum_{n=1}^{\infty}\phi_n \qquad \phi_0 = 0$

using b.c. that $\sigma_{ij} \to 0$ as $y \to -\infty$ pick $\beta_n = D_n = 0$

$\therefore \phi = \sum_{n=1}^{\infty}\phi_n = \sum_{n=1}^{\infty}\sin\frac{n\pi x}{\ell}\left\{\alpha_n e^{\gamma_n y} + C_n y e^{\gamma_n y}\right\} \qquad y<0$

this satisfy compat & equil (since $\nabla^4\phi = 0$ came from there) and b.c. at $-\infty$

now look at b.c. at $y=0$

since $\left.\begin{matrix}\sigma_{xy} = 0 \\ @\, y=0\end{matrix}\right\} \Longrightarrow -\phi_{,xy} = -\sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell}\cdot\left(\frac{n\pi}{\ell}\right)\left\{\alpha_n\gamma_n e^{\gamma_n y} + C_n e^{\gamma_n y} + \gamma_n C_n y e^{\gamma_n y}\right\}$

$$= -\sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell}\left(\frac{n\pi}{\ell}\right)\left\{\alpha_n\gamma_n + C_n\right\} = 0$$

$\Rightarrow \boxed{C_n = -\gamma_n\alpha_n}$

$\therefore \quad \phi = \sum_{n=1}^{\infty}\phi_n = \sum_{n=1}^{\infty}\alpha_n\sin\gamma_n x\,(1-\gamma_n y)e^{\gamma_n y}$

also $\sigma_{yy}(y=0) = \sum_1^{\infty} A_n\sin\frac{n\pi x}{\ell}$, $\quad \phi_{,xx} = -\sum_1^{\infty}\alpha_n\gamma_n^2\sin\gamma_n x\,(1-\gamma_n y)e^{\gamma_n y}$

at $y=0$ $\quad \phi_{,xx} = -\sum^{\infty}\alpha_n\gamma_n^2\sin\gamma_n x$

$\therefore \quad A_n = -\alpha_n \gamma_n^2 \qquad$ or $\quad \boxed{\alpha_n = -\frac{A_n}{\gamma_n^2}}$

$\therefore \phi(x,y) = -\sum_{n=1}^{\infty} \frac{A_n}{\gamma_n^2} \sin \gamma_n x \, (1 - \gamma_n y) e^{\gamma_n y} \qquad$ where $A_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$; $\gamma_n = \frac{n\pi}{L}$

to get displ $\quad \epsilon_{xx} = \frac{\partial u_x}{\partial x} = \frac{1}{E}\left\{\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right\} = -\frac{\nu(1+\nu)}{E}\phi_{,xx} + \frac{1-\nu^2}{E}\phi_{,yy}$

$\epsilon_{zz} = 0 = \frac{\sigma_{zz}}{E} - \frac{\nu}{E}(\sigma_{xx} + \sigma_{yy})$ or $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$

thus $u_x = \int \frac{\partial u_x}{\partial x} dx + f(y, \cancel{z})$ now since we only want $u_x = u(x,y)$ plane strain

Problem #2 $\quad$ on $y = 0 \quad \sigma_{yy} = \frac{a_0}{2} + \sum_1^{\infty} F_n \cos \frac{n\pi x}{L}$

$\sigma_{xy} = 0$

$\sigma_{yy}(x, y=0) = f(x) = \sum_1^{\infty} F_n \cos \gamma_n x + \frac{a_0}{2} \quad$ where $F_n = \frac{2}{L}\int_0^L f(x) \cos \frac{n\pi x}{L} dx$

(2a) if $F_n = 0 \qquad$ look at $\phi_0 = \alpha x^2 \qquad \nabla^4 \phi_0 = 0 \qquad$ and since $\sigma_{yy} = \frac{a_0}{2}$

$\sigma_{yy} = \frac{\partial^2 \phi_0}{\partial x^2} = 2\alpha \Rightarrow 2\alpha = \frac{a_0}{2} \qquad \alpha = \frac{a_0}{4}$

$\sigma_{xx} = 0 \quad \sigma_{xy} = 0 \quad$ since $\phi_0 \neq f(y)$

if load is periodic in direction 1 and $\nabla^4 \phi = 0$ is governing pde in a half-space then $\phi \to 0$ in direction 2, as $|x_2| \to \infty$

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Continuing problem #2

$\sigma_{xy} = T_x = 0$; $\sigma_{yy} = T_y = \frac{a_0}{2} + \sum_{n=1}^{\infty} F_n \cos \gamma_n x$; $\sigma_{yz} = T_z = 0$ on $y = 0 \qquad F_n = \frac{2}{L}\int_0^L f(x) \cos \gamma_n x \, d$

if $\phi_0 = \alpha x^2$ then $\frac{\partial^2 \phi_0}{\partial x^2} = 2\alpha \Rightarrow \alpha = \frac{a_0}{4}$

Try $\phi_n(x,y) = g_n(y) \cos \frac{n\pi x}{L} = g_n(y) \cos \gamma_n x$ (why? because satisfies b.c. $\frac{\partial^2 \phi}{\partial x^2} \sim \cos \gamma_n x$)

$\nabla^4 \phi = \cos \gamma_n x \left(\gamma_n^4 g_n - 2\gamma_n^2 g_n'' + g^{IV}\right) = 0 \Rightarrow g_n(y) = \alpha_n e^{\gamma_n y} + \beta_n e^{-\gamma_n y} + E_n y e^{\gamma_n y} + H_n y e^{-\gamma_n y}$

as $y \to -\infty \quad \sigma \to 0 \Rightarrow \beta_n, H_n \to 0$

$$\phi = \sum_{n=1}^{\infty} \{\alpha_n + E_n y\} e^{\gamma_n y} \cos\gamma_n x$$

$$\sigma_{xy} = -\frac{\partial^2\phi}{\partial x \partial y} = -\sum_1^{\infty} \{-\gamma_n \sin\gamma_n x\}\{(\alpha_n + E_n y)\gamma_n e^{\gamma_n y} + E_n e^{\gamma_n y}\}$$

$$\sigma_{xy}\big|_{y=0} = \sum_1^{\infty} \gamma_n \sin\gamma_n x \{\alpha_n\gamma_n + E_n\} = 0 \quad \forall x \Rightarrow \boxed{E_n = -\alpha_n\gamma_n}$$

$$\therefore \quad \phi = \sum_{n=1}^{\infty} \{1-\gamma_n y\}\alpha_n e^{\gamma_n y}\cos\gamma_n x$$

$$\sigma_{yy} = 0 = \frac{\partial^2\phi}{\partial x^2} = \sum_1^{\infty} (-\gamma_n^2)\{1-\gamma_n y\}\alpha_n e^{\gamma_n y}\cos\gamma_n x = \sum F_n \cos\gamma_n x$$

$$\sigma_{yy}\big|_{y=0} \Rightarrow \sum_1^{\infty} (-\gamma_n^2)\alpha_n\cos\gamma_n x = \sum F_n\cos\gamma_n x$$

take $\boxed{\alpha_n = \frac{-F_n}{\gamma_n^2}}$

$\phi_0 = \frac{a_0 x^2}{4}$ $\quad \phi \sim \cos\frac{n\pi x}{L}$ $\quad \phi \sim \sin\frac{n\pi x}{L}$

Problem 3: if $f(x) = \frac{a_0}{2} + \sum A_n\cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} = \sigma_{yy}$

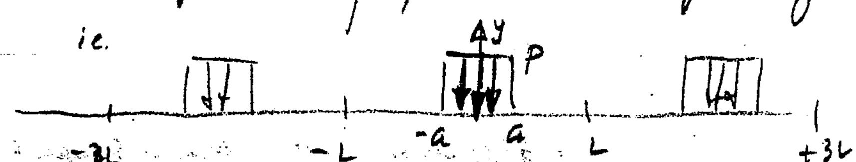
then use superposition of problem #1 & 2

if $f(x) = \frac{a_0}{2} + \sum A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L} = \sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$

take $\phi_0 = \frac{-a_0 xy}{2}$ $\quad \phi \sim \sin\frac{n\pi x}{L}$ $\quad \phi \sim \cos\frac{n\pi x}{L}$ $\quad$ since $\sigma_{xy} \sim \frac{\partial\phi}{\partial x}$ only

Consider a preliminary problem to the point fn.

i.e.



$\sigma_{yy} = -P \quad (y=0); \quad |x|<a$

$\qquad\; 0 \quad ; \quad a<|x|<L$

since this an even problem (symmetric) need only cos series

$$\sigma_{yy}(y=0) = \frac{a_0}{2} + \sum_1^\infty B_n \cos\frac{n\pi x}{L} = -P \qquad \text{since } T_y = \sigma_{yj} n_j = \sigma_{yy}(+1) = -P$$

$$\frac{a_0}{2} = \frac{1}{2L}\int_{-a}^{a}(-P)\,dx \Rightarrow \frac{2}{2L}\int_0^a -P\,dx = -\frac{Pa}{L}$$

$$B_n = \frac{1}{L}\int_{-a}^{a}(-P)\cos\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^a(-P)\cos\frac{n\pi x}{L}\,dx = \frac{-2P}{n\pi}\sin\frac{n\pi a}{L}$$

$$\therefore \quad \therefore \quad -P = -\left\{\frac{Pa}{L} + \sum_1^\infty \frac{2P}{n\pi}\sin\frac{n\pi a}{L}\cos\frac{n\pi x}{L}\right\}$$

$$\sigma_{yy}(y=0) = -\left\{\frac{2Pa}{2L} + \sum_1^\infty \frac{2Pa}{L}\frac{\sin\frac{n\pi a}{L}}{\frac{n\pi a}{L}}\cos\frac{n\pi x}{L}\right\}$$

let $2Pa \to 1$ let $P \to \infty$ $a \to 0$

$$\therefore \sigma_{yy} = -\left\{\frac{1}{2L} + \sum_{n=1}^\infty \frac{1}{L}\cos\frac{n\pi x}{L}\right\} = -\delta(x - 2mL) \qquad m = 0, 1, 2, \ldots$$

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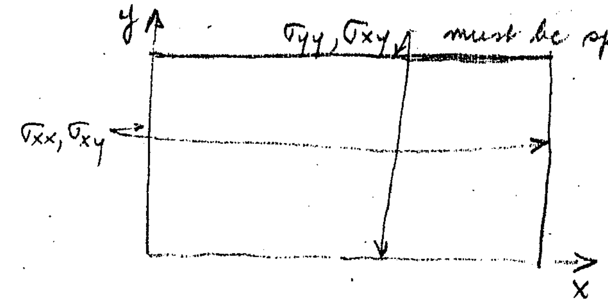
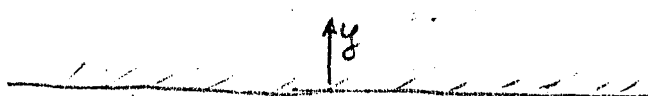
Next order of complexity in problems



P 61 & 62 in Timoshenko
Solution requires eigenfunction expansions
solutions will be of the form
$\sin\beta_n x$ where β_n are complex nos.

Aside:
Why not superposition? It can be done. No reason why not.

Fourier Analysis – Fourier Integrals.
Problem



$\sigma_{xy} = g(x)$ on $y=0$ where $g(x)$ is not periodic
$\sigma_{yy} = f(x)$ on $y=0$

complex imbedding of cosine/sine fn.   basic solutions $\ni$ $|\sigma_{ij}|\to 0$ as $y\to\infty$

try $\underset{\lambda}{\phi}(x,y) = \overbrace{e^{-i\lambda x}}\{\overbrace{Ae^{-\lambda y}+Bye^{-\lambda y}}\}$ w/ $\lambda>0$ $y>0$

then

$\nabla^4\phi = 0$ σ_{ij} are bounded as $y\to+\infty$

since this is not true for neg values of λ then to remove the restriction of $\lambda>0$ take $|\lambda|$

$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A(\lambda)e^{-|\lambda|y} + B(\lambda)ye^{-|\lambda|y}\}d\lambda$$

$$\sigma_{yy}\Big|_{y=0} = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty} e^{-i\lambda x}\{Ae^{-|\lambda|y} + Be^{-|\lambda|y}\cdot y\}d\lambda\Big|_{y=0}\cdot\lambda^2$$

$$= -\int_{-\infty}^{\infty} e^{-i\lambda x}\lambda^2\cdot A(\lambda)\,d\lambda = f(x)$$

$$[\sigma_{xy}]\Big|_{y=0} = -\frac{\partial^2\phi}{\partial x\partial y}\Big|_{y=0} = \int_{-\infty}^{\infty} -i\lambda e^{-i\lambda x}\{-A|\lambda|e^{-|\lambda|y} + Be^{-|\lambda|y} - B|\lambda|ye^{-|\lambda|y}\}d\lambda\Big|_{y=0}$$

$$= \int_{-\infty}^{\infty}\{-i\lambda\}e^{-i\lambda x}\{-|\lambda|A + B\}\Big)\,d\lambda = g(x)$$

Before continuing let's look at Fourier Transforms

1. Fourier Sine Transforms

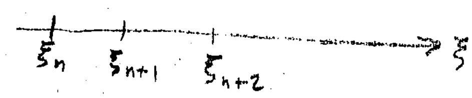
consider $f(x)$ defined on $(-L,L)$ $f(x) \overset{odd}{=} -f(-x)$ w/ period of $2L$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\ ;\ b_n = \frac{2}{L}\int_0^L f(s)\sin\frac{n\pi s}{L}\,ds$$

look at what happens as $L\to\infty$ since $\sin 0 = 0$ $\sum_{n=0}^{\infty} = \sum_{n=1}^{\infty}$

$$f(x) = \sum_{n=0}^{\infty} b_n\left(\sin\frac{n\pi x}{L}\right)\Delta n \qquad (n+1)-n = 1 = \Delta n$$

Let $\xi_n = \frac{n\pi}{L}$; $\Delta\xi_n = \frac{\pi\Delta n}{L}$



then we can write that

$$f(x) = \frac{L}{\pi}\sum_{n=0}^{\infty} b_n(\sin\xi_n x)\,\Delta\xi_n \qquad w/$$

$$b_n = \frac{2}{L}\int_0^L f(s)\sin\xi_n s\,ds$$, Put this back into $f(x)$

$$f(x) = \frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=0}^{\infty}\left\{\int_0^L f(s)\sin\xi_n s\,ds\right\}\sin\xi_n x\;\Delta\xi_n$$

as $L\to\infty$ $\quad \xi_n \to \xi$ $\qquad \xi_n = \frac{n\pi}{L}$; as $L\to\infty$ $\xi_n \to 0$ & $\Delta\xi_n = \frac{\pi\,\Delta n}{L} \to 0$

$$\sum_{n=0}^{\infty}\Delta\xi_n\,(\quad) \longrightarrow \int_0^{\infty}(\quad)\,d\xi$$

$$f(x) = \frac{2}{\pi}\int_0^{\infty} d\xi\left(\int_0^{\infty} ds\, f(s)\sin\xi s\right)\sin\xi x$$

Let $F(\xi) = \cancel{\frac{2}{\pi}}\int_0^{\infty} ds\, f(s)\sin\xi s$ then

$f(x) = \frac{2}{\pi}\int_0^{\infty} d\xi\, F(\xi)\sin\xi x$ $\qquad$ fourier sine transform pairs

$F(\xi)$ is the fourier sine transform of $f(x)$.

2. Fourier Cosine transform

Let $f(x) = f(-x)$ even fn. w/ period $2L$ on $(-L, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}\ ;\quad a_n = \frac{2}{L}\int_0^L f(s)\cos\frac{n\pi s}{L}\,ds$$

we now write this as a $\sum_{-\infty}^{\infty}$ since fn is even.

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} a_n'\cos\frac{n\pi x}{L}\ ;\quad a_0' = a_0\ ;\ a_n' = a_n\ ,\ n\geq 1$$

with $a_{-n}' = a_n' = a_n$

let $\xi_n = \frac{n\pi}{L}$ $\quad \Delta n = 1$ $\quad\therefore$ putting a_n back into the sum

[illegible]

$$f(x) = \frac{1}{2}\cdot\frac{2}{L}\cdot\frac{L}{\pi}\sum_{n=-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\;\Delta\xi_n = \frac{1}{\pi}\sum_{-\infty}^{\infty}\left\{\int_0^L f(s)\cos\xi_n s\,ds\right\}\cos\xi_n x\;\Delta\xi_n$$

as $L\to\infty$ $\quad \xi_n\to\xi \quad \sum_{-\infty}^{\infty} \longrightarrow \int_{-\infty}^{\infty}$

$$\therefore\ f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\xi\cos\xi x\int_0^{\infty} ds\cos\xi s\, f(s) = \frac{2}{\pi}\int_0^{\infty} d\xi\cos\xi x\int_0^{\infty} ds\cos\xi s\, f(s)$$

since things are symmetric

$\therefore$ if $F(\xi) = \frac{2}{\pi}\int_0^\infty ds \cos \xi s\, f(s)$

$f(x) = \int_0^\infty d\xi \cos \xi x\, F(\xi)$ for even fn.

every thing works so long as ① $\int_0^\infty |f(s)|\, ds \leq M$, ② if x_0 is a pt of discontinuity then $f(x_0) = \frac{1}{2} f(x_0^-) + \frac{1}{2} f(x_0^+)$, ③ continuous on intervals, ④ $f'(x^-)$, $f'(x^+)$ exist

3. General Fourier Transform

Let $f(x)$ be defined on $(-\infty, \infty)$; $f(x)$ neither even nor odd – then

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2} = E(x) + O(x)$$

$$E(-x) = E(x) \qquad O(-x) = -O(-x)$$

Call $R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\cos \xi x - i \sin \xi x\right)\left(E(x) + O(x)\right) dx$

since even product $\int_{-\infty}^{\infty} = 2\int_0^\infty = \pi \cdot \frac{2}{\pi}\int_0^\infty$; odd · odd = even f.

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos \xi x\, E(x)\, dx - \frac{i}{2\pi}\int_{-\infty}^{\infty} \sin \xi x\, O(x)\, dx + \frac{i}{2\pi}\int_{-\infty}^{\infty} \underbrace{\begin{matrix}(\sin \xi x)(E(x)) \\ (O(x))(\cos \xi x) \\ \text{odd} \cdot \text{even}\end{matrix}}_{\text{odd}}$$

$= \text{even} \int_{-\infty}^{\infty} = 0$

thus:

$$R(\xi) = \frac{1}{2}\left[U(\xi) + i V(\xi)\right] = \frac{1}{2}\left[U(\xi) - i V(\xi)\right]$$

with

$$U(\xi) = \frac{2}{\pi}\int_0^\infty \cos \xi x\, E(x)\, dx \quad \& \quad V(\xi) = \frac{2}{\pi}\int_0^\infty \sin \xi x\, O(x)\, dx$$

we can define

$$E(x) = \frac{2}{\pi}\int_0^\infty \cos \xi x\, U(\xi)\, d\xi \quad \& \quad O(x) = \frac{2}{\pi}\int_0^\infty \sin \xi x\, V(\xi)\, d\xi$$

Next time we will look at the proof of $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\, d\xi = f(x)$

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Recap:

$$R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx \quad ; \quad f(x) = E(x) + O(x) \qquad E(x) = E(-x), \quad O(x) = -O(-x)$$

$$= \frac{2}{2\pi}\left\{ \int_0^{\infty} \cos\xi x \, E(x)\,dx - i\int_0^{\infty} \sin\xi x \, O(x)\,dx \right\}$$

$$R(\xi) = \frac{1}{2}\left[U(\xi) - i V(\xi) \right] \quad \text{with } U(\xi) = \frac{2}{\pi}\int_0^{\infty} \cos\xi x\, E(x)\,dx, \quad V(\xi) = \frac{2}{\pi}\int_0^{\infty} \sin\xi x\, O(x)\,dx$$

$U(\xi) = U(-\xi)$ since $\cos\xi x$ is even in ξ $\qquad V(\xi) = -V(-\xi)$ since $\sin\xi x$ is odd in ξ

$$E(x) = \frac{2}{\pi}\int_0^{\infty} \cos\xi x\, U(\xi)\,d\xi \quad ; \quad O(x) = \frac{2}{\pi}\int_0^{\infty} \sin\xi x\, V(\xi)\,d\xi$$

Consider: $\displaystyle \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\,d\xi = \frac{1}{2}\int_{-\infty}^{\infty} [\cos\xi x + i\sin\xi x][U(\xi) - i V(\xi)]\,d\xi$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\cos\xi x\, U(\xi)}_{\text{even} \times \text{even} = \text{even}}\,d\xi + \frac{1}{2}\int_{-\infty}^{\infty} \underbrace{\sin\xi x\, V(\xi)}_{\text{odd} \times \text{odd} = \text{even}}\,d\xi + i\underbrace{\int_{-\infty}^{\infty} \left(V(\xi)\cdot\cos\xi x - \sin\xi x\cdot U(\xi)\right)}_{\text{odd}\cdot\text{even} = \text{odd}} \to 0$$

$$= \frac{2}{2}\int_0^{\infty} \left[\cos\xi x\, U(\xi)\,d\xi + \sin\xi x\, V(\xi)\,d\xi\right]$$

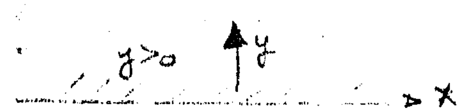
$$= \frac{2}{2}\{E(x) + O(x)\} = \frac{2}{2}\{f(x)\} = f(x)$$

$\therefore$ define $\displaystyle R(\xi) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\xi x} f(x)\,dx$ w/ $f(x) = E(x) + O(x)$

then $\displaystyle f(x) = \int_{-\infty}^{\infty} e^{+i\xi x} R(\xi)\,d\xi$

$R(\xi)$ = Fourier Transform of $f(x)$; $f(x)$ is fourier transform of $R(\xi)$

Return to Half Space problem of 1/17/79

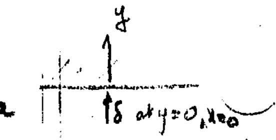


$\sigma_{yy}(y=0) = f(x)$

$\sigma_{xy}(y=0) = g(x)$

$$\phi(x,y) = \int_{-\infty}^{\infty} d\lambda\, e^{-i\lambda x}\left[A(\lambda)\,e^{-|\lambda| y} + y\,B(\lambda)\,e^{-|\lambda| y}\right]$$

Require $\displaystyle f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda \; ; \; g(x) = -i\int_{-\infty}^{\infty} \lambda e^{-i\lambda x}\{-|\lambda|\,A(\lambda) + B(\lambda)\}\,d\lambda$



look at special case where $\sigma_{xy}=0$ and σ_{yy} is the now famous Dirac Delta
take $g(x)=0$ and $\sigma_{yy}(y=0) = f(x) = -\delta(x)$ since $T_y = \sigma_{yj} n_j$ $n_y=-1$ thus
since $\delta(x) = T_y = -\sigma_{yy} \Rightarrow \sigma_{yy} = -\delta(x)$

(note to myself: that T_y the traction in $+y$ direction is $\lim_{\Delta A \to 0} \frac{\Delta F_y}{\Delta A}$ hence direction of traction in this case is in same direction as ΔF_y).

Aside: Delta fns are defined by

$$\begin{cases} (1) & \delta(x-x_0) = 0 \text{ for } x \neq x_0 \qquad \delta(x-x_0)=\infty \text{ for } x=x_0 \\ (2) & \int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1 \\ (3) & \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0) \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \sigma_{yy}(y=0)\,dx = \int_{-\infty}^{\infty} -\delta(x)\,dx = -1 = -\int_{-\infty}^{\infty} T_y\,dx$$

since $f(x) = -\delta(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A\,d\lambda$ we have to represent $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\,d\lambda$

thus using the fourier transforms $R(\lambda) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x}\,\delta(x)\,dx = \frac{1}{2\pi}\, e^{i\lambda\cdot 0} = \frac{1}{2\pi}$

thus $R(\lambda) = \frac{1}{2\pi}$ if $f(x) = \delta(x)$ thus $\delta(x) = \int_{-\infty}^{\infty} e^{-i\lambda x} R(\lambda)\,d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$

and $\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda$ is the Fourier Integral Representation of the δ Function

$$-\delta(x) = f(x) = -\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} A(\lambda)\,d\lambda = \frac{-1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\,d\lambda \Rightarrow \lambda^2 A = \frac{1}{2\pi}$$ and from $g(x)=0$

$-|\lambda| A + B = 0 \quad \therefore \quad A(\lambda) = \frac{1}{2\pi\lambda^2} \qquad B(\lambda) = |\lambda| A(\lambda) = \frac{1}{2\pi|\lambda|}$. Substituting into $\sigma_{yy}(x,y)$

$$\therefore \sigma_{yy}(x,y) = \frac{\partial^2\phi}{\partial x^2} = -\int_{-\infty}^{\infty} d\lambda\, \cancel{\lambda^2} e^{-i\lambda x}\left[\frac{1}{2\pi\cancel{\lambda^2}} e^{-|\lambda| y} + y\,\frac{|\lambda|}{2\pi\cancel{|\lambda|^2}} e^{-|\lambda| y}\right]$$

$$\sigma_{yy}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \underbrace{e^{-i\lambda x}}_{\cos + i\sin}\, \underbrace{e^{-|\lambda| y}}_{\text{even fn of } \lambda} \underbrace{\{1+|\lambda| y\}}_{\text{even as a fn of } \lambda}$$. Since only result exists if $\int_{-\infty}^{\infty}$ even fn

only non zero term is

$$= -\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda\, \cos\lambda x\, e^{-|\lambda| y}\{1+|\lambda| y\} = 2\cdot\frac{-1}{2\pi}\int_0^{\infty} \cos\lambda x\, e^{-\lambda y}\{1+\lambda y\}\,d\lambda$$

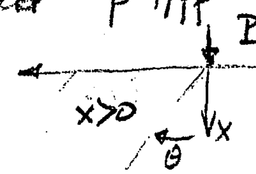
now (1): $\int_0^{\infty} d\lambda \cos\lambda x\, e^{-\lambda y} = \frac{y}{x^2+y^2}$ this is Laplace transf of $\cos\lambda x$ with $y>0$

(2): $y\int_0^\infty d\lambda \cos\lambda x\, e^{-\lambda y}\cdot\lambda$ since integration is wrt λ can take y outside integral. Notice

that $y\frac{\partial}{\partial y}(1) = -(2) = y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) \quad \therefore \quad (2) = -y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)$

$$\sigma_{yy} = -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} - y\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2}$$

Look @ Timoshenko & Goodier P 97ff His system looks like this

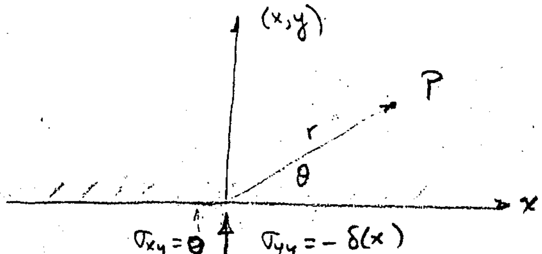


results are same as ours except w/change in x, y and mult by P

Checked out it is correct.

HW find σ_{xy}, σ_{xx} and ϕ for this problem.

1/22/79



$$\sigma_{yy}(x,y) = -\frac{2}{\pi}\frac{y^3}{(x^2+y^2)^2} = -\frac{2}{\pi}\frac{\sin^3\theta}{r}$$

For $\sigma_{yy}(x, y=0) = -f(x)$ $= f(x)\,[-\delta(x)]$ $\quad \sigma_{xy}(x, y=0) = 0$ we can get the answer based on our delta fn result. We know:

1. for a point force applied at at point $x=\xi$ on $y=0$. Then by shift of origin
$$\sigma_{yy}(x, y; \xi) = -\frac{2}{\pi}\frac{y^3}{[(x-\xi)^2+y^2]^2}$$

2. then by the principle of linear superposition with a distributed load $f(x)$
$$\sigma_{yy} = -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{y^3 f(\xi)\,d\xi}{[(x-\xi)^2+y^2]^2}$$

if $f(\xi) = \text{const} = p_0 \quad |x|\le l$

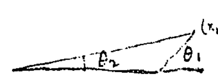
$$= -\frac{2}{\pi}p_0\int_{-l}^{l}\frac{y^3\,d\xi}{((x-\xi)^2+y^2)^2}$$

$$\sigma_{yy} = -\frac{2}{\pi}p_0\int_{x+l}^{x-l}\frac{-y^3\,du}{(u^2+y^2)^2}$$

$$= \frac{2}{\pi}p_0 y^3\left[\frac{u}{2y^2(u^2+y^2)} + \frac{1}{2y^3}\tan^{-1}\frac{u}{y}\right]$$

$$= \frac{2}{\pi}p_0\left\{\frac{y(x-l)}{2((x-l)^2+y^2)} + \frac{1}{2}\tan^{-1}\frac{(x-l)}{y} - \frac{y(x+l)}{2((x+l)^2+y^2)} - \frac{1}{2}\tan^{-1}\frac{(x+l)}{y}\right.$$

$\frac{y^3}{[(x-\xi)^2+y^2]^2}$ is the Green's fn for a half space



y

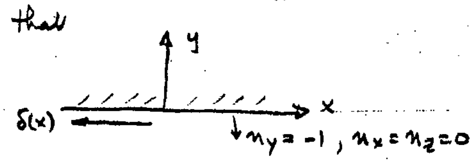
$\sin\theta_1, \cos\theta_1$

x − l

$$\sigma_{yy} = \frac{2}{\pi}p_0\left[\left\{\frac{\sin 2\theta_1}{2} + \theta_1\right\} - \left\{\frac{\sin 2\theta_2}{2} + \theta_2\right\}\right]$$

2. again let $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}\,d\lambda$

we note that



$\therefore\ T_y = \sigma_{yj}n_j = -\sigma_{yy} = 0 \qquad T_x = -\delta(x) = n_j\sigma_{xj} = -\sigma_{xy}$ thus σ_{xy} must be to the left.

Now $\dfrac{\partial\phi}{\partial x} = \int_{-\infty}^{\infty}(-i\lambda)e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$ and $\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}\{A+By\}e^{-|\lambda|y}d\lambda$

since $\sigma_{yy}\Big|_{y=0} = 0 \Rightarrow 0 = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x}A\,d\lambda$. It can be shown that $\int_{-\infty}^{\infty}\lambda^2 e^{-i\lambda x}d\lambda \neq 0 \ \therefore \underline{\underline{A\equiv 0}}$

Now $-\dfrac{\partial}{\partial y}\left(\dfrac{\partial\phi}{\partial x}\right) = +\sigma_{xy} = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}Be^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

But since $\sigma_{xy} = \delta(x) = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}d\lambda = i\int_{-\infty}^{\infty}\lambda e^{-i\lambda x}B\,d\lambda \Rightarrow Bi\lambda = \dfrac{1}{2\pi}$ or $\underline{\underline{B = \dfrac{1}{2\pi i\lambda}}}$

hence $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\dfrac{y}{2\pi i\lambda}e^{-|\lambda|y}d\lambda$; using all this we have

$\sigma_{yy} = \dfrac{\partial^2\phi}{\partial x^2} = \int_{-\infty}^{\infty}\dfrac{-\lambda y}{2\pi i}e^{-i\lambda x}e^{-|\lambda|y}d\lambda$, $\sigma_{xy} = -\dfrac{\partial^2\phi}{\partial x\partial y} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\lambda x}e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda$

Now since $\dfrac{\partial\phi}{\partial y} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{e^{-i\lambda x}}{\lambda}[1-|\lambda|y]e^{-|\lambda|y}d\lambda$ thus $\dfrac{\partial^2\phi}{\partial y^2} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty} -\dfrac{|\lambda|}{\lambda}(2-|\lambda|y)e^{-i\lambda x-|\lambda|y}d\lambda$

using the even/odd argument we then obtain:

a. $\sigma_{yy} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}-\lambda(-i\sin\lambda x)\,y\,e^{-|\lambda|y}d\lambda = \dfrac{y}{\pi}\int_0^{\infty}\lambda\sin\lambda x\,e^{-\lambda y}d\lambda = \dfrac{y}{\pi}\dfrac{-\partial}{\partial x}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right)$

$= \dfrac{-y}{\pi}\dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{-y}{\pi}\left[\dfrac{-2yx}{(x^2+y^2)^2}\right] = \underline{\underline{\dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2} = \sigma_{yy}}}$

b. $\sigma_{xy} = \dfrac{1}{2\pi}\int_{-\infty}^{\infty}\cos\lambda x\,e^{-|\lambda|y}\{1-|\lambda|y\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}[1-\lambda y]d\lambda = \dfrac{1}{\pi}\left[\dfrac{y}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}dy\right)$

$= \dfrac{1}{\pi}\left(\dfrac{y}{x^2+y^2}\right) + \dfrac{y}{\pi}\dfrac{\partial}{\partial y}\left(\dfrac{y}{x^2+y^2}\right) = \underline{\underline{\dfrac{2}{\pi}\dfrac{x^2y}{(x^2+y^2)^2} = \sigma_{xy}}}$

c. $\sigma_{xx} = \dfrac{1}{2\pi i}\int_{-\infty}^{\infty}\dfrac{-i\sin\lambda x}{\lambda}\left\{-|\lambda|e^{-|\lambda|y}[2-|\lambda|y]\right\}d\lambda = \dfrac{1}{\pi}\int_0^{\infty}\sin\lambda x\,e^{-\lambda y}(2-\lambda y)\,d\lambda$

$= \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\int_0^{\infty}\cos\lambda x\,e^{-\lambda y}d\lambda\right] = \dfrac{2}{\pi}\left[\dfrac{x}{x^2+y^2}\right] + \dfrac{y}{\pi}\dfrac{\partial}{\partial x}\left[\dfrac{y}{x^2+y^2}\right] = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}$

or $\underline{\underline{\sigma_{xx} = \dfrac{2}{\pi}\dfrac{x^3}{(x^2+y^2)^2}}}$

d. To obtain ϕ :

$\dfrac{\partial^2\phi}{\partial x^2} = \sigma_{yy} = \dfrac{2}{\pi}\dfrac{y^2x}{(x^2+y^2)^2} \Rightarrow \dfrac{\partial\phi}{\partial x} = \dfrac{-y^2}{\pi}\dfrac{1}{x^2+y^2} + \hat{f}_1(y) \Rightarrow \phi = \dfrac{-y}{\pi}\arctan\dfrac{x}{y} + x\hat{f}_1(y) + \hat{f}_2(y)$

$= y/\pi\,\arctan y/x + x\hat{f}_1(y) + \hat{f}_2(y)$

Florida International University
Department of Mechanical and Materials Engineering

EGM 5315 EXAMINATION 2 Nov 19, 2020

This examination will be a take-home exam due Nov 24 at 2pm. You may use your notes and your book only. You may not discuss the exam with any student, faculty member, visiting or adjunct faculty or seek any outside help. You must upload your signed exam and solution to the exam on canvas. Ensure that your document is either in pdf or word form.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of these instructions will result in failure of the exam.

____________________ ____________________

PRINT NAME SIGN NAME

Problem 1a (25 pts).
Suppose that a rod is such that heat escapes from the lateral boundary, the surfaces perpendicular to the x direction, according to Newton's law of cooling, so that T(x,t) satisfies the equation

$$\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\beta(T-T_o)$$

If $T(x,t=0)=\cos\pi x/L$ and $T(x=0,t)=T_1$ and $T(x=L,t)=T_2$, find the total solution of the problem by the SOV method. Please take β, T_o, T_1, and T_2 as constants.

Problem 2. (25 pts)
Solve the following boundary value problem by Laplace Transforms

$$U_{,tt}=a^2U_{,xx} \text{ in the region } x>0,\ t>0$$

With the following ICs:

$$U(x,0)=0 \text{ and } U_{,t}(x,0)=e^{-x} \text{ when } x>0$$

And the following BCs:

$$U_{,x}(0,t)=U(0,t) \text{ and } U(x,t)\to 0 \text{ as } x\to\infty$$

Problem 3. (25 pts)
Solve the following using a Fourier transform method.

$$U_{,t} = U_{,xx} \text{ in the region } \quad x>0, \quad t>0$$

With the following BCs: $\partial U(0,t)/\partial x = q$, q being a constant, when $t>0$ and
$|U(x,t)|$ is bounded as long as $x>0$ and $t>0$

And the following IC: $U(x,0) = 1$ when $0<x<c$ and is $=0$ for $x>c$

Hint, to transform the IC, put the definition of the IC in your transform to determine its transform function.

Problem 4. (25 points)
For the problem of the stresses on the half space, where $\sigma_{xy}=\delta(x)$ and $\sigma_{yy}=0$ on $y=0$,

Determine the expression for the Airy stress function
Determine the stress σ_{yy} and σ_{xx} everywhere in the half space.
Assume the problem is plane strain, determine the strain ε_{yy} in the half space and also on the surface, $y=0$.
Show your work for each step.

①

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \beta(T - T_0) \qquad \text{let } \hat{T}_p \text{ be a fn of } x \text{ only}$$

$$\therefore \quad \frac{d^2\hat{T}}{dx^2} + \beta\hat{T} = \beta T_0$$

$$\text{let } \hat{T} = A\cos\sqrt{\beta}x + B\sin\sqrt{\beta}x + T_0$$

$$\text{let } \hat{T}(x=0,t) = T_1 = A + T_0 \qquad A = T_1 - T_0$$

$$\hat{T}(x=L,t) = T_2 = A\cos\sqrt{\beta}L + B\sin\sqrt{\beta}L + T_0$$

$$\therefore \quad \frac{T_2 - T_0 - (T_1 - T_0)\cos\sqrt{\beta}L}{\sin\sqrt{\beta}L} = B$$

$$\therefore \quad \hat{T} = (T_1 - T_0)\cos\sqrt{\beta}x + \frac{(T_2 - T_0) - (T_1 - T_0)\cos\sqrt{\beta}L}{\sin\sqrt{\beta}L}\sin\sqrt{\beta}x + T_0$$

We have taken care of the inhomogeneous BC & inhomogeneous PDE with $\hat{T}$

Now let $T = \hat{T} + \tilde{T}$. What equation does $\tilde{T}$ solve?

$$\frac{\partial T}{\partial t} = \cancel{\frac{\partial \hat{T}}{\partial t}} + \frac{\partial \tilde{T}}{\partial t}$$

$$=$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2 \hat{T}}{\partial x^2} + \frac{\partial^2 \tilde{T}}{\partial x^2} = \cancel{-\beta(\hat{T} - T_0)} + \frac{\partial^2 \tilde{T}}{\partial x^2}$$

$$+$$

$$\beta(T - T_0) = \cancel{\beta(\hat{T} - T_0)} + \beta\tilde{T}$$

$$\text{Meaning} \quad \frac{\partial \tilde{T}}{\partial t} = \frac{\partial^2 \tilde{T}}{\partial x^2} + \beta\tilde{T}$$

$$\text{also} \quad T(x=0,t) = \hat{T}(x=0,t) + \tilde{T}(x=0,t) = T_1 + \tilde{T}(x=0,t) = T_1 \Rightarrow \tilde{T}(x=0,t) = 0$$

$$T(x=L,t) = \hat{T}(x=L,t) + \tilde{T}(x=L,t) = T_2 + \tilde{T}(x=L,t) = T_2 \Rightarrow \tilde{T}(x=L,t) = 0$$

$$T(x,t=0) = \hat{T}(x,t=0) + \tilde{T}(x,t=0) = \hat{T} + \tilde{T}(x,t=0) = \cancel{\sin\beta x}\ \cos\frac{\pi x}{L}$$

$$\tilde{T}(x,t=0) = \cancel{\sin\beta x}\ \cos\pi x/L + \frac{(T_1 - T_0)\sin\sqrt{\beta}(L-x) + (T_2 - T_0)\sin\sqrt{\beta}x}{\sin\sqrt{\beta}L}$$

$$\text{Now let } \tilde{T} = \mathcal{X}(x)F(t) \qquad F'\mathcal{X} = F\mathcal{X}'' + \beta F\mathcal{X} \quad \text{or} \quad \frac{F'}{F} = \frac{\mathcal{X}''}{\mathcal{X}} + \beta = -\omega^2$$

$$\text{let } F' + \omega^2 F = 0 \quad F(t) = e^{-\omega^2 t}; \quad \mathcal{X}'' + \underbrace{(\omega^2 + \beta)}_{\gamma^2}\mathcal{X} = 0 \quad \mathcal{X} = C\sin\gamma x + D\cos\gamma x$$

[illegible]

[illegible]

[illegible]

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[illegible]

[illegible]

[illegible]

[illegible]

since $\tilde{T}(x=0,t)=0 \Rightarrow X(0)=0$ or $D=0$

since $\tilde{T}(x=L,t)=0 \Rightarrow X(L)=0$ or $C=0$ or $\gamma L = n\pi$ $\quad \gamma = \frac{n\pi}{L}$

$\therefore$

$$\tilde{T} = \sum e^{-\omega^2 t}\cdot C_n \sin\frac{n\pi x}{L} \qquad \& \ \omega^2 = \gamma^2 - \beta = \frac{n^2\pi^2}{L^2} - \beta$$

$$\tilde{T}(x,t=0) = \sum_n C_n \sin\frac{n\pi x}{L} = \cancel{\sin\beta x}\ \cos\pi x/L + \hat{T} \qquad \text{so } C_n = \frac{2}{L}\int_0^L \left(\cancel{\sin\beta x}\ \cos\pi x/L + \hat{T}\right)\sin\frac{n\pi x}{L}\,dx$$

and $T(x,t) = \tilde{T} + \hat{T}$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

$s^2\tilde U - s\cdot 0 - e^{-x} = a^2\tilde U''$

$a^2\tilde U'' - s^2\tilde U = -e^{-x}$

$\tilde U'' - \frac{s^2}{a^2}\tilde U = \frac{-e^{-x}}{a^2}$ $\tilde U_h = C_1 e^{s/a\,x} + C_2 e^{-s/a\,x}$

$U_p = Ae^{-x}$ $U_p' = -Ae^{-x}$ $U_p'' = Ae^{-x}$

$Ae^{-x} - \frac{s^2}{a^2}Ae^{-x} = \frac{-e^{-x}}{a^2}$

$A\left(1-\frac{s^2}{a^2}\right)e^{-x} = \frac{-e^{-x}}{a^2}$

$A\left(\frac{a^2-s^2}{a^2}\right)e^{-x} = -\frac{1}{a^2}e^{-x}$ $A = \frac{1}{s^2-a^2}$ & $U_p = \frac{e^{-x}}{s^2-a^2}$

$A\left(\frac{a^2-s^2}{a^2}\right) = -\frac{1}{a^2}$

$\tilde U = C_1 e^{+s/a\,x} + C_2 e^{-s/a\,x} + e^{-x}/(s^2-a^2)$

$U \to 0$ as $x\to\infty \Rightarrow \tilde U \to 0$ as $x\to\infty \Rightarrow C_1 = 0$

$\tilde U = C_2 e^{-s/a\,x} + e^{-x}/(s^2-a^2)$

$\tilde U' = \tilde U$ @ $x=0$

$\tilde U' = -\frac{s}{a}C_2 e^{-s/a\,x} - \frac{e^{-x}}{s^2-a^2} = -\frac{s}{a}C_2 - \frac{1}{s^2-a^2}$

$U = C_2 e^{-s/a\,x} + e^{-x}/(s^2-a^2) = C_2 + \frac{1}{s^2-a^2}$

so $-C_2\left(1+\frac{s}{a}\right) = \frac{2}{s^2-a^2}$

$-C_2\left(\frac{a+s}{a}\right) = \frac{2}{s^2-a^2}$

$C_2 = \frac{-2a}{(s+a)^2(s-a)}$

$\tilde U = \frac{-2a}{(s+a)^2(s-a)}e^{-s/a\,x} + \frac{e^{-x}}{s^2-a^2}$

$\frac{-2a}{(s+a)^2(s-a)} = \frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{(s+a)^2}$

$-2a = A(s+a)^2 + B(s^2-a^2) + (Cs+D)(s-a)$

$= A(s^2+2as+a^2) + B(s^2-a^2) + Cs^2 - Cas + Ds - Da$

$\Rightarrow A+B+C=0$ s^2

$-Ca + 2aA + D = 0$ s

$A - B = 0$ a^2

$\cancel{C}-D = -2$ a

$$\begin{bmatrix}1&1&1&0\\2a&0&0&1\\1&-1&0&0\\0&0&+1&+1\end{bmatrix}\begin{pmatrix}A\\B\\C\\D\end{pmatrix}=\begin{pmatrix}0\\0\\0\\+2\end{pmatrix}$$

$$\begin{bmatrix}1&1&1&0\\0&-2a&-2a&1\\0&-2&-1&1\\0&0&1&1\end{bmatrix}\begin{pmatrix}0\\0\\0\\2\end{pmatrix}$$

$$\begin{bmatrix}1&1&1&0\\0&-2a&-2a&1\\0&0&1&-1/a\\0&0&&1+1/a\end{bmatrix}\begin{pmatrix}0\\0\\0\\2\end{pmatrix}$$

$D = \frac{2a}{a+1}$

$C - \frac{1}{a}\left(\frac{2a}{a+1}\right) = 0$

$C = \frac{2}{a+1}$

$-2aB - \frac{2a\cdot 2}{a+1} + \frac{1\cdot 2a}{a+1}$

$B = -\frac{1}{a+1}$

$A - \frac{1}{a+1} + \frac{2}{a+1}$

$A = -\frac{1}{a+1}$

$\frac{A}{s-a} + \frac{B}{s+a} + \frac{C}{(s+a)^2}$

$A(s+a)^2 + B(s^2-a^2) + C(s-a)$

$(s^2+2as+a^2)$

$A+B=0$ s^2

$2aA + C = 0$ s

$Aa^2 - Ba^2 - Ca = -2a$

$C = -2aA$

$B = -A$ $2aA(a+1) = -2a$

$A\cancel{a^2} + A\cancel{a^2} + 2aA = -2a$

$A = \frac{-1}{a+1}$ $C = \frac{+2a}{a+1}$ $B = \frac{+1}{a+1}$

$\frac{-1}{a+1}\frac{1}{s-a}$ $\frac{+1}{a+1}\frac{1}{s+a}$ $\frac{+2a}{(a+1)(s+a)^2}$

$\frac{-1}{a+1}\cdot\frac{1}{s-a} - \frac{1}{a+1}\cdot\frac{1}{s+a} + \left(\frac{2}{a+1}s + \frac{2a}{a+1}\right)\cdot\frac{1}{(s+a)^2}$

$\frac{2(s+a)}{(a+1)}\cdot\frac{1}{(s+a)^2}$

$\frac{-1}{(a+1)}\cdot\frac{1}{(s-a)} + \frac{1}{(a+1)(s+a)}$

$\frac{1}{a+1}\cdot\left[\frac{-1}{s-a} + \frac{1}{s+a}\right]$

$\left(\frac{1}{a+1}\right)\left[\frac{-2a}{s^2-a^2}\right]$

$\frac{1}{a+1}\left[\frac{-1}{s-a} + \frac{1}{s+a}\right]$

$$\frac{-2a}{(s+a)^2(s-a)} = \frac{A}{s-a} + \frac{B}{s+a} + \frac{C}{(s+a)^2}$$

$$A(s^2+2as+a^2) + B(s^2-a^2) + C(s-a) = -2a$$

$A+B=0 \quad s^2$

$2aA+C=0 \quad s$

$Aa^2-Ba^2-Ca=-2a$

$B=-A$

$C=-2aA$

$Aa^2+Aa^2+2a^2A=-2a$

$4Aa^2=-2a$

$A=\frac{-2a}{4a^2}=\frac{-1}{2a}$

$B=\frac{1}{2a}$

$C=1$

$$\left[-\frac{1}{2a}\cdot\frac{1}{s-a} + \frac{1}{2a}\cdot\frac{1}{s+a} + \frac{1}{(s+a)^2}\right]e^{-s/a\,x} + \frac{e^{-x}}{s^2-a^2}$$

$\frac{1}{a}\sinh at\; e^{-x}$

$$-\frac{1}{2a}\left[\frac{1}{s-a} - \frac{1}{s+a}\right] = -\frac{1}{s^2-a^2}e^{-s/a\,x^2}$$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{1}{a}\sinh at$

$\mathcal{L}^{-1}\left\{e^{-bs}f(s)\right\} = F(t-b)\,u(t-b)$

$b=x/a$

$F(t-x/a)\;u(t-x/a)$

$\frac{1}{a}\sinh(at-x)\;u(t-x/a)$

$$\frac{A}{s-a} + \frac{B}{s+a} + \frac{Cs+D}{(s+a)^2}$$

$$A(s^2+2as+a^2) + B(s^2-a^2) + (Cs+D)(s-a)$$

$A+B+C=0 \quad s^2$

$2Aa - Ca + D = 0 \quad s$

$Aa^2 - Ba^2 - Da = -2a$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 2a & 0 & -a & 1 \\ a^2 & -a^2 & 0 & -a \end{bmatrix}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -2a \end{pmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2a & -3a & 1 \\ 0 & -2a^2 & -a^2 & -a \end{bmatrix} \quad \begin{matrix} 0 \\ 0 \\ -2a \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2a & -3a & 1 \\ 0 & 0 & +2a^2 & -2a \end{bmatrix}\begin{matrix} A \\ B \\ C \\ D \end{matrix}\begin{bmatrix} 0 \\ 0 \\ -2a \end{bmatrix}$$

$\cancel{-\frac{1}{2}a - \frac{3}{2}a + Da - Da}$

$2a\left(-\frac{1}{2a}\right) - a\left(-\frac{1}{a}+\frac{D}{a}\right) + D \stackrel{?}{=} 0$

$-1 + 1 - D + D \stackrel{\checkmark}{=} 0$

$a^2\left(-\frac{1}{2a}\right) - a^2\left(\frac{3}{2a}-\frac{D}{a}\right) - Da$

$-\frac{1}{2}a - \frac{3}{2}a + Da - Da = -2a$ ✓

$-\frac{1}{2a} + \frac{3}{2a} - \frac{D}{a} - \frac{2}{2a} + \frac{D}{a} = 0$ ✓

A	$-\frac{1}{2a}$	
B	$3/2a$	$-\frac{D}{a}$
C	$-\frac{1}{a}$	$+D/a$
D	0	$+D$

A	$-5/4a$	$+D/2a$
B	$3/4a$	$+D/2a$
C	$1/2a$	$+0$
D	0	$+D$

$-4a^2C = -2a \qquad C = +\frac{1}{2a}$

$2aB - 3a\cdot\frac{1}{2a} + D = 0$

$2aB = +3/2a - \frac{D}{2a}$

$\therefore$

$A + \frac{3}{4a} - \frac{D}{2a} + \frac{2}{4a} = 0$

$A = -\frac{5}{4a} + \frac{D}{2a}$

$2a^2C - 2aD = -2a$

$C = \frac{-2a+2aD}{2a^2}$

$\boxed{C = -\frac{1}{a} + \frac{D}{a}}$

$-2aB - 3a\left(-\frac{1}{a}+\frac{D}{a}\right) + D = 0$

$-2aB + 3 - 3D + D = 0$

$\boxed{B = \frac{-3+2D}{-2a} = \frac{3}{2a} - \frac{D}{a}}$

$0 = A + \frac{3}{2a} - \frac{D}{a} - \frac{2}{2a} + \frac{D}{a} \Rightarrow A + \frac{1}{2a} = 0 \qquad A = -\frac{1}{2a}$

$\mathcal{F}_c$ $\quad \frac{\partial \tilde{U}}{\partial t} = -\omega^2 \tilde{U} - q$

$\tilde{U}' + \omega^2 U = -q \qquad \tilde{U}_h = Ce^{-\omega^2 t}$

$U_p = \frac{-q}{\omega^2} \qquad \Big\} \; \tilde{U}_{tot} = Ce^{-\omega^2 t} \frac{-q}{\omega^2}$

$\tilde{U}(x, t=0) = \begin{matrix} 1 & x<c \\ 0 & x>c \end{matrix} \qquad \tilde{U} = \int_0^c 1 \cos \omega x \, dx = \frac{\sin \omega x}{\omega}\Big|_0^c = \frac{\sin \omega c}{\omega}$

$\therefore \quad \tilde{U}(x, t=0) = C - q/\omega^2 = \frac{\sin \omega c}{\omega}$

$C = \frac{\sin \omega c}{\omega} + q/\omega^2$

$\tilde{U} = \frac{q}{\omega^2}\left[-1 + e^{-\omega^2 t}\right] + \frac{\sin \omega c}{\omega} e^{-\omega^2 t}$

$\cancel{\frac{-q}{2\sqrt{t}} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right)}$

$U = \frac{1}{2\pi}\int_0^\infty \left\{\frac{q}{\omega^2}[-1 + e^{-\omega^2 t}] + \frac{\sin \omega c}{\omega} e^{-\omega^2 t}\right\} \cos \omega x \, d\omega \qquad t = \frac{1}{4a^2} \quad \frac{1}{2\sqrt{t}} = a$

$\frac{q}{2\sqrt{t}}\left\{\frac{x}{2\sqrt{t}} \operatorname{erfc}\left(\frac{x}{2\sqrt{t}}\right) + \int_0^t [H(\bar{t}) - H(\bar{t} - c)] \frac{\sqrt{\pi}}{4(t-\bar{t})} e^{-\frac{x^2}{4(t-\bar{t})}} d\bar{t}\right. \qquad \int \frac{e^{-\lambda^2/4k}}{\sqrt{\pi k}} \cos \lambda x \, d\lambda = e^{-kx^2}$

p 255

$\int_0^\infty e^{-\lambda^2/4k} \cos \lambda x \, d\lambda = \sqrt{\pi k}\, e^{-kx^2}$

p. 255 $\qquad$ let $\lambda = \omega$

$\frac{1}{4k} = t \qquad k = \frac{1}{4t}$

$\frac{2}{\pi}\int_0^\infty e^{-\omega^2 t} \cos \omega x \, d\omega = \frac{2}{\pi}\sqrt{\frac{\pi}{4t}}\, e^{-x^2/4t}$

$\mathcal{F}_c \frac{\sqrt{\pi}}{\sqrt{4t}} \frac{1}{2t\sqrt{\pi}} e^{-x^2/4t} = e^{-\omega^2 k}$

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Problem Set #2

1. From the lecture we had

a. $$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\{A e^{-|\lambda| y} + B y e^{-|\lambda| y}\} d\lambda \quad \text{with } A = \frac{1}{2\pi\lambda^2} \quad B = \frac{1}{2\pi|\lambda|}$$

$\therefore \phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x}\left\{\frac{1}{2\pi\lambda^2} e^{-|\lambda|y} + \frac{1}{2\pi|\lambda|} y e^{-|\lambda|y}\right\} d\lambda$; By differentiating we obtain

$$\frac{\partial\phi}{\partial y} = -\frac{1}{2\pi}\int_{-\infty}^{\infty} y e^{-i\lambda x - |\lambda| y} d\lambda \; ; \quad \frac{\partial^2\phi}{\partial x \partial y} = \frac{i}{2\pi}\int_{-\infty}^{\infty} \lambda y e^{-i\lambda x - |\lambda| y} d\lambda \; ;$$

thus $\sigma_{xy} = -\frac{\partial^2\phi}{\partial x\partial y} = -\frac{i}{2\pi} y \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} e^{-|\lambda| y} d\lambda$. The only term that will not be zero in the integration is

$$\frac{-i}{2\pi} y \int_{-\infty}^{\infty} (-i \sin\lambda x)\lambda e^{-|\lambda| y} d\lambda = \frac{-y}{2\pi}\int_{-\infty}^{\infty} \lambda \sin\lambda x \, e^{-|\lambda| y} d\lambda = \frac{-y}{\pi}\int_0^{\infty} \lambda \sin\lambda x \, e^{-\lambda y} d\lambda$$

Note that $\frac{\partial}{\partial x}\int_0^{\infty} \cos\lambda x \, e^{-\lambda y} d\lambda = -\int_0^{\infty} \lambda \sin\lambda x \, e^{-\lambda y} d\lambda = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \frac{-2yx}{(x^2+y^2)^2}$

$$\therefore \quad \frac{y}{\pi}\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) = \underline{\underline{-\frac{2}{\pi}\frac{y^2 x}{(x^2+y^2)^2} = \sigma_{xy}}}$$

b. Since $\frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda$. The only term that will not be zero in the integration is

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \cos\lambda x \{1 - |\lambda| y\} e^{-|\lambda| y} d\lambda = -\frac{1}{\pi}\int_0^{\infty} \cos\lambda x \{1 - \lambda y\} e^{-\lambda y} d\lambda$$

$$= -\frac{1}{\pi}\left\{\frac{y}{x^2+y^2} + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\} = \underline{\underline{-\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} = \sigma_{xx}}}$$

here we used $\int_0^{\infty} \cos\lambda x (-\lambda y e^{-\lambda y}) d\lambda = y\frac{\partial}{\partial y}\int_0^{\infty} \cos\lambda x \, e^{-\lambda y} d\lambda$

c. Since

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = -\frac{2}{\pi}\frac{x^2 y}{(x^2+y^2)^2} \Rightarrow \frac{\partial\phi}{\partial y} = \frac{x^2}{\pi}\frac{1}{(x^2+y^2)} + \hat{f}_1(x) \Rightarrow \phi = \frac{x}{\pi}\arctan\frac{y}{x} + \hat{f}_1(x) y + \hat{f}_2(x)$$

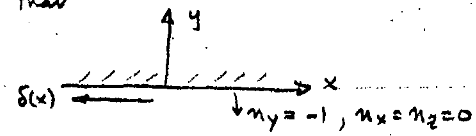
now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right) = \frac{2xy^2}{\pi(x^2+y^2)^2} + \hat{f}_1'(x) = -\sigma_{xy} \Rightarrow \hat{f}_1'(x) = 0$ or $\underline{\hat{f}_1(x) = c_1}$

now $\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial}{\partial x}\left[\frac{1}{\pi}\arctan y/x - \frac{1}{\pi}\frac{yx}{x^2+y^2} + \hat{f}_2'\right] = \frac{-2y^3}{\pi(x^2+y^2)^2} + \hat{f}_2'' = \sigma_{yy} \Rightarrow \hat{f}_2''(x) = 0$

or $\hat{f}_2(x) = c_2 x + c_3 \quad \therefore \quad \underline{\underline{\phi(x,y) = \frac{x}{\pi}\arctan\frac{y}{x} + c_1 y + c_2 x + c_3}}$; the last three terms don't play a role in defining the stresses $\Rightarrow$ we can take $c_1 = c_2 = c_3 = 0$ if we wish

2. again let $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x} \{A + By\} e^{-|\lambda| y} d\lambda$

we note that



$\therefore$ $T_y = \sigma_{yj} n_j = -\sigma_{yy} = 0$ $\quad$ $T_x = -\delta(x) = n_j \sigma_{xj} = -\sigma_{xy}$ thus σ_{xy} must be to the left.

Now $\frac{\partial \phi}{\partial x} = \int_{-\infty}^{\infty} (-i\lambda) e^{-i\lambda x} \{A + By\} e^{-|\lambda| y} d\lambda$ and $\frac{\partial^2 \phi}{\partial x^2} = \sigma_{yy} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} \{A + By\} e^{-|\lambda| y} d\lambda$

since $\sigma_{yy}\Big|_{y=0} = 0 \Rightarrow 0 = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} A \, d\lambda$. It can be shown that $\int_{-\infty}^{\infty} \lambda^2 e^{-i\lambda x} d\lambda \neq 0$ $\therefore \underline{\underline{A \equiv 0}}$

Now $-\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = +\sigma_{xy} = i \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} B e^{-|\lambda| y} \{1 - |\lambda| y\} d\lambda$

But since $\sigma_{xy} = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} d\lambda = i \int_{-\infty}^{\infty} \lambda e^{-i\lambda x} B \, d\lambda \Rightarrow Bi\lambda = \frac{1}{2\pi}$ or $\underline{\underline{B = \frac{1}{2\pi i \lambda}}}$

hence $\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x} \frac{y}{2\pi i \lambda} e^{-|\lambda| y} d\lambda$; using all this we have

$\sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} = \int_{-\infty}^{\infty} \frac{-\lambda y}{2\pi i} e^{-i\lambda x} e^{-|\lambda| y} d\lambda$; $\sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} e^{-|\lambda| y} \{1 - |\lambda| y\} d\lambda$

Now since $\frac{\partial \phi}{\partial y} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{-i\lambda x}}{\lambda} [1 - |\lambda| y] e^{-|\lambda| y} d\lambda$ thus $\frac{\partial^2 \phi}{\partial y^2} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} -\frac{|\lambda|}{\lambda} (2 - |\lambda| y) e^{-i\lambda x - |\lambda| y} d\lambda$

using the even/odd argument we then obtain:

a. $\sigma_{yy} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} -\lambda(-i \sin \lambda x) y e^{-|\lambda| y} d\lambda = \frac{y}{\pi} \int_0^{\infty} \lambda \sin \lambda x \, e^{-\lambda y} d\lambda = \frac{y}{\pi} \frac{-\partial}{\partial x}\left(\int_0^{\infty} \cos \lambda x \, e^{-\lambda y} d\lambda\right)$

$= \frac{-y}{\pi} \frac{\partial}{\partial x}\left(\frac{y}{x^2 + y^2}\right) = \frac{-y}{\pi}\left[\frac{-2yx}{(x^2+y^2)^2}\right] = \underline{\underline{\frac{2}{\pi} \frac{y^2 x}{(x^2+y^2)^2} = \sigma_{yy}}}$

b. $\sigma_{xy} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \cos \lambda x \, e^{-|\lambda| y} \{1 - |\lambda| y\} d\lambda = \frac{1}{\pi} \int_0^{\infty} \cos \lambda x \, e^{-\lambda y} [1 - \lambda y] d\lambda = \frac{1}{\pi}\left[\frac{y}{x^2+y^2}\right] + \frac{y}{\pi} \frac{\partial}{\partial y}\left(\int_0^{\infty} \cos \lambda x \, e^{-\lambda y} d\lambda\right)$

$= \frac{1}{\pi}\left(\frac{y}{x^2+y^2}\right) + \frac{y}{\pi} \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right) = \underline{\underline{\frac{2}{\pi} \frac{x^2 y}{(x^2+y^2)^2} = \sigma_{xy}}}$

c. $\sigma_{xx} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{-i \sin \lambda x}{\lambda} \left\{-|\lambda| e^{-|\lambda| y} [2 - |\lambda| y]\right\} d\lambda = \frac{1}{\pi} \int_0^{\infty} \sin \lambda x \, e^{-\lambda y} (2 - \lambda y) d\lambda$

$= \frac{2}{\pi}\left[\frac{x}{x^2+y^2}\right] + \frac{y}{\pi} \frac{\partial}{\partial x}\left[\int_0^{\infty} \cos \lambda x \, e^{-\lambda y} d\lambda\right] = \frac{2}{\pi}\left[\frac{x}{x^2+y^2}\right] + \frac{y}{\pi} \frac{\partial}{\partial x}\left[\frac{y}{x^2+y^2}\right] = \frac{2}{\pi} \frac{x^3}{(x^2+y^2)^2}$

or $\underline{\underline{\sigma_{xx} = \frac{2}{\pi} \frac{x^3}{(x^2+y^2)^2}}}$

d. To obtain ϕ :

$\frac{\partial^2 \phi}{\partial x^2} = \sigma_{yy} = \frac{2}{\pi} \frac{y^2 x}{(x^2+y^2)^2} \Rightarrow \frac{\partial \phi}{\partial x} = \frac{-y^2}{\pi} \frac{1}{x^2+y^2} + \hat{f}_1(y) \Rightarrow \phi = \frac{-y}{\pi} \arctan \frac{x}{y} + x \hat{f}_1(y) + \hat{f}_2(y)$

$= y/\pi \arctan y/x + x \hat{f}_1(y) + \hat{f}_2(y)$

$$\frac{\partial^2\phi}{\partial x\partial y} = -\sigma_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right) = -\frac{2yx^2}{\pi(x^2+y^2)^2} + \hat{f}_1' \Rightarrow \hat{f}_1'(y)=0 \text{ or } \hat{f}_1(y) = c_1$$

$$\frac{\partial\phi}{\partial y} = \frac{1}{\pi}\arctan y/x + \frac{y}{\pi}\frac{x}{x^2+y^2} + \hat{f}_2' \;;\; \frac{\partial^2\phi}{\partial y^2} = \frac{2x^3}{\pi(x^2+y^2)^2} + \hat{f}_2''(y) = \sigma_{yy} \Rightarrow \hat{f}_2''(y)=0 \text{ or } \hat{f}_2 = c_2y+c_3$$

$$\therefore \underline{\underline{\phi(x,y) = \frac{y}{\pi}\arctan y/x + c_1x + c_2y + c_3}}$$ same argument on c_1, c_2, c_3 as problem 1.

3a For principal stresses in a plane strain problem $\sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$: Define λ_i to be the principal stresses $\therefore \sigma\cdot n = \lambda n$ or $\det\begin{pmatrix}\sigma_{xx}-\lambda & \sigma_{xy} & 0\\ \sigma_{xy} & \sigma_{yy}-\lambda & 0\\ 0 & 0 & \sigma_{zz}-\lambda\end{pmatrix} = 0$ $\therefore \lambda_3 = \sigma_{zz}$ and

$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$ after the plug in and the algebra

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \frac{-2\nu y}{(x^2+y^2)} = \lambda_3 \qquad \lambda_1 = 0 \qquad \lambda_2 = \frac{-2y}{\pi(x^2+y^2)}$$

Since $y>0$ and we assume $0<\nu<1$ the stresses are ordered

$(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

3b. again we obtain for plane strain $\lambda_3 = \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$ and

$$\lambda_{1,2} = \frac{\sigma_{xx}+\sigma_{yy}}{2} \pm \sqrt{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

$$\therefore \nu(\sigma_{xx}+\sigma_{yy}) = \sigma_{zz} = \lambda_3 = \frac{2\nu x}{\pi(x^2+y^2)} \qquad \lambda_1 = 0 \;,\; \lambda_2 = \frac{2x}{\pi(x^2+y^2)}$$

for $x>0$ and assuming $0<\nu<1$ the stresses are ordered $(\lambda_2, \lambda_3, \lambda_1)$ in decreasing tension (from left to right). for $x<0$ the stresses are ordered $(\lambda_1, \lambda_3, \lambda_2)$ in decreasing tension (from left to right)

[redacted]

$\boxed{P120/2}$ $\quad \dfrac{\partial^2 u}{\partial x^2} = \alpha \dfrac{\partial u}{\partial t} \qquad u = u(x,t) = X(x).T(t) \qquad u(x, t=0) = f(x)$

$u(x=0, t) = 0$

$u(x=L, t) = 0$

$\to X''T = \dfrac{1}{\alpha}.\dot{T}X \to \dfrac{X''}{X} = \dfrac{1}{\alpha}.\dfrac{\dot{T}}{T} = -k^2 \to$

$X'' + k^2 X = 0 \to X = A\sin kx + B\cos kx$ ✓

$\dot{T} + \alpha k^2 T = 0 \to T = Ce^{-\alpha k^2 t}$ ✓

$u(x=0,t)=0 \to A\sin k(0) + B\cos k(0) = 0 \to B=0$ ✓

$u(x=L,t)=0 \to A\sin kL = 0 \to k = \dfrac{n\pi}{L}$ ✓

$\to u_n(x,t) = \bar{A}_n \sin(\frac{n\pi}{L}x).e^{-\alpha(\frac{n\pi}{L})^2 t} \to u(x,t) = \sum_{n=1}^{\infty} \bar{A}_n \sin(\frac{n\pi x}{L})e^{-\alpha(\frac{n\pi}{L})^2 t}$

$u(x,t=0) = \sum_{n=1}^{\infty} \bar{A}_n \sin(\frac{n\pi x}{L}) = f(x) \to \bar{A}_n = \dfrac{2}{L}\int_0^L f(x).\sin\frac{n\pi x}{L}.dx$ ✓

$\to u(x,t) = \sum_{n=1}^{\infty}\left[\frac{2}{L}\int_0^L f(x).\sin\frac{n\pi x}{L}dx\right].\sin(\frac{n\pi x}{L}).e^{-\alpha(\frac{n\pi}{L})^2 t}$ ✓ 10

$\boxed{P139/12}$. $\quad \dfrac{\partial u}{\partial t} = k.\dfrac{\partial^2 u}{\partial x^2} - hu + \dfrac{kI^2}{R\sigma A^2} \qquad 0<x<L \quad t>0 \qquad u = u(x,t)$

a) $u(x=0,t)=0 \qquad u(x=L,t)=0 \qquad u(x,t=0)=0$

we assume $u(x,t) = v(x,t) + \psi(x) \to \dfrac{\partial v}{\partial t} = k\left[\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial x^2}\right] - h\left[v+\psi\right] + \dfrac{kI^2}{R\sigma A^2}$

$\to \left[k\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial v}{\partial t} - hv\right] + \underbrace{\left[k\dfrac{\partial^2\psi}{\partial x^2} - h\psi + \dfrac{kI^2}{R\sigma A^2}\right]}_{} = 0 \qquad \xrightarrow{kI^2/R\sigma A^2 = Q}$

we'll choose ψ in such a way that $k\dfrac{d^2\psi}{dx^2} - h\psi + Q = 0$ and

$\psi'' - \dfrac{h}{k}\psi + \dfrac{Q}{k} = 0$ ✓ $\qquad$ @ $x=0 \quad \psi(x)=0$

$k\dfrac{\partial^2 v}{\partial x^2} - \dfrac{\partial v}{\partial t} - hv = 0$ ✓ $\qquad$ @ $x=L \quad \psi'(x)=0 \implies$ ✓

use $\psi_p = D$ and $\quad -\frac{h}{K} D = -\frac{Q}{K}$

$D = \frac{Q}{h}$

$$\psi'' - \frac{h}{K}\psi = -\frac{Q}{K} \quad \xrightarrow{\frac{h}{K} = p^2} \quad \psi'' - p^2\psi = -\frac{Q}{K} \qquad \psi = \psi_h + \psi_p$$

$$\psi_h = c_1 e^{px} + c_2 e^{-px} \qquad W(\psi_1, \psi_2) = \begin{vmatrix} \psi_1 & \psi_1' \\ \psi_2 & \psi_2' \end{vmatrix} = \begin{vmatrix} e^{px} & pe^{px} \\ e^{-px} & -pe^{-px} \end{vmatrix} = -2p$$

$$\psi_p = u_1(x)\psi_1(x) + u_2(x)\psi_2(x) \qquad \longrightarrow \quad \psi_p = \frac{Q}{Kp^2}$$

$$u_1(x) = \int^x \frac{Q/K \cdot e^{-px}}{-2p} = \frac{Q}{2K}\cdot\frac{1}{p^2}\cdot e^{-px} \qquad \psi(x) = c_1 e^{px} + c_2 e^{-px} + \frac{Q}{2Kp^2}$$

$$u_2(x) = \int^x \frac{-Q/K \cdot e^{px}}{-2p} = \frac{Q}{2K}\cdot\frac{1}{p^2}\cdot e^{px} \qquad \begin{array}{ll} @x=0 & \psi(x)=0 \\ @x=L & \psi(L)=0 \end{array} \longrightarrow$$

$$\begin{cases} c_1 + c_2 = -\dfrac{Q}{Kp^2} \\ c_1 e^{pL} + c_2 e^{-pL} = -\dfrac{Q}{Kp^2} \end{cases} \longrightarrow \begin{cases} -c_1 e^{pL} - c_2 e^{pL} = \dfrac{Q}{Kp^2}\cdot e^{pL} \\ c_1 e^{pL} + c_2 e^{-pL} = -\dfrac{Q}{Kp^2} \end{cases} \longrightarrow \begin{array}{l} c_2 = \dfrac{Q}{Kp^2}(e^{pL}-1)\cdot\dfrac{1}{e^{-pL}-e^{pL}} \\ c_1 = -\dfrac{Q}{Kp^2}(e^{-pL}-1)\cdot\dfrac{1}{e^{-pL}-e^{pL}} \end{array}$$

$$\longrightarrow \psi(x) = \frac{Q}{Kp^2}\left[1 - \frac{e^{px}(e^{-pL}-1)}{e^{-pL}-e^{pL}} + \frac{e^{-px}(e^{pL}-1)}{e^{-pL}-e^{pL}}\right] =$$

$$= \frac{Q}{Kp^2}\left[1 - \frac{e^{px} - e^{-p(L-x)} - e^{-px} + e^{+p(L-x)}}{e^{pL} - e^{-pL}}\right] = \frac{Q}{Kp^2}\left[1 - \frac{\sinh(px) + \sinh(p(L-x))}{\sinh(pL)}\right]$$

$$\longrightarrow \psi(x) = \frac{KI^2}{Rh\sigma A^2}\left[1 - \frac{\sinh(\sqrt{\frac{h}{K}}x) + \sinh(\sqrt{\frac{h}{K}}(L-x))}{\sinh(\sqrt{\frac{h}{K}}L)}\right] \quad ✓$$

$$K\frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial t} - hv = 0 \qquad \begin{array}{lll} v(x,t) & @x=0 & =0 \\ v(x,t) & @x=L & =0 \\ v(x,t=0) & = & -\psi(x) \end{array} \qquad v(x,t) = X(x)T(t) \longrightarrow$$

$$K\cdot X''T - X\dot{T} - hXT = 0 \longrightarrow K\cdot\frac{X''}{X} - \frac{\dot{T}}{T} - h = 0 \longrightarrow K\frac{X''}{X} = \frac{\dot{T}}{T} + h = -\alpha \longrightarrow$$

$$\begin{cases} \dot{T} + (\alpha+h)T = 0 & \longrightarrow \quad T(t) = c_1 e^{-(\alpha+h)t} \\ X'' + \frac{\alpha}{K}X = 0 & \longrightarrow \quad X(x) = A\sin\sqrt{\frac{\alpha}{K}}x + B\cos\sqrt{\frac{\alpha}{K}}x \end{cases}$$

Cont'd $\longrightarrow$ 2

[redacted]

$T(x,t)$ @ $x=0$ $=0$ $\longrightarrow$ $X(0) = A \sin\sqrt{\frac{\alpha}{k}}(0) + B\cos\sqrt{\frac{\alpha}{k}}(0) = 0$ $\longrightarrow B = 0$

$T(x,t)$ @ $x = L$ $= 0$ $\longrightarrow$ $X(L) = A\sin\sqrt{\frac{\alpha}{k}}L = 0 \longrightarrow \sqrt{\frac{\alpha}{k}}L = n\pi \longrightarrow \alpha = \frac{n^2\pi^2}{L^2 k}$

$$\longrightarrow V_n(x,t) = \sum_{n=1}^{\infty} C_n e^{-\left(\frac{n^2\pi^2}{kL^2}+h\right)t} \cdot A_n \cdot \sin\left(\frac{n\pi x}{L}\right) \longrightarrow V_n(x,t) = \sum_{n=1}^{\infty} \bar{A}_n e^{-\left(\frac{n^2\pi^2}{kL^2}+h\right)t} \sin\frac{n\pi x}{L}$$

$T(x,t)$ @ $t=0$ $= -\psi(x)$ $\longrightarrow$ $-\psi(x) = \sum_{n=1}^{\infty} \bar{A}_n \sin\frac{n\pi x}{L} \longrightarrow \bar{A}_n = -\frac{2}{L}\int_0^L \psi(x)\sin\frac{n\pi x}{L}\cdot dx$

$$\psi(x) = \frac{KI^2}{kh\sigma A^2}\left[1 - \frac{\sinh(\sqrt{\frac{h}{k}}x) + \sinh(\sqrt{\frac{h}{k}}(L-x))}{\sinh(\sqrt{\frac{h}{k}}L)}\right]$$

$$\longrightarrow \quad \bar{A}_n = -\frac{2}{L}\int_0^L \psi(x)\cdot\sin\frac{n\pi x}{L}\,dx \quad \checkmark$$

$$V_n(x,t) = \sum_{n=1}^{\infty} \bar{A}_n \cdot e^{-\left(\frac{n^2\pi^2}{kL^2}+h\right)t} \sin\frac{n\pi x}{L}$$

$$u(x,t) = V_n(x,t) + \psi(x) \quad \checkmark$$

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$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad 0<x<L \quad t>0$$

$y(0,t) = 0$

$E\cdot\frac{\partial y(L,t)}{\partial x} = F$

$y(x,0) = y_t(x,0) = 0$

$y(x,t) = T(x,t) + \psi(x)$

$$\longrightarrow \frac{\partial^2 T}{\partial t^2} = c^2\left(\frac{\partial^2 T}{\partial x^2} + \frac{d^2\psi}{dx^2}\right) \longrightarrow c^2\frac{\partial^2 T}{\partial x^2} - \frac{\partial^2 T}{\partial t^2} + \underbrace{c^2\frac{d^2\psi}{dx^2}}_{} = 0$$

we will choose $\psi(x)$ in such a way that

$$\frac{d^2\psi}{dx^2} = 0$$

$\psi(x)$ @ $x=0$ $=0$

$\frac{d\psi}{dx}$ @ $x=L$ $= \frac{F}{E}$

$\psi(x) = Ax + B$ $\longleftarrow$

$\psi(0) = 0 \longrightarrow B = 0$ ✓

$\left.\frac{d\psi}{dx}\right|_{x=L} \longrightarrow A = \frac{F}{E}$ ✓

$\longrightarrow \psi(x) = \frac{F}{E}x.$ ✓

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$$c^2 \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial t^2} = 0$$

$v(x,t)$ @ $x=0$ $=0$

$E \cdot \frac{\partial v(x,t)}{\partial x}$ @ $x=L$ $=0$

$v(x,t) = X(x) \cdot T(t)$

$v(x,0) = -\Psi(x)$ $\quad v_t(x,0) = 0$

$\downarrow$

$$c^2 X'' T - \ddot{T} X = 0 \rightarrow c^2 \frac{X''}{X} = \frac{\ddot{T}}{T} = -\omega^2 \rightarrow$$

$\ddot{T} + \omega^2 T = 0 \rightarrow T = A \sin\omega t + B \cos\omega t$ ✓

$X'' + \frac{\omega^2}{c^2} X = 0 \rightarrow X = \bar{C} \sin \frac{\omega}{c} x + D \cos \frac{\omega}{c} x$ ✓

$v(x,t)$ @ $x=0$ $=0$ $\rightarrow D = 0$ ✓

$\frac{\partial v}{\partial x}$ @ $x=L$ $=0$ $\rightarrow \frac{\omega L}{c} = (2n+1) \cdot \frac{\pi}{2} \rightarrow \omega_n = \frac{(2n+1)\pi c}{2L}$ ✓

$v_t(x,t)$ @ $t=0$ $=0$ $\rightarrow A = 0$ ✓

$v(x,t)$ @ $t=0$ $= -\Psi(x)$ $\therefore$ $v_n(x,t) = \sum_{n=0}^{\infty} \bar{B}_n \cos \frac{(2n+1)\pi c t}{2L} \cdot \sin \frac{(2n+1)\pi x}{2L}$

$$v_n(x,0) = \sum_{n=0}^{\infty} \bar{B}_n \cdot \sin \frac{(2n+1)\pi x}{2L} \rightarrow$$

$$\bar{B}_n = \frac{2}{L} \int_0^L -\Psi(x) \cdot \sin \frac{(2n+1)\pi x}{2L} = \frac{2}{L} \int_0^L -\frac{F}{E} x \sin \frac{(2n+1)\pi x}{2L} dx \rightarrow$$

$$\bar{B}_n = -\frac{2F}{EL} \int_0^L \underset{u}{x} \underbrace{\sin \frac{(2n+1)\pi x}{2L} dx}_{dv} = -\frac{2F}{EL} \left[x \cdot \frac{-2L}{(2n+1)\pi} \cos \frac{(2n+1)\pi x}{2L} \Big|_0^L + \int_0^L \frac{2L}{(2n+1)\pi} \cdot \cos \frac{(2n+1)\pi x}{2L} dx \right]$$

$$= -\frac{2F}{EL} \cdot \frac{2L}{(2n+1)\pi} \cdot \frac{2L}{(2n+1)\pi} \sin \frac{(2n+1)\pi x}{2L} \Big|_0^L = -\frac{8FL^2(-1)^n}{E\pi^2(2n+1)^2} \rightarrow$$

$$y_n(x,t) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} 8FL^2}{\pi^2 E(2n+1)^2} \cdot \sin \frac{(2n+1)\pi x}{2L} \cdot \cos \frac{(2n+1)\pi c t}{2L} + \frac{F}{E} x$$

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4

P. 283 / 1 [redacted]

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{K}{R}\, g(x,t) \qquad -\infty < x < +\infty \qquad t>0$$

$$u(x,0) = f(x) \qquad -\infty < x < +\infty$$

$$\mathcal{F}\{u(x,t)\} = \int_{-\infty}^{+\infty} u(x,t)\, e^{-i\omega x}\, dx = \tilde{u}(\omega;t)$$

$$\frac{d\tilde{u}}{dt} = K\left[-\omega^2 \tilde{u}(\omega;t)\right] + \frac{K}{R}\tilde{g}(\omega;t) \longrightarrow \frac{d\tilde{u}}{dt} + K\omega^2\tilde{u} = \frac{K}{R}\tilde{g}(\omega,t) \longrightarrow$$

$$\tilde{u}(\omega;t) = c\, e^{-K\omega^2 t} + \int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})\, e^{-K\omega^2(t-\bar{t})}\, d\bar{t}$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \tilde{u}(\omega,t)\, e^{i\omega x}\, d\omega$$

$$\mathcal{F}\{\text{i.c.}\} \longrightarrow \tilde{u}(\omega,0) = \tilde{f}(\omega) \longrightarrow \tilde{u}(\omega;t) = \tilde{f}(\omega)e^{-K\omega^2 t} + \int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})e^{-K\omega^2(t-\bar{t})}.$$

$$\longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{K\pi t}}\, e^{-(x-u)^2/4Kt}\cdot du + \frac{1}{2\pi}\int_{-\infty}\left\{\int_0^t \frac{K}{R}\tilde{g}(\omega,\bar{t})e^{-K\omega^2(t-\bar{t})}d\bar{t}\right\} e^{+i\omega x}\, d\omega$$

$$\text{if } g(x,t)\equiv 0 \longrightarrow \tilde{g}(\omega,t)\equiv 0 \longrightarrow u(x,t) = \int_{-\infty}^{+\infty} f(u)\cdot\frac{1}{2\sqrt{K\pi t}}\cdot e^{\frac{-(x-u)^2}{4Kt}}\cdot du$$

$$\text{(i)}\quad f(x) = \begin{cases} 1 & |x|<a \\ 0 & |x|>a \end{cases} \longrightarrow u(x,t) = \int_{-a}^{+a}\frac{1}{2\sqrt{K\pi t}}\, e^{\frac{-(x-u)^2}{4Kt}}\cdot du$$

$$\text{let } \frac{x-u}{2\sqrt{Kt}} = v \longrightarrow du = -2\sqrt{Kt}\cdot dv \longrightarrow u(x,t) = -\int_{(x-a)/2\sqrt{Kt}}^{(x+a)/2\sqrt{Kt}} e^{-v^2}\, dv \left(-\frac{1}{\sqrt{\pi}}\right)$$

$$\longrightarrow u(x,t) = +\frac{1}{\sqrt{\pi}}\int_0^{(x+a)/2\sqrt{Kt}} e^{-v^2}dv - \frac{1}{\sqrt{\pi}}\int_0^{(x-a)/2\sqrt{Kt}} e^{-v^2}dv \longrightarrow$$

$$u(x,t) = -\frac{1}{2}\operatorname{erf}\left(\frac{x-a}{2\sqrt{Kt}}\right) + \frac{1}{2}\operatorname{erf}\left(\frac{x+a}{2\sqrt{Kt}}\right)$$

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(ii) $f(x) = \begin{cases} \\ 1 & |x|>a \end{cases} \longrightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}} \int_{-\infty}^{-a} e \, du + \frac{1}{2\sqrt{\kappa\pi t}} \int_{+a} e \, du$

$$\longrightarrow u(x,t) = \frac{1}{2\sqrt{\kappa\pi t}} \left[\int_{-\infty}^{+\infty} e^{-(x-u)^2/4\kappa t} du - \int_{-a}^{a} e^{-(x-u)^2/4\kappa t} du \right]$$

like part (i)

$v = \frac{x-u}{2\sqrt{\kappa t}}$

$$u(x,t) = +\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-v^2} dv - \boxed{\frac{1}{2\sqrt{\kappa\pi t}} \int_{-a}^{a} e^{-(x-u)^2/4\kappa t} du} \longrightarrow$$

we know the answer from part (i)

$$u(x,t) = +1 + \frac{1}{2} \operatorname{erf}\left(\frac{x-a}{2\sqrt{\kappa t}}\right) - \frac{1}{2} \operatorname{erf}\left(\frac{x+a}{2\sqrt{\kappa t}}\right) \checkmark$$

$\frac{20}{20}$

P.284 / 4 $\quad \frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} + \frac{\kappa}{K} g(x,t) \qquad x>0 \quad t>0$

$$u(0,t) = f_1(t) \qquad u(x,0) = f(x)$$

$$\mathcal{F}_s\{u(x,t)\} = \int_0^\infty u(x,t) \sin wx \, dx = \tilde{u}(w,t) \longrightarrow \frac{d\tilde{u}}{dt} = \kappa\left[-w^2\tilde{u} + w u(0,t)\right] + \frac{\kappa}{K}\tilde{g}$$

$$\longrightarrow \frac{d\tilde{u}}{dt} + \kappa w^2 \tilde{u} = \kappa w u(0,t) + \frac{\kappa}{K}\tilde{g}(w,t) \longrightarrow$$

$$\tilde{u}(w,t) = C e^{-\kappa w^2 t} + \int_0^t \left[\frac{\kappa}{K}\tilde{g}(w,\bar{t}) + \kappa w f_1(\bar{t})\right] e^{-\kappa w^2 (t-\bar{t})} d\bar{t}$$

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \tilde{u}(w,t) \sin wx \, dw$$

IC conversion $\tilde{u}(w,0) = C = \tilde{f}(w)$

(b) $g(x,t) \equiv 0, \quad f_1(t) \equiv 0, \quad f(x) = u_0 \longrightarrow \tilde{u}(w,t) = \tilde{f}(w) e^{-\kappa w^2 t} \checkmark$

using Convolution integral $\longrightarrow u(x,t) = \frac{1}{2}\int_0^\infty u_0 \frac{1}{\sqrt{\kappa\pi t}} \left[e^{\frac{-(x-u)^2}{4\kappa t}} - e^{\frac{-(x+u)^2}{4\kappa t}}\right] du \checkmark$

let $\frac{x-u}{2\sqrt{\kappa t}} = v \longrightarrow du = -2\sqrt{\kappa t}\, dv$ & let $z = \frac{x+u}{2\sqrt{\kappa t}} \longrightarrow du = 2\sqrt{\kappa t}\, dz$

$$u(x,t) = -\frac{u_0}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{-\infty} e^{-v^2} dv - \frac{u_0}{\sqrt{\pi}} \int_{x/2\sqrt{\kappa t}}^{+\infty} e^{-z^2} dz$$

Cont. $\longrightarrow$ 2

we know $\frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}du = \text{erfc}(x)$ [redacted]

$$u(x,t) = \frac{u_0}{\sqrt{\pi}}\int_{-x/2\sqrt{\kappa t}}^{\infty} e^{-v^2}dv - \frac{u_0}{\sqrt{\pi}}\int_{x/2\sqrt{\kappa t}}^{\infty} e^{-z^2}dz \rightarrow$$

$$u(x,t) = \frac{u_0}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \quad \checkmark$$

(c) $g(x,t) \equiv 0 \rightarrow \tilde{g}(\omega,t) \equiv 0$

$f(x) \equiv 0 \rightarrow \tilde{f}(\omega) \equiv 0$ ✓

$f_1(t) = \bar{u} \quad \rightarrow \tilde{u}(\omega,t) = \int_0^t \kappa\omega\bar{u}\, e^{-\kappa\omega^2(t-\bar{t})}d\bar{t}$

$$\rightarrow \tilde{u}(\omega,t) = \kappa\omega\bar{u}\cdot\frac{1}{\kappa\omega^2}\cdot e^{-\kappa\omega^2(t-\bar{t})} = \frac{\bar{u}}{\omega}\cdot e^{-\kappa\omega^2(t-\bar{t})}\Big|_0^t$$

$$\rightarrow \tilde{u}(\omega,t) = \frac{\bar{u}}{\omega}(1 - e^{-\kappa\omega^2 t}) \rightarrow \tilde{u}(\omega,t) = \frac{\bar{u}}{\omega} - \bar{u}\,\frac{e^{-\kappa\omega^2 t}}{\omega} \quad \checkmark$$

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\tilde{u}(\omega,t)\cdot \sin\omega x\, d\omega \quad \checkmark$$

$$\mathcal{F}_s^{-1}\left\{\frac{\bar{u}}{\omega}\right\} = \frac{2}{\pi}\int_0^{\infty}\frac{\bar{u}\sin\omega x}{\omega}d\omega = \frac{2}{\pi}\cdot\bar{u}\cdot\frac{\pi}{2} = \bar{u}$$

we know $\int_0^{\infty}\frac{\sin x}{x}dx = \pi/2$

applying convolution on $\bar{u}\,\frac{e^{-\kappa\omega^2 t}}{\omega} \rightarrow$

$$\mathcal{F}^{-1}\left\{\frac{\bar{u}}{\omega}\cdot e^{-\kappa\omega^2 t}\right\} = \frac{1}{2}\int_0^{\infty}\bar{u}\cdot\frac{1}{\sqrt{\kappa\pi t}}\left[e^{\frac{-(x-u)^2}{4\kappa t}} - e^{\frac{-(x+u)^2}{4\kappa t}}\right]du$$

from part (b)

$$= \frac{\bar{u}}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \quad \checkmark \quad \Rightarrow$$

$$u(x,t) = \bar{u} - \frac{\bar{u}}{2}\left[\text{erfc}\left(\frac{-x}{2\sqrt{\kappa t}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{\kappa t}}\right)\right] \quad \checkmark$$

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p.363 / 2 [redacted] [redacted]

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + \frac{K}{k}\, g(x,t) \qquad u(x,0) = f(x)$$

Applying Laplace operator to both side of the equation : [redacted] let $\tilde{u}(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$s\tilde{u}(x;s) - u(x,0) = K\frac{\partial^2}{\partial x^2}\cdot\tilde{u}(x;s) + \frac{K}{k}\tilde{g}(x;s) \longrightarrow$$

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;s) - \frac{s}{K}\tilde{u}(x;s) + \frac{1}{k}\tilde{g}(x;s) + f(x) = 0 \longrightarrow$$

s/B f(x)/K

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;s) - \frac{s}{K}\tilde{u}(x;s) = -\left(\frac{1}{k}\tilde{g}(x;s) + \frac{1}{K}f(x)\right) \qquad \tilde{u}(x;s) = \tilde{u}_h + \tilde{u}_p$$

$$\frac{\partial^2}{\partial x^2}\tilde{u}(x;s) - \frac{s}{K}\tilde{u}(x;s) = 0 \longrightarrow \tilde{u}_h = A\underbrace{e^{\sqrt{\frac{s}{K}}x}}_{\tilde{u}_1} + B\underbrace{e^{-\sqrt{\frac{s}{K}}x}}_{\tilde{u}_2}$$

$$\tilde{u}_p = v_1\tilde{u}_1 + v_2\tilde{u}_2 \qquad W(\tilde{u}_1,\tilde{u}_2) = \begin{vmatrix} e^{\sqrt{\frac{s}{K}}x} & \sqrt{\frac{s}{K}}e^{\sqrt{\frac{s}{K}}x} \\ e^{-\sqrt{\frac{s}{K}}x} & -\sqrt{\frac{s}{K}}e^{-\sqrt{\frac{s}{K}}x} \end{vmatrix} = -2\sqrt{\frac{s}{K}}$$

$$v_1 = \int^x +\left(\frac{1}{K}f(\bar{x}) + \frac{1}{k}\tilde{g}(\bar{x};s)\right)e^{-\sqrt{\frac{s}{K}}\bar{x}}\cdot -\frac{1}{2}\sqrt{\frac{K}{s}}\,d\bar{x}$$

$$v_2 = \int^x -\left(\frac{1}{K}f(\bar{x}) + \frac{1}{k}\tilde{g}(\bar{x};s)\right)e^{\sqrt{\frac{s}{K}}\bar{x}}\cdot -\frac{1}{2}\sqrt{\frac{K}{s}}\,d\bar{x}$$

we can multiple $\tilde{u}_1$, $\tilde{u}_2$ into th functions inside the integral $\longrightarrow$

$$\longrightarrow \tilde{u}_p = \int_0^x -\sqrt{\frac{K}{s}}\left[\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right]\cdot\left[e^{\sqrt{\frac{s}{K}}(x-\bar{x})} - e^{-\sqrt{\frac{s}{K}}(x-\bar{x})}\right]\cdot\frac{1}{2}\,d\bar{x}$$

$$\longrightarrow \tilde{u}_p = -\sqrt{\frac{K}{s}}\int_0^x\left(\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right)\cdot\sinh\left(\sqrt{\frac{s}{K}}(x-\bar{x})\right)d\bar{x} \qquad \tilde{u}_{TOT} = \tilde{u}_h + \tilde{u}_p \longrightarrow$$

$$\tilde{u}(x;s) = Ae^{\sqrt{\frac{s}{K}}x} + Be^{-\sqrt{\frac{s}{K}}x} - \sqrt{\frac{K}{s}}\int^x\left(\frac{f(\bar{x})}{K} + \frac{\tilde{g}(\bar{x};s)}{k}\right)\cdot\sinh\left(\sqrt{\frac{s}{K}}(x-\bar{x})\right)d\bar{x}$$

✓

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what about when the BC is changed to $\frac{\partial U}{\partial x}(0,t) = \frac{-f_1(t)}{K} = -\frac{Q_0}{K}$

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$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \qquad x>0 \quad t>0$$

$$\frac{\partial u(0,t)}{\partial x} = -\frac{f_1(t)}{K} \qquad t>0$$

$$u(x,0) = u_0 = cte \qquad x>0$$

we have the general solution from previous exercise also we know:

$g(x,t) \equiv 0$

$f(x) = u_0$

and the solution shall be bounded as $x \to \infty$

$$\tilde{u}(x;s) = Ae^{\sqrt{\frac{s}{K}}x} + Be^{-\sqrt{\frac{s}{K}}x} - \sqrt{\frac{K}{s}}\int_0^x \frac{u_0}{K}\sinh\left(\sqrt{\frac{s}{K}}(x-\bar{x})\right)d\bar{x}$$

$x \to \infty$, Solution shall be bounded $\to A = 0 \to$

$$\tilde{u}(x;s) = Be^{-\sqrt{\frac{s}{K}}x} - \frac{u_0}{\sqrt{sK}}\int_0^x \sinh\left(\sqrt{\frac{s}{K}}(x-\bar{x})\right)d\bar{x} = Be^{-\sqrt{\frac{s}{K}}x} + \frac{u_0}{\sqrt{sK}}\cdot\sqrt{\frac{K}{s}}\left[\cosh\left(\sqrt{\frac{s}{K}}(x-\bar{x})\right)\right]_0^x \to$$

$$\tilde{u}(x;s) = Be^{-\sqrt{\frac{s}{K}}x} + \frac{u_0}{s}\left(1-\cosh\sqrt{\frac{s}{K}}x\right)$$

$$\mathcal{L}\left\{\frac{\partial}{\partial x}u(0,t)\right\} = \mathcal{L}\left\{-\frac{f_1(t)}{K}\right\} \to \int_0^\infty \frac{\partial}{\partial x}u(0,t)e^{-st}dt = -\frac{\tilde{f}_1(s)}{K} \to$$

$$\frac{\partial}{\partial x}\tilde{u}(0;s) = -\frac{\tilde{f}_1(s)}{K}$$

$$\frac{\partial}{\partial x}\tilde{u}(x;s) = -B\sqrt{\frac{s}{K}}e^{-\sqrt{\frac{s}{K}}x} - \frac{u_0}{s}\cdot\sqrt{\frac{s}{K}}\sinh\sqrt{\frac{s}{K}}x\,\Bigg|_{x=0} = -B\sqrt{\frac{s}{K}} = \frac{\tilde{f}_1(s)}{K} \to$$

$$B = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}$$

$$\to \tilde{u}(x;s) = -\frac{\tilde{f}_1(s)}{\sqrt{sK}}\cdot e^{-\sqrt{\frac{s}{K}}x} + \frac{u_0}{s} - u_0\frac{\cosh\sqrt{\frac{s}{K}}x}{s}$$

$$= \underbrace{-\frac{1}{\sqrt{K}}\cdot\tilde{f}_1(s)\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{\sqrt{s}}}_{\downarrow\,\mathcal{L}^{-1}} + \underbrace{\frac{u_0}{s}}_{\downarrow\,\mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow\,\mathcal{L}^{-1}} - \underbrace{u_0\cdot\frac{e^{-\frac{x}{\sqrt{K}}\cdot\sqrt{s}}}{s}}_{\downarrow\,\mathcal{L}^{-1}}$$

$$-\frac{1}{\sqrt{K}}\int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}}\cdot f_1(a)\cdot da + u_0 - u_0\,\mathrm{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - u_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{Kt}}\right)$$

OK

Cont. $\to$ (2)

[redacted]

$$u(x,t) = -\frac{1}{\sqrt{K}}\int_0^t \frac{e^{-\frac{x^2}{4K(t-a)}}}{\sqrt{\pi(t-a)}} \cdot f_1(a) \cdot da + u_0\left[1 - \text{erfc}\left(\frac{-x}{2\sqrt{Kt}}\right) - \text{erfc}\left(\frac{x}{2\sqrt{Kt}}\right)\right]$$ ✓

P350/28 $\quad y'' + 2y' + y = t \qquad y(0) = 0 \quad y'(0) = 1$

$$\left[s^2 Y(s) - s\,y(t=0^+) - y'(t=0^+)\right] + 2\left[sY(s) - y(t=0^+)\right] + Y(s) = \frac{1}{s^2} \longrightarrow$$

$$s^2 Y(s) + 2sY(s) + Y(s) - 1 = \frac{1}{s^2} \longrightarrow Y(s)\left(s^2 + 2s + 1\right) = \frac{1+s^2}{s^2} \longrightarrow Y(s) = \frac{1+s^2}{s^2(s+1)^2}$$

$$\longrightarrow Y(s) = -\frac{2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}$$

$$\longrightarrow y(t) = -2 + t + 2e^{-t} + 2te^{-t}$$ ✓ [redacted]

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$$\text{let } y(s) = \frac{A}{s} + \frac{Bs+C}{s^2} + \frac{D}{s+1} + \frac{Es+\bar{D}}{(s+1)^2}$$

$$= \frac{As(s+1)^2 + (Bs+C)(s+1)^2 + Ds^2(s+1) + (Es+\bar{D})s^2}{s^2(s+1)^2} = \frac{1+s^2}{s^2(s+1)^2}$$

expand and equate powers of s to find $A, B, C, D, E, \bar{D}$

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for Science and Engineering

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Properties and table of Fourier transforms

	$f(t)$		$F(\omega)=\hat{f}(\omega)$
F1.	$f(t)$		$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt$
F2.	$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega$		$F(\omega)$
F3.	$af(t)+bg(t)$		$aF(\omega)+bG(\omega)$
F4.	$f(at)$	($a\neq 0$ real)	$\frac{1}{\lvert a\rvert}F\left(\frac{\omega}{a}\right)$
F5.	$f(-t)$		$F(-\omega)$
F6.	$\overline{f(t)}$		$\overline{F(-\omega)}$
F7.	$f(t-T)$	(T real)	$e^{-i\omega T}F(\omega)$
F8.	$e^{i\Omega t}f(t)$	(Ω real)	$F(\omega-\Omega)$
F8a.	$f(t)\cos\Omega t$		$\frac{1}{2}(F(\omega-\Omega)+F(\omega+\Omega))$
F8b.	$f(t)\sin\Omega t$		$\frac{1}{2i}(F(\omega-\Omega)-F(\omega+\Omega))$
F9.	$F(t)$		$2\pi f(-\omega)$
F10.	$\left(\frac{d}{dt}\right)^n f(t)$		$(i\omega)^n F(\omega)$
F11.	$(-it)^n f(t)$		$\left(\frac{d}{d\omega}\right)^n F(\omega)$
F12.	$\int_{-\infty}^{t} f(\tau)d\tau$		$\frac{F(\omega)}{i\omega}+\pi F(0)\delta(\omega)$
F13.	$f*g(t)=\int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau$		$F(\omega)G(\omega)$
F14.	$f(t)g(t)$		$\frac{1}{2\pi}F*G(\omega)$
F15.	$\delta(t)$		1
F16.	$\delta^{(n)}(t)$		$(i\omega)^n$
F17.	$\delta^{(n)}(t-T)$		$(i\omega)^n e^{-i\omega T}$ $(n=0,1,2,\ldots)$
F18.	1		$2\pi\delta(\omega)$
F19.	t^n		$2\pi i^n\delta^{(n)}(\omega)$ $(n=1,2,3,\ldots)$
F20.	t^{-n}		$\frac{\pi(-i)^n}{(n-1)!}\omega^{n-1}\operatorname{sgn}\omega$ $(n=1,2,3,\ldots)$

(A-1)

Table of Fourier sine transform

	$f(x),\ x>0$		$F_s(\beta),\ \beta>0$
$F_s1.$	$\begin{cases}1, & x<a\\ 0, & x>a\end{cases}$	$(a>0)$	$\dfrac{1-\cos a\beta}{\beta}$
$F_s2.$	e^{-ax}	$(a>0)$	$\dfrac{\beta}{a^2+\beta^2}$
$F_s3.$	xe^{-ax^2}	$(a>0)$	$\sqrt{\dfrac{\pi}{a}}\,\dfrac{\beta}{4a}\,e^{-\beta^2/4a}$
$F_s4.$	x^{a-1}	$(-1<a<1)$	$\Gamma(a)\beta^{-a}\sin\dfrac{a\pi}{2}$
$F_s5.$	$\cos ax^2$		$\sqrt{\dfrac{\pi}{2a}}\left[\sin\dfrac{\beta^2}{4a}C\left(\dfrac{\beta}{\sqrt{2\pi a}}\right)-\cos\dfrac{\beta^2}{4a}S\left(\dfrac{\beta}{\sqrt{2\pi a}}\right)\right]$
$F_s6.$	$\sin ax^2$		$\sqrt{\dfrac{\pi}{2a}}\left[\cos\dfrac{\beta^2}{4a}C\left(\dfrac{\beta}{\sqrt{2\pi a}}\right)+\sin\dfrac{\beta^2}{4a}S\left(\dfrac{\beta}{\sqrt{2\pi a}}\right)\right]$

Fourier transforms in higher dimensions

Two-dimensional Fourier transforms

Fourier transform $F(u,v)=\iint\limits_{\mathbf{R}^2} f(x,y)e^{-i(ux+vy)}\,dxdy$

Inversion formula $f(x,y)=\dfrac{1}{(2\pi)^2}\iint\limits_{\mathbf{R}^2} F(u,v)e^{i(ux+vy)}\,dudv$

Parseval's formulas $\iint\limits_{\mathbf{R}^2} f(x,y)\,\overline{g(x,y)}\,dxdy=\dfrac{1}{(2\pi)^2}\iint\limits_{\mathbf{R}^2} F(u,v)\,\overline{G(u,v)}\,dudv$

$$\iint\limits_{\mathbf{R}^2}|f(x,y)|^2dxdy=\frac{1}{(2\pi)^2}\iint\limits_{\mathbf{R}^2}|F(u,v)|^2dudv$$

A-4

(A-2)

	$f(t)$	$F(\omega)$
F21.	$\theta(t)=H(t)=\begin{cases}1, & t>0\\ 0, & t<0\end{cases}$	$\frac{1}{i\omega}+\pi\delta(\omega)$
F22.	$t^n\theta(t)$	$\frac{n!}{(i\omega)^{n+1}}+\pi i^n\delta^{(n)}(\omega)$ $(n=1,2,3,\dots)$
F23.	$\mathrm{sgn}\,t=2\theta(t)-1=\begin{cases}1, & t>0\\ -1, & t<0\end{cases}$	$\frac{2}{i\omega}$
F24.	$t^n\,\mathrm{sgn}\,t$	$\frac{2n!}{(i\omega)^{n+1}}$
F25.	$\|t\|=t\,\mathrm{sgn}\,t$	$-\frac{2}{\omega^2}$
F26.	$\|t\|^{2n-1}$	$2(-1)^n\frac{(2n-1)!}{\omega^{2n}}$ $(n=1,2,3,\dots)$
F27.	$\|t\|^{2n}=t^{2n}$	$2\pi(-1)^n\delta^{(2n)}(\omega)$ $(n=1,2,3,\dots)$
F28.	$\|t\|^{p-1}$	$\frac{2\Gamma(p)\cos\frac{p\pi}{2}}{\|\omega\|^p}$ $(p\neq\text{integer})$
F29.	$\|t\|^{p-1}\mathrm{sgn}\,t$	$\frac{-2i\Gamma(p)\sin\frac{p\pi}{2}\mathrm{sgn}\,\omega}{\|\omega\|^p}$ $(p\neq\text{integer})$
F30.	$e^{-at}\theta(t)=\begin{cases}e^{-at}, & t>0\\ 0, & t<0\end{cases}$	$\frac{1}{a+i\omega}$ $(a>0)$
F31.	$e^{at}(1-\theta(t))=e^{at}\theta(-t)=\begin{cases}0, & t>0\\ e^{at}, & t<0\end{cases}$	$\frac{1}{a-i\omega}$ $(a>0)$
F32.	$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$ $(a>0)$
F33.	$e^{-a\|t\|}\mathrm{sgn}\,t$	$-\frac{2i\omega}{a^2+\omega^2}$ $(a>0)$
F34.	$te^{-a\|t\|}$	$-\frac{4ia\omega}{(a^2+\omega^2)^2}$ $(a>0)$
F35.	$\|t\|e^{-a\|t\|}$	$\frac{2(a^2-\omega^2)}{(a^2+\omega^2)^2}$ $(a>0)$

	$f(t)$	$F(\omega)$
F36.	e^{-at^2}	$\sqrt{\frac{\pi}{a}}e^{-\omega^2/4a}$ $(a>0)$
F37.	$\frac{1}{\sqrt{4\pi a}}e^{-t^2/4a}$	$e^{-a\omega^2}$ $(a>0)$
F38.	$\frac{i}{2(n-1)!}(it)^{n-1}e^{ict}\mathrm{sgn}\,t$	$\frac{1}{(\omega-c)^n}$ $(c\text{ real}, n=1,2,3,\dots)$
F39.	$\frac{1}{(n-1)!}t^{n-1}e^{-at}\theta(t)$	$\frac{1}{(a+i\omega)^n}$ $(a>0, n=1,2,3,\dots)$
F40.	$\frac{(-1)^{n-1}}{(n-1)!}t^{n-1}e^{at}(1-\theta(t))$	$\frac{1}{(a-i\omega)^n}$ $(a>0, n=1,2,3,\dots)$
F41a.	$\frac{1}{2a}e^{-a\|t\|}$	$\frac{1}{\omega^2+a^2}$ $(a>0)$
F41b.	$\frac{1}{t^2+a^2}$	$\frac{\pi}{a}e^{-a\|\omega\|}$
F42a.	$\frac{i}{2}e^{-a\|t\|}\mathrm{sgn}\,t$	$\frac{\omega}{\omega^2+a^2}$ $(a>0)$
F42b.	$\frac{t}{t^2+a^2}$	$-i\pi e^{-a\|\omega\|}\mathrm{sgn}\,\omega$
F43.	$\frac{1}{2a}e^{-a\|t\|+ict}(iak\,\mathrm{sgn}\,t+b+kc)$	$\frac{k\omega+b}{(\omega-c)^2+a^2}$ $(a>0, c\text{ real})$
F44.	$\frac{1}{4a^3}e^{-a\|t\|+ict}[ia^2kt+(b+kc)a\|t\|+b+kc]$	$\frac{k\omega+b}{[(\omega-c)^2+a^2]^2}$ $(a>0, c\text{ real})$
F45.	$e^{i\Omega t}$	$2\pi\delta(\omega-\Omega)$
F46.	$\cos\Omega t$	$\pi[\delta(\omega+\Omega)+\delta(\omega-\Omega)]$
F47.	$\sin\Omega t$	$i\pi[\delta(\omega+\Omega)-\delta(\omega-\Omega)]$
F48.	$-\frac{1}{2a}\sin at\,\mathrm{sgn}\,t$	$\frac{1}{\omega^2-a^2}$ $(a>0)$
F49.	$\frac{i}{2}\cos at\,\mathrm{sgn}\,t$	$\frac{\omega}{\omega^2-a^2}$ $(a>0)$
F50.	$\theta(t+a)-\theta(t-a)=\begin{cases}1, & \|t\|<a\\ 0, & \|t\|>a\end{cases}$	$\frac{2\sin a\omega}{\omega}$

$F(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixy} f(y)\,dy$	$f(y) = \int_{-\infty}^{+\infty} e^{ixy} F(x)\,dx$
$\sqrt{a^2+x^2}^{\,\nu} K_\nu(b\sqrt{a^2+x^2})$	$\sqrt{2\pi}\, a\, a^\nu b^\nu \frac{K_{\nu+\frac12}(a\sqrt{b^2+y^2})}{\sqrt{b^2+y^2}^{\,\nu+\frac12}}$
$\frac{H_\nu^{(2)}(b\sqrt{a^2+x^2})}{\sqrt{a^2+x^2}^{\,\nu}}$	$i\,2\sqrt{\frac{2a}{\pi}}\, a^{-\nu} b^{-\nu} \sqrt{y^2-b^2}^{\,\nu-\frac12} K_{\nu-\frac12}(a\sqrt{y^2-b^2})$
$e^{-x^2/2} He_n(x\sqrt{2})$	$\sqrt{2\pi}\, i^n e^{-y^2/2} He_n(y\sqrt{2})$
$e^{-x^2/2} He_n(x) He_m(x)$	$\sqrt{2\pi}\, n!\,(iy)^{m-n} e^{-y^2/2} L_n^{(m-n)}(y^2)$

§ 2. The Laplace-transformation.

The theorems formulated in this section are concerned without exception with the simplest and most important case in which the Laplace-transformation is used; we state two relations between a function $f(t)$ and a function $g(p)$, where the real and imaginary parts of $f(t)$ satisfy the conditions formulated in the preliminary remarks of this chapter, and where moreover:

1a) $f(t)$ is defined for all real values of t; $f(t) = 0$ for $t < 0$.

1b) there exists a real constant c_0, such that the integral

$$\int_0^\infty e^{-c_0 t} |f(t)|\,dt$$

exists.

2a) $g(p)$ is an analytic function of the complex variable $p = \sigma + i\tau$, which is one-valued and regular in an entire half-plane $\sigma \geqq c_0$.

2b) The function $|g(p)|$ is for $\sigma > c_0$ dominated by a function $G(\tau)$ with the property

$$|g(\sigma + i\tau)| \leqq G(\tau) \qquad \text{for } \sigma > c_0$$

such that the integral

$$\int_{-\infty}^{+\infty} G(\tau)\,d\tau$$

exists; it suffices that if this condition be satisfied not by $g(p)$ itself, but only by a function

$$g_0(p) = g(p) - \frac{a_1}{p^{\nu_1}} - \frac{a_2}{p^{\nu_2}} \cdots - \frac{a_n}{p^{\nu_n}}$$

	$f(t)$	$F(\omega)$
F51.	$[\theta(t+a)-\theta(t-a)]\operatorname{sgn} t=\begin{cases}1, & 0<t<a\\ -1, & -a<t<0\\ 0, & \lvert t\rvert>a\end{cases}$	$\dfrac{4\sin^2\frac{a\omega}{2}}{i\omega}$
F52.	$[\theta(t+a)-\theta(t-a)]e^{i\Omega t}=\begin{cases}e^{i\Omega t}, & \lvert t\rvert<a\\ 0, & \lvert t\rvert>a\end{cases}$	$\dfrac{2\sin a(\Omega-\omega)}{\Omega-\omega}$
F53.	$\dfrac{\sin\Omega t}{\pi t}$	$\theta(\omega+\Omega)-\theta(\omega-\Omega)=\begin{cases}1, & \lvert\omega\rvert<\Omega\\ 0, & \lvert\omega\rvert>\Omega\end{cases}$
F54.	$\sin at^2$	$\sqrt{\dfrac{\pi}{a}}\cos\left(\dfrac{\omega^2}{4a}+\dfrac{\pi}{4}\right)\quad (a>0)$
F55.	$\cos at^2$	$\sqrt{\dfrac{\pi}{a}}\cos\left(\dfrac{\omega^2}{4a}-\dfrac{\pi}{4}\right)\quad (a>0)$
F56.	$h_n(t)$	$i^{-n}\sqrt{2\pi}\,h_n(\omega)$
F57.	$l_n(t)\theta(t)$	$\left(i\omega-\frac{1}{2}\right)^n\Big/\left(i\omega+\frac{1}{2}\right)^{n+1}$
F58.	$\begin{cases}(a^2-t^2)^{-1/2}, & \lvert t\rvert<a\\ 0, & \lvert t\rvert>a\end{cases}$	$\pi J_0(a\omega)$
F59.	$\begin{cases}t(a^2-t^2)^{-1/2}, & \lvert t\rvert<a\\ 0, & \lvert t\rvert>a\end{cases}$	$-ia\pi J_1(a\omega)$
F60.	$\dfrac{1}{\cosh t}$	$\dfrac{\pi}{\cosh\frac{\pi\omega}{2}}$
F61.	$\dfrac{1}{\sinh t}$	$-i\pi\tanh\dfrac{\pi\omega}{2}$
F62.	$\dfrac{\sinh at}{\sinh bt}\quad(0<a<b)$	$\dfrac{\pi\sin(a\pi/b)}{b\cosh(\omega\pi/b)+b\cos(a\pi/b)}$
F63.	$\dfrac{\cosh at}{\cosh bt}\quad(0<a<b)$	$\dfrac{2\pi\cos(a\pi/2b)\cosh(\omega\pi/2b)}{b\cosh(\omega\pi/b)+b\cos(a\pi/b)}$
F64.	$\dfrac{1}{2\sqrt{2}a^3}e^{-a\lvert t\rvert/\sqrt{2}}\left(\cos\dfrac{at}{\sqrt{2}}+\sin\dfrac{a\lvert t\rvert}{\sqrt{2}}\right)$	$\dfrac{1}{\omega^4+a^4}\quad(a>0)$

(A-3)

Cosine and sine transforms

$$F_c(\beta)=\int_0^\infty f(x)\cos\beta x\,dx \qquad f(x)=\frac{2}{\pi}\int_0^\infty F_c(\beta)\cos\beta x\,d\beta$$

$$F_s(\beta)=\int_0^\infty f(x)\sin\beta x\,dx \qquad f(x)=\frac{2}{\pi}\int_0^\infty F_s(\beta)\sin\beta x\,d\beta$$

Plancherel's formulas

$$\int_0^\infty |f(x)|^2dx=\frac{2}{\pi}\int_0^\infty |F_c(\beta)|^2d\beta=\frac{2}{\pi}\int_0^\infty |F_s(\beta)|^2d\beta$$

Relations between Fourier transforms

If $F(\beta)$ is the Fourier transform of $f(x)$, $-\infty<x<\infty$, then

$f(x)$ even $\Rightarrow F(\beta)=\ 2F_c(\beta)$

$f(x)$ odd $\Rightarrow F(\beta)=-2iF_s(\beta)$

$\mathcal{F}_c\{f\} = \int_0^\infty f(x)\cos\beta x\,dx$

Table of Fourier cosine transform

	$f(x), x>0$		$F_c(\beta), \beta>0$
F_c1.	$\begin{cases}1, x<a\\ 0, x>a\end{cases}$	$(a>0)$	$\dfrac{\sin a\beta}{\beta}$
F_c2.	e^{-ax}	$(a>0)$	$\dfrac{a}{a^2+\beta^2}$
F_c3.	e^{-ax^2}	$(a>0)$	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\,e^{-\beta^2/4a}$
F_c4.	x^{a-1}	$(0<a<1)$	$\Gamma(a)\beta^{-a}\cos\dfrac{a\pi}{2}$
F_c5.	$\cos ax^2$		$\dfrac{1}{2}\sqrt{\dfrac{\pi}{a}}\cos\left(\dfrac{\beta^2}{4a}-\dfrac{\pi}{4}\right)$
F_c6.	$\sin ax^2$		$\dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}\cos\left(\dfrac{\beta^2}{4a}+\dfrac{\pi}{4}\right)$

Some Useful Mathematical Relationships

$\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$
$\sin(x) = \frac{e^{jx} - e^{-jx}}{2j}$
$\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$
$\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
$\cos(2x) = \cos^2(x) - \sin^2(x)$
$\sin(2x) = 2\sin(x)\cos(x)$
$2\cos^2(x) = 1 + \cos(2x)$
$2\sin^2(x) = 1 - \cos(2x)$
$\cos^2(x) + \sin^2(x) = 1$
$2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)$
$2\sin(x)\sin(y) = \cos(x-y) - \cos(x+y)$
$2\sin(x)\cos(y) = \sin(x-y) + \sin(x+y)$

Useful Integrals

$\int \cos(x)dx$	$\sin(x)$
$\int \sin(x)dx$	$-\cos(x)$
$\int x\cos(x)dx$	$\cos(x) + x\sin(x)$
$\int x\sin(x)dx$	$\sin(x) - x\cos(x)$
$\int x^2\cos(x)dx$	$2x\cos(x) + (x^2 - 2)\sin(x)$
$\int x^2\sin(x)dx$	$2x\sin(x) - (x^2 - 2)\cos(x)$
$\int e^{\alpha x}dx$	$\frac{e^{\alpha x}}{\alpha}$
$\int xe^{\alpha x}dx$	$e^{\alpha x}\left[\frac{x}{\alpha} - \frac{1}{\alpha^2}\right]$
$\int x^2e^{\alpha x}dx$	$e^{\alpha x}\left[\frac{x^2}{\alpha} - \frac{2x}{\alpha^2} - \frac{2}{\alpha^3}\right]$
$\int \frac{dx}{\alpha + \beta x}$	$\frac{1}{\beta}\ln\lvert\alpha + \beta x\rvert$
$\int \frac{dx}{\alpha^2 + \beta^2x^2}$	$\frac{1}{\alpha\beta}\tan^{-1}(\frac{\beta x}{\alpha})$

$F(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixy} f(y)\,dy$	$f(y)=\int_{-\infty}^{+\infty} e^{ixy} F(x)\,dx$
$\frac{1}{\sqrt{\|x\|}}$	$\sqrt{\frac{2\pi}{\|y\|}}$
$\operatorname{sgn} x \frac{1}{\sqrt{\|x\|}}$	$i \operatorname{sgn} y \sqrt{\frac{2\pi}{\|y\|}}$
$\frac{\sin a x}{x}$	$\begin{cases} \pi & \text{for } \|y\| \\ 0 & \text{,, } \|y\| \end{cases}$
$\frac{\sin^2 a x}{x^2}$	$\begin{cases} \pi\left(a-\frac{y}{2}\right) & \text{for } \|y\|< \\ 0 & \text{,, } \|y\|> \end{cases}$
$\left.\begin{matrix} e^{i\omega x} & \text{for } p<x<q \\ 0 & \text{,, } x<p;\ x>q \end{matrix}\right\}$	$i\frac{e^{ip(\omega+y)}-e^{iq(\omega+y)}}{\omega+y}$
$\left.\begin{matrix} e^{-\beta x} e^{i\omega x} & \text{for } x>0 \\ 0 & \text{,, } x<0 \end{matrix}\right\}$	$\frac{i}{\omega+y+i\beta}$
$\frac{1}{\mathfrak{Cof}\, a x}$	$\frac{\pi}{a}\frac{1}{\mathfrak{Cof}\left(\frac{\pi y}{2a}\right)}$
$e^{-\lambda x^2} \quad \operatorname{Re}(\lambda)>0$	$\sqrt{\frac{\pi}{\lambda}}\, e^{-y^2/4\lambda} \quad \operatorname{Re}(\lambda) \; \operatorname{Re}(\sqrt{\lambda})$
$\cos a x^2$	$\sqrt{\frac{\pi}{a}} \cos\left(\frac{y^2}{4a}-\frac{\pi}{4}\right)$
$\sin a x^2$	$\sqrt{\frac{\pi}{a}} \cos\left(\frac{y^2}{4a}+\frac{\pi}{4}\right)$
$\frac{1}{\sqrt{x^2+a^2}}$	$2K_0(a\|y\|)$
$\operatorname{sgn} x \sqrt{\frac{1}{x^2+a^2}}$	$\pi i \operatorname{sgn} y\,[J_0(iay)+i\,\mathrm{H}_0(iay)]$
$\frac{1}{x^2+a^2}$	$\frac{\pi}{a} e^{-a\|y\|}$
$\operatorname{sgn} x \frac{1}{x^2+a^2}$	$\frac{i}{a}[e^{-ay}\operatorname{Ei}(ay)-e^{ay}\operatorname{Ei}(-ay)]$
$\frac{e^{-a\|x\|}}{\sqrt{\|x\|}}$	$\sqrt{2\pi}\,\frac{\sqrt{\sqrt{a^2+y^2}+a}}{\sqrt{a^2+y^2}}$
$\operatorname{sgn} x \frac{e^{-a\|x\|}}{\sqrt{\|x\|}}$	$i\sqrt{2\pi} \operatorname{sgn} y \frac{\sqrt{\sqrt{a^2+y^2}-a}}{\sqrt{a^2+y^2}}$

Note: $\mathfrak{Cof} \equiv \cosh$

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$F(x)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-ixy} f(y)\,dy$	$f(y)=\int_{-\infty}^{+\infty} e^{ixy} F(x)\,dx$
$\frac{\cos a x}{\sqrt{\|x\|}}$	$\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{\|y-a\|}}+\frac{1}{\sqrt{\|y+a\|}}\right)$
$\frac{\sin a x}{\sqrt{\|x\|}}$	$\sqrt{\frac{\pi}{2}}\left(\frac{1}{\sqrt{\|y-a\|}}-\frac{1}{\sqrt{\|y+a\|}}\right)$
$\frac{\sin a x^2}{x}$	$-i\pi\left[S\left(\frac{y}{\sqrt{2\pi a}}\right)-C\left(\frac{y}{\sqrt{2\pi a}}\right)\right]$
$\frac{\sin a x^2}{x^2}$	$\pi y\left[S\left(\frac{y}{\sqrt{2\pi a}}\right)-C\left(\frac{y}{\sqrt{2\pi a}}\right)\right]+2\sqrt{\pi a}\sin\left(\frac{y^2}{4a}+\frac{\pi}{4}\right)$
$\operatorname{sgn} x \cos a x^2$	$i\sqrt{\frac{2\pi}{a}}\left\{\sin\left(\frac{y^2}{4a}\right) C\left(\frac{y}{\sqrt{2\pi a}}\right)-\cos\left(\frac{y^2}{4a}\right) S\left(\frac{y}{\sqrt{2\pi a}}\right)\right\}$
$\operatorname{sgn} x \sin a x^2$	$i\sqrt{\frac{2\pi}{a}}\left\{\cos\left(\frac{y^2}{4a}\right) C\left(\frac{y}{\sqrt{2\pi a}}\right)+\sin\left(\frac{y^2}{4a}\right) S\left(\frac{y}{\sqrt{2\pi a}}\right)\right\}$
$\frac{\sin^2 a x}{x}$	$\begin{cases} \operatorname{sgn} y\, i\frac{\pi}{2} & \text{for } \|y\|<2a \\ 0 & \text{,, } \|y\|>2a \end{cases}$
$\frac{\mathfrak{Cof}\, a x}{\mathfrak{Cof}\, \pi x}$	$\frac{2\cos\frac{a}{2}\mathfrak{Cof}\frac{y}{2}}{\cos a+\mathfrak{Cof}\, y} \quad a<\pi$
$\left.\begin{matrix} \frac{\sin b\sqrt{a^2-x^2}}{\sqrt{a^2-x^2}} & \text{for } \|x\|<a \\ -\frac{e^{-b\sqrt{x^2-a^2}}}{\sqrt{x^2-a^2}} & \text{,, } \|x\|>a \end{matrix}\right\}$	$\pi N_0(a\sqrt{b^2+y^2})$
$\frac{e^{-a\sqrt{\|x\|}}}{\sqrt{\|x\|}}$	$\sqrt{\frac{2\pi}{\|y\|}}(\cos\sqrt{2a\|y\|}-\sin\sqrt{2a\|y\|})$
$\operatorname{sgn} x \frac{e^{-a\sqrt{\|x\|}}}{\sqrt{\|x\|}}$	$i \operatorname{sgn} y \sqrt{\frac{2\pi}{\|y\|}}(\cos\sqrt{2a\|y\|}+\sin\sqrt{2a\|y\|})$
$\left.\begin{matrix} \frac{1}{\sqrt{a^2-x^2}} & \text{for } \|x\|<a \\ 0 & \text{,, } \|x\|>a \end{matrix}\right\}$	$\pi J_0(ay)$
$\left.\begin{matrix} 0 & \text{for } \|x\|<a \\ \frac{\operatorname{sgn} x}{\sqrt{x^2-a^2}} & \text{,, } \|x\|>a \end{matrix}\right\}$	$\pi i \operatorname{sgn} y\, J_0(ay)$

$F(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-ixy} f(y)\,dy$	$f(y)=\int\limits_{-\infty}^{+\infty} e^{ixy} F(x)\,dx$
$\left.\begin{array}{ll} 0 & \text{for } \lvert x\rvert<a \\ -\frac{1}{\sqrt{x^2-a^2}} & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\pi N_0(a\lvert y\rvert)$
$\left.\begin{array}{ll} \frac{\arcsin\left(\frac{x}{a}\right)}{\sqrt{a^2-x^2}} & \text{for } \lvert x\rvert<a \\ \frac{\ln\left[\frac{x}{a}-\sqrt{\left(\frac{x}{a}\right)^2-1}\right]}{\sqrt{x^2-a^2}} & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\operatorname{sgn} y\, N_0(a\lvert y\rvert)\, i\frac{\pi^2}{2}$
$\frac{1}{2}\frac{1}{(x^2+a^2)^{\nu+1/2}}$ $(\operatorname{Re}\nu>-\frac{1}{2})$	$\left(\frac{\lvert y\rvert}{2a}\right)^{\nu}\frac{\Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})}K_{\nu}(a\lvert y\rvert)$ $(\operatorname{Re}\nu>-$
$\frac{\mathfrak{Ar}\,\mathfrak{Sin}\left(\frac{x}{a}\right)}{\sqrt{x^2+a^2}}$	$\pi i \operatorname{sgn} y\, K_0(a\lvert y\rvert)$
e^{iax^3}	$\left\{\begin{array}{ll} \frac{2}{3a}\sqrt{\frac{y}{a}}K_{1/3}\left(\frac{2y}{3a}\sqrt{\frac{y}{3a}}\right) & \text{for } y> \\ \frac{2\pi}{3a}\sqrt{\frac{\lvert y\rvert}{3a}}\left[J_{1/3}\left(\frac{2\lvert y\rvert}{3a}\sqrt{\frac{\lvert y\rvert}{3a}}\right)+J_{-1/3}\left(\frac{2\lvert y\rvert}{3a}\sqrt{\frac{\lvert y\rvert}{3a}}\right.\right. & \text{for } y< \end{array}\right.$
$\frac{\cos\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}$	$\left\{\begin{array}{ll} 2K_0\left(a\sqrt{y^2-b^2}\right) & \text{for } \lvert y\rvert> \\ -\pi N_0\left(a\sqrt{b^2-y^2}\right) & \text{,, } \lvert y\rvert< \end{array}\right.$
$\frac{\sin\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}$	$\left\{\begin{array}{ll} 0 & \text{for } \lvert y\rvert> \\ \pi J_0\left(a\sqrt{b^2-y^2}\right) & \text{,, } \lvert y\rvert< \end{array}\right.$
$\frac{e^{-b\sqrt{a^2+x^2}}}{\sqrt{a^2+x^2}}$	$2K_0\left(a\sqrt{b^2+y^2}\right)$
$\frac{e^{ib\sqrt{a^2+x^2}}}{\sqrt{a^2+x^2}}$	$\pi i H_0^1\left(a\sqrt{b^2-y^2}\right)$
$\left.\begin{array}{ll} \frac{\cos\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} & \text{for } \lvert x\rvert<a \\ 0 & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\pi J_0\left(a\sqrt{b^2+y^2}\right)$
$\left.\begin{array}{ll} -\frac{\sin\left(b\sqrt{x^2-a^2}\right)}{\sqrt{x^2-a^2}} & \text{for } \lvert x\rvert>a \\ +\frac{e^{-b\sqrt{a^2-x^2}}}{\sqrt{a^2-x^2}} & \text{,, } \lvert x\rvert<a \end{array}\right\}$	$\left\{\begin{array}{ll} \pi J_0\left(a\sqrt{y^2-b^2}\right) & \text{for } \lvert y\rvert> \\ 0 & \text{,, } \lvert y\rvert< \end{array}\right.$

Note: $\mathfrak{Sin}$ = sinh

$F(x)=\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty} e^{-ixy} f(y)\,dy$	$f(y)=\int\limits_{-\infty}^{+\infty} e^{ixy} F(x)\,dx$
$\left.\begin{array}{ll} \operatorname{sgn} x\frac{\cos\left(b\sqrt{x^2-a^2}\right)}{\sqrt{x^2-a^2}}; & \lvert x\rvert>a \\ 0; & \lvert x\rvert<a \end{array}\right\}$	$\left\{\begin{array}{ll} \pi i \operatorname{sgn} y\, J_0\left(a\sqrt{y^2-b^2}\right) & \text{for } \lvert y\rvert>b \\ 0 & \text{,, } \lvert y\rvert<b \end{array}\right.$
$\left.\begin{array}{ll} \frac{\mathfrak{Cof}\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} & \text{for } \lvert x\rvert<a \\ 0 & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\pi J_0\left(a\sqrt{y^2-b^2}\right)$
$\frac{\sin\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}\times$ $\times\left[\operatorname{Ci}\left(b\sqrt{a^2+x^2}+bx\right)+\operatorname{Ci}\left(b\sqrt{a^2+x^2}-bx\right)\right]-$ $-\frac{\cos\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}\times$ $\times\left[\operatorname{Si}\left(b\sqrt{a^2+x^2}+bx\right)+\operatorname{Si}\left(b\sqrt{a^2+x^2}-bx\right)\right]$	$\begin{array}{ll} \pi^2 N_0\left(a\sqrt{b^2-y^2}\right) & \text{for } \lvert y\rvert<b \\ 0 & \text{,, } \lvert y\rvert>b \end{array}$
$\frac{\cos\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}\times$ $\times\left[\pi-\operatorname{Si}\left(b\sqrt{a^2+x^2}+bx\right)-\operatorname{Si}\left(b\sqrt{a^2+x^2}-bx\right)\right]+$ $+\frac{\sin\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}}\times$ $\times\left[\operatorname{Ci}\left(b\sqrt{a^2+x^2}+bx\right)+\operatorname{Ci}\left(b\sqrt{a^2+x^2}-bx\right)\right]$	$\left\{\begin{array}{ll} 2\pi K_0\left(a\sqrt{y^2-b^2}\right) & \text{for } \lvert y\rvert>b \\ 0 & \text{,, } \lvert y\rvert<b \end{array}\right.$
$\left.\begin{array}{ll} P_n(x) & \text{for } \lvert x\rvert<1 \\ 0 & \text{,, } \lvert x\rvert>1 \end{array}\right\}$	$2i^n\sqrt{\frac{\pi}{2y}}J_{n+1/2}(y)$
$\left.\begin{array}{ll} \frac{J_{\nu}\left(b\sqrt{a^2-x^2}\right)}{\sqrt{a^2-x^2}} & \text{for } \lvert x\rvert<a \\ 0 & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\pi J_{\nu/2}\left[\frac{a}{2}\left(\sqrt{b^2+y^2}-y\right)\right]\times$ $\times J_{\nu/2}\left[\frac{a}{2}\left(\sqrt{b^2+y^2}+y\right)\right]$
$\frac{J_{\nu}\left(b\sqrt{a^2+x^2}\right)}{\sqrt{a^2+x^2}^{\,\nu}}$	$\left\{\begin{array}{ll} \sqrt{2\pi}\,a^{-\nu}b^{-\nu}\sqrt{b^2-y^2}^{\,\nu-1/2}J_{\nu-1/2}\left(a\sqrt{b^2-y^2}\right) & \text{for } \lvert y\rvert<b \\ 0 & \text{,, } \lvert y\rvert>b \end{array}\right.$
$\left.\begin{array}{ll} \sqrt{a^2-x^2}^{\,\nu}J_{\nu}\left(b\sqrt{a^2-x^2}\right) & \text{for } \lvert x\rvert<a \\ 0 & \text{,, } \lvert x\rvert>a \end{array}\right\}$	$\sqrt{2\pi}\,a^{\nu}b^{\nu}\frac{J_{\nu+1/2}\left(a\sqrt{b^2+y^2}\right)}{\sqrt{b^2+y^2}^{\,\nu+1/2}}$

Table of Fourier Transform Pairs

Function, f(t)	Fourier Transform, F(ω)
Definition of Inverse Fourier Transform $f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(\omega)e^{j\omega t}d\omega$	*Definition of Fourier Transform* $F(\omega)=\int_{-\infty}^{\infty}f(t)e^{-j\omega t}dt$
$f(t-t_0)$	$F(\omega)e^{-j\omega t_0}$
$f(t)e^{j\omega_0 t}$	$F(\omega-\omega_0)$
$f(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}F(\frac{\omega}{\alpha})$
$F(t)$	$2\pi f(-\omega)$
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$
$(-jt)^n f(t)$	$\frac{d^n F(\omega)}{d\omega^n}$
$\int_{-\infty}^{t}f(\tau)d\tau$	$\frac{F(\omega)}{j\omega}+\pi F(0)\delta(\omega)$
$\delta(t)$	1
$e^{j\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$
$\mathrm{sgn}(t)$	$\frac{2}{j\omega}$

Fourier Transform Table

UBC M267 Resources for 2005

$F(t)$	$\widehat{F}(\omega)$	Notes	(0)
$f(t)$	$\int_{-\infty}^{\infty} f(t)e^{-i\omega t}\,dt$	Definition.	(1)
$\frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{f}(\omega)e^{i\omega t}\,d\omega$	$\widehat{f}(\omega)$	Inversion formula.	(2)
$\widehat{f}(-t)$	$2\pi f(\omega)$	Duality property.	(3)
$e^{-at}u(t)$	$\frac{1}{a+i\omega}$	a constant, $\Re(a)>0$	(4)
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+\omega^2}$	a constant, $\Re(a)>0$	(5)
$\beta(t)=\begin{cases}1, & \text{if } \lvert t\rvert<1,\\ 0, & \text{if } \lvert t\rvert>1\end{cases}$	$2\,\mathrm{sinc}(\omega)=2\frac{\sin(\omega)}{\omega}$	Boxcar in time.	(6)
$\frac{1}{\pi}\mathrm{sinc}(t)$	$\beta(\omega)$	Boxcar in frequency.	(7)
$f'(t)$	$i\omega\widehat{f}(\omega)$	Derivative in time.	(8)
$f''(t)$	$(i\omega)^2\widehat{f}(\omega)$	Higher derivatives similar.	(9)
$tf(t)$	$i\frac{d}{d\omega}\widehat{f}(\omega)$	Derivative in frequency.	(10)
$t^2 f(t)$	$i^2\frac{d^2}{d\omega^2}\widehat{f}(\omega)$	Higher derivatives similar.	(11)
$e^{i\omega_0 t}f(t)$	$\widehat{f}(\omega-\omega_0)$	Modulation property.	(12)
$f\left(\frac{t-t_0}{k}\right)$	$ke^{-i\omega t_0}\widehat{f}(k\omega)$	Time shift and squeeze.	(13)
$(f*g)(t)$	$\widehat{f}(\omega)\widehat{g}(\omega)$	Convolution in time.	(14)
$u(t)=\begin{cases}0, & \text{if } t<0\\ 1, & \text{if } t>0\end{cases}$	$\frac{1}{i\omega}+\pi\delta(\omega)$	Heaviside step function.	(15)
$\delta(t-t_0)f(t)$	$e^{-i\omega t_0}f(t_0)$	Assumes f continuous at t_0.	(16)
$e^{i\omega_0 t}$	$2\pi\delta(\omega-\omega_0)$	Useful for $\sin(\omega_0 t)$, $\cos(\omega_0 t)$.	(17)

Convolution: $$(f*g)(t)=\int_{-\infty}^{\infty} f(t-u)g(u)\,du=\int_{-\infty}^{\infty} f(u)g(t-u)\,du.$$

Parseval: $$\int_{-\infty}^{\infty}\lvert f(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left\lvert\widehat{f}(\omega)\right\rvert^2 d\omega.$$

$j\frac{1}{\pi t}$	$\mathrm{sgn}(\omega)$
$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}$	$2\pi\sum_{n=-\infty}^{\infty} F_n\delta(\omega-n\omega_0)$
$rect(\frac{t}{\tau})$	$\tau Sa(\frac{\omega\tau}{2})$
$\frac{B}{2\pi}Sa(\frac{Bt}{2})$	$rect(\frac{\omega}{B})$
$tri(t)$	$Sa^2(\frac{\omega}{2})$
$A\cos(\frac{\pi t}{2\tau})rect(\frac{t}{2\tau})$	$\frac{A\pi}{\tau}\frac{\cos(\omega\tau)}{(\pi/2\tau)^2-\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$u(t)\cos(\omega_0 t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
$u(t)\sin(\omega_0 t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega^2}{\omega_0^2-\omega^2}$
$u(t)e^{-\alpha t}\cos(\omega_0 t)$	$\frac{(\alpha+j\omega)}{\omega_0^2+(\alpha+j\omega)^2}$

$u(t)e^{-\alpha t}\sin(\omega_0 t)$	$\frac{\omega_0}{\omega_0^2+(\alpha+j\omega)^2}$
$e^{-\alpha\lvert t\rvert}$	$\frac{2\alpha}{\alpha^2+\omega^2}$
$e^{-t^2/(2\sigma^2)}$	$\sigma\sqrt{2\pi}\,e^{-\sigma^2\omega^2/2}$
$u(t)e^{-\alpha t}$	$\frac{1}{\alpha+j\omega}$
$u(t)te^{-\alpha t}$	$\frac{1}{(\alpha+j\omega)^2}$

- ***Trigonometric Fourier Series***

$$f(t)=a_0+\sum_{n=1}^{\infty}(a_n\cos(\omega_0 nt)+b_n\sin(\omega_0 nt))$$

where

$$a_0=\frac{1}{T}\int_0^T f(t)dt\ ,\quad a_n=\frac{2}{T}\int_0^T f(t)\cos(\omega_0 nt)dt\ \text{, and}$$

$$b_n=\frac{2}{T}\int_0^T f(t)\sin(\omega_0 nt)dt$$

- ***Complex Exponential Fourier Series***

$$f(t)=\sum_{n=-\infty}^{\infty}F_n e^{j\omega nt}\ \text{, where}\quad F_n=\frac{1}{T}\int_0^T f(t)e^{-j\omega_0 nt}dt$$

Appendix C

Special Fourier Transforms

$$\mathcal{F}\{f\} = \int_{-\infty}^{\infty} f(x) e^{-i\alpha x} dx$$

SPECIAL FOURIER TRANSFORM PAIRS

	$f(x)$	$F(\alpha)$
C-1	$\begin{cases} 1 & \lvert x \rvert < b \\ 0 & \lvert x \rvert > b \end{cases}$	$\dfrac{2 \sin b\alpha}{\alpha}$
C-2	$\dfrac{1}{x^2 + b^2}$	$\dfrac{\pi e^{-b\alpha}}{b}$
C-3	$\dfrac{x}{x^2 + b^2}$	$-\dfrac{\pi i \alpha}{b} e^{-b\alpha}$
C-4	$f^{(n)}(x)$	$i^n \alpha^n F(\alpha)$
C-5	$x^n f(x)$	$i^n \dfrac{d^n F}{d\alpha^n}$
C-6	$f(bx)e^{itx}$	$\dfrac{1}{b} F\left(\dfrac{\alpha - t}{b}\right)$

SPECIAL FOURIER SINE TRANSFORMS

$F_S(\alpha) = \int_0^\infty f(x) \sin \alpha x \, dx$

	$f(x)$	$F_S(\alpha)$
C-21	$\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$	$\frac{1 - \cos b\alpha}{\alpha}$
C-22	x^{-1}	$\frac{\pi}{2}$
C-23	$\frac{x}{x^2 + b^2}$	$\frac{\pi}{2} e^{-b\alpha}$
C-24	e^{-bx}	$\frac{\alpha}{\alpha^2 + b^2}$
C-25	$x^{n-1} e^{-bx}$	$\frac{\Gamma(n) \sin (n \tan^{-1} \alpha/b)}{(\alpha^2 + b^2)^{n/2}}$
C-26	xe^{-bx^2}	$\frac{\sqrt{\pi}}{4b^{3/2}} \alpha e^{-\alpha^2/4b}$
C-27	$x^{-1/2}$	$\sqrt{\frac{\pi}{2\alpha}}$
C-28	x^{-n}	$\frac{\pi \alpha^{n-1} \csc (n\pi/2)}{2\,\Gamma(n)} \quad 0 < n < 2$
C-29	$\frac{\sin bx}{x}$	$\frac{1}{2} \ln \left(\frac{\alpha + b}{\alpha - b} \right)$
C-30	$\frac{\sin bx}{x^2}$	$\begin{cases} \pi\alpha/2 & \alpha < b \\ \pi b/2 & \alpha > b \end{cases}$
C-31	$\frac{\cos bx}{x}$	$\begin{cases} 0 & \alpha < b \\ \pi/4 & \alpha = b \\ \pi/2 & \alpha > b \end{cases}$
C-32	$\tan^{-1} (x/b)$	$\frac{\pi}{2\alpha} e^{-b\alpha}$
C-33	$\csc bx$	$\frac{\pi}{2b} \tanh \frac{\pi\alpha}{2b}$
C-34	$\frac{1}{e^{2\pi x} - 1}$	$\frac{\pi}{4} \coth \left(\frac{\pi\alpha}{2} \right) - \frac{1}{2\alpha}$

$F_c\{f\} = \int_0^\infty f(x) \cos \alpha x \, dx$

SPECIAL FOURIER COSINE TRANSFORMS

	$f(x)$	$F_C(\alpha)$
C-7	$\begin{cases} 1 & 0 < x < b \\ 0 & x > b \end{cases}$	$\dfrac{\sin b\alpha}{\alpha}$
C-8	$\dfrac{1}{x^2 + b^2}$	$\dfrac{\pi e^{-b\alpha}}{2b}$
C-9	e^{-bx}	$\dfrac{b}{\alpha^2 + b^2}$
C-10	$x^{n-1} e^{-bx}$	$\dfrac{\Gamma(n) \cos (n \tan^{-1} \alpha/b)}{(\alpha^2 + b^2)^{n/2}}$
C-11	e^{-bx^2}	$\dfrac{1}{2}\sqrt{\dfrac{\pi}{b}}\, e^{-\alpha^2/4b}$
C-12	$x^{-1/2}$	$\sqrt{\dfrac{\pi}{2\alpha}}$
C-13	x^{-n}	$\dfrac{\pi \alpha^{n-1} \sec (n\pi/2)}{2\,\Gamma(n)}, \quad 0 < n < 1$
C-14	$\ln\left(\dfrac{x^2 + b^2}{x^2 + c^2}\right)$	$\dfrac{e^{-c\alpha} - e^{-b\alpha}}{\pi \alpha}$
C-15	$\dfrac{\sin bx}{x}$	$\begin{cases} \pi/2 & \alpha < b \\ \pi/4 & \alpha = b \\ 0 & \alpha > b \end{cases}$
C-16	$\sin bx^2$	$\sqrt{\dfrac{\pi}{8b}}\left(\cos \dfrac{\alpha^2}{4b} - \sin \dfrac{\alpha^2}{4b}\right)$
C-17	$\cos bx^2$	$\sqrt{\dfrac{\pi}{8b}}\left(\cos \dfrac{\alpha^2}{4b} + \sin \dfrac{\alpha^2}{4b}\right)$
C-18	$\operatorname{sech} bx$	$\dfrac{\pi}{2b} \operatorname{sech} \dfrac{\pi\alpha}{2b}$
C-19	$\dfrac{\cosh (\sqrt{\pi}\, x/2)}{\cosh (\sqrt{\pi}\, x)}$	$\sqrt{\dfrac{\pi}{2}}\, \dfrac{\cosh (\sqrt{\pi}\, \alpha/2)}{\cosh (\sqrt{\pi}\, \alpha)}$
C-20	$\dfrac{e^{-b\sqrt{x}}}{\sqrt{x}}$	$\sqrt{\dfrac{\pi}{2\alpha}}\{\cos (2b\sqrt{\alpha}) - \sin (2b\sqrt{\alpha})\}$

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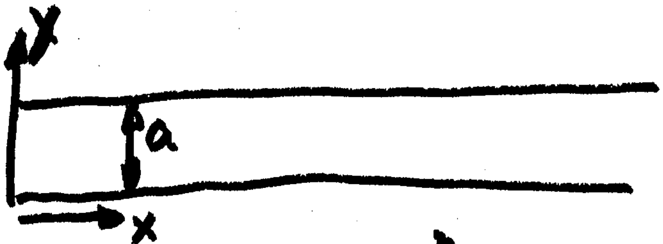
SELF SIMILAR SYSTEMS



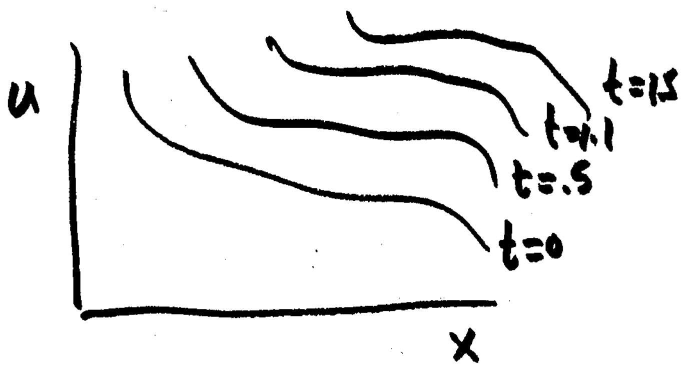
$$z = \frac{y}{a} \qquad w = \frac{x}{a}$$

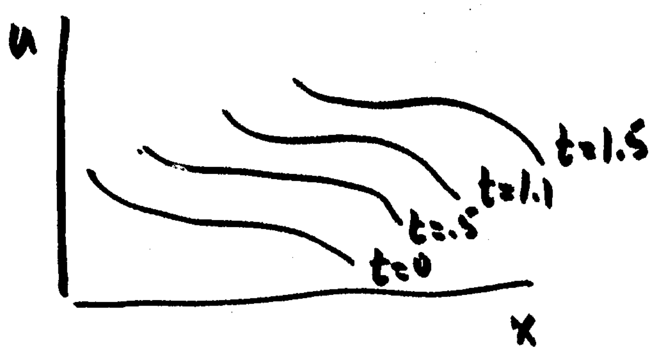
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \Rightarrow \frac{\partial^2 u}{a^2 \partial w^2} = \frac{\partial u}{\partial t}$$

let $\bar{u} = \dfrac{u}{u_0}$ $\qquad \dfrac{u_0\, \partial^2 \bar{u}}{a^2\, \partial w^2} = u_0 \dfrac{\partial \bar{u}}{\partial t} = \dfrac{u_0\, \partial \bar{u}}{a^2\, \partial \bar{t}}$

let $\bar{t} = t/a^2$ $\qquad \dfrac{\partial^2 \bar{u}}{\partial w^2} = \dfrac{\partial \bar{u}}{\partial \bar{t}}$

THE PROBLEM IS SCALE SIMILAR





GIVE PDE HANDOUT

- CERTAIN PROBLEMS DO NOT HAVE NATURAL SCALE FOR INDEPENDENT VARIABLE

FOR EXAMPLE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_i \quad x>0$$
$$T(0,t) = T_s$$

and $x \to \infty \quad T \to T_i$

IF FOR INDEPENDENT VARIABLES OF THE PROBLEM $\nexists$ CLUE TO EXISTENCE

$$\left.\begin{array}{l}\text{• NO CHARACTERISTIC LENGTH} \\ \text{" \quad " \quad TIME}\end{array}\right\} \Rightarrow \text{SELF-SIMILAR SOLUTION.}$$

- SOLUTION OF ALL PHYSICAL PROBLEMS MAY BE EXPRESSED IN DIMENSIONLESS FORM

~~FROM INDEPENDENT VARIABLES~~

- $\Rightarrow$ t, x must form a dimensionless group

FROM PDE : $x^2 = \alpha t$ or $\frac{x^2}{\alpha t}$, $\frac{\alpha t}{x^2}$, or $\frac{x}{\sqrt{\alpha t}}$ or $\frac{\sqrt{\alpha t}}{x}$

- IF SOLN MADE DIMENSIONLESS BY COMBO OF INDEPENDENT VARIABLES INSTEAD OF GEOMETRY, BC OR IC. - PROBLEM IS SELF SIMILAR

- THERE IS A CHARACTERISTIC TEMP TO THE PROBLEM $T_s - T_i$

could guess

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \frac{x}{\sqrt{\alpha t}}\, h\left(\frac{x}{\sqrt{\alpha t}}\right)$$

- Could choose $$\frac{T}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \cdots = h\left(\frac{\alpha t}{x^2}\right)$$

- define $\eta = \frac{x}{\sqrt{\alpha t}}$ SIMILARITY VARIABLE

$\Rightarrow$ • ALL TEMPERATURE PROFILES FALL ONTO ONE GRAPH.

- CHOOSING a solution involving η
  - REDUCES PDE TO AN ODE WHICH IS A FN OF η ONLY

- NOTE x, t (2 indep var.) now becomes η (1 indep var)

  $\Rightarrow$ SELF SIMILARITY REDUCE # OF INDEPENDENT VAR. BY 1

METHOD OF APPROACH

- $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad \text{w/} \quad T(0,t) = T_s$$
  $$T(x,0) = T_i$$
  $$T(x,t) \to T_i \text{ as } x \to \infty$$

<u>IN GENERAL</u>

Choose $\eta = \dfrac{Ax}{t^n}$ $\qquad A, n$ picked to reduce eqn. to ODE

let $\dfrac{T - T_i}{T_s - T_i} = f(\eta)$

- NOTE: SINCE $T_s - T_i$ is a basic aspect of problem

  FORM OF η : SINCE $t = 0$ & $x = \infty$ give T_i

CHOOSE FORM OF η : NUMERATOR – INDEP VAR DIFFERENTIATED MOST OFTEN

DENOMINATOR – LEAST OFTEN DIFFERENTIATED

$$T = T_i + (T_s - T_i) f(\eta)$$

- $$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \; ; \quad \frac{\partial \eta}{\partial x} = \frac{A}{t^n} \; ; \quad \frac{\partial T}{\partial \eta} = (T_s - T_i)\frac{df}{d\eta}$$

$$\therefore \frac{\partial T}{\partial x} = (T_s - T_i) f' \cdot \left(\frac{A}{t^n}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial \eta}\left[(T_s - T_i) f'(\eta) \left(\frac{A}{t^n}\right)\right] \cdot \frac{\partial \eta}{\partial x} = (T_s - T_i)\left(\frac{A^2}{t^{2n}}\right) f''(\eta)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = (T_s - T_i) f' \cdot -\frac{nAx}{t^{n+1}} = (T_s - T_i) f' \left[-\frac{n\eta}{t}\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$(T_s - T_i)\,\frac{A^2}{t^{2n}}\,f''(\eta) = \frac{1}{\alpha}\,(T_s - T_i)\left(\frac{-\eta n}{t}\right) f'$$

$$(T_s - T_i)\frac{A}{t^{2n}}\left[f'' + \frac{n\eta}{\alpha A^2} t^{2n-1} f'\right] = 0$$

FOR ODE ; η, f, f', f'' only $\Rightarrow$ $t^{2n-1} = 1$ $\Rightarrow n = \frac{1}{2}$

$\therefore \quad f'' + \frac{1}{2\alpha A^2}\eta f' = 0$ pick $A \Rightarrow f'' + \eta f' = 0$

$\therefore$ choose $A = \frac{1}{\sqrt{2\alpha}}$

$\therefore \quad \eta = \frac{x}{\sqrt{2\alpha t}}$

LESSON #14

at $x = 0$ $T = T_s$ $\Rightarrow \eta = 0$ $\frac{T_s - T_i}{T_s - T_i} = 1 = f(\eta = 0)$

collapse of 2 conditions to 1
$t = 0$ $T = T_i$ $\Rightarrow \eta = \infty$ $T = T_i \Rightarrow 0 = f(\eta = \infty)$
$x \to \infty$ $T \to T_i$ $\Rightarrow \eta \to \infty$ $T \to T_i$ $0 \leftarrow f(\eta \to \infty)$

note $f'' + \eta f' = 0$ (2^{nd} order ODE)

$\frac{df'}{d\eta} + \eta f' = 0 \Rightarrow \frac{df'}{d\eta} = -\eta f'$ or $\frac{df'}{f'} = -\eta\, d\eta$

$$\ln f' = -\eta^2/2 + \ln C_1$$
$$f' = C_1 e^{-\eta^2/2}$$
$$df = C_1 e^{-\eta^2/2} d\eta \quad \text{or} \quad f = C_1 \int_0^\eta e^{-\sigma^2/2} d\sigma + C_2$$

when $\eta = 0$ $f(\eta = 0) = 1 = C_2$

$f \to 0$ as $\eta \to \infty$ NOW $\int_0^\infty e^{-\sigma^2/2} d\sigma = \sqrt{2}\int_0^\infty e^{-z^2} dz = \sqrt{\frac{\pi}{2}}$

$0 = C_1 \cdot \sqrt{\frac{\pi}{2}} + 1$ $C_1 = -\sqrt{\frac{2}{\pi}}$ $\therefore f = 1 - \sqrt{\frac{2}{\pi}} \int_0^\eta e^{-\sigma^2/2} d\sigma$

$$= 1 - \frac{2}{\sqrt{\pi}} \int_0^{\eta/\sqrt{2}} e^{-z^2} dz$$

$$\frac{T - T_i}{T_s - T_i} = f(\eta) = \text{erfc}(\eta/\sqrt{2})$$

$$\frac{T-T_i}{T_s-T_i} = erfc\left(\frac{x}{2\sqrt{\alpha t}}\right) = 1 - erf\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

1. ALWAYS REMEMBER THAT IF THE PROBLEM IS INDEPENDENT OF LENGTH OR TIME SCALES — WE HAVE SELF SIMILAR SOLUTION

2. SOLUTION WILL REDUCE # OF INDEPENDENT VARIABLES BY 1 & $\underbrace{\text{PDE BECOMES ODE}}_{\text{OF 2 INDEP. VARIABLES}}$

3. ASSUME GENERAL FORM OF TRANSFORMATION BASED ON B.C's & I.C.

4. IN SIMILARITY PARAMETER, η, MOST DIFFERENTIATED VARIABLE SHOULD APPEAR IN NUMERATOR. ie $A\frac{x}{t^n}$

5. TRANSFORM BC & IC USING SIMILARITY PARAMETER AND INSURE THEY ARE SATISFIED

IF NOT ADD ADDITIONAL DEGREES OF FREEDOM

6. CONVERT DE INTO ONE THAT CONTAINS f & ITS DERIV, η and only one of the independent variables.

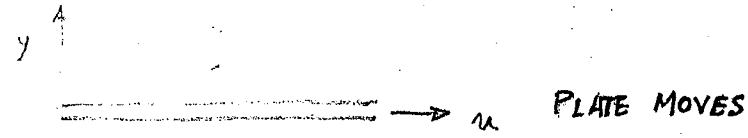
Determine parameters of η to reduce PDE order by one

7. Express BC & IC for reduced problem & solve.

LESSON # 15

MOTION OF A VISCOUS FLUID OVER AN ∞ PLATE

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$



ν kinematic viscosity

let $u = u(y,t)$ $\qquad u(y=0,t) = at^b \qquad a, b$ fixed (1)

BC $\quad u(y,t) \to 0$ as $y \to \infty$ (2)

IC $\quad u(y,0) = 0$ (3)

let $\quad \eta = B\dfrac{y}{t^n} \qquad u = A\,f(\eta) \qquad A, B, n$ constant

- for behavior of $u \to 0$ for y large & for t small $\Rightarrow \eta \to \infty$ for y large, t small

  note that (2) & (3) are collapsed into 1 condition
- Check $u(y=0,t) = A\,f(0) = at^b \quad y=0 \Rightarrow \eta = 0$

  impossible

Must add additional degree of freedom : let $u = At^m f(\eta)$

pick t^m since $u(y=0,t)$ involves t^b

$\therefore \quad u(y=0,t) = At^m f(0) = at^b$ must pick $m = b$

$A f(0) = a \qquad$ may pick $A = a \Rightarrow f(0) = 1$

Guidance : try to get fns $f(0)$, $f(\infty)$ etc to be either $1, 0, \infty$ but not 2.735, 15.2 etc.

$\therefore \quad u = at^b f(\eta) \qquad \eta = By/t^n$

$$\frac{\partial u}{\partial y} = at^b f'(\eta)\frac{d\eta}{dy} = at^b f' \cdot \frac{B}{t^n}$$

$$\partial^2 u/\partial y^2 = at^b f''(\eta)\, B^2/t^{2n}$$

$$\frac{\partial u}{\partial t} = abt^{b-1}f(\eta) + at^b f'(\eta)\cdot\frac{d\eta}{dt}$$

$$= abt^{b-1}f(\eta) + at^b f'(-nBy/t^{n+1})$$

$$\frac{\partial u}{\partial t} = abt^{b-1} f(\eta) + at^{b-1}(-n\eta) f'(\eta); \quad \text{put into } \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

$$\therefore B^2 \nu a \underline{t^{b-2n}} f'' = a\underline{t^{b-1}} [bf + f'(-n\eta)]$$

if ODE, only f, f', f'' & η appears. So...

let $2n = 1 \quad \therefore n = \frac{1}{2}$ $\qquad at^{b-1}$ cancel

$$B^2 \nu f'' + \tfrac{1}{2}\eta f' - bf = 0$$

$$2B^2\nu f'' + \eta f' - 2bf = 0 \qquad \text{let } 2B^2\nu = 1 \implies B = \frac{1}{\sqrt{2\nu}}$$

$$\therefore \quad \eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f'' + \eta f' - 2bf = 0$$

from $u(y=0, t) = at^b = at^b f(0) \implies f(0) = 1$

irrespective of t: $u(y,t) \to 0$ as $y \to \infty \implies f(\eta \to \infty) \to 0$

in order to solve $f'' + \eta f' - 2bf = 0$ we need to know b

Suppose $b = \frac{1}{2}$ $\qquad f'' + \eta f' - f = 0 \implies$ 2nd order ODE $\quad f = c_1 f_1 + c_2 f_2$

Use method of reduction in order - if you know a solution f_1 then $f_2 = f_1(\eta)\, g(\eta) \qquad f_2' = f_1' g + f_1 g' \quad f_2'' = f_1'' g + 2f_1' g' + f_1 g''$

PUT INTO ODE

$$f_1'' g + 2f_1' g_1' + f_1 g'' + \eta [f_1' g + f_1 g'] - f_1 g = 0$$

$$\overset{\checkmark\, 0}{(f_1'' + \eta f_1' - f_1)} g + f_1 g'' + (\eta f_1 + 2f_1')g' = 0$$

$$\therefore \quad \underbrace{g''/g'}_{\frac{d}{d\eta}(\ln g')} = -\left(\frac{\eta f_1 + 2f_1'}{f_1}\right) = -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\frac{d}{d\eta}\left(\eta^2/2 + 2\ln f_1\right)$$

EGM 5315

Inter. Anal. of Mech. Systems

4/14/05

$\frac{\partial u}{\partial t} = abt^{b-1} f + at^{b-1} \cdot -n\eta f'$; put into $\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

$\therefore B^2 \nu a t^{b-2n} f'' = at^{b-1}[bf + f'(-n\eta)]$

if ODE, only f, f', f'' & η appears. So...

let $2n = 1 \quad \therefore n = 1/2 \qquad at^{b-1}$ cancel

$$B^2 \nu f'' + \tfrac{1}{2}\eta f' - bf = 0$$

$$2B^2\nu f'' + \eta f' - 2bf = 0 \qquad \text{let } 2B^2\nu = 1 \implies B = \frac{1}{\sqrt{2\nu}}$$

$$\therefore \quad \eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f'' + \eta f' - 2bf = 0$$

from $u(y=0,t) = at^b = at^b f(0) \implies f(0) = 1$

irrespective of t: $u(y,t) \to 0$ as $y \to \infty \implies f(\eta \to \infty) \to 0$

in order to solve $f'' + \eta f' - 2bf = 0$ we need to know b

Suppose $b = 1/2 \qquad f'' + \eta f' - f = 0 \implies$ 2nd order ODE $\quad f = c_1 f_1 + c_2 f_2$

Use method of reduction in order - if you know a solution f_1 then $f_2 = f_1(\eta)\, g(\eta) \qquad f_2' = f_1' g + f_1 g' \quad f_2'' = f_1'' g + 2f_1' g' + f_1 g''$

PUT INTO ODE

$$f_1'' g + 2f_1' g' + f_1 g'' + \eta[f_1' g + f_1 g'] - f_1 g = 0$$

$$\overbrace{(f_1'' + \eta f_1' - f_1)}^{\checkmark 0} g + f_1 g'' + (\eta f_1 + 2f_1')g' = 0$$

$$\therefore \quad \underbrace{g''/g'}_{\frac{d}{d\eta}(\ln g')} = -\frac{(\eta f_1 + 2f_1')}{f_1} = -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\frac{d}{d\eta}\left(\frac{\eta^2}{2} + 2\ln f_1\right)$$

FIRST ORDER EQN IN g'

NOTICE $f_1 = \eta$ solves $f_1' = 1 \quad f_1'' = 0$

$$\frac{dg'}{g'} = -\left(\eta + \frac{2}{\eta}\right)d\eta \Rightarrow \ln g' = -\eta^2/2 - 2\ln\eta$$

$$g' = \frac{1}{\eta^2} e^{-\eta^2/2} \qquad g(\eta) = \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$$

PICK ∞ as the lower limit since g' is bounded

$$f = C_1\eta + C_2\eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$$

: $f(\eta) \to 0$ as $\eta \to \infty$:

$$\infty > \sigma > \eta$$

$$\infty > \sigma^2 > \eta^2 > \eta \quad \to \quad \frac{1}{\sigma^2} < \frac{1}{\eta^2} < \frac{1}{\eta}$$

$$0 < \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma < \int_\infty^\eta \frac{1}{\eta} e^{-\sigma^2/2} d\sigma \quad \text{and} \quad \frac{1}{\eta}\int_\infty^\eta e^{-\sigma^2/2} d\sigma \text{ is bounded}$$

$$f_2 = \eta g(\eta) < \int_\infty^\eta e^{-\sigma^2/2} d\sigma \qquad \text{as } \eta \to \infty \quad \int \to 0$$

$\therefore \quad f_2 \to 0$ as $\eta \to \infty$ but $f_1 = \eta \to \infty$ as $\eta \to \infty$ $\therefore C_1 = 0$

and $$f = C_2\eta \int_\infty^\eta \frac{1}{\sigma^2} e^{-\sigma^2/2} d\sigma$$

→ INTEGRATE BY PARTS

let $\frac{1}{\sigma^2} d\sigma = dv \quad w = e^{-\sigma^2/2} \quad v = -\frac{1}{\sigma} \quad dw = e^{-\sigma^2/2} \cdot (-\sigma\, d\sigma)$

: $f(\eta = 0) = 1$

$$C_2\eta \left[-\frac{1}{\sigma} e^{-\sigma^2/2} \Big|_\infty^\eta - \int_\infty^\eta \left(-\frac{1}{\sigma}\right)(-\sigma) e^{-\sigma^2/2} d\sigma \right]$$

$$\left[-\frac{1}{\eta} e^{-\eta^2/2} - \int_\infty^\eta e^{-\sigma^2/2} d\sigma \right]$$

$$f = -C_2 e^{-\eta^2/2} - C_2\eta \int_\infty^\eta e^{-\sigma^2/2} d\sigma$$

at $\eta = 0$ $\quad f = -C_2 e^0 - C_2 \cdot 0 \cdot \int_\infty^0 e^{-\sigma^2/2} d\sigma = 1 \qquad C_2 = -1$

$$\therefore \quad f(\eta) = e^{-\eta^2/2} + \eta \int_\infty^\eta e^{-\sigma^2/2} d\sigma$$

let $z = \sigma/\sqrt{2}$ $\quad z^2 = \sigma^2/2$ $\quad dz = \frac{d\sigma}{\sqrt{2}}$

$$\eta \cdot \sqrt{2} \int_{\infty}^{\eta/\sqrt{2}} e^{-z^2} dz \qquad \int_0^1 = \int_0^\infty - \int_1^\infty$$

$$- \frac{\sqrt{\pi}}{2} \operatorname{erfc} \frac{\eta}{\sqrt{2}} \qquad \int_1^\infty = \int_0^\infty - \int_0^1$$

$$f(\eta) = e^{-\eta^2/2} - \eta \sqrt{\frac{\pi}{2}} \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right) \qquad \operatorname{erfc} = 1 - \operatorname{erf}$$

LESSON #[redacted]

LAPLACE TRANSFORMS - HANDOUT TABLES

CAN BE USED TO FIND SOLUTIONS OF PDE's WHEN ONE OR MORE INDEPENDENT VARIABLES CAN RANGE FROM 0 TO ∞

ie. TIME $\quad t \geqslant 0$

$x \quad\quad x \geqslant 0$

used to solve : $\quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

we can define $\quad F(s) = \int_0^\infty f(t)\, e^{-st} dt \qquad s = \text{const.}$

note that $\quad F(s) = \mathcal{L}(f(t))$

for every $\quad f(t) \Leftrightarrow F(s)$

does $F(s)$ exist for any $f(t)$? NO.

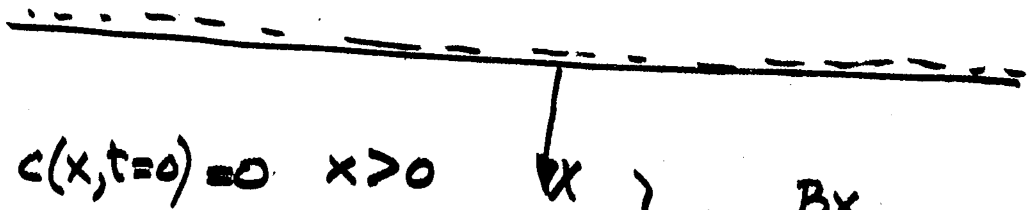
$F(s)$ exists if
- $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \le t \le T$ where $t_1 > 0$
- $t^n | f(t)|$ is bounded near $t = 0$ for some n $\quad n < 1$
- $e^{-s_0 t} | f(t)|$ is bounded for large t, for some value s_0

$|f| < \frac{M}{t^n}$

$\left|\int f e^{-st} dt\right| < \int |f| e^{-st} dt$

$< \int \frac{M}{t^n} e^{-st} dt = M \left[\frac{t^{1-n} e^{-st}}{1-n} - \right.$

$$\alpha \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}$$



initially $c(x, t=0) = 0 \quad x > 0$

for $x \to \infty \quad c(x,t) \to 0$ $\Big\} \; \eta = \dfrac{Bx}{t^m}$

$$\int_0^\infty c(x,t)\,dx = Q$$

$$c = A f(\eta) \qquad \eta = \frac{Bx}{t^m}$$

$$\int_0^\infty c(x,t)\,dx = \int_0^\infty A f(\eta)\,d\eta \cdot \frac{t^m}{B} = Q$$ impossible to satisfy

choose

$$c = A t^n f(\eta)$$

$$\int_0^\infty c(x,t)\,dx = \int_0^\infty A t^n f(\eta) \cdot d\eta \frac{t^m}{B} = Q \implies n = -m$$

$$\frac{\partial c}{\partial x} = \frac{\partial c}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = A t^n f' \cdot \frac{B}{t^m} \; ; \quad \frac{\partial^2 c}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial c}{\partial x}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial c}{\partial x}\right) \cdot \frac{\partial \eta}{\partial x}$$

$$= A t^n f'' \cdot \frac{B^2}{t^{m \cdot 2}}$$

$$\frac{\partial c}{\partial t} = A n t^{n-1} f + A t^n f' \cdot \left(-\frac{mBx}{t^{m+1}}\right)$$

$$= A n t^{n-1} f + A t^n f' \left(-\frac{m\eta}{t}\right)$$

$$\alpha \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} : \quad \alpha A t^n \frac{B^2}{t^{2m}} f'' = A n t^{n-1} f + A t^{n-1} f' \cdot (-m\eta)$$

$$\alpha A t^{-3m} B^2 f'' = -A m t^{-m-1} [f + \eta f']$$

for "t" to disappear $-3m = -m-1$ $\qquad m = \tfrac{1}{2} = -n$

$$\alpha B^2 f'' + \tfrac{1}{2}\left[f + \eta f'\right] = 0$$

$$2\alpha B^2 f'' + \eta f' + f = 0 \quad \Rightarrow \quad B = \frac{1}{\sqrt{2\alpha}}$$

$$\Rightarrow \quad f'' + \underbrace{\eta f' + f}_{} = 0$$

$$f'' + (\eta f)' = 0 \quad \Rightarrow \quad f' + \eta f = C_1$$

$$\left.\begin{array}{l} c(x,t) \rightarrow 0 \text{ as } x \rightarrow \infty \\ c(x,t=0) = 0 \end{array}\right\} \ \eta = \infty \qquad \eta = \frac{Bx}{t^m}$$

$c = At^n f(\eta)$ since $c \rightarrow 0$ as $\eta \rightarrow \infty$ irrespective of t $\Rightarrow$ $\underline{f(\eta \rightarrow \infty) \rightarrow 0}$

f

η

$\Rightarrow f' \rightarrow 0$ as $\eta \rightarrow \infty$

$f' + \eta f = C_1 \quad \Rightarrow \quad C_1 = 0$ since f' & $f \rightarrow 0$ as $\eta \rightarrow \infty$

$$\Rightarrow \quad f' + \eta f = 0 \quad \Rightarrow \quad \frac{df}{f} = -\eta\, d\eta$$

$$\ln f = -\eta^2/2 + \ln C_2$$

$$f = C_2 e^{-\eta^2/2}$$

$$c = A t^{n} f(\eta) \qquad n = -m = -\tfrac{1}{2}$$

$$\eta = \frac{Bx}{t^{m}} \cdot = \frac{x}{\sqrt{2\alpha t}}$$

$$f = C_2 e^{-\eta^2/2}$$

$$\int_0^\infty c(x,t)\,dx = \frac{A}{B}\int_0^\infty f(\eta)\,d\eta = Q$$

$A t^{n} f(\eta)\, dx$

$d\eta = \frac{B}{t^{m}} dx$

choose $f(\eta = 0) = 1 \Longrightarrow C_2 = 1 \Longrightarrow f = e^{-\eta^2/2}$

$$\int_0^\infty c(x,t)\,dx = \frac{A}{B}\int_0^\infty e^{-\eta^2/2}\,d\eta = Q$$

$$= \sqrt{2\alpha}\cdot A \underbrace{\int_0^\infty e^{-z^2}\,dz}_{\frac{\sqrt{\pi}}{2}} \cdot \sqrt{2} = Q$$

$$A = \frac{Q}{\sqrt{\pi\alpha}}$$

$$c = \frac{Q}{\sqrt{\pi\alpha t}}\, e^{-\frac{x^2}{4\alpha t}}$$

- Do 2.1 & 2.2   4/19/05

Exercises:

2.1 The temperature field $T(x,t)$ in a semi-infinite slab with a constant heat flux is described by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad ; \quad T(x,0) = T_i$$

$$T(x,t) \to T_i \text{ as } x \to \infty \quad ; \quad -k \frac{\partial T}{\partial x} = q \text{ at } x = 0$$

Solve for the temperature field for $x \geq 0$, $t \geq 0$.

2.2 The temperature field in the thermal boundary layer that grows within a hydrodynamic boundary layer at a step in wall temperature is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \to T_\infty \text{ as } y \to \infty \quad ; \quad T(x,0) = T_w \ ;$$

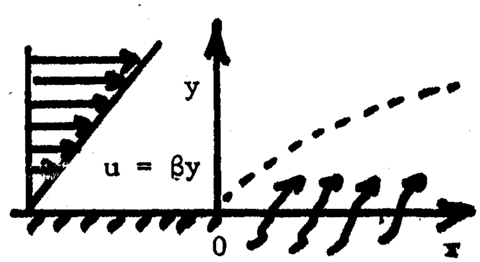
Solve for the temperature field for $x \geq 0$, $y \geq 0$.

2.3 A device for measuring the velocity gradient in flows is shown in the figure. It consists of a heated plate at the wall, over which a thermal boundary layer grows. As long as the thermal boundary layer is confined to the region where the flow velocity u is linear $(u = \beta y)$, the problem is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \to T_\infty \text{ as } y \to \infty \quad ; \quad -k \frac{\partial T}{\partial y} = q \text{ at } y = 0$$

Derive an expression relating the <u>local</u> wall temperature, $T_w(x)$, to the flow parameters and x . Evaluate any constants in this expression.
Hint: Γ .



Florida International University
Department of Mechanical Engineering

EML 5315 EXAMINATION NO. 1 5 November 2001

This examination will be a TAKE HOME examination that will be by 6pm on November 8. You may use your notes and your book and nothing else.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

____________________ PRINT NAME

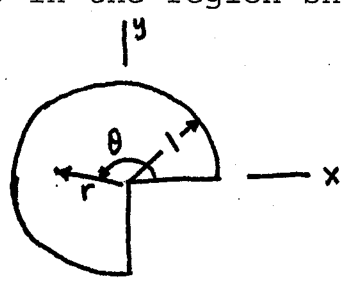
____________________ SIGN NAME

This examination consists of **several problems.** **Do all problems.** Read each question carefully. Show all work!!!!!

1. Classify the following partial differential equation and find its characteristics $\varphi(x,y)$=Constant.

$$2U_{,xx} + 3\,U_{,xy} - 5\,U_{,yy} + 9\,U = 0$$

2. Solve $\nabla^2 W = 0$ in the region shown, given the following boundary conditions:



$W(r,\ 0) = 0$
$W(r,\ 3\pi/2) = 0$
$W(r = 1,\ \theta) = 0$

(HINT: IF $W(r,\theta) = R(r)\,F(\theta)$, SHOW THAT $R(r)$ HAS SOLUTIONS OF THE FORM r^{n} and r^{-n}).

3. Given the telegraph equation

$$v_{,xx} = \alpha v_{,tt} + \beta v_{,t} + \gamma v$$

where $v(x,t)$ is the voltage and $\alpha = CL$, $\beta = CR + LG$, $\gamma = GR$, <u>determine</u>

a) What type of equation this is (e.g. linear, nonlinear, parabolic, elliptic etc.)

b) What are its characteristics $\varphi(x,t) = \text{constant}$

c) The reduced canonical equation (otherwise known as the reduced basic equation).

d) The solution to the equation given the following boundary and initial conditions:

1) if at $x = 0$, the telegraph line is grounded for all time so that there is no voltage ($v(0, t) = 0$) ;

2) if at $x = L$, the telegraph line is "isolated" for all time so that $v_{,x}(L, t) = 0$;

3) initially $v(x, t = 0) = f(x)$ and $v_{,t}(x, t = 0) = 0$.

IN SOLVING PART d, USE ONLY THE METHODS TAUGHT IN CLASS.

4. Use the Method of power series solutions to find the general solution to the following differential equation, valid near $x = 0$:

$x(1-x)\, y'' - 2\, y' + 2\, y = 0$ where $(\ )'$ means $d(\ )/dx$

5. If one of the homogeneous solutions to the equation

$$(1 - x)\, y'' - 2\, x\, y' + 2\, y = 6\,(1 - x)$$

is $y = x$. Find the complete solution.

(HINT: Reduction of order)

#2

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \frac{k}{\kappa} g$$

$$u(x,0) = f(x)$$

$$\mathcal{L}(u) = \int_0^\infty u(x,t)e^{-st}dt \; ; \; \mathcal{L}(g) = \int g(x,t)e^{-st}dt$$

$$sU - u(x,0) = k\frac{d^2U}{dx^2} + \frac{k}{\kappa}G$$

$$\frac{d^2U}{dx^2} - \frac{s}{k}U = -\left[\frac{1}{\kappa}G + \frac{f(x)}{k}\right] = h(x;s)$$

$$\Rightarrow U_H = Ae^{\sqrt{\frac{s}{k}}x} + Be^{-\sqrt{\frac{s}{k}}x}$$

$$U_P = e^{\sqrt{s/k}\,x}v_1 + e^{-\sqrt{s/k}\,x}v_2 \quad \Rightarrow \quad \begin{array}{l} y_1v_1' + y_2v_2' = 0 \quad = e^{\sqrt{s/k}\,x}v_1' + e^{-\sqrt{s/k}\,x}v_2' \\ y_1'v_1' + y_2'v_2' = h \quad = \sqrt{s/k}\left[e^{\sqrt{s/k}\,x}v_1' - e^{-\sqrt{s/k}\,x}v_2'\right]\end{array}$$

$$W[y_1,y_2] = -2\sqrt{s/k}$$

$$v_1' = \frac{\begin{vmatrix} 0 & e^{-\sqrt{}x} \\ h & -\sqrt{s/k}\,e^{-\sqrt{}} \end{vmatrix}}{-2\sqrt{s/k}} = \frac{h(x;s)\,e^{-\sqrt{s/k}\,x}}{2\sqrt{s/k}}$$

$$v_2' = \frac{\begin{vmatrix} e^{\sqrt{}} & 0 \\ \sqrt{s/k}\,e^{\sqrt{}} & h \end{vmatrix}}{-2\sqrt{s/k}} = \frac{-h(x;s)\,e^{\sqrt{s/k}\,x}}{2\sqrt{s/k}}$$

Now $$U = Ae^{\sqrt{s/k}\,x} + Be^{-\sqrt{s/k}\,x} + \underbrace{e^{\sqrt{s/k}\,x}\int^x \frac{h(\bar{x};s)e^{-\sqrt{s/k}\,\bar{x}}}{2\sqrt{s/k}}d\bar{x} - e^{-\sqrt{s/k}\,x}\int^x \frac{h(\bar{x};s)e^{\sqrt{s/k}\,\bar{x}}}{2\sqrt{s/k}}d\bar{x}}$$

$$+ \sqrt{\frac{k}{s}}\int^x h(\bar{x};s)\sinh\sqrt{\frac{s}{k}}(x-\bar{x})\,d\bar{x}$$

$$U = Ae^{\sqrt{s/k}\,x} + Be^{-\sqrt{s/k}\,x} - \sqrt{\frac{k}{s}}\int^x\left[\frac{1}{\kappa}G(\bar{x};s) + \frac{f(\bar{x})}{k}\right]\sinh\sqrt{\frac{s}{k}}(x-\bar{x})d\bar{x}$$

① if $u(x,0) = u_0 = \text{const}$ $\Longrightarrow$ ~~$h(x,s)$~~ $f(x) = U_0$

③ if $u(x,t) = u_0 = \text{const}$ as $x \to \infty$ $\Longrightarrow$ $U(x;s) = \frac{U_0}{s}$ as $x \to \infty$

if $u(0,t) = f_1(t)$ $t>0$ $\mathcal{U}(0;t) = \mathcal{U}(0;s) = \mathcal{F}(s)$ $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st}f(t)dt$

② if $g(x,t)=0$ $\Rightarrow$ $U = Ae^{\sqrt{s/k}\,x} + Be^{-\sqrt{s/k}\,x} - \sqrt{\frac{k}{s}}\int^x\left[\frac{1}{\kappa}\cdot 0 + \frac{U_0}{k}\right]\sinh\sqrt{\frac{s}{k}}(x-\bar{x})\,d\bar{x}$

This is soln when ① & ② applied

as $x \to \infty$ $$U = Ae^{\sqrt{s/k}\,x} + Be^{-\sqrt{s/k}\,x} + \sqrt{\frac{k}{s}}\frac{U_0}{k}\left.\frac{\cosh\sqrt{\frac{s}{k}}(x-\bar{x})}{\sqrt{s/k}}\right|^x \; ; \text{ as } x\to\infty,\; x-\bar{x}\to 0$$

Now — first term $\uparrow\infty$ second to 0 third $\frac{U_0}{s}$

so for bounded solution $A=0$ and ③ is satisfied.

since $\mathcal{F}(s) = \mathcal{U}(0;s) = Be^{-\sqrt{s/k}\,\cancel{x}^{\,0}} - \sqrt{\frac{k}{s}}\int^{0}\frac{U_0}{k}\sinh\sqrt{\frac{s}{k}}(0-\bar{x})\,d\bar{x} = B + \frac{U_0}{s}$

$$B = \mathcal{F}(s) + \sqrt{\frac{k}{s}}\int^{0}\frac{\mathcal{U}_0}{k}\sinh\sqrt{\frac{s}{k}}(0-\bar{x})\,d\bar{x} = \mathcal{F}(s) + \frac{U_0}{s}$$

$$\therefore \quad \mathcal{U} = \mathcal{F}(s)e^{-\sqrt{s/k}\,x} + \sqrt{\frac{k}{s}}e^{-\sqrt{s/k}\,x}\frac{\mathcal{U}_0}{k}\frac{\cosh\sqrt{\frac{s}{k}}(0-\bar{x})}{\sqrt{s/k}}\Bigg|^{0} + \sqrt{\frac{k}{s}}\frac{U_0}{k}\frac{\cosh\sqrt{\frac{s}{k}}(x-\bar{x})}{\sqrt{s/k}}\Bigg|^{x}$$

$$= \left(\mathcal{F}(s) + \frac{U_0}{s}\right)e^{-\sqrt{s/k}\,x} + \frac{U_0}{s}$$

$$\mathcal{L}^{-1}\{\mathcal{U}\} = \underbrace{\mathcal{L}^{-1}\{\mathcal{F}e^{-\sqrt{s/k}\,x}\}}_{\int_0^t f(t-a)\frac{x}{2\sqrt{\pi k a^3}}e^{\frac{-x^2}{4ka}}da} - U_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) + U_0 = u(x,t)$$

when $f_1(t) = \text{const} \Rightarrow \mathcal{L}\{f_1(t)\} = \frac{\bar{U}}{s}$ & $\mathcal{L}^{-1}\left\{\bar{U}\frac{e^{-\sqrt{s/k}\,x}}{s}\right\} = \bar{U}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right)$

and $u(x,t) = (\bar{U} - U_0)\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) + U_0 = \bar{U}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right) + U_0\,\mathrm{erf}\left(\frac{x}{2\sqrt{kt}}\right)$

$$(1-x)y_h'' - 2xy_h' + 2y_h = 0 \qquad \text{if } y_{h_1} = x \implies (1-x)\cdot 0 - 2x\cdot 1 + 2x = 0 \checkmark$$

if $y_2 = y_1 v$ then $y'' + py' + qy = 0$

$y_2' = y_1'v + y_1 v'$

$y_2'' = y_1''v + 2y_1'v' + y_1 v''$

$$\underline{y_1''v} + 2y_1'v' + y_1 v'' + \underline{py_1'v} + py_1v' + \underline{qy_1 v} = 0$$

$$v[y_1'' + py_1' + qy_1] + (2y_1' + py_1)v' + y_1 v'' = 0$$

or
$$\frac{v''}{v'} = -\frac{2y_1' + py_1}{y_1} = -2\frac{y_1'}{y_1} - p$$

$$\frac{dv'}{v'} = -2\frac{dy_1}{y_1} - p\,dx$$

$$\ln v' = -2\ln y_1 - \int p\,dx$$

$$v' = \frac{1}{y_1^2} e^{-\int p\,dx} \quad \& \quad v = \int^x \frac{1}{y_1^2} e^{-\int^{\hat{x}} p\,d\bar{x}}\, d\hat{x}$$

$$\therefore \quad y_h'' - \frac{2x}{1-x}y' + \frac{2y}{1-x} = 6 \qquad p = \frac{-2x}{1-x} \quad q = \frac{2}{1-x} \quad g = 6$$

$$\therefore \quad v = \int^x \frac{1}{y_1^2} e^{-\int^{\hat{x}} p\,d\bar{x}}\, d\hat{x} = \int^x \frac{1}{\hat{x}^2} e^{+\int^{\hat{x}} \frac{2\bar{x}}{1-\bar{x}}d\bar{x}}\, d\hat{x} = \int \frac{1}{\hat{x}^2} e^{-2\hat{x} - 2\ln(1-\hat{x})}\, d\hat{x}$$

$$y_2 = y_1 \cdot v = x \cdot \int^x \frac{1}{\hat{x}^2(1-\hat{x})^2} e^{-2\hat{x}}\, d\hat{x}$$

since $y_{h_1} = x$ $\quad y_{h_2} = xv$ $\quad$ then with $y_p = y_{h_1} v_1 + y_{h_2} v_2$

thus
$$\begin{vmatrix} y_{h_1} & y_{h_2} \\ y_{h_1}' & y_{h_2}' \end{vmatrix} \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$$

$$J(y_{h_1}, y_{h_2}) = y_{h_1}y_{h_2}' - y_{h_2}y_{h_1}' = x[v + xv'] - xv\cdot 1 = x^2 v' = x^2 \cdot \frac{e^{-2x}}{x^2(1-x)^2} = \frac{e^{-2x}}{(1-x)^2}$$

$$v_1' = \frac{-g y_{h_2}}{J} \qquad v_2' = \frac{y_{h_1} g}{J}$$

$$v_1' = \frac{-6xv}{x^2 v'} = -\frac{6v}{xv'} \qquad v_2' = \frac{6x}{x^2 v'} = \frac{6}{xv'}$$

$$v_1 = \int^x \frac{-6v(\bar{x})}{\bar{x}e^{2\bar{x}}}(1-\bar{x})^2\, d\bar{x} \qquad v_2 = \int^x \frac{6\bar{x}(1-\bar{x})^2}{e^{-2\bar{x}}}\, d\bar{x}$$

$$\therefore \quad y_{TOT} = C_1 x + C_2 x \int^x \frac{e^{-2\hat{x}}}{\hat{x}^2(1-\hat{x})^2}\, d\hat{x} + x\int^x \frac{-6v(\bar{x})}{\bar{x}e^{2\bar{x}}}(1-\bar{x})^2\, d\bar{x} + xv(x)\int^x \frac{6\bar{x}(1-\bar{x})^2}{e^{-2\bar{x}}}\, d\bar{x}$$

$2u_{xx} + 3u_{xy} - 5u_{yy} + 9u = 0$

$$\frac{dy}{dx} = \frac{3 \pm \sqrt{9 - 4(2)(5)}}{4} = \frac{3 \pm 7}{4} = 2.5, -1$$

$$\left.\begin{aligned} y - 2.5x &= \xi \\ y + x &= \eta \end{aligned}\right\} \text{ hyperbolic, constant coeff, } 2^{nd} \text{ order PDE, linear}$$

2. $\nabla^2 W = 0 \qquad W = R\Theta$

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = 0 \qquad R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \qquad \frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \alpha^2$$

$$\Theta'' + \alpha^2\Theta = 0 \qquad \Theta = A\sin\alpha\theta + B\cos\alpha\theta$$

$$r^2R'' + rR' - \alpha^2 R = 0 \quad R = \tilde{A}r^{\alpha} + \tilde{B}r^{-\alpha}$$

$$W(r, \theta = 0) = R(r)\Theta(0) = 0 \qquad \Theta(0) = B = 0$$

$$W(r, \theta = 3\pi/2) = R(r)\Theta\left(\tfrac{3\pi}{2}\right) = 0 \qquad \Theta\left(\tfrac{3\pi}{2}\right) = 0 \implies \sin\alpha\cdot\tfrac{3\pi}{2} = 0 \qquad \frac{3\pi\alpha}{2} = n\pi \quad n = 1, 2, \ldots$$

$$\alpha = \frac{2n}{3}$$

what about $\alpha = 0 \implies \Theta'' = 0 \qquad \Theta = \bar{A}\theta + \bar{B} \qquad \Theta(0) = 0 \Rightarrow \bar{B} = 0 \qquad \Theta\left(\tfrac{3\pi}{2}\right) = 0 \Rightarrow \bar{A} = 0$

$\therefore$ $\alpha = 0$ is <u>not</u> a possible solution

Now

$$W(r = 1, \theta) = 0 \implies \left(\tilde{A}r^{\alpha} + \tilde{B}r^{-\alpha}\right)\Big|_{r=1}\Theta(\theta) = 0$$

since origin is included $\tilde{B} = 0 \qquad W(r = 1, \theta) = 0 \implies \tilde{A}\cdot 1^{2n/3}\cdot A_n\sin\frac{2n}{3}\theta = 0$

$$\implies \tilde{\tilde{A}}_n = \tilde{A}\cdot A_n = 0 \quad \text{for all } n$$

$$V_{\xi\eta} + \frac{\beta}{4\sqrt{\alpha}}(V_\eta - V_\xi) - \frac{\gamma}{4}V = 0$$

3. $V_{,xx} - \alpha V_{,tt} - \beta V_{,t} - \gamma V = 0$ linear, constant coeff, 2^{nd} order PDE, hyperbolic

let $V = u e^{\lambda x + \mu t}$

$$V_{xx} = [u_{xx} + 2u_x\lambda + \lambda^2 u]e^{(\ )}$$

$$V_{tt} = [u_{tt} + 2u_t\mu + \mu^2 u]e^{(\ )}$$

$$V_t = [u_t + \mu u]e^{(\ )}$$

eq $\implies \left\{(u_{xx} + 2u_x\lambda + \lambda^2 u) - \alpha(u_{tt} + 2u_t\mu + \mu^2 u) - \beta(u_t + \mu u) - \gamma u\right\}e^{\lambda x + \mu t} = 0$

$$u_{xx} + 2u_x\lambda - \alpha u_{tt} - u_t(2\alpha\mu + \beta) + (\lambda^2 - \alpha\mu^2 - \beta\mu - \gamma)u = 0$$

Choose $\underline{\lambda = 0}$ $\quad \underline{\mu = -\beta/2\alpha}$ $\quad \therefore \lambda^2 - \alpha\mu^2 - \beta\mu - \gamma = 0 - \frac{\alpha\beta^2}{4\alpha^2} + \frac{\beta^2}{2\alpha} - \gamma = +\frac{3}{4}\frac{\beta^2}{\alpha} - \gamma$

$\therefore$ $u_{xx} - \alpha u_{tt} + \left[\frac{3}{4}\frac{\beta^2}{\alpha} - \gamma\right]u = 0$ is governing eq on u

$$\frac{dt}{dx} = \frac{0 \pm \sqrt{0 - 4(1)(-\alpha)}}{2} = \pm\sqrt{\alpha} \qquad \therefore \left.\begin{aligned} t - \sqrt{\alpha}x &= \xi \\ t + \sqrt{\alpha}x &= \eta \end{aligned}\right\} \text{ characteristics}$$

reduced eq becomes $u_{\xi\eta} - \left[\frac{3}{4}\frac{\beta^2}{\alpha} + \gamma\right]u = 0$

or $u_{\xi\xi} - u_{\eta\eta} - \left[\frac{3}{4}\frac{\beta^2}{\alpha} + \gamma\right]u = 0$

$$\frac{\partial \xi}{\partial t} = 1 \qquad \frac{\partial \eta}{\partial t} = 1$$

$$\frac{\partial \xi}{\partial x} = -\sqrt{\alpha} \qquad \frac{\partial \eta}{\partial x} = \sqrt{\alpha}$$

$$V_x = V_\xi \, \xi_x + V_\eta \cdot \eta_x = (-V_\xi + V_\eta)\sqrt{\alpha}$$

$$V_{xx} = (V_x)_\xi \, \xi_x + (V_x)_\eta \cdot \eta_x = \left[(-V_{\xi\xi} + V_{\eta\xi})\sqrt{\alpha}\right](-\sqrt{\alpha}) + \left[(-V_{\xi\eta} + V_{\eta\eta})\sqrt{\alpha}\right]\sqrt{\alpha}$$

$$= \alpha V_{\xi\xi} - 2\alpha V_{\xi\eta} + \alpha V_{\eta\eta}$$

$$V_t = V_\xi \, \xi_t + V_\eta \cdot \eta_t = (V_\xi + V_\eta)$$

$$V_{tt} = (V_t)_\xi \, \xi_t + (V_t)_\eta \cdot \eta_t = V_{\xi\xi} + 2V_{\eta\xi} + V_{\eta\eta}$$

$$\left(\cancel{\alpha V_{\xi\xi}} - 2\alpha V_{\xi\eta} + \cancel{\alpha V_{\eta\eta}}\right) = \cancel{\alpha V_{\xi\xi}} + 2\alpha V_{\eta\xi} + \cancel{\alpha V_{\eta\eta}} + \beta(V_\xi + V_\eta) + \gamma V$$

$$V_{\eta\xi} + \frac{\beta}{4\alpha}(V_\xi + V_\eta) + \frac{\gamma}{4\alpha} V = 0$$

now let $\quad V = u e^{\lambda\xi + \mu\eta}$

$$V_\xi = (u_\xi + \lambda u) e^{\_\_\_}$$

$$V_\eta = (u_\eta + \mu u) e^{\_\_\_}$$

$$V_{\xi\eta} = (u_{\xi\eta} + \lambda u_\eta + \mu u_\xi + \lambda\mu u) e^{\_\_\_}$$

$$\left(u_{\xi\eta} + \underline{\lambda u_\eta} + \underline{\mu u_\xi} + \lambda\mu u\right) + \frac{\beta}{4\alpha}\left(\underline{u_\xi} + \lambda u\right) + \frac{\beta}{4\alpha}\left(\underline{u_\eta} + \mu u\right) + \frac{\gamma}{4\alpha} u = 0$$

$$u_{\xi\eta} + u_\xi\left[\mu + \frac{\beta}{4\alpha}\right] + u_\eta\left[\lambda + \frac{\beta}{4\alpha}\right] + u\left[\lambda\mu + \frac{\lambda\beta}{4\alpha} + \frac{\mu\beta}{4\alpha} + \frac{\gamma}{4\alpha}\right]u$$

$$\mu = -\frac{\beta}{4\alpha} \qquad \lambda = -\beta/4\alpha \qquad \frac{\beta^2}{16\alpha^2} - \frac{\beta^2}{16\alpha^2} - \frac{\beta^2}{16\alpha^2} + \frac{\gamma}{4\alpha}$$

$$-\frac{\beta^2}{16\alpha^2} + \frac{\gamma}{4\alpha}$$

$$u_{\xi\eta} - u\left[\frac{\beta^2}{16\alpha^2} - \frac{\gamma}{4\alpha}\right] = 0$$

Convert ...

$$V(x=0,t)=0 = u(x=0,t)\underbrace{e^{\cdots}}_{\neq 0} \Rightarrow \underline{u(x=0,t)=0}$$

$$\frac{\partial V}{\partial x}(x=L,t)=0 = \left[\frac{\partial u}{\partial x}+\cancel{\gamma}u\right]e^{\cancel{\gamma}x+\mu t} = \frac{\partial u}{\partial x}(x=L,t)\underbrace{e^{\mu t}}_{\neq 0} \Rightarrow \underline{\frac{\partial u}{\partial x}(x=L,t)=0}$$

$$V(x,t=0)=f(x)=u(x,t=0)e^{\cancel{\gamma}x+\mu\cdot\cancel{t}} \Rightarrow \underline{u(x,t=0)=f(x)}$$

$$\frac{\partial V}{\partial t}(x,t=0)=0 = \left[\frac{\partial u}{\partial t}+\mu u\right]e^{\cancel{\gamma}x+\mu\cancel{t}} = \frac{\partial u}{\partial t}(x,t=0)+\mu\underbrace{u(x,t=0)}_{f(x)} = g(x)$$

$$\therefore \underline{\frac{\partial u}{\partial t}(x,t=0)} = g(x)-\mu f(x) = \underline{h(x)}$$

Solve for $u(x,t)$

$\therefore$ let $u(x,t)=F(x)G(t)$ $\qquad u_{xx}-\alpha u_{tt}-\delta u=0 \qquad \delta=\frac{3}{4}\frac{\beta^2}{\alpha}+\gamma$

$F''G-\alpha FG''-\delta FG=0 \qquad \div$ by FG

$$\therefore \frac{F''}{F}-\alpha\frac{G''}{G}-\delta=0 \Rightarrow \frac{1}{\alpha}\frac{F''}{F}-\frac{\delta}{\alpha}=\frac{G''}{G}=-\omega^2 \qquad G''+\omega^2G=0,\quad F''+\underbrace{(\alpha\omega^2-\delta)}_{\epsilon^2}F=0$$

$G = A\cos\omega t + B\sin\omega t$

$F = C\sin\epsilon x + D\cos\epsilon x \qquad$ assume $\epsilon^2>0$

$u=FG \Rightarrow u(x=0,t)=0=F(0)G(t)=0 \Rightarrow F(0)=0$

$\frac{\partial u}{\partial x}(x=L,t)=0=F'(L)G(t)=0 \Rightarrow F'(L)=0$

$F(0)=0 \Rightarrow D=0$

$F'(L)=0 \Rightarrow C\epsilon\cos\epsilon L=0$ $\quad$ either $\epsilon L=(2n-1)\frac{\pi}{2}$ $\quad n=1,2,3,\ldots \quad \therefore \epsilon=\frac{(2n-1)\pi}{2L}$

$$\Rightarrow \omega_n^2=\frac{\epsilon^2+\delta}{\alpha}=\frac{(2n-1)^2\pi^2}{4L^2\alpha}+\frac{3}{4}\frac{\beta^2}{\alpha^2}+\frac{\gamma}{\alpha}>0 \checkmark$$

or $\epsilon=0 \overset{\text{Check}}{\Rightarrow} F''=0 \Rightarrow F=\tilde{C}x+\tilde{D}$

if $\left.\begin{array}{l}F(0)=0\Rightarrow\tilde{D}=0\\ F'(L)=0\Rightarrow\tilde{C}=0\end{array}\right\} F_{\epsilon=0}=0 \quad \therefore \epsilon=0$ gives trivial

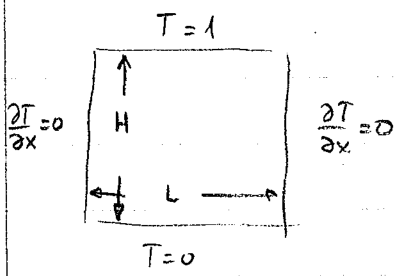
$$\therefore \quad u(x,t)=\sum_{n=1}^{\infty}\sin\frac{2n-1}{2L}\pi x\left[\tilde{A}_n\cos\omega_n t+\tilde{B}_n\sin\omega_n t\right] \quad \text{when } \tilde{A}_n=A\cdot C \quad \tilde{B}_n=B\cdot C$$

$$u(x,t=0)=f(x)=\sum_{n=1}^{\infty}\tilde{A}_n\sin\frac{2n-1}{2L}\pi x \quad \therefore \quad \tilde{A}_n=\frac{2}{L}\int_0^L f(x)\sin\frac{2n-1}{2L}\pi x\,dx$$

$$\frac{\partial u}{\partial t}(x,t=0)=h(x)=\sum_{n=1}^{\infty}\omega_n\tilde{B}_n\sin\frac{2n-1}{2L}\pi x \quad \therefore \quad \tilde{B}_n=\frac{2}{L\omega_n}\int_0^L h(x)\sin\frac{2n-1}{2L}\pi x\,dx$$

Here is V

$$\& \quad V(x,t)=u(x,t)e^{\mu t}=e^{-\frac{\beta t}{2\alpha}}\sum_{n=1}^{\infty}\sin\frac{2n-1}{2L}\pi x\left[\tilde{A}_n\cos\omega_n t+\tilde{B}_n\sin\omega_n t\right]$$



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{let } T = F(x)\,G(y) \qquad 1$$

$$\Rightarrow \quad F''G + FG'' = 0 \quad \text{or} \quad \frac{F''}{F} = -\frac{G''}{G} = -k^2$$

1

From $\frac{F''}{F} = -k^2 \Rightarrow F'' + k^2F = 0$ 1 or $F(x) = A\cos kx + B\sin kx$ if $k \neq 0$

$F'' = 0$ 1 or $F(x) = \bar{A}x + \bar{B}$ if $k = 0$

From $-\frac{G''}{G} = -k^2 \Rightarrow G'' - k^2G = 0$ 1 or $G(y) = C\sinh ky + D\cosh ky$ if $k \neq 0$

$G'' = 0$ 1 or $G(y) = \bar{C}y + \bar{D}$ if $k = 0$

B.C.

$\frac{\partial T}{\partial x} = 0$ on $x = 0$ $\quad \frac{\partial T}{\partial x} = F'(0)\,G(y) = 0 \Rightarrow F'(0) = 0$ 1

$\frac{\partial T}{\partial x} = 0$ on $x = L$ $\quad \frac{\partial T}{\partial x} = F'(L)\,G(y) = 0 \Rightarrow F'(L) = 0$ 1

$T = 0$ on $y = 0$ $\quad T = F(x)\,G(0) = 0 \Rightarrow G(0) = 0$ 1

for $k \neq 0$ $\quad F'(x) = k\left[-A\sin kx + B\cos kx\right]$ $\quad F'(0) = 0 \Rightarrow B = 0$ 1

$F'(L) = 0 \Rightarrow A = 0,\ k = 0$ or $kL = n\pi$ 3

$A = 0$ leads to $T = 0$ everywhere which contradicts B.C. $\therefore F(x) = A\cos\frac{n\pi}{L}x$

Now $k = 0$ is a possibility.

Look at $F(x) = \bar{A}x + \bar{B}$ $\quad F'(x) = \bar{A}$ $\quad F'(0) = 0 \Rightarrow \bar{A} = 0$ 1

with $\bar{A} = 0$ $\quad F'(L) = 0$ automatically $\therefore F(x) = \bar{B}$ for $k = 0$ 1

1

for $k \neq 0$ $\quad G(y) = C\sinh ky + D\cosh ky$ and $G(0) = 0 \Rightarrow D = 0 \quad \therefore G(y) = C\sinh ky$

$k = 0$ $\quad G(y) = \bar{C}y + \bar{D}$ and $G(0) = 0 \Rightarrow \bar{D} = 0 \quad \therefore G(y) = \bar{C}y$ 1

$\therefore$

$$T = \bar{B}\bar{C}y + \sum_{n=1}^{\infty} A_nC_n \cos\frac{n\pi}{L}x \sinh\frac{n\pi}{L}y = \mathscr{C}y + \sum_{n=1}^{\infty} Q_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi}{L}y \qquad 2$$

here $\mathscr{C} = \bar{B}\bar{C}$ and $Q_n = A_nC_n$

we must now satisfy $T=1$ at $y=H$ $\therefore T=1=\mathscr{C}H+\sum_{n=1}^{\infty} Q_n \sinh\frac{n\pi H}{L}\cos\frac{n\pi x}{L}$.

$\mathscr{C}H$ is a constant and by identifying this with the fourier series

$$p(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{L}+\sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}$$

$p(x)=1$ $\quad \frac{a_0}{2}=\mathscr{C}H$ $\quad a_n=Q_n\sinh\frac{n\pi H}{L}$ $\quad b_n=0$

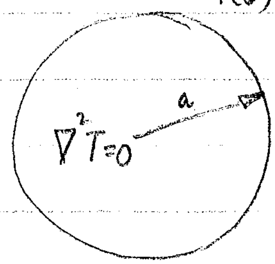
where $\quad a_0=\frac{2}{L}\int_0^L p(x)\,dx,\ a_n=\frac{2}{L}\int_0^L p(x)\cos\frac{n\pi x}{L}\,dx,$

with $p(x)=1$ $\quad a_0=\frac{2}{L}\int_0^L dx=2$ $\quad\therefore \frac{a_0}{2}=\mathscr{C}H=\frac{2}{2}=1$ $\quad\therefore \mathscr{C}=\frac{1}{H}$

$$a_n=\frac{2}{L}\int_0^L 1\cos\frac{n\pi x}{L}\,dx=\frac{2}{L}\frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L=0 \checkmark$$

$$\therefore T(x,y)=\mathscr{C}y=\frac{y}{H}$$

$f(\theta)$ periodic on the boundary



$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

let $T = R(r)G(\theta)$ 1

$$\therefore \quad \nabla^2 T = R''G + \frac{1}{r}R'G + \frac{1}{r^2}RG'' = 0$$

Separate $\therefore\ r^2\frac{R''}{R} + \frac{r^2}{r}\frac{R'}{R} + \frac{G''}{G} = 0$ or $r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{G''}{G} = +k^2$

From this if $k \neq 0$ $\quad G'' + k^2 G = 0 \Rightarrow G = A\cos k\theta + B\sin k\theta$ 1

$r^2R'' + rR' - k^2 R = 0 \Rightarrow R = Cr^k + Dr^{-k}$ 1

if $k = 0$

$G'' = 0 \quad \Rightarrow \quad \bar{A} + \bar{B}\theta$ 1

$r^2R'' + rR' = 0 \quad \Rightarrow \quad \bar{C} + \bar{D}\ln r$ 1

For bounded solutions at $r = 0 \Rightarrow D, \bar{D} = 0$ 2

For periodic solutions i.e. $T(r,\theta) = T(r, \theta + 2\pi) \Rightarrow k = n$ and $\bar{B} = 0$ 2

$$\therefore \quad T(r,\theta) = \bar{A}\bar{C} + \sum_{n=1}^{\infty} r^n \left[CA\cos n\theta + CB\sin n\theta \right]$$ 3

let $\bar{A}\bar{C} = \mathcal{C}_0 \quad CA = \mathcal{C}_n \quad CB = \mathcal{D}_n$

$$T(r,\theta) = \mathcal{C}_0 + \sum_{n=1}^{\infty} r^n \left[\mathcal{C}_n \cos n\theta + \mathcal{D}_n \sin n\theta \right]$$

at $r = a$ $\quad T(a,\theta) = f(\theta) = \mathcal{C}_0 + \sum_{n=1} a^n \left[\mathcal{C}_n \cos n\theta + \mathcal{D}_n \sin n\theta \right]$

from fourier series, for a periodic function

$f(\theta) = \frac{a_0}{2} + \sum_{n=1} a_n \cos n\theta + \sum b_n \sin n\theta$ $\quad$ where $a_0 = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$a_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$ and $b_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$. If we identify

3

$\mathcal{C}_0$ with $\frac{a_0}{2}$ $\quad \mathcal{C}_n a^n$ with a_n and $\mathcal{D}_n a^n$ with b_n and solve for $\mathcal{C}_0$, $\mathcal{C}_n$ and $\mathcal{D}_n$ we have the complete solution.

thus $T(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right]$

where a_0, a_n & b_n are defined as on the other page.

Given

$T = T_1$ at $x=0$; $\quad \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial T}{\partial t}$ on $0 < x < L$; $\quad h\ell \dfrac{\partial T}{\partial x} + (T - T_0) = 0$ at $x = L$; $\quad T = f(x)$ @ $t=0$

find : T

WRITE THIS AS

$T = T_1$ at $x=0$; $\quad \dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial T}{\partial t}$; $\quad h\ell \dfrac{\partial T}{\partial x} + T = T_0$ at $x = L$; $\quad T = f(x)$ @ $t=0$

Now let $\quad T = T_h + T_p \qquad T_h$: homogeneous solution

T_p : particular solution

For particular solution

① $\underline{T_p = T_1}$ at $x=0$; $\quad \underline{h\ell \dfrac{\partial T_p}{\partial x} + T_p = T_0}$ at $x = L$

Note BC here are not zero

For homogeneous

② $\underline{T_h = 0}$ at $x=0$; $\quad \underline{h\ell \dfrac{\partial T_h}{\partial x} + T_h = 0}$ at $x = L$; $\quad p(x) = f(x) - T_p$ @ $t=0$

Note bc here are zero

---

① choose $\underline{T_p = \bar{A}x + \bar{B}}$, Note that $\dfrac{\partial^2 T_p}{\partial x^2} = 0$ & $\dfrac{\partial T_p}{\partial t} = 0 \quad \therefore$ it satisfies $\underline{\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial T}{\partial t} = 0}$

apply BC's of problem ①

$$T_p(x=0) = \bar{B} = T_1 \quad \Rightarrow \quad T_p = \bar{A}x + T_1$$

$$h\ell \frac{\partial T_p}{\partial x} + T_p = h\ell \bar{A} + (\bar{A}\ell + T_1) = T_0 \ \text{ at } x = \ell \quad \Rightarrow \quad \bar{A} = \underline{\frac{T_0 - T_1}{\ell(1+h)}}$$

$$\therefore \quad T_p = T_1 + \frac{T_0 - T_1}{1+h}\frac{x}{\ell} \quad \text{or} \quad \underline{T_1 - \frac{T_1 - T_0}{1+h}\frac{x}{\ell}}.$$

this is also the solution at $t=0$

The object here is to choose a solution that satisfies the BC. regardless of the PDE $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial T}{\partial t}$. If it satisfies it, great! If not, then we would need to do something else. In this case it does, so now we solve part ②

---

② let $T_h = F(x)G(t)$ and put into $\dfrac{\partial^2 T_h}{\partial x^2} = \dfrac{1}{\alpha^2}\dfrac{\partial T_h}{\partial t} \Rightarrow F''G = \dfrac{1}{\alpha^2}FG'$ or $\dfrac{F''}{F} = \dfrac{G'}{\alpha^2 G} = -$

for $k=0$ case $\quad F'' = 0 \Rightarrow F' = \bar{A}$ & $\underline{F(x) = \bar{A}x + \bar{B}}$

$G' = 0 \Rightarrow \underline{G(t) = \bar{C}}$

for $k \neq 0 \quad F'' + k^2 F = 0$ and $\underline{F(x) = A\cos kx + B\sin kx}$

also $G' + \alpha^2 k^2 G = 0 \Rightarrow \underline{G(t) = Ce^{-\alpha^2 k^2 t}}$

Now look at BC's of problem 2

Now $T_h(x=0,t) = 0$; $T_h(x,t) = F(x)G(t)$ & $T_h(0,t) = F(0)G(t) = 0$ for all t

$\Rightarrow \underline{F(0) = 0}$

Now $h\ell \frac{\partial T_h}{\partial x} + T_h = 0$ @ $x = \ell$; $T_h = FG$ $\quad \frac{\partial}{\partial x}T_h = F'G$ since $\frac{\partial}{\partial x}G(t) = 0$

$\therefore h\ell F'(\ell)G + F(\ell)G = G(t)\left[h\ell F'(\ell) + F(\ell)\right] = 0$ for all $t \Rightarrow \underline{h\ell F'(\ell) + F(\ell) = 0}$

12 + 2 fig

<u>For $k \neq 0$ case</u>

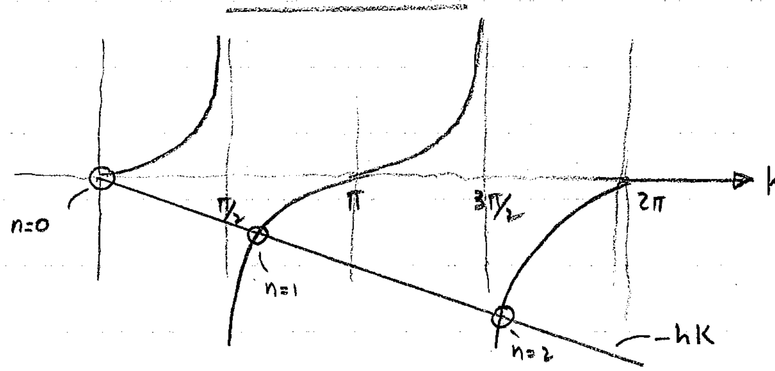
$F(0) = A\cos k\cdot 0 + B\sin k\cdot 0 = \underline{A = 0} \qquad \Rightarrow \qquad F(x) = B\sin kx$

$h\ell F'(\ell) + F(\ell) = h\ell\left[Bk\cos k\ell\right] + B\sin k\ell = B\left[h(k\ell)\cos(k\ell) + \sin(k\ell)\right] = 0$ let $\underline{K = k\ell}$

either $B = 0$ or $hK\cos K + \sin K = 0$. $\quad B = 0 \Rightarrow F(x) = 0 \Rightarrow T(x,t) = 0$

which doesn't satisfy the initial condition. $\therefore hK\cos K + \sin K = 0 \Rightarrow \cos K\left[hK + \tan K\right] =$

$\Longrightarrow \underline{hK + \tan K = 0}$



the circled pts satisfy $\tan K = -hK$

but there are an ∞ no. of solution

$\therefore \tan K_n = -hK_n \quad n = 1, 2, \ldots$

and $K_n = k_n \ell \Rightarrow k_n = \frac{K_n}{\ell}$

$\therefore \quad F(x) = B\sin\frac{K_n}{\ell}x \qquad$ where K_n is a solution to $\tan K = -hK$

$$\therefore \quad T_h(x,t) = \sum_{n=1}^{\infty} BC\, e^{-\frac{\alpha^2 K_n^2}{\ell^2} t^2} \sin\frac{K_n x}{\ell} \qquad K_n \neq 0$$

Now we check the $k = 0$ case to see if it contributes

<u>For $k = 0$ case</u>

$F(0) = 0 \Rightarrow \bar{A}\cdot 0 + \bar{B} = 0$ or $\underline{\bar{B} = 0} \qquad F'(x) = \bar{A}$

$0 = h\ell F'(\ell) + F(\ell) = h\ell\bar{A} + \bar{A}\ell = \bar{A}\,\ell(1+h) \Rightarrow \underline{\bar{A} = 0} \qquad \therefore F(x) = 0$ <u>which is not a solution</u>

$\therefore$ $n = 0$ pt on the graph $\Rightarrow k = 0$ & $K = 0$ is not a solution

Now $\quad T_h = \sum_{n=1}^{\infty} \mathscr{C}_n e^{-\frac{\alpha^2 K_n^2}{\ell} t} \sin \frac{K_n}{\ell} x \qquad$ where $\mathscr{C}_n = BC$

now look at the initial condition

$$T(x,t=0) = f(x) = T_p(x,t=0) + T_h(x,t=0)$$

but $T_p(x,t=0) = T_1 - \left(\frac{T_1 - T_0}{1+h}\right)\frac{x}{\ell} \qquad \therefore \qquad T_h(x,t=0) = f(x) - T_p$

Now $\underline{T_h(x,t=0)} = \sum_{n=1}^{\infty} \mathscr{C}_n e^0 \sin \frac{k_n x}{\ell} = \underline{\sum_{n=1}^{\infty} \mathscr{C}_n \sin \frac{k_n x}{\ell}} = f(x) - T_p = p(x)$

As with Fourier series, we can write $\underline{p(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)}$ if the $\varphi_n(x)$ fns are linearly independent

5 + 1 fig

They are linearly independent if $\underline{\int_0^{\ell} \varphi_n(x)\,\varphi_m(x)\,dx = 0}$ for $m \neq n$

but when is this true? when $\underline{\varphi_n(x)}$ satisfies the condition $\left.\begin{matrix}\varphi = 0 \text{ or} \\ \varphi' = 0 \text{ or} \\ \varphi + \text{const}\cdot\varphi' = 0\end{matrix}\right\}$ at either $x=0$ or $x=\ell$

in our case $\quad \varphi_n(x) = \sin \frac{K_n x}{\ell} \qquad$ which satisfies $\varphi_n(0) = 0$ and $\varphi_n(\ell) + h\varphi_n'(\ell) = 0$

also if $\quad p(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$, then $\int_0^{\ell} p(x)\varphi_n(x)\,dx = a_n \int_0^{\ell} \varphi_n^2(x)\,dx$

and $\quad a_n = \dfrac{\int_0^{\ell} p(x)\varphi_n(x)\,dx}{\int_0^{\ell} \varphi_n^2(x)\,dx} \qquad$ in our case $\quad \underline{a_n = \mathscr{C}_n = \dfrac{\int_0^{\ell} p(x) \sin \frac{K_n}{\ell} x\,dx}{\int_0^{\ell} \sin^2 \frac{K_n x}{\ell}}}$

note that $\int_0^{\ell} \sin^2 \frac{K_n x}{\ell} dx \neq \ell/2$

Read the beginning section 5.7, then read section 5.6 in your books.

TOTAL: 27 gradables

Given $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

also $-k\frac{\partial T}{\partial x} = q$ when $x=0$

$T(x,0) = T_i$

$T(x,t) \to T_i$ as $x \to \infty$ } From these, let $\eta = \frac{Ax}{t^n}$

note for small t and large x $T \to T_i$

also let $T/T_i = Bf(\eta) \Rightarrow T = T_i B f(\eta)$ AS A GUESS. Now look @ $\frac{\partial T}{\partial x}$ @ $x=0$ $(\eta=0)$

$\therefore \frac{\partial T}{\partial x} = T_i B f'(\eta)\cdot\frac{\partial \eta}{\partial x} = T_i B f'(\eta)\cdot\frac{A}{t^n}$; at $x=0$ $\frac{\partial T}{\partial x} = -\frac{q}{K} = BT_i f'(\eta=0)\cdot\frac{A}{t^n}$

Since the left hand side is constant but the right hand side is note (due to t^n, we must add a degree of freedom, let $T/T_i = Bt^m f(\eta)$

Now $\frac{\partial T}{\partial x} = T_i B t^m f'(\eta)\frac{\partial \eta}{\partial x} = T_i B t^m f'(\eta)\cdot\frac{A}{t^n}$; @ $x=0$ $\frac{\partial T}{\partial x} = -\frac{q}{K} = T_i BAf'(0)\frac{t}{t'}$

For both sides to be a constant $m=n$. BUT NOW CHECK TO SEE IF THIS SOLUTION SATISFIES THE OTHER CONDITIONS.

since $T(x,0) = T_i \Rightarrow T/T_i = 1 = B\cdot 0^m f(\eta\to\infty)$ note as $t\to 0$ $\eta = \frac{Ax}{t^n} \to \infty$

$\Rightarrow t^m\cdot f(\eta\to\infty)$ must be a constant (Remember t can be any value - it is a variable)

also since $T(x,t) \to T_i$ as $x\to\infty \Rightarrow T/T_i \to 1 = Bt^m f(\eta\to\infty)$ irrespective of t which would imply again that $t^m f(\eta\to\infty)$ must be a constant.

This causes a problem, for suppose $f(\eta) = A_k\eta^k + A_{k-1}\eta^{k-1} + A_{k-2}\eta^{k-2} + \dots$, A power series in η, that would imply that $t^m f(\eta\to\infty) =$ constant cannot be satisfied. Therefore unlike what I said in class $T/T_i \neq Bt^m f(\eta)$.

NEW GUESS: suppose $\frac{T}{T_i} = C + Bt^m f(\eta)$ then $\frac{\partial T}{\partial x} = Bt^m f'(\eta)\cdot\frac{A}{t^n}$ as before

and

also as $T\to T_i$ for $x\to\infty$ (ie $\eta\to\infty$) $\Rightarrow \frac{T}{T_i} \to 1 = C + Bt^m f(\eta\to\infty)$

but since this must be true for any $t \Rightarrow f(\eta\to\infty) \equiv 0$ and $C = 1$.

Also since at $t=0$ $T(x,0) = T_i$ $T/T_i = 1 = C + B\cdot 0\cdot f(\eta\to\infty) = C \therefore C=1$

Now lets look at the reason why this guess worked. Since the temperature initially was $T = T_i \Rightarrow T(x,t) = T_i +$ something, for $t > 0$. This something is the function

we want. Therefore as a general rule look for the change in temperature $(T-T_i)$ and set that equal to $Bf(\eta)$, if all the conditions are given on T, or set $T-T_i = Bt^m f(\eta)$ if one of the conditions is given on $\frac{\partial T}{\partial x}$.

So for $T/T_i = C + Bt^m f(\eta) = 1 + Bt^m f(\eta)$

$$\frac{\partial T}{\partial x} = T_i B t^m f'(\eta)\frac{\partial \eta}{\partial x}, \text{ where } \frac{\partial \eta}{\partial x} = \frac{A}{t^n}, \text{ and at } x=0\ (\eta=0): \frac{\partial T}{\partial x} = \frac{-q}{K} = T_i B t^m f'(0)\cdot\frac{A}{t^n}$$

so for $\frac{\partial T}{\partial x}$ = constant take $m=n$ and $\left.\frac{\partial T}{\partial x}\right|_{x=0} = \frac{-q}{K} = T_i BA f(0)$

Now choose $T_i BA = -\frac{q}{K}$ and $\Rightarrow f'(0) = 1$. Remember that $f(\eta \to \infty) = 0$ from before. These are the two conditions on f needed.

Now $\frac{\partial^2 T}{\partial x^2} = T_i BA^2 f''\cdot\frac{t^n}{t^{2n}} = T_i BA^2 f'' t^{-n}$ since $m=n$

also

$$\frac{\partial T}{\partial t} = T_i B m t^{m-1} f + Bt^m f'\frac{\partial \eta}{\partial t} \quad \text{and } \frac{\partial \eta}{\partial t} = -\frac{nAx}{t^{n+1}} = -\frac{n\eta}{t}$$

$$= T_i B t^{n-1}[nf - n\eta f'] \quad \text{since } m=n$$

$\therefore$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow T_i BA^2 f'' t^{-n} = \frac{1}{\alpha} T_i B t^{n-1}[nf - n\eta f']$$

for this to be an ODE $\Rightarrow$ powers of t must be the same $\Rightarrow -n = n-1 \Rightarrow$

$$n = \tfrac{1}{2} = m$$

$\Rightarrow T_i B t^{-1/2}\left[A^2 f'' + \frac{1}{2\alpha}\eta f' - \frac{1}{2\alpha} f\right] = 0$ if we choose $\frac{1}{2\alpha A^2} = 1 \Rightarrow A = \frac{1}{\sqrt{2\alpha}}$

and

$f'' + \eta f' - f = 0$ which is the ODE with conditions $f'(0) = 1$ & $f(\infty) = 0$

Now since

$$T_i BA = -q/K \qquad B = \frac{-q}{KT_i A} = -\frac{q\sqrt{2\alpha}}{KT_i}$$

Thus $T/T_i = 1 - \frac{q}{KT_i}\sqrt{2\alpha}\, t^m f(\eta) = 1 - \frac{q}{KT_i}\sqrt{2\alpha t}\, f(\eta)$ since $m = 1/2$

We have already seen the above ODE and we've found the solution as

$$f(\eta) = C_1\eta + C_2\eta\int_\infty^\eta \frac{1}{s^2} e^{-s^2/2} ds$$

By application of $f'(0) = 1$ and $f(\eta=\infty) = 0$ we can find C_1 & C_2.

NOTE
$$\frac{d}{d\eta}\int_\infty^\eta \frac{1}{s^2} e^{-s^2/2} ds = \frac{1}{\eta^2} e^{-\eta^2/2}$$

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PROBLEM: It is required to find the solution of the equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0$$

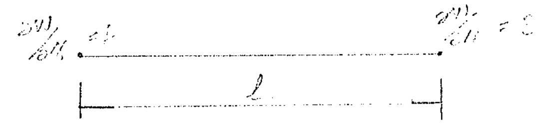
which satisfies the conditions:

$$\left.\begin{aligned} \frac{\partial w}{\partial x}(0,t) &= A \\ \frac{\partial w}{\partial x}(l,t) &= B \end{aligned}\right\} \text{ for all values of } t \quad 0 \le x \le l$$

$$\left.\begin{aligned} w(x,t=0) &= w_0 = 5x \\ \frac{\partial w}{\partial t}(x,t=0) &= w_1 = 0 \end{aligned}\right\} \quad 0 < x < l$$

Solution: since $w(x,t) = w_p(x,t) + w_h(x,t)$

<1> First look at w_p.



assume $\partial w_p / \partial x = Cx + D$

use B.C to determine the C & D

$\partial w_p/\partial x\,(0,t) = C\cdot 0 + D = A \quad \Rightarrow \underline{D = A} \quad \Rightarrow \partial w_p/\partial x = Cx + A$

$\partial w_p/\partial x\,(l,t) = C\cdot l + A = B \quad \Rightarrow C = \frac{B-A}{l}$

$\Rightarrow \underline{\frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A}$ ✓ (It satisfies the B.C)

$\Rightarrow \underline{w_p = \frac{B-A}{2l}x^2 + Ax + f(t)}$ (w_p must satisfies the P.D.E)

$\frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A \qquad \frac{\partial^2 w_p}{\partial x^2} = \frac{B-A}{l}$

$\frac{\partial w_p}{\partial t} = f'(t) \qquad \frac{\partial^2 w_p}{\partial t^2} = f''(t)$

since $\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$

$\Rightarrow \frac{B-A}{l} = \frac{1}{c^2} f''(t) \quad \Rightarrow f'(t) = c^2 \cdot \frac{B-A}{l}t + E$

$\Rightarrow f(t) = \frac{c^2}{2}\frac{B-A}{l}t^2 + Et$

Therefore $\Rightarrow$

$$\underline{w_p = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\frac{B-A}{l}t^2 + Et} \checkmark \qquad \langle\langle 1 \rangle\rangle$$

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1

(w = A) $\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$ ($w = B$)

$\frac{\partial w}{\partial x} = A$ ⟵ l ⟶ $\frac{\partial w}{\partial x} = B$

$w(x, t=0) = w_0(x)$

$\frac{\partial w}{\partial t}(x, t=0) = w_1(x)$ } for while u

||

($w_p = A$) $\frac{\partial^2 w_p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_p}{\partial t^2}$ ($w_p = B$)

$\frac{\partial w_p}{\partial x} = A$ $\frac{\partial w_p}{\partial x} = B$

$w_p(x, t=0) = g_1(x)$

$\frac{\partial w_p}{\partial t}(x, t=0) = g_2(x)$

+

($w_h = 0$) $\frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2}$ $w_h = 0$

$\frac{\partial w_h}{\partial x} = 0$ $\frac{\partial w_h}{\partial x} = 0$

$w_h(x, t=0) = w_0(x) - g_1(x)$

$\frac{\partial w_h}{\partial t}(x, t=0) = w_1(x) - g_2(x)$

$\frac{\partial w}{\partial x} = 0$ — use I.C. when very end

Fn. <<1>> satisfies the B.C. and PDE

(2) look at w_h

we use the method of separation of variable

$$w_h(x,t) = F(x)G(t)$$

$$\frac{\partial w_h}{\partial x} = F'(x)G(t) \quad ; \quad \frac{\partial^2 w_h}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial w_h}{\partial t} = F(x)G'(t) \quad ; \quad \frac{\partial^2 w_h}{\partial t^2} = F(x)G''(t)$$

By $\frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2}$

$$F''(x)G(t) = \frac{1}{c^2}\cdot F(x)G''(t) \Rightarrow \frac{F''(x)G(t)}{F(x)G(t)} = \frac{F(x)G''(t)}{c^2\cdot F(x)G(t)} = -k^2$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''}{G} = -k^2$$

at $k \neq 0$ $\quad F'' + k^2F = 0$, $\quad G'' + c^2k^2F = 0$ $\Rightarrow$ $F(x) = A\cos kx + B\sin kx$, $\quad G(t) = C\cos kct + D\sin kct$

at $k = 0$ $\quad F'' = 0$, $\quad G'' = 0$ $\Rightarrow$ $F(x) = \bar{A}x + \bar{B}$, $\quad G(t) = \bar{C}t + \bar{D}$

look at B.C. first.

$$\frac{\partial w_h}{\partial x}(x=0, t) = 0 = F'(0)G(t) \Rightarrow F'(0) = 0$$

$$\frac{\partial w_h}{\partial x}(x=l, t) = 0 = F'(l)G(t) \Rightarrow F'(l) = 0$$

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at $K=0$

By $F'(0)=0 \Rightarrow \bar{A}=0 \Rightarrow F(x)=\bar{B}$

Then: at $K=0$ $W_h(x,t)=\bar{B}[\bar{C}t+\bar{D}] = \underline{B_1 t + B_2}$

$\boxed{\begin{array}{l} B_1=\bar{B}\cdot\bar{C} \\ B_2=\bar{B}\cdot\bar{D} \end{array}}$

at $K\neq 0$

By $F'(0)=0 = -AK\sin K\cdot 0 + BK\cos K\cdot 0 = 0+BK \Rightarrow B=0$

By $F'(l)=0 = -AK\sin K\cdot l \Rightarrow Kl = n\pi \Rightarrow K=\frac{n\pi}{l}$

$\Rightarrow F(x) = A\cos\frac{n\pi}{l}x$

at $K\neq 0$ $W_h(x,t) = \sum_{n=1}^{\infty} A\cos\frac{n\pi}{l}x\,[C\cos Kct + D\sin Kct]$

$= \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n\sin Kct]$

$\boxed{\begin{array}{l} b_n = AC \\ \mathcal{D}_n = AD \end{array}}$

so that general W_h is

$$W_h = B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n\sin Kct]$$

Therefore The whole fn. of W is:

$$W = W_p + W_h$$

$$= \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + Et + B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n\sin Kct]$$

$$= \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + E_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos\frac{n\pi c}{l}t + \mathcal{D}_n\sin\frac{n\pi c}{l}t]$$

where. $\boxed{E_1 = E + B_1}$

use I.C. to determine the E_1, B_2, b_n and $\mathcal{D}_n$

By $W(x,t=0) = 5x = \frac{B-A}{2l}x^2 + Ax + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot b_n$ «2»

$$\Rightarrow (5-A)x - \frac{B-A}{2l}x^2 = B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot b_n$$

By Fourier series

$$B_2 = \frac{1}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]dx$$

$$b_n = \frac{2}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]\cdot\cos\frac{n\pi}{l}x\,dx$$

By $\frac{\partial w}{\partial t}(x, t=0) = 0 = E_1 + \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x \cdot D_n \cdot \frac{cn\pi}{l}$

$\Rightarrow \quad E_1 = 0 , \quad D_n = 0$

Therefore:

$$w = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cos\frac{cn\pi}{l}t$$

* check the result:

B.C $\quad \frac{\partial w}{\partial x}(0,t) = \frac{B-A}{l}\cdot 0 + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin\frac{n\pi}{l}\cdot 0 = \underline{\underline{A}}$

$\quad \frac{\partial w}{\partial x}(l,t) = \frac{B-A}{l}\cdot l + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin n\pi = \underline{\underline{B}}$

I.C $\quad w(x,0) = \frac{B-A}{2l}x^2 + Ax + 0 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x = \underline{\underline{5x}}$

(according to fn. of «2»)

$\quad \frac{\partial w}{\partial t}(x,0) = 0 + 0 + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x\left(-\frac{cn\pi}{l}\sin\frac{cn\pi}{l}\cdot 0\right) = \underline{\underline{0}}$

PDE $\quad \frac{\partial w}{\partial x} = \frac{B-A}{l}x + A + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t\left(-\frac{n\pi}{l}\sin\frac{n\pi}{l}x\right)$

$$\frac{\partial^2 w}{\partial x^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t\cdot\left[-\left(\frac{n\pi}{l}\right)^2\cos\frac{n\pi}{l}x\right] \qquad «3»$$

$$\frac{\partial w}{\partial t} = c^2\frac{B-A}{l}t + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x\left(-\frac{cn\pi}{l}\cdot\sin\frac{cn\pi}{l}t\right)$$

$$\frac{\partial^2 w}{\partial t^2} = c^2\cdot\frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x\cdot\left[-\left(\frac{cn\pi}{l}\right)^2\cos\frac{cn\pi}{l}t\right]$$

$$\Rightarrow \frac{\partial^2 w}{c^2\,\partial t^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x\cdot\left[-\left(\frac{n\pi}{l}\right)^2\cos\frac{cn\pi}{l}t\right] \qquad «4»$$

By «3» = «4»

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$$

Therefore, Fn. w satisfies the B.C., I.C., and PDE.

So w is the solution of the problem.

zhao xin	Homework	10/2/87

PROBLEM: Infinite string $c = 1$

$w_0(x) = 0$

$w_1(x) = e^{-x^2}$

find $f(x)$, $g(x)$, $f(x+ct)$, $g(x-ct)$, $w(x,t)$

Solution:

① $$\boxed{f(x) = \frac{w_0(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)d\sigma}$$

$$f(x) = 0 + \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}d\sigma = \frac{1}{2}\left\{-\int_0^{x_0} e^{-\sigma^2}d\sigma + \int_0^{x} e^{-\sigma^2}d\sigma\right\}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{2}\left\{-\frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-\sigma^2}d\sigma + \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\sigma^2}d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4}[\operatorname{erf}(x) - \operatorname{erf}(x_0)]$$

② $$\boxed{g(x) = \frac{w_0(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)d\sigma}$$

$$g(x) = 0 - \left[\frac{\sqrt{\pi}}{4}[\operatorname{erf}(x) - \operatorname{erf}(x_0)]\right]$$

$$= \frac{\sqrt{\pi}}{4}[\operatorname{erf}(x_0) - \operatorname{erf}(x)]$$

③ $$\boxed{f(x+ct) = \frac{w_0(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} w_1(\sigma)d\sigma}$$

$$f(x+ct) = 0 + \frac{1}{2c}\int_{x_0}^{x+ct} e^{-\sigma^2}d\sigma$$

$$= \frac{1}{2c}\left\{-\int_0^{x_0} e^{-\sigma^2}d\sigma + \int_0^{x+ct} e^{-\sigma^2}d\sigma\right.$$

$$= \frac{\sqrt{\pi}}{4c}[\operatorname{erf}(x+ct) - \operatorname{erf}(x_0)]$$

④ $$g(x-ct) = \frac{w_0(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} w_1(\sigma)d\sigma$$

$$= \frac{\sqrt{\pi}}{4c}[\operatorname{erf}(x_0) - \operatorname{erf}(x-ct)]$$

⑤ $$\boxed{w(x,t) = \frac{1}{2}[w_0(x+ct) - w_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} w_1(\sigma)d\sigma}$$

$$w(x,t) = 0 + \frac{1}{2c}\left\{-\int_{x_0}^{x-ct} w_1(\sigma)d\sigma + \int_{x_0}^{x+ct} w_1(\sigma)d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4c}[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)]$$

/ -1

7.4. For the line $-\infty < x < +\infty$ where $u_{xx} - u_{tt} = 0$ the solution is given by

$$u(x,t) = \tfrac{1}{2}\left[u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(\sigma)\,d\sigma\right] \quad \text{for } c = 1$$

where $u(x,t=0) = u_0(x)$

$\frac{\partial u}{\partial t}(x,t=0) = u_1(x)$; in our case $u_0(x) = 0$ and $u_1(x) = e^{-x^2}$

$$\therefore\ u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} e^{-\sigma^2}d\sigma = \frac{1}{2}\left[\int_0^{x+t} e^{-\sigma^2}d\sigma - \int_0^{x-t} e^{-\sigma^2}d\sigma\right]$$

with $\frac{2}{\sqrt{\pi}}\int_0^z e^{-\sigma^2}d\sigma = \mathrm{erf}(z) \Rightarrow \int_0^z e^{-\sigma^2}d\sigma = \frac{\sqrt{\pi}}{2}\mathrm{erf}(z)$

$$\therefore\ u(x,t) = \frac{1}{2}\left[\frac{\sqrt{\pi}}{2}\mathrm{erf}(x+t) - \frac{\sqrt{\pi}}{2}\mathrm{erf}(x-t)\right] = \underline{\frac{\sqrt{\pi}}{4}\left[\mathrm{erf}(x+t) - \mathrm{erf}(x-t)\right]}$$

7.5 Given: $u_{xx} - u_{tt} = 0$ with ① $u(x,t=0) = xe^{-x}$ $[= u_0(x)]$ $\Big\}$ $x \geq 0$ I.C.

② $\frac{\partial u}{\partial t}(x,t=0) = 0$ $[= u_1(x)]$

③ with BC $u(x=0,t) = 0$. Find u for $t \geq 0, x \geq 0$

General solution is $u(x,t) = f(x+t) + g(x-t)$ since $c = 1$

now look at ③ : $u(0,t) = 0 = f(t) + g(-t) \Rightarrow g(-\sigma) = -f(\sigma)$ $t \to \sigma$ dum

this equation allows for the definition of g & f for negative arguments

Remember f & g are defined for + arguments only, thus we must extend the definition of f & g over the entire range of arguments (ie $-\infty < \sigma < \infty$) to use the ∞ line solution

$\therefore$ $g(-\sigma) = -f(\sigma)$ when σ is a + argument $\Rightarrow$ g of $-$ argument $= -f$ of + argument. Now replace $-\sigma$ for $\sigma \Rightarrow g(\sigma) = -f(-\sigma)$; here, if σ is +, then f of $-$ argument $= -g$ of + argument. Note also that $u(x,t=0) = f(x) + g(x) = u_0(x) \Rightarrow f(\sigma) = \frac{u_0}{2}(\sigma)$; $g(\sigma) = \frac{u_0}{2}(\sigma)$ ④

Now $u(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right]$ since $u_1 = 0$ &

$f(x+t) = \frac{1}{2}u_0(x+t)$ & $g(x-t) = \frac{1}{2}u_0(x-t)$ from ④

Thus for $\sigma = x+t > 0$ $\quad f(x+t) = \frac{1}{2}u_0(x+t) = \frac{1}{2}(x+t)e^{-(x+t)}$ ⑤

$\sigma = x-t > 0$ $\quad g(x-t) = \frac{1}{2}u_0(x-t) = \frac{1}{2}(x-t)e^{-(x-t)}$ ⑥

for − arguments $\quad f(\sigma) = -g(-\sigma) = -\frac{u_0}{2}(-\sigma) \qquad$ here $-\sigma > 0$

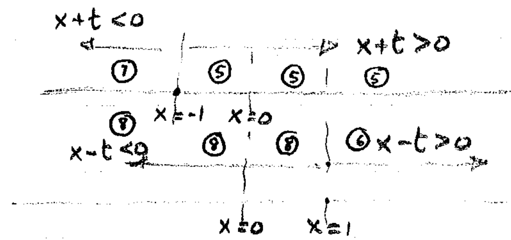
$$g(\sigma) = -f(-\sigma) = -\frac{u_0}{2}(-\sigma)$$

$\therefore$ for $\sigma = x+t < 0 \qquad f(x+t) = -\frac{u_0}{2}(-(x+t)) = -\frac{1}{2}\left[-(x+t)\,e^{(x+t)}\right] \quad ⑦$

$\sigma = x-t < 0 \qquad g(x-t) = -\frac{u_0}{2}(-(x-t)) = -\frac{1}{2}\left[-(x-t)\,e^{(x-t)}\right] \quad ⑧$

at $\quad t=1: \quad x+t>0 \implies x>-1$

$\qquad\qquad\quad x-t>0 \implies x>1$



$x+t>0$
$x-t>0$ $\quad \therefore$ for $t=1$ & $x>1 \qquad u(x,t) = ⑤ + ⑥$

$x+t>0$
$x-t<0$ $\quad t=1$ & $0 \le x < 1 \qquad u(x,t) = ⑤ + ⑧$

remember we want $x \geq 0 \;\; t \geq 0$

$t=1 \quad -1 < x < 0 \qquad u(x,t) = ⑤ + ⑧$

$x < -1 \qquad u(x,t) = ⑦ + ⑧$

for our problem since we want $u(x,t)$ for $x \geq 0$; the first two of these give us want we want. Note that at $\underline{\underline{x=0}}$ & $t=1$ we have the boundary condition

$x-t<0$
$x+t>0$

$$u(x,t) = \underset{⑤}{\frac{1}{2}(x+t)\,e^{-(x+t)}} + \underset{⑧}{\frac{1}{2}(x-t)\,e^{(x-t)}}\Bigg|_{\substack{x=0\\ t=1}} = \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} = 0$$

taking $\mathcal{L}\left\{\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}\right\}$ gives

$$\alpha \frac{d^2}{dx^2} \mathcal{T}(x;s) = s\mathcal{T}(x;s) - T(x,0) \quad ; \text{ we define } \mathcal{T}(x;s) = \int_0^\infty T(x,t)\, e^{-st}\, dt$$

$T(x,0) = T_i$

or $\mathcal{T}'' - \frac{s}{\alpha}\mathcal{T} = \frac{T_i}{\alpha}$ $\qquad \mathcal{T}'' = \frac{d^2}{dx^2}\mathcal{T}$

Now transform the BC's:

@ $x=0$ $\quad -k\frac{\partial T}{\partial x} = q \implies -k\,\mathcal{T}'(0;s) = q/s$ or $\mathcal{T}'(0;s) = -q/ks$ ①

REMEMBER $\int_0^\infty \text{const}\, e^{-st} dt = \text{const}/s$; that's why →

ALSO $T(x,t) \to T_i$ as $x \to \infty \implies \mathcal{T}(x;s) \to T_i/s$ as $x \to \infty$ ②

Since ODE is not HOMOGENEOUS } let $\mathcal{T} = \mathcal{T}_H + \mathcal{T}_p$ where $\mathcal{T}_p'' - \frac{s}{\alpha}\mathcal{T}_p = \frac{T_i}{\alpha}$ choose $\mathcal{T}_p = C$

$0 - \frac{s}{\alpha} C = \frac{T_i}{\alpha} \implies \underline{C = T_i/s = \mathcal{T}_p}$

FOR THE HOMOGENEOUS PART

now $\mathcal{T}_H$ solves $\mathcal{T}_H'' - \frac{s}{\alpha}\mathcal{T}_H = 0 \implies \mathcal{T}_H = C_1 e^{-\sqrt{s/\alpha}\,x} + C_2 e^{\sqrt{s/\alpha}\,x}$

$$\therefore \mathcal{T} = \mathcal{T}_p + \mathcal{T}_H = \frac{T_i}{s} + C_1 e^{-\sqrt{s/\alpha}\,x} + C_2 e^{+\sqrt{s/\alpha}\,x}$$

now from ② $\implies C_2 = 0$ since $\mathcal{T}(x;s) \to \frac{T_i}{s}$ as $x \to \infty$

To use ① take $\frac{d}{dx}\mathcal{T}(x;s)$ and evaluate at $x=0$

$$\left.\frac{d}{dx}\mathcal{T}\right|_{x=0} = 0 + C_1 e^{-\sqrt{s/\alpha}\,x}\cdot\left(-\sqrt{\frac{s}{\alpha}}\right)\Bigg|_{x=0} = -C_1\sqrt{\frac{s}{\alpha}}\, e^0 = -C_1\sqrt{\frac{s}{\alpha}} = -\frac{q}{ks}$$

$$\therefore C_1 = \frac{q}{k}\sqrt{\frac{\alpha}{s^3}}$$

$$\therefore \mathcal{T} = \frac{T_i}{s} + \frac{q}{k}\sqrt{\frac{\alpha}{s^3}}\, e^{-\sqrt{s/\alpha}\,x} = \frac{T_i}{s} + \frac{q\sqrt{\alpha}}{k}\cdot\frac{1}{\sqrt{s^3}}\, e^{-\frac{x}{\sqrt{\alpha}}\cdot\sqrt{s}} \qquad 2$$

To find $T(x,t)$, find $\mathcal{L}^{-1}\{\mathcal{T}(x;s)\}$: $\mathcal{L}^{-1}\{\frac{1}{s}\} = 1$ and

$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^3}} e^{-p\sqrt{s}}\right\} = 2\sqrt{\frac{t}{\pi}}\, e^{-p^2/4t} - p\,\mathrm{erfc}\left(\frac{p}{2\sqrt{t}}\right)$, where $p = \text{const} = \frac{x}{\sqrt{\alpha}}$ using 29.3.85

$$\therefore T(x,t) = T_i\cdot 1 + \frac{q}{k}\sqrt{\alpha}\left[2\sqrt{\frac{t}{\pi}}\, e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{\alpha}}\,\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] \qquad 3$$

21 pts

PROBLEM 108 Heat flow equation:
(Text book, P534)

$$\alpha^2 T_{xx} - T_t = 0 \quad \text{for} \quad \begin{matrix} t \ge 0 \\ 0 \le x \le l \end{matrix}$$

B.C. $T(x=0,t)=0$ I.C. $T(x,t=0)=0$
$T(x=l,t)=1$

when $t>0$, by above condition. $T = A(x,t)$. where $A(x,t)$ is the fuction of which the serise (168) is an expansion.

Solution:

ⓐ let the Laplace Transform $\bar{T}(x,s) = \int_0^\infty T(x,t)e^{-st}dt$

since $\mathcal{L}\{\frac{\partial}{\partial x}T\} = \frac{\partial}{\partial x}\int_0^\infty T(x,t)e^{-st}dt = \frac{d}{dx}\bar{T}(x,s)$

$\mathcal{L}\{\frac{\partial^2}{\partial x}T\} = \frac{d^2}{dx}\bar{T}(x,s)$

$\mathcal{L}\{\frac{\partial}{\partial t}T\} = s\bar{T}(x,s) - T(x,t=0+)$

$\because \mathcal{L}\{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}\} = \frac{d^2}{dx}\bar{T}(x,s) = \frac{1}{\alpha^2}[s\bar{T}(x,s) - \bar{T}(x,t=0+)]$

$\Rightarrow \frac{d^2}{dx^2}\bar{T} - \frac{s}{\alpha^2}\bar{T} = -\frac{1}{\alpha^2}T(x,t=0+)$

By condition I.C. $T(x,t=0)=0$ ①
B.C. $T(x=0,t)=0$ ②
$T(x=l,t)=1$ ③

By ① $T(x,t=0+)=0 \Rightarrow$

$$\underline{\underline{\bar{T}'' - \frac{s}{\alpha^2}\bar{T} = 0}} \quad (\alpha^2\bar{T}_{xx} - s\bar{T} = 0)$$

let $\bar{T} = \bar{T}_P + \bar{T}_H$

first do T_P By ① let $\bar{T}_P = C \Rightarrow$

not necessary since eqn is homogeneous only use T_p when non homogeneous

$\bar{T}_P'' - \frac{s}{\alpha^2}\bar{T}_P = 0$

$\Rightarrow 0 - \frac{s}{\alpha^2}C = 0 \Rightarrow C = 0 \Rightarrow \bar{T}_P = 0$

second do T_H

$\bar{T}_H'' - \frac{s}{\alpha^2}\bar{T}_H = 0 \Rightarrow \bar{T}_H = C_1 e^{-\frac{\sqrt{s}}{\alpha}x} + C_2 e^{\frac{\sqrt{s}}{\alpha}x}$

By ② $x=0$

$$\int_0^\infty T(x=0,t)\,e^{-st}dt = \bar{T}(0;s) = \int_0^\infty 0\cdot e^{-st}dt = 0 \quad \Rightarrow \quad \underline{\underline{\bar{T}(0,s)=0}}$$

By ③ x.

$$\int_0^\infty T(x=l,t)\,e^{-st}dt = \bar{T}(x=l,s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \quad \Rightarrow \quad \underline{\underline{\bar{T}(x=l,s)=\frac{1}{s}}}$$

since $\bar{T} = \bar{T}_P + \bar{T}_H$

$$= 0 + c_1 e^{-\frac{\sqrt{s}}{\alpha}x} + c_2 e^{\frac{\sqrt{s}}{\alpha}x}$$

By $\bar{T}(0,s)=0 \Rightarrow \bar{T}(0,s) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2$ ④

By $\bar{T}(l,s)=1/s$ let $\underline{\underline{s^{\frac{1}{2}}/\alpha = q}} \Rightarrow c_1 e^{-ql} + c_2 e^{ql} = \frac{1}{s}$ ⑤

By ④ & ⑤ $\Rightarrow c_1 = -\frac{1}{2\cdot\sinh ql}\cdot\frac{1}{s}$; $c_2 = \frac{1}{2\sinh ql}\cdot\frac{1}{s}$

Therefore: $\bar{T} = c_1 e^{-\frac{\sqrt{s}}{\alpha}x} + c_2 e^{\frac{\sqrt{s}}{\alpha}x} = \frac{1}{s}\cdot\frac{1\times 2}{2\sinh ql}\left(\frac{e^{qx}-e^{-qx}}{2}\right)$

$$= \underline{\underline{\frac{1}{s}\,\frac{\sinh qx}{\sinh ql}}}$$

$$= \frac{1}{s}\,\frac{e^{qx}-e^{-qx}}{e^{ql}-e^{-ql}} = \frac{1}{s}\,\frac{e^{qx}}{e^{ql}}\,\frac{(1-e^{-2qx})}{(1-e^{-2ql})}$$

$$= \underline{\underline{\frac{1}{s}\,e^{-q(l-x)}\,\frac{1-e^{-2qx}}{1-e^{-2ql}}}}$$

ⓑ The form of a series of ascending power is

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots} \quad -1 < x < 1$$

here $x = -e^{-2ql}$ $\qquad -1 < -e^{-2ql} < 1 \longleftarrow$ ** $\underline{\underline{\text{here } q>0}}$ (complex)

then $\frac{1}{1+(-e^{-2ql})} = 1 + e^{-2ql} + e^{-4ql} + e^{-6ql} + e^{-8ql} + \cdots\cdots$

Therefore: $\bar{T}(x,s) = \frac{1}{s}\cdot\frac{1}{1+(-e^{-2ql})}\cdot e^{-q(l-x)}(1-e^{-2qx})$

$$\Rightarrow \bar{T}(x,s) = \frac{1}{s}\left\{(1+e^{-2ql}+e^{-4ql}+e^{-6ql}+\cdots)(e^{-q(l-x)} - e^{-q(l+x)})\right)$$

$$= \underline{\underline{\frac{1}{s}\cdot e^{-q(l-x)} - \frac{1}{s}e^{-q(l+x)} + \frac{1}{s}e^{-q(3l-x)} - e^{-q(3l+x)} + \cdots\cdots}}$$

(c) By (b)

$$\bar{T}(x,s) = \frac{1}{s} e^{-q(l-x)} - \frac{1}{s} e^{-q(l+x)} + \frac{1}{s} e^{-q(3l-x)} - \frac{1}{s} e^{-q(3l+x)} + \cdots$$

since $q = \frac{\sqrt{s}}{\alpha} \Rightarrow$

$$\bar{T}(x,s) = \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(l-x)\right]} - \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(l+x)\right]} + \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(3l-x)\right]} - \cdots$$

By Laplace transform

$$\bar{T}(x,s) \Longrightarrow A(x,t)$$

By 29.3.83

$$f(s) = \frac{1}{s} e^{-k\sqrt{s}} = \operatorname{erfc} \frac{k}{2\sqrt{t}} = 1 - \operatorname{erf} \frac{k}{2\sqrt{t}}$$

In our problem.

$k = \left[\frac{1}{\alpha}(l-x)\right]$ or $k = \left[\frac{1}{\alpha}(l+x)\right]$, or $\left[\frac{1}{\alpha}(3l-x)\right]$ ……

Therefore By 29.3.83 we get

$$\bar{T}(x,s) \Longrightarrow A(x,t) = \left[1 - \operatorname{erf}\left(\frac{l-x}{2\alpha\sqrt{t}}\right)\right] + \left[1 - \operatorname{erf}\left(\frac{l+x}{2\alpha\sqrt{t}}\right)\right] + \left[1 - \operatorname{erf}\left(\frac{3l-x}{2\alpha\sqrt{t}}\right)\right] + \cdots\cdots$$

$$= T$$

excellent.

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \quad (1)$$

$$u(0,t) = t/\beta \quad (2)$$

$$u(y,0) = 0 \quad (3)$$

$$u(y,t) \to 0 \text{ as } y \to \infty \quad (4)$$

Similar to the 2nd problem we did in class with $a = \frac{1}{\beta}$ and $b = 1$

choose $\eta = \frac{Ay}{t^n}$ note: as $y \to \infty$ & $t \to 0$, $\eta \to \infty$ and $u \to 0$ from (3) & (4)

Choose $u - u_\infty = Bt^m f(\eta)$ where u_∞ is velocity far from plate [ie $u_\infty = 0$ from (4)]

$u(y=0,t) = Bt^m f(\eta)$ as $y \to 0$ $\eta \to 0$ $= Bt^m f(0) = t/\beta$

since $f(0) = $ const. ; let $m = 1$ $B = \frac{1}{\beta}$ $\Rightarrow$ $\underline{\underline{f(0) = 1}}$

From (4): since $u \to 0$ as $y \to \infty$ $u = Bt^m f(\eta) \to Bt^m f(\eta \to \infty)$. Since $u \to 0$ as $y \to \infty$

$\Rightarrow f(\eta \to \infty) \Rightarrow 0$ irrespective of t

Note that if $f(\eta \to \infty) = 0$, then from (3) @ $t = 0$ $u = B \cdot 0^m \cdot f(\eta \to \infty) = B \cdot 0 \cdot 0 = 0$ as requi

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$; $\frac{\partial \eta}{\partial y} = \frac{A}{t^n}$ $\frac{\partial u}{\partial \eta} = Bt^m f'(\eta)$ $\therefore \frac{\partial u}{\partial y} = Bt^m f'(\eta) \cdot \frac{A}{t^n}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left[Bt^m f'(\eta) \cdot \frac{A}{t^n}\right] \cdot \frac{A}{t^n} = Bt^m f''(\eta) \cdot \frac{A^2}{t^{2n}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} ; \frac{\partial \eta}{\partial t} = \frac{-nAy}{t^{n+1}} = -\frac{n}{t}\eta \quad \frac{\partial u}{\partial \eta} = Bt^m f'(\eta) \quad \therefore \frac{\partial u}{\partial t} = -\frac{n\eta}{t} Bt^m f'(\eta) + Bmt^{m-1} f$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \Rightarrow \cancel{B}\cancel{t^m} f'' \frac{A^2}{\cancel{t^{2n}}} = \frac{1}{\alpha}\left[\frac{-n\eta}{\cancel{t}} \cancel{B}\cancel{t^m} f'(\eta)\right] + \frac{1}{\alpha}\cancel{B}m\cancel{t^{m-1}} f$$

for this eqn to be independent for $t \Rightarrow 2n = 1$ $n = 1/2$

or $0 = f'' A^2 + \frac{\eta}{2\alpha} f'(\eta) - \frac{\cancel{B}mt^{0} f}{\alpha} \Rightarrow 0 = f'' + \frac{\eta}{2\alpha A^2} f' - \frac{\cancel{B}mf}{A^2\alpha} \Rightarrow$ $2\alpha A^2 = 1$ $A = \frac{1}{\sqrt{2\alpha}}$

and $f'' + \eta f' - \frac{\cancel{B}mf}{A^2\alpha} = 0$ thus

$$\eta = \frac{Ay}{t^n} = \frac{y}{\sqrt{2\alpha t}} \quad B = \frac{1}{\beta} \quad m = 1$$

$\therefore$ $u = \frac{1}{\beta} t f(\eta)$ where $f(\eta)$ satisfies $f'' + \eta f' - 2f = 0$ with.

$f(0) = 1$ and $f(\eta) \to 0$ as $\eta \to \infty$

$\frac{\partial T}{\partial x} = Q_1$ $\qquad$ $T = T_1$

Choose for $T_p = Ax + B$. $\therefore$ $\frac{\partial T_p}{\partial x} = A$. At $x=0$ $\frac{\partial T_p}{\partial x} = Q_1 = A$ also

$$T_p = Q_1 x + B \quad @ \ x = L \quad T_p = T_1 = Q_1 L + B \quad \therefore \quad B = T_1 - Q_1 L$$

$$\therefore \quad T_p = Q_1 x + T_1 - Q_1 L = Q_1(x-L) + T_1$$

since T_p is linear in x $\quad \frac{\partial T_p}{\partial x} = Q_1 \quad \frac{\partial^2 T_p}{\partial x^2} = 0$ also $\frac{\partial T_p}{\partial t} = 0 \Rightarrow \frac{\partial^2 T_p}{\partial x^2} = 0 = \frac{1}{\alpha}\frac{\partial}{\partial t}$

Also since T_p is not a fn of t $\Rightarrow$ T_p is the solution irrespective of time; thus it must be the solution at $t=0$ $\quad \therefore \quad \underline{T_p(x, t=0) = Q_1(x-L) + T_1}$

for T_h

$\frac{\partial T}{\partial x} = 0$ $\qquad$ $T = 0$

$T(x, t=0) = f(x) - T_p(x, t=0)$

Choose $T_h = F(x)\,G(t)$

$$\therefore \quad \frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t} \Rightarrow F''G = \frac{1}{\alpha}FG'$$

or $\quad \frac{F''}{F} = \frac{G'}{\alpha G} = -k^2$

thus $\left.\begin{array}{l} F'' + k^2 F = 0 \\ G' + k^2 \alpha G = 0 \end{array}\right\}$ for $k \neq 0$ and $\begin{array}{l} F'' = 0 \\ G' = 0 \end{array}$ for $k = 0$

For $k = 0$ $\quad F = \bar{A}x + \bar{B}\,;\ F' = \bar{A}$ $\quad$ for $k \neq 0$ $\quad F = A\cos kx + B\sin kx\,;\ F' = k[-A\sin kx + B\cos kx]$

$\qquad G = \bar{C}$ $\qquad\qquad\qquad\qquad G = Ce^{-\alpha k^2 t}$

<u>BC</u> $\quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = F'(x)\,G(t)\big|_{x=0} = 0$ for all time $\therefore$ $F'(0) = 0$

$\quad T\big|_{x=L} = F(x)\,G(t)\big|_{x=L} = 0$ for all time $\therefore$ $F(L) = 0$

for $k=0$ $\quad F'(0) = \bar{A} = 0 \Rightarrow F = \bar{B}$, also $F(L) = 0 = \bar{B} \Rightarrow F(x) = 0$ for $k=0$
trival solution

for $k \neq 0$ $\quad F'(0) = k[-A\overset{0}{\cancel{\sin 0}} + B\overset{1}{\cancel{\cos 0}}] = 0 \Rightarrow B = 0 \Rightarrow F(x) = A\cos kx$. Also $F(L) = 0 \Rightarrow A\cos kL = 0$

$\Rightarrow kL = \frac{\pi}{2}n$ where n is odd. $\quad \therefore \quad k = \frac{n\pi}{2L}$

$$\therefore \quad T(x,t) = \sum AC\, e^{-\alpha k^2 t}\cos kx = \sum_{n=\text{odd}} \mathcal{C}_n\, e^{-\alpha \frac{n^2\pi^2}{4L^2}t}\cos\frac{n\pi}{2L}x$$

@ $t=0$ $\quad T(x,0) = f(x) - [Q_1(x-L) + T_1] = \sum \mathcal{C}_n \cos\frac{n\pi x}{2L} \Rightarrow \mathcal{C}_n = \frac{2}{L}\int_0^L h(x)\cos\frac{n\pi x}{2L}\,dx$

$\qquad = h(x)$

$\frac{\partial T}{\partial x} = Q_1$ $\qquad$ $T = T_1$

Choose for $T_p = Ax + B$. $\therefore \frac{\partial T_p}{\partial x} = A$. At $x=0$ $\frac{\partial T_p}{\partial x} = \underline{Q_1 = A}$ also

$$T_p = Q_1 x + B \,. \quad @ \; x = L \quad T_p = T_1 = Q_1 L + B \quad \therefore \quad B = T_1 - Q_1 L$$

$$\therefore \quad T_p = Q_1 x + T_1 - Q_1 L = Q_1 (x-L) + T_1$$

since T_p is linear in x $\quad \frac{\partial T_p}{\partial x} = Q_1 \quad \frac{\partial^2 T_p}{\partial x^2} = 0$ also $\frac{\partial T_p}{\partial t} = 0 \Rightarrow \frac{\partial^2 T_p}{\partial x^2} = 0 = \frac{1}{\alpha}\frac{\partial}{\partial}$

Also since T_p is not a fn of t $\Rightarrow$ T_p is the solution irrespective of time; thus it must be the solution at $t=0$ $\quad \therefore \quad \underline{T_p(x, t=0) = Q_1(x-L) + T_1}$

for T_h

$\frac{\partial T}{\partial x} = 0$ $\qquad$ $T = 0$

$T(x, t=0) = f(x) - T_p(x, t=0)$

Chose $T_h = F(x)\,G(t)$

$$\therefore \quad \frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t} \Rightarrow F''G = \frac{1}{\alpha} F G'$$

$$\text{or} \quad \frac{F''}{F} = \frac{G'}{\alpha G} = -k^2$$

thus $\left.\begin{array}{l} F'' + k^2 F = 0 \\ G' + k^2 \alpha G = 0 \end{array}\right\}$ for $k \neq 0$ and $\begin{array}{l} F'' = 0 \\ G' = 0 \end{array}$ for $k = 0$

For $k = 0$ $\quad F = \bar{A}x + \bar{B}$; $F' = \bar{A}$ $\quad$ $G = \bar{C}$

for $k \neq 0$ $\quad F = A\cos kx + B\sin kx$; $F' = k[-A\sin kx + B\cos$ $\quad$ $G = Ce^{-\alpha k^2 t}$

<u>BC</u>

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = F'(x)\,G(t)\Big|_{x=0} = 0 \text{ for all time } \therefore F'(0) = 0$$

$$T\Big|_{x=L} = F(x)\,G(t)\Big|_{x=L} = 0 \text{ for all time } \therefore F(L) = 0$$

for $k=0$ $\quad F'(0) = \bar{A} = 0 \Rightarrow F = \bar{B}$, also $F(L) = 0 = \bar{B} \Rightarrow F(x) = 0$ for $k = 0$

trival solution

for $k \neq 0$ $\quad F'(0) = k[-A\cancel{\sin 0}^{\,0} + B\cancel{\cos 0}^{\,1}] = 0 \Rightarrow B = 0 \Rightarrow F(x) = A\cos kx$. Also $F(L) = 0 \Rightarrow A\cos kL =$

$$\Rightarrow kL = \frac{\pi}{2} n \quad \text{where } n \text{ is odd.} \quad \therefore \quad k = \frac{n\pi}{2L}$$

$$\therefore \quad T(x,t) = \sum AC\, e^{-\alpha k^2 t} \cos kx = \sum_{n=\text{odd}} \mathcal{C}_n\, e^{-\alpha \frac{n^2\pi^2}{4L^2} t} \cos \frac{n\pi}{2L} x$$

@ $t=0$ $\quad T(x,0) = f(x) - [Q_1(x-L) + T_1] = \sum \mathcal{C}_n \cos\frac{n\pi x}{2L} \Rightarrow \mathcal{C}_n = \frac{2}{L}\int_0^L h(x) \cos\frac{n\pi x}{2L}\, dx$

$= h(x)$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \quad (1)$$

$$u(0,t) = t/\beta \quad (2)$$

$$u(y,0) = 0 \quad (3)$$

$$u(y,t) \to 0 \text{ as } y \to \infty \quad (4)$$

Similar to the 2nd problem we did in class with $a = \frac{1}{\beta}$ and $b = 1$

choose $\eta = \frac{Ay}{t^n}$ note: as $y \to \infty$ & $t \to 0$, $\eta \to \infty$ and $u \to 0$ from (3) & (4)

Choose $u - u_\infty = Bt^m f(\eta)$ where u_∞ is velocity far from plate [ie $u_\infty = 0$ from (4)]

$\boxed{u(y=0,t) = Bt^m f(\eta)}$ as $y \to 0$ $\eta \to 0$ $= Bt^m f(0) = t/\beta$

since $f(0) = \text{const.}$; let $m = 1$ $B = \frac{1}{\beta}$ $\Rightarrow$ $\underline{\underline{f(0) = 1}}$

From (4): since $u \to 0$ as $y \to \infty$ $u = Bt^m f(\eta) \to Bt^m f(\eta \to \infty)$. Since $u \to 0$ as $y \to \infty$

$\Rightarrow f(\eta \to \infty) \equiv 0$ irrespective of t

Note that if $f(\eta \to \infty) = 0$, then from (3) @ $t = 0$ $u = B \cdot 0^m \cdot f(\eta \to \infty) = B \cdot 0 \cdot 0 = 0$ as req.

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$; $\frac{\partial \eta}{\partial y} = \frac{A}{t^n}$ $\frac{\partial u}{\partial \eta} = Bt^m f'(\eta)$ $\therefore \frac{\partial u}{\partial y} = Bt^m f'(\eta)\cdot\frac{A}{t^n}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left[Bt^m f'(\eta)\cdot\frac{A}{t^n}\right]\cdot\frac{A}{t^n} = Bt^m f''(\eta)\cdot\frac{A^2}{t^{2n}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} ; \quad \frac{\partial \eta}{\partial t} = \frac{-nAy}{t^{n+1}} = -\frac{n}{t}\eta \quad \frac{\partial u}{\partial \eta} = Bt^m f'(\eta) \quad \therefore \frac{\partial u}{\partial t} = -\frac{n\eta}{t}Bt^m f'(\eta) + Bm\ldots$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \Rightarrow \cancel{B}\cancel{t^m} f''\frac{A^2}{\cancel{t^{2n}}} = \frac{1}{\alpha}\left[\frac{-n\eta}{\cancel{t}}\cancel{B}\cancel{t^m}f'(\eta)\right] + \frac{1}{\alpha}\cancel{B}m\cancel{t^{m-1}}f$$

for this eqn to be independent for $t \Rightarrow 2n = 1$ $n = 1/2$

or $0 = f''A^2 + \frac{\eta}{2\alpha}f'(\eta) - \frac{\cancel{B}mt}{\alpha}f \Rightarrow 0 = f'' + \frac{\eta}{2\alpha A^2}f' - \frac{\cancel{B}mf}{A^2\alpha} \Rightarrow$ $2\alpha A^2 = 1$ $A = \frac{1}{\sqrt{2\alpha}}$

and $f'' + \eta f' - \frac{\cancel{B}mf}{A^2\alpha} = 0$ thus $\eta = \frac{Ay}{t^n} = \frac{y}{\sqrt{2\alpha t}}$ $B = \frac{1}{\beta}$ $m = 1$

$\therefore$ $u = \frac{1}{\beta}t f(\eta)$ where $f(\eta)$ satisfies $f'' + \eta f' - 2f = 0$ with. $f(0) = 1$ and $f(\eta) \to 0$ as $\eta \to$

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PROBLEM: Infinite string $c = 1$

$w_0(x) = 0$

$w_1(x) = e^{-x^2}$

find $f(x)$, $g(x)$, $f(x+ct)$, $g(x-ct)$, $w(x,t)$

Solution:

① $\boxed{f(x) = \frac{w_0(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)d\sigma}$

$$f(x) = 0 + \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}d\sigma = \frac{1}{2}\left\{-\int_0^{x_0} e^{-\sigma^2}d\sigma + \int_0^{x} e^{-\sigma^2}d\sigma\right\}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{2}\left\{-\frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-\sigma^2}d\sigma + \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\sigma^2}d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4}[\mathrm{erf}(x) - \mathrm{erf}(x_0)]$$

② $\boxed{g(x) = \frac{w_0(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)d\sigma}$

$$g(x) = 0 - \left[\frac{\sqrt{\pi}}{4}[\mathrm{erf}(x) - \mathrm{erf}(x_0)]\right]$$

$$= \frac{\sqrt{\pi}}{4}[\mathrm{erf}(x_0) - \mathrm{erf}(x)]$$

③ $\boxed{f(x+ct) = \frac{w_0(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} w_1(\sigma)d\sigma}$

$$f(x+ct) = 0 + \frac{1}{2c}\int_{x_0}^{x+ct} e^{-\sigma^2}d\sigma$$

$$= \frac{1}{2c}\left\{-\int_0^{x_0} e^{-\sigma^2}d\sigma + \int_0^{x+ct} e^{-\sigma^2}d\sigma\right.$$

$$= \frac{\sqrt{\pi}}{4c}[\mathrm{erf}(x+ct) - \mathrm{erf}(x_0)]$$

④ $g(x-ct) = \frac{w_0(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} w_1(\sigma)d\sigma$

$$= \frac{\sqrt{\pi}}{4c}[\mathrm{erf}(x_0) - \mathrm{erf}(x-ct)]$$

⑤ $\boxed{w(x,t) = \frac{1}{2}[w_0(x+ct) - w_0(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} w_1(\sigma)d\sigma}$

$$w(x,t) = 0 + \frac{1}{2c}\left\{-\int_{x_0}^{x-ct} w_1(\sigma)d\sigma + \int_{x_0}^{x+ct} w_1(\sigma)d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4c}[\mathrm{erf}(x+ct) - \mathrm{erf}(x-ct)]$$

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at $K=0$

By $F'(0)=0 \Rightarrow \bar{A}=0 \Rightarrow F(x)=\bar{B}$

Then, at $K=0$ $W_h(x,t) = \bar{B}[\bar{C}t+\bar{D}] = \underline{B_1 t + B_2}$

$\boxed{\begin{aligned} B_1 &= \bar{B}\bar{C} \\ B_2 &= \bar{B}\cdot\bar{D} \end{aligned}}$

at $K\neq 0$

By $F'(0)=0 = -AK\sin K\cdot 0 + BK\cos K\cdot 0 = 0 + BK \Rightarrow B=0$

By $F'(l)=0 = -AK\sin K\cdot l \Rightarrow Kl = n\pi \Rightarrow K=\frac{n\pi}{l}$

$\Rightarrow F(x) = A\cos\frac{n\pi}{l}x$

at $K\neq 0$ $W_h(x,t) = \sum_{n=1}^{\infty} A\cos\frac{n\pi}{l}x\,[C\cos Kct + D\sin Kct]$

$= \underline{\sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x\,[\mathcal{C}_n\cos Kct + \mathcal{D}_n\sin Kct]}$

$\boxed{\begin{aligned} \mathcal{C}_n &= AC \\ \mathcal{D}_n &= AD \end{aligned}}$

so that general W_h is

$$W_h = B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{C}_n\cos Kct + \mathcal{D}_n\sin Kct]$$

Therefore The whole fn. of W is:

$$W = W_p + W_h$$

$$= \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + Et + B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{C}_n\cos Kct + \mathcal{D}_n\sin Kct]$$

$$= \underline{\frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + E_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{C}_n\cos\frac{n\pi c}{l}t + \mathcal{D}_n\sin\frac{n\pi c}{l}t]}$$

where. $\boxed{E_1 = E + B_1}$

use I.C. to determine the E_1, B_2, $\mathcal{C}_n$ and $\mathcal{D}_n$

<<2>>

By $W(x,t=0) = 5x = \frac{B-A}{2l}x^2 + Ax + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot\mathcal{C}_n$

$\Rightarrow (5-A)x - \frac{B-A}{2l}x^2 = B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot\mathcal{C}_n$

By Fourier series

$$B_2 = \frac{1}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]dx$$

$$\mathcal{C}_n = \frac{2}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]\cdot\cos\frac{n\pi}{l}x\,dx$$

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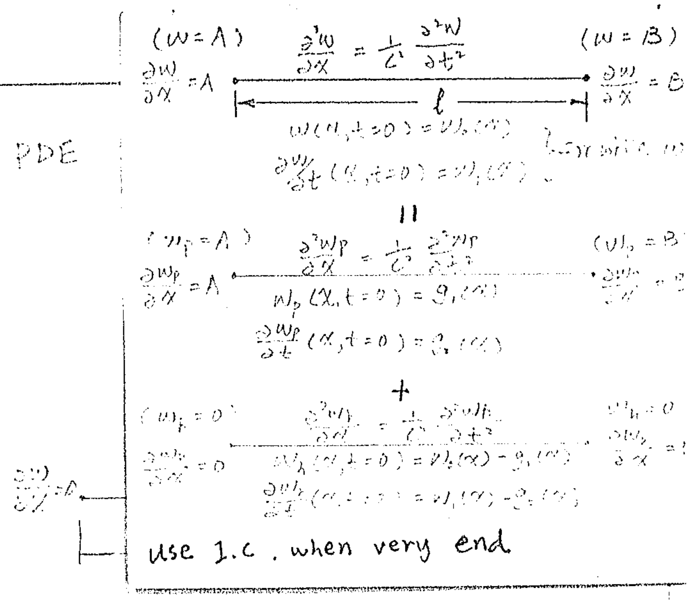
Fn. «1» satisfies the B.C. and PDE



(2) look at w_h

we use the method of separation of variable

$$w_h(x,t) = F(x)G(t)$$

$$\frac{\partial w_h}{\partial x} = F'(x)G(t) \quad ; \quad \frac{\partial^2 w_h}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial w_h}{\partial t} = F(x)G'(t) \quad ; \quad \frac{\partial^2 w_h}{\partial t^2} = F(x)G''(t)$$

By $\frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2}$

$$F''(x)G(t) = \frac{1}{c^2}\cdot F(x)G''(t) \Rightarrow \frac{F''(x)G(t)}{F(x)G(t)} = \frac{F(x)G''(t)}{c^2\cdot F(x)G(t)} = -K^2$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''}{G} = -K^2$$

at $K \neq 0$ $\quad F'' + K^2F = 0, \quad G'' + c^2K^2F = 0 \Rightarrow F(x) = A\cos Kx + B\sin Kx, \quad G(t) = C\cos Kct + D\sin Kct$

at $K = 0$ $\quad F'' = 0, \quad G'' = 0 \Rightarrow F(x) = \bar{A}x + \bar{B}, \quad G(t) = \bar{C}t + \bar{D}$

look at B.C. first

$$\frac{\partial w_h}{\partial x}(x=0,t) = 0 = F'(0)G(t) \Rightarrow F'(0) = 0$$

$$\frac{\partial w_h}{\partial x}(x=l,t) = 0 = F'(l)G(t) \Rightarrow F'(l) = 0$$

By $\frac{\partial w}{\partial t}(x, t=0) = 0 = E_1 + \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x \cdot D_n \cdot \frac{cn\pi}{l}$

$$\Rightarrow \quad E_1 = 0 \ , \quad D_n = 0$$

Therefore :

$$w = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cos\frac{cn\pi}{l}t$$

* check the result :

B.C $\quad \frac{\partial w}{\partial x}(0,t) = \frac{B-A}{l}\cdot 0 + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin\frac{n\pi}{l}\cdot 0 = A$

$\quad \frac{\partial w}{\partial x}(l,t) = \frac{B-A}{l}\cdot l + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin n\pi = B$

I.C $\quad w(x,0) = \frac{B-A}{2l}x^2 + Ax + 0 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x = 5x$

(according to fn. of «2»)

$\quad \frac{\partial w}{\partial t}(x,0) = 0 + 0 + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l}\sin\frac{cn\pi}{l}\cdot 0\right) = 0$

PDE $\quad \frac{\partial w}{\partial x} = \frac{B-A}{l}x + A + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \left(-\frac{n\pi}{l}\sin\frac{n\pi}{l}x\right)$

$$\frac{\partial^2 w}{\partial x^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi}{l}x\right] \qquad \text{«3»}$$

$$\frac{\partial w}{\partial t} = c^2\frac{B-A}{l}t + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l}\cdot\sin\frac{cn\pi}{l}t\right)$$

$$\frac{\partial^2 w}{\partial t^2} = c^2\cdot\frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{cn\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right]$$

$$\Rightarrow \frac{\partial^2 w}{c^2\partial t^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right] \qquad \text{«4»}$$

By «3» = «4»

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$$

Therefore, Fn. w satisfies the B.C., I.C., and PDE.

So w is the solution of the problem.

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PROBLEM: It is required to find the solution of the equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0$$

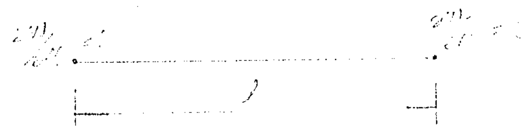
which satisfies the conditions:

$$\frac{\partial w}{\partial x}(0,t) = A \quad \text{for all values of } t \quad 0 \le x \le l$$
$$\frac{\partial w}{\partial x}(l,t) = B$$

$$w(x, t=0) = w_0 = 5x \qquad 0 < x < l$$
$$\frac{\partial w}{\partial t}(x, t=0) = w_1 = 0$$

solution: since $w(x,t) = w_p(x,t) + w_h(x,t)$

<1> First look at w_p.



assume $\partial w_p/\partial x = Cx + D$

use B.C to determine the C & D

$\partial w_p/\partial x\,(0,t) = C\cdot 0 + D = A \Rightarrow \underline{D = A} \Rightarrow \frac{\partial w_p}{\partial x} = Cx + A$

$\partial w_p/\partial x\,(l,t) = C\cdot l + A = B \Rightarrow C = \frac{B-A}{l}$

$$\Rightarrow \frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A \checkmark \quad (\text{it satisfies the B.C})$$

$$\Rightarrow w_p = \frac{B-A}{2l}x^2 + Ax + f(t) \qquad (w_p \text{ must satisfies the PDE})$$

$$\frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A \qquad \frac{\partial^2 w_p}{\partial x^2} = \frac{B-A}{l}$$

$$\frac{\partial w_p}{\partial t} = f'(t) \qquad \frac{\partial^2 w_p}{\partial t^2} = f''(t)$$

since $\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$

$$\Rightarrow \frac{B-A}{l} = \frac{1}{c^2}f''(t) \Rightarrow f'(t) = c^2\cdot\frac{B-A}{l}t + E$$

$$\Rightarrow f(t) = \frac{c^2}{2}\frac{B-A}{l}t^2 + Et$$

Therefore $\Rightarrow$

$$w_p = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\frac{B-A}{l}t^2 + Et \checkmark \qquad <<1>>$$

Cesar for your information. Esam Hamad

EGM 3311 Analysis of Engineering Systems

FALL 2002

Final Exam

Name:________________________

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1. A mechanical part has an exponential time-to-failure distribution with mean time to failure of 10,000 hours. The part has already lasted for 15,000 hours. What is the probability that it will fail by 20,000 hours?

2. The outer diameters of 10 piston rings are found to be (in mm)

121.5 119.4 126.7 117.9 120.2 124.3 122.5 120.8 121.9 123.6

If the diameters follow normal distribution, find the 95% confidence interval for the outer diameters of the entire population of piston rings.

3. A new filtering device is being tested. Before its installation, a random sample yielded the following information about the percentage of impurity: $\overline{x_1} = 12.5$, $s_1^2 = 101.17$ and $n_1 = 8$. After installation, a random sample yielded $\overline{x_2} = 10.2$, $s_2^2 = 94.73$ and $n_2 = 9$.

a) Can you conclude that the two variances are equal?
b) Has the filtering device reduced the percentage of impurity significantly?

4. Find the solution of the following set of equations using the Gauss elimination method:

$$
\begin{aligned}
-5\,x_1 - \;\; x_2 + 2x_3 &= 1; \\
2\,x_1 + 6x_2 - 3x_3 &= 2; \\
2\,x_1 + \;\; x_2 + 7x_3 &= 32;
\end{aligned}
$$

5. A metal rod of length 1 m is initially at 100° C. The steady-state temperatures of the left and right ends of the rod are 150° C and 25° C, respectively. Using $\alpha^2=0.2$, $\Delta t=0.05$ min and $\Delta t=0.2$ m, determine the temperature distribution in the rod at t=0.1 min. The temperature is governed by the following partial differential equation:

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$