The following tables presenting SI (Systems International) units and their conversion to SI units conclude this introduction.

Si Prefixes

Multiplication Factor	*Prefix*	*Symbol*
10^{15}	peta	P
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{-1}	deci	d
10^{-2}	centi	c
10^{-3}	mili	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f

SI base units

Quantity	*Name*	*Symbol*
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

SI Derived Units

Quantity	*Name*	*Symbol*	*In Terms of Other Units*
area	square meter		m^2
volume	cubic meter		m^3
velocity	meter per second		m/s
acceleration	meter per second squared		m/s^2
density	kilogram per cubic meter		kg/m^3
specific volume	cubic meter per kilogram		m^3/kg
frequency	hertz	Hz	s^{-1}
force	newton	N	$m \cdot kg/s^2$
pressure, stress	pascal	Pa	$kg/(m \cdot s^2)$
energy, work, heat	joule	J	N·m
power	watt	W	J/s
electric charge	coulomb	C	A·s
electric potential	volt	V	W/A
capacitance	farad	F	C/V
electric resistance	ohm	Ω	V/A
conductance	siemens	S	A/V
magnetic flux	weber	Wb	V·s
inductance	henry	H	Wb/A
viscosity	pascal second		Pa·s
moment (torque)	meter newton		N·m
heat flux	watt per square meter		W/m^2
entropy	joule per kelvin		J/K
specific heat	joule per kilogram-kelvin		J/(kg·K)
conductivity	watt per meter-kelvin		W/(m·K)

Conversion Factors to SI Units

English	SI	SI Symbol	To Convert from English to SI Multiply by
	Area		
square inch	square centimeter	cm^2	6.452
square foot	square meter	m^2	0.09290
acre	hectare	ha	0.4047
	Length		
inch	centimeter	cm	2.54
foot	meter	m	0.3048
mile	kilometer	km	1.6093
	Volume		
cubic inch	cubic centimeter	cm^3	16.387
cubic foot	cubic meter	m^3	0.02832
gallon	cubic meter	m^3	0.004546
gallon	liter	ℓ	3.785
	Mass		
pound mass	kilogram	kg	0.4536
slug	kilogram	kg	14.59
	Force		
pound	newton	N	4.448
kip (1000 lb)	newton	N	4448
	Density		
pound/cubic foot	kilogram/cubic meter	kg/m^3	16.02
pound/cubic foot	grams/liter	g/ℓ	16.02
	Work, Energy, Heat		
foot-pound	joule	J	1.356
BTU	joule	J	1055
BTU	kilowatt-hour	kWh	0.000293
therm	kilowatt-hour	kWh	29.3
quad	giga joule	GJ	1.055×10^9

Conversion Factors to SI Units (continued)

English	*SI*	*SI Symbol*	*To Convert from English to SI Multiply by*
	Power, Heat Rate		
horsepower	watt	W	745.7
foot pound/sec	watt	W	1.356
BTU/hour	watt	W	0.2931
BTU/hour-ft²-°F	watt/meter squared-degree celsius	$W/m^2\cdot°C$	5.678
tons of refrig.	kilowatts	kW	3.517
	Pressure		
pound/square inch	kilopascal	kPa	6.895
pound/square foot	kilopascal	kPa	0.04788
inches of H_2O	kilopascal	kPa	0.2486
inches of Hg	kilopascal	kPa	3.374
one atmosphere	kilopascal	kPa	101.3
	Temperature		
Fahrenheit	Celsius	°C	5/9(°F-32)
Fahrenheit	kelvin	K	5/9(°F+460)
	Velocity		
foot/second	meter/second	m/s	0.3048
mile/hour	meter/second	m/s	0.4470
mile/hour	kilometer/hour	km/h	1.609
	Acceleration		
foot/second squared	meter/second squared	m/s^2	0.3048
	Torque		
pound-foot	newton-meter	N·m	1.356
pound-inch	newton-meter	N·m	0.1130
	Viscosity, Kinematic Viscosity		
pound-sec/square foot	newton-sec/square meter	$N\cdot s/m^2$	47.88
square foot/second	square meter/second	m^2/s	0.09290
	Flow Rate		
cubic foot/second	cubic meter/second	m/s	0.0004719
cubic foot/second	liter/second	ℓ/s	0.4719
	Frequency		
cycles/second	hertz	Hz	1.00

Conversion Factors

Length

1 cm	= 0.3937 in
1 m	= 3.281 ft
1 yd	= 3 ft
1 mi	= 5280 ft
1 mi	= 1760 yd
1 km	= 3281 ft

Area

1 cm^2	= 0.155 in^2
1 m^2	= 10.76 ft^2
1 ha	= 10^4 m^2
1 are	= 100 m^2
1 acre	= 4047 m^2
1 acre	= 43560 ft^2

Volume

1 ft^3	= 28.32 ℓ
1 ℓ	= 0.03531 ft^3
1 ℓ	= 0.2642 gal
1 m^3	= 264.2 gal
1 ft^3	= 7.481 gal
1 m^3	= 35.31 ft^3
1 acre-ft	= 43560 ft^2

Velocity

1 m/s	= 3.281 ft/s
1 mph	= 1.467 ft/s
1 mph	= 0.8684 knot
1 knot	= 1.688 ft/s
1 km/h	= 0.2778 m/s
1 km/h	= 0.6214 mph

Force

1 lb	= 4.448 x 10^5 dyne
1 lb	= 32.17 pdl
1 lb	= 0.4536 kg
1 N	= 10^5 dyne
1 N	= 0.2248 lb
1 kip	= 1000 lb

Mass

1 oz	= 23.35 g
1 lb	= 0.4536 kg
1 kg	= 2.205 lb
1 slug	= 14.59 kg
1 slug	= 32.17 lb

Work and Heat

1 Btu	= 778.2 ft-lb
1 Btu	= 1055 J
1 Cal	= 3.088 ft-lb
1 J	= 10^7 ergs
1 kJ	= 0.9478 ft-lb
1 Btu	= 0.2929 W · hr
1 ton	= 12000 Btu
1 kWh	= 3414 Btu
1 quad	= 10^{15} Btu
1 therm	= 10^5 Btu

Power

1 Hp	= 550 ft-lb/s
1 HP	= 33000 ft-lb/min
1 Hp	= 0.7067 Btu/s
1 Hp	= 2545 Btu/hr
1 Hp	= 745.7 W
1 W	= 3.414 Btu/hr
1 kW	= 1.341 Hp

Volume Flow Rate

1 cfm	= 7.481 gal/min
1 cfm	= 0.4719 ℓ/s
1 m^3/s	= 35.31 ft^3/s
1 m^3/s	= 2119 cfm
1 gal/min	= 0.1337 cfm

Torque

1 N · m	= 10^7 dyne · cm
1 N · m	= 0.7376 lb-ft
1 N · m	= 10197 g · cm
1 lb-ft	= 1.356 N · m

Viscosity

1 lb-s/ft^2	= 478 poise
1 poise	= 1 g/cm · s
1 N · s/m^2	= 0.02089 lb-s/ft^2

Pressure

1 atm	= 14.7 psi
1 atm	= 29.92 in Hg
1 atm	= 33.93 ft H_20
1 atm	= 1.013 bar
1 atm	= 1.033 kg/cm^2
1 atm	= 101.3 kPa
1 psi	= 2.036 in Hg
1 psi	= 6.895 kPa
1 psi	= 68950 dyne/cm^2
1 ft H_20	= 0.4331 psi

The length of the curve between the two points is expressed as

$$L = \int_a^b (1 + y'^2)^{1/2}\, dx. \tag{1.6.10}$$

Volumes of various objects are also found by an appropriate integration.

If the integral has limits, it is a *definite integral;* if it does not have limits, it is an *indefinite integral* and a constant is always added. Some common indefinite integrals follow:

$$\int dx = x + C.$$

$$\int cy\,dx = c \int y\,dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int x^{-1} dx = \ln x + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C \tag{1.6.11}$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \cos^2 x\, dx = \frac{x}{2} + \frac{1}{4} \sin 2x + C$$

$$\int u\, dv = uv - \int v\, du.$$

This last integral is often referred to as "integration by parts." If the integrand (the coefficient of the differential) is not one of the above, then, in the last integral, $\int v\,du$ may in fact, be integrable.

1.7 Differential Equations

A differential equation is *linear* if no term contains the dependent variable to a power other than one (terms that do not contain the dependent variable are not considered in the test of linearity). For example,

$$y'' + 2xy' - y \sin x = 3x^2 \tag{1.7.1}$$

is a linear differential equation; the dependent variable is y and the independent variable is x. If a term contained y'^2, or $y^{1/2}$, or $\sin y$ the equation would be nonlinear.

A differential equation is *homogeneous* if all of its terms contain the dependent variable. Eq. 1.7.1 is nonhomogeneous because of the term $3x^2$.

The *order* of a differential equation is established by its highest order derivative. Eq. 1.7.1 is a second order differential equation.

The general solution of a differential equation involves a number of arbitrary constants equal to the order of the equation. If conditions are specified, the arbitrary constants may be calculated.

1.7.1 First Order

A first order differential equation is *separable* if it can be expressed as

$$M(x)\,dx + N(y)\,dy = 0. \tag{1.7.2}$$

The solution follows by integrating each of the terms.

If $M = M(x,y)$ and $N = N(x,y)$, the solution $F(x,y) = C$ can be found if Eq. 1.7.2 is *exact*, that is $\partial M/\partial y = \partial N/\partial x$; then $M = \partial F/\partial x$ and $N = \partial F/\partial y$. Note that Eq. 1.7.2 is, in general, nonlinear.

The linear, first order differential equation

$$y' + h(x)\,y = g(x) \tag{1.7.3}$$

has the solution

$$y(x) = \frac{1}{u}\int u\,g(x)\,dx + \frac{C}{u} \tag{1.7.4}$$

where

$$u(x) = e^{\int h(x)\,dx} \tag{1.7.5}$$

The function $u(x)$ is called an integrating factor.

1.7.2 Second Order, Linear, Homogeneous, with Constant Coefficients

The general form of a second order, linear, homogeneous differential equation with constant coefficients is

$$y'' + Ay' + By = 0. \tag{1.7.6}$$

To find a solution we must first solve the characteristic equation

$$m^2 + Am + B = 0. \tag{1.7.7}$$

If $m_1 \neq m_2$ and both are real, the general solution is

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{1.7.8}$$

If $m_1 = m_2$, the general solution is

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}. \tag{1.7.9}$$

Finally, if $m_1 = a + ib$ and $m_2 = a - ib$, the general solution is

$$y(x) = (c_1 \sin bx + c_2 \cos bx)\,e^{ax}. \tag{1.7.10}$$

$$a = -\frac{A}{2} \qquad b = \frac{\sqrt{4B - A^2}}{2}$$

NOTE: EQUATIONS 1.7.8 – 1.7.10 ARE THE HOMOGENEOUS SOLUTIONS $y_h(x)$ USED IN THE NEXT SECTION

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1.7.3 Linear, Nonhomogeneous, with Constant Coefficients

If Eq. 1.7.6 were nonhomogeneous, it would be written as

$$y'' + Ay' + By = g(x). \qquad (1.7.11)$$

The general solution is found by finding the solution $y_h(x)$ to the homogeneous equation (simply let the right-hand side be zero and solve the equation as in Section 1.7.2) and adding to it a particular solution $y_p(x)$ found by using Table 1.2.

TABLE 1.2 Particular Solutions

g(x)	$y_p(x)$	Provisions
a	C	
$ax + b$	$Cx + D$	
$ax^2 + bx + c$	$Cx^2 + Dx + E$	
e^{ax}	Ce^{ax}	if m_1 or $m_2 \neq a$
	Cxe^{ax}	if m_1 or $m_2 = a$
$b \sin ax$	$C \sin ax + D \cos ax$	if $m_{1,2} \neq \pm ai$
	$Cx \sin ax + Dx \cos ax$	if $m_{1,2} = \pm ai$
$b \cos ax$	(same as above)	

NOTE b, a are general constants; not the same as those in equation 1.7.10

1.8 Probability and Statistics

Events are independent if the probability of occurrence of one event does not influence the probability of occurrence of other events. The number of permutations of n things taken r at a time is

$$p(n,r) = \frac{n!}{(n-r)!}. \qquad (1.8.1)$$

If the starting point is unknown, as in a ring, the *ring permutation* is

$$p(n,r) = \frac{(n-1)!}{(n-r)!}. \qquad (1.8.2)$$

ORDER DEPENDENT

The number of *combinations* of n things taken r at a time (it is not an order-conscious arrangement) is given by

$$C(n,r) = \frac{n!}{r!(n-r)!}. \qquad (1.8.3)$$

$y' = x - y$ near $x = x_0$

$$y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$$

$$y' = \sum_{n=1} A_n \cdot n (x-x_0)^{n-1}$$

$$y' + y - x = \sum_{n=1} A_n \cdot n (x-x_0)^{n-1} + \sum_{n=0} A_n (x-x_0)^n - x = 0$$

let $m = n-1$ $n = m+1$

$$y' + y - x = \sum_{m=0} A_{m+1} \cdot (m+1)(x-x_0)^m + \sum_{n=0} A_n (x-x_0)^n - x = 0 = \sum_{n=0}^{\infty} 0 \cdot (x-x_0)^n$$

since n & m are dummy

$$\sum_{n=0} \left[A_{n+1}(n+1) + A_n\right](x-x_0)^n - x = \sum 0 (x-x_0)^n$$

$\Rightarrow n=0: A_1 \cdot 1 + A_0 = 0$ $\qquad A_1 = -A_0$

$n=1: A_2 \cdot 2 + A_1 - 1 = 0$ $\qquad A_2 = \frac{1-A_1}{2} = \frac{1+A_0}{2}$

$n=2: A_3 \cdot 3 + A_2 = 0$ $\qquad A_3 = -\frac{A_2}{3} = -\frac{(1+A_0)}{3 \cdot 2}$

$n=k: A_{k+1}(k+1) + A_k = 0$

recursion formula $A_{k+1} = \frac{A_k}{k+1} = \cdots = \frac{(-1)^{k+1}(1+A_0)}{(k+1)!}$

$$\therefore \quad y = \sum A_n (x-x_0)^n = A_0 + A_1(x-x_0) + A_2(x-x_0)^2 + \cdots$$

$$= A_0\left[1 - (x-x_0) + \tfrac{1}{2}(x-x_0)^2 - \tfrac{1}{3!}(x-x_0)^3 \cdots\right] + \tfrac{1}{2}(x-x_0)^2 - \tfrac{1}{3!}(x-x_0)^3 + \cdots$$

$$y = A_0 e^{-(x-x_0)} + e^{-(x-x_0)} - 1 + (x-x_0)$$

$$= (A_0+1) e^{-(x-x_0)} - 1 + (x-x_0)$$

since $y = y_0$ when $x = x_0$ $\Rightarrow y_0 = (A_0+1)e^0 - 1 + 0 = A_0$ and

$$\underline{y = (1+y_0) e^{-(x-x_0)} - 1 + (x-x_0)}$$

what about convergence $\left|\frac{A_{k+1}(x-x_0)^{k+1}}{A_k (x-x_0)^k}\right| = \left|\frac{1}{k+1}(x-x_0)\right| < 1$ as $k \to \infty$ $\Rightarrow |x-x_0| < \infty$

since $\frac{1}{k+1} \to 0$ irrespective of x

[illegible]

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1 O.D.E. REVIEW

- ORDER OF DIFF. EQ. = HIGHEST DERIV.

$$y^{IV} + 2y'' = 0 \qquad y^{IV} = \frac{d^4y}{dx^4}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^2 w}{\partial x^2} = 0 \qquad w = w(x,y)$$

- IF COEFFS OF DERIVATIVES ARE CONST. ⇒ CONSTANT COEFF DIFF. EQ.

$$u''(t) + c u'(t) + m u(t) = 0 \qquad m, c \text{ CONST.}$$

- IF FN IS FN OF ONE INDEP. VARIABLE ie $y = y(x)$

 ⇒ ORDINARY DIFF. EQ.

$$a(x) y'' + b(x) y' + c(x) y = 0 \qquad y = y(x)$$

- LINEAR IF Power OF y or its derivatives are $y^1, (y')^1$ etc only

$$a(x) y'' + b(x) y' + c(x) y = 0$$

- NON LINEAR IF products of y & derivs or powers of y

$$a(x) y'' + b(x) y y' = 0$$

$$a(x) y'' + b(x)(y')^2 + c(x) = 0$$

- HOMOGENEOUS IF TERM THAT DOESN'T INVOLVE y or derivs $= 0$

$$a(x) y'' + b(x) y' + c(x) y + d(x) = 0 \qquad \text{HOMOG IF } d(x) = 0$$

- FIRST ORDER ODE : FIND IN ANY O.D.E. BOOK.

$$a(x) y' + b(x) y = c(x)$$

$$\Rightarrow \quad y' + p(x) y = h(x)$$

- SOLUTION use integrating factor $\mu(x) = e^{\int^x p(t)\,dt}$

$$y(x) = \frac{1}{\mu(x)} \int^x \mu(t) h(t)\, dt + \frac{\text{const}}{\mu(x)}$$

- need one condition $y = y_0$ at $x = x_0$ to define const

- WHAT IF IT IS HARD TO FIND SOLUTION BY CLOSED FORM

- PICARD'S METHOD   Good for $|x|<a$ if y & $\partial f/\partial y$ are continuous in $|x|\le a$ $|y|\le b$

WRITE $y' = f(x,y)$ with $y(x=x_0) = y_0$

$$\Rightarrow \quad y - y_0 = \int_{x_0}^{x} f(\bar{x}, y)\, d\bar{x} \quad \text{when we integrate}$$

Picard says: define sequence $y_0, y_1, y_2, \dots y_n \rightarrow y$ in limit

$$y_1 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_0)\, d\bar{x}$$
$$y_2 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_1)\, d\bar{x}$$
$$\vdots$$
$$y_n = y_0 + \int_{x_0}^{x} f(\bar{x}, y_{n-1})\, d\bar{x}$$

- unsatisfactory on practical reasons it is hard to integrate

- example $y' = x - y$ $y(x=0) = 1$ $y_0 = 1$ $x_0 = 0$

$f(x,y) = x - y$

$$y_1 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_0)\, d\bar{x} = 1 + \int_0^x (\bar{x} - 1)\, d\bar{x}$$

$$= 1 + \frac{(\bar{x}-1)^2}{2}\Big|_0^x = 1 + \frac{(x-1)^2}{2} - \frac{(-1)^2}{2} = 1 + \frac{x^2}{2} - x = 1 - x + \frac{x^2}{2}$$

$$y_2 = y_0 + \int_{x_0}^{x} f(\bar{x}, y_1)\, d\bar{x} = 1 + \int_0^x \left[\bar{x} - \left(1 - \bar{x} + \frac{\bar{x}^2}{2}\right)\right] d\bar{x}$$

$$= 1 + \left[-\bar{x} + \bar{x}^2 - \frac{\bar{x}^3}{6}\right]_0^x = 1 - x + x^2 - \frac{x^3}{6}$$

- actual : $\mu(x) = e^{\int^x 1\,dt} = e^x$ $y' + y = x$ $p(x) = 1$ $h(x) =$

$$y(x) = \frac{1}{e^x} \int^x e^t\, t\, dt + \frac{C}{e^x}$$
$$= \frac{1}{e^x}\left[te^t - e^t\right]^x + C/e^x = x - 1 + Ce^{-x},$$

when $x=0$ $y=1$ $\Rightarrow C = 2$

3 AT $x=0$ $y=1$ $\Rightarrow$ $C=2$

$$y(x) = x-1+2e^{-x}$$
$$= x-1+2\left[1-x+\frac{x^2}{2}-\frac{x^3}{3!}+\dots\right]$$
$$= 1-x+x^2-\frac{x^3}{3}+\dots$$

y_1 error in last term

y_2 error in last term

but sequence tends to y

HW Use Picard's method to find the solution to $y'-x^2y=x$ with $y(x=0)=1$ at $x=.2$ (ie $y(x=.2)=?$) Solution must be accurate to 5 decimal places

- Picard's method is good but unwieldy

Series solutions

First Order Equations

- Given $y' = f(x,y)$ A solution exists if $f(x,y)$ is continuous & single valued over the region of interest
  - $\partial f/\partial y$ exists & is continuous

- if so we can assume $y = \sum_{n=0}^{\infty} A_n x^n$ and all the A_n's can be determined in terms of A_0 & A_0 can be found if an initial value is given. if $y = y_0$ at $x = x_0$ use $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ & $A_0 = y_0$

$s_i = \sum_{j=0}^{i} a_j$

- Radius of convergence $\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| |x-x_0| < 1$

$\lim_{i\to\infty} s_i = s$ if $\sum |a_i|$ converges & if $\left|\frac{a_{i+1}}{a_i}\right| < 1$

here $a_i = A_i (x-x_0)^i$

- example $y' = x - y$ $y(x=0) = 1$ $f(x,y) = x-y$ $\frac{\partial f}{\partial y} = -1$

let $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ $x_0 = 0$ $y_0 = 1$

@ $x = x_0 = 0$ $y = y_0 = 1 = A_0$

$$y' = \sum_{n=1}^{\infty} A_n n (x-x_0)^{n-1}$$

$$y' - x + y = A_1 + 2A_2 x + 3A_3 x^2 + \dots - x + [A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots] = 0$$

$$(A_1 + A_0) + (2A_2 - 1 + A_1)x + (A_2 + 3A_3)x^2 + \dots + (nA_n + A_{n-1})x^n + \dots = 0$$

$= 0 + 0 \cdot x + 0 \cdot x^2 + \dots$

$$A_1 = -A_0$$

$$A_2 = \frac{1-A_1}{2} = \frac{1}{2} + \frac{A_0}{2} = \frac{1}{2}(1-A_1)$$

$$A_3 = -\frac{A_2}{3} = -\frac{1}{3}\cdot\frac{1}{2}(1-A_1)$$

$$A_4 = -\frac{A_3}{4} = +\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}(1-A_1)$$

$\vdots$

$$A_n = -\frac{A_{n-1}}{n} = (-1)^n \cdot \frac{1}{n!}(1-A_1)$$

$$y = \left[+\frac{x^2}{2} - \frac{x^3}{3!} + \dots\right](1-A_1) + A_0 - A_0 x \qquad 1 - A_1 = 1 + A_0 = 2$$

$$= 1 - x + 2\left[\frac{x^2}{2} - \frac{x^3}{3!} + \dots\right] = -(1-x) + 2\left[1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$

$$y = 2e^{-x} - 1 + x$$

radius of convergence $\left|\frac{A_{n+1}}{A_n}\right| |x-x_0| < 1 \Rightarrow \frac{1}{(n+1)!} / \frac{1}{n!} \; |x| \to 0 \Rightarrow |x| < \infty$

HW $(1-x)y' = 2x - y$ $y = y_0$ when $x = 0$

Linear Equations of order 2

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

IF $P_0(a) \neq 0$ at $x=a \Rightarrow x=a$ is an ordinary point then $y(x) = \sum_{n=0}^{\infty} a_n (x-a)$

and $y = c_1 y_1(x) + c_2 y_2(x)$ $\quad y_1(x) = \sum_{n=0}^{\infty} A_n (x-a)^n$

$y_2(x) = \sum_{n=0}^{\infty} B_n (x-a)^n$

$y_1(x)$ and $y_2(x)$ are linearly independent & analytic at $x=a$

IF $P_0(a) = 0 \Rightarrow x=a$ is a singular point

- a is a regular singular point if

$\frac{P_1(x)}{P_0(x)}(x-a)$ and $\frac{P_2(x)}{P_0(x)}(x-a)^2$ can be expanded in Power Taylor series about $x=a$

$\sum_{n=0}^{\infty} C_n (x-a)^n \qquad \sum_{n=0}^{\infty} D_n (x-a)^n$

- a is an irregular singular point if cannot be expanded in Taylor series in terms of $x-a$

radius of Convergence is $|x| = a$

- <u>Example</u> Regular singular point

$(1+x)y'' + 2xy' - 3y = 0$ at $x = -1$

$\frac{P_1}{P_0}(x+1) = 2x = 2(x+1) - 2 \qquad C_0 = -2,\ C_1 = 2,\ C_2 \cdots C_\infty =$

$\frac{P_2}{P_0}(x+1)^2 = -3(x+1) \qquad D_0 = 0 \quad D_1 = -3 \quad D_2 \cdots D_\infty = 0$

if we assume $y(x) = \sum_{n=0}^{\infty} A_n (x-a)^{n+r_1} + \sum_{n=0}^{\infty} B_n (x-a)^{n+r_2} = y_1(x) + y_2(x)$

for a regular singular pt

$\Rightarrow y_2(x) = 0 \Rightarrow B_n\text{'s} = 0$ only one solution will be obtained & of the form $y(x) = (x-a)^m \sum_{n=0}^{\infty} A_n (x-a)^n$

- Use $y_2(x) = y_1(x) \int^x \frac{C e^{-\int^s \frac{P_1(t)}{P_0(t)} dt}}{y_1^2(s)} ds$ To find 2nd solution

Cannot use since $P_0(x) = 0$ & must divide by $P_0(x)$ to use this.

- m may be real or complex

- FOR $x=a$ BEING AN IRREGULAR SINGULAR PT – A SOLUTION IN POWER series form may or may not exist.

- FOR REGULAR SINGULAR PTS USE METHOD OF FROBENIUS (ABOUT $x=0$)
  - COEFF OF FIRST TERM IN POWER SERIES EXPANSION = INDICIAL EQN
  - INDICIAL EQUATION GIVES m (FOR 2^{nd} order m_1 & m_2)
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ is not integer

    $\Rightarrow$ 2 DISTINCT SOLNS EACH OF FORM $x^{m_1}\left[\sum_{n=0}^{\infty} A_n x^n\right]$ & $x^{m_2}\left[\sum_{n=0}^{\infty} B_n x^n\right]$

    where A_n depends on m_1 & B_n depends on m_2
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ IS an integer
    - $\Rightarrow$ LARGER ROOT ALWAYS GIVES SOLN $m=m_1$ $\quad y_1 = x^{m_1}\sum_{n=0}^{\infty} A_n x^n$
    - $y_2(x) = A\, y_1(x) \ln x + \bar{y}_2(x)$ $\quad \bar{y}_2 = x^{m_2}\sum_{n=0}^{\infty} B_n x^n$

      $y(x) = C_1 y_1(x) + C_2 y_2(x)$

    ($m=m_2$ (2^{nd}) can yield a soln, no soln or a general soln)
  - IF $m_1 = m_2$

    $y_2(x) = y_1(x) \ln x + \bar{y}(x)$ $\quad \bar{y} = x^{m_1}\sum_{n=0}^{\infty} B_n x^n$

    $y(x) = C_1 y_1(x) + C_2 y_2(x)$

  - AN EASIER FORM IS IF you know $y_1(x)$

$$y_2(x) = \left.\frac{\partial y_1}{\partial m}\right|_{m=m_1}$$

- WHAT IF $a \neq 0$ let $t = x-a$ $\Rightarrow$ $x=a$ $t=0$

$$\frac{d}{dx}(\;) = \frac{d}{dt}(\;)\cdot\frac{dt}{dx} = \frac{d}{dt}(\;)$$

- WHAT IF $a=\infty$ let $t = \frac{1}{x}$ $\Rightarrow$ $x\to\infty$ $t \to 0$

Regular Singular

Example $\quad xy'' + y' - y = 0$ $\qquad \frac{P_1}{P_0}x = \frac{1}{x}\cdot x = 1 \qquad \frac{P_2}{P_0}\cdot x^2 = \frac{-1}{x}\cdot x^2 = -x$

let $y = x^m \sum_{n=0}^{\infty} A_n x^n$

$y' = m x^{m-1}\sum_{n=0}^{\infty} A_n x^n + x^m \sum_{n=1}^{\infty} A_n n x^{n-1}$

$y'' = m(m-1)x^{m-2}\sum A_n x^n + 2m x^{m-1}\sum A_n n x^{n-1} + x^m \sum A_n n(n-1) x^{n-2}$

$$xy'' + y' - y = m^2 A_0 x^{m-1} + [(m+1)^2 A_1 - A_0]x^m + [(m+2)^2 A_2 - A_1]x^{m+1} + \dots + [(m+n)^2 A_n - A_{n-1}]x^{m+n-1} + \dots = 0$$

$$\Rightarrow \quad A_n = \frac{A_{n-1}}{(m+n)^2} \qquad n = 1 \dots \infty$$

$$\Rightarrow \quad A_0 = 0 \quad \text{or} \quad m = 0 \quad \text{equal roots}$$

$$A_n = \frac{A_{n-1}}{(m+n)^2} = \frac{A_{n-2}}{(m+n)^2(m+n-1)^2} = \dots \frac{A_0}{[(m+n)(m+n-1)\cdots(m+1)]^2}$$

$$\therefore \quad y_1 = x^m A_0 \sum \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \dots \right] = x^m \bar{y}_1$$

$$\frac{x^3}{(m+1)^2(m+2)^2(m+3)^2}$$

for $m = 0 \qquad y_1 = x^m A_0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = x^m \bar{y}_1$

$$y_2 = \frac{\partial y_1}{\partial m} = x^m \ln x \, \bar{y}_1 + x^m A_0 \sum \left[\frac{-2x}{(m+1)^3} - \left\{\frac{2}{(m+1)^3(m+2)^2} + \frac{2}{(m+1)^2(m+2)^3}\right\}x^2 \right.$$

@ $m = 0 \qquad y_2 = \bar{y}_1 \ln x + A_0 \sum \left[\frac{-2x}{1} - \left[\frac{2}{4} + \frac{2}{8}\right]x^2 - \left[\frac{2}{6} + \dots + \dots\right]x^3 + \dots\right.$

$$= \bar{y}_1 \ln x - 2A_0 \left[x + \frac{1}{2!}\left(1 + \frac{1}{2}\right)x^2 + \frac{1}{3!}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^3 + \dots\right]$$

$$y = C_1 y_1(x) + C_2 y_2(x)$$

- Find a soln to $y'' = xy$ about $x = 1$ — ordinary

- if x & x^2 are solns to a differential eqn ~~what~~ how can we define the d.e. about $x=0$ what about $x = \infty$

  what is the radius of convergence

- How can you classify $2(x-2)^2 x y'' + 3xy' + (x-2)y = 0$