4) $(1-x)x\,y'' - 2y' + 2y = 0$ near $x=0$

$P(x=0)=0$ singular point $\therefore$ $\lim_{x\to 0} x\dfrac{Q}{P} = \dfrac{-2x}{x(1-x)} = -2$ ; $\lim_{x\to 0} \dfrac{x^2\, 2}{x(1-x)} = 0$ $\Big\}\to x=0$ is a regular singular pt

$$\therefore\ y = x^{\alpha}\sum_{n=0}^{\infty} A_n x^n = A_0x^{\alpha} + A_1x^{\alpha+1} + A_2x^{\alpha+2} + A_3x^{\alpha+3} + A_4x^{\alpha+4}+\dots$$

$$y' = A_0\alpha x^{\alpha-1} + A_1(\alpha+1)x^{\alpha} + A_2(\alpha+2)x^{\alpha+1} + A_3(\alpha+3)x^{\alpha+2} + A_4(\alpha+4)x^{\alpha+3}+\dots$$

$$y'' = A_0\,\alpha(\alpha-1)x^{\alpha-2} + A_1(\alpha+1)\alpha x^{\alpha-1} + A_2(\alpha+2)(\alpha+1)x^{\alpha} + A_3(\alpha+3)(\alpha+2)x^{\alpha+1} + A_4(\alpha+4)(\alpha+3)x^{\alpha+2}+\dots$$

$$xy'' = A_0\alpha(\alpha-1)x^{\alpha-1} + A_1(\alpha+1)\alpha x^{\alpha} + A_2(\alpha+2)(\alpha+1)x^{\alpha+1} + A_3(\alpha+3)(\alpha+2)x^{\alpha+2} + A_4(\alpha+4)(\alpha+3)x^{\alpha+3}+\dots$$

$$-x^2y'' = \qquad - A_0\alpha(\alpha-1)x^{\alpha} - A_1(\alpha+1)\alpha x^{\alpha+1} - A_2(\alpha+2)(\alpha+1)x^{\alpha+2} - A_3(\alpha+3)(\alpha+2)x^{\alpha+3}+\dots$$

$$-2y' = -2A_0\alpha x^{\alpha-1} - 2A_1(\alpha+1)x^{\alpha} - 2A_2(\alpha+2)x^{\alpha+1} - 2A_3(\alpha+3)x^{\alpha+2} - 2A_4(\alpha+4)x^{\alpha+3}+\dots$$

$$+2y = \qquad 2A_0x^{\alpha} + 2A_1x^{\alpha+1} + 2A_2x^{\alpha+2} + 2A_3x^{\alpha+3}+\dots$$

$$0 = A_0x^{\alpha-1}\left[\alpha(\alpha-1)-2\alpha\right] + \Big\{A_1\left[(\alpha+1)\alpha-2(\alpha+1)\right] - A_0\left[\alpha(\alpha-1)-2\right]\Big\}x^{\alpha} + \Big\{A_2\left[(\alpha+2)(\alpha+1)-2(\alpha+2)\right] - A_1\left[(\alpha+1)(\alpha)-2\right]\Big\}x^{\alpha+1}$$
$$+ \Big\{A_3\left[(\alpha+3)(\alpha+2)-2(\alpha+3)\right] - A_2\left[(\alpha+2)(\alpha+1)-2\right]\Big\}x^{\alpha+2} + \Big\{A_4\left[(\alpha+4)(\alpha+3)-2(\alpha+4)\right] - A_3\left[(\alpha+3)(\alpha+2)-2\right]\Big\}x^{\alpha+3}$$

either $A_0=0$ or $\alpha^2-3\alpha=0 \Rightarrow \alpha=3$ & $\alpha=0$ ; if $A_0=0$ & $\alpha$ is anything $\Rightarrow A_1, A_2, \dots, A_n = 0\dots$

if $A_0\neq 0$ then $$A_n = A_{n-1}\frac{\left([\alpha+n-1][\alpha+n-2]-2\right)}{(\alpha+n)(\alpha+n-1)-2(\alpha+n)} = A_{n-1}\frac{(\alpha+n)(\alpha+n-3)}{(\alpha+n)(\alpha+n-3)} = A_{n-1} \quad n\geq 1$$

A

so if $\alpha_1=0$
$-2A_1+2A_0=0$ $\quad A_0=A_1$
$-2A_2+2A_1=0$ $\quad A_2=A_1=A_0$
$0A_3 - A_2\cdot 0 = 0$ $\quad A_3$ is anything
$4A_4 - A_3\cdot 4 = 0$ $\quad A_4=A_3$
$10A_5 - 10A_4 = 0$ $\quad A_5=A_4=A_3$ and so on $\Rightarrow$ $y_1 = A_0\sum_{n=0}^{\infty} x^n$

if $\alpha_2=3$ apparent $y_{2a} = \tilde{A}_0\sum_{n=0}^{\infty} x^{n+3}$ since $|\alpha_2-\alpha_1|=3 \Rightarrow$

$\therefore$ $y_2 = a y_1\ln|x| + \tilde{A}_0\sum_{n=0}^{\infty} x^{n+3}$ & $y_2' = a y_1'\ln|x| + a y_1/x + y_{2a}'$
$y_2'' = a y_1''\ln|x| + 2a y_1'/x - a y_1/x^2 + y_{2a}''$

$$(1-x)x\,y_2'' - 2y_2' + 2y_2 = (1-x)x\,a y_1''\ln|x| + (1-x)x\cdot 2a y_1'/x - (1-x)x\,a y_1/x^2 + (1-x)x\,y_{2a}'' - 2a y_1'\ln|x| - 2a y_1/x - 2y_{2a}'$$
$$+ 2a y_1\ln|x| + 2y_{2a} = a\ln|x|\left[(1-x)y_1'' - 2y_1' + 2y_1\right] + \left[(1-x)x\,y_{2a}'' - 2y_{2a}' + 2y_{2a}\right]$$
$$+ 2a y_1' - 2ax y_1' - a y_1/x + a y_1 - 2y_{2a}' = 0$$

$$\Rightarrow\quad a = \frac{2y_{2a}'}{2(1-x)y_1' - y_1\frac{(1-x)}{x}}$$