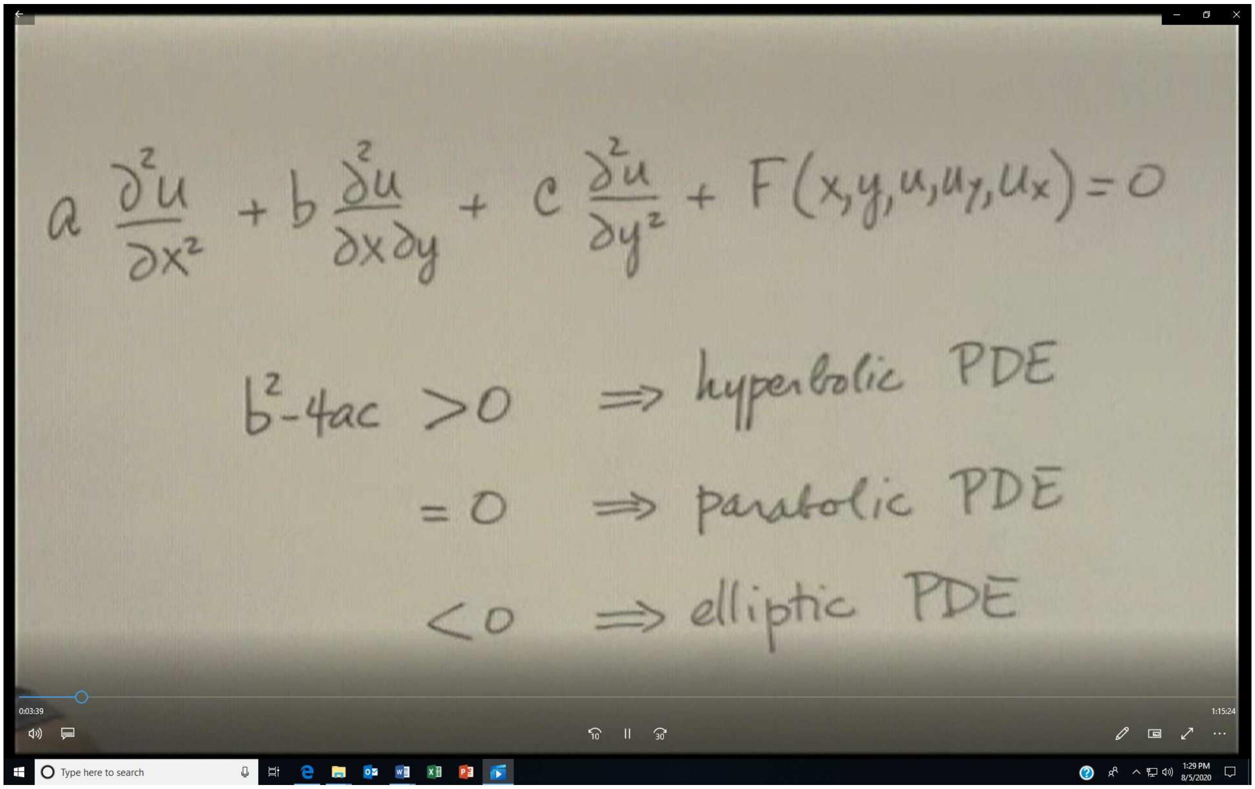
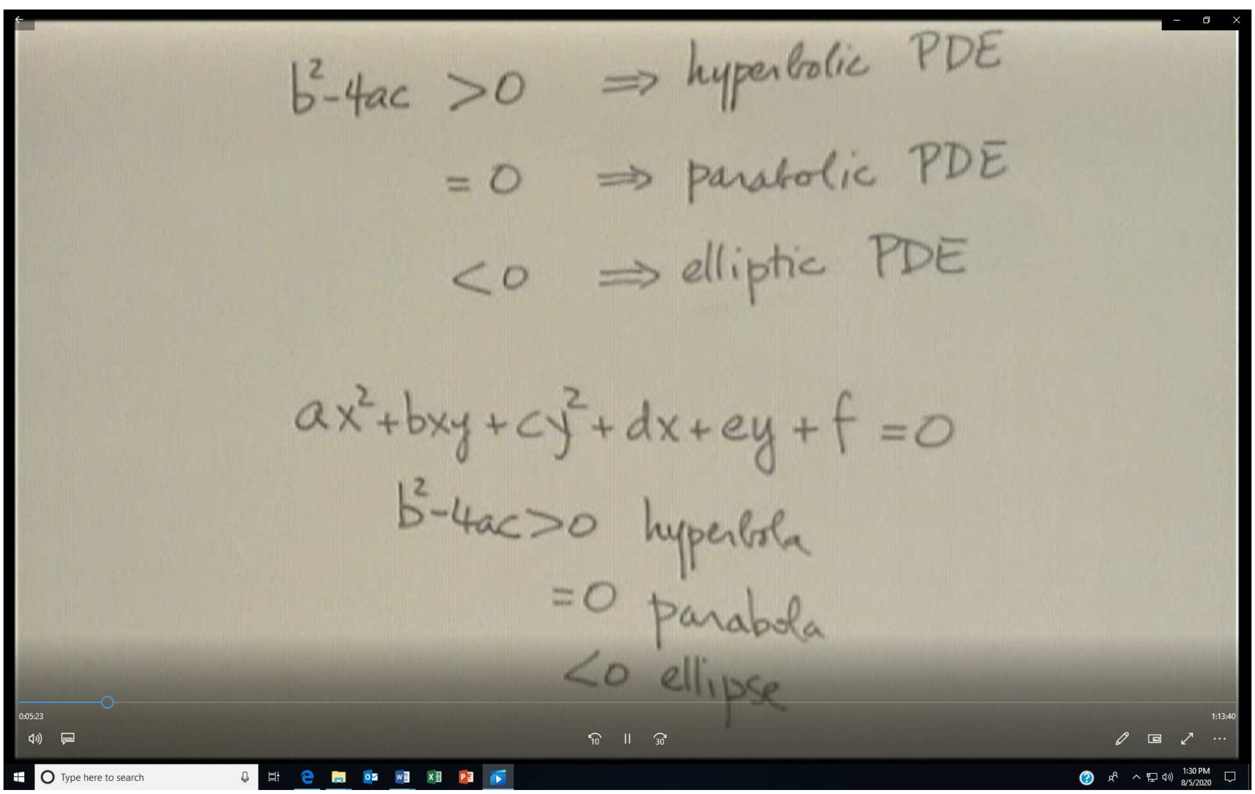
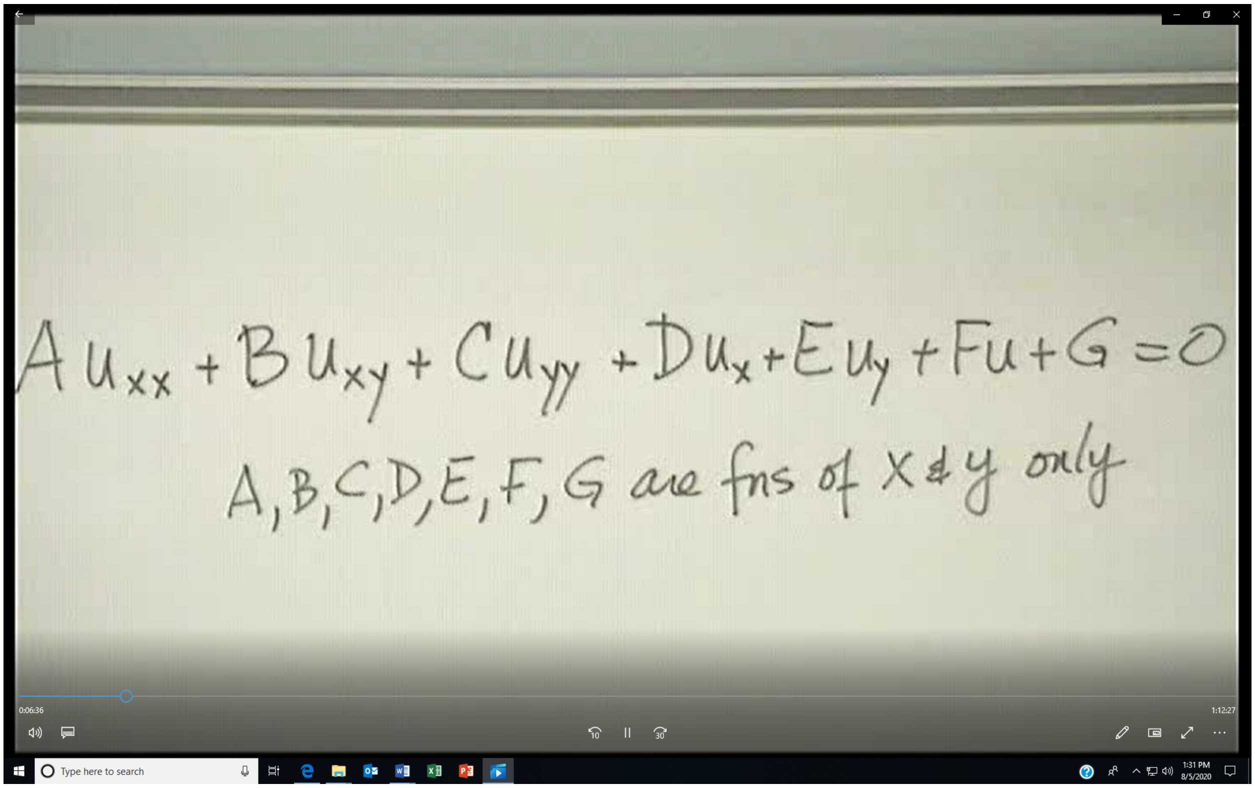
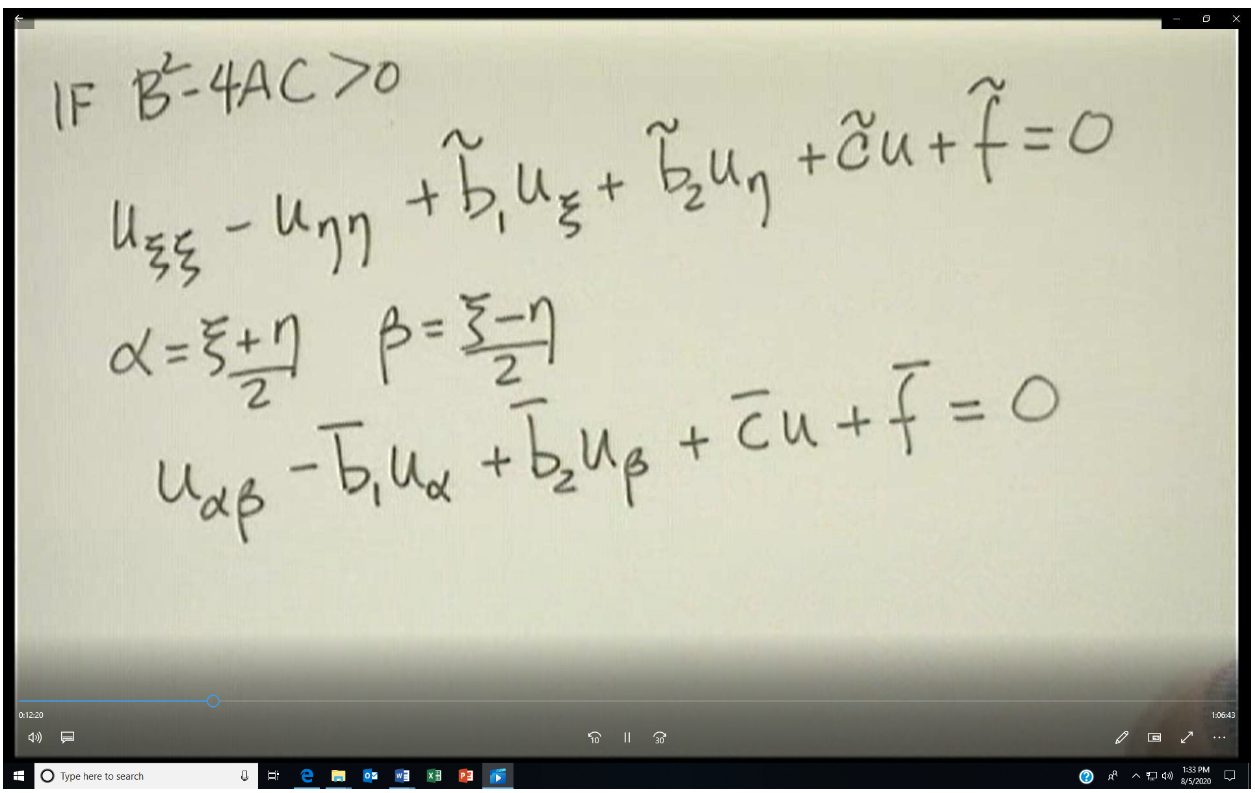


Reminiscent of conic sections



Linear PDE since all are functions of x and y only, then





All tilde variables and all barred variables are functions of ξ and η only for all the cases we discuss

- $u_{\alpha\beta} - \bar{b}_1 u_\alpha + \bar{b}_2 u_\beta + \bar{c}u + \bar{f} = 0$

IF $B^2-4AC=0$ PARABOLIC PDE

$$u_{\xi\xi} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \hat{c}u + \tilde{f} = 0$$

IF $B^2-4AC<0$ ELLIPTIC TYPE

$$u_{\xi\xi} + u_{\eta\eta} + \tilde{b}_1 u_\xi + \hat{b}_2 u_\eta + \tilde{c}u + \bar{f} = 0$$

Example problem

$$\underset{\overset{\downarrow}{A}}{3u_{xx}} + \underset{\overset{\downarrow}{B}}{10u_{xy}} + \underset{\overset{\downarrow}{C}}{3u_{yy}} = 0$$

$$B^2-4AC = 10^2-4(3)(3) = 64 > 0$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2-4AC}}{2A}$$

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{10 \pm 8}{6} = 3, \frac{1}{3}$$

$$\frac{dy}{dx} = 3 \implies y = 3x + c$$

$$y - 3x = \xi$$

$$\frac{dy}{dx} = \frac{1}{3} \implies y = \frac{1}{3}x + c$$

$$y - \frac{1}{3}x = \eta$$

$$u_x = u_\xi \cdot \xi_x + u_\eta \cdot \eta_x$$

$$u_x = u_\xi \cdot (-3) + u_\eta (-1/3)$$

$$u_y = u_\xi \cdot \xi_y + u_\eta \cdot \eta_y$$

$$= u_\xi \cdot 1 + u_\eta \cdot 1$$

$$u_y = u_\xi \cdot \xi_y + u_\eta \, \eta_y$$

$$= u_\xi \cdot 1 + u_\eta \cdot 1$$

$$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial \xi}(\quad)\cdot \xi_x + \frac{\partial}{\partial \eta}(\quad)\cdot \eta_x$$

$$u_{xx} = \frac{\partial}{\partial \xi}\left[-3u_\xi - \tfrac{1}{3}u_\eta\right](-3) + \frac{\partial}{\partial \eta}\left[-3u_\xi - \tfrac{1}{3}u_\eta\right](-\tfrac{1}{3})$$

$$u_{xx} = 9u_{\xi\xi} + 1\,u_{\xi\eta} + u_{\xi\eta} + \tfrac{1}{9}u_{\eta\eta} = 9u_{\xi\xi} + 2u_{\xi\eta} + \tfrac{1}{9}$$

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$$u_{xx} = \frac{\partial}{\partial \xi}\left[-3u_\xi - \tfrac{1}{3}u_\eta\right](-3) + \frac{\partial}{\partial \eta}\left[-3u_\xi - \tfrac{1}{3}u_\eta\right](-\tfrac{1}{3})$$

$$\boxed{u_{xx} = 9u_{\xi\xi} + 1\,u_{\xi\eta} + u_{\xi\eta} + \tfrac{1}{9}u_{\eta\eta} = 9u_{\xi\xi} + 2u_{\xi\eta} + \tfrac{1}{9}u_{\eta\eta}}$$

$$u_{yy} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial \xi}(\quad)\cdot \xi_y + \frac{\partial}{\partial \eta}(\quad)\cdot \eta_y$$

$$\boxed{u_{yy} = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}}$$

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$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial x}(u_y)$$

$$= \frac{\partial}{\partial \xi}(\quad)\cdot \xi_y + \frac{\partial}{\partial \eta}(\quad)\cdot \eta_y$$

$$u_{xy} = \frac{\partial}{\partial y}(u_x)$$

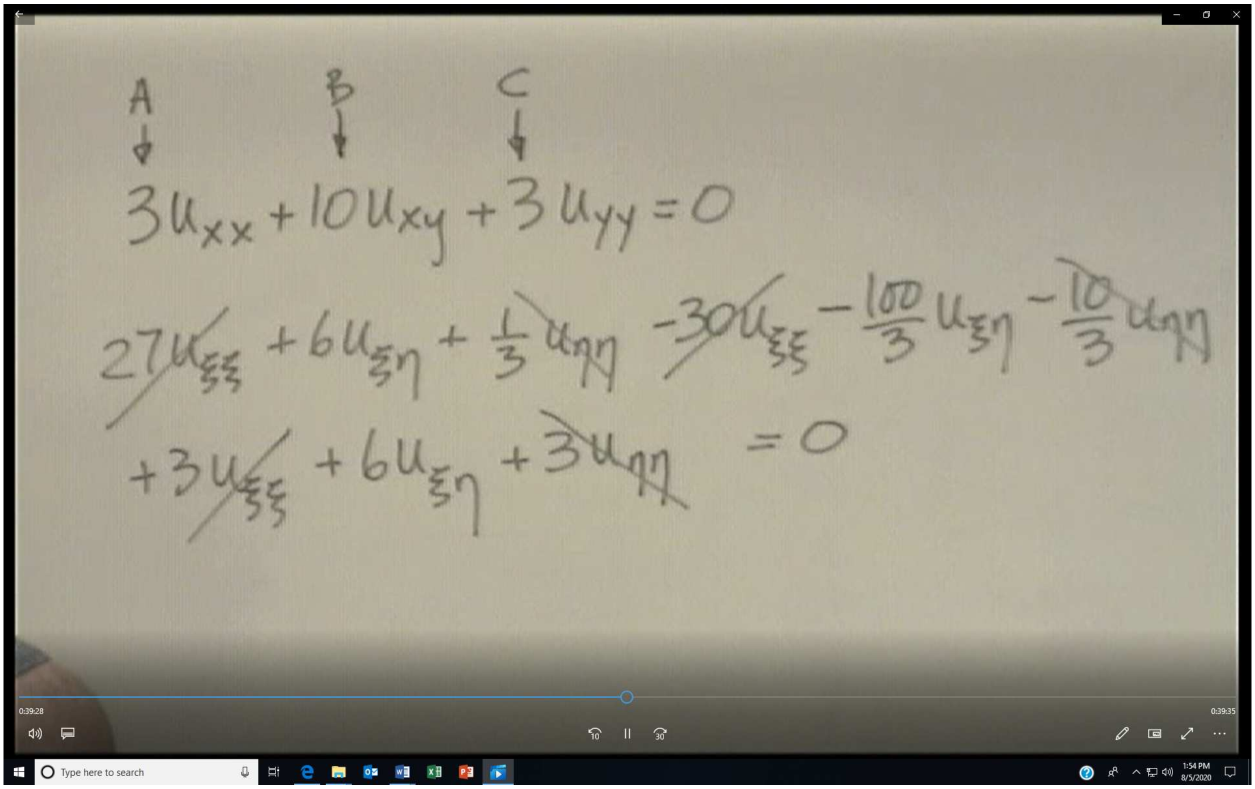
$$= \frac{\partial}{\partial \xi}(\quad)\cdot \xi_y + \frac{\partial}{\partial \eta}(\quad)\cdot \eta_y$$

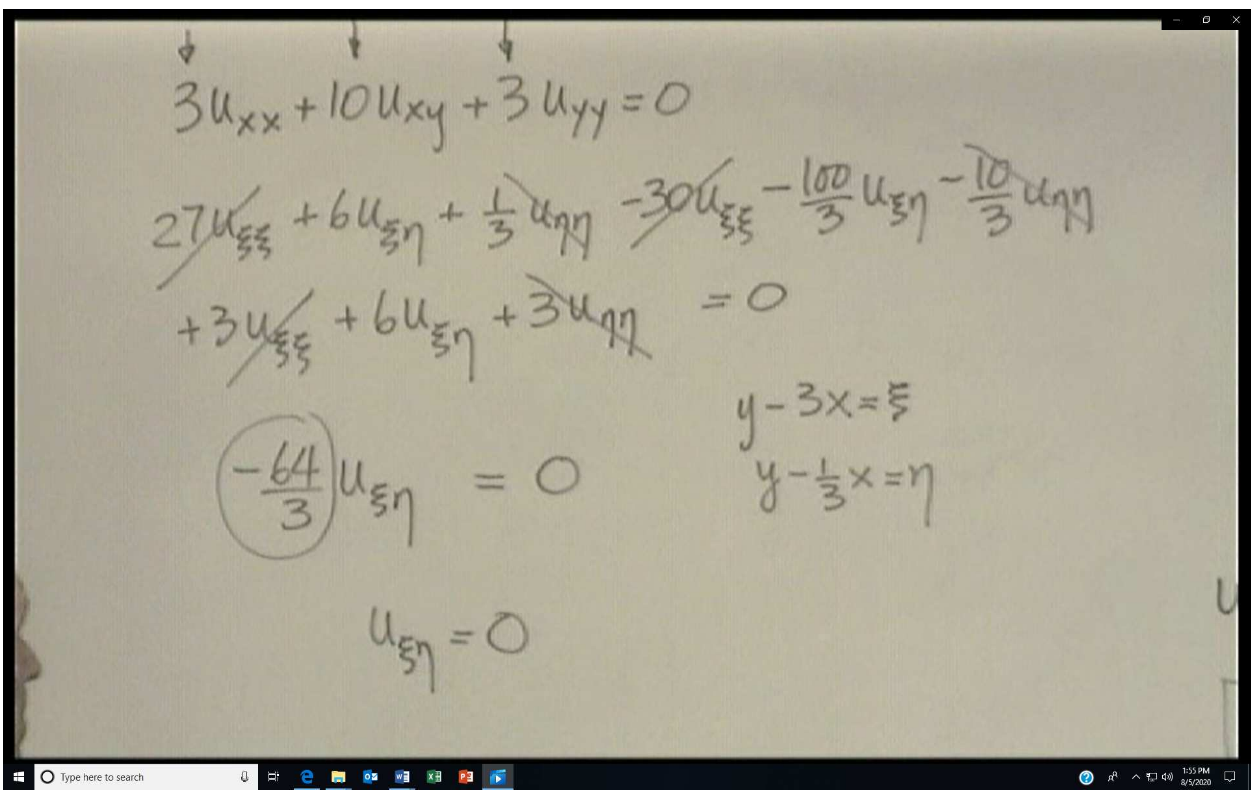
$$= \frac{\partial}{\partial \xi}\left[-3u_\xi - \frac{1}{3}u_\eta\right]\cdot 1 + \frac{\partial}{\partial \eta}\left[-3u_\xi - \frac{1}{3}u_\eta\right]\cdot 1$$

$$= -3u_{\xi\xi} - \frac{1}{3}u_{\eta\xi} - 3u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}$$

$$= -3u_{\xi\xi} - \frac{10}{3}u_{\xi\eta} - \frac{1}{3}u_{\eta\eta}$$

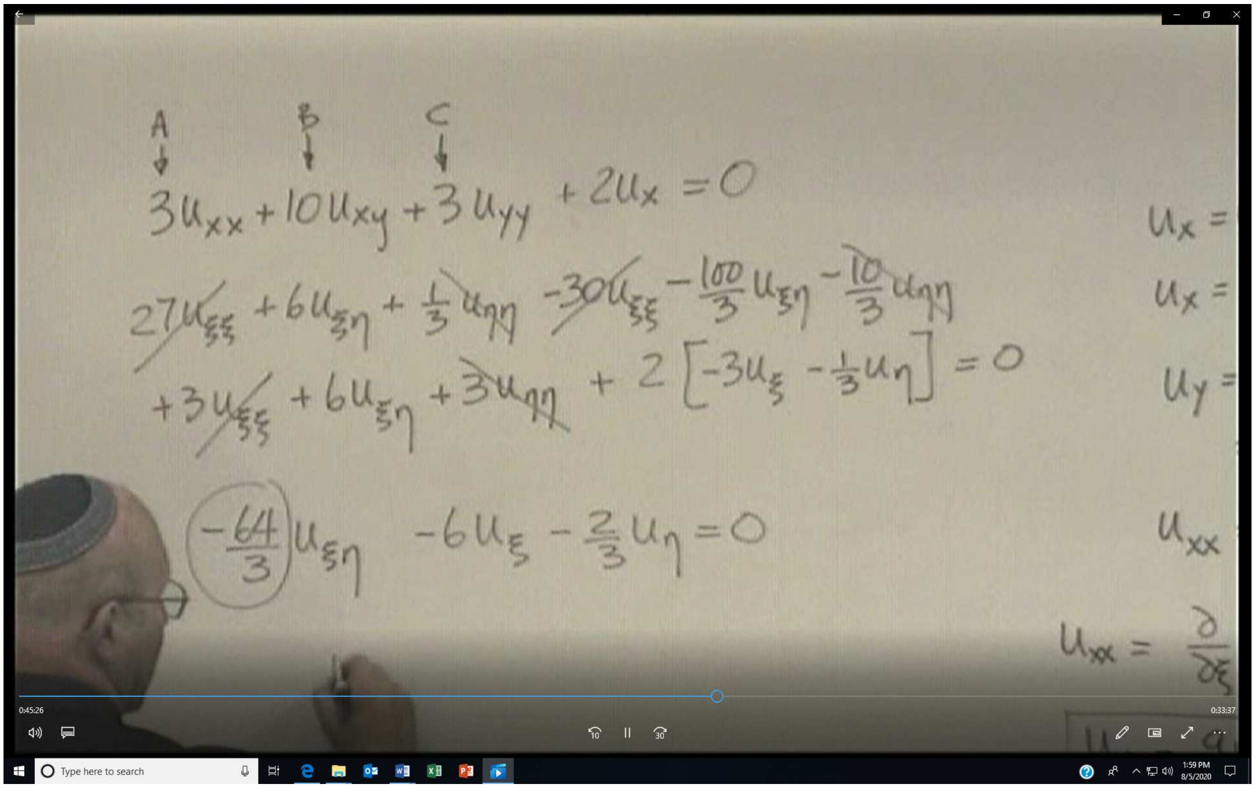
Take the expressions and put them in the original equation to get



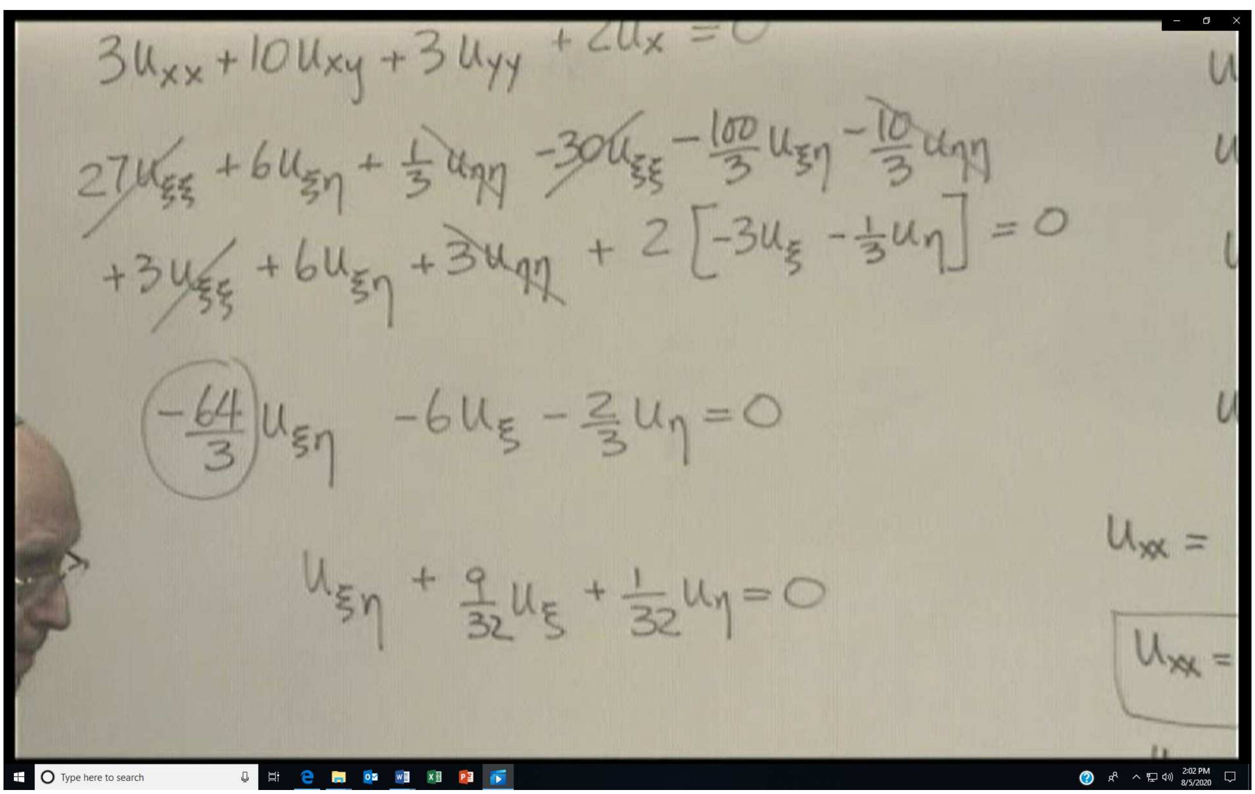


This is the canonical form for the example problem.

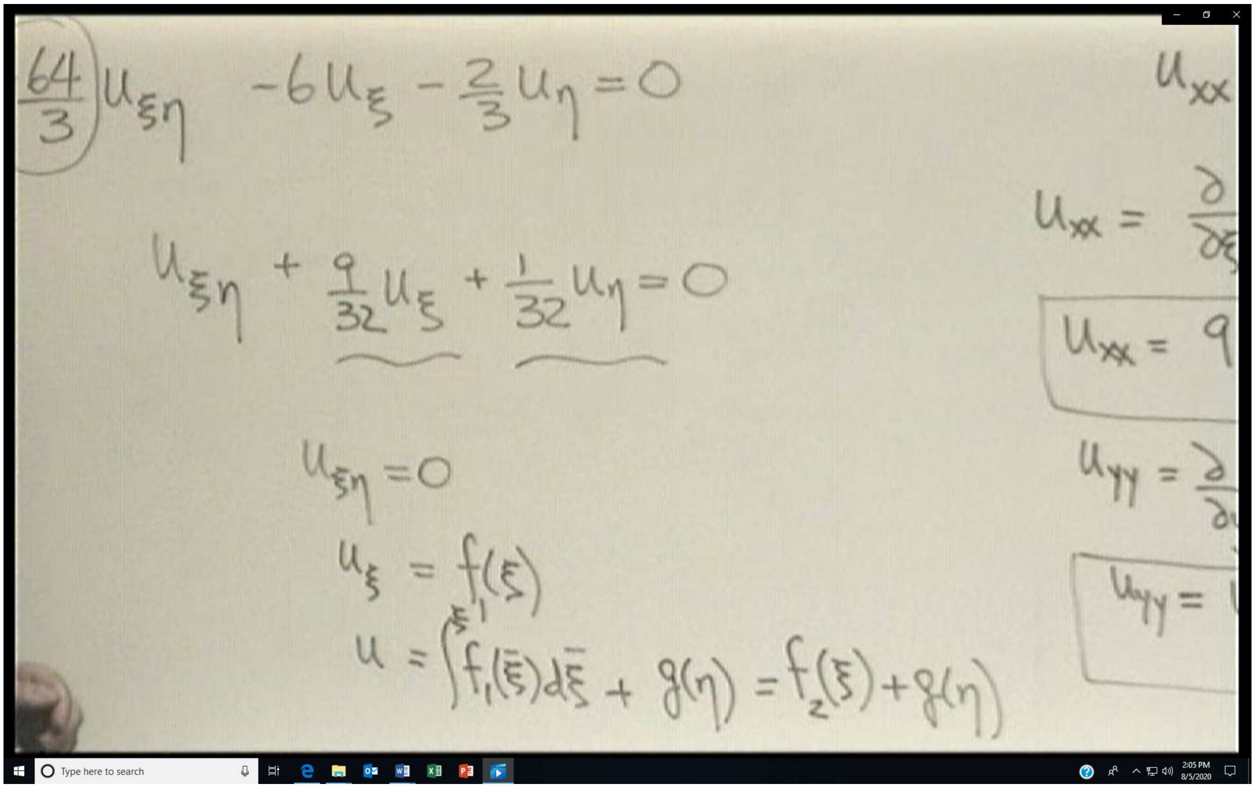
Now suppose we add a $2u_x$ to the original equation



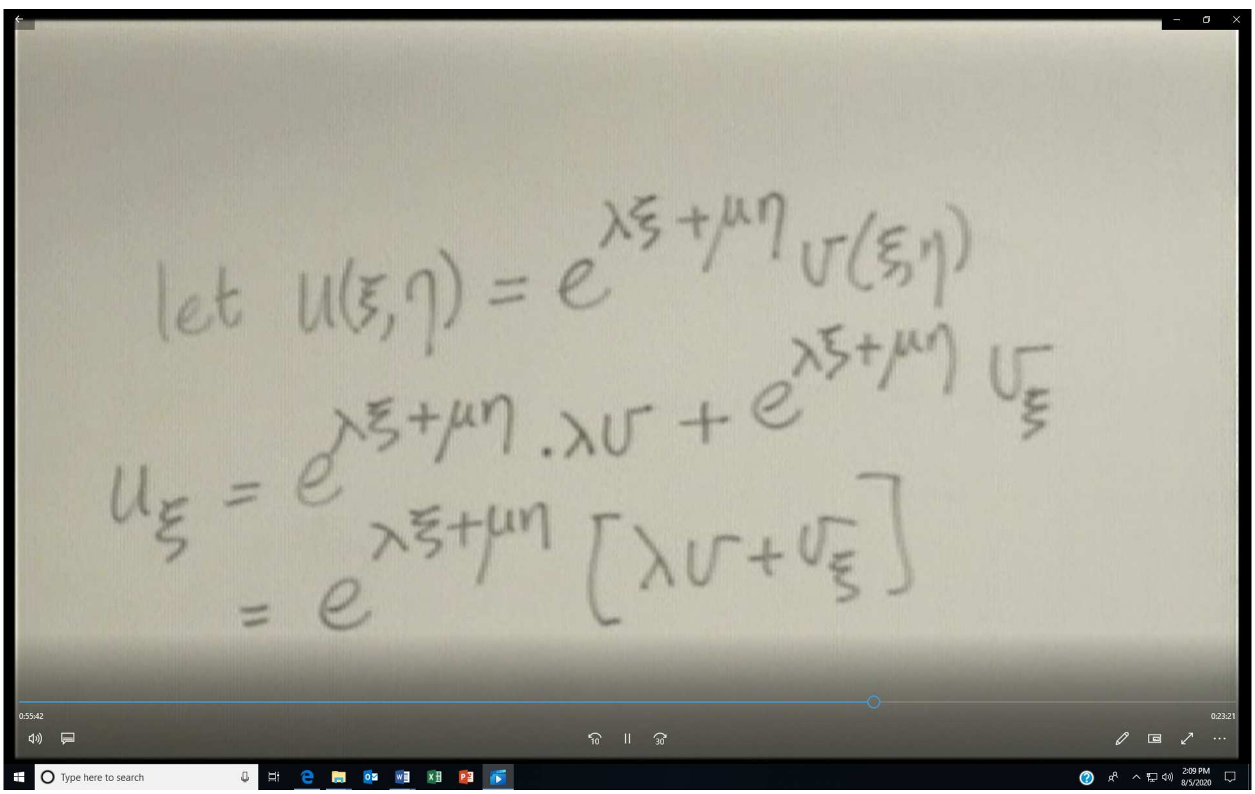
By making the first coefficient equal to one we get



If we had the first problem, integrating along the characteristics given



Creating the reduced canonical forms (if coeffs of first derivatives are constant)



$$= e^{\lambda\xi+\mu\eta}\left[\lambda v + v_\xi\right]$$

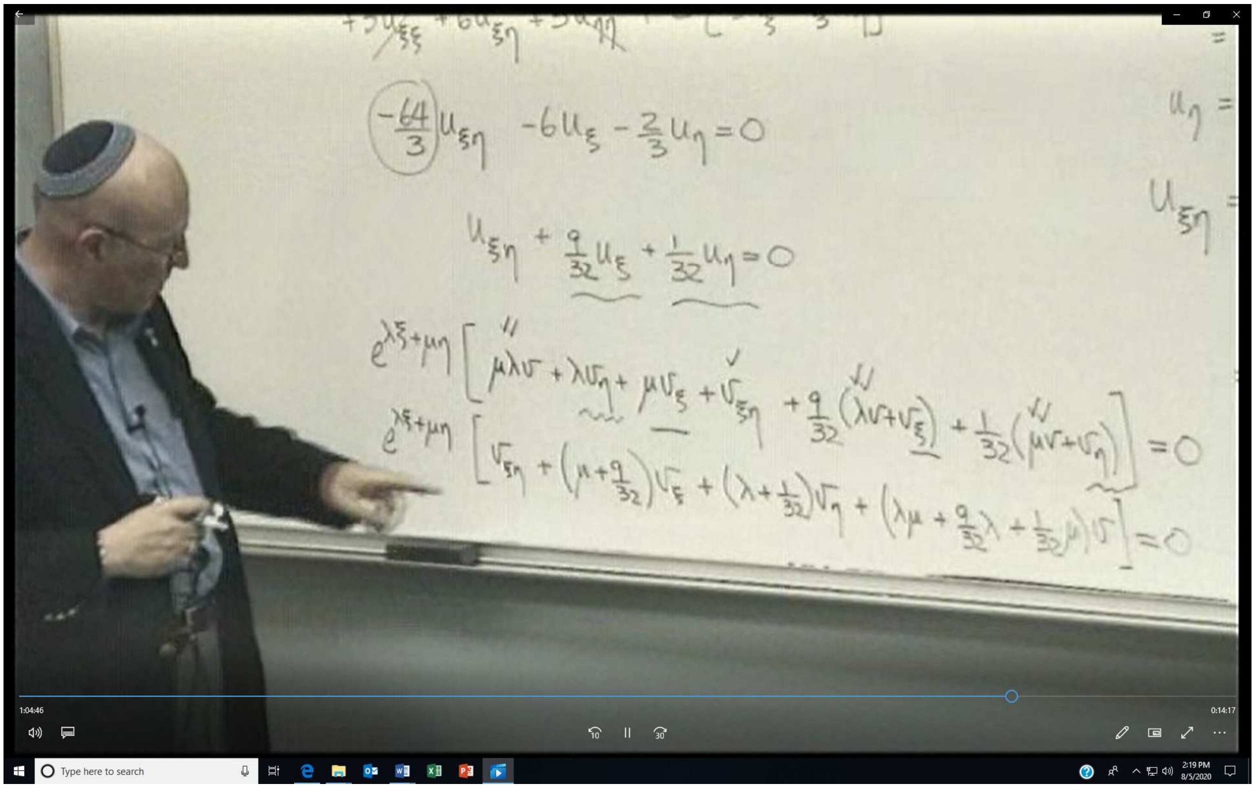
$$u_\eta = e^{\lambda\xi+\mu\eta}\left[\mu v + v_\eta\right]$$

$$u_{\xi\eta} = \frac{\partial}{\partial\xi}\left[u_\eta\right] = \lambda e^{\lambda\xi+\mu\eta}\left[\mu v + v_\eta\right] + \left[\mu v_\xi + v_{\eta\xi}\right]e^{\lambda\xi+\mu\eta}$$

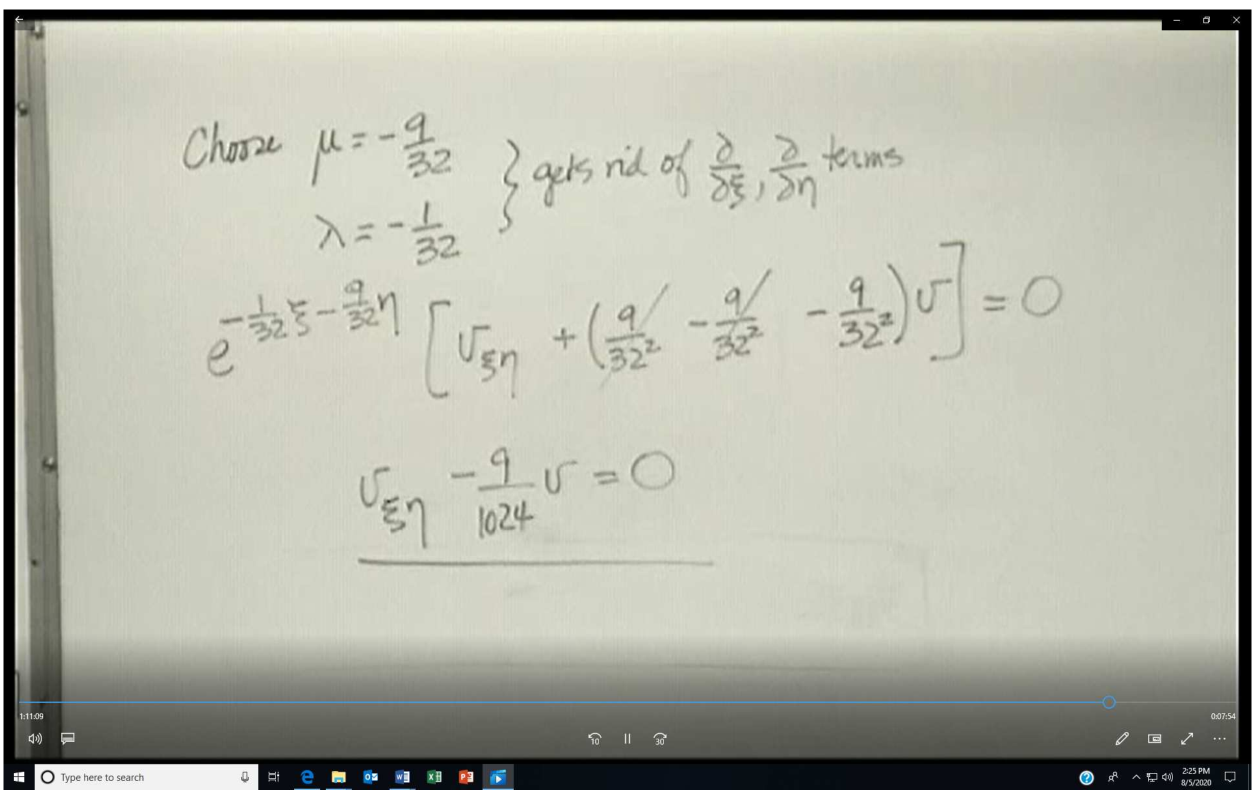
$$\xi\eta = \frac{\partial}{\partial\xi}\left[u_\eta\right] = \lambda e \left[\mu v + v_\eta\right] + \left[\mu v_\xi + v_{\eta\xi}\right]e^{\lambda\xi+\mu\eta}$$

$$= e^{\lambda\xi+\mu\eta}\left[\mu\lambda v + \lambda v_\eta + \mu v_\xi + v_{\eta\xi}\right]$$

When we apply to the second problem



By choosing the coefficients of the first derivative terms to be equal to zero, we solve for λ and μ



We can divide by the exponential since it is not equal to zero. What is left is called the reduced canonical form.

If this problem had boundary and initial conditions on u, if you plan to use the reduced canonical form, then you much convert the boundary and initial conditions to conditions on v. This means that the BCs and the ICs may become very complicated.