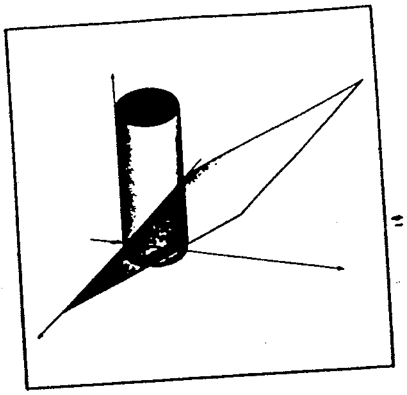


CHAPTER

S I X

Finite Fourier Transforms and Nonhomogeneous Problems

6.1 Finite Fourier Transforms

In Section 3.3 we used transformations and eigenfunction expansions to solve nonhomogeneous (initial) boundary value problems. These problems were relatively straightforward, principally because they contained only one spatial variable. When nonhomogeneities were time independent, the solution was represented as the sum of steady-state and transient parts. The steady-state portion was determined by an ODE, and the transient portion satisfied a homogeneous problem. (In problems with two or three spatial variables, the steady-state part will satisfy two- or three-dimensional boundary value problems.) When nonhomogeneities were time dependent, the method of eigenfunction expansions had to be used. The corresponding homogeneous problem was solved, and arbitrary constants were then replaced by functions of time.

In this chapter we present an alternative technique for solving nonhomogeneous (initial) boundary value problems, namely, finite Fourier transforms. They handle time-dependent and time-independent nonhomogeneities in exactly the same way and adapt to problems in higher dimensions very easily.

This function is identical to $g(x)$ except at $x = L/2$, where its value is $1/2$. In addition, because the series converges to zero at $x = 0$ and $x = L$, we may write

$$h(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(n\pi/2)}{n}\sin\frac{n\pi x}{L}, \qquad 0 \le x \le L,$$

where $h(x)$ is the function in Figure 6.2(b).

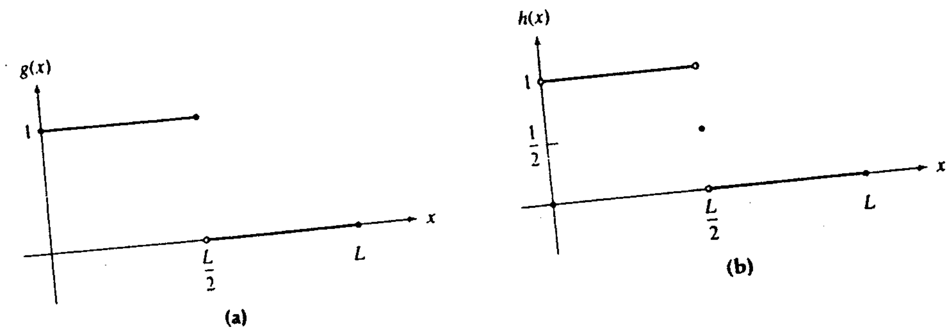


Figure 6.2

When solving (initial) boundary value problems by finite Fourier transforms, it is often necessary to answer questions like that posed in the following example.

Example 1: Given that the finite Fourier transform of a function $f(x)$ with respect to the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L,$$
$$X(0) = 0 = X(L)$$

is

$$\tilde{f}(\lambda_n) = \frac{\sqrt{2L}\,[1 + (-1)^{n+1}] + (2L)^{3/2}(-1)^{n+1}}{n\pi},$$

find $f(x)$.

Solution: Eigenvalues of the Sturm-Liouville system are $\lambda_n^2 = n^2\pi^2/L^2$, with corresponding normalized eigenfunctions $X_n(x) = \sqrt{2/L}\,\sin(n\pi x/L)$. When $g(x) = x$,

$$\tilde{g}(\lambda_n) = \int_0^L x\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx = \sqrt{\frac{2}{L}}\left\{\frac{-Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_0^L$$
$$= \frac{\sqrt{2L^3}}{n\pi}(-1)^{n+1}.$$

In addition, if $h(x) = 1$,

$$\tilde{h}(\lambda_n) = \int_0^L \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx = \sqrt{\frac{2}{L}}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_0^L = \frac{\sqrt{2L}}{n\pi}[1 + (-1)^{n+1}].$$

Since $\tilde{f}(\lambda_n) = 2\tilde{g}(\lambda_n) + \tilde{h}(\lambda_n)$, it follows that $f(x) = 2g(x) + h(x) = 2x + 1$. ■

Exercises 6.1

In Exercises 1–10, find the finite Fourier transform of the function $f(x)$, defined on the interval $0 \le x \le L$, with respect to the given Sturm-Liouville system.

1. $f(x) = x^2 - 2x;\quad X'' + \lambda^2 X = 0,\quad X(0) = X'(L) = 0$
2. $f(x) = 5;\quad X'' + \lambda^2 X = 0,\quad X(0) = X(L) = 0$
3. $f(x) = 5;\quad X'' + \lambda^2 X = 0,\quad X'(0) = X'(L) = 0$
4. $f(x) = x;\quad X'' + \lambda^2 X = 0,\quad X(0) = 0,\quad l_2 X'(L) + h_2 X(L) = 0$
5. $f(x) = L - x;\quad X'' + \lambda^2 X = 0,\quad X'(0) = 0,\quad l_2 X'(L) + h_2 X(L) = 0$
6. $f(x) = \sin x;\quad X'' + \lambda^2 X = 0,\quad X'(0) = X(L) = 0$
7. $f(x) = e^x;\quad X'' + \lambda^2 X = 0,\quad X'(0) = X'(L) = 0$
8. $f(x) = \begin{cases} x^2 & 0 \le x \le L/2 \\ 0 & L/2 < x \le L \end{cases};\quad X'' + \lambda^2 X = 0,\quad X(0) = X'(L) = 0$
9. $f(x) = \sin(\pi x/L)\cos(\pi x/L);\quad X'' + \lambda^2 X = 0,\quad X(0) = X(L) = 0$
10. $f(x) = 1;\quad X'' + 2X' + \lambda^2 X = 0,\quad X'(0) = X'(L) = 0$

In Exercises 11–14, find, in closed form, the inverse finite Fourier transform for $\tilde{f}(\lambda_n)$ with respect to the given Sturm-Liouville system.

11. $\tilde{f}(\lambda_n) = (-1)^{n+1}(2L)^{3/2}/(n\pi);\quad X'' + \lambda^2 X = 0,\quad X(0) = X(L) = 0$
12. $\tilde{f}(\lambda_n) = \dfrac{3\sqrt{2}L^{5/2}(-1)^n}{n\pi} + \dfrac{6\sqrt{2}L^{5/2}[1 + (-1)^{n+1}]}{n^3\pi^3};\quad X'' + \lambda^2 X = 0,\quad X(0) = X(L) = 0$
13. $\tilde{f}(\lambda_n) = \begin{cases} 2\sqrt{2L} & n = 0 \\ 0 & n > 0 \end{cases};\quad X'' + \lambda^2 X = 0,\quad X'(0) = X'(L) = 0$
14. $\tilde{f}(\lambda_n) = \dfrac{(2L-1)\sqrt{2/L}(-1)^{n+1}}{\lambda_n} - \dfrac{2\sqrt{2/L}}{\lambda_n^2};\quad X'' + \lambda^2 X = 0,\quad X'(0) = X(L) = 0$

6.2 Nonhomogeneous Problems in Two Variables

We now show how finite Fourier transforms can be used to solve (initial) boundary value problems. Every initial boundary value problem that we have solved by separation of variables can also be solved using transforms. There is little advantage, however, in using transforms for homogeneous problems; their power is realized when the PDE and/or the boundary conditions are nonhomogeneous. Nonetheless, we choose to introduce the method with problem (8) of Section 3.2, a problem with homogeneous PDE and homogeneous boundary conditions. We do this because the application of finite Fourier transforms to initial boundary value problems always follows the same pattern whether the problem is homogeneous or nonhomogeneous. As a result, we can clearly illustrate the technique in a homogeneous problem without added complications due to nonhomogeneities.

The separation method on

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{4a}$$

$$y(0, t) = 0, \qquad t > 0, \tag{4b}$$

$$y(L,t) = 0, \qquad t > 0, \tag{4c}$$
$$y(x,0) = f(x), \qquad 0 < x < L, \tag{4d}$$
$$y_t(x,0) = 0, \qquad 0 < x < L \tag{4e}$$

determines separated functions $y(x,t) = X(x)T(t)$, which satisfy (4a, b, c, e). The result is a Sturm-Liouville system in $X(x)$ and an ordinary differential equation in $T(t)$:

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L, \tag{5a}$$
$$X(0) = 0, \tag{5b}$$
$$X(L) = 0; \tag{5c}$$

$$T'' + c^2\lambda^2 T = 0, \qquad t > 0, \tag{6a}$$
$$T'(0) = 0. \tag{6b}$$

From these, separated functions take the form $C\sqrt{2/L}\sin(n\pi x/L)\cos(n\pi ct/L)$ for arbitrary C. The solution of problem (4) is obtained by superposing these functions,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}, \tag{7a}$$

and imposing (4d) to give

$$c_n = \int_0^L f(x)\sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}\,dx. \tag{7b}$$

To solve this problem by finite Fourier transforms, we note that the transform associated with Sturm-Liouville system (5) is

$$\tilde{f}(\lambda_n) = \int_0^L f(x)X_n(x)\,dx, \tag{8}$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenvalues and orthonormal eigenfunctions. If we apply this transform to both sides of PDE (4a),

$$\int_0^L \frac{\partial^2 y}{\partial t^2} X_n(x)\,dx = c^2 \int_0^L \frac{\partial^2 y}{\partial x^2} X_n(x)\,dx. \tag{9}$$

We interchange orders of integration with respect to x and differentiation with respect to t on the left side of this equation. Integration by parts on the right, together with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\frac{\partial^2}{\partial t^2}\int_0^L yX_n\,dx = c^2\left\{\frac{\partial y}{\partial x}X_n\right\}_0^L - c^2\int_0^L \frac{\partial y}{\partial x}X_n'\,dx = -c^2\int_0^L \frac{\partial y}{\partial x}X_n'\,dx. \tag{10}$$

The integral on the left of this equation is the definition of $\tilde{y}(\lambda_n, t)$, the finite Fourier transform of $y(x,t)$. Integration by parts once again on the right yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\tilde{y}(\lambda_n,t) &= -c^2\{yX_n'\}_0^L + c^2\int_0^L yX_n''\,dx \\ &= -c^2 y(L,t)X_n'(L) + c^2 y(0,t)X_n'(0) + c^2\int_0^L yX_n''\,dx.\end{aligned} \tag{11}$$

Now, boundary conditions (4b, c) imply that the first two terms on the right vanish. Further, equation (5a) may be used to replace X_n'' with $-\lambda_n^2 X_n$, with the result

$$\frac{\partial^2 \tilde{y}(\lambda_n, t)}{\partial t^2} = c^2 \int_0^L y(-\lambda_n^2 X_n)\,dx = -c^2\lambda_n^2 \int_0^L yX_n\,dx = -c^2\lambda_n^2\tilde{y}(\lambda_n, t). \qquad (12)$$

Because $\tilde{y}(\lambda_n, t)$ is a function of only one variable, t, and a parameter, λ_n, the partial derivative may be replaced by an ordinary derivative,

$$\frac{d^2\tilde{y}}{dt^2} = -c^2\lambda_n^2\tilde{y}. \qquad (13a)$$

This is an ordinary differential equation for $\tilde{y}(\lambda_n, t)$. When we take finite Fourier transforms of initial conditions (4d, e), we obtain initial conditions for ODE (13a):

$$\tilde{y}(\lambda_n, 0) = \tilde{f}(\lambda_n), \qquad (13b)$$

$$\frac{d\tilde{y}(\lambda_n, 0)}{dt} = 0. \qquad (13c)$$

What the finite Fourier transform has done, therefore, is replace initial boundary value problem (4) for $y(x, t)$ with initial value problem (13) for $\tilde{y}(\lambda_n, t)$; a PDE has been reduced to an ODE. In actual fact, (13) is an infinite system of ODEs ($n = 1, 2, \ldots$), but because all differential equations have exactly the same form, solving one solves them all.

The general solution of (13a) is

$$\tilde{y}(\lambda_n, t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t, \qquad (14)$$

where A_n and B_n are constants. Initial conditions (13b, c) require these constants to satisfy

$$A_n = \tilde{f}(\lambda_n), \qquad 0 = c\lambda_n B_n, \qquad (15)$$

and therefore

$$\tilde{y}(\lambda_n, t) = \tilde{f}(\lambda_n)\cos c\lambda_n t. \qquad (16)$$

The inverse transform defines the solution of problem (4) as

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} \tilde{y}(\lambda_n, t) X_n(x) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n)\cos c\lambda_n t X_n(x) \\ &= \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \tilde{f}(\lambda_n) \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}, \end{aligned} \qquad (17)$$

a solution identical to that obtained by separation of variables.

Briefly, the transform technique applied to the PDE replaces the PDE in $y(x, t)$ with an ordinary differential equation in its transform $\tilde{y}(\lambda_n, t)$. Once the differential equation for $\tilde{y}(\lambda_n, t)$ is solved, the inverse transform yields $y(x, t)$. A number of aspects of the method deserve special mention:

(1) Not just any finite Fourier transform will yield a solution to this initial boundary value problem. It must be the transform associated with Sturm-Liouville

system (5); that is, it must be the transform associated with the Sturm-Liouville system $X(x)$ that would result if separation of variables were applied to the problem. In nonhomogeneous problems, we use the transform associated with the Sturm-Liouville system that would result were separation used on the corresponding homogeneous problem. Apparently, then, to use transforms effectively, we must be able to recognize quickly the Sturm-Liouville system that would result were we to use separation of variables.

(2) Boundary conditions on $y(x, t)$ are incorporated in the simplification leading to the ordinary differential equation in $\tilde{y}(\lambda_n, t)$.

(3) Initial conditions on $y(x, t)$ are converted by the transform into initial conditions on $\tilde{y}(\lambda_n, t)$.

(4) Finite Fourier transforms always give a solution in the form of an infinite series (the inverse transform). It may happen that part or all of the solution is the eigenfunction expansion of a simple function. In particular, when nonhomogeneities are time independent, part of the solution is always representable in closed form. However, considerable ingenuity may be required to discover this function. The next example illustrates this point.

It is probably fair to say that the transform technique applied to the above problem is more involved than the separation method. This is in agreement with our earlier statement that the transform method shows its true versatility in problems with nonhomogeneous PDE and/or boundary conditions. To illustrate this, consider problem (34) of Section 3.3, where gravity introduces a nonhomogeneity into the PDE:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g, \qquad 0 < x < L, \qquad t > 0, \qquad (g < 0), \tag{18a}$$

$$y(0, t) = 0, \qquad t > 0, \tag{18b}$$

$$y(L, t) = 0, \qquad t > 0, \tag{18c}$$

$$y(x, 0) = f(x), \qquad 0 < x < L, \tag{18d}$$

$$y_t(x, 0) = 0, \qquad 0 < x < L. \tag{18e}$$

In Section 3.3 we expressed the solution in the form $y(x, t) = z(x, t) + \psi(x)$, where $\psi(x) = [g/(2c^2)](Lx - x^2)$ is the solution of the corresponding static deflection problem. The function $z(x, t)$ must then satisfy the homogeneous problem

$$\frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{19a}$$

$$z(0, t) = 0, \qquad t > 0, \tag{19b}$$

$$z(L, t) = 0, \qquad t > 0, \tag{19c}$$

$$z(x, 0) = f(x) - \frac{g}{2c^2}(Lx - x^2), \qquad 0 < x < L, \tag{19d}$$

$$z_t(x, 0) = 0, \qquad 0 < x < L. \tag{19e}$$

Separation of variables on (19) gives

$$z(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{20a}$$

where

$$c_n = \frac{2}{L}\int_0^L \left(f(x) - \frac{g}{2c^2}(Lx - x^2)\right)\sin\frac{n\pi x}{L}\,dx. \tag{20b}$$

The final solution is

$$y(x,t) = z(x,t) + \frac{g}{2c^2}(Lx - x^2). \tag{21}$$

Consider now the finite Fourier transform technique applied to this problem. The transform associated with this problem is again (9), where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of (5) (this being the Sturm-Liouville system that would result were separation of variables used on the corresponding homogeneous problem). If we apply the transform to PDE (18a),

$$\int_0^L \frac{\partial^2 y}{\partial t^2} X_n(x)\,dx = \int_0^L \left(c^2\frac{\partial^2 y}{\partial x^2} + g\right) X_n(x)\,dx. \tag{22}$$

Integration by parts on the right, along with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\int_0^L yX_n\,dx &= c^2\left\{\frac{\partial y}{\partial x}X_n\right\}_0^L - c^2\int_0^L \frac{\partial y}{\partial x}X_n'\,dx + g\tilde{1}\\ &= -c^2\int_0^L \frac{\partial y}{\partial x}X_n'\,dx + g\tilde{1},\end{aligned} \tag{23}$$

where $\tilde{1}$ is the transform of the function identically equal to unity,

$$\tilde{1} = \int_0^L X_n(x)\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx = \frac{\sqrt{2L}}{n\pi}[1 + (-1)^{n+1}]. \tag{24}$$

Integration by parts again and boundary conditions (18b, c) yield

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\tilde{y}(\lambda_n, t) &= -c^2\{yX_n'\}_0^L + c^2\int_0^L yX_n''\,dx + g\tilde{1}\\ &= c^2\int_0^L y(-\lambda_n^2 X_n)\,dx + g\tilde{1}\end{aligned}$$

or

$$\frac{d^2\tilde{y}}{dt^2} = -c^2\lambda_n^2\tilde{y} + g\tilde{1}. \tag{25a}$$

This is an ordinary differential equation for $\tilde{y}(\lambda_n, t)$. Transforms of initial conditions (18d, e) require $\tilde{y}(\lambda_n, t)$ to satisfy the initial conditions

$$\tilde{y}(\lambda_n, 0) = \tilde{f}(\lambda_n), \tag{25b}$$

$$\frac{d\tilde{y}(\lambda_n, 0)}{dt} = 0. \tag{25c}$$

The general solution of (25a) is

$$\tilde{y}(\lambda_n, t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t + \frac{g\tilde{1}}{c^2\lambda_n^2}, \tag{26}$$

where A_n and B_n are constants. Initial conditions (25b, c) require these constants to satisfy

$$\tilde{f}(\lambda_n) = A_n + \frac{g\tilde{1}}{c^2\lambda_n^2}, \qquad 0 = c\lambda_n B_n \tag{27}$$

and therefore

$$\tilde{y}(\lambda_n, t) = \left(\tilde{f}(\lambda_n) - \frac{g\tilde{1}}{c^2\lambda_n^2}\right)\cos c\lambda_n t + \frac{g\tilde{1}}{c^2\lambda_n^2}. \tag{28}$$

The inverse transform now defines the solution of (18) as

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} \tilde{y}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} X_n(x)\left(\left[\tilde{f}(\lambda_n) - \frac{g\tilde{1}}{c^2\lambda_n^2}\right]\cos c\lambda_n t + \frac{g\tilde{1}}{c^2\lambda_n^2}\right) \\ &= \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(\left[\tilde{f}(\lambda_n) - \frac{g\tilde{1}}{c^2\lambda_n^2}\right]\cos\frac{n\pi ct}{L} + \frac{g\tilde{1}}{c^2\lambda_n^2}\right). \end{aligned} \tag{29}$$

To show that this solution is identical to that obtained by separation, we calculate that for $\psi(x) = [g/(2c^2)](Lx - x^2)$,

$$\tilde{\psi}(\lambda_n) = \int_0^L \psi(x) X_n(x)\,dx = \frac{g}{c^2 n^3 \pi^3}\sqrt{2L^5}\,[1 + (-1)^{n+1}] = \frac{g\tilde{1}}{c^2\lambda_n^2}. \tag{30}$$

Consequently, the last term of the series in (29) can be expressed as

$$\sum_{n=1}^{\infty} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}\left(\frac{g\tilde{1}}{c^2\lambda_n^2}\right) = \sum_{n=1}^{\infty} \tilde{\psi}(\lambda_n) X_n(x) = \psi(x).$$

Solution (29) can therefore be written in the form

$$y(x, t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}(\tilde{f}(\lambda_n) - \tilde{\psi}(\lambda_n))\cos\frac{n\pi ct}{L} + \psi(x), \tag{31}$$

which is clearly identical to that obtained by separation.

The transform method applied to this problem with a nonhomogeneous PDE is essentially the same as when applied to the homogeneous problem (4). This is the advantage of the transform method: it does not require homogeneous PDEs or boundary conditions. To illustrate the method applied to nonhomogeneous boundary conditions, we consider Example 4 in Section 3.3:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{32a}$$

$$U(0, t) = U_0, \qquad t > 0, \tag{32b}$$

$$U(L,t) = U_L, \qquad t > 0, \tag{32c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{32d}$$

The finite Fourier transform for this problem is once again (9), where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of Sturm-Liouville system (5) [obtained by separation when (32b, c) are homogeneous]. If we apply this transform to (32a),

$$\int_0^L \frac{\partial U}{\partial t} X_n(x)\,dx = k\int_0^L \frac{\partial^2 U}{\partial x^2} X_n(x)\,dx. \tag{33}$$

Integration by parts on the right, together with the fact that $X_n(0) = X_n(L) = 0$, gives

$$\frac{\partial}{\partial t}\int_0^L U X_n\,dx = k\left\{\frac{\partial U}{\partial x} X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x} X_n'\,dx = -k\int_0^L \frac{\partial U}{\partial x} X_n'\,dx. \tag{34}$$

Another integration by parts yields

$$\frac{\partial}{\partial t}\tilde{U}(\lambda_n, t) = -k\{U X_n'\}_0^L + k\int_0^L U X_n''\,dx$$

$$= -k\big(U(L,t)X_n'(L) - U(0,t)X_n'(0)\big) + k\int_0^L U(-\lambda_n^2 X_n)\,dx,$$

in which we may use boundary conditions (32b, c):

$$\frac{d\tilde{U}}{dt} = -kU_L\sqrt{\frac{2}{L}}\lambda_n(-1)^n + kU_0\sqrt{\frac{2}{L}}\lambda_n - k\lambda_n^2\tilde{U}$$

$$= -k\lambda_n^2\tilde{U} + k\sqrt{\frac{2}{L}}\lambda_n[U_0 + U_L(-1)^{n+1}]. \tag{35a}$$

Accompanying this ODE in $\tilde{U}(\lambda_n, t)$ is the transform of initial condition (32d),

$$\tilde{U}(\lambda_n, 0) = \tilde{f}(\lambda_n). \tag{35b}$$

The general solution of (35a) is

$$\tilde{U}(\lambda_n, t) = A_n e^{-k\lambda_n^2 t} + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}], \tag{36}$$

where A_n is a constant. Initial condition (35b) requires that

$$\tilde{f}(\lambda_n) = A_n + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}], \tag{37}$$

and therefore

$$\tilde{U}(\lambda_n, t) = e^{-k\lambda_n^2 t}\left(\tilde{f}(\lambda_n) - \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]\right)$$

$$+ \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]. \tag{38}$$

Inverse transform (3b) defines the solution of problem (32) as

$$U(x,t) = \sum_{n=1}^{\infty} \tilde{U}(\lambda_n, t) X_n(x)$$
$$= \sum_{n=1}^{\infty} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}\left(e^{-n^2\pi^2kt/L^2}\left[\tilde{f}(\lambda_n) - \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]\right]\right.$$
$$\left. + \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]\right). \tag{39}$$

To show that this solution is identical to that obtained by separation of variables in Example 4 of Section 3.3, we calculate that for $\psi(x) = U_0 + (U_L - U_0)x/L$,

$$\tilde{\psi}(\lambda_n) = \int_0^L \psi(x) X_n(x)\,dx = \lambda_n^{-1}\sqrt{\frac{2}{L}}[U_0 + U_L(-1)^{n+1}]. \tag{40}$$

Solution (39) can therefore be written in the form

$$U(x,t) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} e^{-n^2\pi^2kt/L^2}(\tilde{f}(\lambda_n) - \tilde{\psi}(\lambda_n)) \sin\frac{n\pi x}{L} + \psi(x), \tag{41}$$

identical to that obtained by separation of variables.

If we set $x = 0$ and $x = L$ in (39), we obtain $U(0,t) = U(L,t) = 0$, whereas $x = 0$ and $x = L$ in (41) give $U(0,t) = U_0$ and $U(L,t) = U_L$. In other words, the function in (39) does not satisfy boundary conditions (32b, c), but (41) does. This is because the series expansion of $\psi(x)$ in (39) is a Fourier sine series, and as such it converges to the odd extension of $\psi(x)$ to a function of period $2L$. At $x = 0$ and $x = L$, this extension (see Figure 6.3) is discontinuous, and the series therefore converges to the average value of the right and left limits, namely zero. For any other value of x between 0 and L, solutions (39) and (41) give identical results.

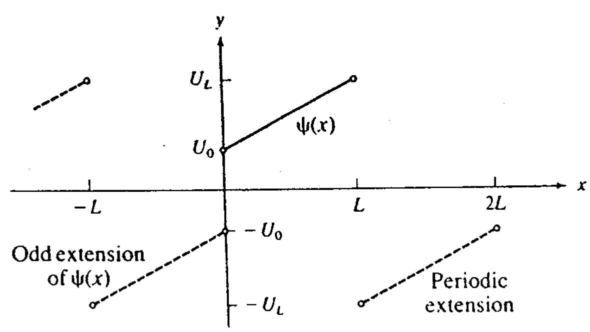


Figure 6.3

Parts of a finite Fourier transform solution that can be expressed in closed form should always be so represented. An additional reason for doing this is that the rate of convergence of the series is enhanced when the closed-form portion is extracted.

In the remainder of this section we consider two additional problems that have nonhomogeneities of a more general nature.

ample 2: Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{42a}$$

$$U(0,t) = f_1(t), \quad t > 0, \tag{42b}$$

$$U_x(L,t) = -\kappa^{-1}f_2(t), \qquad t > 0, \tag{42c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{42d}$$

Physically described is a rod of length L with insulated sides that at time $t = 0$ has temperature $f(x)$. For $t > 0$, the temperature of its left end is a prescribed $f_1(t)$, and heat is transferred across the right end at a rate $f_2(t)$. When $f_2(t)$ is positive, heat is being removed from the rod, and when $f_2(t)$ is negative, heat is being added.

Solution: Were separation of variables to be applied to the associated homogeneous problem [with $f_1(t) = f_2(t) = 0$], the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \qquad X(0) = 0, \qquad X'(L) = 0$$

would result. Eigenvalues are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$, with corresponding eigenfunctions $X_n(x) = \sqrt{2/L}\sin\lambda_n x$. If we apply the finite Fourier transform associated with this system to (42a),

$$\int_0^L \frac{\partial U}{\partial t} X_n\,dx = k\int_0^L \frac{\partial^2 U}{\partial x^2} X_n\,dx.$$

Integration by parts on the right integral gives

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^L UX_n\,dx &= k\left\{\frac{\partial U}{\partial x}X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x}X_n'\,dx \\ &= kU_x(L,t)X_n(L) - k\{UX_n'\}_0^L + k\int_0^L UX_n''\,dx \\ &= k\left(U_x(L,t)X_n(L) + U(0,t)X_n'(0) + \int_0^L -\lambda_n^2 X_n U\,dx\right).\end{aligned} \tag{43}$$

When we use (42c) in the first term and (42b) in the second, we may write

$$\frac{d\tilde{U}}{dt} = k[-\kappa^{-1}f_2(t)X_n(L) + f_1(t)X_n'(0) - \lambda_n^2\tilde{U}(\lambda_n,t)].$$

Thus, $\tilde{U}(\lambda_n,t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda_n^2\tilde{U} = A(\lambda_n,t), \tag{44a}$$

where

$$\begin{aligned}A(\lambda_n,t) &= k[-\kappa^{-1}f_2(t)X_n(L) + f_1(t)X_n'(0)] \\ &= k\sqrt{\frac{2}{L}}[(-1)^n\kappa^{-1}f_2(t) + \lambda_n f_1(t)]\end{aligned} \tag{44b}$$

subject to the transform of (42d),

$$\tilde{U}(\lambda_n,0) = \tilde{f}(\lambda_n). \tag{44c}$$

The general solution of (44a) is

$$\tilde{U}(\lambda_n, t) = e^{-k\lambda_n^2 t} \int A(\lambda_n, t) e^{k\lambda_n^2 t}\, dt,$$

but, in order to incorporate initial condition (44c), it is advantageous to express this solution as a definite integral:

$$\begin{aligned}\tilde{U}(\lambda_n, t) &= e^{-k\lambda_n^2 t}\left(\int_0^t A(\lambda_n, u) e^{k\lambda_n^2 u}\, du + C_n\right) \\ &= C_n e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)}\, du.\end{aligned}$$

Condition (44c) now requires that $\tilde{f}(\lambda_n) = C_n$, and therefore

$$\tilde{U}(\lambda_n, t) = \tilde{f}(\lambda_n) e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)}\, du. \tag{45}$$

The solution to problem (42) is defined by the inverse finite Fourier transform,

$$\begin{aligned}U(x, t) &= \sum_{n=1}^{\infty} \tilde{U}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} \left(\tilde{f}(\lambda_n) e^{-k\lambda_n^2 t} + \int_0^t A(\lambda_n, u) e^{k\lambda_n^2 (u-t)}\, du\right)\sqrt{\frac{2}{L}} \sin \lambda_n x.\end{aligned} \tag{46}$$

As a specific example, suppose the rod is initially at temperature zero $[f(x) \equiv 0]$, its right end is insulated $[f_2(t) \equiv 0]$, and its left end is held at constant temperature 100°C. According to (44b) and (45),

$$\tilde{U}(\lambda_n, t) = \int_0^t k\sqrt{\frac{2}{L}}\lambda_n (100) e^{k\lambda_n^2 (u-t)}\, du = \frac{100\sqrt{2/L}}{\lambda_n}(1 - e^{-k\lambda_n^2 t}),$$

and hence

$$U(x, t) = \sum_{n=1}^{\infty} \frac{100\sqrt{2/L}}{\lambda_n}(1 - e^{-k\lambda_n^2 t})\sqrt{\frac{2}{L}} \sin \frac{(2n-1)\pi x}{2L}.$$

The solution may be simplified by noting that when $h(x) \equiv 100$,

$$\tilde{h}(\lambda_n) = \int_0^L 100\sqrt{\frac{2}{L}} \sin \lambda_n x\, dx = \frac{100\sqrt{2/L}}{\lambda_n}.$$

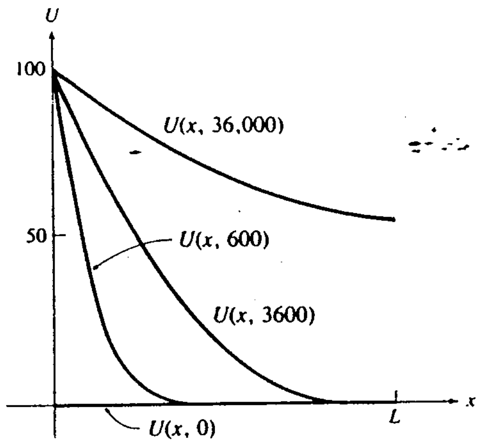
Thus,

$$h(x) = 100 = \sum_{n=1}^{\infty} \frac{100\sqrt{2/L}}{\lambda_n}\sqrt{\frac{2}{L}} \sin \lambda_n x,$$

and it follows that

$$U(x, t) = 100 - \frac{400}{\pi}\sum_{n=1}^{\infty} \frac{e^{-(2n-1)^2\pi^2 kt/(4L^2)}}{2n-1} \sin \frac{(2n-1)\pi x}{2L}.$$

This function is plotted for various values of t in Figure 6.4 (assuming a thermal diffusivity of $k = 12 \times 10^{-6}$ m²/s).



;ure 6.4

■

ample 3: A taut string has one end, at $x = 0$, fixed on the x-axis while the other end, at $x = L$, is forced to undergo periodic vertical motion described by $g(t) = A \sin \omega t$, $t \geq 0$ (A a constant). If the string is initially at rest on the x-axis, find its subsequent displacement.

Solution: The initial boundary value problem for displacements $y(x, t)$ of points on the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{47a}$$

$$y(0, t) = 0, \qquad t > 0, \tag{47b}$$

$$y(L, t) = g(t), \qquad t > 0, \tag{47c}$$

$$y(x, 0) = 0, \qquad 0 < x < L, \tag{47d}$$

$$y_t(x, 0) = 0, \qquad 0 < x < L. \tag{47e}$$

The finite Fourier transform associated with x is

$$\tilde{f}(\lambda_n) = \int_0^L f(x) X_n(x)\, dx,$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L} \sin \lambda_n x$. Application of the transform to PDE (47a) leads to the following ODE in $\tilde{y}(\lambda_n, t)$:

$$\frac{d^2 \tilde{y}}{dt^2} + c^2 \lambda_n^2 \tilde{y} = -c^2 X_n'(L) g(t) \tag{48a}$$

subject to

$$\tilde{y}(\lambda_n, 0) = \tilde{y}'(\lambda_n, 0) = 0. \tag{48b}$$

Variation of parameters on problem (48) gives the solution in the form

$$\tilde{y}(\lambda_n, t) = \frac{-cX_n'(L)}{\lambda_n} \int_0^t g(u) \sin c\lambda_n(t - u)\, du. \tag{49}$$

This is a general formula valid for any function $g(t)$ whatsoever. In this problem, $g(t) = A \sin \omega t$, so that $\bar{y}(\lambda_n, t)$ could be obtained by evaluation of integral (49). (Try it.) Alternatively, if we return to (48a), the general solution when $g(t) = A \sin \omega t$ is

$$\bar{y}(\lambda_n, t) = B_n \cos c\lambda_n t + D_n \sin c\lambda_n t - \frac{Ac^2 X_n'(L)}{c^2\lambda_n^2 - \omega^2} \sin \omega t, \tag{50}$$

provided $\omega \neq c\lambda_n$ for any integer n. Initial conditions (48b) imply that

$$0 = B_n, \qquad 0 = c\lambda_n D_n - \frac{Ac^2\omega X_n'(L)}{c^2\lambda_n^2 - \omega^2},$$

from which

$$\bar{y}(\lambda_n, t) = \frac{Ac\omega X_n'(L)}{\lambda_n(c^2\lambda_n^2 - \omega^2)} \sin c\lambda_n t - \frac{Ac^2 X_n'(L)}{c^2\lambda_n^2 - \omega^2} \sin \omega t. \tag{51}$$

Thus,

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} \bar{y}(\lambda_n, t) X_n(x) \\ &= \sum_{n=1}^{\infty} \frac{AcX_n'(L)}{c^2\lambda_n^2 - \omega^2} \left(\frac{\omega}{\lambda_n} \sin c\lambda_n t - c \sin \omega t \right) X_n(x) \\ &= 2cA \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2 L^2} \left(\omega L \sin \frac{n\pi ct}{L} - n\pi c \sin \omega t \right) \sin \frac{n\pi x}{L}. \end{aligned} \tag{52a}$$

This is the solution of problem (47), provided $\omega \neq c\lambda_n$; that is, provided ω is not equal to a natural frequency of the vibrating string. If this solution is separated into two series,

$$\begin{aligned} y(x, t) &= 2\omega cLA \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2L^2} \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L} \\ &\quad + 2\pi c^2 A \sin \omega t \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2c^2 - \omega^2L^2} \sin \frac{n\pi x}{L}, \end{aligned}$$

it is not unreasonable to expect that the second series, since it is void of t, is the Fourier expansion for some function. Indeed, it is straightforward to show that the series represents $(2\pi c^2)^{-1} \sin(\omega x/c)/\sin(\omega L/c)$. In other words, the solution may be expressed in the simplified form

$$y(x, t) = \frac{A \sin(\omega x/c) \sin \omega t}{\sin(\omega L/c)} + 2\omega cLA \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2L^2} \sin \frac{n\pi ct}{L} \sin \frac{n\pi x}{L}. \tag{52b}$$

We now investigate what happens when ω is equal to a natural frequency of the vibrating string; that is, suppose $\omega = m\pi c/L$ for some integer m. When $n \neq m$, solution (51) of (48) is unchanged. But for $n = m$, $\bar{y}(\lambda_m, t)$ must satisfy

$$\frac{d^2\bar{y}}{dt^2} + c^2\lambda_m^2 \bar{y} = -c^2 X_m'(L) A \sin c\lambda_m t, \tag{53a}$$

$$\bar{y}(\lambda_m, 0) = \bar{y}'(\lambda_m, 0) = 0. \tag{53b}$$

The general solution of (53a) is

$$\bar{y}(\lambda_m, t) = B_m \cos c\lambda_m t + D_m \sin c\lambda_m t + \frac{AcX_m'(L)}{2\lambda_m} t \cos c\lambda_m t. \tag{54}$$

Initial conditions (53b) imply that

$$0 = B_m, \qquad 0 = c\lambda_m D_m + \frac{AcX'_m(L)}{2\lambda_m},$$

from which

$$\tilde{y}(\lambda_m, t) = \frac{-AX'_m(L)}{2\lambda_m^2}\sin c\lambda_m t + \frac{AcX'_m(L)}{2\lambda_m} t\cos c\lambda_m t. \tag{55}$$

In other words, when $\omega = c\lambda_m = m\pi c/L$, the sequence $\{\tilde{y}(\lambda_n, t)\}$ remains unchanged except for the mth term, $\tilde{y}(\lambda_m, t)$. The inverse transform now gives

$$y(x, t) = \tilde{y}(\lambda_m, t)X_m(x) + \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \tilde{y}(\lambda_n, t)X_n(x),$$

and substitutions from (55) and (51) lead to

$$\begin{aligned} y(x, t) &= \frac{A(-1)^m}{L}\left(ct\cos\frac{m\pi ct}{L} - \frac{L}{m\pi}\sin\frac{m\pi ct}{L}\right)\sin\frac{m\pi x}{L} \\ &\quad + \frac{2A}{\pi}\sum_{\substack{n=1 \\ n\neq m}}^{\infty}\frac{(-1)^n}{n^2 - m^2}\left(m\sin\frac{n\pi ct}{L} - n\sin\frac{m\pi ct}{L}\right)\sin\frac{n\pi x}{L}. \end{aligned} \tag{56}$$

For large t, the first term in (56) becomes unbounded. This phenomenon is known as *resonance*. When the forcing frequency (ω) is equal to a natural frequency ($c\lambda_n$) of the vibrating system, oscillations may become excessive and destroy the system. ∎

Further instances of resonance are discussed in Exercises 23–32 and 35. In some applications, resonance is disastrous for the system; in others, resonance is exactly what is desired.

In this section we have dealt with initial boundary value problems. Finite Fourier transforms can also be used to solve nonhomogeneous boundary value problems in Cartesian coordinates x and y. We have already suggested (see Section 5.3) that when nonhomogeneities occur only in boundary conditions, the problem can easily be solved by subdivision into homogeneous problems. In other words, finite Fourier transforms need only be used to accommodate nonhomogeneities in the PDE. This is illustrated in Exercises 42–47.

Exercises 6.2

Use finite Fourier transforms to solve all problems in this set of exercises.

Part A—Heat Conduction

1. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $f(x)$, $0 \leq x \leq L$, at time $t = 0$. For time $t > 0$, the end $x = 0$ is held at 0°C and the end $x = L$ is held at constant temperature U_L°C. What is the temperature in the rod for $0 < x < L$ and $t > 0$?
2. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature 10°C throughout ($0 \leq x \leq L$). For time $t > 0$, its ends ($x = 0$ and $x = L$) are held at temperature 0°C.

At each position x in the rod, heat generation occurs and is defined by $g(x,t) = e^{-\alpha t}$, $\alpha > 0, t > 0$, $0 < x < L$. Find the temperature in the rod as a function of x and t. Assume that $\alpha \neq n^2\pi^2k/L^2$ for any integer n.

3. Solve Exercise 1 in Section 3.3.
4. Solve Exercise 6 in Section 3.3.
5. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature $U_0(1 - x/L)$, U_0 a constant. For time $t > 0$, the end $x = 0$ is maintained at temperature U_0 and end $x = L$ is insulated. Find the temperature in the rod for $0 < x < L$ and $t > 0$.
6. Solve the initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides and ends held at temperature zero. Heat generation is defined at position x and time t by $g(x,t)$, and the initial temperature of the rod is described by $f(x)$.
7. Repeat Exercise 6 if the ends of the rod are insulated.
8. (a) Show that finite Fourier transforms for the problem in Exercise 5 of Section 3.3 leads to the following solution:

$$U(x,t) = 200 \sum_{n=1}^{\infty} \left[\left(\frac{[1 + (-1)^{n+1}]}{n\pi} + \frac{n\pi k(-1)^n}{n^2\pi^2 k - L^2} \right) e^{-n^2\pi^2 kt/L^2} + \frac{n\pi k(-1)^{n+1}}{n^2\pi^2 k - L^2} e^{-t} \right] \sin\frac{n\pi x}{L}.$$

(b) Simplify this solution by finding the transform of the function $f(x) = x$ and using the partial fraction decomposition

$$\frac{1}{n(n^2\pi^2 k - L^2)} = \frac{-1/L^2}{n} + \frac{n\pi^2 k/L^2}{n^2\pi^2 k - L^2}$$

on the last term.

9. We have claimed that to solve an initial boundary value problem with finite Fourier transforms, it is necessary to use the transform associated with the Sturm-Liouville system that would result were separation of variables used on the corresponding homogeneous problem. To illustrate this, apply the finite Fourier transform associated with Sturm-Liouville system (2) of Chapter 4 to Exercise 2. Show that an insoluble problem in $\tilde{y}(\lambda_n, t)$ is obtained.
10. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature zero throughout. For times $t > 0$, there is located at cross section $x = b$ $(0 < b < L)$ a plane heat source of constant strength g. If the ends $x = 0$ and $x = L$ of the rod are kept at zero temperature, the initial boundary value problem for temperature in the rod is

$$\frac{\partial^2 U}{\partial x^2} = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{g}{\kappa}\delta(x - b), \qquad 0 < x < L, \qquad t > 0,$$

$$U(0,t) = 0, \qquad t > 0,$$

$$U(L,t) = 0, \qquad t > 0,$$

$$U(x,0) = 0, \qquad 0 < x < L,$$

where $\delta(x - b)$ is the Dirac delta function. Solve this problem for $U(x,t)$, using the fact that

$$\int_0^L f(x)\delta(x - b)\,dx = f(b).$$

11. Solve Exercise 8 in Section 3.3.

12. Repeat Exercise 5 if the temperature of the end $x = 0$ is $U_0 e^{-\alpha t}$ ($\alpha > 0$ a constant). To simplify the solution, use the technique of Exercise 8(b) with $f(x) = 1$. Assume that $\alpha \neq n^2\pi^2 k/L^2$ for any integer n.

13. If the ends $x = 0$ and $x = L$ of the thin-wire problem in Exercise 3 of Section 5.2 are kept at constant temperatures U_0 and U_L, respectively, and the initial temperature is zero throughout, show that

$$U(x,t) = \frac{U_0 \sinh\sqrt{h/k}(L-x) + U_L \sinh\sqrt{h/k}\,x}{\sinh\sqrt{h/k}\,L} - 2k\pi e^{-ht}\sum_{n=1}^{\infty}\frac{n[U_0 + (-1)^{n+1}U_L]}{hL^2 + n^2\pi^2 k}e^{-n^2\pi^2 kt/L^2}\sin\frac{n\pi x}{L}.$$

14. Repeat Exercise 5 if heat is added uniformly over the end $x = L$ at a constant rate q W/m^2.

15. (a) A cylindrical, homogeneous, isotropic rod with insulated sides is initially at constant temperature U_0 throughout. For time $t > 0$, the right end, $x = L$, continues to be held at temperature U_0. Heat is added uniformly over the left end, $x = 0$, at a constant rate q W/m^2 for the first t_0 seconds, and the end is insulated thereafter. Find the temperature in the rod for $0 < x < L$ and $0 < t < t_0$.
    (b) Assuming that $U(x,t)$ must be continuous at time t_0, find $U(x,t)$ for $0 < x < L$ and $t > t_0$.
    (c) What is the steady-state solution?

16. Repeat Exercise 15 if the end $x = L$ is insulated.

17. Find a formula for the solution of the general one-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{kg(x,t)}{\kappa}, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(t), \qquad x = L, \qquad t > 0,$$

$$U(x,0) = f(x), \qquad 0 < x < L.$$

18. The general thin-wire problem (see Exercise 31 in Section 1.2) is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - h(U - U_m) + \frac{k}{\kappa}g(x,t), \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(t), \qquad x = L, \qquad t > 0,$$

$$U(x,0) = f(x), \qquad 0 < x < L.$$

(a) Show that the change of dependent variable $\bar{U}(x,t) = e^{ht}U(x,t)$ leads to the initial boundary value problem

$$\frac{\partial \bar{U}}{\partial t} = k\frac{\partial^2 \bar{U}}{\partial x^2} + \left(hU_m + \frac{k}{\kappa}g(x,t)\right)e^{ht}, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial \bar{U}}{\partial x} + h_1\bar{U} = e^{ht}f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial \bar{U}}{\partial x} + h_2\bar{U} = e^{ht}f_2(t), \qquad x = L, \qquad t > 0,$$

$$\bar{U}(x,0) = f(x), \qquad 0 < x < L.$$

(b) Use the results of Exercise 17 to find $\bar{U}(x,t)$ and hence $U(x,t)$.

Part B—Vibrations

19. Solve Exercise 13 in Section 3.3.
20. Solve Exercise 14 in Section 3.3.
21. The end $x = 0$ of a horizontal elastic bar of length L is kept fixed, and the other end, $x = L$, is subjected to a constant force per unit area F acting parallel to the bar. If the bar is initially unstrained and at rest, the initial boundary value problem for longitudinal displacement $y(x,t)$ of the cross section originally at position x is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$

$$y(0,t) = 0, \qquad t > 0,$$

$$E\frac{\partial y(L,t)}{\partial x} = F, \qquad t > 0,$$

$$y(x,0) = 0, \qquad 0 < x < L,$$

$$y_t(x,0) = 0, \qquad 0 < x < L,$$

where $E/\rho = c^2$ (E = Young's modulus of elasticity and ρ = density).

(a) Show that the solution to this problem is

$$y(x,t) = \frac{8LF}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\left(1 - \cos\frac{(2n-1)c\pi t}{2L}\right)\sin\frac{(2n-1)\pi x}{2L}.$$

(b) Find the finite Fourier transform of the function $M(x) = x, 0 < x < L$, and use the result to write $y(x,t)$ in the form

$$y(x,t) = \frac{F}{E}M(x) - \frac{8LF}{E\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)c\pi t}{2L}\sin\frac{(2n-1)\pi x}{2L}.$$

(c) Show that $y(x,t)$ can now be expressed in the form

$$y(x,t) = \frac{F}{E}\left(M_L(x) - \frac{1}{2}M_L(x+ct) - \frac{1}{2}M_L(x-ct)\right),$$

where $M_L(x)$ is the extension of $M(x)$ to an odd, odd-harmonic function (see Exercise 21 in Section 2.2).

(d) Evaluate $y(L, t)$ and draw its graph as a function of t to illustrate the motion of the end $x = L$ of the bar.

22. A horizontal elastic bar of natural length L lies along the x-axis between $x = 0$ and $x = L$. At time $t = 0$, it is stretched so that the displacement of the cross section at position x is given by the function kx, $k > 0$ a constant, $0 \le x \le L$. The bar is released from rest at this position. If a constant force per unit area F acts parallel to the bar on the end $x = 0$, find subsequent displacements of cross sections of the bar.

23. A taut string initially at rest along the x-axis has its ends $x = 0$ and $x = L$ fixed on the axis. If a periodic external force $F_0 \sin \omega t$, $t \ge 0$, per unit x-length acts at every point on the string, find the displacement of the string. Include a discussion of resonance.

24. A taut string initially at rest along the x-axis has its end $x = 0$ fixed on the x-axis. The end $x = L$ is forced to undergo periodic vertical motion $A \sin \omega t$, $t \ge 0$ (A and ω constants). Find the displacement of the string. Include a discussion of resonance.

In Exercises 25–32, determine frequencies of the applied force that will produce resonance. Do not determine the solution to the initial boundary value problem, only the frequencies.

25. The string in Example 4 if the end $x = L$ is free to slide vertically and an external force $F_0 \sin \omega t$, $t \ge 0$, per unit x-length acts at every point on the string.

26. The string in Example 4 if both ends are free to slide vertically and an external force $F_0 \sin \omega t$, $t \ge 0$, per unit x-length acts at every point on the string. [Find the solution $y(x, t)$ in this case.]

27. The bar in Exercise 21 if the force is $F = F_0 \sin \omega t$.

28. The bar in Exercise 21 if the end $x = 0$ is free and $F = F_0 \sin \omega t$.

29. The bar in Exercise 21 if the end $x = L$ has a prescribed displacement $A_0 \sin \omega t$.

30. The bar in Exercise 21 if the end $x = 0$ is free and the end $x = L$ has a prescribed displacement $A_0 \sin \omega t$.

31. The bar in Exercise 21 if the ends $x = 0$ and $x = L$ have prescribed displacements $A_0 \sin \omega t$ and $B_0 \sin \phi t$, respectively.

32. The bar in Exercise 21 if the ends $x = 0$ and $x = L$ are subjected to forces $F_0 \sin \omega t$ and $G_0 \sin \phi t$ (per unit area), respectively.

33. An elastic bar of natural length L is clamped along its length, turned to the vertical position, and hung from its end $x = 0$. At time $t = 0$, the clamp is removed and gravity is therefore permitted to act on the bar.

    (a) Show that vertical displacements of cross sections of the bar are given by

$$y(x, t) = \frac{gx}{2c^2}(2L - x) - \frac{16gL^2}{c^2\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \cos\frac{(2n-1)c\pi t}{2L} \sin\frac{(2n-1)\pi x}{2L}.$$

    (b) Find a closed-form solution for $y(x, t)$. [*Hint:* See part (c) of Exercise 21.]

    (c) Sketch a graph of $y(L, t)$. Does the end $x = L$ of the bar oscillate about its equilibrium position, that is, the position of the lower end of the bar if the bar were to hang motionless under its own weight? (See Exercise 13 in Section 1.3.)

34. (a) Find displacements in the bar of Exercise 33 if the top of the bar is attached to a spring with constant k. Let $x = 0$ correspond to the top end of the bar when the spring is in the unstretched position.

    (b) Does the lower end of the bar oscillate about its equilibrium position? (See Exercise 14 in Section 1.3.)

35. Repeat Example 3 if a damping force $-\beta\partial y/\partial t$, proportional to velocity, acts at every point on the string. Assume that $\beta < 2\pi\rho c/L$. Can resonance with unbounded oscillations occur?

36. (a) The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest along the axis and then is allowed to drop under its own weight. Find a series representation for the displacement of the string.

(b) Show that the solution in (a) can be expressed in the closed form

$$y(x,t) = M(x) - \frac{1}{2}[M(x+ct) + M(x-ct)],$$

where $M(x)$ is the odd, $2L$-periodic extension of the function $g(Lx - x^2)/(2c^2)$.

37. Repeat Exercise 36 if the string has an initial displacement $f(x)$.

38. The ends of a taut string are looped around smooth vertical supports at $x = 0$ and $x = L$. If the string falls from rest along the x-axis, and a constant vertical force F_0 acts on the loop at $x = L$, find displacements of the string. Take gravity into account.

39. A motionless, horizontal beam has its ends simply supported at $x = 0$ and $x = L$. At time $t = 0$, a concentrated force of magnitude A is suddenly applied at the midpoint.

(a) If the weight per unit length of the beam is negligible compared with A, show that the initial boundary value problem for transverse displacements $y(x,t)$ is

$$\frac{\partial^2 y}{\partial t^2} + c^2\frac{\partial^4 y}{\partial x^4} = -\frac{A}{\rho}\delta\left(x - \frac{L}{2}\right), \qquad 0 < x < L, \qquad t > 0,$$

$$y(0,t) = y(L,t) = 0, \qquad t > 0,$$

$$y_{xx}(0,t) = y_{xx}(L,t) = 0, \qquad t > 0,$$

$$y(x,0) = y_t(x,0) = 0, \qquad 0 < x < L,$$

where $c^2 = EI/\rho$ and ρ is the linear density of the beam.

(b) Solve this problem using the finite Fourier transform associated with Sturm-Liouville system (1) of Chapter 4. [See Exercise 10 for the requisite property of the delta function $\delta(x - L/2)$.]

40. Find a formula for the solution of the general one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial y}{\partial x} + h_1 y = f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial y}{\partial x} + h_2 y = f_2(t), \qquad x = L, \qquad t > 0,$$

$$y(x,0) = f(x), \qquad 0 < x < L,$$

$$y_t(x,0) = g(x), \qquad 0 < x < L.$$

41. The end $x = 0$ of a horizontal elastic bar of length L is kept fixed, and the other end has a mass m attached to it. The mass m is then subjected to a horizontal periodic force $F = F_0 \sin \omega t$. If the bar is initially unstrained and at rest, set up the initial boundary value problem for longitudinal displacements in the bar. Can we solve this problem with finite Fourier transforms?

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

42. A charge distribution with density $\sigma(x, y)$ coulombs per cubic meter occupies the volume R in space bounded by the planes $x = 0$, $y = 0$, $x = L$, and $y = L'$, and these planes are all held at potential zero.

(a) Use finite Fourier transforms to find the potential $V(x, y)$ in R when σ is constant. Find two series, one by transforming the x-variable and the other by transforming the y-variable.

(b) If $\sigma = \sigma(x)$ is a function of x only, find $V(x, y)$.

(c) Find $V(x, y)$ when $\sigma = xy$.

43. A uniform charge distribution of density σ coulombs per cubic meter occupies the volume R bounded by the planes $x = 0, y = 0, x = L$, and $y = L'$. If the electrostatic potential on the planes $x = 0$, $y = 0$, and $y = L'$ is zero and that on $x = L$ is $f(y)$, find the potential in R.

44. Repeat Exercise 43 when planes $x = 0$, $x = L$, and $y = L'$ are held at zero potential and $y = 0$ is at potential $g(x)$.

45. Repeat Exercise 43 when planes $x = L$ and $y = L'$ are held at zero potential and $x = 0$ and $y = 0$ are at $f(y)$ and $g(x)$, respectively.

46. Find a formula for the solution of the general two-dimensional Dirichlet boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \qquad 0 < x < L, \qquad 0 < y < L',$$

$$V(0, y) = f_1(y), \qquad 0 < y < L',$$

$$V(L, y) = f_2(y), \qquad 0 < y < L',$$

$$V(x, 0) = g_1(x), \qquad 0 < x < L,$$

$$V(x, L') = g_2(x), \qquad 0 < x < L.$$

47. We suggested at the end of this section that two-dimensional boundary value problems on rectangles with four nonhomogeneous boundary conditions and homogeneous PDEs can be subdivided into two problems, each of which has two homogeneous and two nonhomogeneous boundary conditions. There is an exception to this, namely the Neumann problem. For example, the Neumann problem associated with Laplace's equation is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad 0 < x < L, \qquad 0 < y < L',$$

$$\frac{\partial V(0, y)}{\partial x} = f_1(y), \qquad 0 < y < L',$$

$$\frac{\partial V(L, y)}{\partial x} = f_2(y), \qquad 0 < y < L',$$

$$\frac{\partial V(x, 0)}{\partial y} = g_1(x), \qquad 0 < x < L,$$

$$\frac{\partial V(x, L')}{\partial y} = g_2(x), \qquad 0 < x < L,$$

where the nonhomogeneities must satisfy the consistency condition

$$\int_0^L [g_2(x) - g_1(x)]\,dx + \int_0^{L'} [f_2(y) - f_1(y)]\,dy = 0.$$

Our previous suggestion would indicate that $V(x, y)$ should be set equal to $V(x, y) = V_1(x, y) + V_2(x, y)$, where V_1 and V_2 satisfy Laplace's equation on the rectangle and the following boundary conditions:

$$\frac{\partial V_1(0, y)}{\partial x} = f_1(y), \quad 0 < y < L', \qquad \frac{\partial V_2(0, y)}{\partial x} = 0, \quad 0 < y < L',$$

$$\frac{\partial V_1(L, y)}{\partial x} = f_2(y), \quad 0 < y < L', \qquad \frac{\partial V_2(L, y)}{\partial x} = 0, \quad 0 < y < L',$$

$$\frac{\partial V_1(x, 0)}{\partial y} = 0, \quad 0 < x < L, \qquad \frac{\partial V_2(x, 0)}{\partial y} = g_1(x), \quad 0 < x < L,$$

$$\frac{\partial V_1(x, L')}{\partial y} = 0, \quad 0 < x < L; \qquad \frac{\partial V_2(x, L')}{\partial y} = g_2(x), \quad 0 < x < L.$$

But these Neumann problems must satisfy the consistency conditions

$$\int_0^{L'} [f_2(y) - f_1(y)]\,dy = 0 \quad \text{and} \quad \int_0^L [g_2(x) - g_1(x)]\,dx = 0.$$

The difficulty is that the combined consistency condition on f_1, f_2, g_1, and g_2 may not imply these separately. In general, then, solutions for V_1 and V_2 may not exist. With finite Fourier transforms, this difficulty presents no problem. Find $V(x, y)$ using such a transform.

6.3 Higher-Dimensional Problems in Cartesian Coordinates

To solve nonhomogeneous initial boundary value problems in three and four variables, we can once again remove space variables from the problem with finite Fourier transforms, leaving an ODE in the transform function regarded only as a function of time. There are two ways to do this. Successive finite Fourier transforms, each a transform in only one space variable, can be applied to the PDE. This corresponds to successively separating off space variables in homogeneous problems. Alternatively, multidimensional finite Fourier transforms associated with multidimensional eigenvalue problems (see Section 5.5) can be introduced. We take the former approach. To illustrate, consider the following initial boundary value problem.

Example 4: Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{57a}$$

$$U(0, y, t) = U_1, \quad 0 < y < L', \quad t > 0, \tag{57b}$$

$$U(L, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{57c}$$

$$U(x, 0, t) = U_2, \quad 0 < x < L, \quad t > 0, \tag{57d}$$

$$U_y(x, L', t) = 0, \quad 0 < x < L, \quad t > 0, \tag{57e}$$
$$U(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'. \tag{57f}$$

Physically described is a horizontal plate that is insulated top and bottom and along the edge $y = L'$. Initially the temperature is zero throughout the plate, and for $t > 0$, faces $x = 0$, $x = L$, and $y = 0$ are held at constant temperatures U_1, 0, and U_2, respectively.

Solution: The finite Fourier transform associated with the x-variable is

$$\tilde{f}(\lambda_n) = \int_0^L f(x)X_n(x)\,dx, \tag{58}$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = X(L) = 0.$$

This is the system that would result were separation of variables applied to problem (57) with homogeneous boundary conditions. If we apply this transform to PDE (57a), and use integration by parts,

$$\int_0^L \frac{\partial U}{\partial t} X_n\,dx = k\int_0^L \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) X_n\,dx$$
$$= k\frac{\partial^2}{\partial y^2}\int_0^L UX_n\,dx + k\left\{\frac{\partial U}{\partial x}X_n\right\}_0^L - k\int_0^L \frac{\partial U}{\partial x}X_n'\,dx.$$

Since $X_n(0) = X_n(L) = 0$,

$$\frac{\partial}{\partial t}\int_0^L UX_n\,dx = k\frac{\partial^2 \tilde{U}(\lambda_n, y, t)}{\partial y^2} - k\{UX_n'\}_0^L + k\int_0^L UX_n''\,dx.$$

Boundary conditions (57b, c) and the fact that $X_n'' = -\lambda_n^2 X_n$ now give

$$\frac{\partial \tilde{U}}{\partial t} = k\frac{\partial^2 \tilde{U}}{\partial y^2} + kU_1X_n'(0) + k\int_0^L U(-\lambda_n^2 X_n)\,dx.$$

Thus, $\tilde{U}(\lambda_n, y, t)$ must satisfy the PDE

$$\frac{\partial \tilde{U}}{\partial t} = k\frac{\partial^2 \tilde{U}}{\partial y^2} + kU_1X_n'(0) - k\lambda_n^2\tilde{U}, \quad 0 < y < L', \quad t > 0 \tag{59a}$$

subject to the transforms of conditions (57d–f),

$$\tilde{U}(\lambda_n, 0, t) = U_2\tilde{1}_n, \quad t > 0, \tag{59b}$$
$$\tilde{U}_y(\lambda_n, L', t) = 0, \quad t > 0, \tag{59c}$$
$$\tilde{U}(\lambda_n, y, 0) = 0, \quad 0 < y < L', \tag{59d}$$

where
$$\tilde{1}_n = \int_0^L X_n\,dx = \int_0^L \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx = \frac{\sqrt{2L}[1 + (-1)^{n+1}]}{n\pi}. \tag{59e}$$

The finite Fourier transform associated with the y-variable in problem (59) is

$$\tilde{f}(\mu_m) = \int_0^{L'} f(y) Y_m(y)\, dy, \tag{60}$$

where $\mu_m^2 = (2m-1)^2\pi^2/(4L'^2)$ and $Y_m(y) = \sqrt{2/L'}\sin[(2m-1)\pi y/(2L')]$ are eigenpairs of the Sturm-Liouville system

$$Y'' + \mu^2 Y = 0, \qquad 0 < y < L',$$
$$Y(0) = Y'(L') = 0.$$

If we apply this transform to PDE (59a),

$$\int_0^{L'} \frac{\partial \tilde{U}}{\partial t} Y_m\, dy = k \int_0^{L'} \frac{\partial^2 \tilde{U}}{\partial y^2} Y_m\, dy + \int_0^{L'} [kU_1 X_n'(0) - k\lambda_n^2 \tilde{U}]\, Y_m\, dy,$$

and use integration by parts,

$$\frac{\partial \tilde{\tilde{U}}(\lambda_n, \mu_m, t)}{\partial t} - kU_1 X_n'(0)\tilde{1}_m + k\lambda_n^2 \tilde{\tilde{U}} = k\left\{\frac{\partial \tilde{U}}{\partial y} Y_m\right\}_0^{L'} - k\int_0^{L'} \frac{\partial \tilde{U}}{\partial y} Y_m'\, dy,$$

where $$\tilde{1}_m = \int_0^{L'} Y_m\, dy = \int_0^{L'} \sqrt{\frac{2}{L'}}\sin\frac{(2m-1)\pi y}{2L'}\, dy = \frac{2\sqrt{2L'}}{(2m-1)\pi}. \tag{61}$$

Since $Y_m(0) = 0$ and $\partial \tilde{U}(\lambda_n, L', t)/\partial y = 0$,

$$\frac{\partial \tilde{\tilde{U}}}{\partial t} - kU_1 X_n'(0)\tilde{1}_m + k\lambda_n^2 \tilde{\tilde{U}} = -k\{\tilde{U} Y_m'\}_0^{L'} + k\int_0^{L'} \tilde{U} Y_m''\, dy.$$

Boundary condition (59b) and the facts that $Y_m'(L') = 0$ and $Y_m'' = -\mu_m^2 Y_m$ yield

$$\frac{\partial \tilde{\tilde{U}}}{\partial t} - kU_1 X_n'(0)\tilde{1}_m + k\lambda_n^2 \tilde{\tilde{U}} = kU_2 Y_m'(0)\tilde{1}_n + k\int_0^{L'} \tilde{U}(-\mu_m^2 Y_m)\, dy$$

or $$\frac{d\tilde{\tilde{U}}}{dt} + k(\lambda_n^2 + \mu_m^2)\tilde{\tilde{U}} = k[U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m]. \tag{62a}$$

Accompanying this ODE in $\tilde{\tilde{U}}(\lambda_n, \mu_m, t)$ is the transform of initial condition (59d),

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, 0) = 0. \tag{62b}$$

Because the right side of (62a) is a constant with respect to t, a general solution of this ODE is

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, t) = A_{mn} e^{-k(\lambda_n^2 + \mu_m^2)t} + \frac{U_2 Y_m'(0)\,\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m}{\lambda_n^2 + \mu_m^2},$$

where the A_{mn} are constants. Initial condition (62b) requires that

$$0 = A_{mn} + \frac{U_2 Y_m'(0)\tilde{1}_n + U_1 X_n'(0)\tilde{1}_m}{\lambda_n^2 + \mu_m^2},$$

and therefore

$$\tilde{\tilde{U}}(\lambda_n, \mu_m, t) = \frac{U_2 Y_m'(0)\bar{1}_n + U_1 X_n'(0)\bar{1}_m}{\lambda_n^2 + \mu_m^2}(1 - e^{-k(\lambda_n^2+\mu_m^2)t}). \quad (63)$$

To find $U(x, y, t)$, we now invert transforms (60) and (58):

$$U(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \tilde{\tilde{U}}(\lambda_n, \mu_m, t) Y_m(y) X_n(x).$$

Substitutions for $\tilde{\tilde{U}}(\lambda_n, \mu_m, t)$, $Y_m(y)$, and $X_n(x)$ lead to

$$U(x, y, t) = 8 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} B_{mn}(1 - e^{-[4n^2\pi^2L'^2+(2m-1)^2\pi^2L^2]kt/(4L^2L'^2)}) \times \sin\frac{n\pi x}{L}\sin\frac{(2m-1)\pi y}{2L'}, \quad (64a)$$

where

$$B_{mn} = \frac{[1+(-1)^{n+1}](2m-1)^2L^2U_2 + 4n^2L'^2U_1}{n(2m-1)[4n^2\pi^2L'^2 + (2m-1)^2\pi^2L^2]}. \quad ■ \quad (64b)$$

As a second example, we consider a boundary value problem in three dimensions.

Example 5: Find the potential inside the region bounded by the planes $x = 0, x = L, y = 0, y = L'$, $z = 0$, and $z = L''$ if all such planes are held at potential zero and the region contains a uniform charge distribution with density σ coulombs per cubic meter.

Solution: The boundary value problem for potential $V(x, y, z)$ in the region is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\sigma}{\varepsilon}, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \quad (65a)$$

$$V(0, y, z) = 0, \quad 0 < y < L', \quad 0 < z < L'', \quad (65b)$$

$$V(L, y, z) = 0, \quad 0 < y < L', \quad 0 < z < L'', \quad (65c)$$

$$V(x, 0, z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \quad (65d)$$

$$V(x, L', z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \quad (65e)$$

$$V(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L', \quad (65f)$$

$$V(x, y, L'') = 0, \quad 0 < x < L, \quad 0 < y < L'. \quad (65g)$$

The finite Fourier transform associated with the x-variable is

$$\tilde{f}(\lambda_n) = \int_0^L f(x) X_n(x)\, dx, \quad (66)$$

where $\lambda_n^2 = n^2\pi^2/L^2$ and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$ are the eigenpairs of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X(0) = 0 = X(L).$$

When we apply this transform to PDE (65a) and use integration by parts,

$$\int_0^L \left(\frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + \frac{\sigma}{\varepsilon}\right) X_n\,dx = -\int_0^L \frac{\partial^2 V}{\partial x^2} X_n\,dx$$

$$= -\left\{\frac{\partial V}{\partial x} X_n\right\}_0^L + \int_0^L \frac{\partial V}{\partial x} X_n'\,dx$$

[and since $X_n(0) = X_n(L) = 0$]

$$= \{VX_n'\}_0^L - \int_0^L VX_n''\,dx$$

[and since $V(L, y, z) = V(0, y, z) = 0$]

$$= -\int_0^L V(-\lambda_n^2 X_n)\,dx$$

$$= \lambda_n^2 \tilde{V}(\lambda_n, y, z).$$

Thus, $\tilde{V}(\lambda_n, y, z)$ must satisfy the PDE

$$\frac{\partial^2 \tilde{V}}{\partial y^2} + \frac{\partial^2 \tilde{V}}{\partial z^2} - \lambda_n^2 \tilde{V} = -\frac{\sigma}{\varepsilon}\tilde{1}_n, \qquad 0 < y < L', \qquad 0 < z < L'', \tag{67a}$$

subject to the boundary conditions

$$\tilde{V}(\lambda_n, 0, z) = 0, \qquad 0 < z < L'', \tag{67b}$$

$$\tilde{V}(\lambda_n, L', z) = 0, \qquad 0 < z < L'', \tag{67c}$$

$$\tilde{V}(\lambda_n, y, 0) = 0, \qquad 0 < y < L', \tag{67d}$$

$$\tilde{V}(\lambda_n, y, L'') = 0, \qquad 0 < y < L' \tag{67e}$$

and

$$\tilde{1}_n = \int_0^L 1X_n\,dx = \frac{\sqrt{2L}[1 + (-1)^{n+1}]}{n\pi}. \tag{67f}$$

To eliminate y from problem (67), we use the finite Fourier transform

$$\tilde{f}(\mu_m) = \int_0^{L'} f(y) Y_m(y)\,dy, \tag{68}$$

where $\mu_m^2 = m^2\pi^2/L'^2$ and $Y_m(y) = \sqrt{2/L'}\sin(m\pi y/L')$ are eigenpairs of the Sturm-Liouville system

$$Y'' + \mu^2 Y = 0, \qquad 0 < y < L',$$

$$Y(0) = 0 = Y(L').$$

Application of this transform to (67a) yields

$$\int_0^{L'} \left(\frac{\partial^2 \tilde{V}}{\partial z^2} - \lambda_n^2 \tilde{V} + \frac{\sigma}{\varepsilon}\tilde{1}_n\right) Y_m\,dy = -\int_0^L \frac{\partial^2 \tilde{V}}{\partial y^2} Y_m\,dy$$

$$= -\left\{\frac{\partial \tilde{V}}{\partial y} Y_m\right\}_0^L + \int_0^{L'} \frac{\partial \tilde{V}}{\partial y} Y_m'\,dy$$

[and since $Y_m(0) = Y_m(L') = 0$]

$$= \{\tilde{V}Y'_m\}_0^{L'} - \int_0^{L'} \tilde{V}Y''_m\,dy$$

[and since $\tilde{V}(\lambda_n, 0, z) = \tilde{V}(\lambda_n, L', z) = 0$]

$$= \int_0^{L'} \tilde{V}(-\mu_m^2 Y_m)\,dy$$

$$= \mu_m^2 \tilde{\tilde{V}}(\lambda_n, \mu_m, z).$$

Thus, $\tilde{\tilde{V}}(\lambda_n, \mu_m, z)$ must satisfy the ODE

$$\frac{d^2\tilde{\tilde{V}}}{dz^2} - (\lambda_n^2 + \mu_m^2)\tilde{\tilde{V}} = -\frac{\sigma}{\varepsilon}\tilde{\tilde{1}}_{nm}, \quad 0 < z < L'', \tag{69a}$$

subject to

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, 0) = 0, \tag{69b}$$

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, L'') = 0 \tag{69c}$$

and

$$\tilde{\tilde{1}}_{nm} = \int_0^{L'} \tilde{1}_n Y_m\,dy = \frac{2\sqrt{LL'}[1 + (-1)^{n+1}][1 + (-1)^{m+1}]}{mn\pi^2}. \tag{69d}$$

The general solution of (69a) is

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, z) = A_{mn}\cosh\sqrt{\lambda_n^2 + \mu_m^2}\,z + B_{mn}\sinh\sqrt{\lambda_n^2 + \mu_m^2}\,z + \frac{(\sigma/\varepsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2}. \tag{70}$$

Boundary conditions (69b, c) require that

$$0 = A_{mn} + \frac{(\sigma/\varepsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2},$$

$$0 = A_{mn}\cosh\sqrt{\lambda_n^2 + \mu_m^2}\,L'' + B_{mn}\sinh\sqrt{\lambda_n^2 + \mu_m^2}\,L'' + \frac{(\sigma/\varepsilon)\tilde{\tilde{1}}_{nm}}{\lambda_n^2 + \mu_m^2}.$$

When these are solved for A_{mn} and B_{mn} and the results are substituted into (70), $\tilde{\tilde{V}}(\lambda_n, \mu_m, z)$ simplifies to

$$\tilde{\tilde{V}}(\lambda_n, \mu_m, z) = \frac{-(\sigma/\varepsilon)\tilde{\tilde{1}}_{nm}}{(\lambda_n^2 + \mu_m^2)\sinh\sqrt{\lambda_n^2 + \mu_m^2}\,L''}\Big(\sinh\sqrt{\lambda_n^2 + \mu_m^2}\,(L'' - z) + \sinh\sqrt{\lambda_n^2 + \mu_m^2}\,z - \sinh\sqrt{\lambda_n^2 + \mu_m^2}\,L''\Big). \tag{71}$$

The solution of problem (65) is therefore

$$V(x, y, z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \tilde{\tilde{V}}(\lambda_n, \mu_m, z)X_n(x)Y_m(y). \quad \blacksquare \tag{72}$$

Exercises 6.3

Part A—Heat Conduction

1. An isotropic, homogeneous, horizontal plate has its top and bottom faces insulated. Edges $x = 0$, $x = L$, $y = 0$, and $y = L'$ are all held at constant temperatures U_1, U_2, U_3, and U_4, respectively,

for time $t > 0$. If the temperature in the plate at time $t = 0$ is $f(x, y), 0 \le x \le L, 0 \le y \le L'$, find its temperature thereafter.

2. (a) Solve the following heat conduction problem:

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$U(0, y, t) = U_1, \quad 0 < y < L', \quad t > 0,$$
$$U(L, y, t) = U_2, \quad 0 < y < L', \quad t > 0,$$
$$U_y(x, 0, t) = \kappa_1^{-1}\phi_1, \quad 0 < x < L, \quad t > 0,$$
$$U_y(x, L', t) = -\kappa_2^{-1}\phi_2, \quad 0 < x < L, \quad t > 0,$$
$$U(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L',$$

where U_1, U_2, ϕ_1, and ϕ_2 are constants. Interpret the problem physically.

(b) What is the solution when $\phi_1 = \phi_2 = 0$?

3. Repeat Exercise 2(a) when U_1, U_2, ϕ_1, and ϕ_2 are functions of time t.

4. Find a formula for the solution of the general two-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{kg(x, y, t)}{\kappa}, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$
$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(y, t), \quad x = 0, \quad 0 < y < L', \quad t > 0,$$
$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(y, t), \quad x = L, \quad 0 < y < L', \quad t > 0,$$
$$-l_3\frac{\partial U}{\partial y} + h_3 U = f_3(x, t), \quad y = 0, \quad 0 < x < L, \quad t > 0,$$
$$l_4\frac{\partial U}{\partial y} + h_4 U = f_4(x, t), \quad y = L', \quad 0 < x < L, \quad t > 0,$$
$$U(x, y, 0) = f(x, y), \quad 0 < x < L, \quad 0 < y < L'.$$

Part B—Vibrations

5. A rectangular membrane of side lengths L and L' has its edges fixed on the xy-plane. If it is released from rest at a displacement given by $f(x, y)$, find subsequent displacements of the membrane if gravity is taken into account.

6. A square membrane of side length L, which is initially at rest on the xy-plane, has its edges fixed on the xy-plane. If a periodic force per unit area $A\cos(\omega t)$, $t > 0$ (A a constant) acts at every point in the membrane, find displacements in the membrane. Assume that $\omega \ne c\pi\sqrt{n^2 + m^2}/L$ for any positive integers m and n.

7. Repeat Exercise 6 if $\omega = \sqrt{2}\pi c/L$.

8. Repeat Exercise 6 if $\omega = \sqrt{17}\pi c/L$.

9. Repeat Exercise 6 if $\omega = \sqrt{65}\pi c/L$.

10. Repeat Exercise 6 if $\omega = \sqrt{10}\pi c/L$.

11. Repeat Exercise 6 if $\omega = \sqrt{130}\pi c/L$.

12. Find a formula for the solution of the general two-dimensional vibration problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F(x,y,t)}{\rho}, \qquad 0 < x < L, \qquad 0 < y < L', \qquad t > 0,$$

$$-l_1 \frac{\partial z}{\partial x} + h_1 z = f_1(y,t), \qquad x = 0, \qquad 0 < y < L', \qquad t > 0,$$

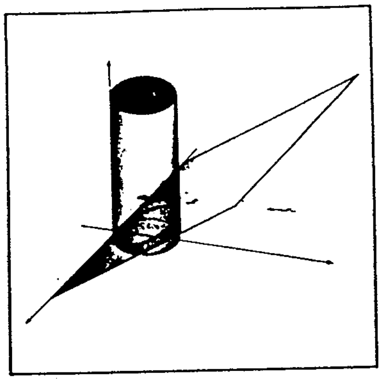
$$l_2 \frac{\partial z}{\partial x} + h_2 z = f_2(y,t), \qquad x = L, \qquad 0 < y < L', \qquad t > 0,$$

$$-l_3 \frac{\partial z}{\partial y} + h_3 z = f_3(x,t), \qquad y = 0, \qquad 0 < x < L, \qquad t > 0,$$

$$l_4 \frac{\partial z}{\partial y} + h_4 z = f_4(x,t), \qquad y = L', \qquad 0 < x < L, \qquad t > 0,$$

$$z(x,y,0) = g(x,y), \qquad 0 < x < L, \qquad 0 < y < L',$$

$$z_t(x,y,0) = h(x,y), \qquad 0 < x < L, \qquad 0 < y < L'.$$



CHAPTER
SEVEN

Problems on Infinite Spatial Domains

7.1 Introduction

In Chapters 2–6 we restricted consideration to problems on bounded spatial domains, but many important problems take place on infinite or semi-infinite domains. For example, suppose a rod of infinite length is initially at temperature $f(x)$, $-\infty < x < \infty$. The initial value problem for temperature $U(x, t)$ in the rod when the sides are insulated is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0, \tag{1a}$$

$$U(x, 0) = f(x), \qquad -\infty < x < \infty. \tag{1b}$$

It may be argued that there is no such thing as an infinite rod. Physically it must be finite, and therefore boundary effects must be taken into account. This can be countered by stating that the rod may be so long that boundary effects are negligibly small in that part of the rod under consideration. Consequently, if there is a simple solution to the infinite problem that is an excellent approximation to the Fourier series solution of the bounded problem, then clearly there is an advantage in considering the infinite problem.

In this chapter we illustrate that separation of variables on problems with infinite spatial domains leads to integral representations of the solution called *Fourier integrals*. The Fourier integral replaces the Fourier series representation for finite intervals; it is a direct result of the fact that eigenvalues of the separated equation form a continuous, rather than discrete, set. When the solution of an infinite spatial problem is known to be even or odd, the Fourier integral takes on a simplified form called the *Fourier cosine* or *sine integral*. These integrals also arise naturally in problems on semi-infinite intervals ($0 < x < \infty$) when the boundary condition at $x = 0$ is Neumann or Dirichlet. Generalized Fourier integrals arise when the boundary condition at $x = 0$ is of Robin type. Associated with each Fourier integral is an integral transform that provides a convenient alternative to separation of variables. These transforms are as valuable for homogeneous problems as they are for nonhomogeneous problems (unlike finite Fourier transforms, which are not normally used on homogeneous problems).

We begin by illustrating the continuous nature of "eigenvalues" for infinite spatial problems. Separation of variables $U(x,t) = X(x)T(t)$ in problem (1) yields

$$X'' + \alpha X = 0, \qquad T' + k\alpha T = 0, \qquad \alpha = \text{constant}. \tag{2}$$

The solution for $T(t)$ is $Ce^{-k\alpha t}$, which clearly indicates that α must be nonnegative. We therefore set $\alpha = \lambda^2$, in which case

$$X(x) = A\cos\lambda x + B\sin\lambda x. \tag{3}$$

[Alternatively, we could argue that the solution $X(x)$ of $X'' + \alpha X = 0$ must be bounded as $x \to \pm\infty$, and this would again imply that α be nonnegative.] Thus, any function of the form

$$e^{-k\lambda^2 t}(A\cos\lambda x + B\sin\lambda x)$$

for arbitrary A, B, and λ satisfies PDE (1a). For problems on bounded intervals, boundary conditions determine a discrete set of eigenvalues λ_n and equations expressing A and B in terms of λ_n. Separated functions are then superposed as infinite series. For infinite intervals, no boundary conditions exist, and hence A, B, and λ are all arbitrary. But suppose for the moment that A and B are functions of λ. It is straightforward to show that when the integral

$$U(x,t) = \int_0^\infty e^{-k\lambda^2 t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda \tag{4}$$

is suitably convergent so that integrations with respect to λ may be interchanged with differentiations with respect to x and y, such a function satisfies (1a) (see Exercise 2). This integral is a superposition of separated functions over all values of the parameter λ, and it satisfies (1a) for arbitrary $A(\lambda)$ and $B(\lambda)$. To determine these functions, we demand that (4) satisfy initial condition (1b):

$$f(x) = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda, \qquad -\infty < x < \infty. \tag{5}$$

The solution of (1) is therefore defined by improper integral (4), provided we can find functions $A(\lambda)$ and $B(\lambda)$ satisfying (5). Equation (5) is called the Fourier integral representation of $f(x)$; it is the integral analog of the Fourier series of a periodic function. In Section 7.2 we investigate conditions under which a function has a Fourier integral representation, and we determine formulas for $A(\lambda)$ and $B(\lambda)$.

Exercises 7.1

1. Why does the integral superposition in equation (4) not extend over the interval $-\infty < \lambda < \infty$?
2. Show that if partial derivatives of the improper integral in (4) with respect to x and y may be interchanged with the λ-integration, then $U(x, t)$ satisfies PDE (1a).

7.2 The Fourier Integral Formulas

To state conditions under which the Fourier integral of a function does indeed represent the function, we require the concept of absolute integrability.

Definition 1

A function $f(x)$ is said to be absolutely integrable on the interval $-\infty < x < \infty$ if

$$\int_{-\infty}^{\infty} |f(x)|\, dx$$

converges.

For example, the functions e^{-x^2} and $(x^2 + 1)^{-1}$ are absolutely integrable on $-\infty < x < \infty$, but x, $\sin x$, and $1/\sqrt{|x|}$ are not.

Corresponding to Theorem 2 in Section 2.1 for Fourier series, we have the following result for Fourier integrals.

Theorem 1

If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right and left derivative,

$$\frac{f(x+) + f(x-)}{2} = \int_0^{\infty} [A(\lambda)\cos \lambda x + B(\lambda)\sin \lambda x]\, d\lambda \tag{6a}$$

$$\text{when} \quad A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos \lambda x\, dx, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin \lambda x\, dx. \tag{6b}$$

Equation (6) is called the *Fourier integral formula* for the function $f(x)$. It is proved in Appendix *B*. Since functions that are piecewise smooth must have right and left derivatives, we may state the following corollary to Theorem 1.

Corollary

If $f(x)$ is absolutely integrable on $-\infty < x < \infty$ and is piecewise smooth on every finite interval, then $f(x)$ can be expressed in Fourier integral form (6).

One of the most important functions that we encounter in this chapter is contained in the following example.

Example 1: Find the Fourier integral representation of the Gaussian $f(x) = e^{-kx^2}$, $k > 0$ a constant.

Solution: Since this function and its derivative are continuous, and the function is absolutely integrable, we may write

$$e^{-kx^2} = \int_0^\infty [A(\lambda)\cos \lambda x + B(\lambda)\sin \lambda x]\, dx,$$

where $\quad A(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} e^{-kx^2}\cos \lambda x\, dx \quad$ and $\quad B(\lambda) = \dfrac{1}{\pi}\displaystyle\int_{-\infty}^{\infty} e^{-kx^2}\sin \lambda x\, dx.$

To evaluate $A(\lambda)$, we note that the presence of the exponential e^{-kx^2} permits differentiation under the integral to obtain

$$\frac{dA}{d\lambda} = \frac{1}{\pi}\int_{-\infty}^{\infty} -xe^{-kx^2}\sin \lambda x\, dx.$$

Integration by parts now gives

$$\frac{dA}{d\lambda} = \frac{1}{\pi}\left\{\frac{e^{-kx^2}}{2k}\sin \lambda x\right\}_{-\infty}^{\infty} - \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{e^{-kx^2}}{2k}\lambda \cos \lambda x\, dx = -\frac{\lambda}{2k}A(\lambda).$$

In other words, $A(\lambda)$ must satisfy the ODE

$$\frac{dA}{d\lambda} + \frac{\lambda}{2k}A = 0.$$

An initial condition for this differential equation is

$$A(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-kx^2}\, dx = \frac{1}{\sqrt{k\pi}}$$

(see Exercise 24 for the value of this integral). The solution of this problem is

$$A(\lambda) = \frac{1}{\sqrt{k\pi}} e^{-\lambda^2/(4k)}.$$

Because $e^{-kx^2}\sin \lambda x$ is an odd function, we quickly conclude that $B(\lambda) = 0$. We may therefore write

$$e^{-kx^2} = \int_0^\infty \frac{e^{-\lambda^2/(4k)}}{\sqrt{k\pi}}\cos \lambda x\, d\lambda.$$

An alternative derivation of $A(\lambda)$ using complex contour integrals is given in Exercise 25. ∎

When a function $f(x)$ satisfying the conditions of Theorem 1 (or its corollary) is even, it is obvious that

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x\,dx, \qquad B(\lambda) = 0, \tag{7b}$$

in which case

$$\boxed{\frac{f(x+)+f(x-)}{2} = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda.} \tag{7a}$$

This result is called the *Fourier cosine integral formula.* The function e^{-kx^2} in Example 1 is represented in the form of a Fourier cosine integral.

Example 2: Find an integral representation for the function

$$f(x) = \begin{cases} k(L-|x|)/L & |x| \le L \\ 0 & |x| > L \end{cases}.$$

Solution: Because $f(x)$ is even (Figure 7.1), it has a cosine integral representation, where

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x\,dx = \frac{2}{\pi}\int_0^L \frac{k}{L}(L-x)\cos\lambda x\,dx$$

$$= \frac{2k}{\pi L}\left\{\frac{L-x}{\lambda}\sin\lambda x - \frac{1}{\lambda^2}\cos\lambda x\right\}_0^L = \frac{2k}{\pi L\lambda^2}(1-\cos\lambda L).$$

Since $f(x)$ is continuous, we may write

$$f(x) = \int_0^\infty \frac{2k}{\pi L\lambda^2}(1-\cos\lambda L)\cos\lambda x\,d\lambda = \frac{2k}{\pi L}\int_0^\infty \frac{1-\cos\lambda L}{\lambda^2}\cos\lambda x\,d\lambda.$$

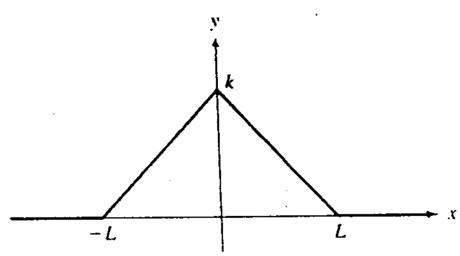


Figure 7.1

■

When $f(x)$ is an odd function, coefficient $A(\lambda) = 0$, and $f(x)$ may be represented by the *Fourier sine integral formula*

$$\boxed{\frac{f(x+)+f(x-)}{2} = \int_0^\infty B(\lambda)\sin\lambda x\,d\lambda,} \tag{8a}$$

where

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\sin\lambda x\,dx. \tag{8b}$$

nple 3: Find an integral representation for the function

$$(\operatorname{sgn} x)e^{-|x|} = \begin{cases} e^{-x} & x > 0 \\ -e^{x} & x < 0 \end{cases}.$$

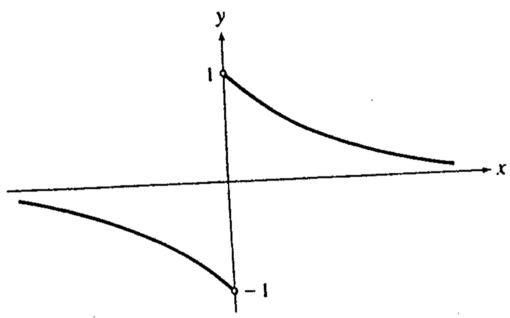
lution: Because $(\operatorname{sgn} x)e^{-|x|}$ is odd (Figure 7.2), it has a sine integral representation, where

$$B(\lambda) = \frac{2}{\pi}\int_0^\infty e^{-x}\sin\lambda x\,dx = \frac{2}{\pi}\left\{\frac{-e^{-x}}{1+\lambda^2}(\sin\lambda x + \lambda\cos\lambda x)\right\}_0^\infty = \frac{2\lambda}{\pi(1+\lambda^2)}.$$

Hence,

$$(\operatorname{sgn} x)e^{-|x|} = \int_0^\infty \frac{2\lambda}{\pi(1+\lambda^2)}\sin\lambda x\,d\lambda,$$

provided the function is assigned the value zero at $x = 0$.



igure 7.2

■

The Fourier sine and cosine integral formulas also provide integral representations for functions that are defined only for $0 < x < \infty$. Indeed, when $f(x)$ is absolutely integrable on $0 < x < \infty$, and $f(x)$ is piecewise smooth on every finite interval $0 \le x \le X$, integrals (7) and (8) converge to $[f(x+) + f(x-)]/2$ for $x > 0$. For $x < 0$, they converge to the even and odd extensions of $f(x)$, respectively. At $x = 0$, the Fourier cosine integral converges to $f(0+)$, and the sine integral yields the value zero.

Theorem 1 would seem to eliminate many functions that we might wish to represent in the form of a Fourier integral. For instance, it would be quite reasonable to have a sinusoidal initial temperature distribution $f(x)$ in problem (1). But such a function is not absolutely integrable on $-\infty < x < \infty$; absolutely integrable functions must necessarily have limit zero as $x \to \pm\infty$. Thus, Fourier integrals cannot presently be used to solve problem (1) when $f(x)$ is sinusoidal. "Generalized functions," the class of functions that contain the Dirac delta function as a special case (see Chapter 11) can be used to weaken the condition of absolute integrability. In this chapter, however, we shall maintain this restriction unless otherwise specified and concentrate our attention on how Fourier integrals and Fourier transforms are used to solve problems, rather than attempt to enlarge the class of problems to which the techniques can be applied.

Fourier integral formula (6) can be used, in conjunction with separation of variables, to solve problems with spatial domain $-\infty < x < \infty$ (see Example 4). In many of these problems, Fourier integral (6) reduces to the cosine or sine integral (7) or (8). Additionally, sine and cosine integrals are useful for problems on the semi-infinite domain $0 < x < \infty$ when the boundary condition at $x = 0$ is homogeneous and of Dirichlet or Neumann type. We illustrate this in Examples 5 and 6.

Example 4: Solve heat conduction problem (1) when

$$f(x) = \begin{cases} x(L-x) & 0 \le x \le L \\ 0 & \text{otherwise} \end{cases}.$$

Solution: Separation of variables and superposition lead to solution (4),

$$U(x,t) = \int_0^{\infty} e^{-k\lambda^2 t}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda,$$

where boundary condition (1b) requires (5):

$$f(x) = \int_{-\infty}^{\infty} [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda, \qquad -\infty < x < \infty.$$

Consequently, $A(\lambda)$ and $B(\lambda)$ are coefficients in the Fourier integral representation of $f(x)$, defined by (6b):

$$\begin{aligned} A(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx = \frac{1}{\pi}\int_0^L x(L-x)\cos\lambda x\,dx \\ &= \frac{1}{\pi}\left\{\frac{x(L-x)}{\lambda}\sin\lambda x + \frac{L-2x}{\lambda^2}\cos\lambda x + \frac{2}{\lambda^3}\sin\lambda x\right\}_0^L \\ &= \frac{-L}{\pi\lambda^2}(1+\cos\lambda L) + \frac{2\sin\lambda L}{\pi\lambda^3}; \\ B(\lambda) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx = \frac{1}{\pi}\int_0^L x(L-x)\sin\lambda x\,dx \\ &= \frac{1}{\pi}\left\{\frac{x(x-L)}{\lambda}\cos\lambda x + \frac{L-2x}{\lambda^2}\sin\lambda x - \frac{2}{\lambda^3}\cos\lambda x\right\}_0^L \\ &= \frac{-L}{\pi\lambda^2}\sin\lambda L + \frac{2}{\pi\lambda^3}(1-\cos\lambda L). \end{aligned}$$

Thus,
$$\begin{aligned} U(x,t) = \int_0^{\infty} \frac{e^{-k\lambda^2 t}}{\pi\lambda^3}&([-\lambda L(1+\cos\lambda L) + 2\sin\lambda L]\cos\lambda x \\ &+ [-\lambda L\sin\lambda L + 2(1-\cos\lambda L)]\sin\lambda x)\,d\lambda. \end{aligned}$$ ■

This particular representation is of little practical use. With the Fourier transform of Section 7.3, we derive a simpler representation in Section 7.4. (See also Exercise 5 in this section.)

Example 5: A taut string of semi-infinite length is given an initial displacement $f(x)$, $x > 0$, but no initial velocity. If the end $x = 0$ is free to move vertically for $t > 0$, find an integral representation for subsequent displacements of points on the string.

Solution: The initial boundary value problem for displacements $y(x,t)$ is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad x > 0, \qquad t > 0, \tag{9a}$$

$$y_x(0,t) = 0, \qquad t > 0. \tag{9b}$$

$$y(x,0) = f(x), \qquad x > 0, \tag{9c}$$
$$y_t(x,0) = 0, \qquad x > 0. \tag{9d}$$

Separation of variables $y(x,t) = X(x)T(t)$ on (9a, b, d) leads to the ODEs

$$X'' + \lambda^2 X = 0, \quad x > 0, \qquad T'' + c^2\lambda^2 T = 0, \quad t > 0,$$
$$X'(0) = 0; \qquad\qquad T'(0) = 0.$$

These yield $X(x) = A\cos\lambda x$ and $T(t) = D\cos c\lambda t$. Superposition of separated functions in integral form gives

$$y(x,t) = \int_0^\infty A(\lambda)\cos\lambda x\cos c\lambda t\,d\lambda. \tag{10}$$

Initial condition (9c) requires $A(\lambda)$ to satisfy

$$f(x) = \int_0^\infty A(\lambda)\cos\lambda x\,d\lambda; \tag{11}$$

that is, $A(\lambda)$ is the coefficient in the Fourier cosine integral representation of $f(x)$,

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(x)\cos\lambda x\,dx.$$

If we replace the variable of integration by u and substitute into (10), the solution of problem (9) is

$$\begin{aligned} y(x,t) &= \int_0^\infty \left(\frac{2}{\pi}\int_0^\infty f(u)\cos\lambda u\,du\right)\cos\lambda x\cos c\lambda t\,d\lambda \\ &= \frac{2}{\pi}\int_0^\infty\int_0^\infty f(u)\cos\lambda u\cos\lambda x\cos c\lambda t\,du\,d\lambda. \end{aligned} \tag{12}$$

This is not a particularly useful representation for $y(x,t)$. If we return to equation (10), we can obtain the solution in closed form:

$$\begin{aligned} y(x,t) &= \int_0^\infty A(\lambda)\left(\frac{1}{2}\right)(\cos\lambda(x+ct) + \cos\lambda(x-ct))\,d\lambda \\ &= \frac{1}{2}\int_0^\infty A(\lambda)\cos\lambda(x+ct)\,d\lambda + \frac{1}{2}\int_0^\infty A(\lambda)\cos\lambda(x-ct)\,d\lambda. \end{aligned}$$

But if equation (11) is a representation of $f(x)$, these integrals must represent $f(x+ct)$ and $f(x-ct)$. In other words,

$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)]. \tag{13}$$

Although $f(x)$ is defined only for $x > 0$, the fact that it has been represented in cosine integral form requires that for this solution, it must be extended as an even function. This form for $y(x,t)$ is d'Alembert's representation on the interval $x > 0$. It can be interpreted in exactly the same way as d'Alembert's solution for the finite string in Section 1.7. Geometrically, the position of the string at any given time t is the algebraic sum of one-half the original displacement $f(x)$, $-\infty < x < \infty$, shifted ct units to the right, $[f(x - ct)]/2$, and one-half the same curve shifted ct units to the left, $[f(x + ct)]/2$. From a physical standpoint, initial displacement $f(x)$, $0 < x < \infty$, separates into two equal disturbances $f(x)/2$, one of which travels with speed c to the right, $[f(x - ct)]/2$, and the other of which travels with speed c to the left, $[f(x + ct)]/2$. The left-traveling wave is reflected at $x = 0$ (with no reversal in sign) and combines with what remains of this same wave. A specific example is discussed in Exercise 8. In Exercise 9, the motion of individual particles is examined. ■

Example 6: Find an integral representation for electrostatic potential in the source-free region $0 < x < L$, $y > 0$ (Figure 7.3) when potential along $y = 0$ is zero and potentials along $x = 0$ and $x = L$ are arbitrarily specified functions.

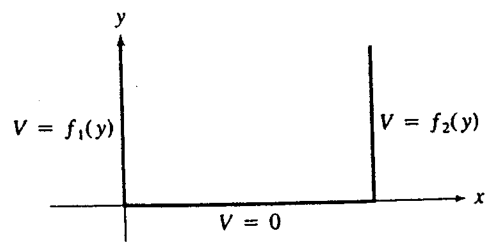


Figure 7.3

Solution: The boundary value problem for $V(x, y)$ is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad 0 < x < L, \qquad y > 0, \tag{14a}$$

$$V(0, y) = f_1(y), \qquad y > 0, \tag{14b}$$

$$V(L, y) = f_2(y), \qquad y > 0, \tag{14c}$$

$$V(x, 0) = 0, \qquad 0 < x < L. \tag{14d}$$

Separation of variables $V(x, y) = X(x)Y(y)$ on (14a, d) leads to the ODEs

$$X'' - \lambda^2 X = 0, \qquad 0 < x < L; \qquad Y'' + \lambda^2 Y = 0, \qquad y > 0,$$
$$Y(0) = 0.$$

Solutions of these are $X(x) = A\cosh\lambda x + B\sinh\lambda x$ and $Y(y) = D\sin\lambda y$, which we superpose in integral form:

$$V(x, y) = \int_0^\infty [A(\lambda)\cosh\lambda x + B(\lambda)\sinh\lambda x]\sin\lambda y\, d\lambda. \tag{15}$$

Boundary conditions (14b, c) require that

$$f_1(y) = \int_0^\infty A(\lambda)\sin\lambda y\, d\lambda, \tag{16a}$$

$$f_2(y) = \int_0^\infty [A(\lambda)\cosh\lambda L + B(\lambda)\sinh\lambda L]\sin\lambda y\, d\lambda. \tag{16b}$$

These are Fourier sine integral representations of $f_1(y)$ and $f_2(y)$; hence, by (8b),

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f_1(y)\sin\lambda y\,dy,$$

$$A(\lambda)\cosh\lambda L + B(\lambda)\sinh\lambda L = \frac{2}{\pi}\int_0^\infty f_2(y)\sin\lambda y\,dy.$$

When these are solved for $A(\lambda)$ and $B(\lambda)$ and substituted into (15), the solution of problem (14) is

$$V(x,y) = \int_0^\infty \left(\frac{2\cosh\lambda x}{\pi}\int_0^\infty f_1(u)\sin\lambda u\,du + \frac{2\sinh\lambda x}{\pi\sinh\lambda L}\int_0^\infty f_2(u)\sin\lambda u\,du \right.$$
$$\left. - \frac{2\sinh\lambda x\cosh\lambda L}{\pi\sinh\lambda L}\int_0^\infty f_1(u)\sin\lambda u\,du\right)\sin\lambda y\,d\lambda$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda y}{\sinh\lambda L}\left(\int_0^\infty\left[f_1(u)(\cosh\lambda x\sinh\lambda L - \sinh\lambda x\cosh\lambda L)\right.\right.$$
$$\left.\left. + f_2(u)\sinh\lambda x\right]\sin\lambda u\,du\right)d\lambda \quad ■ \quad (17)$$
$$= \frac{2}{\pi}\int_0^\infty \frac{\sin\lambda y}{\sinh\lambda L}\left(\int_0^\infty\left[f_1(u)\sinh\lambda(L-x) + f_2(u)\sinh\lambda x\right]\sin\lambda u\right)d\lambda.$$

When the nonhomogeneity in Example 6 is along $y = 0$ instead of $x = 0$ and $x = L$, Fourier integrals are not needed. This is illustrated in the next example.

ple 7: Solve the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad 0 < x < L, \qquad y > 0, \tag{18a}$$
$$V(0,y) = 0, \qquad y > 0, \tag{18b}$$
$$V(L,y) = 0, \qquad y > 0, \tag{18c}$$
$$V(x,0) = f(x), \qquad 0 < x < L. \tag{18d}$$

ion: Separation of variables $V(x,y) = X(x)Y(y)$ on (18a–c) leads to

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L, \qquad Y'' - \lambda^2 Y = 0, \qquad y > 0.$$
$$X(0) = 0 = X(L);$$

Eigenfunctions of the Sturm-Liouville system are $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$, and the corresponding solutions for $Y(y)$ are

$$Y(y) = Ae^{-n\pi y/L} + Be^{n\pi y/L}.$$

For the solution to remain bounded for large y, we must set $B = 0$, in which case superposition of separated functions gives

$$V(x,y) = \sum_{n=1}^{\infty} A_n e^{-n\pi y/L} X_n(x). \tag{19}$$

The boundary condition along $y = 0$ requires that

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x), \qquad 0 < x < L,$$

and therefore

$$A_n = \int_0^L f(x) X_n(x)\, dx.$$

Exercises 7.2

Do the exercises in Part D first.

Part A—Heat Conduction

1. Solve Example 4 if the rod occupies only the region $x \geq 0$ and the end $x = 0$ of the rod is held at temperature zero.
2. Solve Exercise 1 if the end $x = 0$ of the rod is insulated.
3. Repeat Example 4 if $f(x) = e^{-ax^2}$, $a > 0$ constant.
4. (a) Show that the solution of problem (1) can be expressed in the form

$$U(x, t) = \frac{1}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} e^{-k\lambda^2 t} f(u) \cos \lambda(u - x)\, du\, d\lambda.$$

   (b) Formally interchange orders of integration and use the result of Example 1 to replace this iterated integral with the single integral

$$U(x, t) = \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(u) e^{-(u-x)^2/(4kt)}\, du.$$

5. Use the result of Exercise 4 to simplify the solution to Example 4.
6. Use the technique of Exercise 4 to solve problem (1) on the semi-infinite interval $x \geq 0$ if the end $x = 0$ of the rod is held at temperature 0°C.
7. Repeat Exercise 6 if the end $x = 0$ is insulated.

Part B—Vibrations

8. Suppose the initial displacement of the string in Example 5 is

$$f(x) = \begin{cases} k - k|x - a|/\varepsilon & |x - a| \leq \varepsilon \\ 0 & |x - a| > \varepsilon \end{cases},$$

where a, ε, and k are positive constants with $a > 2\varepsilon$.

   (a) Use the geometric interpretation of solution (13) as the superposition of $f(x)/2$ shifted ct units to the left and right to draw the position of the string for the following times:

   (i) $\varepsilon/(2c)$ (ii) ε/c (iii) $(2a - \varepsilon)/(2c)$
   (iv) a/c (v) $(2a + \varepsilon)/(2c)$ (vi) $(a + \varepsilon)/c$.

   Describe the position of the string for $t > (a + \varepsilon)/c$.

(b) What difference to the analysis in (a) occurs if the physical interpretation of left- and right-traveling waves is used?

9. (a) Discuss the motion of the end of the string in Example 5.
   (b) Discuss the motion of other points on the string.
10. (a) Repeat Example 5 if end $x = 0$ of the string is held fixed on the x-axis.
    (b) Discuss the motion of points on the string.
11. Repeat Exercise 10(a) if the string has no initial displacement but is given an initial velocity $g(x)$, where $g(0) = 0$.

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

12. Solve Example 6 if the boundary condition along $y = 0$ is homogeneous Neumann.
13. (a) Prove that for $0 \le r < 1$,

$$\sum_{n=1}^{\infty} \frac{r^{2n-1}}{2n-1} \sin(2n-1)\theta = \frac{1}{2}\text{Tan}^{-1}\left(\frac{2r\sin\theta}{1-r^2}\right).$$

    (b) Use this result to find a closed-form solution to Example 7 when $f(x) = V_0 =$ constant.

Part D—General Results

In Exercises 14–18, find the Fourier integral representation of the function. Sketch a graph of the function to which the integral converges.

14. $f(x) = e^{-a|x|}$, $a > 0$ constant
15. $f(x) = H(x-a) - H(x-b)$, $b > a$ constants. $H(x-a)$ is the *Heaviside unit step function*, defined as

$$H(x-a) = \begin{cases} 0 & x < a \\ 1 & x > a \end{cases}.$$

16. $f(x) = \begin{cases} (b/a)(a-|x|) & |x| < a \\ 0 & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants
17. $f(x) = \begin{cases} b(a^2 - x^2)/a^2 & |x| < a \\ 0 & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants
18. $f(x) = e^{-ax}H(x)$, $a > 0$ constant
19. What is the Fourier cosine integral for the function $f(x) = e^{-kx^2}$ $(k > 0)$, defined only for $x \ge 0$?

In Exercises 20–23, $f(x)$ is defined only for $x \ge 0$. Find its Fourier sine and cosine integral representations. To what does each integral converge at $x = 0$?

20. $f(x) = H(x-a) - H(x-b)$, $b > a > 0$ constants
21. $f(x) = \begin{cases} (b/a)(a-|x-c|) & |x-c| < a \\ 0 & |x-c| > a \end{cases}$, a, b, and c all positive constants with $c > a > 0$
22. $f(x) = e^{-ax}\cos bx$, $a > 0$, $b > 0$ constants
23. $f(x) = e^{-ax}\sin bx$, $a > 0$, $b > 0$ constants

24. To evaluate

$$I = \int_{-\infty}^{\infty} e^{-kx^2}\,dx = 2\int_0^{\infty} e^{-kx^2}\,dx,$$

we write

$$\frac{I^2}{4} = \left(\int_0^{\infty} e^{-kx^2}\,dx\right)\left(\int_0^{\infty} e^{-ky^2}\,dy\right) = \int_0^{\infty}\int_0^{\infty} e^{-k(x^2+y^2)}\,dy\,dx$$

and transform the double integral into polar coordinates. Show that $I = \sqrt{\pi/k}$.

25. In this exercise we use complex contour integrals to evaluate $A(\lambda)$ in Example 1.

(a) Transform the complex combination of real integrals

$$I = \int_{-\infty}^{\infty} e^{-kx^2} e^{i\lambda x}\,dx$$

by means of $z = x - i\lambda/(2k)$ into the contour integral

$$I = e^{-\lambda^2/(4k)} \int_C e^{-kz^2}\,dz$$

along the line $\operatorname{Im}(z) = -i\lambda/(2k)$.

(b) Use the contour integral

$$\oint_{C'} e^{-kz^2}\,dz,$$

where C' is the rectangle in Figure 7.4, to find I.

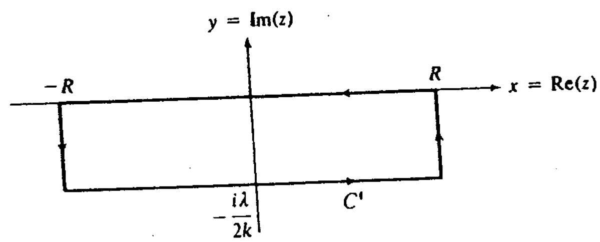


Figure 7.4

(c) Take real and imaginary parts of I to find $A(\lambda)$ and $B(\lambda)$.

7.3 Fourier Transforms

In Section 7.2 we used separation of variables to solve homogeneous problems on infinite and semi-infinite intervals. In this section we develop Fourier transforms in order to handle nonhomogeneities. We shall also find that the transforms yield solutions to homogeneous problems that are often simpler than those obtained by separation of variables. We begin with the transform associated with Fourier integral (6) for a function $f(x)$ absolutely integrable on $-\infty < x < \infty$. To obtain the transform, we express integral (6a) in complex form, reminiscent of the complex form for Fourier series (see Exercise 27 in Section 2.1):

$$\begin{aligned}\frac{f(x+)+f(x-)}{2} &= \int_0^\infty \left[A(\lambda)\left(\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}\right)+B(\lambda)\left(\frac{e^{i\lambda x}-e^{-i\lambda x}}{2i}\right)\right]d\lambda \\ &= \int_0^\infty \left[e^{i\lambda x}\left(\frac{A(\lambda)-iB(\lambda)}{2}\right)+e^{-i\lambda x}\left(\frac{A(\lambda)+iB(\lambda)}{2}\right)\right]d\lambda \\ &= \int_0^\infty e^{i\lambda x}\left(\frac{A(\lambda)-iB(\lambda)}{2}\right)d\lambda + \int_0^{-\infty} e^{i\lambda x}\left(\frac{A(-\lambda)+iB(-\lambda)}{2}\right)(-d\lambda) \\ &= \int_0^\infty C(\lambda)e^{i\lambda x}\,d\lambda + \int_{-\infty}^0 C(\lambda)e^{i\lambda x}\,d\lambda = \int_{-\infty}^\infty C(\lambda)e^{i\lambda x}\,d\lambda,\end{aligned}$$

where
$$C(\lambda) = \begin{cases}[A(\lambda)-iB(\lambda)]/2 & \lambda > 0\\ [A(-\lambda)+iB(-\lambda)]/2 & \lambda < 0\end{cases}$$

But using (6b), we may write, for $\lambda > 0$,

$$C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\cos\lambda x\,dx - \frac{i}{2\pi}\int_{-\infty}^\infty f(x)\sin\lambda x\,dx = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}\,dx,$$

and for $\lambda < 0$,

$$\begin{aligned}C(\lambda) &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)\cos(-\lambda x)\,dx + \frac{i}{2\pi}\int_{-\infty}^\infty f(x)\sin(-\lambda x)\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}\,dx.\end{aligned}$$

If, as has been our custom, we define, or redefine if necessary, $f(x)$ as the average value of left- and right-hand limits at any point of discontinuity, we have shown that Fourier integral (6) may be expressed in the complex form

$$f(x) = \int_{-\infty}^\infty C(\lambda)e^{i\lambda x}\,d\lambda, \tag{20a}$$

where
$$C(\lambda) = \frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-i\lambda x}\,dx. \tag{20b}$$

A somewhat more critical analysis of the improper integrals leading to (20a) indicates that the integral should be taken in the sense of Cauchy's principal value,

$$f(x) = \lim_{R\to\infty}\int_{-R}^R C(\lambda)e^{i\lambda x}\,dx \tag{20c}$$

(see Exercise 23). We shall continue to write (20a) for brevity, but if convergence difficulties arise, we shall replace (20a) with (20c).

It is clear that by redefining $C(\lambda)$, we could also write

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^\infty C(\lambda)e^{i\lambda x}\,d\lambda, \tag{21a}$$

where

$$C(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx, \tag{21b}$$

or

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} C(\lambda)e^{i\lambda x}\,d\lambda \tag{22a}$$

with

$$C(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}\,dx. \tag{22b}$$

Any of the pairs (20), (21), or (22) can be used to define the Fourier transform; we pick (21) simply because it involves the factor 2π only in the latter stages of applications. It is customary to use ω in place of λ for Fourier transforms.

Definition 2

The Fourier transform of a function $f(x)$ is defined as

$$\boxed{\mathscr{F}\{f(x)\} = \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx.} \tag{23a}$$

The associated inverse transform is

$$\boxed{\mathscr{F}^{-1}\{\tilde{f}(\omega)\} = f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{i\omega x}\,d\omega.} \tag{23b}$$

The transform of $f(x)$ exists if the function is piecewise smooth on every finite interval and absolutely integrable on $-\infty < x < \infty$. Once again, we point out that (23b) should be interpreted as Cauchy's principal value,

$$f(x) = \frac{1}{2\pi} \lim_{R\to\infty} \int_{-R}^{R} \tilde{f}(\omega)e^{i\omega x}\,d\omega. \tag{23c}$$

Definition 2 should be compared with equation (3) in Chapter 6 for the finite Fourier transform. Finite Fourier transforms are associated with Sturm-Liouville systems. When $[\lambda_n, y_n(x)]$ are eigenpairs of Sturm-Liouville system (3) in Chapter 4, the finite Fourier transform of a function $f(x)$, defined on $0 < x < L$, is

$$\tilde{f}(\lambda_n) = \int_0^L p(x)f(x)y_n(x)\,dx,$$

and the inverse transform is

$$f(x) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_n)y_n(x).$$

The finite Fourier transform is a sequence of numbers $\{\tilde{f}(\lambda_n)\}$, or a discrete function defined only for integers n; the inverse transform is a superposition over all eigenintegral definitions when solving (initial) boundary value problems. This makes it much simpler to apply Fourier transforms.

We shall state elementary properties of the Fourier transforms, leaving verifications to the exercises, and concentrate on those aspects that are crucial to our discussions of PDEs. All three Fourier transforms and their inverses are linear operators. For example,

$$\mathscr{F}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1 \mathscr{F}\{f_1(x)\} + c_2 \mathscr{F}\{f_2(x)\}, \tag{26}$$

and similar results apply to $\mathscr{F}_C$ and $\mathscr{F}_S$ and their inverses (see Exercise 1).

When a is a real constant

$$\mathscr{F}\{f(ax)\} = \frac{1}{|a|}\mathscr{F}\{f(x)\}_{|\omega/a} \tag{27a}$$

and

$$\mathscr{F}^{-1}\{\tilde{f}(a\omega)\} = \frac{1}{|a|}\mathscr{F}^{-1}\{\tilde{f}(\omega)\}_{|x/a} \tag{27b}$$

(see Exercise 2). Similar properties hold for $\mathscr{F}_S$ and $\mathscr{F}_C$, but in these cases $a > 0$, since $f(x)$ is not defined for $x < 0$.

Translation of a function $f(x)$ along the x-axis by an amount a results in its Fourier transform being multiplied by $e^{-ia\omega}$:

$$\mathscr{F}\{f(x-a)\} = e^{-ia\omega}\mathscr{F}\{f(x)\}, \tag{28a}$$

$$\mathscr{F}^{-1}\{e^{-ia\omega}\tilde{f}(\omega)\} = \mathscr{F}^{-1}\{\tilde{f}(\omega)\}_{|x-a} \tag{28b}$$

(see Exercise 3). Properties similar to (28a) hold for $\mathscr{F}_C$ and $\mathscr{F}_S$ in the case in which $a > 0$, provided $f(x-a)$ is multiplied by the Heaviside function $H(x-a)$:

$$\mathscr{F}_S\{f(x-a)H(x-a)\} = (\cos a\omega)\mathscr{F}_S\{f(x)\} + (\sin a\omega)\mathscr{F}_C\{f(x)\}, \tag{28c}$$

$$\mathscr{F}_C\{f(x-a)H(x-a)\} = (\cos a\omega)\mathscr{F}_C\{f(x)\} - (\sin a\omega)\mathscr{F}_S\{f(x)\}. \tag{28d}$$

[Once again, the presence of $H(x-a)$ is attributable to the fact that $f(x)$ need not be defined for $x < 0$.]

Multiplication of a function $f(x)$ by an exponential e^{-ax} $(a > 0)$ results in a "translation" of its Fourier transform,

$$\mathscr{F}\{e^{-ax}f(x)\} = \tilde{f}(\omega - ai) \tag{29}$$

(see Exercise 4). No such property holds for the sine and cosine transforms.

The following theorem and its corollary eliminate much of the work when Fourier transforms are applied to (initial) boundary value problems.

…orem 2

Suppose $f(x)$ is continuous for $-\infty < x < \infty$ and $f'(x)$ is piecewise continuous on every finite interval. If both functions are absolutely integrable on $-\infty < x < \infty$,

$$\mathscr{F}\{f'(x)\} = i\omega\mathscr{F}\{f(x)\}, \tag{30a}$$

$$\mathscr{F}^{-1}\{i\omega\tilde{f}(\omega)\} = \frac{d}{dx}(\mathscr{F}^{-1}\{\tilde{f}(\omega)\}). \tag{30b}$$

Proof: When integration by parts is used on the definition of $\mathscr{F}\{f'(x)\}$,

$$\mathscr{F}\{f'(x)\} = \int_{-\infty}^{\infty} f'(x)e^{-i\omega x}\,dx = \{f(x)e^{-i\omega x}\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)(-i\omega)e^{-i\omega x}\,dx$$

$$= i\omega \int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx = i\omega\mathscr{F}\{f(x)\}. \blacksquare$$

It is straightforward to extend this result to second derivatives (see the corollary below) and higher-order derivatives (see Exercise 5).

Corollary

Suppose $f(x)$ and $f'(x)$ are continuous for $-\infty < x < \infty$ and $f''(x)$ is piecewise continuous on every finite interval. If all three functions are absolutely integrable on $-\infty < x < \infty$,

$$\mathscr{F}\{f''(x)\} = -\omega^2\mathscr{F}\{f(x)\}, \tag{31a}$$

$$\mathscr{F}^{-1}\{-\omega^2\tilde{f}(\omega)\} = \frac{d^2}{dx^2}(\mathscr{F}^{-1}\{\tilde{f}(\omega)\}). \blacksquare \tag{31b}$$

Results corresponding to (30a) and (31a) for the sine and cosine transforms are

$$\mathscr{F}_C\{f'(x)\} = \omega\mathscr{F}_S\{f(x)\} - f(0+), \tag{31c}$$

$$\mathscr{F}_C\{f''(x)\} = -\omega^2\mathscr{F}_C\{f(x)\} - f'(0+), \tag{31d}$$

$$\mathscr{F}_S\{f'(x)\} = -\omega\mathscr{F}_C\{f(x)\}, \tag{31e}$$

$$\mathscr{F}_S\{f''(x)\} = -\omega^2\mathscr{F}_S\{f(x)\} + \omega f(0+). \tag{31f}$$

The limits in (31c, d, f) allow for the possibility of $f(x)$ being undefined at $x = 0$ (but its right-hand limit must exist).

In applications of Fourier transforms to initial boundary value problems, it is often necessary to find the inverse transform of the product of two functions $\tilde{f}(\omega)$ and $\tilde{g}(\omega)$, both of whose inverse transforms are known; that is, we require $\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\}$, knowing that $\mathscr{F}^{-1}\{\tilde{f}(\omega)\} = f(x)$ and $\mathscr{F}^{-1}\{\tilde{g}(\omega)\} = g(x)$. In Theorem 3 it is shown that

$$\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \int_{-\infty}^{\infty} f(u)g(x-u)\,du.$$

This integral, called the *convolution* of the functions $f(x)$ and $g(x)$, is often given the notation $f * g$:

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)\,du. \tag{32}$$

Theorem 3

Suppose that $f(x)$ and $g(x)$ and their first derivatives are piecewise continuous on every finite interval and that $f(x)$ and $g(x)$ are absolutely integrable on $-\infty < x < \infty$. If either $\tilde{f}(\omega)$ or $\tilde{g}(\omega)$ is absolutely integrable on $-\infty < \omega < \infty$, then

$$\boxed{\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = f * g = \int_{-\infty}^{\infty} f(u)g(x-u)\,du.} \tag{33}$$

roof: Let us assume that $\tilde{g}(\omega)$ is absolutely integrable. [The proof is similar if $\tilde{f}(\omega)$ is absolutely integrable.] By definition (23b),

$$\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}(\omega)e^{i\omega x}\,d\omega,$$

and when we substitute the integral definition of $\tilde{f}(\omega)$,

$$\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} f(u)e^{-i\omega u}\,du\right)\tilde{g}(\omega)e^{i\omega x}\,d\omega.$$

The fact that $f(x)$ and $\tilde{g}(\omega)$ are both absolutely integrable permits us to interchange the order of integration and write

$$\begin{aligned}\mathscr{F}^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} &= \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{g}(\omega)e^{i\omega(x-u)}\,d\omega\right)f(u)\,du\\ &= \int_{-\infty}^{\infty} f(u)g(x-u)\,du.\end{aligned}$$

■

The simplicity of the proof of Theorem 3 is a direct result of the assumption that $\tilde{g}(\omega)$ is absolutely integrable. This condition can be weakened, but because functions that we encounter satisfy this condition, we pursue the discussion no further.

By making a change of variable of integration in (32), it is easily shown that convolutions are symmetric; that is, $f * g = g * f$. Other properties of convolutions are discussed in Exercise 7. An example of convolutions that we encounter in heat conduction problems is

$$\mathscr{F}^{-1}(\tilde{f}(\omega)e^{-k\omega^2 t}),$$

where $\tilde{f}(\omega)$ is the transform of an initial temperature distribution, k is thermal diffusivity, and t is time. According to Example 8, $\mathscr{F}^{-1}(e^{-k\omega^2 t}) = [1/(2\sqrt{k\pi t})]e^{-x^2/(4kt)}$, and hence convolutions yield

$$\begin{aligned}\mathscr{F}^{-1}(\tilde{f}(\omega)e^{-k\omega^2 t}) &= \int_{-\infty}^{\infty} f(u)\frac{1}{2\sqrt{k\pi t}}e^{-(x-u)^2/(4kt)}\,du\\ &= \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}\,du.\end{aligned}$$

The following convolution properties for sine and cosine transforms are verified in Exercise 8.

When $f(x) = \mathscr{F}_C^{-1}\{\tilde{f}(\omega)\}$ and $g(x) = \mathscr{F}_C^{-1}\{\tilde{g}(\omega)\}$,

$$\mathscr{F}_C^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \frac{1}{2}\int_0^{\infty} f(u)[g(x-u) + g(x+u)]\,du, \tag{34a}$$

$$= \frac{1}{2}\int_0^{\infty} g(u)[f(x-u) + f(x+u)]\,du, \tag{34b}$$

provided $f(x)$ and $g(x)$ are extended as even functions for $x < 0$.

When $f(x) = \mathscr{F}_S^{-1}\{\tilde{f}(\omega)\}$ and $g(x) = \mathscr{F}_C^{-1}\{\tilde{g}(\omega)\}$ [note that $\tilde{g}(\omega)$ is a Fourier cosine transform],

$$\mathscr{F}_S^{-1}\{\tilde{f}(\omega)\tilde{g}(\omega)\} = \frac{1}{2}\int_0^{\infty} f(u)[g(x-u) - g(x+u)]\,du, \tag{34c}$$

$$= \frac{1}{2}\int_0^{\infty} g(u)[f(x-u) + f(x+u)]\,du, \tag{34d}$$

provided $f(x)$ and $g(x)$, respectively, are extended as odd and even functions for $x < 0$.

In Section 7.4 we make use of these properties when Fourier transforms are applied to (initial) boundary value problems.

Exercises 7.3

1. Verify that the Fourier transforms and their inverses are linear operators.
2. Verify properties (27) and similar properties for the sine and cosine transforms.
3. Verify properties (28).
4. Verify property (29).
5. (a) Extend the result of Theorem 1 to nth derivatives.
   (b) Verify the transforms in (31).
6. (a) Verify that for $n \geq 1$,

$$\mathscr{F}\{x^n f(x)\} = i^n \frac{d^n}{d\omega^n}\mathscr{F}\{f(x)\}, \tag{35a}$$

$$\mathscr{F}^{-1}\{\tilde{f}^{(n)}(\omega)\} = (-ix)^n \mathscr{F}^{-1}\{\tilde{f}(\omega)\}. \tag{35b}$$

   (b) What are the results corresponding to (35a) for $\mathscr{F}_S$ and $\mathscr{F}_C$ when $n = 1$ and $n = 2$?
7. Verify the following properties for convolution (32):
   (a) $f * g = g * f$ (36a)
   (b) $f * (kg) = (kf) * g = k(f * g)$, $k =$ constant (36b)
   (c) $(f * g) * h = f * (g * h)$ (36c)
   (d) $f * (g + h) = f * g + f * h$ (36d)
8. (a) Verify convolution properties (34a, b) for the Fourier cosine transform.
   (b) Verify convolution properties (34c, d) for the Fourier sine transform.

9. **(a)** Prove that when $f(x)$ is an even function with a Fourier transform,

$$\mathscr{F}\{f(x)\} = 2\mathscr{F}_C\{f(x)\}. \tag{37a}$$

(b) Prove that when $f(x)$ is an odd function with a Fourier transform,

$$\mathscr{F}\{f(x)\} = -2i\mathscr{F}_S\{f(x)\}. \tag{37b}$$

10. **(a)** Show that when $\tilde{f}(\omega) = \mathscr{F}\{f(x)\}$,

$$\mathscr{F}\{\tilde{f}(x)\} = 2\pi f(-\omega). \tag{38}$$

(b) What are corresponding results for Fourier sine and cosine transforms?

11. **(a)** Show that

$$\mathscr{F}\left(\int_{-\infty}^{x} f(u)\,du\right) = \frac{\tilde{f}(\omega)}{i\omega}, \tag{39a}$$

provided the integral does have a transform, and

$$\mathscr{F}^{-1}\left(\frac{\tilde{f}(\omega)}{\omega}\right) = i\int_{-\infty}^{x} f(u)\,du. \tag{39b}$$

(b) What are corresponding results for the sine and cosine transforms?

In Exercises 12–17, find the Fourier transform of the function.

12. $f(x) = e^{-a|x|}$, $a > 0$ constant

13. $f(x) = x^n e^{-ax} H(x)$, $a > 0$ constant, $n \geq 0$ an integer

14. $f(x) = H(x-a) - H(x-b)$, $b > a$ constants

15. $f(x) = \dfrac{\sin ax}{x}$, $a > 0$ constant. (*Hint:* Use Exercises 10 and 14.)

16. $f(x) = \begin{cases} (b/a)(a-|x|) & |x| < a \\ 0 & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants

17. $f(x) = \begin{cases} b(a^2 - x^2)/a^2 & |x| < a \\ 0 & |x| > a \end{cases}$, $a > 0$, $b > 0$ constants

In Exercises 18–20, find the Fourier sine and cosine transforms of the function.

18. $f(x) = e^{-ax^2}$, $a > 0$ constant. (*Hint:* See Exercises 24 and 25 in Section 7.2.)

19. $f(x) = H(x-a) - H(x-b)$, $b > a > 0$ constants

20. $f(x) = \begin{cases} (b/a)(a-|x-c|) & |x-c| < a \\ 0 & |x-c| > a \end{cases}$, a, b, and c all positive constants with $c > a$

21. The *error function*, $\operatorname{erf}(x)$, is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du.$$

Because this function is increasing for $x > 0$ and $\lim_{x\to\infty}\operatorname{erf}(x) = 1$, it does not have Fourier transforms. The *complementary error function*, $\operatorname{erfc}(x)$, defined by

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-u^2}\,du,$$

does have Fourier transforms. Use properties (31) and Exercise 18 to derive the following results:

(a) $\mathscr{F}_S\{\mathrm{erfc}(ax)\} = (1 - e^{-\omega^2/(4a^2)})/\omega$, $a > 0$ constant

(b) $\mathscr{F}_C(ax\,\mathrm{erfc}(ax) - (1/\sqrt{\pi})e^{-a^2x^2}) = (a/\omega^2)(-1 + e^{-\omega^2/(4a^2)})$, $a > 0$ constant

22. Verify formally each of the following results, often called *Parseval's relations*:

(a) $$\int_{-\infty}^{\infty} \tilde{f}(x)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\tilde{g}(x)\,dx \tag{40a}$$

(b) $$2\pi \int_{-\infty}^{\infty} f(x)g(x)\,dx = \int_{-\infty}^{\infty} \tilde{f}(\omega)\tilde{g}(-\omega)\,d\omega \tag{40b}$$

(c) $$2\pi \int_{-\infty}^{\infty} [f(x)]^2\,dx = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2\,d\omega \tag{40c}$$

23. Verify that improper integral (20a) should be taken in the sense of Cauchy's principal value (20c).

24. When the boundary condition at $x = 0$ for an initial boundary value problem on the semi-infinite interval $x > 0$ is of Robin type, separation of variables leads to the system

$$X'' + \omega^2 X = 0, \quad x > 0,$$
$$-lX' + hX = 0, \quad x = 0,$$
$$X(x) \text{ bounded as } x \to \infty.$$

Eigenfunctions of this system are

$$X_\omega(x) = \frac{1}{\sqrt{1 + [h/(\omega l)]^2}}\left(\cos\omega x + \frac{h}{\omega l}\sin\omega x\right) \tag{41}$$

for arbitrary ω, which we take as positive. Associated therewith is a generalized Fourier integral formula that states that a function $f(x)$ satisfying the conditions of Theorem 1 can be represented in the form

$$\frac{f(x+) + f(x-)}{2} = \frac{2}{\pi}\int_0^\infty G(\omega)X_\omega(x)\,d\omega, \tag{42a}$$

where $$G(\omega) = \int_0^\infty f(x)X_\omega(x)\,dx. \tag{42b}$$

From this formula we define a generalized Fourier transform,

$$\tilde{f}(\omega) = \mathscr{G}\{f(x)\} = \int_0^\infty f(x)X_\omega(x)\,dx, \tag{43a}$$

and an inverse transform,

$$f(x) = \mathscr{G}^{-1}\{\tilde{f}(\omega)\} = \frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)X_\omega(x)\,d\omega. \tag{43b}$$

Find transforms of the following functions:

(a) $f(x) = e^{-ax}$, $a > 0$ constant

(b) $f(x) = H(x - a) - H(x - b)$, $b > a > 0$ constants

The following exercises should be attempted only by readers who are already familiar with the Laplace transform. In these exercises $\mathscr{L}\{f(x)\}$ denotes the Laplace transform of a function $f(x)$.

25. (a) Show that when $f(x)$ is absolutely integrable on $0 < x < \infty$, and $f(x) = 0$ for $x < 0$,

$$\mathscr{F}\{f(x)\} = \mathscr{L}\{f(x)\}_{|s=i\omega}, \tag{44a}$$

$$\mathscr{F}_S\{f(x)\} = -\mathrm{Im}(\mathscr{L}\{f(x)\}_{|s=i\omega}), \tag{44b}$$

$$\mathscr{F}_C\{f(x)\} = \mathrm{Re}(\mathscr{L}\{f(x)\}_{|s=i\omega}). \tag{44c}$$

(b) Use the results in (a) to calculate Fourier transforms for the following:

(i) $f(x)$ in Exercise 14. (ii) $f(x)$ in Exercise 13.

(c) Use the results in (a) to calculate Fourier sine and cosine transforms for the following:

(i) $f(x)$ in Exercise 19. (ii) $f(x)$ in Exercise 20.

26. (a) The inverse result of (44a) can be stated as follows: Suppose that when ω in a function $\tilde{f}(\omega)$ is replaced by $-is$, the function $\tilde{f}(-is)$ has no poles on the imaginary s-axis or in the right half-plane. If $\tilde{f}(-is)$ has an inverse Laplace transform, this is also the inverse Fourier transform of $\tilde{f}(\omega)$,

$$\mathscr{F}^{-1}\{\tilde{f}(\omega)\} = \begin{cases} \mathscr{L}^{-1}\{\tilde{f}(-is)\} & x > 0 \\ 0 & x < 0 \end{cases}. \tag{45}$$

Use (45) to find inverse Fourier transforms for the following:

(i) $\tilde{f}(\omega) = 1/(8 + i\omega)^3$

(ii) $\tilde{f}(\omega) = (b/a)[(1 - e^{-i\omega a})/\omega^2 - ia/\omega]$, $a > 0$, $b > 0$ constants

(b) Can the result in (45) be used to find $\mathscr{F}^{-1}\{(i/\omega)e^{-ia\omega}\}$?

27. (a) Show that when $f(x)$ is absolutely integrable on $-\infty < x < 0$, and $f(x) = 0$ for $x > 0$,

$$\mathscr{F}\{f(x)\} = \mathscr{L}\{f(-x)\}_{|s=-i\omega}. \tag{46}$$

(b) Use (46) to find Fourier transforms for the following:

(i) $f(x) = \begin{cases} -x(x+L) & -L \le x \le 0 \\ 0 & \text{otherwise} \end{cases}$

(ii) $f(x) = e^{cx}[H(x-a) - H(x-b)]$, $a < b < 0$, $c > 0$

28. (a) Let $f(x)$ be a function that has a Fourier transform. Denote by $f^+(x)$ and $f^-(x)$ the right and left halves respectively, of $f(x)$:

$$f^+(x) = \begin{cases} 0 & x < 0 \\ f(x) & x > 0 \end{cases}; \qquad f^-(x) = \begin{cases} f(x) & x < 0 \\ 0 & x > 0 \end{cases}.$$

Show that

$$\mathscr{F}\{f(x)\} = \mathscr{F}\{f^+(x)\} + \mathscr{F}\{f^-(x)\}.$$

(b) Use the result in (a) in conjunction with the results of (44a) and (46) to find Fourier transforms for the following:

(i) $f(x)$ in Exercise 16.

(ii) $f(x) = \sin(ax)[H(x + 2\pi n/a) - H(x - 2\pi n/a)]$, $n > 0$ an integer, $a > 0$.

7.4 Applications of Fourier Transforms to Initial Boundary Value Problems

Fourier transform (23) is an alternative to separation of variables in problems over infinite intervals. We begin with heat conduction problem (1). When we apply Fourier transform (23) to PDE (1a),

$$\int_{-\infty}^{\infty} \frac{\partial U}{\partial t} e^{-i\omega x}\, dx = k \int_{-\infty}^{\infty} \frac{\partial^2 U}{\partial x^2} e^{-i\omega x}\, dx.$$

When we interchange the operations of integration with respect to x and differentiation with respect to t on the left, and use property (31a) for the transform on the right,

$$\frac{d\tilde{U}}{dt} = -k\omega^2 \tilde{U}(\omega, t).$$

We should not forget that use of (31a) assumes that U, $\partial U/\partial x$, and $\partial^2 U/\partial x^2$ are all absolutely integrable, that U and $\partial U/\partial x$ are continuous, and that $\partial^2 U/\partial x^2$ is piecewise continuous on every finite interval. The general solution of this ODE in $\tilde{U}(\omega, t)$ is

$$\tilde{U}(\omega, t) = Ce^{-k\omega^2 t}.$$

The Fourier transform of initial condition (1b) is $\tilde{U}(\omega, 0) = \tilde{f}(\omega)$, and this condition requires that $C = \tilde{f}(\omega)$. Thus,

$$\tilde{U}(\omega, t) = \tilde{f}(\omega) e^{-k\omega^2 t}$$

and

$$U(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(\omega) e^{-k\omega^2 t} e^{i\omega x}\, d\omega. \tag{47a}$$

A much more useful form of the solution, which expresses $U(x, t)$ as a real integral involving $f(x)$, rather than a complex integral in $\tilde{f}(\omega)$, can be obtained with convolutions. Because the inverse transform of $e^{-k\omega^2 t}$ is $1/(2\sqrt{k\pi t})e^{-x^2/(4kt)}$ (see Example 8), convolution property (33) yields

$$\begin{aligned} U(x, t) &= \int_{-\infty}^{\infty} f(u) \frac{1}{2\sqrt{k\pi t}} e^{-(x-u)^2/(4kt)}\, du \\ &= \frac{1}{2\sqrt{k\pi t}} \int_{-\infty}^{\infty} f(u) e^{-(x-u)^2/(4kt)}\, du. \end{aligned} \tag{47b}$$

This form of the solution clearly indicates the dependence of $U(x, t)$ on the initial temperature distribution $f(x)$. It also has another advantage. Because (47b) does not contain the Fourier transform of $f(x)$, it may represent a solution to (1) even when $f(x)$ has no Fourier transform. Indeed, provided $f(x)$ is piecewise continuous on some bounded interval, and continuous and bounded outside this interval, it can be shown that $U(x, t)$ so defined satisfies (1). This is illustrated in the first two special cases that follow on the next page.

Case 1: $f(x) = U_0$**, a constant.**
In this case, we would expect that $U(x,t) = U_0$ for all x and t. That (47b) gives this result is easily demonstrated by setting $v = (x-u)/(2\sqrt{kt})$ and $dv = -du/(2\sqrt{kt})$:

$$U(x,t) = \frac{U_0}{2\sqrt{k\pi t}} \int_{\infty}^{-\infty} e^{-v^2}(-2\sqrt{kt}\,dv) = \frac{U_0}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2}\,dv = U_0$$

(see Exercise 24 in Section 7.2 for the value of this integral). Thus, integral (47b) has given the correct solution in spite of the fact that the function $f(x) = U_0$ does not have a Fourier transform.

Case 2: $f(x) = U_0 H(x)$**.**
In this case, we set $v = (x-u)/(2\sqrt{kt})$ and $dv = -du/(2\sqrt{kt})$ in

$$U(x,t) = \frac{U_0}{2\sqrt{k\pi t}} \int_0^{\infty} e^{-(x-u)^2/(4kt)}\,du$$

to obtain

$$\begin{aligned} U(x,t) &= \frac{U_0}{2\sqrt{k\pi t}} \int_{x/(2\sqrt{kt})}^{-\infty} e^{-v^2}(-2\sqrt{kt}\,dv) \\ &= \frac{U_0}{\sqrt{\pi}} \int_{-\infty}^{x/(2\sqrt{kt})} e^{-v^2}\,dv \\ &= \frac{U_0}{\sqrt{\pi}} \left(\int_{-\infty}^{0} e^{-v^2}\,dv + \int_0^{x/(2\sqrt{kt})} e^{-v^2}\,dv \right) \\ &= \frac{U_0}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} + \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} \right) \right] = \frac{U_0}{2} \left[1 + \operatorname{erf}\left(\frac{x}{2\sqrt{kt}} \right) \right] \end{aligned}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\,du \tag{48}$$

is the error function. This solution indicates how heat that is concentrated in one-half of a rod diffuses into the other half. It indicates, in particular, that temperature at every point in the left half of the rod ($x < 0$) is positive for every $t > 0$. This substantiates our claim in Section 5.6 that heat propagates with infinite speed.

Case 3: $f(x) = x(L-x)$**,** $0 \le x \le L$**, and vanishes otherwise.**
This is the initial temperature distribution in Example 4. In this case, (47b) gives

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}} \int_0^L u(L-u) e^{-(x-u)^2/(4kt)}\,du,$$

an integral representation that is preferable to that in Example 4, principally because it is not improper.

In the following example, heat is generated over the interval $-x_0 \le x \le x_0$ at a constant rate.

Example 10: Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}[H(x + x_0) - H(x - x_0)], \qquad -\infty < x < \infty, \qquad t > 0, \quad \textbf{(49a)}$$

$$U(x, 0) = f(x), \qquad -\infty < x < \infty. \quad \textbf{(49b)}$$

Solution: When we take Fourier transforms of the PDE [and use (31a) and Exercise 14 in Section 7.3],

$$\frac{d\tilde{U}}{dt} = -k\omega^2\tilde{U} + \frac{2k}{\kappa\omega}\sin x_0\omega. \quad \textbf{(50a)}$$

The transform $\tilde{U}(\omega, t)$ must satisfy this ODE subject to the transform of (49b),

$$\tilde{U}(\omega, 0) = \tilde{f}(\omega). \quad \textbf{(50b)}$$

The general solution of (50a) is

$$\tilde{U} = Ce^{-k\omega^2 t} + \frac{2}{\kappa\omega^3}\sin x_0\omega,$$

and condition (50b) requires that

$$\tilde{f}(\omega) = C + \frac{2}{\kappa\omega^3}\sin x_0\omega.$$

Thus,
$$\tilde{U}(\omega, t) = \left(\tilde{f}(\omega) - \frac{2}{\kappa\omega^3}\sin x_0\omega\right)e^{-k\omega^2 t} + \frac{2}{\kappa\omega^3}\sin x_0\omega,$$

and $U(x, t)$ is the inverse transform thereof. According to (33), the inverse transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$ can be expressed as

$$\frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}\,du,$$

and therefore

$$U(x, t) = \frac{1}{2\sqrt{k\pi t}}\int_{-\infty}^{\infty} f(u)e^{-(x-u)^2/(4kt)}\,du - \frac{1}{\kappa\pi}\int_{-\infty}^{\infty}\frac{1}{\omega^3}(1 - e^{-k\omega^2 t})\sin x_0\omega\, e^{i\omega x}\,d\omega. \quad \blacksquare \quad \textbf{(51}$$

Fourier sine and cosine transforms are used to solve problems on the semi-infinit interval $x > 0$ in Examples 11 and 12. The sine transform is applied to problems with Dirichlet boundary condition at $x = 0$. This is because separation of variables on suc a problem leads to the ODE

$$X'' + \omega^2 X = 0, \qquad x > 0, \quad \textbf{(52a}$$

$$X(0) = 0, \quad \textbf{(52b}$$

and the only bounded solution of this equation is $\sin \omega x$. Similarly, on Neumann problems, we use the cosine transform, since separation leads to

$$X'' + \omega^2 X = 0, \qquad x > 0, \tag{53a}$$

$$X'(0) = 0, \tag{53b}$$

the bounded solution of which is $\cos \omega x$.

Example 11: Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad x > 0, \qquad t > 0, \tag{54a}$$

$$y(0, t) = f_1(t), \qquad t > 0, \tag{54b}$$

$$y(x, 0) = f(x), \qquad x > 0, \tag{54c}$$

$$y_t(x, 0) = g(x), \qquad x > 0, \tag{54d}$$

for displacement of a semi-infinite bar (or string) with prescribed motion at its one end, $x = 0$.

Solution: We apply the Fourier sine transform to the PDE and use property (31f) for the transform of $\partial^2 y/\partial x^2$:

$$\frac{d^2 \tilde{y}}{dt^2} = -\omega^2 c^2 \tilde{y}(\omega, t) + \omega c^2 f_1(t).$$

Thus, the Fourier sine transform $\tilde{y}(\omega, t)$ of $y(x, t)$ must satisfy the ODE

$$\frac{d^2 \tilde{y}}{dt^2} + \omega^2 c^2 \tilde{y} = \omega c^2 f_1(t) \tag{55a}$$

subject to transforms of initial conditions (54c, d):

$$\tilde{y}(\omega, 0) = \tilde{f}(\omega), \tag{55b}$$

$$\tilde{y}'(\omega, 0) = \tilde{g}(\omega). \tag{55c}$$

Variation of parameters leads to the following general solution of ODE (55a):

$$\tilde{y}(\omega, t) = A \cos c\omega t + B \sin c\omega t + c \int_0^t f_1(u) \sin c\omega(t - u)\, du.$$

Initial conditions (55b, c) require the constants A and B to satisfy

$$\tilde{f}(\omega) = A, \qquad \tilde{g}(\omega) = c\omega B.$$

Hence,

$$\tilde{y}(\omega, t) = \tilde{f}(\omega) \cos c\omega t + \frac{\tilde{g}(\omega)}{c\omega} \sin c\omega t + c \int_0^t f_1(u) \sin c\omega(t - u)\, du, \tag{56}$$

and $y(x, t)$ is the inverse transform of this function:

$$y(x, t) = \frac{2}{\pi} \int_0^\infty \tilde{y}(\omega, t) \sin \omega x \, d\omega. \tag{57}$$

The first term in this integral is

$$\frac{2}{\pi}\int_0^\infty \tilde{f}(\omega)\cos c\omega t \sin \omega x\, d\omega = \frac{2}{\pi}\int_0^\infty \frac{1}{2}\tilde{f}(\omega)(\sin \omega(x-ct) + \sin \omega(x+ct))\, d\omega$$
$$= \frac{1}{2}[f(x-ct) + f(x+ct)],$$

provided $f(x)$ is extended as an odd function.

According to Exercise 19 in Section 7.3, the Fourier cosine transform of $H(x) - H(x-ct)$ is $(\sin c\omega t)/\omega$. Consequently, convolution identity (34d) implies that the inverse sine transform of $[\tilde{g}(\omega)/(c\omega)]\sin c\omega t$ is

$$\frac{1}{2c}\int_0^\infty [H(u) - H(u-ct)][g(x+u) + g(x-u)]\, du$$
$$= \frac{1}{2c}\left(\int_0^{ct} g(x+u)\, du + \int_0^{ct} g(x-u)\, du\right),$$

provided $g(x)$ is extended as an odd function for $x < 0$. When we set $v = x + u$ and $v = x - u$, respectively, in these integrals, the result is

$$\frac{1}{2c}\left(\int_x^{x+ct} g(v)\, dv + \int_x^{x-ct} g(v)(-dv)\right) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(v)\, dv.$$

The inverse transform of the integral term in $\tilde{y}(\omega, t)$ can also be expressed in closed form if we set $v = c(t-u)$:

$$c\int_0^t f_1(u)\sin c\omega(t-u)\, du = c\int_{ct}^0 f_1\left(t - \frac{v}{c}\right)\sin \omega v\left(-\frac{dv}{c}\right)$$
$$= \int_0^{ct} f_1\left(t - \frac{v}{c}\right)\sin \omega v\, dv.$$

But this is the Fourier sine transform of the function

$$\begin{cases} f_1\left(t - \dfrac{x}{c}\right) & x < ct \\ 0 & x > ct \end{cases}$$

or

$$\begin{cases} 0 & t < x/c \\ f_1\left(t - \dfrac{x}{c}\right) & t > x/c \end{cases} = f_1\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right).$$

The solution is therefore

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du + f_1\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right). \quad \text{(58)}$$

The first two terms constitute the d'Alembert part of the solution (see also Section 1.7). The last term is due to the nonhomogeneity at the end $x = 0$; it can be interpreted physically, and this is most easily done when $f(x) = g(x) = 0$. In this case, the complete solution is

$$y(x,t) = f_1\left(t - \frac{x}{c}\right)H\left(t - \frac{x}{c}\right).$$

A point x on the string remains at rest until time $t = x/c$, when it begins to execute the same motion as the end $x = 0$. The time x/c taken by the disturbance to reach x is called *retarded time*. The disturbance $f_1(t)$ at $x = 0$ therefore travels down the string with velocity c.

The solution of the original problem is a superposition of the d'Alembert displacement and the displacement due to the end effect at $x = 0$. ■

Example 12: The temperature of a semi-infinite rod at time $t = 0$ is $f(x)$, $x \geq 0$. For time $t > 0$, heat is added to the rod uniformly over the end $x = 0$ at a variable rate $f_1(t)$ W/m^2. Find the temperature in the rod.

Solution: The initial boundary value problem for temperature $U(x,t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0, \tag{59a}$$

$$U_x(0,t) = -\kappa^{-1} f_1(t), \qquad t > 0, \tag{59b}$$

$$U(x,0) = f(x), \qquad x > 0. \tag{59c}$$

When we apply the Fourier cosine transform to the PDE and use property (31d),

$$\frac{d\tilde{U}}{dt} = -k\omega^2 \tilde{U}(\omega,t) + \kappa^{-1} f_1(t).$$

Thus, the Fourier cosine transform $\tilde{U}(\omega,t)$ of $U(x,t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\omega^2 \tilde{U} = \kappa^{-1} f_1(t) \tag{60a}$$

subject to the transform of (59c),

$$\tilde{U}(\omega,0) = \tilde{f}(\omega). \tag{60b}$$

The general solution of (60a) is

$$\tilde{U}(\omega,t) = Ce^{-k\omega^2 t} + \frac{1}{\kappa}\int_0^t e^{-k\omega^2(t-u)} f_1(u)\,du,$$

and condition (60b) requires that $\tilde{f}(\omega) = C$. Consequently,

$$\tilde{U}(\omega,t) = \tilde{f}(\omega)e^{-k\omega^2 t} + \frac{1}{\kappa}\int_0^t e^{-k\omega^2(t-u)} f_1(u)\,du, \tag{61}$$

and the required temperature is the inverse transform of $\tilde{U}(\omega,t)$,

$$U(x,t) = \frac{2}{\pi}\int_0^\infty \tilde{U}(\omega,t)\cos\omega x\,d\omega. \tag{62}$$

The first term in this integral is the inverse cosine transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$. According to Exercise 18 in Section 7.3, the Fourier cosine transform of e^{-ax^2} is $(1/2)\sqrt{\pi/a}\,e^{-\omega^2/(4a)}$, or, conversely, the inverse Fourier cosine transform of $e^{-k\omega^2 t}$ is $1/(\sqrt{k\pi t})e^{-x^2/(4kt)}$. Convolution property (34a) therefore gives the inverse cosine transform of $\tilde{f}(\omega)e^{-k\omega^2 t}$ as

$$\frac{1}{2}\int_0^\infty f(u)\frac{1}{\sqrt{k\pi t}}(e^{-(x-u)^2/(4kt)} + e^{-(x+u)^2/(4kt)})\,du$$
$$= \frac{1}{2\sqrt{k\pi t}}\int_0^\infty f(u)(e^{-(x-u)^2/(4kt)} + e^{-(x+u)^2/(4kt)})\,du.$$

Finally,

$$U(x,t) = \frac{1}{2\sqrt{k\pi t}}\int_0^\infty f(u)(e^{-(x-u)^2/(4kt)} + e^{-(x+u)^2/(4kt)})\,du + \frac{2}{\kappa\pi}\int_0^\infty \left(\int_0^t e^{-k\omega^2(t-u)} f_1(u)\,du\right)\cos\omega x\,d\omega. \quad ■ \quad (63)$$

Example 13: Solve Laplace's equation for the quarter-plane $x > 0$, $y > 0$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad x > 0, \qquad y > 0, \tag{64a}$$

subject to the boundary conditions

$$V(0, y) = g(y), \qquad y > 0, \tag{64b}$$
$$V_y(x, 0) = f(x), \qquad x > 0. \tag{64c}$$

Solution: Superposition can be used to express $V(x, y)$ as the sum of functions $V_1(x, y)$ and $V_2(x, y)$, satisfying

$$\frac{\partial^2 V_1}{\partial x^2} + \frac{\partial^2 V_1}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \tag{65a}$$
$$V_1(0, y) = g(y), \quad y > 0, \tag{65b}$$
$$\frac{\partial V_1(x, 0)}{\partial y} = 0, \quad x > 0; \tag{65c}$$

$$\frac{\partial^2 V_2}{\partial x^2} + \frac{\partial^2 V_2}{\partial y^2} = 0, \quad x > 0, \quad y > 0, \tag{66a}$$
$$V_2(0, y) = 0, \quad y > 0, \tag{66b}$$
$$\frac{\partial V_2(x, 0)}{\partial y} = f(x), \quad x > 0. \tag{66c}$$

To find $V_1(x, y)$, we apply Fourier cosine transform (24a) (with respect to y) to PDE (65a) and use property (31d):

$$\frac{d^2 \tilde{V}_1}{dx^2} - \omega^2 \tilde{V}_1(x, \omega) = 0, \qquad x > 0. \tag{67a}$$

This transform function $\tilde{V}_1(x, \omega)$ is also subject to

$$\tilde{V}_1(0, \omega) = \tilde{g}(\omega). \tag{67b}$$

The general solution of (67a) is

$$\tilde{V}_1(x, \omega) = Ae^{\omega x} + Be^{-\omega x}.$$

For $\tilde{V}_1(x, \omega)$ to remain bounded as $x \to \infty$, A must be zero, and the boundary condition $\tilde{V}_1(0, \omega) = \tilde{g}(\omega)$ then implies that $B = \tilde{g}(\omega)$. Hence,

$$\tilde{V}_1(x, \omega) = \tilde{g}(\omega)e^{-\omega x}. \tag{68}$$

To invert this transform, we first recall from Example 9 that the Fourier cosine transform of e^{-ay} is $a/(\omega^2 + a^2)$. The result of Exercise 10(b) in Section 7.3 implies, then, that the Fourier cosine transform of $a/(y^2 + a^2)$ is $(\pi/2)e^{-a\omega}$. In other words, the inverse cosine transform of $e^{-\omega x}$ is $(2/\pi)x/(y^2 + x^2)$. Convolution property (34b) now gives

$$V_1(x, y) = \frac{1}{2}\int_0^\infty g(u)\left(\frac{2}{\pi}\right)\left(\frac{x}{(y-u)^2 + x^2} + \frac{x}{(y+u)^2 + x^2}\right) du$$
$$= \frac{x}{\pi}\int_0^\infty g(u)\left(\frac{1}{x^2 + (y-u)^2} + \frac{1}{x^2 + (y+u)^2}\right) du. \tag{69}$$

Taking Fourier sine transforms with respect to x in order to find $V_1(x, y)$ leads to a nonhomogeneous ODE in $\tilde{V}_1(\omega, y)$ that is more difficult to solve.

To find $V_2(x, y)$, we apply Fourier sine transform (25a) (with respect to x) to PDE (66a) and use property (31f):

$$-\omega^2\tilde{V}_2(\omega, y) + \frac{d^2\tilde{V}_2}{dy^2} = 0. \tag{70a}$$

The transform of (66c) requires that

$$\frac{d\tilde{V}_2(\omega, 0)}{dy} = \tilde{f}(\omega). \tag{70b}$$

The general solution of (70a) is

$$\tilde{V}_2(\omega, y) = Ae^{\omega y} + Be^{-\omega y}.$$

For $\tilde{V}_2(\omega, y)$ to remain bounded as $y \to \infty$, A must be zero, and the boundary condition on $\tilde{V}_2$ then implies that $B = -\tilde{f}(\omega)/\omega$. Hence,

$$\tilde{V}_2(\omega, y) = -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y} \tag{71}$$

and

$$V_2(x, y) = \frac{2}{\pi}\int_0^\infty -\frac{\tilde{f}(\omega)}{\omega}e^{-\omega y}\sin \omega x\, d\omega. \tag{72}$$

The final solution is therefore $V(x, y) = V_1(x, y) + V_2(x, y)$. ■

Exercises 7.4

Part A—Heat Conduction

1. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x, t), \qquad -\infty < x < \infty, \qquad t > 0,$$
$$U(x, 0) = f(x), \qquad -\infty < x < \infty.$$

(b) Simplify the solution in (a) in the case that $g(x,t) \equiv 0$ and

(i) $f(x) = \begin{cases} 1 & |x| < a \\ 0 & |x| > a \end{cases}$; (ii) $f(x) = \begin{cases} 0 & |x| < a \\ 1 & |x| > a \end{cases}$.

2. (a) Use the Fourier sine transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$
$$U(0,t) = \bar{U} = \text{constant}, \qquad t > 0,$$
$$U(x,0) = 0, \qquad x > 0.$$

(*Hint:* See Exercise 21 in Section 7.3 when inverting the transform.)

(b) Comment on the possibility of using the transformation $W = U - \bar{U}$ to remove the nonhomogeneity from the boundary condition.

3. Use the Fourier cosine transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$
$$U_x(0,t) = -\kappa^{-1}Q_0 = \text{constant}, \qquad t > 0,$$
$$U(x,0) = 0, \qquad x > 0.$$

(*Hint:* See Exercise 21 in Section 7.3 when inverting the transform.)

4. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \qquad x > 0, \qquad t > 0,$$
$$U(0,t) = f_1(t), \qquad t > 0,$$
$$U(x,0) = f(x), \qquad x > 0.$$

(b) Simplify the solution in (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 = \text{constant}$.

(c) Simplify the solution in (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = \bar{U} = \text{constant}$. Is it the solution of Exercise 2?

5. (a) Use a Fourier transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \qquad x > 0, \qquad t > 0,$$
$$U_x(0,t) = -\kappa^{-1}f_1(t), \qquad t > 0,$$
$$U(x,0) = f(x), \qquad x > 0.$$

(b) Simplify the solution in (a) when $g(x,t) \equiv 0$, $f_1(t) \equiv 0$, and $f(x) = U_0 = \text{constant}$.

(c) Simplify the solution in (a) when $g(x,t) \equiv 0$, $f(x) \equiv 0$, and $f_1(t) = Q_0 = \text{constant}$. Is it th solution of Exercise 3?

6. Use the Fourier transform of Exercise 24 in Section 7.3 to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$

$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu U_m = \text{constant}, \qquad t > 0,$$

$$U(x,0) = 0, \qquad x > 0.$$

7. Use the Fourier transform of Exercise 24 in Section 7.3 to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \qquad x > 0, \qquad t > 0,$$

$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu f_1(t), \qquad t > 0,$$

$$U(x,0) = f(x), \qquad x > 0.$$

Part B—Vibrations

8. Repeat Example 11 if the Dirichlet boundary condition at $x = 0$ is replaced by the Neumann condition

$$y_x(0,t) = -\tau^{-1}f_1(t), \qquad \tau = \text{constant}.$$

Part C—Potential, Steady-State Heat Conduction, Static Deflection of Membranes

9. Solve Example 6 in Section 7.2 using Fourier transforms.

10. (a) Solve the boundary value problem for potential in the semi-infinite strip $0 \le y \le L'$, $x \ge 0$ when
    (i) potential on $y = 0$ and $y = L'$ is zero and that on $x = 0$ is $f(y)$.
    (ii) potential on $x = 0$ and $y = 0$ is zero and that on $y = L'$ is $g(x)$.
    (iii) potential on $x = 0$ and $y = L'$ is zero and that on $y = 0$ is $g(x)$.
    (iv) potentials on $x = 0$, $y = 0$, and $y = L'$ are $f(y)$, $g_1(x)$, and $g_2(x)$, respectively. [*Hint:* Superpose solutions of the types in (i), (ii), and (iii).]

    (b) Try to solve the problem in (iv) by using
    (i) a Fourier sine transform on x. (ii) a finite Fourier transform on y.

11. A thin plate has edges along $y = 0$, $y = L'$, and $x = 0$ for $0 \le y \le L'$. The other edge is so far to the right that its effect may be considered negligible. Assuming no heat flow in the z-direction, find the steady-state temperature inside the plate (for $x > 0$, $0 < y < L'$) if side $y = 0$ is held at temperature 0°C, side $y = L'$ is insulated, and, along $x = 0$,
    (a) temperature is held at a constant U_0°C.
    (b) heat is added to the plate at a constant rate $Q_0 > 0$ W/m^2 over the interval $0 < y < L'/2$ and extracted at the same rate for $L'/2 < y < L'$.
    (c) heat is transferred to a medium at constant temperature U_m according to Newton's law of cooling.

12. What are the solutions to Exercise 11 if edge $y = 0$ is insulated instead of held at temperature $0°C$?

13. Does the function

$$U(x, y) = \begin{cases} -Q_0 x/\kappa & 0 < y < L'/2 \\ Q_0 x/\kappa & L'/2 < y < L' \end{cases}$$

satisfy the PDE and the boundary conditions on $x = 0, y = 0$, and $y = L'$ in Exercise 11(b)? Why is this not the solution?

14. (a) A uniform charge distribution of density σ coulombs per cubic meter occupies the region bounded by the planes $x = 0, y = 0$, and $x = L$ $(y \geq 0)$. If the planes $x = 0$ and $y = 0$ are kept at zero potential and $x = L$ is maintained at a constant potential V_L, find the potential between the planes using

(i) a finite Fourier transform.

(ii) a transformation to remove the constant nonhomogeneities σ and V_L.

(b) Can we apply a Fourier sine transform with respect to y?

15. If the charge distribution in Exercise 14 is a function of y, $\sigma(y) = e^{-y}$, find the potential between the plates.

16. Solve Exercise 15 when $V_L = 0$, using

(a) a finite Fourier transform. (b) the Fourier sine transform.

17. (a) Show that the Fourier sine transform with respect to x of the solution of the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad x > 0, \qquad y > 0,$$

$$V(0, y) = 0, \qquad y > 0,$$

$$V(x, 0) = f(x), \qquad x > 0,$$

is

$$\tilde{V}(\omega, y) = \tilde{f}(\omega)e^{-\omega y}.$$

(b) Use Example 9 and the result of Exercise 10(b) in Section 7.3 to show that

$$\mathscr{F}_C\left(\frac{y}{x^2 + y^2}\right) = \frac{\pi}{2}e^{-\omega y}, \qquad y > 0.$$

(c) Now use convolution property (34c) to show that

$$V(x, y) = \frac{y}{\pi}\int_0^\infty f(u)\left(\frac{1}{(x - u)^2 + y^2} - \frac{1}{(x + u)^2 + y^2}\right) du.$$

(d) Simplify the solution in (c) when $f(x) \equiv 1$.

18. (a) Use the technique of Exercise 17 to solve Laplace's equation for the upper half-plane subject to the condition that $V(x, 0) = f(x)$. The result is called *Poisson's integral formula for the half-plane*.

(b) What is the solution when $f(x) = H(x)$?

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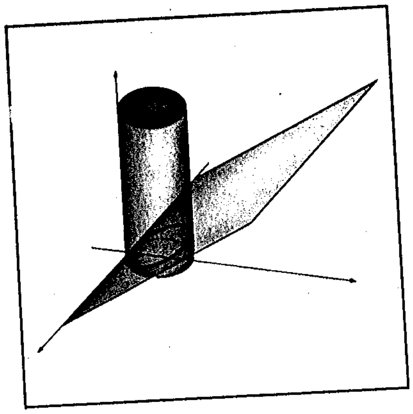


CHAPTER

EIGHT

Special Functions

8.1 Introduction

In Chapters 3–7, discussions have been confined to (initial) boundary value problems expressed in Cartesian coordinates (with the exception of Laplace's equation in polar coordinates in Section 5.3). When separation of variables, finite Fourier transforms, and Laplace transforms are applied to initial boundary value problems in polar, cylindrical, and spherical coordinates, new functions arise, namely, Bessel functions and Legendre functions. In Sections 8.3 and 8.5, we introduce these functions as solutions of ordinary differential equations, as this is how they arise in the context of PDEs. Bessel's differential equation and Legendre's differential equation are homogeneous, second-order, linear differential equations with variable coefficients. The most general form of such an equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0. \tag{1}$$

A point x_0 is said to be an *ordinary point* of this differential equation when the functions $Q(x)/P(x)$ and $R(x)/P(x)$ have convergent Taylor series about x_0; otherwise, x_0 is called a *singular point*. When x_0 is an ordinary point of (1), there exist two independent solutions $y_1(x)$ and $y_2(x)$, both with Taylor series convergent in some interval $|x - x_0| < \delta$. A general solution of the differential equation valid in this interval is $c_1 y_1(x) + c_2 y_2(x)$, where c_1 and c_2 are constants.

When x_0 is a singular point of (1), independent solutions in the form of power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ about x_0 may not exist. In this case, it is customary to search for solutions in the form

$$(x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} a_n(x - x_0)^{n+r}, \tag{2}$$

called *Frobenius* solutions. Solutions of this type may or may not exist, depending on the severity of the singularity. A singular point x_0 is said to be *regular* if

$$(x - x_0)\frac{Q(x)}{P(x)} \quad \text{and} \quad (x - x_0)^2 \frac{R(x)}{P(x)}$$

both have Taylor series expansions about x_0. Otherwise, x_0 is said to be an *irregular* singular point.

When x_0 is a regular singular point of (1), a Frobenius solution (2) always leads to a quadratic equation for the unknown index r. Depending on the nature of the roots of this quadratic, called the *indicial equation*, three situations arise; they are summarized in the following theorem.

Theorem 1

Let r_1 and r_2 be the indicial roots for a Frobenius solution of (1) about a regular singular point x_0. To find linearly independent solutions of (1), it is necessary to consider the cases in which the difference $r_1 - r_2$ is not an integer, is zero, or is a positive integer.

Case 1: $r_1 \neq r_2$ **and** $r_1 - r_2 \neq$ **integer.**
In this case, two linearly independent solutions,

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{with } a_0 = 1 \tag{3a}$$

and

$$y_2(x) = (x - x_0)^{r_2} \sum_{n=0}^{\infty} b_n(x - x_0)^n \quad \text{with } b_0 = 1, \tag{3b}$$

always exist.

Case 2: $r_1 = r_2 = r$.
In this case, one Frobenius solution,

$$y_1(x) = (x - x_0)^r \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{with} \quad a_0 = 1, \tag{4a}$$

is obtained. A second (independent) solution exists in the form

$$y_2(x) = y_1(x)\ln(x - x_0) + (x - x_0)^r \sum_{n=1}^{\infty} A_n(x - x_0)^n, \qquad x > x_0. \tag{4b}$$

Case 3: $r_1 - r_2$ = positive integer.
In this case, one Frobenius solution can always be obtained from the larger root r_1:

$$y_1(x) = (x - x_0)^{r_1} \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad \text{with } a_0 = 1. \tag{5a}$$

The smaller root r_2 may yield no solution, one solution, or a general solution. In the event that it yields no solution, a second (independent) solution can always be found in the form

$$y_2(x) = Ay_1(x)\ln(x - x_0) + (x - x_0)^{r_2} \sum_{n=0}^{\infty} A_n(x - x_0)^n \quad \text{with } A_0 = 1, x > x_0. \tag{5b}$$

In all cases, a general solution of the differential equation is

$$y(x) = c_1 y_1(x) + c_2 y_2(x).$$

8.2 Gamma Function

The gamma function is a generalization of the factorial operation to noninteger values. For $\nu > 0$, it is defined by the convergent improper integral

$$\boxed{\Gamma(\nu) = \int_0^\infty x^{\nu-1} e^{-x}\, dx.} \tag{6}$$

Integration by parts yields the recursive formula

$$\Gamma(\nu + 1) = \nu\Gamma(\nu), \qquad \nu > 0. \tag{7a}$$

With this formula, and the fact that the gamma function is well tabulated in many references[†] for $1 \le \nu < 2$, $\Gamma(\nu)$ can be calculated quickly for all $\nu > 0$. We note, in particular, that

$$\Gamma(1) = \int_0^\infty e^{-x}\, dx = 1, \tag{8}$$

and hence for ν a positive integer,

$$\Gamma(\nu + 1) = \nu!. \tag{9}$$

…mple 1: Evaluate $\Gamma(4.2)$.

…lution: With recursive formula (7a),

$$\Gamma(4.2) = (3.2)\Gamma(3.2) = (3.2)(2.2)\Gamma(2.2)$$
$$= (3.2)(2.2)(1.2)\Gamma(1.2).$$

[†] See, for example, M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions* (New York: Dover, 1965).

But from tables, $\Gamma(1.2) = 0.918169$, and therefore

$$\Gamma(4.2) = (3.2)(2.2)(1.2)(0.918169) = 7.7567.$$

If $v \leq 0$, the improper integral in (1) diverges (at $x = 0$), so the integral cannot be used to define $\Gamma(v)$ for $v \leq 0$. Instead we reverse recursive formula (7a),

$$\Gamma(v) = \frac{\Gamma(v+1)}{v}, \tag{7b}$$

and iterate to define

$$\Gamma(v) = \frac{\Gamma(v+k)}{v(v+1)(v+2)\cdots(v+k-1)}, \tag{10}$$

where k is chosen such that $1 < v + k < 2$. With (10) as the definition of $\Gamma(v)$ for $v < 1$, $\Gamma(v)$ is now defined for all v except $v = 0, -1, -2, \ldots$, and its graph is as shown in Figure 8.1.

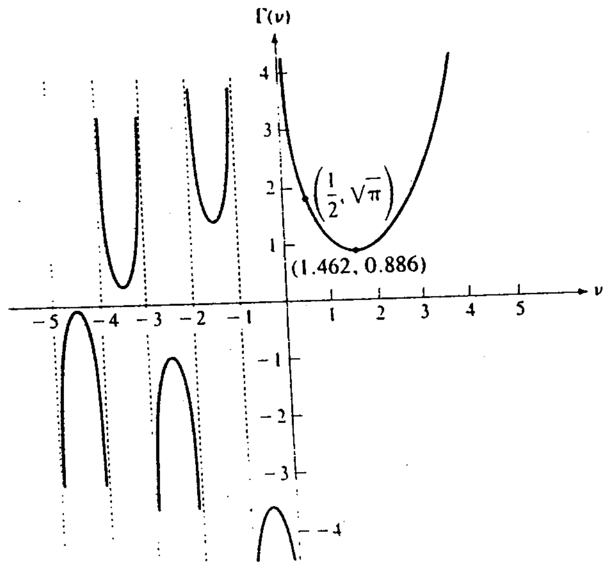


Figure 8.1

Example 2: Evaluate $\Gamma(-2.3)$.

Solution: We use (7b) to write

$$\Gamma(-2.3) = \frac{\Gamma(-1.3)}{-2.3} = \frac{\Gamma(-0.3)}{(-2.3)(-1.3)} = \frac{\Gamma(0.7)}{(-2.3)(-1.3)(-0.3)}$$

$$= \frac{\Gamma(1.7)}{(-2.3)(-1.3)(-0.3)(0.7)}.$$

But from tables, $\Gamma(1.7) = 0.908639$, and therefore

$$\Gamma(-2.3) = \frac{0.908639}{(-2.3)(-1.3)(-0.3)(0.7)} = -1.4471.$$

Exercises 8.2

1. Use tables for the gamma function, or otherwise, to evaluate the following:

(a) $\Gamma(6)$ (b) $\Gamma(3.4)$ (c) $\Gamma(4.16)$

(d) $\Gamma(-0.8)$ (e) $\Gamma(-3.2)$ (f) $\Gamma(-2.44)$

2. Show that

$$\int_0^\infty x^v e^{-\alpha x}\,dx = \frac{\Gamma(v+1)}{\alpha^{v+1}}, \qquad v > -1, \qquad \alpha > 0.$$

3. By definition,

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty x^{-1/2} e^{-x}\,dx.$$

Set $x = y^2$ to show that

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy,$$

and use the result of Exercise 24 in Section 7.2 to obtain $\Gamma(1/2) = \sqrt{\pi}$.

4. Prove that for n a positive integer,

$$\Gamma\left(n + \frac{1}{2}\right) = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}.$$

8.3 Bessel Functions

Bessel functions arise when separation of variables is applied to initial boundary value problems expressed in polar, cylindrical, and spherical coordinates. They are solutions of the linear, homogeneous, second-order ODE

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - v^2)y = 0, \qquad v \ge 0, \tag{11}$$

called *Bessel's differential equation of order v*. When we assume a Frobenius solution $y(x) = \sum_{n=0}^{\infty} a_n x^{n+r}$ ($x = 0$ being a regular singular point for the differential equation), we obtain the indicial equation

$$r^2 - v^2 = 0, \tag{12a}$$

and from the remaining coefficients,

$$a_1[(r+1)^2 - v^2] = 0, \tag{12b}$$

$$a_n[(n+r)^2 - v^2] + a_{n-2} = 0, \qquad n \ge 2. \tag{12c}$$

For the nonnegative indicial root $r = \nu$, we must choose $a_1 = 0$, and iteration of (12c) yields, for $n > 0$,

$$a_{2n+1} = 0, \tag{13a}$$

$$a_{2n} = \frac{(-1)^n a_0}{2^{2n} n!(\nu+1)(\nu+2)\cdots(\nu+n)}. \tag{13b}$$

If we choose $a_0 = 1/[2^\nu \Gamma(\nu+1)]$, the particular solution of Bessel's differential equation corresponding to the indicial root $r = \nu$ is denoted by

$$\boxed{J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}} \tag{14}$$

and is called the *Bessel function of the first kind of order* ν. The ratio test shows that this series converges for all x, and hence $J_\nu(x)$ is a solution of Bessel's differential equation for all x (provided, of course, that x^ν is defined).

When ν is a nonnegative integer, the gamma function can be expressed as a factorial:

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+\nu)!} \left(\frac{x}{2}\right)^{2n}, \qquad \nu = 0, 1, 2, \ldots. \tag{15}$$

Graphs of $J_\nu(x)$ for $\nu = 0, 1, 2$ are shown in Figure 8.2.

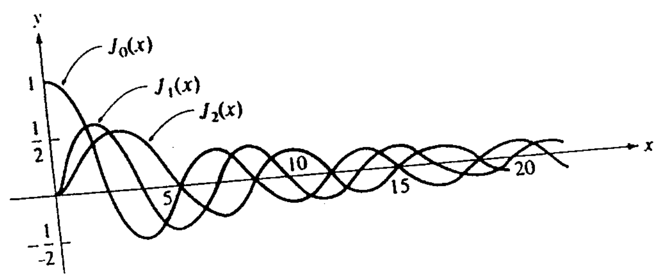


Figure 8.2

To obtain a second independent solution of Bessel's differential equation, three different cases arise, depending on whether ν is not an integer, ν is zero, or ν is a positive integer.

Case 1: ν is not an integer.
We could iterate recursive relation (12c) with the negative indicial root $r = -\nu$ (see Exercise 1), but there is a more direct route to the same solution. We examine the function obtained by replacing ν by $-\nu$ in $J_\nu(x)$:

$$J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n-\nu+1)} \left(\frac{x}{2}\right)^{2n}. \tag{16}$$

It is clear that this function also satisfies Bessel's differential equation (since the differential equation involves only v^2). Further, it is independent of $J_v(x)$, since $J_v(0) = 0$ and $\lim_{x\to 0^+} J_{-v}(x) = \infty$. Thus, if v is not an integer, a general solution of Bessel's differential equation is

$$y(x) = AJ_v(x) + BJ_{-v}(x), \tag{17}$$

which certainly is valid for $x > 0$ (and may or may not be valid for $x < 0$, depending on the value of v). In the special case that v is one-half an odd integer (1/2, 3/2, 5/2, etc.), the indicial roots differ by an integer, and this general solution is generated by the negative indicial root alone. The solutions in this case are called *spherical Bessel functions* (see Exercise 6).

Case 2: $v = 0$.
When $v = 0$, the indicial roots are equal, and a solution of Bessel's differential equation of order zero,

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0, \tag{18}$$

independent of

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \tag{19}$$

can be found in the form

$$y(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} A_n x^n$$

(see Case 2 of Theorem 1 in Section 8.1). Substitution of this solution into Bessel's differential equation leads to

$$2xJ_0' + \sum_{n=1}^{\infty} n(n-1)A_n x^n + \sum_{n=1}^{\infty} nA_n x^n + \sum_{n=1}^{\infty} A_n x^{n+2} = 0.$$

When $J_0'(x)$ is calculated from (19) and the remaining three summations are combined, the result is

$$A_1 x + 4A_2 x^2 + \sum_{n=3}^{\infty} (n^2 A_n + A_{n-2})x^n + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!(n-1)!2^{2n-2}} x^{2n} = 0.$$

Evidently, $A_1 = 0$, and if n is odd, the recursive formula

$$n^2 A_n + A_{n-2} = 0$$

yields $A_{2n+1} = 0$ for $n > 0$. From the terms in x^2, $A_2 = 1/4$, and from those in x^{2n}, $n \geq 2$,

$$(2n)^2 A_{2n} + A_{2n-2} + \frac{(-1)^n}{n!(n-1)!2^{2n-2}} = 0. \tag{20}$$

Iteration of this result gives

$$A_{2n} = \frac{(-1)^{n+1}}{2^{2n}(n!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right), \qquad n \geq 1. \tag{21}$$

With the notation

$$\phi(n) = \sum_{r=1}^{n} \frac{1}{r}, \tag{22}$$

we obtain the independent solution

$$y(x) = J_0(x)\ln x + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\phi(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}, \tag{23}$$

called *Neumann's Bessel function (of the second kind) of order zero.* The series in (23) converges for all x, but the logarithm term restricts the function to $x > 0$. Any linear combination of this solution and $J_0(x)$,

$$aJ_0(x) + by(x),$$

constitutes a general solution of Bessel's differential equation of order zero. Often taken are

$$a = A + \frac{2B}{\pi}(\gamma - \ln 2), \qquad b = \frac{2}{\pi}B,$$

where γ is Euler's constant, defined by

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n\right), \tag{24}$$

and A and B are arbitrary constants. In this case, the general solution of Bessel's differential equation of order zero is

$$y(x) = AJ_0(x) + BY_0(x), \tag{25a}$$

where

$$\boxed{Y_0(x) = \frac{2}{\pi}\left\{J_0(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\phi(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n}\right\}.} \tag{25b}$$

The solution $Y_0(x)$ is called *Weber's Bessel function (of the second kind) of order zero.*

Case 3: ν is a positive integer.
When ν is a positive integer, the indicial roots differ by an integer, and we find that $r = -\nu$ once again yields $J_\nu(x)$ (see Exercise 2). A second solution can be found in the form

$$y(x) = AJ_\nu(x)\ln x + \sum_{n=0}^{\infty} A_n x^{n-\nu} \tag{26}$$

(see Case 3 in Theorem 1 of Section 8.1). Substitution of this series into Bessel's differential equation (11) gives

$$2AxJ_\nu' + \sum_{n=0}^{\infty}(n-\nu)(n-\nu-1)A_n x^{n-\nu} + \sum_{n=0}^{\infty}(n-\nu)A_n x^{n-\nu} + (x^2 - \nu^2)\sum_{n=0}^{\infty} A_n x^{n-\nu} = 0,$$

and, if this equation is multiplied by x^ν and the summations are combined,

$$(1-2\nu)A_1x+\sum_{n=2}^{\infty}[n(n-2\nu)A_n+A_{n-2}]x^n+\sum_{n=0}^{\infty}\frac{(-1)^nA(2n+\nu)}{n!(n+\nu)!2^{2n+\nu-1}}x^{2n+2\nu}=0.$$

Evidently, $A_1=0$, and if n is odd, the recursive formula

$$n(n-2\nu)A_n+A_{n-2}=0$$

requires that $A_{2n+1}=0$ for $n>0$. Since this recursive formula is also valid for even n and $0<n<2\nu$, iteration gives

$$A_{2n}=\frac{A_0(\nu-n-1)!}{2^{2n}n!(\nu-1)!},\qquad 0<n<\nu. \tag{27}$$

From the coefficient of $x^{2\nu}$,

$$A_{2\nu-2}+\frac{A\nu}{\nu!2^{\nu-1}}=0,$$

which can be solved for

$$A=\frac{-A_0}{2^{\nu-1}(\nu-1)!}. \tag{28}$$

From the terms in $x^{2n+2\nu}$, $n>0$,

$$2n(2n+2\nu)A_{2n+2\nu}+A_{2n+2\nu-2}+\frac{(-1)^nA(2n+\nu)}{n!(n+\nu)!2^{2n+\nu-1}}=0.$$

Iteration of this result gives

$$A_{2n+2\nu}=\frac{(-1)^{n+1}A[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!2^{2n+\nu+1}},\qquad n>0, \tag{29a}$$

provided we make the choice

$$A_{2\nu}=\frac{-A\phi(\nu)}{2^{\nu+1}\nu!}. \tag{29b}$$

Finally, then, the solution is

$$y(x)=AJ_\nu(x)\ln x+x^{-\nu}\left(\sum_{n=0}^{\nu-1}\frac{A_0(\nu-n-1)!}{n!(\nu-1)!}\left(\frac{x}{2}\right)^{2n}-\frac{A\phi(\nu)}{2^{\nu+1}\nu!}x^{2\nu}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{(-1)^{n+1}A[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!2^{2n+\nu+1}}x^{2n+2\nu}\right). \tag{30}$$

The particular solution obtained by setting $A_0=-2^{\nu-1}(\nu-1)!$ is

$$y(x)=J_\nu(x)\ln x-\frac{1}{2}\left(\frac{x}{2}\right)^{-\nu}\sum_{n=0}^{\nu-1}\frac{(\nu-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n}$$
$$-\frac{1}{2}\left(\frac{x}{2}\right)^{\nu}\sum_{n=0}^{\infty}\frac{(-1)^n[\phi(n)+\phi(n+\nu)]}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n}. \tag{31}$$

where we have adopted the convention that $\phi(0) = 0$. This solution is called *Neumann's Bessel function (of the second kind) of order* ν. Any linear combination of this solution and $J_\nu(x)$,

$$aJ_\nu(x) + by(x),$$

constitutes a general solution of Bessel's differential equation of order ν, ν a positive integer. Often taken are a and b, as in the $\nu = 0$ case, in which case the general solution of Bessel's differential equation of positive integer order ν is

$$y(x) = AJ_\nu(x) + BY_\nu(x), \tag{32a}$$

where

$$Y_\nu(x) = \frac{2}{\pi}\left\{J_\nu(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] - \frac{1}{2}\left(\frac{x}{2}\right)^{-\nu} \sum_{n=0}^{\nu-1} \frac{(\nu - n - 1)!}{n!}\left(\frac{x}{2}\right)^{2n} - \frac{1}{2}\left(\frac{x}{2}\right)^{\nu} \sum_{n=0}^{\infty} \frac{(-1)^n[\phi(n) + \phi(n+\nu)]}{n!(n+\nu)!}\left(\frac{x}{2}\right)^{2n}\right\}. \tag{32b}$$

The solution $Y_\nu(x)$ is called *Weber's Bessel function (of the second kind) of order* ν.

Notice that in the special case that $\nu = 0$, $Y_\nu(x)$ reduces to $Y_0(x)$, provided we stipulate that the first sum vanish. Graphs of $Y_0(x)$ and $Y_1(x)$ are shown in Figure 8.3.

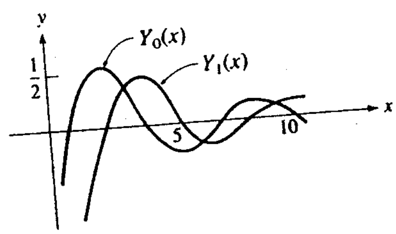


Figure 8.3

For nonnegative integer values of ν, a general solution of Bessel's differential equation has been obtained in the form $y(x) = AJ_\nu(x) + BY_\nu(x)$, and, for noninteger ν, the solution is $y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$. This situation is not completely satisfactory because the second solution is defined differently, depending on whether ν is an integer. To provide uniformity of formalism and numerical tabulation, a form of the second solution valid for all orders is sometimes preferable. Such a form is contained in

$$Y_\nu(x) = \frac{1}{\sin \nu\pi}(J_\nu(x)\cos \nu\pi - J_{-\nu}(x)), \qquad \nu \neq \text{integer}, \tag{33a}$$

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x), \qquad n = \text{integer}. \tag{33b}$$

If ν is not an integer, $Y_\nu(x)$ is simply a linear combination of $J_\nu(x)$ and $J_{-\nu}(x)$, and since $J_\nu(x)$ and $Y_\nu(x)$ must therefore be independent,

$$AJ_\nu(x) + BY_\nu(x) \tag{34}$$

is a general solution of Bessel's differential equation. It can be shown that as ν approaches n, $Y_\nu(x)$ is also given by (25b) or (32b). Consequently, a general solution of Bessel's differential equation (11) is (34), where $J_\nu(x)$ is given by (14) and $Y_\nu(x)$ is given by (33). When ν is an integer, $Y_\nu(x)$ is also given by (25b) or (32b).

Recurrence Relations

Bessel functions of lower orders are well tabulated.[†] With recurrence relations, it is then possible to evaluate Bessel functions of higher orders. We now develop some of these relations.

Using series (14),

$$\begin{aligned}
J_{\nu-1}(x) + J_{\nu+1}(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+2)}\left(\frac{x}{2}\right)^{2n+\nu+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu)}\left(\frac{x}{2}\right)^{2n+\nu-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \frac{1}{\Gamma(\nu)}\left(\frac{x}{2}\right)^{\nu-1} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!\,\Gamma(n+\nu+1)}(-(n+\nu)+n)\left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \frac{\nu}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^{\nu-1} + \sum_{n=1}^{\infty} \frac{(-1)^n \nu}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n \nu}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu-1} \\
&= \frac{2\nu}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)}\left(\frac{x}{2}\right)^{2n+\nu} = \frac{2\nu}{x} J_\nu(x).
\end{aligned}$$

Thus, we have the recurrence relation

$$J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x), \qquad \nu \geq 1, \tag{35}$$

which allows evaluation of Bessel functions of higher order by means of Bessel functions of lower orders.

In addition to this functional relation, there exist many relationships among the Bessel functions and their derivatives. A derivation similar to the above yields

$$2J'_\nu(x) = J_{\nu-1}(x) - J_{\nu+1}(x), \qquad \nu \geq 1, \tag{36}$$

(see Exercise 5). This result combines with recurrence relation (35) to give

$$J'_\nu(x) = -\frac{\nu}{x} J_\nu(x) + J_{\nu-1}(x), \qquad \nu \geq 1, \tag{37}$$

and

$$J'_\nu(x) = \frac{\nu}{x} J_\nu(x) - J_{\nu+1}(x), \qquad \nu \geq 0. \tag{38}$$

[†] Ibid.

Further, multiplication of these equations by x^{ν} and $x^{-\nu}$, respectively, implies that

$$\frac{d}{dx}(x^{\nu}J_{\nu}(x)) = x^{\nu}J_{\nu-1}(x), \qquad \nu \geq 1, \tag{39}$$

and

$$\frac{d}{dx}(x^{-\nu}J_{\nu}(x)) = -x^{-\nu}J_{\nu+1}(x), \qquad \nu \geq 0. \tag{40}$$

The results in (35)–(40) are also valid for $Y_{\nu}(x)$.

Zeros of Bessel Functions

Zeros of Bessel functions play an important role in Sturm-Liouville systems involving Bessel's differential equation (see Section 8.4). We shall show that $J_{\nu}(x)$ has an infinite number of positive zeros and that these zeros cannot be contained in an interval of finite length; that is, there must be arbitrarily large zeros of $J_{\nu}(x)$. [The results will also be valid for $Y_{\nu}(x)$, but our interest is in $J_{\nu}(x)$, and we shall therefore deal directly with $J_{\nu}(x)$.] We begin by changing dependent variables in Bessel's differential equation (11) according to $R = \sqrt{x}\,y(x)$ for $x > 0$ (see Exercise 7). The result is

$$\frac{d^2R}{dx^2} + \left(1 + \frac{1/4 - \nu^2}{x^2}\right)R = 0, \qquad x > 0, \tag{41}$$

and $R(x) = \sqrt{x}\,J_{\nu}(x)$ is a solution of this equation. When $0 < \varepsilon < 1$, the differential equation

$$\frac{d^2R}{dx^2} + \varepsilon^2 R = 0, \qquad x > 0, \tag{42}$$

has general solution $R(x) = A\sin(\varepsilon x + \phi)$, where A and ϕ $(0 < \phi < \pi)$ are arbitrary constants, and this solution has an infinity of positive zeros, $x = (n\pi - \phi)/\varepsilon$ $(n > 0)$.

According to the Sturm comparison theorem in Section 4.3, if $1 + (1/4 - \nu^2)/x^2$ is greater than or equal to ε^2, every solution of (41) has a zero between every consecutive pair of zeros of $A\sin(\varepsilon x + \phi)$. But

$$1 + \frac{1/4 - \nu^2}{x^2} > \varepsilon^2 \tag{43}$$

if, and only if,

$$x^2 > \frac{\nu^2 - 1/4}{1 - \varepsilon^2}.$$

When $0 \leq \nu \leq 1/2$, this is valid for all $x > 0$. When $\nu > 1/2$, this is valid for all $x > x_0$ if $x_0 = \sqrt{(\nu^2 - 1/4)/(1 - \varepsilon^2)}$. In other words, it is always possible to find an interval $x > x_0 \geq 0$ on which inequality (43) is valid. On this interval, then, $R(x)$, and therefore $J_{\nu}(x)$, has at least one zero between every consecutive pair of zeros of $A\sin(\varepsilon x + \phi)$. Since the zeros $x = (n\pi - \phi)/\varepsilon$ of $A\sin(\varepsilon x + \phi)$ become indefinitely large with increasing n, it follows that $J_{\nu}(x)$ must also have arbitrarily large zeros. The first five zeros of $J_0(x)$ and $J_1(x)$ are shown in Figure 8.2.

Exercises 8.3

1. Show that when ν is not an integer, solution (16) of Bessel's differential equation can be obtained from the negative indicial root.
2. Show that when ν is a positive integer, the solution obtained from the negative indicial root $r = -\nu$ is $J_\nu(x)$.
3. Use series (15) to find values of the following, correct to four decimals.

   (a) $J_0(0.4)$ (b) $J_0(1.3)$ (c) $J_1(0.8)$ (d) $J_1(3.6)$

   (e) $J_2(3.6)$ (f) $J_2(6.2)$ (g) $J_3(4.1)$ (h) $J_4(2.9)$
4. Calculate the following, using recurrence relation (35) and tabulated values of J_0 and J_1:

   (a) $J_2(3.6)$ (b) $J_2(6.2)$ (c) $J_3(4.1)$ (d) $J_4(2.9)$
5. Verify identity (36).
6. Bessel functions of the first kind of order $\pm(n + 1/2)$, n a nonnegative integer, are called *spherical Bessel functions*. They can be expressed in terms of sines and cosines.

   (a) Use series (14) and the result of Exercise 4 in Section 8.2 to show that

   $$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x.$$

   (b) Use (39) and (40) to show that

   $$\left(\frac{1}{x}\frac{d}{dx}\right)^n (x^{-\nu}J_\nu(x)) = (-1)^n x^{-\nu-n}J_{\nu+n}(x)$$

   and

   $$\left(\frac{1}{x}\frac{d}{dx}\right)^n (x^{\nu}J_\nu(x)) = x^{\nu-n}J_{\nu-n}(x),$$

   where the left sides mean to apply the operator $x^{-1}d/dx$ successively n times.

   (c) Prove that for $n = 0, 1, 2, \ldots$,

   $$J_{n+1/2}(x) = (-1)^n\sqrt{\frac{2}{\pi}}\,x^{n+1/2}\left(\frac{1}{x}\frac{d}{dx}\right)^n\left(\frac{\sin x}{x}\right),$$

   $$J_{1/2-n}(x) = \sqrt{\frac{2}{\pi}}\,x^{n-1/2}\left(\frac{1}{x}\frac{d}{dx}\right)^n(\sin x).$$
7. Show that the change of dependent variable $R(x) = \sqrt{x}\,y(x)$ transforms Bessel's differential equation into equation (41).
8. Show that the function $e^{x(t-1/t)/2}$ can be expressed as the product of the series

   $$e^{x(t-1/t)/2} = \left(\sum_{k=0}^{\infty}\left(\frac{x}{2}\right)^k\frac{t^k}{k!}\right)\left(\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n\frac{t^{-n}}{n!}\right)$$

   and that the product can be rearranged into the form

   $$e^{x(t-1/t)/2} = J_0(x) + \sum_{m=1}^{\infty}\left[J_m(x)t^m + (-1)^m J_m(x)t^{-m}\right].$$

   Because of this, $e^{x(t-1/t)/2}$ is said to be a *generating function* for $J_m(x)$, m a nonnegative integer.

9. Use integration by parts and the facts that $d[xJ_1(x)]/dx = xJ_0(x)$ and $dJ_0(x)/dx = -J_1(x)$ [see identities (39) and (38)] to derive the reduction formula

$$\int x^n J_0(x)\,dx = x^n J_1(x) + (n-1)x^{n-1}J_0(x) - (n-1)^2 \int x^{n-2}J_0(x)\,dx, \qquad n \geq 2.$$

10. (a) The differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + v^2)y = 0, \qquad v \geq 0,$$

is called ***Bessel's modified differential equation of order*** v. Show that the change of independent variable $z = ix$ reduces it to Bessel's differential equation of order v.

(b) Verify that the function $I_v(x) = i^{-v}J_v(ix)$, called the ***modified Bessel function of the first kind of order*** v, is a solution of Bessel's modified differential equation. Find the Maclaurin series for $I_v(x)$ to illustrate why the factor i^{-v} is included in its definition.

(c) Sketch graphs of $I_0(x)$ and $I_1(x)$ for $x \geq 0$.

(d) A second (linearly independent) solution of the modified equation is called the ***modified Bessel function of the second kind of order*** v. Its definition is analogous to definition (33) for $Y_v(x)$:

$$K_v(x) = \frac{\pi}{2\sin v\pi}[I_{-v}(x) - I_v(x)], \qquad v \neq \text{integer},$$

$$K_n(x) = \lim_{v \to n} K_v(x), \qquad n = \text{integer}.$$

It can be shown that this definition leads to the following expressions for $K_v(x)$ when v is an integer:

$$K_0(x) = -I_0(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \sum_{n=1}^{\infty}\frac{\phi(n)}{(n!)^2}\left(\frac{x}{2}\right)^{2n},$$

$$K_v(x) = (-1)^{v+1}I_v(x)\left[\ln\left(\frac{x}{2}\right) + \gamma\right] + \frac{1}{2}\left(\frac{x}{2}\right)^{-v}\sum_{n=0}^{v-1}\frac{(-1)^n(v-n-1)!}{n!}\left(\frac{x}{2}\right)^{2n}$$

$$+ \frac{1}{2}\left(-\frac{x}{2}\right)^{v}\sum_{n=0}^{\infty}\frac{[\phi(n) + \phi(n+v)]}{n!(v+n)!}\left(\frac{x}{2}\right)^{2n}, \qquad v > 0.$$

Express $K_v(x)$ in terms of $J_v(ix)$ and $Y_v(ix)$ when v is an integer.

(e) Show that $K_v(x)$ is unbounded near $x = 0$ when v is an integer.

8.4 Sturm-Liouville Systems and Bessel's Differential Equation

When separation of variables is applied to initial boundary value problems in polar and cylindrical coordinates (and we shall do this in Chapter 9), both regular and singular Sturm-Liouville systems in the radial coordinate r occur. Regular systems take the form

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{v^2}{r}\right)R = 0, \qquad 0 < r_1 < r < r_2, \tag{44a}$$

$$-l_1R'(r_1) + h_1R(r_1) = 0, \tag{44b}$$

$$l_2R'(r_2) + h_2R(r_2) = 0; \tag{44c}$$

singular systems appear as

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \qquad 0 < r < r_2, \tag{45a}$$

$$l_2 R'(r_2) + h_2 R(r_2) = 0, \tag{45b}$$

where l_1, l_2, h_1, h_2, and ν are nonnegative constants. Eigenvalues have been represented as λ^2, since (44) is a proper Sturm-Liouville system (the eigenvalues of which must be nonnegative).

Properties of system (44) are a straightforward application of the general theory in Section 4.1. Although we make limited use of the results, we include a brief discussion for two reasons. First, it affords us the opportunity to review the theory of Sturm-Liouville systems; second, the notation introduced and some of the results obtained are useful in the discussion of singular system (45).

We begin by making a change of independent variable $x = \lambda r$ in (44a). Since $d/dr = \lambda d/dx$, the resulting differential equation is

$$\lambda \frac{d}{dx}\left(x\frac{dR}{dx}\right) + \left(\lambda x - \frac{\lambda}{x}\nu^2\right)R = 0,$$

or

$$x^2\frac{d^2R}{dx^2} + x\frac{dR}{dx} + (x^2 - \nu^2)R = 0, \tag{46}$$

Bessel's differential equation of order ν. According to equation (34), the general solution of this equation is

$$R = AJ_\nu(x) + BY_\nu(x), \tag{47}$$

where A and B are arbitrary constants and J_ν and Y_ν are Bessel functions of the first and second kind of order ν. Consequently, the general solution of (44a) is

$$R(\lambda, r) = AJ_\nu(\lambda r) + BY_\nu(\lambda r). \tag{48}$$

If we let J'_ν denote the derivative of J_ν with respect to its argument, that is, if

$$J'_\nu(x) = \frac{d}{dx}J_\nu(x),$$

then

$$\frac{d}{dr}J_\nu(\lambda r) = \lambda J'_\nu(\lambda r).$$

With this notation, boundary conditions (44b, c) require that

$$-l_1\lambda[AJ'_\nu(\lambda r_1) + BY'_\nu(\lambda r_1)] + h_1[AJ_\nu(\lambda r_1) + BY_\nu(\lambda r_1)] = 0, \tag{49a}$$

$$l_2\lambda[AJ'_\nu(\lambda r_2) + BY'_\nu(\lambda r_2)] + h_2[AJ_\nu(\lambda r_2) + BY_\nu(\lambda r_2)] = 0. \tag{49b}$$

From (49b),

$$B = -A\left(\frac{\lambda l_2 J'_\nu(\lambda r_2) + h_2 J_\nu(\lambda r_2)}{\lambda l_2 Y'_\nu(\lambda r_2) + h_2 Y_\nu(\lambda r_2)}\right),$$

which, substituted into (49a), yields

$$\frac{-\lambda l_1 J'_\nu(\lambda r_1) + h_1 J_\nu(\lambda r_1)}{-\lambda l_2 J'_\nu(\lambda r_2) + h_2 J_\nu(\lambda r_2)} = \frac{-\lambda l_1 Y'_\nu(\lambda r_1) + h_1 Y_\nu(\lambda r_1)}{\lambda l_2 Y'_\nu(\lambda r_2) + h_2 Y_\nu(\lambda r_2)}. \tag{50}$$

This is the eigenvalue equation, the equation defining eigenvalues of Sturm-Liouville system (44). Because values of λ will depend on the value of ν in differential equation (44a), we denote eigenvalues of (50) by $\lambda_{\nu n}$ $(n = 1, 2, \ldots)$ [although, in fact, $(\lambda_{\nu n})^2$ are the eigenvalues of the Sturm-Liouville system]. Corresponding eigenfunctions can be expressed in the form

$$R_{\nu n}(r) = R(\lambda_{\nu n}, r) = \frac{1}{N}\left(\frac{J_\nu(\lambda_{\nu n} r)}{\lambda_{\nu n} l_2 J'_\nu(\lambda_{\nu n} r_2) + h_2 J_\nu(\lambda_{\nu n} r_2)} - \frac{Y_\nu(\lambda_{\nu n} r)}{\lambda_{\nu n} l_2 Y'_\nu(\lambda_{\nu n} r_2) + h_2 Y_\nu(\lambda_{\nu n} r_2)}\right), \tag{51a}$$

where the normalizing factor N^{-1} is given by

$$N^2 = \int_{r_1}^{r_2} r\left(\frac{J_\nu(\lambda_{\nu n} r)}{\lambda_{\nu n} l_2 J'_\nu(\lambda_{\nu n} r_2) + h_2 J_\nu(\lambda_{\nu n} r_2)} - \frac{Y_\nu(\lambda_{\nu n} r)}{\lambda_{\nu n} l_2 Y'_\nu(\lambda_{\nu n} r_2) + h_2 Y_\nu(\lambda_{\nu n} r_2)}\right)^2 dr. \tag{51b}$$

This integral is evaluated in Exercise 1. We end our discussion of system (44) by noting that according to Theorem 2 in Section 4.2, functions of r can be expressed in terms of the orthonormal eigenfunctions $R_{\nu n}(r)$. Indeed, when $f(r)$ is piecewise smooth for $r_1 \le r \le r_2$, we find that at any point in the open interval $r_1 < r < r_2$,

$$\frac{f(r+) + f(r-)}{2} = \sum_{n=1}^{\infty} c_n R_{\nu n}(r), \tag{52a}$$

where

$$c_n = \int_{r_1}^{r_2} r f(r) R_{\nu n}(r)\, dr. \tag{52b}$$

This is often called the Fourier-Bessel series for $f(r)$. It is important to remember that ν has been fixed throughout this discussion; that is, for a fixed value of $\nu \ge 0$, there is a sequence of eigenvalues $\{\lambda_{\nu n}^2\}$ of (44) together with corresponding orthonormal eigenfunctions $R_{\nu n}(r)$ and an eigenfunction expansion (52). Changing the value of ν results in another set of eigenpairs and a new eigenfunction expansion.

More important for our discussions is singular Sturm-Liouville system (45); we consider it in detail. The system is singular because no boundary condition exists at $r = 0$. Notice also that $q(r) = -\nu^2/r$ is not continuous at $r = 0$.

We are not really justified in denoting eigenvalues of a singular system by λ^2, since we cannot yet be sure that eigenvalues are nonnegative. However, because we shall show shortly that all eigenvalues must indeed be nonnegative, and because use of λ^2 has the immediate advantage of avoiding square roots in subsequent discussions, it is convenient to adopt this notation. Since the coefficient function of $R'(r)$ vanishes at $r = 0$, the corollary to Theorem 1 in Section 4.1 indicates that a boundary condition at $r = 0$ is unnecessary for that theorem. Examination of the proof of the theorem also indicates that continuity of $q(r)$ at $r = 0$ is unnecessary. Consequently, eigenvalues of this singular system are real and corresponding eigenfunctions are orthogonal. As in the discussion of system (44), the change in independent variable $x = \lambda r$ leads to the general solution

$$R = AJ_\nu(\lambda r) + BY_\nu(\lambda r) \tag{53}$$

of (45a). Because $Y_\nu(\lambda r)$ is unbounded near $r = 0$, B must be set equal to zero, and we take

$$R = AJ_\nu(\lambda r). \tag{54}$$

Boundary condition (45b) yields the eigenvalue equation

$$l_2\lambda J'_\nu(\lambda r_2) + h_2 J_\nu(\lambda r_2) = 0, \tag{55}$$

where once again the prime in the first term indicates differentiation of J_ν with respect to its argument.

Because the Sturm-Liouville system is singular, we cannot quote the results of Theorem 2 in Section 4.2; we must verify that the theorem is indeed valid for this system. We first show that there is an infinity of eigenvalues, all of which are positive (except when $\nu = h_2 = 0$, in which case zero is also an eigenvalue). We subdivide our discussion into three cases, depending on whether $l_2 = 0$, $h_2 = 0$, or $h_2 l_2 \neq 0$.

Case 1: $l_2 = 0$.
In this case, we set $h_2 = 1$, and from equation (55) eigenvalues are defined by

$$J_\nu(\lambda r_2) = 0; \tag{56}$$

that is, eigenvalues are the zeros of Bessel function $J_\nu(x)$ divided by r_2. In Section 8.3 we verified that Bessel functions have an infinity of positive zeros.

Case 2: $h_2 = 0$.
In this case, we set $l_2 = 1$, and eigenvalues are defined by the equation

$$J'_\nu(\lambda r_2) = 0; \tag{57}$$

that is, eigenvalues are critical values of Bessel function $J_\nu(x)$ divided by r_2. Since $J_\nu(x)$ has a continuous first derivative, Rolle's theorem from elementary calculus indicates that between every pair of zeros of $J_\nu(x)$, there is at least one point at which its derivative vanishes. Hence, (57) has an infinity of positive solutions. [The first few positive critical values of $J_0(x)$ and $J_1(x)$ are shown in Figure 8.2.]

Case 3: $h_2 l_2 \neq 0$.
In this case, eigenvalues are defined by (55). If we set $x = \lambda r_2$, eigenvalues are roots of the equation

$$Q(x) = xJ'_\nu(x) + \frac{r_2 h_2}{l_2} J_\nu(x) = 0 \tag{58}$$

divided by r_2. When x_j and x_{j+1} are consecutive positive zeros of $J_\nu(x)$, $Q(x)$ has one sign at x_j and the opposite sign at x_{j+1}. Because $Q(x)$ is continuous, it must have at least one zero between x_j and x_{j+1}. It follows, therefore, that equation (58) must have an infinity of positive solutions.

We have shown that each of the eigenvalue equations (55), (56), and (57) has an infinity of positive solutions λ. These solutions define positive eigenvalues λ^2 of the singular Sturm-Liouville system. To show that the system can have no negative eigenvalues, we set $\lambda = i\phi$ (ϕ real and not equal to zero). Equation (55) with $\lambda = i\phi$ then reads

$$il_2\phi J'_\nu(i\phi r_2) + h_2 J_\nu(i\phi r_2) = 0.$$

If we replace J'_ν by J_ν and $J_{\nu+1}$ according to equation (38), this equation becomes

$$[r_2 h_2 + \nu l_2]J_\nu(i\phi r_2) - i\phi r_2 l_2 J_{\nu+1}(i\phi r_2) = 0.$$

We now express $J_\nu(i\phi r_2)$ and $J_{\nu+1}(i\phi r_2)$ in terms of their power series; the result is

$$0 = \left(\frac{i\phi r_2}{2}\right)^\nu \left((r_2 h_2 + \nu l_2) \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+1)} \left(\frac{\phi r_2}{2}\right)^{2n} + \frac{\phi^2 r_2^2 l_2}{2} \sum_{n=0}^{\infty} \frac{1}{n!\,\Gamma(n+\nu+2)} \left(\frac{\phi r_2}{2}\right)^{2n}\right).$$

Because $r_2 h_2 + \nu l_2 \geq 0$ and $l_2 \geq 0$, and both series contain only positive terms, there can be no solution ϕ. Thus, all eigenvalues of (55) must be nonnegative.

We now show that $\lambda = 0$ is an eigenvalue only when $h_2 = \nu = 0$. Since the eigenfunction corresponding to an eigenvalue λ is always $J_\nu(\lambda r)$, it is clear that the eigenfunction will be identically zero if $\lambda = 0$ is an eigenvalue, except when $\nu = 0$ [when $\nu = 0$, the eigenfunction corresponding to $\lambda = 0$ is $J_0(0) = 1$]. Because $J_0(0) \neq 0$ and $J_0'(0) = 0$, it follows that $\lambda = 0$ is an eigenvalue of equation (57) but not of (55) or (56). Thus, there is only one possibility for a zero eigenvalue—both h_2 and ν must be equal to zero.

Only one last point remains to be cleared up. If ν is such that J_ν is defined for negative arguments, then for every positive solution λ of (55), (56), and (57), $-\lambda$ is also a solution. However, the power series expansion for J_ν clearly indicates that the eigenfunction $J_\nu(-\lambda r)$ is, except for a multiplicative constant, identical to $J_\nu(\lambda r)$. Thus, negative solutions of the eigenvalue equations lead to the same eigenvalues λ^2 of the Sturm-Liouville system and the same eigenfunctions.

We have now shown that singular Sturm-Liouville system (45) has an infinity of eigenvalues, all of which are positive (except when $\nu = h_2 = 0$, in which case zero is also an eigenvalue). If we denote these eigenvalues by $\lambda_{\nu n}$ ($n = 1, 2, \ldots$), then from (54), corresponding orthonormal eigenfunctions are

$$R_{\nu n}(r) = R(\lambda_{\nu n}, r) = \frac{1}{N} J_\nu(\lambda_{\nu n} r), \tag{59a}$$

where

$$N^2 = \int_0^{r_2} r[J_\nu(\lambda_{\nu n} r)]^2\, dr. \tag{59b}$$

To avoid direct integration of J_ν, we note that any function R satisfying differential equation (45) also satisfies

$$0 = 2rR'(rR')' + \left(\lambda^2 r - \frac{\nu^2}{r}\right) 2rRR'$$
$$= \frac{d}{dr}(rR')^2 + (\lambda^2 r^2 - \nu^2)\frac{d}{dr}(R^2).$$

Integration of this equation with respect to r from $r = 0$ to $r = r_2$ gives

$$0 = \{(rR')^2 - \nu^2 R^2\}_0^{r_2} + \lambda^2 \int_0^{r_2} r^2 \frac{d}{dr}(R^2)\, dr$$
$$= \{(rR')^2 - \nu^2 R^2\}_0^{r_2} + \lambda^2\{r^2 R^2\}_0^{r_2} - \lambda^2 \int_0^{r_2} 2rR^2\, dr,$$

and when this is solved for the remaining integral,

$$2\lambda^2 \int_0^{r_2} rR^2\,dr = \{(rR')^2 - \nu^2R^2 + \lambda^2 r^2 R^2\}_0^{r_2}.$$

If we now replace λ with $\lambda_{\nu n}$ and R with the corresponding solution $J_\nu(\lambda_{\nu n} r)$ of (45a),

$$2\lambda_{\nu n}^2 \int_0^{r_2} r[J_\nu(\lambda_{\nu n} r)]^2\,dr = r_2^2\lambda_{\nu n}^2[J'_\nu(\lambda_{\nu n} r_2)]^2 + (\lambda_{\nu n}^2 r_2^2 - \nu^2)[J_\nu(\lambda_{\nu n} r_2)]^2,$$

from which

$$\begin{aligned}2N^2 &= 2\int_0^{r_2} r[J_\nu(\lambda_{\nu n} r)]^2\,dr \\ &= r_2^2[J'_\nu(\lambda_{\nu n} r_2)]^2 + \left(r_2^2 - \frac{\nu^2}{\lambda_{\nu n}^2}\right)[J_\nu(\lambda_{\nu n} r_2)]^2 \\ &= r_2^2\left(\frac{-h_2 J_\nu(\lambda_{\nu n} r_2)}{\lambda_{\nu n} l_2}\right)^2 + \left[r_2^2 - \left(\frac{\nu}{\lambda_{\nu n}}\right)^2\right][J_\nu(\lambda_{\nu n} r_2)]^2 \\ &= r_2^2\left[1 - \left(\frac{\nu}{\lambda_{\nu n} r_2}\right)^2 + \left(\frac{h_2}{\lambda_{\nu n} l_2}\right)^2\right][J_\nu(\lambda_{\nu n} r_2)]^2.\end{aligned}$$

Summarizing our results, orthonormal eigenfunctions of singular Sturm-Liouville system (45) are

$$R_{\nu n}(r) = \frac{1}{N} J_\nu(\lambda_{\nu n} r), \tag{60a}$$

where

$$2N^2 = r_2^2\left[1 - \left(\frac{\nu}{\lambda_{\nu n} r_2}\right)^2 + \left(\frac{h_2}{\lambda_{\nu n} l_2}\right)^2\right][J_\nu(\lambda_{\nu n} r_2)]^2 \tag{60b}$$

and eigenvalues $\lambda_{\nu n}$ are defined by the equation

$$l_2 \lambda J'_\nu(\lambda r_2) + h_2 J_\nu(\lambda r_2) = 0. \tag{60c}$$

There are three possible boundary conditions at $r = r_2$, depending on whether $l_2 = 0$, $h_2 = 0$, or $l_2 h_2 \neq 0$. The results for all three cases are listed in Table 8.1.

Table 8.1 *Eigenpairs for Sturm-Liouville System* $(rR')' + (\lambda^2 r - \nu^2/r)R = 0,\ 0 < r < r_2,$ $l_2 R'(r_2) + h_2 R(r_2) = 0$

Condition at $r = r_2$	Eigenvalue Equation	$NR_{\nu n}$	$2N^2$
$h_2 l_2 \neq 0$	$l_2\lambda J'_\nu(\lambda r_2) + h_2 J_\nu(\lambda r_2) = 0$	$J_\nu(\lambda_{\nu n} r)$	$r_2^2\left\{1 - \left(\frac{\nu}{\lambda_{\nu n} r_2}\right)^2 + \left(\frac{h_2}{\lambda_{\nu n} l_2}\right)^2\right\}[J_\nu(\lambda_{\nu n} r_2)]^2$
$h_2 = 0$	$J'_\nu(\lambda r_2) = 0$	$J_\nu(\lambda_{\nu n} r)$	$r_2^2\left\{1 - \left(\frac{\nu}{\lambda_{\nu n} r_2}\right)^2\right\}[J_\nu(\lambda_{\nu n} r_2)]^2$
$l_2 = 0$	$J_\nu(\lambda r_2) = 0$	$J_\nu(\lambda_{\nu n} r)$	$r_2^2[J'_\nu(\lambda_{\nu n} r_2)]^2 = r_2^2[J_{\nu+1}(\lambda_{\nu n} r_2)]^2$

According to the following theorem, piecewise smooth functions of r can be expanded in Fourier Bessel series of these eigenfunctions.

Theorem 2

If a function $f(r)$ is piecewise smooth on the interval $0 \le r \le r_2$, then for each r in $0 < r < r_2$,

$$\frac{f(r+) + f(r-)}{2} = \sum_{n=1}^{\infty} c_n R_{\nu n}(r), \tag{61a}$$

where

$$c_n = \int_0^{r_2} r f(r) R_{\nu n}(r)\, dr. \tag{61b}$$

Example 3: Expand the function $f(r) = r^2$ in terms of the eigenfunctions of Sturm-Liouville system (45) when $r_2 = 1$, $l_2 = 0$, $h_2 = 1$, and $\nu = 0$.

Solution: Orthonormal eigenfunctions of

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \lambda^2 r R = 0, \qquad 0 < r < 1,$$

$$R(1) = 0$$

are

$$R_n(r) = \frac{\sqrt{2}\, J_0(\lambda_n r)}{J_1(\lambda_n)},$$

where eigenvalues λ_n are solutions of $J_0(\lambda) = 0$. The eigenfunction expansion of r^2 is

$$r^2 = \sum_{n=1}^{\infty} c_n R_n(r),$$

where

$$c_n = \int_0^1 r^3 R_n(r)\, dr = \frac{\sqrt{2}}{J_1(\lambda_n)} \int_0^1 r^3 J_0(\lambda_n r)\, dr.$$

To evaluate this integral, we first set $x = \lambda_n r$, in which case

$$c_n = \frac{\sqrt{2}}{J_1(\lambda_n)} \int_0^{\lambda_n} \left(\frac{x}{\lambda_n}\right)^3 J_0(x) \frac{dx}{\lambda_n} = \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)} \int_0^{\lambda_n} x^3 J_0(x)\, dx.$$

We now use the reduction formula in Exercise 9 of Section 8.3:

$$c_n = \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)}\left(\{x^3 J_1(x) + 2x^2 J_0(x)\}_0^{\lambda_n} - 4\int_0^{\lambda_n} x J_0(x)\, dx\right)$$

$$= \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)}\left(\lambda_n^3 J_1(\lambda_n) - 4\int_0^{\lambda_n} \frac{d}{dx}[x J_1(x)]\, dx\right) \quad \text{[see identity (39) with } \nu = 1]$$

$$= \frac{\sqrt{2}}{\lambda_n^4 J_1(\lambda_n)}(\lambda_n^3 J_1(\lambda_n) - 4\lambda_n J_1(\lambda_n))$$

$$= \frac{\sqrt{2}(\lambda_n^2 - 4)}{\lambda_n^3}.$$

Consequently,

$$r^2 = \sum_{n=1}^{\infty} \frac{\sqrt{2}(\lambda_n^2 - 4)}{\lambda_n^3} \frac{\sqrt{2} J_0(\lambda_n r)}{J_1(\lambda_n)}$$

$$= 2 \sum_{n=1}^{\infty} \frac{\lambda_n^2 - 4}{\lambda_n^3 J_1(\lambda_n)} J_0(\lambda_n r), \qquad 0 < r < 1. \qquad \blacksquare$$

Exercises 8.4

1. Use the following argument to evaluate the normalizing factor N^{-1} in (51b).

(a) Show that any solution of (44a) also satisfies

$$\frac{d}{dr}(rR')^2 + (\lambda^2 r^2 - \nu^2)\frac{d}{dr}R^2 = 0.$$

(b) Integrate this equation from r_1 to r_2 to obtain

$$2\lambda^2 \int_{r_1}^{r_2} rR^2\,dr = \{(rR')^2 + (\lambda^2 r^2 - \nu^2)R^2\}_{r_1}^{r_2}.$$

(c) Use boundary conditions (44b, c) to write this expression in the form

$$2\lambda^2 \int_{r_1}^{r_2} rR^2\,dr = [r_2 R(r_2)]^2\left[\lambda^2 - \left(\frac{\nu}{r_2}\right)^2 + \left(\frac{h_2}{l_2}\right)^2\right] - [r_1 R(r_1)]^2\left[\lambda^2 - \left(\frac{\nu}{r_1}\right)^2 + \left(\frac{h_1}{l_1}\right)^2\right].$$

(d) Substitute $\lambda = \lambda_{\nu n}$ and $R = R_{\nu n}$ [from (51a), without the normalizing factor N^{-1}] to obtain an expression for N^{-1}.

2. Expand the function r^ν ($\nu \geq 1$) in terms of the eigenfunctions of Sturm-Liouville system (45) when (a) $l_2 = 0$ and (b) $h_2 = 0$.

3. Expand the function $f(r) = 1$ in terms of the eigenfunctions of Sturm-Liouville system (45) when $\nu = 0$.

4. Show that eigenpairs for the singular Sturm-Liouville system

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \qquad 0 < r < r_2, \qquad \nu > 0,$$

$$l_2 R'(r_2) - h_2 R(r_2) = 0,$$

where $l_2 > 0$ and $h_2 > 0$, are also given in the first line of Table 8.1 (with h_2 replaced by $-h_2$).

5. The singular Sturm-Liouville system

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \lambda^2 r^2 R = 0, \qquad 0 < r < r_2,$$

$$R(r_2) = 0$$

arises when separation of variables is applied to heat conduction problems in a sphere, when temperature is a function of only radial distance r and time.

(a) Use a Frobenius series to obtain the general solution

$$R(r) = \frac{1}{r}(A\cos \lambda r + B\sin \lambda r)$$

of the differential equation.

(b) Find eigenvalues and normalized eigenfunctions of the Sturm-Liouville system.

(c) An alternative way to find eigenfunctions of this Sturm-Liouville system is to make a change of dependent variable $Z(r) = \sqrt{\lambda r}\,R(r)$. Show that this leads to the differential equation

$$\frac{d}{dr}\left(r\frac{dZ}{dr}\right) + \left(\lambda^2 r - \frac{1/4}{r}\right)Z = 0, \qquad 0 < r < r_2,$$

and the solutions $R(r) = (A\cos \lambda r + B\sin \lambda r)/r$.

6. (a) Show that when the boundary condition in Exercise 5 is $R'(r_2) = 0$, eigenvalues are nonnegative solutions of

$$\tan \lambda r_2 = \lambda r_2.$$

(b) Find normalized eigenfunctions.

7. (a) Show that when the boundary condition in Exercise 5 is $l_2 R'(r_2) + h_2 R(r_2) = 0$, eigenvalues are positive solutions of

$$\left(1 - \frac{h_2 r_2}{l_2}\right)\tan \lambda r_2 = \lambda r_2.$$

(b) Find normalized eigenfunctions when $h_2 r_2 / l_2 > 1$.

8. Use the technique of Exercise 5(c) to find normalized eigenfunctions of the singular Sturm-Liouville system

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + (\lambda^2 r^2 - m(m+1))R = 0, \qquad 0 < r < r_2,$$

$$l_2 R'(r_2) + h_2 R(r_2) = 0,$$

where $m \geq 0$ is an integer and $l_2 \geq 0$ and $h_2 \geq 0$. Tabulate the results for the three cases $l_2 = 0$, $h_2 = 0$, and $l_2 h_2 \neq 0$.

8.5 Legendre Functions

Legendre functions arise when separation of variables is applied to (initial) boundary value problems expressed in spherical coordinates. They are solutions of the linear, homogeneous, second-order differential equation

$$(1 - x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0, \tag{62}$$

called *Legendre's differential equation*. If we assume a power series solution $y(x) = \sum_{k=0}^{\infty} a_k x^k$ ($x = 0$ being an ordinary point of the differential equation), we obtain arbitrary a_0 and a_1 and the recurrence relation

$$a_k = -\frac{(n-k+2)(n+k-1)}{k(k-1)}a_{k-2}, \qquad k \geq 2. \tag{63}$$

Iteration of this result leads to the general solution

$$y(x) = a_0\left(1 + \sum_{k=1}^{\infty}(-1)^k\frac{(n-2k+2)\cdots(n-2)n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!}x^{2k}\right)$$
$$+ a_1\left(x + \sum_{k=1}^{\infty}(-1)^k\frac{(n-2k+1)\cdots(n-3)(n-1)(n+2)(n+4)\cdots(n+2k)}{(2k+1)!}x^{2k+1}\right), \tag{64}$$

which converges for $|x| < 1$.

When n is a nonnegative integer, one of these series reduces to a polynomial while the other remains an infinite series. In particular, if n is an even integer, all terms in the first series vanish for $2k > n$, and if n is odd, all terms in the second series vanish for $2k + 1 > n$. Thus, in either case, the solution defines a polynomial of degree n. To express these polynomials compactly, we reverse (63) to write

$$a_{k-2} = -\frac{k(k-1)}{(n-k+2)(n+k-1)}a_k$$

and iterate to obtain

$$a_{n-2k} = \frac{(-1)^k n(n-1)(n-2)\cdots(n-2k+1)}{2^k k!(2n-1)(2n-3)\cdots(2n-2k+1)}a_n. \tag{65}$$

When we choose $a_n = (2n)!/[2^n(n!)^2]$, (65) becomes

$$a_{n-2k} = \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}, \qquad k = 1, 2, \ldots, [n/2], \tag{66}$$

where $[n/2]$ denotes the integer part of $n/2$. With this choice for a_n, the particular polynomial solution of (62) is called the *Legendre polynomial of degree n*, denoted by

$$\boxed{P_n(x) = \sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!}x^{n-2k}.} \tag{67}$$

The first five Legendre polynomials are

$$P_0(x) = 1, \qquad P_1(x) = x, \qquad P_2(x) = \frac{3x^2-1}{2},$$
$$P_3(x) = \frac{5x^3-3x}{2}, \qquad P_4(x) = \frac{35x^4-30x^2+3}{8}.$$

The remaining solution of (62) for n a nonnegative integer is in the form of an infinite series valid for $|x| < 1$. When n is even, and a_1 is chosen as $(-1)^{n/2}2^n[(n/2)!]^2/n!$, the series solution is denoted by

$$Q_n(x) = \frac{(-1)^{n/2}2^n[(n/2)!]^2}{n!}$$
$$\times\left(x + \sum_{k=1}^{\infty}\frac{(-1)^k(n-2k+1)\cdots(n-3)(n-1)(n+2)(n+4)\cdots(n+2k)}{(2k+1)!}x^{2k+1}\right). \tag{68a}$$

When n is odd, and a_0 is set equal to $(-1)^{(n+1)/2}2^{n-1}([(n-1)/2]!)^2/n!$, the series solution is

$$Q_n(x) = \frac{(-1)^{(n+1)/2}2^{n-1}([(n-1)/2]!)^2}{n!} \times \left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k(n-2k+2)\cdots(n-2)n(n+1)(n+3)\cdots(n+2k-1)}{(2k)!} x^{2k}\right). \tag{68b}$$

These solutions are called *Legendre functions of the second kind of order n*. Closed-form representations are discussed in Exercise 10; they are unbounded near $x = \pm 1$.

In summary, the general solution of Legendre's differential equation (62) for n a nonnegative integer is

$$y(x) = AP_n(x) + BQ_n(x), \tag{69}$$

where A and B are arbitrary constants. Legendre polynomials $P_n(x)$ are given by (67), and Legendre functions $Q_n(x)$ of the second kind are defined by (68). Our discussions concentrate on Legendre polynomials.

Generating Function for Legendre Polynomials

When the binomial expansion is applied to the function $(1 - 2xt + t^2)^{-1/2}$,

$$\frac{1}{(1-2xt+t^2)^{1/2}} = 1 + \sum_{m=1}^{\infty} \frac{(1/2)(3/2)\cdots(1/2+m-1)}{m!}(2xt - t^2)^m,$$

and the binomial theorem is then used on $(2xt - t^2)^m$:

$$\frac{1}{(1-2xt+t^2)^{1/2}} = 1 + \sum_{m=1}^{\infty} \frac{(1)(3)(5)\cdots(2m-1)}{2^m m!} \sum_{k=0}^{m} (-1)^k \binom{m}{k} (2x)^{m-k} t^{m+k}.$$

Terms in t^n occur when $k + m = n$, and since k ranges from 0 to m, it follows that the coefficient of t^n is

$$\sum_{m=[(n+1)/2]}^{n} \frac{(1)(3)(5)\cdots(2m-1)}{2^m m!} (-1)^{n-m} \binom{m}{n-m} (2x)^{2m-n}.$$

If we set $k = n - m$ in this summation, the coefficient of t^n is

$$\sum_{k=n-[(n+1)/2]}^{0} \frac{(1)(3)(5)\cdots(2n-2k-1)}{2^{n-k}(n-k)!} (-1)^k \binom{n-k}{k} (2x)^{n-2k},$$

and this immediately reduces to

$$\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!} x^{n-2k};$$

that is,

$$\frac{1}{(1-2xt+t^2)^{1/2}} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/2]} \frac{(-1)^k(2n-2k)!}{2^n k!(n-2k)!(n-k)!} x^{n-2k} \right) t^n. \tag{70}$$

The coefficient of t^n is $P_n(x)$, and we say that $(1 - 2xt + t^2)^{-1/2}$ is a generating function for $P_n(x)$:

$$\frac{1}{\sqrt{1 - 2xt + t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n. \tag{71}$$

Recurrence Relations

When we differentiate (71) with respect to t,

$$(1 - 2xt + t^2)^{-3/2}(x - t) = \sum_{n=0}^{\infty} nP_n(x)t^{n-1}, \tag{72}$$

from which

$$(x - t)\sum_{n=0}^{\infty} P_n(x)t^n = (1 - 2xt + t^2)\sum_{n=0}^{\infty} nP_n(x)t^{n-1}.$$

Equating coefficients of like powers of t gives the recurrence relation

$$(n + 1)P_{n+1}(x) - (2n + 1)xP_n(x) + nP_{n-1}(x) = 0, \qquad n \geq 1, \tag{73}$$

which permits evaluation of Legendre polynomials of higher orders in terms of those of lower orders. Useful relations among the derivatives of Legendre polynomials also exist. Differentiation of (71) with respect to x gives

$$\frac{t}{(1 - 2xt + t^2)^{3/2}} = \sum_{n=0}^{\infty} P_n'(x)t^n,$$

which, together with (72), implies that

$$t\sum_{n=0}^{\infty} nP_n(x)t^{n-1} = (x - t)\sum_{n=0}^{\infty} P_n'(x)t^n. \tag{74}$$

Equating coefficients yields

$$xP_n'(x) - P_{n-1}'(x) - nP_n(x) = 0, \qquad n \geq 1. \tag{75}$$

Differentiation of (73) gives

$$(n + 1)P_{n+1}'(x) - (2n + 1)P_n(x) - (2n + 1)xP_n'(x) + nP_{n-1}'(x) = 0, \qquad n \geq 1. \tag{76}$$

Elimination of $P_n'(x)$ between (75) and (76) yields

$$P_{n+1}'(x) - P_{n-1}'(x) = (2n + 1)P_n(x), \qquad n \geq 1 \tag{77}$$

and, in addition,

$$P_{n+1}'(x) - xP_n'(x) = (n + 1)P_n(x), \qquad n \geq 0. \tag{78}$$

We now show that $P_n(x)$ is a constant multiple of $d^n[(x^2 - 1)^n]/dx^n$. We first note that

$$\frac{d}{dx}(x^2 - 1)^n = 2nx(x^2 - 1)^{n-1}$$

or
$$(x^2-1)\frac{d}{dx}(x^2-1)^n = 2nx(x^2-1)^n.$$

Differentiation of this equation $n+1$ times with Leibniz's rule† gives

$$\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}(x^2-1)\frac{d^{n-k+2}}{dx^{n-k+2}}(x^2-1)^n = 2n\sum_{k=0}^{n+1}\binom{n+1}{k}\frac{d^k}{dx^k}x\frac{d^{n-k+1}}{dx^{n-k+1}}(x^2-1)^n,$$

but only the first three terms on the left and the first two terms on the right do not vanish. When these terms are written out and rearranged,

$$(1-x^2)\frac{d^2}{dx^2}\left(\frac{d^n}{dx^n}(x^2-1)^n\right) - 2x\frac{d}{dx}\left(\frac{d^n}{dx^n}(x^2-1)^n\right) + n(n+1)\left(\frac{d^n}{dx^n}(x^2-1)^n\right) = 0.$$

This equation indicates that the function $d^n[(x^2-1)^n]/dx^n$ satisfies Legendre's differential equation (62). Since the function is a polynomial in x, it follows that

$$P_n(x) = A\frac{d^n}{dx^n}(x^2-1)^n.$$

To obtain the constant A, we equate coefficients of x^n on each side:

$$\frac{(2n)!}{2^n(n!)^2} = A(2n)(2n-1)\cdots(n+1).$$

Thus, $A = 1/(2^n n!)$, and we obtain Rodrigues' formula,

$$\boxed{P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.} \tag{79}$$

Rodrigues' formula is useful in the evaluation of definite integrals involving Legendre's polynomials. In addition, it quickly yields values for $P_n(\pm 1)$. With x^2-1 in factored form, Leibniz's rule gives

$$P_n(\pm 1) = \frac{1}{2^n n!}\left(\frac{d^n}{dx^n}(x^2-1)^n\right)_{|x=\pm 1}$$
$$= \frac{1}{2^n n!}\left(\sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(x+1)^n\frac{d^{n-k}}{dx^{n-k}}(x-1)^n\right)_{|x=\pm 1}.$$

The only term in this summation that does not involve $x-1$ occurs when $k=0$, and therefore

$$P_n(1) = \frac{\binom{n}{0}2^n n!}{2^n n!} = 1. \tag{80a}$$

† Leibniz's rule for the nth derivative of a product is

$$\frac{d^n}{dx^n}[f(x)g(x)] = \sum_{r=0}^{n}\binom{n}{r}\left[\frac{d^r}{dx^r}f(x)\right]\left[\frac{d^{n-r}}{dx^{n-r}}g(x)\right].$$

Similarly, because $k = n$ is the only term without a factor $x + 1$,

$$P_n(-1) = \frac{\binom{n}{n} n!(-2)^n}{2^n n!} = (-1)^n. \tag{80b}$$

Associated Legendre Functions

Legendre's associated differential equation is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left(n(n+1) - \frac{m^2}{1-x^2}\right)y = 0, \tag{81}$$

where m is some given nonnegative integer. When $m = 0$, it reduces to Legendre's differential equation (62). It is straightforward to show (see Exercise 9) that when $y(x)$ is a solution of (62), $(1 - x^2)^{m/2}d^my/dx^m$ is a solution of (81). This means that a general solution of (81) is

$$y(x) = (1 - x^2)^{m/2}\left(A\frac{d^mP_n(x)}{dx^m} + B\frac{d^mQ_n(x)}{dx^m}\right), \tag{82}$$

where $P_n(x)$ are Legendre polynomials and $Q_n(x)$ are Legendre functions of the second kind. The functions

$$P_{mn}(x) = (1 - x^2)^{m/2}\frac{d^mP_n(x)}{dx^m} \tag{83a}$$

and

$$Q_{mn}(x) = (1 - x^2)^{m/2}\frac{d^mQ_n(x)}{dx^m} \tag{83b}$$

are called *associated Legendre functions of degree n and order m of the first and second kind.* Since $P_n(x)$ is a polynomial of degree n, it follows that $P_{mn}(x)$ is nonvanishing only when $n \geq m$.

Exercises 8.5

1. Calculate the first seven Legendre polynomials, using (a) equation (79); (b) equation (67).
2. Show that Legendre polynomials $P_n(x)$ are even when n is even and odd when n is odd.
3. Use $P_0(x) = 1$, $P_1(x) = x$, and recurrence relation (73) to obtain $P_2(x)$, $P_3(x)$, $P_4(x)$, $P_5(x)$, and $P_6(x)$.
4. Prove the following:

   (a) $P_{2n+1}(0) = 0$

   (b) $P_{2n}(0) = \dfrac{(-1)^n(2n)!}{2^{2n}(n!)^2}$

   (c) $P'_{2n}(0) = 0$

   (d) $P'_{2n+1}(0) = \dfrac{(-1)^n(2n+1)!}{2^{2n}(n!)^2}$

   (e) $P'_n(1) = \dfrac{n(n+1)}{2}$

   (f) $P'_n(-1) = \dfrac{(-1)^{n-1}n(n+1)}{2}$

5. Verify the following identities for Legendre's polynomials:

(a) $nP_{n-1}(x) - P'_n(x) + xP'_{n-1}(x) = 0, \quad n > 0$

Hint: Show that the generating function for $P_n(x)$ satisfies

$$t\frac{\partial}{\partial t}\left(\frac{t}{\sqrt{1-2xt+t^2}}\right) + (tx-1)\frac{\partial}{\partial x}\left(\frac{1}{\sqrt{1-2xt+t^2}}\right) = 0.$$

(b) $(1-x^2)P'_n(x) = nP_{n-1}(x) - nxP_n(x), \quad n > 0$

(c) $nP_n(x) = nxP_{n-1}(x) + (x^2-1)P'_{n-1}(x), \quad n > 0$

6. Verify that when $f(x)$ has continuous derivatives of orders up to and including n,

$$\int_{-1}^{1} f(x)P_n(x)\,dx = \frac{(-1)^n}{2^n n!}\int_{-1}^{1} f^{(n)}(x)(x^2-1)^n\,dx.$$

7. Verify the following results:

(a) $\displaystyle\int_{-1}^{1} P_n(x)\,dx = \begin{cases} 2 & n = 0 \\ 0 & n \neq 0 \end{cases}$

(b) $\displaystyle\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & n \neq m \\ \dfrac{2}{2n+1} & n = m \end{cases}$

(*Hint:* Use Exercise 6.)

(c) $\displaystyle\int_{-1}^{1} xP_n(x)P'_n(x)\,dx = \frac{2n}{2n+1}, \quad n \geq 0$

(d) $\displaystyle\int_{-1}^{1} xP_n(x)P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}, \quad n > 0$

(e) $\displaystyle\int_{-1}^{1} P_n(x)P'_{n+1}(x)\,dx = 2, \quad n > 0$

(f) $\displaystyle\int_{-1}^{1} x^m P_n(x)\,dx = \begin{cases} 0 & m < n \\ \dfrac{2^{n+1}(n!)^2}{(2n+1)!} & m = n \\ 0 & m - n > 0 \text{ is odd} \\ \dfrac{2^{n+1}m!\left(\frac{m+n}{2}\right)!}{(m+n+1)!\left(\frac{m-n}{2}\right)!} & m - n > 0 \text{ is even} \end{cases}$

(*Hint:* Use Exercise 6.)

8. Verify that

(a) $\displaystyle\int_{0}^{1} P_n(x)\,dx = \begin{cases} 1 & n = 0 \\ 0 & n > 0 \text{ even} \\ \dfrac{(-1)^{(n-1)/2}(n-1)!}{2^n\left(\frac{n+1}{2}\right)!\left(\frac{n-1}{2}\right)!} & n \text{ odd} \end{cases}$

(b) $$\int_0^1 xP_n(x)\,dx = \begin{cases} 0 & n \geq 3 \text{ odd} \\ 1/2 & n = 0 \\ 1/3 & n = 1 \\ \dfrac{(-1)^{(n-2)/2}(n-2)!}{2^n\left(\dfrac{n-2}{2}\right)!\left(\dfrac{n+2}{2}\right)!} & n \geq 2 \text{ even} \end{cases}$$

9. Verify that when $y(x)$ is a solution of Legendre's differential equation (62), $(1-x^2)^{m/2}d^my/dx^m$ is a solution of Legendre's associated equation (81).

10. (a) Use series (68a, b) to show that

$$Q_0(x) = \text{Tanh}^{-1}\,x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \quad \text{and} \quad Q_1(x) = xQ_0(x) - 1.$$

(b) Assuming that the $Q_n(x)$ also satisfy recurrence relation (73), express $Q_2(x)$, $Q_3(x)$, and $Q_4(x)$ in terms of $Q_0(x)$.

(c) Express $Q_n(x)$ $(n = 2, 3, 4)$ in terms of $Q_0(x)$ and $P_n(x)$.

11. Prove the following recurrence relations for $P_{mn}(x)$:

(a) $P_{m+1,n+1}(x) - P_{m+1,n-1}(x) = (2n+1)\sqrt{1-x^2}\,P_{mn}(x)$

(b) $xP_{m+1,n}(x) - P_{m+1,n-1}(x) = (n-m)\sqrt{1-x^2}\,P_{mn}(x)$

(c) $(n-m+1)P_{m,n+1}(x) - (2n+1)xP_{mn}(x) + (n+m)P_{m,n-1}(x) = 0$

8.6 Sturm-Liouville Systems and Legendre's Differential Equation

When separation of variables is applied to (initial) boundary value problems expressed in spherical coordinates, the following singular Sturm-Liouville system often results:

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \left(\lambda\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \qquad 0 < \phi < \pi, \tag{84}$$

where m is some given nonnegative integer. The system is singular because there are no boundary conditions and also because $q(\phi) = -m^2/\sin\phi$ is not continuous at $\phi = 0$ and $\phi = \pi$. Because the coefficient $\sin\phi$ of $d\Phi/d\phi$ vanishes at $\phi = 0$ and $\phi = \pi$, the corollary to Theorem 1 of Section 4.1 indicates that boundary conditions at $\phi = 0$ and $\phi = \pi$ are unnecessary for that theorem.Examination of the proof of the theorem also indicates that continuity of $q(\phi)$ at $\phi = 0$ and $\phi = \pi$ is not necessary. Consequently, eigenvalues of this singular system are real, and corresponding eigenfunctions are orthogonal.

If we make the change of independent variable $\mu = \cos\phi$, then $d/d\mu = -(\sin\phi)^{-1}d/d\phi$ and (84) is replaced by

$$\frac{d}{d\mu}\left((1-\mu^2)\frac{d\Phi}{d\mu}\right) + \left(\lambda - \frac{m^2}{1-\mu^2}\right)\Phi = 0$$

or

$$(1-\mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + \left(\lambda - \frac{m^2}{1-\mu^2}\right)\Phi = 0, \qquad -1 < \mu < 1, \tag{85}$$

Legendre's associated differential equation. When λ is set equal to $n(n+1)$, where $n \geq m$ is an integer, this equation has general solution

$$\Phi = AP_{mn}(\mu) + BQ_{mn}(\mu), \tag{86}$$

where A and B are arbitrary constants and P_{mn} and Q_{mn} are associated Legendre functions of degree n and order m of the first and second kind. Since $Q_{mn}(\mu)$ is unbounded near $\mu = \pm 1$, bounded solutions are

$$\Phi = AP_{mn}(\mu). \tag{87}$$

In other words, $\lambda_{mn} = n(n+1)$, where $n \geq m$, are eigenvalues of this singular Sturm-Liouville system, with corresponding orthonormal eigenfunctions

$$\Phi_{mn}(\phi) = \Phi(\lambda_{mn}, \phi) = \frac{1}{N} P_{mn}(\cos\phi), \tag{88a}$$

where
$$N^2 = \int_0^\pi \sin\phi\,[P_{mn}(\cos\phi)]^2\,d\phi = \int_{-1}^1 [P_{mn}(\mu)]^2\,d\mu. \tag{88b}$$

To evaluate N, we proceed as follows. Since

$$P_{mn}(\mu) = (1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_n(\mu),$$

where $P_n(\mu)$ is the Legendre polynomial of degree n, differentiation with respect to μ yields

$$\frac{d}{d\mu}P_{mn}(\mu) = -\mu m(1-\mu^2)^{m/2-1}\frac{d^m}{d\mu^m}P_n(\mu) + (1-\mu^2)^{m/2}\frac{d^{m+1}}{d\mu^{m+1}}P_n(\mu).$$

Multiplication of this result by $(1-\mu^2)^{1/2}$ gives

$$\begin{aligned}(1-\mu^2)^{1/2}\frac{d}{d\mu}P_{mn}(\mu) &= \frac{-\mu m(1-\mu^2)^{m/2}}{(1-\mu^2)^{1/2}}\frac{d^m}{d\mu^m}P_n(\mu) + (1-\mu^2)^{(m+1)/2}\frac{d^{m+1}}{d\mu^{m+1}}P_n(\mu)\\ &= \frac{-\mu m}{(1-\mu^2)^{1/2}}P_{mn}(\mu) + P_{m+1,n}(\mu).\end{aligned}$$

When this equation is solved for $P_{m+1,n}(\mu)$, squared, and integrated between the limits $\mu = \pm 1$,

$$\int_{-1}^1 (P_{m+1,n})^2\,d\mu = \int_{-1}^1 (1-\mu^2)\left(\frac{d}{d\mu}P_{mn}\right)^2 d\mu + 2m\int_{-1}^1 \mu P_{mn}\frac{d}{d\mu}P_{mn}\,d\mu + m^2\int_{-1}^1 \frac{\mu^2}{1-\mu^2}(P_{mn})^2\,d\mu.$$

Integration by parts on the first two integrals on the right gives

$$\begin{aligned}\int_{-1}^1 (P_{m+1,n})^2\,d\mu &= \left\{(1-\mu^2)\frac{dP_{mn}}{d\mu}P_{mn}\right\}_{-1}^1 - \int_{-1}^1 P_{mn}\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_{mn}}{d\mu}\right)d\mu\\ &\quad + 2m\left\{\frac{\mu}{2}(P_{mn})^2\right\}_{-1}^1 - 2m\int_{-1}^1 \frac{1}{2}(P_{mn})^2\,d\mu + m^2\int_{-1}^1 \frac{\mu^2}{1-\mu^2}(P_{mn})^2\,d\mu\\ &= \int_{-1}^1 P_{mn}\left[-\frac{d}{d\mu}\left((1-\mu^2)\frac{dP_{mn}}{d\mu}\right) - mP_{mn} + \frac{m^2\mu^2}{1-\mu^2}P_{mn}\right]d\mu,\end{aligned}$$

since $P_{mn}(\pm 1) = 0$ for $m > 0$; $P_{0n}(1) = P_n(1) = 1$; and $P_{0n}(-1) = P_n(-1) = (-1)^n$. Now, using Legendre's associated differential equation (81), we obtain

$$\int_{-1}^{1} (P_{m+1,n})^2\, d\mu = \int_{-1}^{1} P_{mn}\left[\left(\frac{-m^2}{1-\mu^2} + n(n+1)\right)P_{mn} - mP_{mn} + \left(\frac{m^2}{1-\mu^2} - m^2\right)P_{mn}\right]d\mu$$

$$= \int_{-1}^{1} (P_{mn})^2\left(n(n+1) - m - m^2\right)d\mu$$

or

$$\int_{-1}^{1} (P_{mn})^2\, d\mu = \frac{1}{(n-m)(n+m+1)}\int_{-1}^{1} (P_{m+1,n})^2\, d\mu.$$

Iteration of this result on m from m to n gives

$$\int_{-1}^{1} (P_{mn})^2\, d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\int_{-1}^{1} (P_{nn})^2\, d\mu. \tag{89}$$

Now,

$$P_{nn} = (1-\mu^2)^{n/2}\frac{d^n}{d\mu^n}P_n = (1-\mu^2)^{n/2}\frac{d^n}{d\mu^n}\left(\frac{1}{2^n n!}\frac{d^n}{d\mu^n}(\mu^2-1)^n\right)$$

$$= \frac{(1-\mu^2)^{n/2}}{2^n n!}\frac{d^{2n}}{d\mu^{2n}}(\mu^2-1)^n = \frac{(2n)!}{2^n n!}(1-\mu^2)^{n/2},$$

and substitution of this into (89) yields

$$\int_{-1}^{1} (P_{mn})^2\, d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\frac{[(2n)!]^2}{2^{2n}(n!)^2}\int_{-1}^{1} (1-\mu^2)^n\, d\mu.$$

In elementary calculus [see also Exercise 7(a) in Section 8.5], it is shown that

$$\int_{-1}^{1} (1-\mu^2)^n\, d\mu = \frac{2^{2n+1}(n!)^2}{(2n+1)!},$$

and therefore

$$N^2 = \int_{-1}^{1} [P_{mn}(\mu)]^2\, d\mu = \frac{(n+m)!}{(n-m)!(2n)!}\frac{[(2n)!]^2}{2^{2n}(n!)^2}\frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

$$= \frac{(n+m)!}{(n-m)!}\frac{2}{2n+1}. \tag{90}$$

Summarizing our results, orthonormal eigenfunctions of (84) are

$$\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}\, P_{mn}(\cos\phi), \tag{91}$$

corresponding to eigenvalues $\lambda_{mn} = n(n+1)$, where n is an integer greater than or equal to m.

Because the Sturm-Liouville system is singular, we cannot quote the results of Theorem 2 in Section 4.2. We have already shown that there is an infinite number of eigenvalues, all of which are positive, except when $m = 0$, in which case $\lambda = 0$ is also an eigenvalue. According to the following theorem, piecewise smooth functions can be expanded in terms of these eigenfunctions.

Theorem 3

If a function $f(\phi)$ is piecewise smooth on the interval $0 \le \phi \le \pi$, then for each ϕ in $0 < \phi < \pi$,

$$\frac{f(\phi+) + f(\phi-)}{2} = \sum_{n=m}^{\infty} c_n \Phi_{mn}(\phi), \tag{92a}$$

where

$$c_n = \int_0^\pi \sin\phi\, f(\phi)\Phi_{mn}(\phi)\, d\phi. \tag{92b}$$

Example 4: Expand the function

$$f(\phi) = \begin{cases} 1 & 0 \le \phi < \pi/2 \\ 0 & \phi = \pi/2 \\ -1 & \pi/2 < \phi \le \pi \end{cases}$$

in terms of the eigenfunctions of Sturm-Liouville system (84) when $m = 0$.

Solution: Orthonormal eigenfunctions of

$$\frac{d}{d\phi}\left(\sin\phi \frac{d\Phi}{d\phi}\right) + \lambda \sin\phi\, \Phi = 0, \qquad 0 < \phi < \pi,$$

are Legendre polynomials

$$\Phi_{0n}(\phi) = \sqrt{\frac{2n+1}{2}}\, P_n(\cos\phi), \qquad n \ge 0.$$

The eigenfunction expansion of $f(\phi)$ is

$$f(\phi) = \sum_{n=0}^{\infty} c_n \Phi_{0n}(\phi),$$

where

$$c_n = \int_0^\pi \sin\phi\, f(\phi)\Phi_{0n}(\phi)\, d\phi.$$

When we set $\mu = \cos\phi$,

$$\begin{aligned} \sqrt{\frac{2}{2n+1}}\, c_n &= \int_1^{-1} f[\phi(\mu)] P_n(\mu)(-d\mu) \\ &= -\int_{-1}^{0} P_n(\mu)\, d\mu + \int_0^1 P_n(\mu)\, d\mu \\ &= \begin{cases} 0 & n \text{ even} \\ 2\displaystyle\int_0^1 P_n(\mu)\, d\mu & n \text{ odd} \end{cases} \end{aligned}$$

$$= \begin{cases} 0 & n \text{ even} \\ \dfrac{(-1)^{(n-1)/2}(n-1)!}{2^{n-1}\left(\dfrac{n+1}{2}\right)!\left(\dfrac{n-1}{2}\right)!} & n \text{ odd} \end{cases}$$

(see Exercise 8 in Section 8.5). Consequently,

$$c_{2n-1} = \sqrt{\frac{4n-1}{2}}\,\frac{(-1)^{n-1}(2n-2)!}{2^{2n-2}n!(n-1)!}$$

and

$$f(\phi) = \sum_{n=1}^{\infty} \sqrt{\frac{4n-1}{2}}\,\frac{(-1)^{n-1}(2n-2)!}{2^{2n-2}n!(n-1)!}\sqrt{\frac{4n-1}{2}}\,P_{2n-1}(\cos\phi)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!(4n-1)}{2^{2n-1}n!(n-1)!} P_{2n-1}(\cos\phi).$$ ■

Exercises 8.6

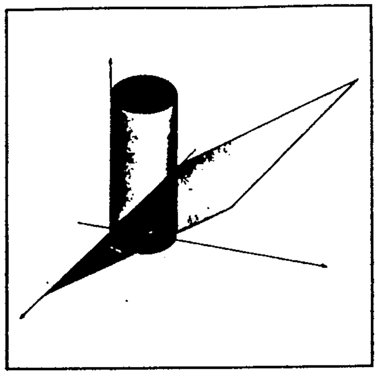
In Exercises 1–4, expand the function in terms of the orthonormal eigenfunctions of Sturm-Liouville system (84) when m = 0.

1. $f(\phi) = \begin{cases} 1 & 0 \le \phi < \pi/2 \\ 0 & \pi/2 < \phi < \pi \end{cases}$
2. $f(\phi) = \cos^4\phi$
3. $f(\phi) = \begin{cases} \cos\phi & 0 \le \phi \le \pi/2 \\ 0 & \pi/2 < \phi \le \pi \end{cases}$
4. $f(\phi) = \begin{cases} \cos\phi & 0 \le \phi \le \pi/2 \\ -\cos\phi & \pi/2 < \phi \le \pi \end{cases}$
5. Find eigenvalues and orthonormal eigenfunctions of the Sturm-Liouville system

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \lambda\sin\phi\,\Phi = 0, \qquad 0 < \phi < \pi/2,$$

$$\Phi\left(\frac{\pi}{2}\right) = 0.$$

6. Repeat Exercise 5 if the boundary condition is $\Phi'(\pi/2) = 0$.



C H A P T E R

N I N E

Problems in Polar, Cylindrical, and Spherical Coordinates

9.1 Homogeneous Problems in Polar, Cylindrical, and Spherical Coordinates

In Section 5.3, separation of variables was used to solve homogeneous boundary value problems expressed in polar coordinates. With the results of Chapter 8, we are in a position to tackle boundary value problems in cylindrical and spherical coordinates and initial boundary value problems in all three coordinate systems. Homogeneous problems are discussed in this section; nonhomogeneous problems are discussed in Section 9.2.

We begin with the following heat conduction problem.

Example 1: An infinitely long cylinder of radius r_2 is initially at temperature $f(r) = r_2^2 - r^2$, and for time $t > 0$, the boundary $r = r_2$ is insulated. Find the temperature in the cylinder for $t > 0$.

Solution: The initial boundary value problem for $U(r,t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \qquad 0 < r < r_2, \qquad t > 0, \tag{1a}$$

$$\frac{\partial U(r_2,t)}{\partial r} = 0, \qquad t > 0, \tag{1b}$$

$$U(r,0) = r_2^2 - r^2, \qquad 0 < r < r_2. \tag{1c}$$

When a function $U(r,t) = R(r)T(t)$ with variables separated is substituted into PDE (1a) and the equation is divided by kRT, there results

$$\frac{T'}{kT} = \frac{R''}{R} + \frac{R'}{rR} = \alpha = \text{constant independent of } r \text{ and } t.$$

This equation and boundary condition (1b) yield the Sturm-Liouville system

$$(rR')' - \alpha r R = 0, \qquad 0 < r < r_2, \tag{2a}$$

$$R'(r_2) = 0. \tag{2b}$$

This singular system was discussed in Section 8.4 (see Table 8.1 with $\nu = 0$). If we set $\alpha = -\lambda^2$, eigenvalues are defined by the equation $J_1(\lambda r_2) = 0$, and normalized eigenfunctions are

$$R_n(r) = \frac{\sqrt{2}\,J_0(\lambda_n r)}{r_2 J_0(\lambda_n r_2)}, \qquad n \geq 0. \tag{3}$$

(For simplicity of notation, we have dropped the zero subscript on R_{0n} and λ_{0n}.)

The differential equation

$$T' + k\lambda_n^2 T = 0 \tag{4}$$

has general solution

$$T(t) = Ce^{-k\lambda_n^2 t}. \tag{5}$$

In order to satisfy initial condition (1c), we superpose separated functions and take

$$U(r,t) = \sum_{n=0}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r), \tag{6}$$

where the C_n are constants. Condition (1c) requires these constants to satisfy

$$r_2^2 - r^2 = \sum_{n=0}^{\infty} C_n R_n(r), \qquad 0 < r < r_2. \tag{7}$$

Thus, the C_n are coefficients in the Fourier Bessel series of $r_2^2 - r^2$, and, according to equation (61b) in Chapter 8,

$$C_n = \int_0^{r_2} r(r_2^2 - r^2)R_n(r)\,dr = \frac{\sqrt{2}}{r_2 J_0(\lambda_n r_2)}\int_0^{r_2} r(r_2^2 - r^2)J_0(\lambda_n r)\,dr.$$

To evaluate this integral when $n > 0$, we set $u = \lambda_n r$, in which case

$$C_n = \frac{\sqrt{2}}{r_2 J_0(\lambda_n r_2)} \int_0^{\lambda_n r_2} \left(\frac{r_2^2 u}{\lambda_n} - \frac{u^3}{\lambda_n^3}\right) J_0(u) \frac{du}{\lambda_n}$$
$$= \frac{\sqrt{2}}{\lambda_n^4 r_2 J_0(\lambda_n r_2)} \int_0^{\lambda_n r_2} (r_2^2 \lambda_n^2 u - u^3) J_0(u)\, du.$$

For the term involving u^3, we use the reduction formula in Exercise 9 of Section 8.3:

$$C_n = \frac{\sqrt{2}}{\lambda_n^4 r_2 J_0(\lambda_n r_2)} \left(r_2^2 \lambda_n^2 \int_0^{\lambda_n r_2} u J_0(u)\, du - \{u^3 J_1(u)\}_0^{\lambda_n r_2} \right.$$
$$\left. - \{2u^2 J_0(u)\}_0^{\lambda_n r_2} + 4 \int_0^{\lambda_n r_2} u J_0(u)\, du \right).$$

If we recall the eigenvalue equation $J_1(\lambda r_2) = 0$, and equation (39) in Section 8.3 with $v = 1$, we may write

$$C_n = \frac{\sqrt{2}}{\lambda_n^4 r_2 J_0(\lambda_n r_2)} \left(-2\lambda_n^2 r_2^2 J_0(\lambda_n r_2) + (r_2^2 \lambda_n^2 + 4) \int_0^{\lambda_n r_2} \frac{d}{du}[u J_1(u)]\, du \right)$$
$$= \frac{\sqrt{2}}{\lambda_n^4 r_2 J_0(\lambda_n r_2)} (-2\lambda_n^2 r_2^2 J_0(\lambda_n r_2) + (r_2^2 \lambda_n^2 + 4)\{u J_1(u)\}_0^{\lambda_n r_2})$$
$$= \frac{-2\sqrt{2}\, r_2}{\lambda_n^2}.$$

When $n = 0$, $R_0(r) = \sqrt{2}/r_2$, and

$$C_0 = \int_0^{r_2} r(r_2^2 - r^2) R_0(r)\, dr = \frac{\sqrt{2}}{r_2} \left(\frac{r_2^2 r^2}{2} - \frac{r^4}{4} \right)_0^{r_2} = \frac{\sqrt{2}\, r_2^3}{4}.$$

The solution of problem (1) is therefore

$$U(r,t) = \frac{\sqrt{2}\, r_2^3}{4} \left(\frac{\sqrt{2}}{r_2} \right) + \sum_{n=1}^{\infty} \frac{-2\sqrt{2}\, r_2}{\lambda_n^2} e^{-k\lambda_n^2 t} \frac{\sqrt{2}\, J_0(\lambda_n r)}{r_2 J_0(\lambda_n r_2)}$$
$$= \frac{r_2^2}{2} - 4 \sum_{n=1}^{\infty} \frac{e^{-k\lambda_n^2 t}}{\lambda_n^2} \frac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}. \tag{8}$$

Notice that for large t, the limit of this solution is $r_2^2/2$, and this is the average value of $r_2^2 - r^2$ over the circle $r \le r_2$. ■

In the following heat conduction problem, we add angular dependence to the temperature function.

Example 2: An infinitely long rod with semicircular cross section is initially ($t = 0$) at a constant nonzero temperature throughout. For $t > 0$, its flat side is held at temperature 0°C while its round side is insulated. Find temperature in the rod for $t > 0$.

Solution: Temperature in that half of the rod for which $x < 0$ in Figure 9.1 is identical to that in the half for which $x \ge 0$; no heat crosses the $x = 0$ plane. As a result, the temperature function $U(r, \theta, t)$ (and it is independent of z) must satisfy the initial boundary value problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), \quad 0 < r < r_2, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9a}$$

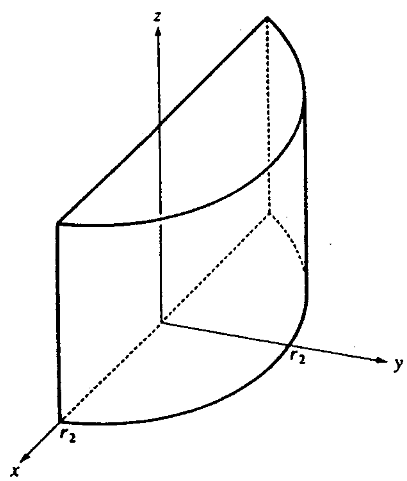
$$U(r, 0, t) = 0, \quad 0 < r < r_2, \quad t > 0, \tag{9b}$$

$$U_\theta\left(r, \frac{\pi}{2}, t\right) = 0, \quad 0 < r < r_2, \quad t > 0, \tag{9c}$$

$$U_r(r_2, \theta, t) = 0, \quad 0 < \theta < \frac{\pi}{2}, \quad t > 0, \tag{9d}$$

$$U(r, \theta, 0) = U_0, \quad 0 < r < r_2, \quad 0 < \theta < \frac{\pi}{2}. \tag{9e}$$

[In Exercise 4, the problem is solved for $0 < \theta < \pi$ with the condition $U(r, \pi, t) = 0$ in place of (9c).]



9.1

When a function with variables separated, $U(r, \theta, t) = R(r)H(\theta)T(t)$, is substituted into PDE (9a),

$$RHT' = k(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or
$$-\frac{H''}{H} = \frac{r^2R''}{R} + \frac{rR'}{R} - \frac{r^2T'}{kT} = \alpha = \text{constant independent of } r, \theta, \text{ and } t.$$

When boundary conditions (9b, c) are imposed on the separated function, a Sturm-Liouville system in $H(\theta)$ results:

$$H'' + \alpha H = 0, \quad 0 < \theta < \frac{\pi}{2}, \tag{10a}$$

$$H(0) = 0 = H'\left(\frac{\pi}{2}\right). \tag{10b}$$

This system was discussed in Section 4.2. If we set $\alpha = \nu^2$, then, according to Table 4.1, eigenvalues are $\nu_m^2 = (2m-1)^2$ $(m = 1, 2, \ldots)$, with orthonormal eigenfunctions

$$H_m(\theta) = \frac{2}{\sqrt{\pi}} \sin(2m-1)\theta. \tag{11}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{\nu_m^2}{r^2} = \frac{T'}{kT} = \beta = \text{constant independent of } r \text{ and } t.$$

Boundary condition (9d) leads to the Sturm-Liouville system

$$(rR')' + \left(-\beta r - \frac{(2m-1)^2}{r}\right)R = 0, \qquad 0 < r < r_2, \tag{12a}$$

$$R'(r_2) = 0. \tag{12b}$$

This is Sturm-Liouville system (45) of Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues are defined by the equation

$$J'_{2m-1}(\lambda r_2) = 0 \tag{13}$$

with corresponding eigenfunctions

$$R_{mn}(r) = \frac{1}{N} J_{2m-1}(\lambda_{mn} r), \tag{14a}$$

where $$2N^2 = r_2^2\left[1 - \left(\frac{2m-1}{\lambda_{mn} r_2}\right)^2\right][J_{2m-1}(\lambda_{mn} r_2)]^2. \tag{14b}$$

The differential equation

$$T' = -k\lambda_{mn}^2 T \tag{15}$$

has general solution

$$T(t) = Ce^{-k\lambda_{mn}^2 t}. \tag{16}$$

To satisfy initial condition (9e), we superpose separated functions and take

$$U(r, \theta, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} e^{-k\lambda_{mn}^2 t} R_{mn}(r) H_m(\theta), \tag{17}$$

where c_{mn} are constants. Initial condition (9e) requires these constants to satisfy

$$U_0 = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} C_{mn} R_{mn}(r) H_m(\theta), \qquad 0 < r < r_2, \qquad 0 < \theta < \frac{\pi}{2}. \tag{18}$$

If we multiply this equation by $H_i(\theta)$ and integrate with respect to θ from $\theta = 0$ to $\theta = \pi/2$, orthogonality of the eigenfunctions in θ gives

$$\sum_{n=1}^{\infty} C_{in} R_{in}(r) = \int_0^{\pi/2} U_0 H_i(\theta)\, d\theta = U_0 \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} \sin(2i-1)\theta\, d\theta$$

$$= \frac{2U_0}{\sqrt{\pi}} \left\{\frac{-1}{2i-1} \cos(2i-1)\theta\right\}_0^{\pi/2} = \frac{2U_0}{(2i-1)\sqrt{\pi}}.$$

But this equation implies that the C_{in} are Fourier Bessel coefficients for the function $2U_0/[(2i-1)\sqrt{\pi}]$, that is,

$$C_{in} = \int_0^{r_2} \frac{2U_0}{(2i-1)\sqrt{\pi}} rR_{in}(r)\,dr.$$

Thus, the solution of (9) for $0 \le \theta \le \pi/2$ is (17), where

$$C_{mn} = \frac{2U_0}{(2m-1)\sqrt{\pi}} \int_0^{r_2} rR_{mn}(r)\,dr. \tag{19}$$

For an angle θ between $\pi/2$ and π. we should evaluate $U(r, \pi-\theta, t)$. Since

$$H_m(\pi-\theta) = \frac{2}{\sqrt{\pi}}\sin(2m-1)(\pi-\theta) = \frac{2}{\sqrt{\pi}}\sin(2m-1)\theta,$$

it follows that $U(r,\pi-\theta,t) = U(r,\theta,t)$. Hence, solution (17) is valid for $0 \le \theta \le \pi$. ∎

Our next example is a vibration problem.

nple 3: Solve the initial boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right), \quad 0 < r < r_2, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{20a}$$

$$z(r_2,\theta,t) = 0, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{20b}$$

$$z(r,\theta,0) = f(r,\theta), \quad 0 < r < r_2, \quad -\pi < \theta \le \pi, \tag{20c}$$

$$z_t(r,\theta,0) = 0, \quad 0 < r < r_2, \quad -\pi < \theta \le \pi. \tag{20d}$$

Physically described is a membrane stretched over the circle $r \le r_2$ that has an initial displacement $f(r,\theta)$ and zero initial velocity. Boundary condition (20b) states that the edge of the membrane is fixed on the xy-plane.

lution: When a function $z(r,\theta,t)$, separated in the form $z(r,\theta,t) = R(r)H(\theta)T(t)$, is substituted into PDE (20a),

$$RHT'' = c^2(R''HT + r^{-1}R'HT + r^{-2}RH''T)$$

or $$-\frac{H''}{H} = r^2\left(\frac{R''+r^{-1}R'}{R} - \frac{T''}{c^2T}\right) = \alpha = \text{constant independent of } r,\theta, \text{ and } t.$$

Since the solution and its first derivative with respect to θ must be 2π-periodic in θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \quad -\pi < \theta \le \pi, \tag{21a}$$

$$H(-\pi) = H(\pi), \tag{21b}$$

$$H'(-\pi) = H'(\pi). \tag{21c}$$

This system was discussed in Chapter 4 [Example 2 and equation (21)]. The eigenvalues are $\alpha = m^2$, m a nonnegative integer, with orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\sin m\theta, \quad \frac{1}{\sqrt{\pi}}\cos m\theta. \tag{22}$$

Continued separation of the equation in $R(r)$ and $T(t)$ gives

$$\frac{R'' + r^{-1}R'}{R} - \frac{m^2}{r^2} = \frac{T''}{c^2T} = \beta = \text{constant independent of } r \text{ and } t.$$

When boundary condition (20b) is imposed on the separated function, a Sturm-Liouville system in $R(r)$ results:

$$(rR')' + \left(-\beta r - \frac{m^2}{r}\right)R = 0, \qquad 0 < r < r_2, \tag{23a}$$

$$R(r_2) = 0. \tag{23b}$$

This is, once again, singular system (45) in Section 8.4. If we set $\beta = -\lambda^2$, eigenvalues λ_{mn} are defined by

$$J_m(\lambda r_2) = 0, \tag{24}$$

with corresponding orthonormal eigenfunctions

$$R_{mn}(r) = \frac{\sqrt{2}\,J_m(\lambda_{mn}r)}{r_2 J_{m+1}(\lambda_{mn}r_2)} \tag{25}$$

(see Table 8.1).

The differential equation

$$T'' + (c\lambda_{mn})^2T = 0 \tag{26}$$

has general solution

$$T(t) = d\cos c\lambda_{mn}t + b\sin c\lambda_{mn}t, \tag{27}$$

where d and b are constants. Initial condition (20d) implies that $b = 0$, and hence

$$T(t) = d\cos c\lambda_{mn}t. \tag{28}$$

In order to satisfy the final initial condition (20c), we superpose separated functions and take

$$\begin{aligned} z(r, \theta, t) &= \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}}\cos c\lambda_{0n}t \\ &\quad + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right)\cos c\lambda_{mn}t, \end{aligned} \tag{29}$$

where d_{mn} and f_{mn} are constants. Condition (20c) requires these constants to satisfy

$$f(r, \theta) = \sum_{n=1}^{\infty} d_{0n}\frac{R_{0n}(r)}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} R_{mn}(r)\left(d_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + f_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \tag{30}$$

for $0 < r < r_2$, $-\pi < \theta \leq \pi$. If we multiply this equation by $(1/\sqrt{\pi})\cos i\theta$ and integrate with respect to θ from $\theta = -\pi$ to $\theta = \pi$, orthogonality of the eigenfunctions in θ gives

$$\int_{-\pi}^{\pi} f(r, \theta)\frac{\cos i\theta}{\sqrt{\pi}}\,d\theta = \sum_{n=1}^{\infty} d_{in}R_{in}(r).$$

Multiplication of this equation by $rR_{ij}(r)$ and integration with respect to r from $r = 0$ to $r = r_2$ yields (because of orthogonality of the R_{ij} for fixed i)

$$\int_0^{r_2}\int_{-\pi}^{\pi} rf(r,\theta)R_{ij}\frac{\cos i\theta}{\sqrt{\pi}}\,d\theta\,dr = d_{ij};$$

that is,

$$d_{mn} = \int_{-\pi}^{\pi}\int_0^{r_2} rR_{mn}\frac{\cos m\theta}{\sqrt{\pi}} f(r,\theta)\,dr\,d\theta. \tag{31a}$$

Similarly,

$$f_{mn} = \int_{-\pi}^{\pi}\int_0^{r_2} rR_{mn}\frac{\sin m\theta}{\sqrt{\pi}} f(r,\theta)\,dr\,d\theta \tag{31b}$$

and

$$d_{0n} = \int_{-\pi}^{\pi}\int_0^{r_2} rR_{0n}(r)\frac{f(r,\theta)}{\sqrt{2\pi}}\,dr\,d\theta. \tag{31c}$$

The solution of (20) is therefore (29), where d_{mn} and f_{mn} are defined by (31). ■

Coefficients d_{mn} and f_{mn} in this example were calculated by first using orthogonality of the trigonometric eigenfunctions and then using orthogonality of the $R_{mn}(r)$. An alternative procedure is to determine the multidimensional eigenfunctions for problem (20). This approach is discussed in Exercise 23.

Our final example on separation is a potential problem.

xample 4: Find the electrostatic potential interior to a sphere when the potential is given on the sphere.

Solution: The boundary value problem for the potential $V(r,\theta,\phi)$ is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 V}{\partial\theta^2} = 0,$$

$$0 < r < r_2, \qquad -\pi < \theta \le \pi, \qquad 0 < \phi < \pi, \tag{32a}$$

$$V(r_2,\theta,\phi) = f(\theta,\phi), \qquad -\pi < \theta \le \pi, \qquad 0 \le \phi \le \pi. \tag{32b}$$

When a function with variables separated, $V(r,\theta,\phi) = R(r)H(\theta)\Phi(\phi)$, is substituted into (32a),

$$R''H\Phi + \frac{2}{r}R'H\Phi + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}(\sin\phi\, RH\Phi') + \frac{RH''\Phi}{r^2\sin^2\phi} = 0$$

or

$$r^2\sin^2\phi\left(\frac{R''}{R} + \frac{2R'}{rR} + \frac{1}{r^2\sin\phi\,\Phi}\frac{d}{d\phi}(\sin\phi\,\Phi')\right) = -\frac{H''}{H}$$

$$= \alpha = \text{constant independent of } r,\ \phi, \text{ and } \theta.$$

Because $V(r,\theta,\phi)$ must be 2π-periodic in θ, as must its first derivative with respect to θ, it follows that $H(\theta)$ must satisfy the periodic Sturm-Liouville system

$$H'' + \alpha H = 0, \qquad -\pi < \theta \le \pi, \tag{33a}$$

$$H(-\pi) = H(\pi), \tag{33b}$$

$$H'(-\pi) = H'(\pi). \tag{33c}$$

This is Sturm-Liouville system (21) with eigenvalues $\alpha = m^2$ and orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}} \quad \text{and} \quad \frac{1}{\sqrt{\pi}}\cos m\theta, \qquad \frac{1}{\sqrt{\pi}}\sin m\theta.$$

Continued separation of the equation in $R(r)$ and $\Phi(\phi)$ gives

$$\frac{r^2R''}{R} + \frac{2rR'}{R} = \frac{m^2}{\sin^2\phi} - \frac{1}{\Phi\sin\phi}\frac{d}{d\phi}(\sin\phi\,\Phi') = \beta = \text{constant independent of } r \text{ and } \phi.$$

Thus, $\Phi(\phi)$ must satisfy the singular Sturm-Liouville system

$$\frac{d}{d\phi}\left(\sin\phi\frac{d\Phi}{d\phi}\right) + \left(\beta\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \qquad 0 < \phi < \pi. \tag{34}$$

According to the results of Section 8.6, eigenvalues are $\beta = n(n+1)$, where $n \geq m$ is an integer, with orthonormal eigenfunctions

$$\Phi_{mn}(\phi) = \sqrt{\frac{(2n+1)(n-m)!}{2(n+m)!}}\,P_{mn}(\cos\phi). \tag{35}$$

The remaining differential equation,

$$r^2R'' + 2rR' - n(n+1)R = 0, \tag{36}$$

is a Cauchy-Euler equation that can be solved by setting $R(r) = r^s$, s an unknown constant. This results in the general solution

$$R(r) = \frac{C}{r^{n+1}} + Ar^n. \tag{37}$$

For $R(r)$ to remain bounded as r approaches zero, we must set $C = 0$. Superposition of separated functions now yields

$$V(r,\theta,\phi) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r^n\Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} r^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right), \tag{38}$$

where A_{mn} and B_{mn} are constants. Boundary condition (32b) requires these constants to satisfy

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\frac{1}{\sqrt{2\pi}}A_{0n}r_2^n\Phi_{0n}(\phi) + \sum_{m=1}^{\infty}\sum_{n=m}^{\infty} r_2^n\Phi_{mn}(\phi)\left(A_{mn}\frac{\cos m\theta}{\sqrt{\pi}} + B_{mn}\frac{\sin m\theta}{\sqrt{\pi}}\right) \tag{39}$$

for $-\pi < \theta \leq \pi$, $0 \leq \phi \leq \pi$. Because of orthogonality of eigenfunctions in θ and ϕ, multiplication by $(1/\sqrt{2\pi})\sin\phi\,\Phi_{0j}(\phi)$ and integration with respect to θ and ϕ give

$$A_{0j} = \frac{1}{r_2^n}\int_0^{\pi}\int_{-\pi}^{\pi} f(\theta,\phi)\frac{1}{\sqrt{2\pi}}\sin\phi\,\Phi_{0j}(\phi)\,d\theta\,d\phi. \tag{40a}$$

Similarly,

$$A_{mn} = \frac{1}{r_2^n} \int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) \frac{\cos m\theta}{\sqrt{\pi}} \sin \phi \, \Phi_{mn}(\phi) \, d\theta \, d\phi \tag{40b}$$

and

$$B_{mn} = \frac{1}{r_2^n} \int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) \frac{\sin m\theta}{\sqrt{\pi}} \sin \phi \, \Phi_{mn}(\phi) \, d\theta \, d\phi. \tag{40c}$$

Notice that the potential at the center of the sphere is

$$\begin{aligned} V(0, \theta, \phi) &= \frac{1}{\sqrt{2\pi}} A_{00} \Phi_{00}(\phi) \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) \frac{1}{\sqrt{2\pi}} \sin \phi \, \Phi_{00}(\phi) \, d\theta \, d\phi \right) \Phi_{00}(\phi). \end{aligned}$$

Since $\Phi_{00}(\phi) = 1/\sqrt{2}$,

$$\begin{aligned} V(0, \theta, \phi) &= \frac{1}{4\pi} \int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) \sin \phi \, d\theta \, d\phi \\ &= \frac{1}{4\pi r_2^2} \int_0^\pi \int_{-\pi}^{\pi} f(\theta, \phi) r_2^2 \sin \phi \, d\theta \, d\phi, \end{aligned}$$

and this is the average value of $f(\theta, \phi)$ over the sphere. ■

Exercises 9.1

Part A—Heat Conduction

1. (a) The initial temperature of an infinitely long cylinder of radius r_2 is $f(r)$. If, for time $t > 0$, the outer surface is held at 0°C, find the temperature in the cylinder.
   (b) Simplify the solution in (a) when $f(r)$ is a constant U_0.
   (c) Find the solution when $f(r) = r_2^2 - r^2$.
2. A long cylinder of radius r_2 is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = r_2$ is insulated.
   (a) Find the temperature $U(r, t)$ in the cylinder.
   (b) What is the limit of $U(r, t)$ for large t?
3. A thin circular plate of radius r_2 is insulated top and bottom. At time $t = 0$ its temperature is $f(r, \theta)$. If the temperature of its edge is held at 0°C for $t > 0$, find its interior temperature for $t > 0$.
4. Solve Example 2 using the boundary condition $U(r, \pi, t) = 0$ in place of $\partial U(r, \pi/2, t)/\partial\theta = 0$.
5. A flat plate is in the form of a sector of a circle of radius 1 and angle α. At time $t = 0$, the temperature of the plate increases linearly from 0°C at $r = 0$ to a constant value $\bar{U}$ at $r = 1$ (and is therefore independent of θ). If, for $t > 0$, the rounded edge is insulated and the straight edges are held at temperature 0°C, find the temperature in the plate for $t > 0$. Prove that heat never crosses the line $\theta = \alpha/2$.
6. Find the temperature in the plate of Exercise 5 if the initial temperature is $f(r)$, the straight sides are insulated, and the curved edge is held at temperature 0°C.

7. Repeat Exercise 6 if the initial temperature is a function of r and θ, $f(r,\theta)$.
8. A cylinder occupies the region $r \le r_2$, $0 \le z \le L$. It has temperature $f(r,z)$ at time $t = 0$. For $t > 0$, its end $z = 0$ is insulated, and the remaining two surfaces are held at temperature 0°C. Find the temperature in the cylinder.
9. Solve Exercise 1(a), (b) if heat is transferred at $r = r_2$ according to Newton's law of cooling to an environment at temperature zero.
10. (a) A sphere of radius r_2 is initially at temperature $f(r)$ and, for time $t > 0$, the boundary $r = r_2$ is held at temperature zero. Find the temperature in the sphere for $t > 0$. (You will need the results of Exercise 5 in Section 8.4.)
    (b) Simplify the solution when $f(r) = U_0$, a constant.
11. Repeat Exercise 10 if the surface of the sphere is insulated. (See Exercise 6 in Section 8.4.) What is the temperature for large t?
12. Repeat Exercise 10 if the surface transfers heat to an environment at temperature zero according to Newton's law of cooling; that is, take as boundary condition

$$\kappa \frac{\partial U(r_2,t)}{\partial r} + \mu U(r_2,t) = 0, \qquad t > 0.$$

(Assume that $\mu r_2 > \kappa$ and see Exercise 7 in Section 8.4.)
13. Repeat Exercise 10(a) if the initial temperature is also a function of ϕ. (You will need the results of Exercise 8 in Section 8.4.)
14. (a) Repeat Exercise 10(a) if the initial temperature is also a function of ϕ and the surface of the sphere is insulated. (You will need the results of Exercise 8 in Section 8.4.)
    (b) What is the limit of the solution for large t?
15. The result of this exercise is analogous to that in Exercise 9 of Section 5.4. Show that the solution of the homogeneous heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \qquad 0 < r < r_2, \qquad 0 < z < L, \qquad t > 0,$$

$$-l_1 \frac{\partial U}{\partial z} + h_1 U = 0, \qquad z = 0, \qquad 0 < r < r_2, \qquad t > 0,$$

$$l_2 \frac{\partial U}{\partial z} + h_2 U = 0, \qquad z = L, \qquad 0 < r < r_2, \qquad t > 0,$$

$$l_3 \frac{\partial U}{\partial r} + h_3 U = 0, \qquad r = r_2, \qquad 0 < z < L, \qquad t > 0,$$

$$U(r,z,0) = f(r)g(z), \qquad 0 < r < r_2, \qquad 0 < z < L,$$

where the initial temperature is the product of a function of r and a function of z, is the product of the solutions of the problems

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \qquad 0 < r < r_2, \qquad t > 0,$$

$$l_3 \frac{\partial U(r_2,t)}{\partial r} + h_3 U(r_2,t) = 0, \qquad t > 0,$$

$$U(r,0) = f(r), \qquad 0 < r < r_2$$

and
$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial z^2}, \qquad 0 < z < L, \qquad t > 0,$$
$$-l_1 \frac{\partial U(0,t)}{\partial z} + h_1 U(0,t) = 0, \qquad t > 0,$$
$$l_2 \frac{\partial U(L,t)}{\partial z} + h_2 U(L,t) = 0, \qquad t > 0,$$
$$U(z,0) = g(z), \qquad 0 < z < L.$$

16. Solve the heat conduction problem
$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2}\right), \qquad 0 < r < r_2, \qquad 0 < z < L, \qquad t > 0,$$
$$U_z(r,0,t) = 0, \qquad 0 < r < r_2, \qquad t > 0,$$
$$U(r,L,t) = 0, \qquad 0 < r < r_2, \qquad t > 0,$$
$$U_r(r_2,z,t) = 0, \qquad 0 < z < L, \qquad t > 0,$$
$$U(r,z,0) = (r_2^2 - r^2)(L - z), \qquad 0 < r < r_2, \qquad 0 < z < L,$$

(a) by using the results of Exercise 15 and Example 1 in this section and that of Exercise 1(a) in Section 5.2.

(b) by separation of variables.

Part B—Vibrations

17. (a) A vibrating circular membrane of radius r_2 is given an initial displacement that is a function only of r, namely, $f(r)$, $0 \le r \le r_2$, and zero initial velocity. Show that subsequent displacements of the membrane, if its edge $r = r_2$ is fixed on the xy-plane, are of the form
$$z(r,t) = \frac{\sqrt{2}}{r_2}\sum_{n=1}^{\infty} A_n \cos c\lambda_n t \frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}.$$
What is A_n?

(b) The first term in the series in (a), called the *fundamental mode of vibration* for the membrane, is
$$H_1(r,t) = \frac{\sqrt{2}}{r_2} A_1 \cos c\lambda_1 t \frac{J_0(\lambda_1 r)}{J_1(\lambda_1 r_2)}.$$
Simplify and describe this mode when $r_2 = 1$. Does $H_1(r,t)$ have nodal curves?

(c) Repeat part (b) for the second mode of vibration.

(d) Are frequencies of modes of vibration for a circular membrane integer multiples of the frequency of the fundamental mode? Were they for a vibrating string with fixed ends?

18. A circular membrane of radius r_2 has its edge fixed on the xy-plane. In addition, a clamp holds the membrane on the xy-plane along a radial line from the center to the circumference. If the membrane is released from rest at a displacement $f(r,\theta)$, find subsequent displacements. [For consistency, we would require $f(r,\theta)$ to vanish along the clamped radial line.]

19. Simplify the solution in part (a) of Exercise 17 when $f(r) = r_2^2 - r^2$. (See Example 1.)

20. All points in a circular membrane of radius r_2 are given the same initial velocity v_0 but no initial displacement (except points on the edge). If its edge is fixed on the xy-plane, find subsequent displacements of points in the membrane.

21. Equation (29) with coefficients defined in (31) describes displacements of a circular membrane with fixed edge when oscillations are initiated from rest at some prescribed displacement. In this exercise we examine nodal curves for various modes of vibration.
   (a) The first mode of vibration is the term $(d_{01}/\sqrt{2\pi})R_{01}(r)\cos c\lambda_{01}t$. Show that this mode has no nodal curves.
   (b) Show that the mode $(d_{02}/\sqrt{2\pi})R_{02}(r)\cos c\lambda_{02}t$ has one nodal curve, a circle.
   (c) Show that the mode $(d_{03}/\sqrt{2\pi})R_{03}(r)\cos c\lambda_{03}t$ has two circular nodal curves.
   (d) On the basis of (a), (b), and (c), what are the nodal curves for the mode $(d_{0n}/\sqrt{2\pi})R_{0n}(r)\cos c\lambda_{0n}t$?
   (e) Corresponding to $n = m = 1$ there are two modes of vibration, $(d_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\cos\theta$ and $(f_{11}/\sqrt{\pi})R_{11}(r)\cos c\lambda_{11}t\sin\theta$. Show that each of these modes has only one nodal curve, a straight line.
   (f) Find nodal curves for the modes $(d_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\cos\theta$ and $(f_{12}/\sqrt{\pi})R_{12}(r)\cos c\lambda_{12}t\sin\theta$.
   (g) Find nodal curves for the modes $(d_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\cos 2\theta$ and $(f_{22}/\sqrt{\pi})R_{22}(r)\cos c\lambda_{22}t\sin 2\theta$.
   (h) On the basis of (e), (f), and (g), what are the nodal curves for the modes $(d_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\cos m\theta$ and $(f_{mn}/\sqrt{\pi})R_{mn}(r)\cos c\lambda_{mn}t\sin m\theta$?

22. The initial boundary value problem for small horizontal displacements of a suspended cable when gravity is the only force acting on the cable is

$$\frac{\partial^2 y}{\partial t^2} = -g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right), \qquad 0 < x < L, \qquad t > 0,$$
$$y(L,t) = 0, \qquad t > 0,$$
$$y(x,0) = f(x), \qquad 0 < x < L,$$
$$y_t(x,0) = h(x), \qquad 0 < x < L$$

(see Exercise 20 in Section 1.3).
   (a) Show that when a new independent variable $z = \sqrt{-4x/g}$ is introduced, $y(z,t)$ must satisfy

$$\frac{\partial^2 y}{\partial t^2} = \frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial y}{\partial z}\right), \qquad 0 < z < M, \qquad t > 0,$$
$$y(M,t) = 0, \qquad t > 0,$$
$$y(z,0) = f\left(\frac{-gz^2}{4}\right), \qquad 0 < z < M,$$
$$y_t(z,0) = h\left(\frac{-gz^2}{4}\right), \qquad 0 < z < M,$$

   where $M = \sqrt{-4L/g}$.
   (b) Solve this problem by separation of variables, and hence find $y(x,t)$.

23. Multidimensional eigenfunctions for problem (20) are solutions of the two-dimensional

eigenvalue problem

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} + \lambda^2 W = 0, \qquad 0 < r < r_2, \qquad -\pi < \theta \le \pi,$$

$$W(r_2, \theta) = 0, \qquad -\pi < \theta \le \pi.$$

(a) Find eigenfunctions (normalized with respect to the unit weight function over the circle $r \le r_2$).

(b) Use the eigenfunctions in (a) to solve problem (20).

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

24. **(a)** Solve the following boundary value problem associated with the Helmholtz equation on a circle

$$\nabla^2 V + k^2 V = 0, \qquad 0 < r < r_2, \qquad -\pi < \theta \le \pi \quad (k > 0 \text{ a constant})$$

$$V(r_2, \theta) = f(\theta), \qquad -\pi < \theta \le \pi.$$

(b) Is $V(0, \theta)$ the average value of $f(\theta)$ on $r = r_2$?

(c) What is the solution when $f(\theta) = 1$?

25. Solve the following problem for potential in a cylinder:

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} = 0, \qquad 0 < r < r_2, \qquad 0 < z < L,$$

$$V(r_2, z) = 0, \qquad 0 < z < L,$$

$$V(r, 0) = 0, \qquad 0 < r < r_2,$$

$$V(r, L) = f(r), \qquad 0 < r < r_2.$$

26. Find the potential inside a cylinder of length L and radius r_2 when potential on the curved surface is zero and potentials on the flat ends are nonzero.

27. **(a)** Find the steady-state temperature in a cylinder of radius r_2 and length L if the end $z = 0$ is maintained at temperature $f(r)$, the end $z = L$ is kept at temperature zero, and heat is transferred on $r = r_2$ to a medium at temperature zero according to Newton's law of cooling.

(b) Simplify the solution when $f(r) = U_0$, a constant.

28. Find the potential inside a hemisphere $r \le r_2$, $z \ge 0$ when the potential on $z = 0$ is zero and that on $r = r_2$ is a function of ϕ only. (*Hint:* See the results of Exercise 5 in Section 8.6.)

29. Find the potential interior to a sphere of radius r_2 when the potential on the upper half is a constant V_0 and the potential on the lower half is zero.

30. Use the result of Exercise 29 to find the potential inside a sphere of radius r_2 when potentials on the top and bottom halves are constant values V_0 and V_1, respectively.

31. Find the potential in the region between two concentric spheres when the potential on each sphere is a function only of ϕ.

32. What is the potential exterior to a sphere when the potential is given on the sphere, if the potential must vanish at infinity?

33. Consider the following boundary value problem for steady-state temperature inside a cylinder of length L and radius r_2 when temperatures of its ends are zero:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad 0 < r < r_2, \qquad 0 < z < L,$$

$$U(r, 0) = 0, \qquad 0 < r < r_2,$$

$$U(r, L) = 0, \qquad 0 < r < r_2,$$

$$U(r_2, z) = f(z), \qquad 0 < z < L.$$

(a) Verify that separation of variables $U(r, z) = R(r)Z(z)$ leads to a Sturm-Liouville system in $Z(z)$ and the following differential equation in $R(r)$:

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} - \lambda^2 rR = 0, \qquad 0 < r < r_2.$$

(b) Show that the change of variable $x = \lambda r$ leads to Bessel's modified differential equation of order zero,

$$x\frac{d^2R}{dx^2} + \frac{dR}{dx} - xR = 0.$$

(See Exercise 10 in Section 8.3.)

(c) Find functions $R_n(r)$ corresponding to eigenvalues λ_n, and use superposition to solve the boundary value problem.

(d) Simplify the solution in (c) in the case that $f(z)$ is a constant value U_0.

34. Solve the boundary value problem in Exercise 33 if the ends of the cylinder are insulated.

35. (a) A charge Q is distributed uniformly around a thin ring of radius a in the xy-plane with center at the origin (Figure 9.2). Show that the potential at every point on the z-axis due to this charge is

$$V = \frac{Q}{4\pi\varepsilon_0\sqrt{a^2 + r^2}}.$$

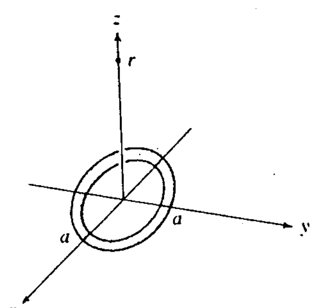


Figure 9.2

(b) The potential at other points in space must be independent of the spherical coordinate θ. Show that $V(r, \phi)$ must be of the form

$$V(r, \phi) = \sum_{n=0}^{\infty}\left(A_n r^n + \frac{B_n}{r^{n+1}}\right)\sqrt{\frac{2n+1}{2}}\,P_n(\cos\phi).$$

What does this result predict for potential at points on the positive z-axis?

(c) Equate expressions from (a) and (b) for V on the positive z-axis and expand $1/\sqrt{a^2 + r^2}$ in powers of r/a and a/r to find $V(r, \phi)$.

36. Repeat Exercise 35 in the case that charge Q is distributed uniformly over a disc of radius a in the xy-plane with center at the origin (Figure 9.3).

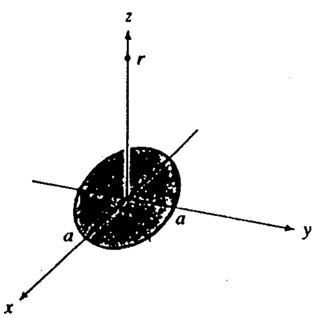


Figure 9.3

9.2 Nonhomogeneous Problems in Polar, Cylindrical, and Spherical Coordinates

Nonhomogeneities in problems expressed in polar, cylindrical, or spherical coordinates can be treated in the same way that they were treated in Cartesian coordinates—separate off "steady-state" or "static deflection" solutions, or use eigenfunction expansions or finite Fourier transforms. We begin our discussions with finite Fourier transforms.

With each of the Sturm-Liouville systems in Sections 8.4 and 8.6 we associate a finite Fourier transform. In particular, for the singular system

$$(rR')' + \left(\lambda^2 r - \frac{\nu^2}{r}\right)R = 0, \qquad 0 < r < r_2, \tag{41a}$$

$$l_2 R'(r_2) + h_2 R(r_2) = 0, \tag{41b}$$

with eigenvalues and eigenfunctions in Table 8.1, we define the transform

$$\tilde{f}(\lambda_{\nu n}) = \int_0^{r_2} r f(r) R_{\nu n}(r)\, dr, \tag{42a}$$

called the finite *Hankel* transform. It associates with a function $f(r)$, the sequence $\{\tilde{f}(\lambda_{\nu n})\}$ of coefficients in the eigenfunction expansion of $f(r)$ in terms of the $R_{\nu n}(r)$. The inverse transform of (42a) is this eigenfunction expansion,

$$f(r) = \sum_{n=1}^{\infty} \tilde{f}(\lambda_{\nu n}) R_{\nu n}(r), \qquad 0 < r < r_2, \tag{42b}$$

[provided, of course, that $f(r)$ is defined as the average of right and left limits at any point of discontinuity]. The finite Hankel transform is used to eliminate the r-variable from initial boundary value problems in polar, cylindrical, and spherical coordinates.

With the singular Sturm-Liouville system

$$(\sin\phi\,\Phi')' + \left(\lambda\sin\phi - \frac{m^2}{\sin\phi}\right)\Phi = 0, \qquad 0 < \phi < \pi \tag{43}$$

($m \geq 0$ an integer) is associated the *Legendre* transform,

$$\tilde{f}(m, n) = \int_0^\pi \sin\phi f(\phi)\Phi_{mn}(\phi)\,d\phi, \tag{44a}$$

where eigenvalues are $\lambda_{mn} = n(n+1)$ ($n \geq m$ an integer), and Φ_{mn} are normalized associated Legendre functions of the first kind [see equation (91) in Section 8.6]. The inverse transform is

$$f(\phi) = \sum_{n=m}^{\infty} \tilde{f}(m, n)\Phi_{mn}(\phi). \tag{44b}$$

This transform removes the ϕ-variable from problems in spherical coordinates.

To complete the set of finite Fourier transforms, we associate a transform with the periodic Sturm-Liouville system

$$H'' + \lambda^2 H = 0, \qquad -\pi < \theta \leq \pi, \tag{45a}$$

$$H(-\pi) = H(\pi), \tag{45b}$$

$$H'(-\pi) = H'(\pi), \tag{45c}$$

which arises in so many of our problems. Eigenvalues of this system are $\lambda_m^2 = m^2$, m a nonnegative integer, with orthonormal eigenfunctions

$$\frac{1}{\sqrt{2\pi}} \leftrightarrow \lambda_0 = 0; \qquad \frac{1}{\sqrt{\pi}}\cos m\theta, \quad \frac{1}{\sqrt{\pi}}\sin m\theta \longleftrightarrow \lambda_m, \qquad m > 0.$$

Periodic functions $f(\theta)$ may be expressed in terms of these eigenfunctions as ordinary trigonometric Fourier series:

$$f(\theta) = \frac{a_0}{\sqrt{2\pi}} + \sum_{m=1}^{\infty}\left(a_m\frac{\cos m\theta}{\sqrt{\pi}} + b_m\frac{\sin m\theta}{\sqrt{\pi}}\right), \tag{46a}$$

where

$$a_0 = \int_{-\pi}^{\pi}\frac{f(\theta)}{\sqrt{2\pi}}\,d\theta, \qquad a_m = \int_{-\pi}^{\pi} f(\theta)\frac{\cos m\theta}{\sqrt{\pi}}\,d\theta, \tag{46b}$$

$$b_m = \int_{-\pi}^{\pi} f(\theta)\frac{\sin m\theta}{\sqrt{\pi}}\,d\theta.$$

The complex representation of this series in Exercise 27 of Section 2.1 provides the finite Fourier transform. We may rewrite (46) in the form

$$f(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} C_m e^{im\theta}, \tag{47a}$$

where

$$C_m = \int_{-\pi}^{\pi} f(\theta)e^{-im\theta}\,d\theta. \tag{47b}$$

[We took the liberty in Exercise 27 of Section 2.1 of incorporating the 2π-factor into the series rather than the coefficient C_m. The series representation of $f(\theta)$ is the same in either case.] Associated with this representation is the finite Fourier transform of 2π-periodic functions

$$\tilde{f}(m) = \int_{-\pi}^{\pi} f(\theta)e^{-im\theta}\,d\theta \tag{48a}$$

and its inverse,

$$f(\theta) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} \tilde{f}(m)e^{im\theta}. \tag{48b}$$

[The exponentials in equations (48) could be interchanged to give an alternative transform; this uses the complex representation of equation (16) in Section 2.1.] The similarity between this finite Fourier transform and Fourier transform (23) in Chapter 7 is unmistakable.

The following examples illustrate how these transforms facilitate the solution of (initial) boundary value problems that are nonhomogeneous.

xample 5: A circular plate of radius r_2 is insulated at its top and bottom. At time $t = 0$, its temperature is 0°C throughout. If, for $t > 0$, all points on the edge of the plate have the same temperature $\bar{U}$, find the temperature in the plate for $t > 0$.

Solution: The initial boundary value problem for $U(r, t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \qquad 0 < r < r_2, \qquad t > 0, \tag{49a}$$

$$U(r_2, t) = \bar{U}, \qquad t > 0, \tag{49b}$$

$$U(r,0) = 0, \qquad 0 < r < r_2. \tag{49c}$$

To eliminate r from the problem, we use the finite Hankel transform

$$\tilde{f}(\lambda_n) = \int_0^{r_2} rf(r)R_n(r)\,dr, \tag{50}$$

where $R_n(r) = \sqrt{2}\,J_0(\lambda_n r)/[r_2 J_1(\lambda_n r_2)]$ are eigenfunctions of the Sturm-Liouville system

$$(rR')' + \lambda^2 rR = 0, \qquad 0 < r < r_2, \tag{51a}$$

$$R(r_2) = 0. \tag{51b}$$

(This is the system that would result were separation of variables applied to the corresponding homogeneous problem.) Application of the transform to PDE (49a) gives

$$\int_0^{r_2} r\frac{\partial U}{\partial t}R_n\,dr = k\int_0^{r_2} r\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)R_n\,dr.$$

An interchange of differentiation with respect to t and integration with respect to r on the left, and integration by parts on the right, yield

$$\begin{aligned}\frac{\partial \tilde{U}}{\partial t} &= k\left\{r\frac{\partial U}{\partial r}R_n\right\}_0^{r_2} + k\int_0^{r_2}\frac{\partial U}{\partial r}\left(-\frac{d}{dr}(rR_n) + R_n\right)dr \\ &= -k\int_0^{r_2} r\frac{\partial U}{\partial r}R_n'\,dr \quad \text{[because of (51b)]} \\ &= -k\left\{UrR_n'\right\}_0^{r_2} + k\int_0^{r_2} U(rR_n')'\,dr \quad \text{(by a second integration by parts)} \\ &= -kr_2R_n'(r_2)\bar{U} + k\int_0^{r_2} U[-\lambda_n^2 rR_n]\,dr \quad \text{[from (49b) and (51a)]} \\ &= -k\bar{U}r_2R_n'(r_2) - k\lambda_n^2\tilde{U}.\end{aligned}$$

Thus, $\tilde{U}(\lambda_n, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda_n^2\tilde{U} = -k\bar{U}r_2R_n'(r_2) \tag{52a}$$

subject to the transform of initial condition (49c),

$$\tilde{U}(\lambda_n, 0) = 0. \tag{52b}$$

Since the solution of (52) is

$$\tilde{U}(\lambda_n, t) = \frac{\bar{U}r_2R_n'(r_2)}{\lambda_n^2}\left(-1 + e^{-k\lambda_n^2 t}\right), \tag{53}$$

we obtain

$$\begin{aligned}U(r,t) &= \sum_{n=1}^{\infty}\tilde{U}(\lambda_n, t)R_n(r) \\ &= \sum_{n=1}^{\infty}\frac{\bar{U}r_2\sqrt{2}\,\lambda_n J_0'(\lambda_n r_2)}{r_2J_1(\lambda_n r_2)\lambda_n^2}\left(e^{-k\lambda_n^2 t} - 1\right)\frac{\sqrt{2}J_0(\lambda_n r)}{r_2J_1(\lambda_n r_2)} \\ &= \frac{2\bar{U}}{r_2}\sum_{n=1}^{\infty}\frac{-J_1(\lambda_n r_2)}{\lambda_n[J_1(\lambda_n r_2)]^2}\left(e^{-k\lambda_n^2 t} - 1\right)J_0(\lambda_n r) \\ &= \frac{2\bar{U}}{r_2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_1(\lambda_n r_2)}\left(1 - e^{-k\lambda_n^2 t}\right)J_0(\lambda_n r).\end{aligned} \tag{54}$$

The limit of this temperature function for large t is

$$\lim_{t\to\infty} U(r,t) = \frac{2\bar{U}}{r_2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_2)}.$$

The transform $\tilde{1}$ of the function $f(r) \equiv 1$ is

$$\tilde{1} = \int_0^{r_2} r\frac{\sqrt{2}J_0(\lambda_n r)}{r_2J_1(\lambda_n r_2)}\,dr = \frac{\sqrt{2}}{r_2J_1(\lambda_n r_2)}\int_0^{\lambda_n r_2}\left(\frac{u}{\lambda_n}\right)J_0(u)\left(\frac{du}{\lambda_n}\right)$$

$$= \frac{\sqrt{2}}{r_2 \lambda_n^2 J_1(\lambda_n r_2)} \int_0^{\lambda_n r_2} \frac{d}{du}[uJ_1(u)]\,du \quad \text{[see identity (39) in Section 8.3 with } \nu = 1\text{]}$$

$$= \frac{\sqrt{2}}{r_2 \lambda_n^2 J_1(\lambda_n r_2)} \Big\{ uJ_1(u) \Big\}_0^{\lambda_n r_2}$$

$$= \frac{\sqrt{2}}{\lambda_n}.$$

Consequently,

$$1 = \sum_{n=1}^{\infty} \frac{\sqrt{2}}{\lambda_n} \frac{\sqrt{2} J_0(\lambda_n r)}{r_2 J_1(\lambda_n r_2)} = \frac{2}{r_2} \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_2)}, \tag{55}$$

and it follows that

$$\lim_{t \to \infty} U(r, t) = \bar{U},$$

as expected. Furthermore, this suggests that we write $U(r, t)$ in the form

$$U(r, t) = \bar{U} - \frac{2\bar{U}}{r_2} \sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n r_2)} e^{-k\lambda_n^2 t} J_0(\lambda_n r). \tag{56}$$

Because the nonhomogeneity in boundary condition (49b) is independent of time, we could have begun by separating off the steady-state solution; that is, we could set $U(r, t) = V(r, t) + \psi(r)$, where $\psi(r)$ is the solution of

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = 0, \qquad 0 < r < r_2, \tag{57a}$$

$$\psi(r_2) = \bar{U}. \tag{57b}$$

The only bounded solution of this system is $\psi(r) = \bar{U}$. With this steady-state solution, $V(r, t)$ must satisfy the homogeneous problem

$$\frac{\partial V}{\partial t} = k\left(\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right), \qquad 0 < r < r_2, \qquad t > 0, \tag{58a}$$

$$V(r_2, t) = 0, \qquad t > 0, \tag{58b}$$

$$V(r, 0) = -\bar{U}, \qquad 0 < r < r_2. \tag{58c}$$

Separation $V(r, t) = R(r)T(t)$ leads to Sturm-Liouville system (51) in $R(r)$ and the ODE

$$T' + k\lambda^2 T = 0, \qquad t > 0. \tag{59}$$

Eigenvalues are defined by $J_0(\lambda r_2) = 0$, and normalized eigenfunctions are $R_n(r) = \sqrt{2} J_0(\lambda_n r)/[r_2 J_1(\lambda_n r_2)]$. Corresponding solutions of (59) are

$$T(t) = Ce^{-k\lambda_n^2 t}. \tag{60}$$

Superposition of separated functions yields

$$V(r, t) = \sum_{n=1}^{\infty} C_n e^{-k\lambda_n^2 t} R_n(r), \tag{61}$$

and initial condition (58c) requires that

$$-\bar{U} = \sum_{n=1}^{\infty} C_n R_n(r). \tag{62}$$

The C_n are therefore Fourier coefficients in the eigenfunction expansion of the function $-\bar{U}$; that is,

$$C_n = \int_0^{r_2} r(-\bar{U})R_n(r)\,dr = -\bar{U}\int_0^{r_2} r\frac{\sqrt{2}J_0(\lambda_n r)}{r_2 J_1(\lambda_n r_2)}\,dr = \frac{-\sqrt{2}\bar{U}}{\lambda_n}.$$

(This integral was evaluated in the above transform solution.) Consequently,

$$U(r,t) = \bar{U} + \sum_{n=1}^{\infty} \frac{-\sqrt{2}\bar{U}}{\lambda_n} e^{-k\lambda_n^2 t} \frac{\sqrt{2}J_0(\lambda_n r)}{r_2 J_1(\lambda_n r_2)}$$

$$= \bar{U} - \frac{2\bar{U}}{r_2}\sum_{n=1}^{\infty} \frac{1}{\lambda_n J_1(\lambda_n r_2)} e^{-k\lambda_n^2 t} J_0(\lambda_n r),$$

the same solution as that obtained by finite Fourier transforms. ■

Our next example is a vibration problem.

Example 6: A circular membrane of radius r_2 has an initial displacement at time $t = 0$ described by the function $f(r,\theta)$, $0 \le r \le r_2$, $-\pi < \theta \le \pi$, but no initial velocity. For time $t > 0$, its edge $r = r_2$ is forced to undergo periodic oscillations described by $A \sin \omega t$, A a constant. [For consistency, we assume that $f(r_2,\theta) = 0$.] Find its displacement as a function of r, θ, and t.

Solution: The initial boundary value problem for $z(r,\theta,t)$ is

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right), \quad 0 < r < r_2, \quad -\pi < \theta \le \pi, \quad t > 0, \tag{63a}$$

$$z(r_2,\theta,t) = A\sin\omega t, \qquad -\pi < \theta \le \pi, \qquad t > 0, \tag{63b}$$

$$z(r,\theta,0) = f(r,\theta), \qquad 0 < r < r_2, \qquad -\pi < \theta \le \pi, \tag{63c}$$

$$z_t(r,\theta,0) = 0, \qquad 0 < r < r_2, \qquad -\pi < \theta \le \pi. \tag{63d}$$

To remove θ from the problem, we apply transform (48a) to PDE (63a):

$$\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial t^2} e^{-im\theta}\,d\theta = c^2 \int_{-\pi}^{\pi}\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right)e^{-im\theta}\,d\theta.$$

Integrations with respect to θ and differentiations with respect to t and r may be interchanged, with the result that

$$\frac{\partial^2 \tilde{z}}{\partial t^2} - c^2\left(\frac{\partial^2 \tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{z}}{\partial r}\right) = \frac{c^2}{r^2}\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial \theta^2} e^{-im\theta}\,d\theta.$$

Integration by parts on the remaining integral gives

$$\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial \theta^2} e^{-im\theta}\,d\theta = \left\{\frac{\partial z}{\partial \theta} e^{-im\theta}\right\}_{-\pi}^{\pi} + \int_{-\pi}^{\pi} im\frac{\partial z}{\partial \theta} e^{-im\theta}\,d\theta$$

$$= \frac{\partial z(r,\pi,t)}{\partial \theta}\cos(-m\pi) - \frac{\partial z(r,-\pi,t)}{\partial \theta}\cos m\pi + im\int_{-\pi}^{\pi}\frac{\partial z}{\partial \theta} e^{-im\theta}\,d\theta.$$

Because $\partial z/\partial\theta$ must be 2π-periodic, it follows that $\partial z(r,\pi,t)/\partial\theta = \partial z(r,-\pi,t)/\partial\theta$, and therefore

$$\begin{aligned}\int_{-\pi}^{\pi} \frac{\partial^2 z}{\partial\theta^2} e^{-im\theta}\,d\theta &= im\int_{-\pi}^{\pi} \frac{\partial z}{\partial\theta} e^{-im\theta}\,d\theta \\ &= im\{ze^{-im\theta}\}_{-\pi}^{\pi} + im\int_{-\pi}^{\pi} imze^{-im\theta}\,d\theta \\ &= im(z(r,\pi,t)\cos(-m\pi) - z(r,-\pi,t)\cos m\pi) - m^2\int_{-\pi}^{\pi} ze^{-im\theta}\,d\theta \\ &= -m^2\tilde{z},\end{aligned}$$

since $z(r,\theta,t)$ must also be 2π-periodic. Consequently, $\tilde{z}(r,m,t)$ must satisfy the PDE

$$\frac{\partial^2\tilde{z}}{\partial t^2} = c^2\left(\frac{\partial^2\tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{z}}{\partial r} - \frac{m^2}{r^2}\tilde{z}\right), \qquad 0 < r < r_2, \qquad t > 0 \tag{64a}$$

subject to the transforms of (63b–d),

$$\tilde{z}(r_2,m,t) = A\sin\omega t\,\tilde{1}, \qquad t > 0, \tag{64b}$$

$$\tilde{z}(r,m,0) = \tilde{f}(r,m), \qquad 0 < r < r_2, \tag{64c}$$

$$\tilde{z}_t(r,m,0) = 0, \qquad 0 < r < r_2, \tag{64d}$$

where

$$\tilde{1} = \int_{-\pi}^{\pi} e^{-im\theta}\,d\theta = \begin{cases}2\pi & m = 0\\ 0 & m \neq 0\end{cases}. \tag{64e}$$

To eliminate r from problem (64), we use the finite Hankel transform

$$\tilde{f}(\lambda_{mn}) = \int_0^{r_2} rf(r)R_{mn}(r)\,dr, \tag{65}$$

where $R_{mn}(r)$ are the orthonormal eigenfunctions of the Sturm-Liouville system

$$(rR')' + \left(\lambda^2 r - \frac{m^2}{r}\right)R = 0, \qquad 0 < r < r_2, \tag{66a}$$

$$R(r_2) = 0 \tag{66b}$$

[the system that would result were separation performed on problem (64) with the homogeneous version of (64b)]. Application of (65) to (64a) and integration by parts give

$$\begin{aligned}\frac{\partial^2\tilde{\tilde{z}}}{\partial t^2} &= c^2\int_0^{r_2} r\left(\frac{\partial^2\tilde{z}}{\partial r^2} + \frac{1}{r}\frac{\partial\tilde{z}}{\partial r} - \frac{m^2}{r^2}\tilde{z}\right)R_{mn}\,dr \\ &= c^2\left\{rR_{mn}\frac{\partial\tilde{z}}{\partial r}\right\}_0^{r_2} + c^2\int_0^{r_2}\left(-\frac{\partial\tilde{z}}{\partial r}(rR_{mn})' + \frac{\partial\tilde{z}}{\partial r}R_{mn} - \frac{m^2}{r}\tilde{z}R_{mn}\right)dr \\ &= c^2\int_0^{r_2}\left(-r\frac{\partial\tilde{z}}{\partial r}R'_{mn} - \frac{m^2}{r}\tilde{z}R_{mn}\right)dr \quad [\text{since } R_{mn}(r_2) = 0] \\ &= c^2\{-r\tilde{z}R'_{mn}\}_0^{r_2} + c^2\int_0^{r_2}\left(\tilde{z}(rR'_{mn})' - \frac{m^2}{r}\tilde{z}R_{mn}\right)dr\end{aligned}$$

$$= -r_2c^2A\sin\omega t\,\bar{1}R'_{mn}(r_2) + c^2\int_0^{r_2}\tilde{z}\left((rR'_{mn})' - \frac{m^2}{r}R_{mn}\right)dr \quad \text{[by (64b)]}$$

$$= -r_2c^2A\bar{1}R'_{mn}(r_2)\sin\omega t + c^2\int_0^{r_2}\tilde{z}(-\lambda_{mn}{}^2 r)R_{mn}\,dr \quad \text{[by (66a)]}$$

$$= -r_2c^2A\bar{1}R'_{mn}(r_2)\sin\omega t - c^2\lambda_{mn}^2\tilde{\tilde{z}}.$$

Thus, $\tilde{\tilde{z}}(\lambda_{mn}, m, t)$ must satisfy the ODE

$$\frac{d^2\tilde{\tilde{z}}}{dt^2} + c^2\lambda_{mn}^2\tilde{\tilde{z}} = -r_2c^2A\bar{1}R'_{mn}(r_2)\sin\omega t \tag{67a}$$

subject to

$$\tilde{\tilde{z}}(\lambda_{mn}, m, 0) = \tilde{\tilde{f}}(\lambda_{mn}, m), \tag{67b}$$

$$\tilde{\tilde{z}}_t(\lambda_{mn}, m, 0) = 0. \tag{67c}$$

The general solution of (67a) is

$$\tilde{\tilde{z}}(\lambda_{mn}, m, t) = \begin{cases} B_{0n}\cos c\lambda_{0n}t + D_{0n}\sin c\lambda_{0n}t + \dfrac{2\pi r_2c^2AR'_{0n}(r_2)\sin\omega t}{\omega^2 - c^2\lambda_{0n}^2} & m = 0 \\ B_{mn}\cos c\lambda_{mn}t + D_{mn}\sin c\lambda_{mn}t & m \neq 0 \end{cases} \tag{68}$$

provided $\omega \neq c\lambda_{0n}$ for any n. Discussion of this special case is given in Exercise 18. Initial conditions (67b, c) yield

$$\tilde{\tilde{z}}(\lambda_{mn}, m, t) = \begin{cases} \tilde{\tilde{f}}(\lambda_{0n}, 0)\cos c\lambda_{0n}t + \dfrac{2\pi Ar_2cR'_{0n}(r_2)}{\lambda_{0n}(\omega^2 - c^2\lambda_{0n}^2)}(c\lambda_{0n}\sin\omega t - \omega\sin c\lambda_{0n}t) & m = 0 \\ \tilde{\tilde{f}}(\lambda_{mn}, m)\cos c\lambda_{mn}t & m \neq 0 \end{cases} \tag{69}$$

The inverse transform now yields

$$z(r,\theta,t) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\sum_{n=1}^{\infty}\tilde{\tilde{z}}(\lambda_{mn}, m, t)R_{mn}(r)e^{im\theta}$$

$$= \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\tilde{\tilde{f}}(\lambda_{0n}, 0)\cos c\lambda_{0n}t + \frac{2\pi Ar_2cR'_{0n}(r_2)}{\lambda_{0n}(\omega^2 - c^2\lambda_{0n}^2)}(c\lambda_{0n}\sin\omega t - \omega\sin c\lambda_{0n}t)\right)R_{0n}(r)$$

$$+ \frac{1}{2\pi}\sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty}\sum_{n=1}^{\infty}\tilde{\tilde{f}}(\lambda_{mn}, m)\cos c\lambda_{mn}t\,R_{mn}(r)e^{im\theta}. \tag{70}$$

We can reduce the second double summation by noting that $\lambda_{-mn} = \lambda_{mn}$, $R_{-mn}(r) = R_{mn}(r)$, and $\tilde{\tilde{f}}(\lambda_{-mn}, -m) = \overline{\tilde{\tilde{f}}(\lambda_{mn}, m)}$ [the complex conjugate of $\tilde{\tilde{f}}(\lambda_{mn}, m)$]. Then

$$z(r,\theta,t) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\tilde{\tilde{f}}(\lambda_{0n}, 0)\cos c\lambda_{0n}t + \frac{2\pi Ar_2cR'_{0n}(r_2)}{\lambda_{0n}(\omega^2 - c^2\lambda_{0n}^2)}(c\lambda_{0n}\sin\omega t - \omega\sin c\lambda_{0n}t)\right)R_{0n}(r)$$

$$+ \frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\tilde{\tilde{f}}(\lambda_{mn}, m)e^{im\theta} + \overline{\tilde{\tilde{f}}(\lambda_{mn}, m)}e^{-im\theta}\right)\cos c\lambda_{mn}t\,R_{mn}(r)$$

or

$$z(r,\theta,t) = \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\tilde{\tilde{f}}(\lambda_{0n},0)\cos c\lambda_{0n}t + \frac{2\pi A r_2 c R'_{0n}(r_2)}{\lambda_{0n}(\omega^2 - c^2\lambda_{0n}^2)}(c\lambda_{0n}\sin\omega t - \omega\sin c\lambda_{0n}t)\right)R_{0n}(r)$$
$$+\frac{1}{2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} 2\,\mathrm{Re}[\tilde{\tilde{f}}(\lambda_{mn},m)e^{im\theta}]\cos c\lambda_{mn}t\,R_{mn}(r). \quad \blacksquare \quad (71)$$

Our final example is a potential problem.

nple 7: Find the potential inside a sphere if the potential on the sphere is only a function $g(\phi)$ of angle ϕ and the region contains a constant charge with density σ.

olution: The boundary value problem is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right) = -\frac{\sigma}{\varepsilon}, \qquad 0 < r < r_2, \qquad 0 < \phi < \pi, \quad (72a)$$

$$V(r_2,\phi) = g(\phi), \qquad 0 < \phi < \pi. \quad (72b)$$

To remove ϕ from the problem, we use the Legendre transform

$$\tilde{f}(n) = \int_0^\pi \sin\phi\, f(\phi)\Phi_n(\phi)\,d\phi, \quad (73)$$

where $\Phi_n(\phi) = \sqrt{(2n+1)/2}P_n(\cos\phi)$ are orthonormal eigenfunctions of the Sturm-Liouville system

$$(\sin\phi\,\Phi')' + n(n+1)\sin\phi\,\Phi = 0, \qquad 0 < \phi < \pi \quad (74)$$

[the system that would result were separation of variables applied to the homogeneous version of (72a)]. Application of (73) to (72a) and integration by parts give

$$\frac{d^2\tilde{V}}{dr^2} + \frac{2}{r}\frac{d\tilde{V}}{dr} + \frac{\sigma}{\varepsilon}\tilde{1} = \frac{-1}{r^2}\int_0^\pi \frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial V}{\partial\phi}\right)\Phi_n(\phi)\,d\phi$$
$$= \frac{-1}{r^2}\left(\left\{\sin\phi\frac{\partial V}{\partial\phi}\Phi_n\right\}_0^\pi - \int_0^\pi \sin\phi\frac{\partial V}{\partial\phi}\Phi'_n\,d\phi\right)$$
$$= \frac{1}{r^2}\left(\left\{\sin\phi\, V\Phi'_n\right\}_0^\pi - \int_0^\pi V(\sin\phi\,\Phi'_n)'\,d\phi\right)$$
$$= \frac{-1}{r^2}\int_0^\pi V[-n(n+1)\sin\phi\,\Phi_n]\,d\phi \quad [\text{by } (74)]$$
$$= \frac{n(n+1)}{r^2}\tilde{V}.$$

Thus, $\tilde{V}(r,n)$ must satisfy the ODE

$$\frac{d^2\tilde{V}}{dr^2} + \frac{2}{r}\frac{d\tilde{V}}{dr} - \frac{n(n+1)}{r^2}\tilde{V} = -\frac{\sigma}{\varepsilon}\tilde{1}, \quad (75a)$$

where

$$\tilde{1} = \int_0^\pi \sin\phi\,\Phi_n\,d\phi = \begin{cases}\sqrt{2}, & n = 0\\ 0 & n > 0\end{cases} \quad (75b)$$

subject to

$$\tilde{V}(r_2, n) = \tilde{g}(n). \tag{75c}$$

The general solution of (75a) is

$$\tilde{V}(r, n) = \begin{cases} A_0 + \dfrac{B_0}{r} - \dfrac{\sqrt{2}\sigma r^2}{6\varepsilon} & n = 0 \\ A_n r^n + \dfrac{B_n}{r^{n+1}}, & n > 0 \end{cases}. \tag{76}$$

The only bounded solution satisfying (75c) is

$$\tilde{V}(r, n) = \begin{cases} \tilde{g}(0) + \dfrac{\sqrt{2}\sigma}{6\varepsilon}(r_2^2 - r^2) & n = 0 \\ \dfrac{\tilde{g}(n)}{r_2^n} r^n & n > 0 \end{cases}, \tag{77}$$

and therefore

$$\begin{aligned} V(r, \phi) &= \sum_{n=0}^{\infty} \tilde{V}(r, n)\Phi_n(\phi) \\ &= \frac{\tilde{g}(0)}{\sqrt{2}} + \frac{\sigma}{6\varepsilon}(r_2^2 - r^2) + \sum_{n=1}^{\infty} \left(\frac{r}{r_2}\right)^n \tilde{g}(n)\Phi_n(\phi) \\ &= \frac{\sigma}{6\varepsilon}(r_2^2 - r^2) + \sum_{n=0}^{\infty} \left(\frac{r}{r_2}\right)^n \tilde{g}(n)\Phi_n(\phi). \end{aligned} \tag{78}$$

In retrospect, notice that $\sigma(r_2^2 - r^2)/(6\varepsilon)$ satisfies (72a) and a homogeneous (72b), while the series part of $V(r, \phi)$ satisfies (72b) and a homogeneous (72a). ■

Exercises 9.2

Part A—Heat Conduction

1. Solve Example 5 if the temperature of the edge $r = r_2$ is a function $f(t)$ of time.
2. (a) Solve Example 5 if heat is transferred to the plate along its edge $r = r_2$ at a rate $f_1(t)$ W/m^2 equally all around.
   (b) Simplify the solution when $f_1(t) = Q$, a constant.
3. (a) A very long cylinder of radius r_2 is initially at temperature $f(r)$. For time $t > 0$, its edge $r = r_2$ is held at 0°C. If heat generation within the cylinder is $g(r, t)$, find the temperature for $0 \le r < r_2$ and $t > 0$.
   (b) Simplify the solution in (a) when $f(r) \equiv 0$ and $g(r, t)$ is constant.
   (c) Solve the problem in (b) by separating off the steady-state solution.
4. Repeat Exercise 3(a) and (b) if the boundary $r = r_2$ is insulated.
5. Repeat Exercise 3 if heat is transferred at $r = r_2$ to a medium at constant temperature U_m according to Newton's law of cooling.

6. (a) A sphere of radius r_2 is initially at temperature $f(r)$. For $t > 0$, its surface is held at temperature $f_1(t)$, and heat is generated at a rate $g(r,t)$. Find the temperature in the sphere. (See Exercise 5 in Section 8.4 for the appropriate finite Fourier transform.)
   (b) Simplify the solution when $f(r) \equiv 0$, $f_1(t) \equiv 0$, and $g(r,t)$ is constant.
   (c) Simplify the solution when $f(r) \equiv 0$, $g(r,t) \equiv 0$, and $f_1(t)$ is constant.
7. (a) A sphere of radius r_2 is initially at temperature $f(r)$. For $t > 0$, heat is added to its surface at a rate $f_1(t)$ W/m^2, and heat is generated at a rate $g(r,t)$ W/m^3. Find the temperature in the sphere. (See Exercise 6 in Section 8.4 for the appropriate finite Fourier transform.)
   (b) Simplify the solution when $f(r) \equiv 0$, $g(r,t) \equiv 0$, and $f_1(t)$ is constant.
8. A cylinder of length L and radius r_2 is initially at temperature $f(r,z)$, $0 \le r \le r_2$, $0 \le z \le L$. For time $t > 0$, the face $z = 0$ is insulated, face $z = L$ has a time-dependent temperature $f_1(t)$, and the round surface $r = r_2$ has temperature $f_2(t)$. Find the temperature of the cylinder for $t > 0$.
9. A hemisphere $x^2 + y^2 + z^2 \le r_2^2$, $z \ge 0$, is initially at temperature zero throughout. For time $t > 0$, its base $z = 0$ continues to be held at temperature zero, but the surface of the hemisphere has a time-dependent temperature $f_1(t)$. Find a series representation for temperature inside the hemisphere. (*Hint:* You will need the eigenfunctions from Exercise 8 in Section 8.4.)
10. Solve Example 5 if the constant temperature on $r = r_2$ is replaced by $f(\theta) = \sin\theta$.
11. (a) Solve Example 5 when the initial temperature of the plate is $f(r,\theta)$.
   (b) Does the solution reduce to that of Example 5 when $f(r,\theta) = 0$?
12. Solve Exercise 2 when the initial temperature of the plate is $f(r,\theta)$.
13. Solve Example 5 if heat is exchanged with a constant-temperature environment along the edge $r = r_2$ according to Newton's law of cooling and the initial temperature of the plate is $f(r,\theta)$.

Part B—Vibrations

14. (a) Find the displacement of a circular membrane of radius r_2 that is initially ($t = 0$) at rest but is displaced according to $f(r,\theta)$, the boundary of which is displaced permanently according to $f_1(\theta)$.
   (b) Simplify the solution when $f(r,\theta)$ and $f_1(\theta)$ are independent of θ.
15. Solve the following nonhomogeneous version of Exercise 22 in Section 9.1:

$$\frac{\partial^2 y}{\partial t^2} = -g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) + \frac{F(x,t)}{\rho}, \qquad 0 < x < L, \qquad t > 0,$$
$$y(L,t) = 0, \qquad t > 0,$$
$$y(x,0) = f(x), \qquad 0 < x < L,$$
$$y_t(x,0) = h(x), \qquad 0 < x < L.$$

16. A circular membrane of radius r_2 is initially at rest on the xy-plane. For time $t > 0$, its edge is forced to undergo periodic oscillations described by $A\sin\omega t$, A a constant. Use finite Fourier transforms to find its displacement as a function of r and t. Include a discussion of resonance.
17. A circular membrane of radius r_2 is initially at rest on the xy-plane. For time $t > 0$, a periodic vertical force per unit area $A\sin\omega t$ (A a constant) acts at every point in the membrane. If its edge $r = r_2$ is fixed on the xy-plane, find its displacement.
18. Discuss the solution of Example 6 when $\omega = c\lambda_{0k}$ for some k.
19. Do the solutions of Example 6 and Exercise 18 reduce to those of Exercise 16 when $f(r,\theta) \equiv 0$?

Part C—Potential, Steady-State Heat Conduction, and Static Deflections of Membranes

20. A solid cylinder is bounded by the planes $\theta = 0$ and $\theta = \beta$ and the curved surface $r = r_2$ $(0 \le \theta \le \beta)$. A constant charge density σ exists inside the cylinder. If the three bounding surfaces are all held at potential zero, find the potential interior to the cylinder. Special consideration is required for the cases $\beta = \pi/2$, π, and $3\pi/2$.
21. An infinite cylinder of radius r_2 has charge density kr^n, $k > 0$ and $n > 0$ constants. If the surface of the cylinder has potential $f(\theta)$, what is the interior potential?
22. A hemisphere $x^2 + y^2 + z^2 \le r_2^2$, $z \ge 0$, has a constant charge σ throughout. If potentials on the rounded and flat surfaces are both specified constants, but different ones, find the potential inside. (You will need the results of Exercise 5 in Section 8.6 and Exercise 8 in Section 8.5.)
23. A thin plate is in the shape of a sector of a circle bounded by the lines $\theta = 0$ and $\theta = \beta < \pi$ and the arc $r = r_2$, $0 \le \theta \le \beta$. Edge $\theta = \beta$ is insulated, as are the top and bottom of the plate. Heat is removed from the plate along the edge $\theta = 0$ at a constant rate $q > 0$ W/m^2. Along the curved edge $r = r_2$, heat is also removed at a constant rate $Q > 0$ W/m^2. Heat is being generated at each point in the plate at a uniform rate of g W/m^3.
    (a) Formulate the boundary value problem for steady-state temperature in the plate. (See Exercises 16 and 17 in Section 1.2 for the boundary conditions along $\theta = 0$ and $r = r_2$.) What condition must q, Q, and g satisfy?
    (b) Solve the problem in (a).

9.3 Hankel Transforms

Fourier transforms have been used to remove Cartesian coordinates on infinite intervals from (initial) boundary value problems; Fourier sine and cosine transforms are applicable to Cartesian coordinates on semi-infinite intervals. For problems in polar and cylindrical coordinates wherein the radial coordinate has range $r \ge 0$, the Hankel transform is prominent. It is based on Bessel's differential equation

$$\frac{d}{dr}\left(r\frac{dR}{dr}\right) + \left(\lambda^2 r - \frac{v^2}{r}\right)R = 0, \qquad r > 0, \qquad v \ge 0. \tag{79}$$

We have already seen that solutions of this differential equation that are bounded near $r = 0$ are multiples of

$$R(r) = J_v(\lambda r). \tag{80}$$

In order to associate a transform with $J_v(\lambda r)$, we must be aware of the behavior of Bessel functions for large r. It is shown in the theory of asymptotics that $J_v(r)$ may be approximated for large r by

$$J_v(r) \approx \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4} - \frac{v\pi}{2}\right), \tag{81}$$

the approximation being better the larger the value of r. This means that for large r, $J_v(r)$ is oscillatory with an amplitude that decays at the same rate as $1/\sqrt{r}$.

Corresponding to the corollary of Theorem 1 in Section 7.2, we have the following *Hankel integral formula.*

heorem 1

If $\sqrt{r}f(r)$ is absolutely integrable on $0 < r < \infty$, and $f(r)$ is piecewise smooth on every finite interval, then for $0 < r < \infty$,

$$\frac{f(r+)+f(r-)}{2} = \int_0^\infty \lambda A(\lambda) J_\nu(\lambda r)\, d\lambda \tag{82a}$$

where

$$A(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\, dr. \tag{82b}$$

In view of the asymptotic behavior of $J_\nu(r)$ in expression (81), it is clear that absolute integrability of $\sqrt{r}f(r)$ guarantees convergence of (82b). Associated with this integral formula is the Hankel transform $\tilde{f}_\nu(\lambda)$ of a function $f(r)$,

$$\tilde{f}_\nu(\lambda) = \int_0^\infty r f(r) J_\nu(\lambda r)\, dr, \tag{83a}$$

and its inverse,

$$f(r) = \int_0^\infty \lambda \tilde{f}_\nu(\lambda) J_\nu(\lambda r)\, d\lambda, \tag{83b}$$

where it is understood in (83b) that $f(r)$ is defined as the average of left and right limits at points of discontinuity. We place a subscript ν on $\tilde{f}_\nu(\lambda)$ to remind ourselves that the Hankel transform is dependent on the choice of ν in (79); changing ν changes the transform.

xample 8: Find the Hankel transform $\tilde{f}_\nu(\lambda)$ of

$$f(r) = \begin{cases} r^\nu & 0 < r < a \\ 0 & r > a \end{cases}.$$

Solution: By definition (83a),

$$\tilde{f}_\nu(\lambda) = \int_0^\infty r^{\nu+1} J_\nu(\lambda r)\, dr = \int_0^a r^{\nu+1} J_\nu(\lambda r)\, dr.$$

If we set $u = \lambda r$, then

$$\begin{aligned}\tilde{f}_\nu(\lambda) &= \int_0^{\lambda a} \left(\frac{u}{\lambda}\right)^{\nu+1} J_\nu(u) \frac{du}{\lambda} = \frac{1}{\lambda^{\nu+2}} \int_0^{\lambda a} u^{\nu+1} J_\nu(u)\, du \\ &= \frac{1}{\lambda^{\nu+2}} \int_0^{\lambda a} \frac{d}{du}[u^{\nu+1} J_{\nu+1}(u)]\, du \quad \text{[see equation (39) in Section 8.3]} \\ &= \frac{1}{\lambda} a^{\nu+1} J_{\nu+1}(\lambda a).\end{aligned}$$

The inverse Hankel transform (83b) then gives

$$\int_0^\infty \lambda \left(\frac{1}{\lambda} a^{\nu+1} J_{\nu+1}(\lambda a)\right) J_\nu(\lambda r)\, d\lambda = \begin{cases} r^\nu & 0 < r < a \\ a^\nu/2 & r = a, \\ 0 & r > a \end{cases}$$

and from this we obtain the following useful integration formula:

$$\int_0^\infty J_{\nu+1}(\lambda a)J_\nu(\lambda r)\,d\lambda = \begin{cases} \dfrac{1}{a}\left(\dfrac{r}{a}\right)^\nu & 0 < r < a \\ 1/(2a) & r = a. \\ 0 & r > a \end{cases}$$

Example 9: Use the Hankel transform to find an integral representation for the solution of the heat conduction problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \qquad r > 0, \qquad t > 0, \tag{84a}$$

$$U(r,0) = f(r), \qquad r > 0. \tag{84b}$$

Solution: Because the Bessel function $J_0(r)$ results when separation of variables is performed on the PDE, we apply the Hankel transform associated with $J_0(r)$, namely,

$$\tilde{f}(\lambda) = \int_0^\infty rf(r)J_0(\lambda r)\,dr,$$

where we have suppressed the zero subscript on $\tilde{f}(\lambda)$. Application of this transform to the PDE gives

$$\begin{aligned}
\frac{d\tilde{U}}{dt} &= k\int_0^\infty r\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right)J_0(\lambda r)\,dr \\
&= k\left\{r\frac{\partial U}{\partial r}J_0(\lambda r)\right\}_0^\infty - k\int_0^\infty \frac{\partial U}{\partial r}\left(\frac{d}{dr}[rJ_0(\lambda r)] - J_0(\lambda r)\right)dr \\
&= -k\int_0^\infty \frac{\partial U}{\partial r}r\frac{d}{dr}[J_0(\lambda r)]\,dr \quad \left(\text{provided } \lim_{r\to\infty}\sqrt{r}\frac{\partial U}{\partial r} = 0\right) \\
&= -k\left\{Ur\frac{d}{dr}[J_0(\lambda r)]\right\}_0^\infty + k\int_0^\infty U\frac{d}{dr}\left(r\frac{dJ_0(\lambda r)}{dr}\right)dr \\
&= k\int_0^\infty U(-\lambda^2 rJ_0(\lambda r))\,dr \quad \left(\text{provided } \lim_{r\to\infty}\sqrt{r}\,U = 0\right) \\
&= -k\lambda^2\tilde{U}.
\end{aligned}$$

Thus, $\tilde{U}(\lambda, t)$ must satisfy the ODE

$$\frac{d\tilde{U}}{dt} + k\lambda^2\tilde{U} = 0 \tag{85a}$$

subject to the transform of (84b),

$$\tilde{U}(\lambda, 0) = \tilde{f}(\lambda) = \int_0^\infty rf(r)J_0(\lambda r)\,dr. \tag{85b}$$

The solution of this problem is

$$\tilde{U}(\lambda, t) = \tilde{f}(\lambda)e^{-k\lambda^2 t}, \tag{86}$$

and therefore

$$U(r,t) = \int_0^\infty \lambda \tilde{f}(\lambda)e^{-k\lambda^2 t} J_0(\lambda r)\, d\lambda. \quad \blacksquare \tag{87}$$

Exercises 9.3

Part A—Heat Conduction

1. Heat is generated at a constant rate g W/m^2 inside the cylinder $0 < r < a$ for time $t > 0$. If the temperature of space is zero at time $t = 0$, find the temperature at all points for $t > 0$.
2. An infinite wedge is bounded by the straight edges $\theta = 0$ and $\theta = \alpha (0 < \alpha < 2\pi)$. At time $t = 0$, its temperature is zero throughout, and for $t > 0$, its edges $\theta = 0$ and $\theta = \alpha$ are held at constant temperature $\bar{U}$. Find the temperature in the wedge for $t > 0$. *Hint:* Apply a finite Fourier transform with respect to θ and a Hankel transform with respect to r. You will need the result that

$$\int_0^\infty \frac{J_\nu(x)}{x}\,dx = \frac{1}{\nu}.$$

Part B—Vibrations

3. (a) A very large membrane is given an initial displacement that is only a function $f(r)$ of distance r from some fixed point but has no initial velocity. Find an integral representation for its subsequent displacement.

   (b) Use the result that

$$\int_0^\infty e^{-a\lambda} J_0(\lambda r)\, d\lambda = \frac{1}{\sqrt{r^2 + a^2}}$$

   to simplify the solution when $f(r) = A/\sqrt{1 + (r/a)^2}$, a and A positive constants, to

$$z(r,t) = aA \int_0^\infty e^{-a\lambda} \cos c\lambda t\, J_0(\lambda r)\, d\lambda.$$

   (c) By expressing $\cos c\lambda t$ as the real part of $e^{ic\lambda t}$, show that the solution can be expressed in the form

$$z(r,t) = \frac{aA\sqrt{\sqrt{(r^2 + a^2 - c^2t^2)^2 + 4a^2c^2t^2} + (r^2 + a^2 - c^2t^2)}}{\sqrt{2}\sqrt{(r^2 + a^2 - c^2t^2)^2 + 4a^2c^2t^2}}.$$

4. Repeat part (a) of Exercise 3 when $f(r)$ is the initial velocity of the membrane and it has no initial displacement.

Part C—Potential, Steady-State Heat Conduction, and Static Deflections of Membranes

5. A disc $0 \le r < a$ in the xy-plane emits heat into the region $z > 0$ at a constant rate Q W/m^2. If the remainder ($r > a$) of the plane is insulated, the steady-state temperature in $z > 0$ must satisfy

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = 0, \qquad r > 0, \qquad z > 0,$$

$$\frac{\partial U(r,0)}{\partial z} = \begin{cases} -Q/\kappa & 0 \le r < a \\ 0 & r > a \end{cases}.$$

Find $U(r, z)$.

6. Repeat Exercise 5 if the disc is held at constant temperature $\bar{U}$ and the remainder of the xy-plane is held at temperature zero.