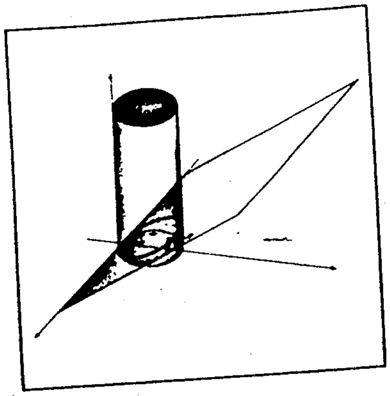


CHAPTER THREE

Separation of Variables

3.1 Linearity and Superposition

Separation of variables is one of the most fundamental techniques for solving PDEs. It is a method that can by itself yield solutions to many initial boundary value problems; in addition, it is the basis for more sophisticated techniques that must be used on more complicated problems. Separation of variables is applied to linear PDEs. A PDE is said to be *linear* if it is linear in the unknown function and all its derivatives (but not necessarily in the independent variables). For example, the most general linear second-order PDE for a function $u(x, y)$ of two independent variables is

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2} + A(x, y)\frac{\partial u}{\partial x} + B(x, y)\frac{\partial u}{\partial y} + C(x, y)u = F(x, y); \tag{1}$$

it is a linear combination of u and its partial derivatives, the coefficients being functions of only the independent variables x and y. All linear PDEs may be represented symbolically in the form

$$Lu = F, \tag{2}$$

where L is a linear* differential operator. In particular, for PDE (1), $L = a\partial^2/\partial x^2 + b\partial^2/\partial x\partial y + c\partial^2/\partial y^2 + A\partial/\partial x + B\partial/\partial y + C$.

When $F(x, y) \equiv 0$ in (1), the PDE is said to be *homogeneous*; otherwise, it is said to be *nonhomogeneous*.

The study of linear ordinary differential equations is based on the idea of *superposition*—that when solutions to a linear, homogeneous ODE are added together, new solutions are obtained. These same principles are the basis for separation of variables in PDEs. We set them forth in the following two theorems.

Theorem 1 (Superposition Principle 1)

If u_j ($j = 1, \ldots, n$) are solutions of the same linear, homogeneous PDE, then so also is any linear combination of the u_j,

$$u = \sum_{j=1}^{n} c_j u_j, \qquad c_j = \text{constants}.$$

Furthermore, if each u_j satisfies the same linear, homogeneous boundary and/or initial conditions, then so also does u.

For example, if $y_1(x, t)$ and $y_2(x, t)$ are solutions of the one-dimensional wave equation $y_{tt} = (\tau/\rho)y_{xx}$ and the boundary conditions $y(0, t) = 0$ and $y(L, t) = 0$, then for $y(x, t) = c_1 y_1 + c_2 y_2$,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2}{\partial t^2}(c_1 y_1 + c_2 y_2) = c_1 \frac{\partial^2 y_1}{\partial t^2} + c_2 \frac{\partial^2 y_2}{\partial t^2}$$

$$= c_1 \frac{\tau}{\rho}\frac{\partial^2 y_1}{\partial x^2} + c_2 \frac{\tau}{\rho}\frac{\partial^2 y_2}{\partial x^2} = \frac{\tau}{\rho}\frac{\partial^2}{\partial x^2}(c_1 y_1 + c_2 y_2) = \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2}$$

and

$$y(0, t) = c_1 y_1(0, t) + c_2 y_2(0, t) = 0,$$

$$y(L, t) = c_1 y_1(L, t) + c_2 y_2(L, t) = 0.$$

Thus $y(x, t)$ satisfies the same linear, homogeneous PDE and boundary conditions as y_1 and y_2.

In short, superposition principle 1 states that linear combinations of solutions to linear, homogeneous PDEs and linear, homogeneous subsidiary conditions are solutions of the same PDE and conditions. Superposition principle 2 addresses nonhomogeneous PDEs. It states that nonhomogeneous terms in a PDE may be handled individually, if it is desirable to do so.

Theorem 2 (Superposition Principle 2)

If u_j ($j = 1, \ldots, n$) are, respectively, solutions of linear, nonhomogeneous PDEs $Lu = F_j$, then $u = \sum_{j=1}^{n} u_j$ is a solution of $Lu = \sum_{j=1}^{n} F_j$.

* An operator L is linear if for any two functions $u(x, y)$ and $v(x, y)$ and any constants C_1 and C_2,

$$L(C_1 u + C_2 v) = C_1(Lu) + C_2(Lv).$$

For example, if $U_1(x, y, t)$ and $U_2(x, y, t)$ satisfy the two-dimensional heat conduction equations

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa} g_1(x, y, t), \qquad \frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa} g_2(x, y, t),$$

respectively, then $U(x, y, t) = U_1(x, y, t) + U_2(x, y, t)$ satisfies

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{k}{\kappa}[g_1(x, y, t) + g_2(x, y, t)].$$

This principle can also be extended to incorporate nonhomogeneous boundary conditions. To illustrate, consider the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \quad 0 < x < L, \quad 0 < y < L',$$
$$V(0, y) = g_1(y), \quad 0 < y < L',$$
$$V(L, y) = g_2(y), \quad 0 < y < L',$$
$$V(x, 0) = h_1(x), \quad 0 < x < L,$$
$$V(x, L') = h_2(x), \quad 0 < x < L,$$

for potential in the rectangle of Figure 3.1. The solution is the sum of the functions $V_1(x, y)$, $V_2(x, y)$, and $V_3(x, y)$, satisfying the PDEs in Figure 3.2 together with the

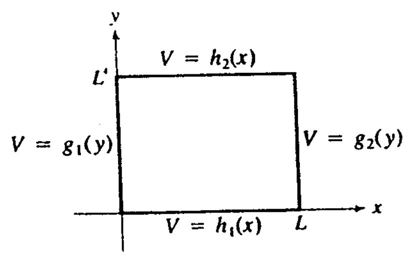


Figure 3.1

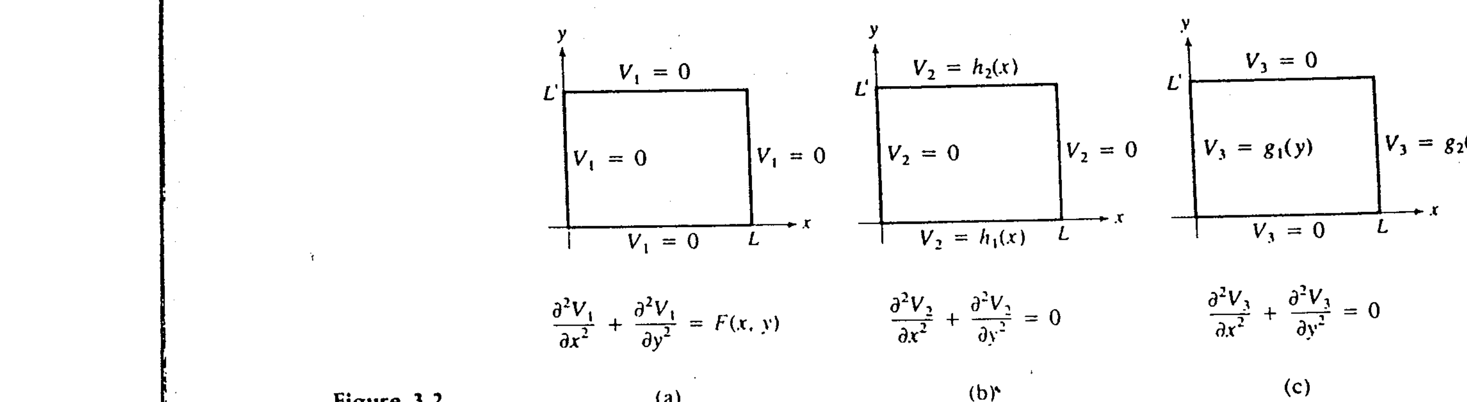


Figure 3.2

indicated boundary conditions. The problem in Figure 3.2(b) could be further subdivided into two problems, each of which contained only one nonhomogeneous boundary condition [as could the problem in Figure 3.2(c)]. In Section 3.2 we show that this is not necessary.

Exercises 3.1

In Exercises 1–10, determine whether the PDE is linear. Which of the linear equations are homogeneous and which are nonhomogeneous?

1. $\dfrac{\partial^2 y}{\partial x^2} = \dfrac{\partial^2 y}{\partial t^2} + \dfrac{\partial y}{\partial t} + y$

2. $\dfrac{\partial^2 U}{\partial x^2} = 3\dfrac{\partial U}{\partial t} + U^2 + t^2 x$

3. $\dfrac{\partial^2 y}{\partial x^2}\dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial y}{\partial t} + \dfrac{\partial y}{\partial x}$

4. $\dfrac{\partial^2 y}{\partial x^2} + \dfrac{\partial^2 y}{\partial t^2} = \dfrac{\partial y}{\partial t}\dfrac{\partial y}{\partial x}$

5. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = F(x, y, z)V$

6. $x^2\dfrac{\partial V}{\partial x} + x\dfrac{\partial^2 V}{\partial y^2} = xy$

7. $2\dfrac{\partial y}{\partial t} = xt\dfrac{\partial^2 y}{\partial x^2} + e^t\dfrac{\partial y}{\partial x} + t$

8. $\dfrac{\partial^2 U}{\partial t^2} + 2\dfrac{\partial^2 U}{\partial x \partial t} + \dfrac{\partial^2 U}{\partial x^2} = U\left(\dfrac{\partial U}{\partial x} + \dfrac{\partial U}{\partial t}\right)$

9. $\dfrac{\partial^2 U}{\partial x^2} - \dfrac{\partial^2 U}{\partial y^2} = 0$

10. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial V}{\partial x} - \dfrac{\partial V}{\partial y} = 3V$

11. Prove Theorem 1.

12. Prove Theorem 2.

13. Based on superposition principle 2, how would you subdivide the problem consisting of Poisson's equation $\nabla^2 V = F(x, y, z)$ inside the box $0 < x < L, 0 < y < L', 0 < z < L''$, subject to the following boundary conditions?

$$\begin{aligned}
V(0, y, z) &= f_1(y, z), & 0 < y < L', && 0 < z < L'',\\
V(L, y, z) &= f_2(y, z), & 0 < y < L', && 0 < z < L'',\\
V(x, 0, z) &= g_1(x, z), & 0 < x < L, && 0 < z < L'',\\
V(x, L', z) &= g_2(x, z), & 0 < x < L, && 0 < z < L'',\\
V(x, y, 0) &= h_1(x, y), & 0 < x < L, && 0 < y < L',\\
V(x, y, L'') &= h_2(x, y), & 0 < x < L, && 0 < y < L'.
\end{aligned}$$

14. (a) Show that $u_1(x, y) = e^{x+y}$ and $u_2(x, y) = e^{x-y}$ are solutions of the nonlinear PDE

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 2u^2.$$

(b) Is $u_1 + u_2$ a solution?

3.2 Separation of Variables

Before considering specific initial boundary value problems, we illustrate the basic idea of separation of variables on the PDE

$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial y}{\partial t}. \tag{3}$$

Separation of variables assumes that functions $y(x, t)$ satisfying (3) can be found that are functions $X(x)$ of x multiplied by functions $T(t)$ of t; that is, it assumes that there are functions satisfying (3) that are of the form

$$y(x,t) = X(x)T(t). \tag{4}$$

When this representation for $y(x, t)$ is substituted into the PDE,

$$\frac{d^2X}{dx^2}T(t) = X(x)\frac{dT}{dt},$$

and division by $X(x)T(t)$ gives

$$\frac{1}{X(x)}\frac{d^2X}{dx^2} = \frac{1}{T(t)}\frac{dT}{dt}. \tag{5}$$

The right side of this equation is a function of t only, and the left side is a function of x only. In other words, variables x and t have been *separated* from each other. Now, the only way this equation can hold for a range of values of x and t is for both sides to be equal to some constant, say α, which we take as real†; that is, we may write

$$\frac{1}{X}\frac{d^2X}{dx^2} = \alpha = \frac{1}{T}\frac{dT}{dt}. \tag{6}$$

We call this the *separation principle.*‡ Equation (6) gives rise to two ordinary differential equations for $X(x)$ and $T(t)$,

$$\frac{d^2X}{dx^2} - \alpha X = 0 \quad \text{and} \quad \frac{dT}{dt} - \alpha T = 0. \tag{7}$$

Thus, by assuming that a function $y(x, t) = X(x)T(t)$ with variables separated satisfies (3), the PDE is replaced by the two ODEs (7). Boundary and/or initial conditions accompanying PDE (3) may give rise to subsidiary conditions to accompany ODEs (7). We shall see these in the examples to follow.

There is no reason to expect *a priori* that the solution to an initial boundary value problem should separate in form (4). In fact, separation of variables, by itself, seldom yields the solution to an initial boundary value problem. However, separated functions can often be combined to yield the solution to an initial boundary value problem. We illustrate these ideas with the initial boundary value problem for transverse vibrations of a taut string with fixed ends (Figure 3.3):

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{8a}$$

$$y(0, t) = 0, \qquad t > 0. \tag{8b}$$

† That α must be real for the problems of this chapter is proved in Exercise 30. That α must always be real is verified in Chapter 4.

‡ That the separation principle is valid can also be seen by differentiating (5) with respect to x. The result is

$$\frac{d}{dx}\left(\frac{1}{X}\frac{d^2X}{dx^2}\right) = 0,$$

and this implies that $(1/X)d^2X/dx^2$ must be equal to a numerical constant.

$$y(L,t) = 0, \quad t > 0, \tag{8c}$$

$$y(x,0) = f(x), \quad 0 < x < L, \tag{8d}$$

$$y_t(x,0) = 0, \quad 0 < x < L, \tag{8e}$$

where $c^2 = \tau/\rho$. Conditions (8d, e) indicate an initial displacement defined by $f(x)$ and zero initial velocity. We solve this problem for three initial displacement functions:

$$\text{(a) } 3\sin\frac{\pi x}{L}, \qquad \text{(b) } 3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L}, \qquad \text{(c) } x(L-x).$$

We begin by searching for separated functions that satisfy the (linear, homogeneous) PDE, the (linear) homogeneous boundary conditions (8b, c), and the (linear) homogeneous initial condition (8e). We do not consider initial condition (8d); it is nonhomogeneous. As a general principle, then, separated functions are sought to satisfy only linear and homogeneous PDEs, boundary conditions, and initial conditions.

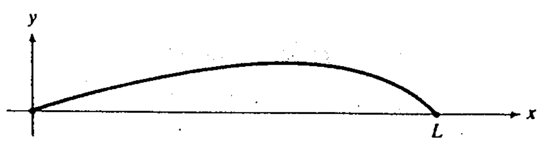


Figure 3.3

When we substitute a separated function $y(x,t) = X(x)T(t)$ into (8a),

$$XT'' = c^2X''T \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{c^2T},$$

where the $''$ on X'' indicates derivatives with respect to x, whereas on T'', it represents derivatives with respect to t. By the separation principle, we may set each side of this equation equal to a constant, say α, which is independent of both x and t. This results in two ODEs for $X(x)$ and $T(t)$,

$$X'' - \alpha X = 0, \qquad T'' - \alpha c^2 T = 0. \tag{9}$$

Homogeneous boundary condition (8b) implies that

$$X(0)T(t) = 0, \qquad t > 0.$$

Because $T(t) \not\equiv 0$ (why not?), it follows that $X(0) = 0$. Similarly, homogeneous boundary condition (8c) and initial condition (8e) require that $X(L) = 0$ and $T'(0) = 0$. Thus, $X(x)$ and $T(t)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \tag{10a}$$

$$X(0) = 0, \tag{10b}$$

$$X(L) = 0; \tag{10c}$$

$$T'' - \alpha c^2 T = 0, \quad t > 0, \tag{11a}$$

$$T'(0) = 0. \tag{11b}$$

Notice once again that we do not consider nonhomogeneous condition (8d) at this time. For a separated function $y(x,t) = X(x)T(t)$, it would imply that $X(x)T(0) = f(x)$, but this would give no information about $X(x)$ and $T(t)$ separately. This is always the situation; nonhomogeneous boundary and/or initial conditions are never considered in conjunction with separation of the PDE.

Solutions of ODEs (10) and (11) depend on whether α is positive, zero, or negative. On purely physical grounds, a positive or zero value can be eliminated, for in these cases the time dependence of y is given by

$$T(t) = Ae^{c\sqrt{\alpha}t} + Be^{-c\sqrt{\alpha}t} \quad \text{and} \quad T(t) = At + B,$$

respectively, and these certainly do not yield oscillatory motions. Alternatively, for positive α and zero α, the general solution of (10a) is

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x} \quad \text{and} \quad X(x) = Ax + B.$$

But boundary conditions (10b, c) imply that $A = B = 0$, and this in turn implies that $y(x, t) = 0$. Because α must therefore be negative, we set $\alpha = -\lambda^2$ ($\lambda > 0$) and replace systems (10) and (11) with

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \quad \text{(12a)} \qquad T'' + c^2\lambda^2 T = 0, \quad t > 0, \quad \text{(13a)}$$

$$X(0) = 0, \quad \text{(12b)} \qquad T'(0) = 0. \quad \text{(13b)}$$

$$X(L) = 0; \quad \text{(12c)}$$

Boundary conditions (12b, c) on the general solution

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

of (12a) yield

$$0 = A, \qquad 0 = B\sin\lambda L.$$

Since we cannot set $B = 0$ [else $X(x) = 0$], we must therefore set $\sin\lambda L = 0$, and this implies that $\lambda L = n\pi$, n an integer. Thus,

$$X(x) = B\sin\frac{n\pi x}{L}.$$

Condition (13b) on the general solution

$$T(t) = F\cos\frac{n\pi ct}{L} + G\sin\frac{n\pi ct}{L}$$

of (13a) yields

$$0 = \frac{n\pi c}{L}G \quad \text{or} \quad G = 0.$$

We have now determined that the separated function

$$y(x, t) = X(x)T(t) = \left(B\sin\frac{n\pi x}{L}\right)\left(F\cos\frac{n\pi ct}{L}\right) = b\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \tag{14}$$

for an arbitrary constant b and any integer n is a solution of the one-dimensional wave equation (8a) and conditions (8b, c, e). The final condition (8d) requires b and n to satisfy

$$f(x) = b\sin\frac{n\pi x}{L}, \qquad 0 < x < L. \tag{15}$$

We now consider the three cases for $f(x)$ following equation (8e), namely, $3\sin(\pi x/L)$, $3\sin(\pi x/L) - \sin(2\pi x/L)$, and $x(L-x)$.

(a) When $f(x) = 3\sin(\pi x/L)$, this condition becomes

$$3\sin\frac{\pi x}{L} = b\sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

Obviously, we should choose $b = 3$ and $n = 1$, in which case the solution of initial boundary value problem (8) is

$$y(x,t) = 3\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L}.$$

This function is drawn for various values of t in Figure 3.4. The string oscillates back and forth between its initial position and the negative thereof, doing so once every $2L/c$ seconds.

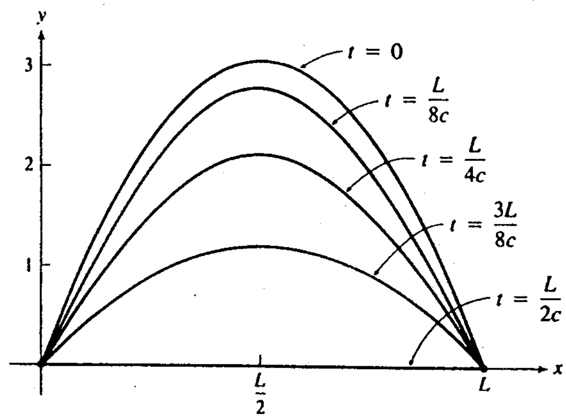


Figure 3.4

We have shown, then, that when the initial position of the string is $3\sin(\pi x/L)$, separation of variables leads to the solution of problem (8).

(b) When $f(x) = 3\sin(\pi x/L) - \sin(2\pi x/L)$, condition (15) is

$$3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L} = b\sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

It is not possible to choose values for b and n to satisfy this equation. In other words, the solution of (8) is not separable when $f(x) = 3\sin(\pi x/L) - \sin(2\pi x/L)$. Does this mean that we must abandon separation? Fortunately, the answer is no. Because PDE (8a), boundary conditions (8b, c), and initial condition (8e) are all linear and homogeneous, superposition principle 1 states that linear combinations of solutions of (8a, b, c, e) are also solutions. In particular, the function

$$y(x,t) = b\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} + d\sin\frac{m\pi x}{L}\cos\frac{m\pi ct}{L}$$

satisfies (8a, b, c, e) for arbitrary integers n and m and any constants b and d. If we apply initial condition (8d) to this function, b, d, n, and m must satisfy

$$3\sin\frac{\pi x}{L} - \sin\frac{2\pi x}{L} = b\sin\frac{n\pi x}{L} + d\sin\frac{m\pi x}{L}, \qquad 0 < x < L.$$

Clearly, we should choose $b = 3$, $d = -1$, $n = 1$, and $m = 2$, in which case the solution of (8) is

$$y(x,t) = 3\sin\frac{\pi x}{L}\cos\frac{\pi ct}{L} - \sin\frac{2\pi x}{L}\cos\frac{2\pi ct}{L}.$$

This is not a separated solution; it is the sum of two separated functions. The motion of the string in this case has two terms, called *modes*. The first term, $3\sin(\pi x/L) \times \cos(\pi ct/L)$, called the fundamental mode, is shown in Figure 3.4. The second mode, $-\sin(2\pi x/L)\cos(2\pi ct/L)$, is illustrated in Figure 3.5 for the same times. Oscillations of this mode occur twice as fast as those for the fundamental mode. The addition of these two modes gives the position of the string in Figure 3.6.

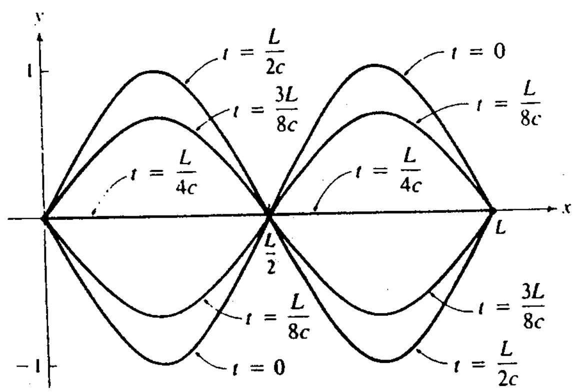


Figure 3.5

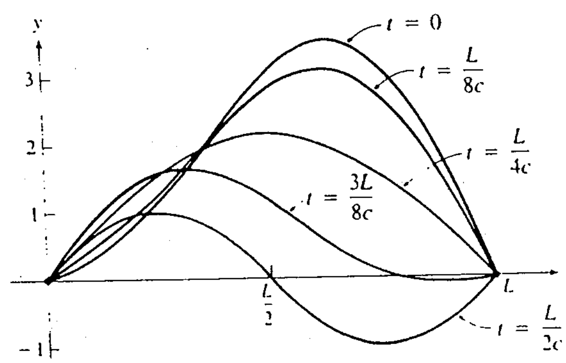


Figure 3.6

(c) Finally, we consider the case in which the initial displacement in the string is parabolic, $f(x) = x(L - x)$. It is definitely impossible to satisfy (15),

$$x(L - x) = b\sin\frac{n\pi x}{L}, \qquad 0 < x < L,$$

for any choice of b and n. Furthermore, for no finite linear combination of terms of the form $b\sin(n\pi x/L)$ can coefficients (b) and integers (n) be chosen to satisfy this condition. Does this mean the ultimate demise of separation of variables? Again, the answer is no. We superpose an infinity of separated functions in the form

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{16}$$

where the constants b_n are arbitrary. No advantage is gained by including terms with negative values of n, for if we had a term in $-n$ (n positive), say

$$X_{-n}(x) = b_{-n}\sin\left(\frac{-n\pi x}{L}\right),$$

we could combine it with

$$X_n(x) = b_n \sin\frac{n\pi x}{L}$$

and write

$$X_n + X_{-n} = (b_n - b_{-n})\sin\frac{n\pi x}{L} = B_n \sin\frac{n\pi x}{L},$$

which is of the same form as $X_n(x)$.

Initial condition (8d) requires the b_n in (16) to satisfy

$$x(L - x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L. \tag{17}$$

This equation is satisfied if the b_n are chosen as the coefficients in the Fourier sine series of the odd extension of $x(L - x)$ to a function of period $2L$. According to equation (18b) in Chapter 2,

$$b_n = \frac{2}{L}\int_0^L x(L - x)\sin\frac{n\pi x}{L}\,dx,$$

and integration by parts leads to

$$b_n = \frac{4L^2[1 + (-1)^{n+1}]}{n^3\pi^3}$$

(see Exercise 10 in Section 2.2). Substitution of these into (16) gives displacements of the string when the initial position is $f(x) = x(L - x)$:

$$\begin{aligned} y(x, t) &= \sum_{n=1}^{\infty} \frac{4L^2[1 + (-1)^{n+1}]}{n^3\pi^3}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L} \\ &= \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n - 1)^3}\sin\frac{(2n - 1)\pi x}{L}\cos\frac{(2n - 1)\pi ct}{L}. \end{aligned} \tag{18}$$

Each term in this series is called a mode of vibration of the string. The position of the string is the sum of an infinite number of modes, lower modes contributing more significantly than higher ones. We shall have more to say about them in Section 5.2.

You would be wise in questioning whether (18) is really a solution of problem (8). Certainly it satisfies boundary conditions (8b, c), and, because $x(L - x)$ is continuously differentiable, our theory of Fourier series implies that initial condition (8d) must also be satisfied. Conditions (8a) and (8e) present difficulties, however. First of all, because (16) is the superposition of an infinity of separated functions, and superposition principle 1 discusses only finite combinations, an infinite combination must be suspect. Second, because (18) is an infinite series, there is a question of its convergence. Does it, for instance, converge for $0 < x < L$ and $t > 0$, and do its derivatives satisfy wave equation (8a) and initial condition (8e)? Each of these questions must be answered, and we shall do so, but not at this time. In this chapter, we wish to illustrate the technique of separation of variables and some of its adaptations to more difficult problems. Verification that the resulting series are truly solutions of initial boundary value problems is discussed in Sections 5.6–5.8. To remind us that these series have not yet been verified as solutions to their respective problems, we call them *formal* solutions.

The one-dimensional wave equation (8a) is a hyperbolic second-order equation (see Section 1.8). In the following two examples we show that separation of variables can be used on parabolic and elliptic equations as well.

Example 1: Solve the following initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides and no internal heat generation (Figure 3.7):

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{19a}$$

$$U_x(0,t) = 0, \qquad t > 0, \tag{19b}$$

$$U_x(L,t) = 0, \qquad t > 0, \tag{19c}$$

$$U(x,0) = x, \qquad 0 < x < L. \tag{19d}$$

The ends of the rod are also insulated [conditions (19b, c)], and its initial temperature increases linearly from $U = 0$ at $x = 0$ to $U = L$ at $x = L$.

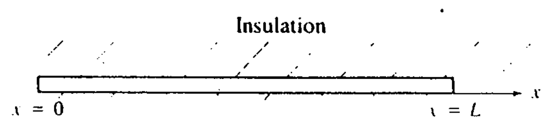


Figure 3.7

Solution: The assumption of a separated function $U(x,t) = X(x)T(t)$ satisfying (19a) leads to

$$XT' = kX''T \quad \text{or} \quad \frac{X''}{X} = \frac{T'}{kT}.$$

The separation principle implies that both sides of the last equation must be equal to a constant, say α, in which case

$$X'' - \alpha X = 0, \qquad T' - \alpha kT = 0.$$

Homogeneous boundary conditions (19b, c) imply that $X'(0) = 0 = X'(L)$, so that $X(x)$ and $T(t)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \quad \textbf{(20a)} \qquad T' - \alpha kT = 0. \qquad \textbf{(21)}$$
$$X'(0) = 0, \qquad \textbf{(20b)}$$
$$X'(L) = 0; \qquad \textbf{(20c)}$$

For positive α, the general solution of (20a) is

$$X(x) = Ae^{\sqrt{\alpha}x} + Be^{-\sqrt{\alpha}x},$$

and boundary conditions (20b, c) require that

$$0 = A - B, \qquad 0 = Ae^{\sqrt{\alpha}L} - Be^{-\sqrt{\alpha}L}.$$

From these, $A = B = 0$, and therefore α cannot be positive. For $\alpha = 0$, we obtain $X(x) = Ax + B$, and the boundary conditions imply that $A = 0$. Thus when $\alpha = 0$, solutions of (20) and (21) are

$$X(x) = B = \text{constant} \quad \text{and} \quad T(t) = D = \text{constant}.$$

What we have shown, then, is that $U(x, t) = X(x)T(t) = \text{constant}$ satisfies PDE (19a) and boundary conditions (19b, c).

When α is negative, for convenience we set $\alpha = -\lambda^2$ ($\lambda > 0$), in which case (20) and (21) are replaced by

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \quad \textbf{(22a)} \qquad T' + k\lambda^2 T = 0, \quad t > 0. \qquad \textbf{(23)}$$
$$X'(0) = 0, \qquad \textbf{(22b)}$$
$$X'(L) = 0; \qquad \textbf{(22c)}$$

Boundary conditions (22b, c) on the general solution

$$X(x) = A\cos\lambda x + B\sin\lambda x$$

of (22a) require that

$$0 = B, \qquad 0 = \lambda A\sin\lambda L.$$

Since we cannot set $A = 0$ [else $X(x) = 0$], we must therefore set $\sin\lambda L = 0$, and this implies that $\lambda L = n\pi$, n an integer. Thus,

$$X(x) = A\cos\frac{n\pi x}{L}.$$

The general solution of (23) is

$$T(t) = De^{-n^2\pi^2kt/L^2}.$$

Consequently, besides constant functions, we also have separated functions,

$$X(x)T(t) = \left(A\cos\frac{n\pi x}{L}\right)\left(De^{-n^2\pi^2kt/L^2}\right) = ae^{-n^2\pi^2kt/L^2}\cos\frac{n\pi x}{L},$$

which satisfy (19a–c) for integers $n > 0$ and arbitrary a. Notice that when $n = 0$, this function reduces to the constant function corresponding to $\alpha = 0$. In other words, all separated functions satisfying (19a–c) can be expressed in the form

$$ae^{-n^2\pi^2kt/L^2}\cos\frac{n\pi x}{L}, \qquad n \geq 0.$$

(It is not necessary to include $n < 0$, since a is arbitrary.) Initial condition (19d) would require a separated function to satisfy

$$x = a\cos\frac{n\pi x}{L}, \qquad 0 < x < L,$$

an impossibility. But because the heat equation and boundary conditions are linear and homogeneous, we superpose separated functions and take

$$U(x, t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-n^2\pi^2kt/L^2}\cos\frac{n\pi x}{L}$$

with arbitrary constants a_n. Initial condition (19d) requires the a_n to satisfy

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad 0 < x < L.$$

This equation is satisfied if the a_n are chosen as the coefficients in the Fourier cosine series of the even extension of the function $f(x) = x$ to a function of period $2L$. According to equation (17b) in Chapter 2,

$$a_n = \frac{2}{L}\int_0^L x\cos\frac{n\pi x}{L}\,dx,$$

and integration gives

$$a_0 = L, \qquad a_n = \frac{2L[(-1)^n - 1]}{n^2\pi^2}, \qquad n > 0.$$

The formal solution of heat conduction problem (19) is therefore

$$\begin{aligned} U(x, t) &= \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L[(-1)^n - 1]}{n^2\pi^2} e^{-n^2\pi^2kt/L^2}\cos\frac{n\pi x}{L} \\ &= \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/L^2}\cos\frac{(2n-1)\pi x}{L}. \end{aligned} \tag{24}$$

An interesting feature of this solution is its limit as time t becomes very large:

$$\lim_{t\to\infty} U(x, t) = \frac{L}{2}.$$

In other words, for large times, the temperature of the rod becomes constant throughout. But this is exactly what we should expect. Because the rod is totally insulated after $t = 0$, the original amount of heat in the rod will redistribute itself until a steady-state situation is achieved, the steady-state temperature being a constant value equal to the average of the initial temperature distribution. Since initially the temperature varies linearly from $U = 0$ at one end to $U = L$ at the other, its average value is $L/2$, precisely that predicted by the above limit. ∎

For a copper rod of length 1 m and diffusivity $k = 114 \times 10^{-6}$ m^2/s, (24) becomes

$$U(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-114\times 10^{-6}(2n-1)^2\pi^2 t}\cos(2n-1)\pi x.$$

This function is plotted in Figure 3.8 for various values of t to illustrate the transition from initial temperature $U(x, 0) = x$ to final temperature 1/2. These curves indicate that $U(x, t)$ is always an increasing function of x, and therefore heat always flows from right to left. Notice also that each curve is horizontal at $x = 0$ and $x = 1$. This reflects boundary conditions (19b, c).

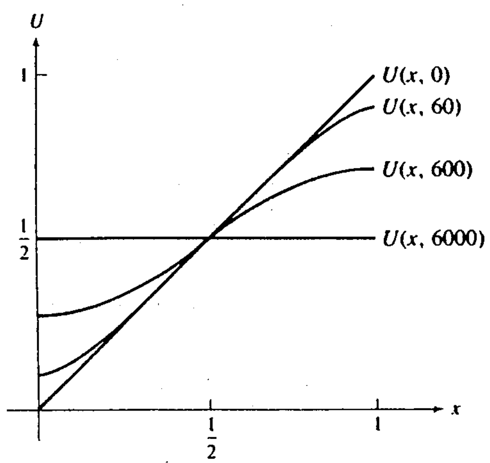


Figure 3.8

Example 2: Solve the following boundary value problem for potential in the rectangular plate of Figure 3.9 when the sides are maintained at the potentials shown:

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{25a}$$

$$V(0, y) = 0, \quad 0 < y < L', \tag{25b}$$

$$V(L, y) = 0, \quad 0 < y < L', \tag{25c}$$

$$V(x, L') = 0, \quad 0 < x < L, \tag{25d}$$

$$V(x, 0) = 1, \quad 0 < x < L. \tag{25e}$$

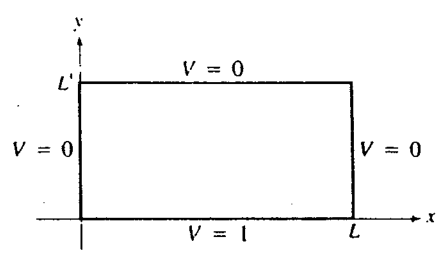


Figure 3.9

Solution: When we assume that a function with variables separated, $V(x, y) = X(x)Y(y)$, satisfies (25a),

$$X''Y + XY'' = 0 \quad \text{or} \quad \frac{X''}{X} = -\frac{Y''}{Y}.$$

The separation principle requires X''/X and $-Y''/Y$ both to equal a constant α, so that

$$X'' - \alpha X = 0, \qquad Y'' + \alpha Y = 0.$$

Homogeneous boundary conditions (25b–d) imply that $X(0) = X(L) = Y(L') = 0$, and therefore $X(x)$ and $Y(y)$ must satisfy

$$X'' - \alpha X = 0, \quad 0 < x < L, \quad \text{(26a)} \qquad Y'' + \alpha Y = 0, \quad 0 < y < L', \quad \text{(27a)}$$
$$X(0) = 0, \quad \text{(26b)} \qquad Y(L') = 0. \quad \text{(27b)}$$
$$X(L) = 0; \quad \text{(26c)}$$

System (26) is identical to (10); nontrivial solutions exist only when α is negative. If we set $\alpha = -\lambda^2$ ($\lambda > 0$), then $\lambda = n\pi/L$, and the solution of (26) is

$$X(x) = B\sin\frac{n\pi x}{L}$$

for arbitrary B and n an integer. With $\alpha = -\lambda^2 = -n^2\pi^2/L^2$, the general solution of (27a) is

$$Y(y) = D\cosh\frac{n\pi y}{L} + E\sinh\frac{n\pi y}{L},$$

and (27b) requires that

$$0 = D\cosh\frac{n\pi L'}{L} + E\sinh\frac{n\pi L'}{L}.$$

We solve this for E in terms of D, in which case

$$\begin{aligned} Y(y) &= D\cosh\frac{n\pi y}{L} - D\frac{\cosh(n\pi L'/L)}{\sinh(n\pi L'/L)}\sinh\frac{n\pi y}{L} \\ &= \frac{D}{\sinh(n\pi L'/L)}\left(\sinh\frac{n\pi L'}{L}\cosh\frac{n\pi y}{L} - \cosh\frac{n\pi L'}{L}\sinh\frac{n\pi y}{L}\right) \\ &= F\sinh\frac{n\pi(L' - y)}{L}, \qquad F = \frac{D}{\sinh(n\pi L'/L)}. \end{aligned}$$

We have now determined that separated functions

$$X(x)Y(y) = b\sin\frac{n\pi x}{L}\sinh\frac{n\pi(L' - y)}{L} \qquad (b = BF)$$

for any constant b and any integer n are solutions of Laplace's equation (25a) and boundary conditions (25b–d). Since these conditions and this PDE are linear and homogeneous, we superpose separated functions and take

$$V(x, y) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi(L' - y)}{L} \tag{28}$$

with arbitrary constants b_n. Boundary condition (25e) requires the b_n to satisfy

$$1 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi L'}{L} = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L,$$

where $C_n = b_n \sinh(n\pi L'/L)$. But this equation is satisfied if the numbers C_n are chosen as the coefficients in the Fourier sine series of the odd extension of the function $f(x) = 1$ to a function of period $2L$. Hence

$$C_n = b_n \sinh\frac{n\pi L'}{L} = \frac{2}{L}\int_0^L (1)\sin\frac{n\pi x}{L}\,dx = \frac{2[1 + (-1)^{n+1}]}{n\pi}.$$

Formal solution (28) of potential problem (25) is therefore

$$V(x, y) = \sum_{n=1}^{\infty} \frac{2[1 + (-1)^{n+1}]}{n\pi \sinh\frac{n\pi L'}{L}} \sin\frac{n\pi x}{L} \sinh\frac{n\pi(L' - y)}{L}$$

$$= \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n - 1)\sinh\frac{(2n - 1)\pi L'}{L}} \sin\frac{(2n - 1)\pi x}{L} \sinh\frac{(2n - 1)\pi(L' - y)}{L}. \tag{29}$$

■

These three examples have illustrated the essentials of the method of separation of variables and Fourier series for boundary value and initial boundary value problems. In each, functions with variables separated are found to satisfy the linear, homogeneous PDE and the linear, homogeneous boundary and/or initial conditions. These separated functions invariably involve an arbitrary multiplicative constant and an integer parameter. To satisfy the one nonhomogeneous boundary or initial condition, these functions are superposed into an infinite series.

Our next example illustrates that separation of variables is not restricted to second-order PDEs.

Example 3: Transverse vibrations of a uniform beam with simply supported ends (Figure 3.10) are described by the initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2} + c^2\frac{\partial^4 y}{\partial x^4} = 0, \qquad 0 < x < L, \qquad t > 0, \tag{30a}$$

$$y(0, t) = 0, \qquad t > 0, \tag{30b}$$

$$y(L, t) = 0, \qquad t > 0, \tag{30c}$$

$$y_{xx}(0, t) = 0, \qquad t > 0, \tag{30d}$$

$$y_{xx}(L, t) = 0, \qquad t > 0, \tag{30e}$$

$$y(x, 0) = x \sin\frac{\pi x}{L}, \qquad 0 < x < L, \tag{30f}$$

$$y_t(x, 0) = 0, \qquad 0 < x < L, \tag{30g}$$

where $c^2 = EI/\rho$. The force of gravity on the beam has been assumed negligible relative to internal forces (see Section 1.5). Conditions (30f, g) indicate an initial displacement $x\sin(\pi x/L)$ and zero initial velocity. Solve this problem.

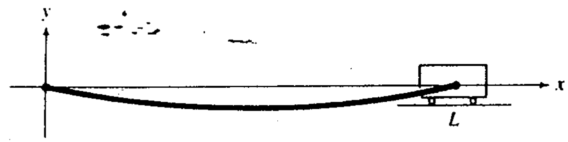


Figure 3.10

Solution: Substitution of a function $y(x,t) = X(x)T(t)$ with variables separated into (30a) gives

$$XT'' + c^2X''''T = 0 \quad \text{or} \quad \frac{X''''}{X} = \frac{-T''}{c^2T}.$$

The separation principle implies that

$$X'''' - \alpha X = 0 \quad \text{and} \quad T'' + \alpha c^2 T = 0$$

for some constant α. When $\alpha < 0$ and $\alpha = 0$, general solutions for $T(t)$ are

$$T(t) = A\cosh c\sqrt{-\alpha}t + B\sinh c\sqrt{-\alpha}t \quad \text{and} \quad T(t) = At + B,$$

respectively. Because the motion of the beam must be oscillatory, and neither of these functions displays this characteristic, we conclude that α must be positive. [The same conclusion can be obtained from the ODE $X'''' - \alpha X = 0$ in conjunction with boundary conditions (30b–e), but not so easily.] When we set $\alpha = \lambda^4$ ($\lambda > 0$) and use separation on homogeneous boundary conditions (30b–e) and initial condition (30g), $X(x)$ and $T(t)$ must satisfy the systems

$$X'''' - \lambda^4 X = 0, \quad 0 < x < L, \quad \text{(31a)} \qquad T'' + c^2\lambda^4 T = 0, \quad t > 0, \quad \text{(32a)}$$
$$X(0) = 0, \quad \text{(31b)} \qquad T'(0) = 0. \quad \text{(32b)}$$
$$X(L) = 0, \quad \text{(31c)}$$
$$X''(0) = 0, \quad \text{(31d)}$$
$$X''(L) = 0; \quad \text{(31e)}$$

Boundary conditions (31b–e) on the general solution

$$X(x) = A\cos\lambda x + B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x$$

of (31a) yield

$$0 = A + C,$$
$$0 = A\cos\lambda L + B\sin\lambda L + C\cosh\lambda L + D\sinh\lambda L,$$
$$0 = -\lambda^2 A + \lambda^2 C,$$
$$0 = -\lambda^2 A\cos\lambda L - \lambda^2 B\sin\lambda L + \lambda^2 C\cosh\lambda L + \lambda^2 D\sinh\lambda L.$$

The first and third of these imply that $A = C = 0$, while the second and fourth require that

$$B\sin\lambda L = 0, \qquad D\sinh\lambda L = 0.$$

Since $\lambda > 0$, we must set $D = 0$, in which case $B \neq 0$. It follows, then, that $\lambda L = n\pi$, n an integer, and

$$X(x) = B\sin\frac{n\pi x}{L}.$$

Condition (32b) on the general solution —

$$T(t) = E\cos\frac{n^2\pi^2 ct}{L^2} + F\sin\frac{n^2\pi^2 ct}{L^2}$$

of (32a) yields

$$0 = \frac{n^2\pi^2 c}{L^2}F,$$

from which $F = 0$. We have now determined that separated functions

$$X(x)T(t) = b\sin\frac{n\pi x}{L}\cos\frac{n^2\pi^2 ct}{L^2}$$

for an arbitrary constant b and any integer n are solutions of PDE (30a), its boundary conditions (30b–e), and initial condition (30g). Since the PDE and these conditions are linear and homogeneous, we superpose separated functions and take

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n^2\pi^2 ct}{L^2}$$

with arbitrary constants b_n. Condition (30f) requires the b_n to satisfy

$$x\sin\frac{\pi x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

The b_n are therefore the coefficients in the Fourier sine series of the odd extension of $x\sin(\pi x/L)$ to a function of period $2L$. Hence,

$$b_n = \frac{2}{L}\int_0^L x\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx,$$

and integration leads to

$$b_1 = \frac{L}{2}, \qquad b_n = \frac{-4nL[1 + (-1)^n]}{(n^2 - 1)^2\pi^2}, \qquad n > 1.$$

Transverse vibrations of the beam are therefore described formally by

$$\begin{aligned} y(x, t) &= \frac{L}{2}\sin\frac{\pi x}{L}\cos\frac{\pi^2 ct}{L^2} + \sum_{n=2}^{\infty} \frac{-4nL[1 + (-1)^n]}{(n^2 - 1)^2\pi^2}\sin\frac{n\pi x}{L}\cos\frac{n^2\pi^2 ct}{L^2} \\ &= \frac{L}{2}\sin\frac{\pi x}{L}\cos\frac{\pi^2 ct}{L^2} - \frac{16L}{\pi^2}\sum_{n=1}^{\infty}\frac{n}{(4n^2 - 1)^2}\sin\frac{2n\pi x}{L}\cos\frac{4n^2\pi^2 ct}{L^2}. \end{aligned} \qquad (33)$$

■

Exercises 3.2

Part A—Heat Conduction

1. Determine $U(x, t)$ in Example 1 if the initial temperature is constant throughout.
2. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $f(x), 0 \leq x \leq L$, at time $t = 0$. For time $t > 0$, its ends (at $x = 0$ and $x = L$) are held at temperature 0°C. Find a formula for the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.
3. (a) Use the result in Exercise 2 to find $U(x, t)$ when

$$f(x) = \begin{cases} x & 0 \leq x \leq L/2 \\ L - x & L/2 \leq x \leq L \end{cases}.$$

   (b) The amount of heat per unit area per unit time flowing from left to right across the cross section of the rod at position x and time t is the x-component of the heat flux vector (this being the only component) $q(x, t) = -\kappa \partial U/\partial x$ (see Section 1.2). Find the heat flow rate for cross sections at positions $x = 0$, $x = L/2$, and $x = L$ by calculating

$$\lim_{x \to 0^+} q(x, t), \qquad q\left(\frac{L}{2}, t\right), \qquad \lim_{x \to L^-} q(x, t).$$

   (c) Calculate limits of the heat flows in (b) as $t \to 0^+$ and $t \to \infty$.
4. Repeat parts (a), (b), and (c) of Exercise 3 if $f(x) = 10, 0 \leq x \leq L$. In addition,
   (d) Calculate

$$\lim_{x \to 0^+} U(x, 0) \quad \text{and} \quad \lim_{t \to 0^+} U(0, t).$$

   (e) Sketch what you feel $U(x, t)$ would look like as a function of x for various fixed values of t.
5. (a) Find the rate of flow of heat across the cross section at position $x = L/2$ for the rod in Example 1.
   (b) What is the limit of your answer in (a) as $t \to 0^+$?
6. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature $L - x$, $0 \leq x \leq L$, at time $t = 0$. For time $t > 0$, its right end, $x = L$, is held at temperature zero and its left end, $x = 0$, is insulated. Use the result of Exercise 22 in Section 2.2 to find the temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.

Part B—Vibrations

7. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement

$$f(x) = \begin{cases} x/5 & 0 \leq x \leq L/2 \\ (L - x)/5 & L/2 \leq x \leq L \end{cases}$$

   at time $t = 0$, but no initial velocity. Find its displacement for $t > 0$ and $0 < x < L$.
8. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial velocity $g(x) = x(L - x), 0 \leq x \leq L$ at time $t = 0$, but no initial displacement. Find its displacement for $t > 0$ and $0 < x < L$.
9. If the string in Exercises 7 and 8 is given both the initial displacement $f(x)$ and the initial velocity $g(x)$ at time $t = 0$, what is its displacement for $t > 0$ and $0 < x < L$?
10. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If it is given an initial displacement $f(x)$ and an initial velocity $g(x)$ at time $t = 0$, find a formula for its subsequent displacement in terms of integrals of $f(x)$ and $g(x)$.

11. Solve Exercise 10 if an external force (per unit x-length) $F = -ky$ ($k > 0$) acts at each point in the string.
12. Solve Exercise 11 if the external force $F = -ky$ is replaced by $F = -\beta \partial y/\partial t$. Assume that $\beta < 2\rho\pi c/L$.
13. A taut string is given an initial displacement (at time $t = 0$) of $f(x)$, $0 \le x \le L$ and initial velocity $g(x)$, $0 \le x \le L$. If the ends $x = 0$ and $x = L$ of the string are free to slide vertically without friction, find $y(x, t)$.
14. (a) What is the solution in Exercise 10 when $g(x) \equiv 0$?
    (b) Show that the series solution in (a) can be expressed in the form
$$y(x,t) = \frac{1}{2}[f(x+ct) + f(x-ct)],$$
    provided $f(x)$ is extended outside the interval $0 \le x \le L$ as an odd function of period $2L$. Is this the result predicted by d'Alembert's formula [(119)] in Section 1.7?
15. (a) What is the solution in Exercise 10 when $f(x) \equiv 0$?
    (b) Show that the solution in (a) can be expressed in the form
$$y(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta,$$
    provided $g(x)$ is extended outside the interval $0 \le x \le L$ as an odd function of period $2L$. Is this the result predicted by d'Alembert's formula [(119)] in Section 1.7?
16. A circular bar of natural length L is clamped at both ends and stretched until its length is L^*. At time $t = 0$ the left end of the bar is at position $x = 0$ and the clamps are removed. If horizontal vibrations occur along a frictionless surface, find displacements of cross sections of the bar.

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

17. A region A (in the xy-plane) is bounded by the lines $x = 0$, $y = 0$, $x = L$, and $y = L'$. If the edges $y = 0$, $y = L'$, and $x = L$ are held at potential zero, and $x = 0$ is at potential equal to 100, find the potential in A.
18. Solve Exercise 17 if edges $x = 0$ and $y = 0$ are at potential 100, while $x = L$ and $y = L'$ are at zero potential. (*Hint:* See the extension of superposition principle 2 in Figure 3.2.)
19. Solve Exercise 17 if edges $x = 0$ and $x = L$ are at potential 100 while $y = 0$ and $y = L'$ are at zero potential.
20. Solve Exercise 17 if the condition $V(0, y) = 100$ along $x = 0$ is replaced by $\partial V(0, y)/\partial x = 100$, $0 < y < L'$.
21. Solve Exercise 17 if the boundary conditions are
$$\frac{\partial V(0,y)}{\partial x} = 100, \qquad 0 < y < L',$$
$$\frac{\partial V(L,y)}{\partial x} = 100, \qquad 0 < y < L',$$
$$\frac{\partial V(x,0)}{\partial y} = \frac{\partial V(x,L')}{\partial y} = 0, \qquad 0 < x < L.$$
    Is the solution unique? What is the solution if $V(L/2, L'/2) = 0$?

22. Can Exercise 21 be solved if the condition along $x = L$ is $\partial V(L, y)/\partial x = -100$, $0 < y < L'$? Explain.
23. A thin rectangular plate occupies the region described by $0 \le x \le L$, $0 \le y \le L'$. Its top and bottom surfaces are insulated. If edges $x = 0$ and $x = L$ are held at temperature 0°C, while $y = 0$ and $y = L'$ have temperatures $x(L - x)$ and $-x(L - x)$, respectively, what is the steady-state temperature of the plate?
24. Solve Exercise 23 if edges $x = 0$, $x = L$, and $y = L'$ are held at temperature 0°C while heat is added along the edge $y = 0$ at a constant rate q W/m^2.
25. Solve Exercise 24 if heat is added to both edges $y = 0$ and $y = L'$ at rate q W/m^2 while edges $x = 0$ and $x = L$ are held at temperature 10°C.
26. A membrane is stretched tightly over the rectangle $0 \le x \le L$, $0 \le y \le L'$. Its edges are given deflections described by the following boundary conditions:

$$z(0, y) = kL(y - L')/L', \quad 0 < y < L',$$
$$z(L, y) = 0, \quad 0 < y < L',$$
$$z(x, 0) = k(x - L), \quad 0 < x < L,$$
$$z(x, L') = 0, \quad 0 < x < L$$

($k > 0$ a constant). Find static deflections of the membrane when all external forces are negligible compared with tensions in the membrane.
27. Find a formula for the solution of Laplace's equation inside the rectangle $0 \le x \le L$, $0 \le y \le L'$ of Figure 3.1 when
    (a) $g_1(y) = g_2(y) = h_2(x) = 0$;
    (b) $g_1(y) = g_2(y) = h_1(x) = 0$;
    (c) $g_1(y) = g_2(y) = 0$;
    (d) $h_1(x) = h_2(x) = 0$.
28. Solve Exercise 23 if edges $x = 0$ and $y = L'$ are insulated, $x = L$ is held at temperature 0°C, and $y = 0$ has temperature $(L - x)^2$, $0 \le x \le L$. (*Hint:* Use Exercise 22 in Section 2.2.)

Part D—General Results

29. Prove that a second-order, linear, homogeneous PDE in two independent variables with constant coefficients is always separable. (A more general result is proved in Exercise 10 of Section 4.3.)
30. Verify that we cannot have a complex separation constant α for the two problems (10) and (20).

3.3 Nonhomogeneities and Eigenfunction Expansions

In Section 3.2 we stressed the fact that separation of variables is carried out on linear, *homogeneous* PDEs and linear, *homogeneous* boundary and/or initial conditions. Separated functions are then superposed in order to satisfy nonhomogeneous conditions. When nonhomogeneities are present in PDE, or in the boundary conditions of time-dependent problems, separation by itself fails. To illustrate, we reconsider vibration problem (8) for displacement of a taut string with fixed end points, but take gravity into account:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + g, \quad 0 < x < L, \quad t > 0, \quad (g = -9.81), \tag{34a}$$

$$y(0,t) = 0, \qquad t > 0, \tag{34b}$$

$$y(L,t) = 0, \qquad t > 0, \tag{34c}$$

$$y(x,0) = f(x), \qquad 0 < x < L, \tag{34d}$$

$$y_t(x,0) = 0, \qquad 0 < x < L. \tag{34e}$$

Only the partial differential equation is affected; it becomes nonhomogeneous. The boundary conditions remain homogeneous. Substitution of a separated function $y(x,t) = X(x)T(t)$ into (34a) gives

$$XT'' = c^2X''T + g.$$

Our usual procedure of dividing by $X(x)T(t)$ would not lead to a separated equation; in fact, this equation cannot be separated. Likewise, were (34b) not homogeneous, say $y(0,t) = f(t)$, in which case the left end of the string would be forced to undergo specific motion, substitution of $y(x,t) = X(x)T(t)$ would not lead to information about $X(x)$ and $T(t)$ separately.

In this section we illustrate two methods for handling nonhomogeneities. The first method uses steady-state solutions for heat conduction problems and static deflections for vibration problems. It applies, however, only to time-independent nonhomogeneities. The second method is called *eigenfunction expansion*; it applies to time-dependent as well as time-independent nonhomogeneities.

Time-Independent Nonhomogeneities

Partial differential equation (34a) has a time-independent nonhomogeneity (it is also independent of x, but that is immaterial). To solve this problem, we define a new dependent variable $z(x,t)$ according to

$$y(x,t) = z(x,t) + \psi(x), \tag{35}$$

where $\psi(x)$ is the solution of the corresponding static-deflection problem

$$0 = c^2\frac{d^2\psi}{dx^2} + g, \qquad 0 < x < L, \tag{36a}$$

$$\psi(0) = 0, \qquad \psi(L) = 0. \tag{36b}$$

Differential equation (36a) implies that

$$\psi(x) = \frac{-g}{2c^2}x^2 + Ax + B,$$

and boundary conditions (36b) require that

$$0 = B, \qquad 0 = \frac{-g}{2c^2}L^2 + AL + B.$$

From these we obtain the position of the string were it to hang motionless under gravity:

$$\psi(x) = \frac{-g}{2c^2}x^2 + \frac{gL}{2c^2}x = \frac{gx}{2c^2}(L - x). \tag{37}$$

We expect that the string will vibrate about this position and that $z(x,t)$ represents displacements from this position. A PDE satisfied by $z(x,t)$ can be found by substituting (35) into (34a):

$$\frac{\partial^2}{\partial t^2}[z(x,t) + \psi(x)] = c^2\frac{\partial^2}{\partial x^2}[z(x,t) + \psi(x)] + g.$$

This equation simplifies to the following homogeneous PDE when we note that $\psi(x)$ is only a function of x that satisfies (36a):

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0. \tag{38a}$$

Boundary conditions for $z(x,t)$ are obtained by setting $x = 0$ and $x = L$ in (35) and using (34b, c):

$$z(0,t) = y(0,t) - \psi(0) = 0, \qquad t > 0, \tag{38b}$$

$$z(L,t) = y(L,t) - \psi(L) = 0, \qquad t > 0. \tag{38c}$$

Finally, by setting $t = 0$ in (35) and its partial derivative with respect to t, and using (34d, e), we obtain initial conditions for $z(x,t)$:

$$z(x,0) = y(x,0) - \psi(x) = f(x) - \frac{gx}{2c^2}(L - x), \qquad 0 < x < L, \tag{38d}$$

$$z_t(x,0) = y_t(x,0) = 0, \qquad 0 < x < L. \tag{38e}$$

We have therefore replaced problem (34), which has a nonhomogeneous PDE, with (38), which has a homogeneous PDE. We have complicated one of the initial conditions, but this is a small price to pay. As for problem (8), if a function with variables separated is to satisfy PDE (38a), boundary conditions (38b, c), and initial condition (38e), it must be of the form

$$b\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L},$$

for arbitrary b and n an integer. Because PDE (38a) and conditions (38b, c, e) are linear and homogeneous, we superpose these functions and take

$$z(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \tag{39}$$

Initial condition (38d) requires the constants b_n to satisfy

$$f(x) - \frac{gx}{2c^2}(L - x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

Consequently, the b_n are coefficients in the Fourier sine series of the odd extension of $f(x) - gx(L - x)/(2c^2)$ to a function of period $2L$; that is,

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - \frac{gx}{2c^2}(L - x)\right)\sin\frac{n\pi x}{L}\,dx. \tag{40}$$

The formal solution of vibration problem (34) is therefore

$$y(x,t) = \frac{gx}{2c^2}(L-x) + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{41}$$

where the b_n are given by (40).

This technique of separating off static deflections can be applied to any nonhomogeneity that is only a function of position, be it in the PDE or in a boundary condition. We illustrate nonhomogeneities in boundary conditions in the following example.

Example 4: Solve the initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides when the ends of the rod are held at constant nonzero temperatures

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{42a}$$

$$U(0,t) = U_0, \qquad t > 0, \tag{42b}$$

$$U(L,t) = U_L, \qquad t > 0, \tag{42c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{42d}$$

Solution: We define a new dependent variable $V(x,t)$ by

$$U(x,t) = V(x,t) + \psi(x), \tag{43}$$

where $\psi(x)$ is the solution of the associated steady-state problem

$$0 = k\frac{d^2\psi}{dx^2}, \qquad 0 < x < L, \tag{44a}$$

$$\psi(0) = U_0, \tag{44b}$$

$$\psi(L) = U_L. \tag{44c}$$

Differential equation (44a) implies that

$$\psi(x) = Ax + B,$$

and boundary conditions (44b, c) require that

$$U_0 = B, \qquad U_L = AL + B.$$

From these, we obtain the steady-state solution

$$\psi(x) = U_0 + \frac{(U_L - U_0)x}{L} \tag{45}$$

(the temperature in the rod after a very long time). With this choice for $\psi(x)$, the PDE for $V(x,t)$ can be found by substituting (43) into (42a):

$$\frac{\partial}{\partial t}[V(x,t) + \psi(x)] = k\frac{\partial^2}{\partial x^2}[V(x,t) + \psi(x)].$$

Because $\psi(x)$ is only a function of x that has a vanishing second derivative, this equation simplifies to

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0. \tag{46a}$$

Boundary conditions for $V(x,t)$ are obtained from (43) and (42b, c):

$$V(0,t) = U(0,t) - \psi(0) = U_0 - U_0 = 0, \qquad t > 0, \tag{46b}$$

$$V(L,t) = U(L,t) - \psi(L) = U_L - U_L = 0, \qquad t > 0. \tag{46c}$$

Finally, $V(x,t)$ must satisfy the initial condition

$$V(x,0) = U(x,0) - \psi(x) = f(x) - U_0 - \frac{(U_L - U_0)x}{L}, \qquad 0 < x < L. \tag{46d}$$

Separation of variables $V(x,t) = X(x)\,T(t)$ on (46a–c) leads to the ordinary differential equations

$$X'' + \lambda^2 X = 0, \quad 0 < x < L \quad \text{(47a)} \qquad T' + k\lambda^2 T = 0, \quad t > 0. \quad \text{(48)}$$

$$X(0) = X(L) = 0; \quad \text{(47b)}$$

These give separated functions

$$be^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$$

for arbitrary b and n an integer. To satisfy the initial condition, we superpose separated functions and take

$$V(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}. \tag{49}$$

Initial condition (46d) requires the constants b_n to satisfy

$$f(x) - U_0 - \frac{(U_L - U_0)x}{L} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L.$$

Consequently, the b_n are the coefficients in the Fourier sine series of the odd extension of $f(x) - U_0 - (U_L - U_0)x/L$ to a function of period $2L$:

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - U_0 - \frac{(U_L - U_0)x}{L}\right)\sin\frac{n\pi x}{L}\,dx. \tag{50}$$

The formal solution of (42) is therefore

$$U(x,t) = V(x,t) + U_0 + \frac{(U_L - U_0)x}{L}, \tag{51}$$

where $V(x,t)$ is given by (49) and b_n by (50). ■

It is interesting and informative to analyze solution (51) further for two specific initial temperature distributions $f(x)$. First, suppose that the initial temperature of the rod is 0°C throughout; that is, $f(x) \equiv 0$. In this case, equations (49)–(51) yield, for the temperature in the rod,

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L},$$

where $b_n = \frac{2}{L}\int_0^L \left(-U_0 - (U_L - U_0)\frac{x}{L}\right)\sin\frac{n\pi x}{L}\,dx = \frac{-2}{n\pi}[U_0 + (-1)^{n+1}U_L].$

This function is plotted for various fixed values of t in Figure 3.11 (using a diffusivity of $k = 12.4 \times 10^{-6}\,\text{m}^2/\text{s}$). What is important to notice is the smooth transition from initial temperature 0°C to final (steady-state) temperature at every point in the rod except for its ends, $x = 0$ and $x = L$. Here the transition is instantaneous, as is dictated by problem (42) when $f(x)$ is chosen to vanish identically. Physically, this is an impossibility, but the mathematics required to describe a very quick but smooth change in temperature from 0°C at $x = 0$ and $x = L$ to U_0 and U_L would complicate the problem enormously. In practice, then, we are willing to live with the anomaly of the solution at time $t = 0$ for $x = 0$ and $x = L$ in order to avoid these added complications. This anomaly is manifested in the heat transfer across the ends of the rod at time $t = 0$. According to equation (13) in Section 1.2, the amount of heat flowing left to right through any cross section of the rod is

$$q(x,t) = -\kappa\frac{\partial U}{\partial x} = -\kappa\left(\frac{U_L - U_0}{L} + \frac{\pi}{L}\sum_{n=1}^{\infty} nb_n e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}\right)$$

$$= \frac{\kappa}{L}\left(U_0 - U_L + 2\sum_{n=1}^{\infty}[U_0 + (-1)^{n+1}U_L]e^{-n^2\pi^2 kt/L^2}\cos\frac{n\pi x}{L}\right).$$

The series in this expression diverges (to infinity) when $x = 0$ and $t = 0$. In other words, the instantaneous temperature change at time $t = 0$ from 0°C to U_0°C is predicated on an infinite heat flux at that time. A similar situation occurs at the end $x = L$.

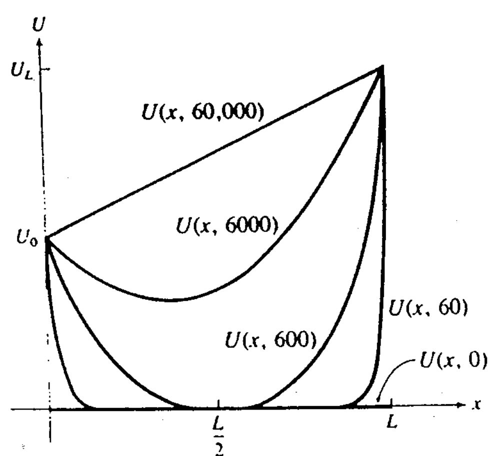


Figure 3.11

The second initial temperature function we consider is $f(x) = U_0(1 - x^2/L^2) + U_L x/L$, a distribution that does not give rise to abrupt temperature changes at time

$t = 0$ since $f(0) = U_0$ and $f(L) = U_L$. In this case, coefficients b_n in (50) are $4U_0[1 + (-1)^{n+1}]/(n^3\pi^3)$, and

$$U(x,t) = U_0 + \frac{(U_L - U_0)x}{L} + \frac{8U_0}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)^2\pi^2kt/L^2} \sin\frac{(2n-1)\pi x}{L}$$

As shown in Figure 3.12, the transition from initial to steady-state temperatu smooth for all $0 \le x \le L$. Supporting this is the heat flux vector

$$q(x,t) = \frac{\kappa}{L}\left(U_0 - U_L - \frac{8U_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2kt/L^2} \cos\frac{(2n-1)\pi x}{L}\right).$$

The series herein converges uniformly for $0 \le x \le L$ and $t \ge 0$. If we take limit $x \to 0^+$ and $t \to 0^+$, we find the initial heat flux across the end $x = 0$,

$$q(0+,0+) = \frac{\kappa}{L}\left(U_0 - U_L - \frac{8U_0}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\right)$$

$$= \frac{\kappa}{L}\left[U_0 - U_L - \frac{8U_0}{\pi^2}\left(\frac{\pi^2}{8}\right)\right] = -\frac{\kappa U_L}{L}$$

[since $\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$]. Perhaps unexpectedly, we find that the directior heat flow across $x = 0$ at time $t = 0$ is completely dictated by the sign of U_L. W $U_L < 0$, heat flows into the rod, and when $U_L > 0$, heat flows out. This is most ea seen by calculating the derivative of the initial temperature distribution in the roc $x = 0$, $f'(0) = U_L/L$. If $U_L < 0$, points in the rod near $x = 0$ have temperature less tl those in the end $x = 0$, and heat flows into the rod; if $U_L > 0$, points near $x = 0$ are ε higher temperature than those at $x = 0$, and heat flows out of the rod.

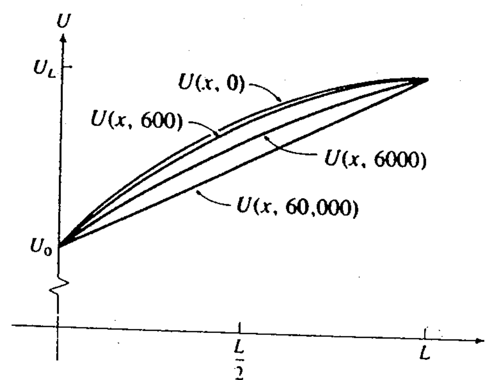


Figure 3.12

Time-Dependent Nonhomogeneities

When the nonhomogeneity in a PDE is time dependent, it is necessary to adopt different approach. The technique used resembles the method of variation c parameters for ODEs. Because variation of parameters for ODEs is used in a forr perhaps different from that which many readers might have seen, and because it lead into the method of eigenfunction expansion for PDEs, we digress to review th technique quickly. Consider the ODE

$$y'' + y = f(x), \tag{52a}$$

where $f(x)$ is as yet an unspecified function. The general solution of the associated homogeneous equation $y'' + y = 0$ is $y(x) = A\cos x + B\sin x$, to which must be added a particular solution of (52a). When $f(x)$ is a polynomial, an exponential, a sine, a cosine, or a combination of these, various techniques (such as undetermined coefficients, or operators) yield this particular solution. Variation of parameters also gives a particular solution in these cases, but it realizes its true potential when $f(x)$ is not one of these, or when a general solution is required for arbitrary $f(x)$. The method assumes that a general solution of (52a) can be found in the form $A\cos x + B\sin x$, but where A and B are functions of x; that is, it assumes that the general solution of the nonhomogeneous equation is $y(x) = A(x)\cos x + B(x)\sin x$. To obtain $A(x)$ and $B(x)$, this function is substituted into the differential equation. Because this imposes only one condition on two functions $A(x)$ and $B(x)$, the opportunity is taken to impose a second condition, and this condition is always taken as $A'(x)\cos x + B'(x)\sin x = 0$. The result is the following system of linear equations in $A'(x)$ and $B'(x)$:

$$A'(x)\cos x + B'(x)\sin x = 0, \tag{53a}$$

$$-A'(x)\sin x + B'(x)\cos x = f(x). \tag{53b}$$

These can be solved for

$$A'(x) = -f(x)\sin x \quad \text{and} \quad B'(x) = f(x)\cos x,$$

from which

$$A(x) = -\int f(x)\sin x\,dx + C_1 \quad \text{and} \quad B(x) = \int f(x)\cos x\,dx + C_2,$$

where C_1 and C_2 are constants of integration. The general solution of (52a) is therefore

$$y(x) = \left(C_1 - \int f(x)\sin x\,dx\right)\cos x + \left(C_2 + \int f(x)\cos x\,dx\right)\sin x. \tag{54}$$

(If C_1 and C_2 are omitted, this is a particular solution of the differential equation.) A simplified form results if we express the antiderivatives as definite integrals:

$$\begin{aligned} y(x) &= C_1\cos x + C_2\sin x - \cos x\int_0^x f(t)\sin t\,dt + \sin x\int_0^x f(t)\cos t\,dt \\ &= C_1\cos x + C_2\sin x + \int_0^x f(t)\sin(x-t)\,dt. \end{aligned} \tag{55}$$

In this form, any initial conditions

$$y(0) = y_0 \quad \text{and} \quad y'(0) = y_0', \tag{52b}$$

that might accompany ODE (52a) are easily incorporated. They require that

$$y_0 = C_1 \qquad y_0' = C_2,$$

and therefore the final solution of differential equation (52a) subject to initial conditions (52b) is

$$y(x) = y_0\cos x + y_0'\sin x + \int_0^x f(t)\sin(x-t)\,dt. \tag{56}$$

We now develop the analogous method for solving initial boundary value problems that have time-dependent nonhomogeneities in their PDEs. The one-dimensional vibration problem for displacements of a taut string with a time-dependent forcing function $F(x,t) = e^{-t}$ is a convenient vehicle:

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{e^{-t}}{\rho}, \qquad 0 < x < L, \qquad t > 0, \tag{57a}$$

$$y(0,t) = 0, \qquad t > 0, \tag{57b}$$

$$y(L,t) = 0, \qquad t > 0, \tag{57c}$$

$$y(x,0) = f(x), \qquad 0 < x < L, \tag{57d}$$

$$y_t(x,0) = 0, \qquad 0 < x < L. \tag{57e}$$

We have taken a forcing function that does not depend on x to simplify calculations, but the technique works when the forcing function is a function of x as well as t. If the forcing term were absent, the PDE would be homogeneous, and, according to our solution of problem (8), separation of variables on (57a, b, c, e) would lead to a superposed solution of the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L},$$

where the C_n are arbitrary constants [see equation (16)]. To incorporate a nonzero forcing term, we use a method called eigenfunction expansion. (In Chapter 5, when we consider more general problems, we learn the significance of this name.) This method is much like variation of parameters for ODEs; we attempt to find a solution in this form, but where $C_n = C_n(t)$ are functions of t,

$$y(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \tag{58}$$

Because at this point $C_n(t)$ is an unknown function, it is more convenient, and no less general, to group $C_n(t)$ and $\cos(n\pi ct/L)$ together as the unknown function, say $d_n(t) = C_n(t)\cos(n\pi ct/L)$. In other words, we replace (58) with

$$y(x,t) = \sum_{n=1}^{\infty} d_n(t) \sin\frac{n\pi x}{L}. \tag{59}$$

It is this series that is called an eigenfunction expansion. The $\sin(n\pi x/L)$ are the eigenfunctions, and $d_n(t)$ are coefficients in the expansion of $y(x,t)$ in terms of these eigenfunctions.

For any choice of $d_n(t)$ whatsoever, the representation in (59) satisfies boundary conditions (57b, c). To satisfy initial condition (57e), we must have

$$\sum_{n=1}^{\infty} d_n'(0) \sin\frac{n\pi x}{L} = 0, \qquad 0 < x < L.$$

This requires the unknown functions $d_n(t)$ to have vanishing first derivatives at $t = 0$, $d_n'(0) = 0$.

To determine whether a function of form (59) can satisfy (57a), we substitute (59) into (57a) and formally differentiate term by term:

$$\sum_{n=1}^{\infty} d_n''(t)\sin\frac{n\pi x}{L} = c^2 \sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} d_n(t)\sin\frac{n\pi x}{L} + \frac{e^{-t}}{\rho}. \tag{60}$$

In its present form, this equation is intractable, but the fact that two of the terms are series in $\sin(n\pi x/L)$ suggests that the function e^{-t}/ρ be expressed in this way also; that is, we should write

$$\frac{e^{-t}}{\rho} = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi x}{L}. \tag{61a}$$

We have seen equations of this form before; they are Fourier sine series representations for the function on the left. However, should not the function on the left be a function of x, not t? Indeed it should, but e^{-t}/ρ is trivially a function of x, and in addition it is a function of t. In other words, it is not that e^{-t}/ρ is a function of the wrong variable; it is a function of both x and t, and we wish to express this function of x and t as a Fourier sine series in x for any given t. Clearly, this can happen only if coefficients are functions of t; that is, we really want to express e^{-t}/ρ in the form

$$\frac{e^{-t}}{\rho} = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi x}{L}. \tag{61b}$$

For each fixed t, (61b) is the Fourier sine series of the odd, $2L$-periodic extension of the constant function (of x) e^{-t}/ρ. According to equation (18b) in Section 2.2, then,

$$F_n(t) = \frac{2}{L}\int_0^L \frac{1}{\rho} e^{-t} \sin\frac{n\pi x}{L}\,dx = \frac{2e^{-t}[1+(-1)^{n+1}]}{n\pi\rho},$$

and therefore

$$\frac{e^{-t}}{\rho} = \frac{2e^{-t}}{\rho\pi}\sum_{n=1}^{\infty} \frac{[1+(-1)^{n+1}]}{n}\sin\frac{n\pi x}{L}. \tag{62}$$

If (62) is now substituted into (60), the result is

$$\sum_{n=1}^{\infty} d_n''(t)\sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} -\frac{n^2\pi^2c^2}{L^2} d_n(t)\sin\frac{n\pi x}{L} + \frac{2e^{-t}}{\rho\pi}\sum_{n=1}^{\infty}\frac{[1+(-1)^{n+1}]}{n}\sin\frac{n\pi x}{L}$$

or

$$\sum_{n=1}^{\infty}\left(d_n''(t) + \frac{n^2\pi^2c^2}{L^2}d_n(t) - \frac{2e^{-t}[1+(-1)^{n+1}]}{\rho\pi n}\right)\sin\frac{n\pi x}{L} = 0.$$

But for each fixed t, the series on the left of this equation is the Fourier sine series of the function on the right, the function that is identically zero. It follows that all coefficients must be zero; that is,

$$d_n''(t) + \frac{n^2\pi^2c^2}{L^2}d_n(t) - \frac{2e^{-t}[1+(-1)^{n+1}]}{n\pi\rho} = 0.$$

In other words, each unknown function $d_n(t)$ must satisfy the differential equation

$$\frac{d^2d_n}{dt^2} + \frac{n^2\pi^2c^2}{L^2}d_n = \frac{2[1+(-1)^{n+1}]e^{-t}}{n\pi\rho}.$$

The general solution of this equation is

$$d_n(t) = b_n \cos\frac{n\pi ct}{L} + a_n \sin\frac{n\pi ct}{L} + \frac{2L^2[1 + (-1)^{n+1}e^{-t}}{n\pi\rho(L^2 + n^2\pi^2c^2)},$$

where a_n and b_n are constants. The condition $d_n'(0) = 0$ implies that

$$a_n = \frac{2L^3[1 + (-1)^{n+1}]}{n^2\pi^2\rho c(L^2 + n^2\pi^2c^2)},$$

and therefore

$$d_n(t) = b_n \cos\frac{n\pi ct}{L} + \frac{2L^2[1 + (-1)^{n+1}]}{n^2\pi^2\rho c(L^2 + n^2\pi^2c^2)}\left(n\pi ce^{-t} + L\sin\frac{n\pi ct}{L}\right).$$

Substitution of this expression into (59) gives

$$y(x, t) = \sum_{n=1}^{\infty}\left[b_n \cos\frac{n\pi ct}{L} + \frac{2L^2[1 + (-1)^{n+1}]}{n^2\pi^2\rho c(L^2 + n^2\pi^2c^2)}\left(n\pi ce^{-t} + L\sin\frac{n\pi ct}{L}\right)\right]\sin\frac{n\pi x}{L}. \tag{63}$$

Initial condition (57d) requires that

$$f(x) = \sum_{n=1}^{\infty}\left(b_n + \frac{2L^2[1 + (-1)^{n+1}]}{n\pi\rho(L^2 + n^2\pi^2c^2)}\right)\sin\frac{n\pi x}{L}, \qquad 0 < x < L,$$

from which

$$b_n + \frac{2L^2[1 + (-1)^{n+1}]}{n\pi\rho(L^2 + n^2\pi^2c^2)} = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{64}$$

The formal solution of (57) is now complete; it is (63) with the b_n defined by (64).

Perhaps a summary of the eigenfunction expansion technique would be valuable at this juncture. When a PDE has a nonhomogeneity, the method proceeds as follows:

(1) Find separated functions satisfying the homogeneous boundary conditions (and homogeneous initial conditions) and the corresponding *homogeneous* PDE. Suppose we denote the functions of x by $X_n(x)$. [$X_n(x) = \sin(n\pi x/L)$ in our previous problem.]

(2) Represent the unknown function in a series of the form

$$\sum_{n=1}^{\infty} d_n(t)X_n(x)$$

with unknown coefficients $d_n(t)$.

(3) Substitute the eigenfunction expansion of step (2) into the PDE, at the same time expanding the nonhomogeneity in terms of the functions $X_n(x)$.

(4) Obtain and solve ordinary differential equations for the $d_n(t)$.

(5) Use initial conditions on the PDE to determine any constants of integration in step (4).

When time-dependent nonhomogeneities are present in boundary conditions, they are transformed into nonhomogeneities in the PDE. They can then be handled by an eigenfunction expansion. This is illustrated in the following example.

Example 5: Solve the following initial boundary value problem for temperature in a homogeneous, isotropic rod with insulated sides:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{65a}$$

$$U(0,t) = \phi_0(t), \qquad t > 0, \tag{65b}$$

$$U(L,t) = \phi_L(t), \qquad t > 0, \tag{65c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{65d}$$

The rod is free of internal heat generation, and its ends are kept at prescribed temperatures.

Solution: We define a new dependent variable $V(x,t)$ according to

$$U(x,t) = V(x,t) + \psi(x,t), \tag{66}$$

where $\psi(x,t)$ is to be chosen so that $V(x,t)$ will satisfy homogeneous boundary conditions. Boundary conditions (65b, c) require that

$$V(0,t) = \phi_0(t) - \psi(0,t) \quad \text{and} \quad V(L,t) = \phi_L(t) - \psi(L,t).$$

Consequently, $V(x,t)$ will satisfy homogeneous boundary conditions

$$V(0,t) = 0, \qquad t > 0, \tag{67a}$$

$$V(L,t) = 0, \qquad t > 0 \tag{67b}$$

if $\psi(x,t)$ is chosen so that

$$\psi(0,t) = \phi_0(t), \qquad \psi(L,t) = \phi_L(t).$$

These are accommodated if $\psi(x,t)$ is chosen as

$$\psi(x,t) = \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]. \tag{68}$$

This is not the only choice for $\psi(x,t)$, but it is perhaps the simplest. With this choice,

$$U(x,t) = V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]. \tag{69}$$

The PDE for $V(x,t)$ can be obtained by substituting (69) into (65a):

$$\frac{\partial}{\partial t}\left(V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]\right) = k\frac{\partial^2}{\partial x^2}\left(V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)]\right)$$

or

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + G(x,t), \tag{67c}$$

where

$$G(x,t) = -\phi_0'(t) - \frac{x}{L}[\phi_L'(t) - \phi_0'(t)]. \tag{67d}$$

Initial condition (65d) yields the initial condition for $V(x,t)$,

$$V(x,0) = f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)], \qquad 0 < x < L. \tag{67e}$$

Our problem now is to solve PDE (67c, d) subject to homogeneous boundary conditions (67a, b) and initial condition (67e); that is, $V(x,t)$ must satisfy

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} + G(x,t), \qquad 0 < x < L, \qquad t > 0, \tag{70a}$$

$$V(0,t) = 0, \qquad t > 0, \tag{70b}$$

$$V(L,t) = 0, \qquad t > 0, \tag{70c}$$

$$V(x,0) = f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)], \qquad 0 < x < L, \tag{70d}$$

where

$$G(x,t) = -\phi_0'(t) - \frac{x}{L}[\phi_L'(t) - \phi_0'(t)]. \tag{70e}$$

What we have done is transform the nonhomogeneities in boundary conditions (65b, c) into PDE (70a). But this presents no difficulty; eigenfunction expansions handle nonhomogeneous PDEs. Were $G(x,t)$ not present, separation of variables would lead to a solution in the form

$$V(x,t) = \sum_{n=1}^{\infty} C_n e^{-n^2\pi^2 kt/L^2} \sin\frac{n\pi x}{L}.$$

We therefore assume a solution for nonhomogeneous problem (70) in the form

$$V(x,t) = \sum_{n=1}^{\infty} C_n(t) \sin\frac{n\pi x}{L}, \tag{71}$$

where the exponential has been absorbed into the unknown function $C_n(t)$. This function satisfies boundary conditions (70b, c) and will satisfy PDE (70a) if

$$\sum_{n=1}^{\infty} C_n'(t)\sin\frac{n\pi x}{L} = k\sum_{n=1}^{\infty} -\frac{n^2\pi^2}{L^2} C_n(t)\sin\frac{n\pi x}{L} + G(x,t). \tag{72}$$

To simplify this equation, we extend $G(x,t)$ as an odd, $2L$-periodic function and expand it in a Fourier sine series

$$G(x,t) = \sum_{n=1}^{\infty} G_n(t)\sin\frac{n\pi x}{L}, \tag{73a}$$

where

$$G_n(t) = \frac{2}{L}\int_0^L G(x,t)\sin\frac{n\pi x}{L}\,dx. \tag{73b}$$

Substitution of this series into (72) gives

$$\sum_{n=1}^{\infty}\left(C_n'(t) + \frac{n^2\pi^2 k}{L^2} C_n(t) - G_n(t)\right)\sin\frac{n\pi x}{L} = 0.$$

But for each fixed t, the series on the left of this equation is the Fourier sine series of the function on the right, the function that is identically zero. It follows that all coefficients must vanish; that is,

$$C_n'(t) + \frac{n^2\pi^2 k}{L^2} C_n(t) = G_n(t).$$

The general solution of this linear, first-order ODE is

$$C_n(t) = b_n e^{-n^2\pi^2 kt/L^2} + \int_0^t G_n(u) e^{n^2\pi^2 k(u-t)/L^2}\, du,$$

where b_n is a constant. Substitution of this into (71) gives

$$V(x,t) = \sum_{n=1}^{\infty} \left(b_n e^{-n^2\pi^2 kt/L^2} + \int_0^t G_n(u) e^{n^2\pi^2 k(u-t)/L^2}\, du \right) \sin\frac{n\pi x}{L}. \qquad (74)$$

To satisfy initial condition (70d), we must have

$$f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)] = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad 0 < x < L,$$

and this implies that

$$b_n = \frac{2}{L}\int_0^L \left(f(x) - \phi_0(0) - \frac{x}{L}[\phi_L(0) - \phi_0(0)] \right) \sin\frac{n\pi x}{L}\, dx. \qquad (75)$$

The formal solution of (65) is therefore

$$U(x,t) = V(x,t) + \phi_0(t) + \frac{x}{L}[\phi_L(t) - \phi_0(t)],$$

with $V(x,t)$ given by (74), (75), and (73b). ■

Let us summarize the techniques for handling nonhomogeneities.

(1) When nonhomogeneous boundary conditions are associated with Laplace's equation, all that is needed is superposition principle 2. The problem is divided into two or more subproblems, each of which can be solved by separation of variables, and the solutions of these subproblems are then added together. [For example, when $F(x,y) \equiv 0$ in the problem of Figure 3.1, $V(x,y)$ is the sum of $V_1(x,y)$ and $V_2(x,y)$.] Nonhomogeneities that turn Laplace's equation into Poisson's equation require eigenfunction expansions (see Exercise 20).

(2) When time-independent nonhomogeneities occur in initial boundary value problems (be they in the boundary conditions or in the PDE), it is advantageous to separate out steady-state or static solutions. The remaining part of the solution then satisfies a homogeneous PDE and homogeneous boundary conditions.

(3) When nonhomogeneities in boundary conditions of initial boundary value problems are time dependent, they can be transformed into time-dependent nonhomogeneities in the PDE. [See, for example, transformation (69) in Example 5.] Eigenfunction expansions then take care of time-dependent nonhomogeneities in PDEs.

Because time-independent nonhomogeneities [in technique (2)] are trivially functions of time, it is natural to ask whether technique (2) is necessary now that we have technique (3). To answer this question, we use technique (3) on problem (34). Separation of variables on (34a, b, c, e) in the absence of the nonhomogeneity leads to a superposition of separated functions in the form

$$y(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

Eigenfunction expansions suggest a solution of (34) (with g now present) in the form

$$y(x,t) = \sum_{n=1}^{\infty} d_n(t)\sin\frac{n\pi x}{L}.$$

When this solution is pursued, the result obtained is

$$y(x,t) = \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi ct}{L} + \frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\left(1-\cos\frac{n\pi ct}{L}\right)\right]\sin\frac{n\pi x}{L}, \quad \text{(76a)}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \quad \text{(76b)}$$

This does not appear to be the same as solution (41) of (34),

$$y(x,t) = \frac{gx(L-x)}{2c^2} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \quad \text{(41)}$$

where
$$b_n = \frac{2}{L}\int_0^L\left(f(x) - \frac{gx(L-x)}{2c^2}\right)\sin\frac{n\pi x}{L}\,dx. \quad \text{(40)}$$

They do, however, represent the same function as we now show. Integration by parts gives

$$\begin{aligned} b_n &= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx - \frac{2}{L}\int_0^L\frac{gx(L-x)}{2c^2}\sin\frac{n\pi x}{L}\,dx \\ &= a_n - \frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2} \end{aligned}$$

and therefore (41) may be expressed as

$$y(x,t) = \frac{gx(L-x)}{2c^2} + \sum_{n=1}^{\infty}\left(a_n - \frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\right)\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}.$$

If we divide the summation in (76a) into two parts, this function may be written in the form

$$\begin{aligned} y(x,t) = &\sum_{n=1}^{\infty}\frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\sin\frac{n\pi x}{L} \\ &+ \sum_{n=1}^{\infty}\left(a_n - \frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2}\right)\cos\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}. \end{aligned}$$

These expressions are indeed identical, since the first series in the latter equation is the Fourier sine series of the odd, $2L$-periodic extension of $gx(L-x)/(2c^2)$,

$$\frac{gx(L-x)}{2c^2} = \sum_{n=1}^{\infty} \frac{2gL^2[1+(-1)^{n+1}]}{n^3\pi^3c^2} \sin\frac{n\pi x}{L}, \qquad 0 \le x \le L.$$

Although this example illustrates that eigenfunction expansions can also be used to solve problems when nonhomogeneities are time independent, we would not suggest abandoning technique (2). There is a definite advantage to solution (41) over (76). Contained in (41) is a closed-form part, $gx(L-x)/(2c^2)$. This is also a part of (76a), but it is in the form of a Fourier sine series. This is the advantage of technique (2); it always separates out, in closed form, a steady-state or static part of the solution. Technique (3) does not; it delivers steady-state or static parts in series form. Given only the Fourier series for steady-state and static solutions, it could be very difficult to recognize their closed forms.

In Sections 3.2 and 3.3 we have shown how the method of separation of variables leads to the use of Fourier series in the solution of various initial boundary value problems. We have considered problems with one and more than one nonhomogeneous condition, many second-order equations, and one fourth-order equation. All equations contained two independent variables in order that the method not be obscured by overly complicated calculations. Certainly, however, the method can, and will, be used for problems in several independent variables.

We do not yet know whether we have solved any of the initial boundary value problems in these sections; we have found only what we call *formal* solutions. They are formal because of the questionable validity of superposing an infinity of separated functions. Each formal solution must therefore be verified as a valid solution to its initial boundary value problem. We do this in Sections 5.6–5.8 when we take up detailed analyses of convergence properties of formal solutions.

In problems (8), (25), (34), (42), (57), and (65), separation of variables led to the system

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L,$$
$$X(0) = 0 = X(L)$$

and in problem (19) to the system

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L,$$
$$X'(0) = 0 = X'(L).$$

Each of these problems is a special case of a general mathematical system called a *Sturm-Liouville system.* It consists of an ordinary differential equation

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + \{\lambda p(x) - q(x)\}y = 0 \tag{77a}$$

on some interval $a < x < b$, together with two boundary conditions

$$-l_1 y'(a) + h_1 y(a) = 0, \tag{77b}$$
$$l_2 y'(b) + h_2 y(b) = 0, \tag{77c}$$

where λ is a parameter and h_1, h_2, l_1, and l_2 are constants.

In Chapter 4 we discuss Sturm-Liouville systems in a general context an properties of solutions of such systems. These systems lead to *generalized Fouri* containing not only trigonometric functions but many other types of functions. Bessel functions and Legendre functions.

Finally, it is obvious that the steps in the solutions of boundary value an boundary value problems in Sections 3.2 and 3.3, and even the wording of the st almost identical. Surely, then, we should be able to devise a method that eliminate the tedious repetition of these steps in every problem. Indeed, *finite transforms* associated with Sturm-Liouville systems can be used for this purpos are discussed in Chapter 6.

Exercises 3.3

Part A—Heat Conduction

1. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature 20°C thro $(0 \le x \le L)$ at time $t = 0$. For $t > 0$, a constant electric current I is passed along the len the rod, creating heat generation $g(x,t) = I^2/(A^2\sigma)$, where σ is the electrical conductivity rod and A is its cross-sectional area (see Exercise 32 in Section 1.2). If the ends of the rod a at temperature zero for $t > 0$, find the temperature in the rod for $t > 0$ and $0 < x < L$.
2. Repeat Exercise 1 if the ends of the rod are held at temperature 100°C for $t > 0$.
3. Repeat Exercise 1 if the ends $x = 0$ and $x = L$ are held at constant temperatures U_0 ar respectively, for $t > 0$.
4. Repeat Exercise 1 if the electric current is a function of time $I = e^{-\alpha t}$.
5. A cylindrical, homogeneous, isotropic rod with insulated sides has temperature 100°C thr out $(0 \le x \le L)$ at time $t = 0$. For $t > 0$, its left end $(x = 0)$ is held at temperature zero a right end has temperature $100e^{-t}$. Find the temperature in the rod for $t > 0$ and $0 < x$ Assume that $k \ne L^2/(n^2\pi^2)$ for any integer n.
6. Repeat Exercise 1 if the ends of the rod are insulated for $t > 0$.
7. Repeat Exercise 1 if the ends of the rod are insulated and $I = e^{-\alpha t}$.
8. A cylindrical, homogeneous, isotropic rod with insulated sides is initially at temperature throughout $(0 \le x \le L)$. For time $t > 0$, its ends $x = 0$ and $x = L$ continue to be hel temperature zero, and heat generation at each point of the rod is described by $g(x,$ $e^{-\alpha t}\sin(m\pi x/L)$, where $\alpha > 0$ and m is a positive integer. Find the temperature in the rod function of x and t.
9. Repeat Exercise 8 if $g(x,t) = e^{-\alpha t}$, $\alpha > 0$, and the initial temperature in the rod is 10°C thro out. Assume that $\alpha \ne n^2\pi^2 k/L^2$ for any integer n.
10. The general one-dimensional heat conduction problem for a homogeneous, isotropic rod w insulated sides is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t), \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = f_2(t), \qquad x = L, \qquad t > 0,$$

$$U(x,0) = f(x), \qquad 0 < x < L.$$

Show that when the nonhomogeneities $g(x,t)$, $f_1(t)$, and $f_2(t)$ are independent of time, the change of dependent variable $U(x,t) = V(x,t) + \psi(x)$, where $\psi(x)$ is the solution of the corresponding steady-state problem, leads to an initial boundary value problem in $V(x,t)$ that has a homogeneous PDE and homogeneous boundary conditions.

11. Explain how to solve Exercise 1 if the current is turned on for only 100 s beginning at time $t = 0$. Do not solve the problem; just explain the steps you would take to solve it.

12. Suppose that heat generation in the thin wire of Exercise 31 in Section 1.2 is caused by an electric current I. When the temperature of the material surrounding the wire is a constant 0°C and σ is the electrical conductivity of the material in the wire, temperature at points in the wire must satisfy the PDE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - hU + \frac{kI^2}{\kappa\sigma A^2}, \qquad 0 < x < L, \qquad t > 0$$

(see Exercise 32 in Section 1.2).

(a) Assuming that the ends of the wire are held at temperature 0°C and the initial temperature in the wire at time $t = 0$ is also 0°C, show that when $U(x,t)$ is separated into steady-state and transient parts, $U(x,t) = V(x,t) + \psi(x)$:

$$\psi(x) = \frac{kI^2}{\kappa h\sigma A^2}\left(1 - \frac{\sinh\sqrt{h/k}\,x + \sinh\sqrt{h/k}(L-x)}{\sinh\sqrt{h/k}\,L}\right).$$

(b) Find $V(x,t)$ and hence $U(x,t)$.

Part B — Vibrations

13. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at time $t = 0$ of $f(x)$, $0 \le x \le L$, and an initial velocity $g(x)$, $0 \le x \le L$. If an external force per unit length of constant magnitude acts vertically downward at every point on the string, find displacements in the string for $t > 0$ and $0 < x < L$.

14. A taut string has an end at $x = 0$ fixed on the x-axis, but the end at $x = L$ is removed a small amount y_L away from the x-axis and kept at this position. If it has initial position $f(x)$ and velocity $g(x)$ (at time $t = 0$), find displacements for $t > 0$ and $0 < x < L$.

15. A horizontal cylindrical bar is originally at rest and unstrained along the x-axis between $x = 0$ and $x = L$. For time $t > 0$, the left end is fixed and the right end is subjected to a constant elongating force per unit area F parallel to the bar. Displacements $y(x,t)$ of cross sections then satisfy the initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$

$$y(0,t) = 0, \qquad t > 0,$$

$$E\frac{\partial y(L,t)}{\partial x} = F, \qquad t > 0,$$

$$y(x,0) = y_t(x,0) = 0, \qquad 0 < x < L.$$

(a) Can this problem be solved by separation [$y(x,t) = X(x)T(t)$] and superposition? It has o one nonhomogeneous condition.

(b) Replace this initial boundary value problem by one in $z(x,t)$ in which $y(x,t) = z(x,t) + \psi$ and $\psi(x)$ is the solution of the associated static deflection problem.

(c) If separation of variables and superposition are used on the problem for $z(x,t)$, what fc does the series take? Finish the problem using the result of Exercise 21 in Section 2.2.

16. A beam of uniform cross section and length L has its ends simply supported at $x = 0$ and $x =$ The beam has constant density ρ (in kilograms per meter) and is subjected to an additio uniform loading of k kg/m. If the beam is released from rest at a horizontal position at time $t =$ find subsequent displacements.

17. Repeat Exercise 16 if the beam is at rest at time $t = 0$ with displacement $f(x)$, $0 \le x \le L$.

18. Repeat Exercise 10 for the general one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial y}{\partial x} + h_1 y = f_1(t), \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial y}{\partial x} + h_2 y = f_2(t), \qquad x = L, \qquad t > 0,$$

$$y(x,0) = f(x), \qquad 0 < x < L,$$

$$y_t(x,0) = g(x), \qquad 0 < x < L.$$

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

19. Find a formula for the solution of Laplace's equation inside the rectangle $0 \le x \le L, 0 \le y$: when the boundary conditions are as indicated in Figure 3.1.

20. Nonhomogeneities in Laplace's equation $\nabla^2 V = 0$ convert it into Poisson's equation. example, suppose a charge distribution with density $\sigma(x,y)$ coulombs per cubic meter occu the volume R in space bounded by the planes $x = 0$, $y = 0$, $x = L$, and $y = L'$.

(a) If the bounding planes are maintained at zero potential, what is the boundary value prob for potential in R?

(b) Use eigenfunction expansions to solve the boundary value problem in (a) when σ is const Find two series, one in terms of $\sin(n\pi x/L)$ and the other is terms of $\sin(n\pi y/L')$. Is ei series preferred?

(c) Solve the problem in (a) when σ is constant by setting $V(x,y) = U(x,y) + \psi(x)$, where satisfies

$$\frac{d^2\psi}{dx^2} = \frac{-\sigma}{\varepsilon_0}, \qquad 0 < x < L,$$

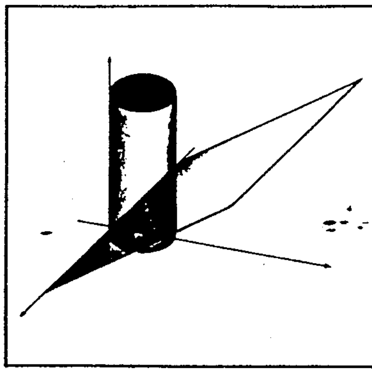
$$\psi(0) = \psi(L) = 0.$$

Is this the same solution as in (b)?

(d) If $\sigma = \sigma(x)$ is a function of x only, which type of expansion in (b) is preferred? Find potential in this case.

(e) Find the potential when $\sigma = xy$.

21. Solve Exercise 28 in Section 3.2 if heat is generated at a constant rate at every point in the pl.



C H A P T E R

F O U R

Sturm-Liouville Systems

4.1 Eigenvalues and Eigenfunctions

In Chapter 3, separation of variables on linear (initial) boundary value problems led to what are called Sturm-Liouville systems. In particular, we obtained two Sturm-Liouville systems,

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad 0 < x < L, \tag{1a}$$

$$X(0) = 0, \tag{1b}$$

$$X(L) = 0; \tag{1c}$$

and

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad 0 < x < L, \tag{2a}$$

$$X'(0) = 0, \tag{2b}$$

$$X'(L) = 0. \tag{2c}$$

In this chapter we undertake a general study of Sturm-Liouville systems. The results obtained are then applied to Sturm-Liouville systems that arise from more difficult problems associated with the (initial) boundary value problems of Chapter 1.

Nontrivial solutions of Sturm-Liouville systems (1) and (2) do not exist for arbitrary λ. On the contrary, only for specific values of λ, namely $\lambda = n\pi/L$, do nontrivial solutions exist, and to each such value there corresponds a solution (unique to a multiplicative constant). Because the solution depends on the value of λ chosen, it is customary to indicate this dependence by writing $X(\lambda, x)$ instead of $X(x)$. For system (1), the solution is $X(\lambda, x) = C \sin \lambda x$, and for system (2), it is $X(\lambda, x) = C \cos \lambda x$, C an arbitrary constant.

In general, a Sturm-Liouville system consists of a second-order, homogeneous differential equation of the following form, together with two linear, homogeneous boundary conditions for an unknown function $y(\lambda, x)$:

$$\frac{d}{dx}\left(r(x)\frac{dy(\lambda, x)}{dx}\right) + \{\lambda p(x) - q(x)\}y(\lambda, x) = 0, \qquad a < x < b, \tag{3a}$$

$$-l_1 y'(\lambda, a) + h_1 y(\lambda, a) = 0, \tag{3b}$$

$$l_2 y'(\lambda, b) + h_2 y(\lambda, b) = 0. \tag{3c}$$

The constants h_1, h_2, l_1, and l_2 are real and independent of the parameter λ. When the functions p, q, r, and r' are real and continuous for $a \le x \le b$, and $p > 0$ and $r > 0$ for $a \le x \le b$, the Sturm-Liouville system is said to be *regular*. The negative signs in (3a, b) are chosen simply as a matter of convenience for applications.

No matter what the value of λ, the trivial function $y(\lambda, x) \equiv 0$ always satisfies (3), but for certain values of λ, called *eigenvalues*, the system has nontrivial solutions. We shall see that there is always a countable (but infinite) number of such eigenvalues, which we denote by λ_n $(n = 1, 2, \ldots)$. A solution of (3) corresponding to an eigenvalue λ_n is called an *eigenfunction* and is denoted by

$$y_n(x) = y(\lambda_n, x). \tag{4}$$

Eigenfunctions are to satisfy the usual conditions for solutions of second-order differential equations, namely that y_n and dy_n/dx be continuous for $a \le x \le b$.

When $\lambda = 0$ is an eigenvalue of a Sturm-Liouville system, it is customary to denote it by $\lambda_0 = 0$. Such is the case for system (2).

The eigenfunctions $\sin(n\pi x/L)$ of system (1) form the basis for Fourier sine series, and in Chapter 2 we saw that they were orthogonal on the interval $0 \le x \le L$. The eigenfunctions $\cos(n\pi x/L)$ of system (2) are also orthogonal on this interval. This is not coincidence; the following theorem verifies orthogonality for eigenfunctions of every regular Sturm-Liouville system.

Theorem 1

All eigenvalues of a regular Sturm-Liouville system are real, and eigenfunctions corresponding to distinct eigenvalues are orthogonal with respect to the weight function $p(x)$,

$$\int_a^b p(x) y_n(x) y_m(x)\, dx = 0. \tag{5}$$

[See equation (6) in Section 2.1 for the definition of orthogonality of a sequence of functions.]

Proof: If $[\lambda_n, y_n(x)]$ and $[\lambda_m, y_m(x)]$ are eigenpairs of Sturm-Liouville system (3), where $\lambda_n \neq \lambda_m$, then

$$(ry_n')' = -(\lambda_n p - q)y_n, \qquad (ry_m')' = -(\lambda_m p - q)y_m.$$

Multiplication of the first by y_m and of the second by y_n, and subtraction of the two equations, eliminates q:

$$y_m(ry_n')' - y_n(ry_m')' = -\lambda_n p y_n y_m + \lambda_m p y_m y_n$$

or

$$(\lambda_n - \lambda_m)py_n y_m = (ry_m')'y_n - (ry_n')'y_m.$$

The expression on the right can be expressed as a total derivative if we simultaneously add and subtract the term $ry_m'y_n'$:

$$\begin{aligned}(\lambda_n - \lambda_m)py_n y_m &= [(ry_m')'y_n + (ry_m')y_n'] - [(ry_n')'y_m + (ry_n')y_m'] \\ &= (ry_m'y_n)' - (ry_n'y_m)' \\ &= (ry_m'y_n - ry_n'y_m)'.\end{aligned}$$

Integration of this equation with respect to x from $x = a$ to $x = b$ gives

$$\int_a^b (\lambda_n - \lambda_m)py_n y_m\,dx = \int_a^b \frac{d}{dx}(ry_m'y_n - ry_n'y_m)\,dx = \{ry_m'y_n - ry_n'y_m\}_a^b.$$

The right side of this result may be expressed as the difference in the values of two determinants:

$$(\lambda_n - \lambda_m)\int_a^b py_n y_m\,dx = r(b)\begin{vmatrix} y_n(b) & y_m(b) \\ y_n'(b) & y_m'(b)\end{vmatrix} - r(a)\begin{vmatrix} y_n(a) & y_m(a) \\ y_n'(a) & y_m'(a)\end{vmatrix}.$$

Since $y_n(x)$ and $y_m(x)$ both satisfy boundary condition (3b),

$$-l_1 y_n'(a) + h_1 y_n(a) = 0,$$
$$-l_1 y_m'(a) + h_1 y_m(a) = 0.$$

Because at least one of h_1 and l_1 is not zero, these equations (regarded as homogeneous, linear equations in l_1 and h_1) must have nontrivial solutions. Consequently, the determinant of their coefficients must vanish:

$$\begin{vmatrix} y_n'(a) & y_n(a) \\ y_m'(a) & y_m(a)\end{vmatrix} = 0.$$

A similar discussion with boundary condition (3c) indicates that

$$\begin{vmatrix} y_n'(b) & y_n(b) \\ y_m'(b) & y_m(b)\end{vmatrix} = 0.$$

It follows now that

$$(\lambda_n - \lambda_m)\int_a^b p(x)y_n(x)y_m(x)\,dx = 0,$$

and, because $\lambda_n \neq \lambda_m$, (5) has been established.

To prove that eigenvalues are real, we assume that $\lambda = \alpha + i\beta$ ($\beta \neq 0$) is a complex eigenvalue with eigenfunction $y(\lambda, x)$. This eigenfunction could be complex, but if it is, it is a complex-valued function of the real variable x. If we divide $y(\lambda, x)$ into real and imaginary parts,

$y(\lambda, x) = u(\lambda, x) + iv(\lambda, x)$, the complex conjugate of dy/dx is

$$\overline{\frac{dy}{dx}} = \overline{\frac{d}{dx}(u + iv)} = \overline{\frac{du}{dx} + i\frac{dv}{dx}} = \frac{du}{dx} - i\frac{dv}{dx} = \frac{d}{dx}(u - iv) = \frac{d\bar{y}}{dx}.$$

With this result, it is straightforward to take complex conjugates of (3). Because the functions $r(x)$, $p(x)$, and $q(x)$ are real, as are the constants h_1, h_2, l_1, and l_2, we find that $\bar{\lambda}$ and $\overline{y(\lambda, x)}$ must satisfy

$$(r\bar{y}')' + (\bar{\lambda}p - q)\bar{y} = 0,$$
$$-l_1\overline{y(\lambda, a)}' + h_1\overline{y(\lambda, a)} = 0, \qquad l_2\overline{y(\lambda, b)}' + h_2\overline{y(\lambda, b)} = 0.$$

These imply that $\overline{y(\lambda, x)}$ is an eigenfunction of (3) corresponding to the eigenvalue $\bar{\lambda}$. Since $\lambda \neq \bar{\lambda}$, $y(x, \lambda)$ and $\overline{y(x, \lambda)}$ must therefore be orthogonal; that is,

$$\int_a^b p(x)\overline{y(\lambda, x)}y(\lambda, x)\,dx = 0.$$

But this is impossible because $p(x) > 0$ for $a < x < b$, and $\overline{y(\lambda, x)}y(\lambda, x) = |y(\lambda, x)|^2 \geq 0$. Consequently, λ cannot be complex. ■

It is evident from the above proof that Theorem 1 is also valid under the circumstances in the following corollary.

Corollary

The results of Theorem 1 are valid when

(1) $r(a) = 0$ [boundary condition (3b) then being unnecessary];

(2) $r(b) = 0$ [boundary condition (3c) then being unnecessary];

(3) $r(a) = r(b)$ if boundary conditions (3b, c) are replaced by the periodic conditions

$$y(a) = y(b), \qquad y'(a) = y'(b). \tag{6}$$

A Sturm-Liouville system is said to be *singular* if either or both of its boundary conditions is absent; it is said to be *periodic* if $r(a) = r(b)$ and boundary conditions (3b, c) are replaced by periodic conditions (6). Theorem 1 and its corollary state that eigenfunctions of regular and periodic Sturm-Liouville systems are always orthogonal. They are also orthogonal for singular systems when boundary conditions (3b) or (3c) or both are absent, provided either $r(a) = 0$ or $r(b) = 0$, or both, respectively. We consider only regular and periodic Sturm-Liouville systems in this chapter; singular systems are discussed in Chapter 8.

Example 1: Find eigenvalues and eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda X = 0, \qquad 0 < x < L,$$
$$X(0) = 0 = X'(L).$$

Solution: When $\lambda < 0$, the general solution of the differential equation is

$$X(x) = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}.$$

The boundary conditions require that

$$0 = X(0) = A + B, \qquad 0 = X'(L) = A\sqrt{-\lambda}\,e^{\sqrt{-\lambda}L} - B\sqrt{-\lambda}\,e^{-\sqrt{-\lambda}L},$$

the only solution of which is $A = B = 0$.

When $\lambda = 0$, $X(x) = Ax + B$, and the boundary conditions once again imply that $A = B = 0$.

Thus, eigenvalues of the Sturm-Liouville system must be positive, and when $\lambda > 0$, the boundary conditions require constants A and B in the general solution $X(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x$ of the differential equation to satisfy

$$0 = X(0) = A, \qquad 0 = X'(L) = -A\sqrt{\lambda}\sin\sqrt{\lambda}L + B\sqrt{\lambda}\cos\sqrt{\lambda}L.$$

With A vanishing, the second condition reduces to $B\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$. Since neither B nor λ can vanish, $\cos\sqrt{\lambda}L$ must be zero. Hence, $\sqrt{\lambda}L$ must be equal to $-\pi/2$ plus an integer multiple of π; that is, permissible values of λ are λ_n where $\sqrt{\lambda_n}L = n\pi - \pi/2$, n an integer. Corresponding functions are

$$X_n(x) = B\sin\sqrt{\lambda_n}x = B\sin\frac{(2n-1)\pi x}{2L}.$$

But the set of functions for $n \le 0$ is identical to that for $n > 0$. In other words, eigenvalues of the Sturm-Liouville system are $\lambda_n = (2n-1)^2\pi^2/(4L^2)$, $n \ge 1$, with corresponding eigenfunctions $X_n(x) = B\sin[(2n-1)\pi x/(2L)]$. ∎

Example 2: Discuss the periodic Sturm-Liouville system

$$y'' + \lambda y = 0, \qquad -L < x < L, \tag{7a}$$

$$y(-L) = y(L), \tag{7b}$$

$$y'(-L) = y'(L). \tag{7c}$$

Solution: If $\lambda > 0$, the general solution of (7a) is

$$y(\lambda, x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x.$$

Conditions (7b, c) require that

$$A\cos\sqrt{\lambda}L - B\sin\sqrt{\lambda}L = A\cos\sqrt{\lambda}L + B\sin\sqrt{\lambda}L,$$

$$\sqrt{\lambda}A\sin\sqrt{\lambda}L + \sqrt{\lambda}B\cos\sqrt{\lambda}L = -\sqrt{\lambda}A\sin\sqrt{\lambda}L + \sqrt{\lambda}B\cos\sqrt{\lambda}L.$$

These equations require that $\sin\sqrt{\lambda}L = 0$, and this implies that $\sqrt{\lambda}L = n\pi$. In other words, eigenvalues of the Sturm-Liouville system are $\lambda_n = n^2\pi^2/L^2$, where n is an integer that we take as positive. Corresponding to these eigenvalues are the eigenfunctions

$$y_n(x) = y(\lambda_n, x) = A\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L}.$$

When $\lambda = 0$, $y(0, x) = A + Bx$, and the boundary conditions require that $B = 0$. Thus, corresponding to the eigenvalue $\lambda_0 = 0$, we have the eigenfunction $y_0(x) = A$.

The only solution of (7) when $\lambda < 0$ is the trivial solution.

Theorem 1 guarantees that for nonnegative integers m and n ($m \neq n$), the eigenfunctions

$$y_n(x) = A\cos\frac{n\pi x}{L} + B\sin\frac{n\pi x}{L} \quad \text{and} \quad y_m(x) = C\cos\frac{m\pi x}{L} + D\sin\frac{m\pi x}{L}$$

are orthogonal over the interval $-L \leq x \leq L$. It is true, however, that all functions in the set

$$\left\{1, \cos\frac{n\pi x}{L}, \sin\frac{n\pi x}{L}\right\}$$

are orthogonal. These are precisely the "eigenfunctions" found in the Fourier series expansion of a function of period $2L$. We shall return to this point in Section 4.2. ∎

Because differential equation (3a) and boundary conditions (3b, c) are homogeneous, if $[\lambda_n, y_n(x)]$ is an eigenpair for a Sturm-Liouville system, then so also is $[\lambda_n, cy_n(x)]$ for any constant $c \neq 0$. In other words, eigenfunctions are not unique; if $y_n(x)$ is an eigenfunction corresponding to an eigenvalue λ_n, then any constant times $y_n(x)$ is also an eigenfunction corresponding to the same λ_n. This fact is reflected in Example 1, where eigenfunctions were determined only to multiplicative constants. In this example, there is, except for the multiplicative constant, only one eigenfunction $\sin[(2n-1)\pi x/(2L)]$, corresponding to each eigenvalue. This is not the case in Example 2. Corresponding to each positive eigenvalue in Example 2 there are two linearly independent eigenfunctions, $\sin(n\pi x/L)$ and $\cos(n\pi x/L)$. The difference is that in Example 1 the Sturm-Liouville system is regular, but in Example 2 it is periodic. It can be shown (see Exercise 12) that in a regular Sturm-Liouville system, there cannot be two linearly independent eigenfunctions corresponding to the same eigenvalue.

In regular Sturm-Liouville systems, it is customary to single out one of the eigenfunctions $y_n(x)$ corresponding to an eigenvalue as special and refer all other eigenfunctions to it. The one that is chosen is an eigenfunction with "length" unity, that is, an eigenfunction $y_n(x)$ satisfying

$$\|y_n(x)\| = \sqrt{\int_a^b p(x)[y_n(x)]^2\,dx} = 1.$$

Normalized eigenfunctions can always be found by dividing nonnormalized eigenfunctions by their lengths. Consider, for example, Sturm-Liouville system (1). Since $\sin(n\pi x/L)$ is an eigenfunction of this system corresponding to the eigenvalue $\lambda_n^2 = n^2\pi^2/L^2$, so also is $c\sin(n\pi x/L)$ for any constant $c \neq 0$. The normalized eigenfunction corresponding to this eigenvalue is

$$\frac{\sin(n\pi x/L)}{\|\sin(n\pi x/L)\|},$$

where
$$\left\|\sin\frac{n\pi x}{L}\right\|^2 = \int_0^L \sin^2\frac{n\pi x}{L}\,dx = \frac{L}{2}.$$

Thus, with each eigenvalue $\lambda_n^2 = n^2\pi^2/L^2$ of the Sturm-Liouville system, we associate the normalized eigenfunction

$$X_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L}.$$

All other eigenfunctions for λ_n^2 are then $cX_n(x)$.

Similarly, normalized eigenfunctions for Sturm-Liouville system (2) are

$$X_0(x) = \frac{1}{\sqrt{L}} \quad \text{corresponding to } \lambda_0^2 = 0$$

and $$X_n(x) = \sqrt{\frac{2}{L}} \cos\frac{n\pi x}{L} \quad \text{corresponding to } \lambda_n^2 = n^2\pi^2/L^2, \qquad n > 0.$$

In general, if $y_n(x)$ is an eigenfunction of Sturm-Liouville system (3), we replace it by the normalized eigenfunction

$$\frac{1}{N} y_n(x) \tag{8a}$$

where $$N^2 = \|y_n(x)\|^2 = \int_a^b p(x)[y_n(x)]^2\,dx. \tag{8b}$$

The complete set of normalized eigenfunctions, one for each eigenvalue, then constitutes a set of orthonormal eigenfunctions for the Sturm-Liouville system. Unless otherwise stated, we shall always regard $y_n(x)$ as normalized eigenfunctions of a Sturm-Liouville system. Notice that any number of the $y_n(x)$ could be replaced by $-y_n(x)$, and the new set would also be orthonormal. In other words, orthonormal eigenfunctions are determined only to a factor of ± 1.

Example 3: Find eigenvalues and normalized eigenfunctions of the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + \lambda y = 0, \qquad 0 < x < 1,$$

$$y(0) = 0 = y(1).$$

Solution: Roots of the auxiliary equation $m^2 + m + \lambda = 0$ associated with the differential equation are $m = (-1 \pm \sqrt{1 - 4\lambda})/2$. When $\lambda < 1/4$, these roots are real; denote them by $\omega_1 = (-1 + \sqrt{1 - 4\lambda})/2$ and $\omega_2 = (-1 - \sqrt{1 - 4\lambda})/2$. The general solution of the differential equation in this case is $y(x) = Ae^{\omega_1 x} + Be^{\omega_2 x}$, and the boundary conditions require that

$$0 = A + B, \qquad 0 = Ae^{\omega_1} + Be^{\omega_2}.$$

The only solution of these equations is $A = B = 0$, leading to the trivial solution $y(x) \equiv 0$.

When $\lambda = 1/4$, the auxiliary equation has equal roots, and $y(x) = (A + Bx)e^{-x/2}$. Once again, the boundary conditions require that $A = B = 0$.

Consequently, λ must be greater than $1/4$, in which case we set $m = -1/2 \pm i\omega$, where $\omega = \sqrt{4\lambda - 1}/2$. The boundary conditions require constants A and B in the general solution $y(x) = e^{-x/2}(A\cos\omega x + B\sin\omega x)$ to satisfy

$$0 = A, \qquad 0 = e^{-1/2}(A\cos\omega + B\sin\omega).$$

With vanishing A, the second condition requires that $\sin\omega = 0$, that is, that $\omega = n\pi$ n an integer. In other words, eigenvalues of the Sturm-Liouville system are given by $\sqrt{4\lambda_n - 1}/2 = n\pi$, or

$$\lambda_n = \frac{1}{4} + n^2\pi^2.$$

Except for the multiplicative constant B, corresponding eigenfunctions are $e^{-x/2}\sin n\pi x$. Clearly, we need only take $n > 0$. To normalize these functions, we express the differential equation in standard Sturm-Liouville form (3a). This can be done by multiplying by e^x (see Exercise 1):

$$0 = e^x\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} + \lambda e^x y = \frac{d}{dx}\left(e^x\frac{dy}{dx}\right) + \lambda e^x y.$$

With the weight function now identified as $p(x) = e^x$, we calculate lengths of the eigenfunctions:

$$\|e^{-x/2}\sin n\pi x\|^2 = \int_0^1 e^x(e^{-x/2}\sin n\pi x)^2\,dx = \int_0^1 \sin^2 n\pi x\,dx = \frac{1}{2}.$$

Normalized eigenfunctions are therefore

$$y_n(x) = \sqrt{2}\,e^{-x/2}\sin n\pi x.$$

Exercises 4.1

1. (a) Show that when the differential equation

$$\alpha(x)\frac{d^2y}{dx^2} + \beta(x)\frac{dy}{dx} + [\gamma(x) + \lambda\delta(x)]y = 0, \qquad a < x < b,$$

is multiplied by the "integrating factor"

$$r(x) = e^{\int \beta(x)/\alpha(x)\,dx},$$

it can immediately be expressed in standard Sturm-Liouville form (3a). Notice that $\alpha(x)$ must not vanish for $a \le x \le b$.

(b) In view of Example 3, what is the importance of this result?

In Exercises 2–9 find eigenvalues and orthonormal eigenfunctions for the given Sturm-Liouville system.

2. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 3, \quad y(0) = 0 = y(3)$

3. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 4, \quad y'(0) = 0 = y'(4)$

4. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 9, \quad y(0) = 0 = y'(9)$

5. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < 1, \quad y'(0) = 0 = y(1)$

6. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y(L)$

7. $\dfrac{d^2y}{dx^2} + \lambda y = 0, \quad 1 < x < 10, \quad y(1) = 0 = y(10)$
   (Do this directly and also by making the change of independent variable $z = x - 1$.)

8. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + \lambda y = 0, \quad 0 < x < 1, \quad y(0) = 0 = y(1)$

9. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \lambda y = 0, \quad 1 < x < 5, \quad y'(1) = 0 = y'(5)$
   (*Hint:* Use the change of variable $z = x - 1$.)

10. Find eigenvalues and eigenfunctions of the periodic Sturm-Liouville system

$$y'' + \lambda y = 0, \qquad 0 < x < 2L,$$
$$y(0) = y(2L),$$
$$y'(0) = y'(2L).$$

11. Consider the Sturm-Liouville system

$$\frac{d^2y}{dx^2} + 4\lambda y = 0, \qquad 0 < x < L,$$
$$y(0) = 0 = y(L).$$

We could regard this system as one with eigenvalues λ and weight function $p(x) = 4$, or, alternatively, as one with eigenvalues 4λ and weight function $p(x) = 1$. Is there a difference as far as normalized eigenfunctions are concerned?

12. In this exercise we prove that a regular Sturm-Liouville system cannot have two linearly independent eigenfunctions corresponding to the same eigenvalue.
    (a) Suppose that $y(x)$ and $z(x)$ are eigenfunctions of (3) corresponding to the same eigenvalue λ. Show that $w(x) \equiv y'(a)z(x) - z'(a)y(x)$ satisfies (3a) and that $w(a) = w'(a) = 0$. This implies that $w(x) \equiv 0$ [and therefore that $y(x)$ and $z(x)$ are linearly dependent] unless $y'(a) = z'(a) = 0$.
    (b) If $y'(a) = z'(a) = 0$, then $h_1 = 0$. Define $w(x) = y(a)z(x) - z(a)y(x)$ to show once again that $w(x) \equiv 0$.

13. Use the result of Exercise 12 to show that up to a multiplicative constant, eigenfunctions of regular Sturm-Liouville systems are real.

14. In Exercises 7 and 9 we suggested the change of variable $z = x - 1$ in order to find eigenfunctions of the Sturm-Liouville system. Does it make any difference whether normalization is carried out in the z-variable or in the x-variable?

4.2 Eigenfunction Expansions

In Chapters 2 and 3 we learned how to express functions $f(x)$, which are piecewis smooth on the interval $0 \le x \le L$, in the form of Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \quad (9a$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \quad (9b$$

We regard the Fourier coefficients b_n as the components of the function $f(x)$ witl respect to the basis functions $\{\sin(n\pi x/L)\}$. In Section 4.1 we discovered that th $\sin(n\pi x/L)$ are eigenfunctions of Sturm-Liouville system (1), and it has becom our practice to replace eigenfunctions with normalized eigenfunctions, namel $\sqrt{2/L}\sin(n\pi x/L)$. Representation (9) can easily be replaced by an equivalent expressio in terms of these normalized eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} c_n\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right), \quad (10.$$

where

$$c_n = \int_0^L f(x)\left(\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\right)dx. \quad (10l$$

Coefficients c_n are components of $f(x)$ with respect to the orthonormal bas $\{\sqrt{2/L}\sin(n\pi x/L)\}$. Equation (9) should be compared with equation (3) in Section 2. together with the fact that the length of $\sin(n\pi x/L)$ is $\sqrt{L/2}$. Equation (10) is analogo to equation (1) in Section 2.1.

The same function $f(x)$ can be represented by a Fourier cosine series in terms normalized eigenfunctions of system (2):

$$f(x) = \frac{c_0}{\sqrt{L}} + \sum_{n=1}^{\infty} c_n\left(\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right) \quad (11$$

where

$$c_0 = \int_0^L f(x)\left(\frac{1}{\sqrt{L}}\right)dx \quad \text{and} \quad c_n = \int_0^L f(x)\left(\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\right)dx, \qquad n > 0. \quad (11$$

A natural question to ask now is the following: Given a function $f(x)$, defined the interval $a \le x \le b$, and given a Sturm-Liouville system on the same interval, i always possible to express $f(x)$ in terms of the orthonormal eigenfunctions of Sturm-Liouville system? It is still not clear that every Sturm-Liouville system has infinity of eigenfunctions, but, as we shall see, this is indeed the case. We wish then investigate the possibility of finding coefficients c_n such that on $a \le x \le b$

$$f(x) = \sum_{n=1}^{\infty} c_n y_n(x), \quad ($$

where $y_n(x)$ are the orthonormal eigenfunctions of Sturm-Liouville system (3). If formally multiply equation (12) by $p(x)y_m(x)$, and integrate term by term between $x =$

and $x = b$,

$$\int_a^b p(x)f(x)y_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b p(x)y_n(x)y_m(x)\,dx.$$

Because of the orthogonality of eigenfunctions, only the mth term in the series does not vanish, and therefore

$$\int_a^b p(x)f(x)y_m(x)\,dx = c_m. \tag{13}$$

This has been strictly a formal procedure. It has illustrated that *if* $f(x)$ can be represented in form (12), and *if* the series is suitably convergent, coefficients c_n must be calculated according to (13). What we must answer is the converse question: if coefficients c_n are calculated according to (13), where $y_n(x)$ are orthonormal eigenfunctions of a Sturm-Liouville system, does series (12) converge to $f(x)$? This question is answered in the following theorem.

Theorem 2

Let p, q, r, r', and $(pr)''$ be real and continuous functions of x for $a \le x \le b$, and let $p > 0$ and $r > 0$ for $a \le x \le b$. Let l_1, l_2, h_1, and h_2 be real constants independent of λ. Then Sturm-Liouville system (3) has a countable infinity of eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$ (all real), not more than a finite number of which are negative, and $\lim_{n\to\infty} \lambda_n = \infty$. Corresponding orthonormal eigenfunctions $y_n(x)$ are such that $y_n(x)$ and $y_n'(x)$ are continuous and $|y_n(x)|$ and $|\lambda_n^{-1/2} y_n'(x)|$ are uniformly bounded with respect to x and n. If $f(x)$ is piecewise smooth on $a \le x \le b$, then for any x in $a < x < b$,

$$\frac{f(x+) + f(x-)}{2} = \sum_{n=1}^{\infty} c_n y_n(x), \tag{14a}$$

where

$$c_n = \int_a^b p(x)f(x)y_n(x)\,dx. \tag{14b}$$

■

Series (14a) is called the *generalized Fourier series* for $f(x)$ with respect to the eigenfunctions $y_n(x)$, and the c_n are the *generalized Fourier coefficients*. They are the components of $f(x)$ with respect to the orthonormal basis of eigenfunctions $\{y_n(x)\}$. Notice the similarity between this theorem and Theorem 2 in Section 2.1 for Fourier series. Both guarantee pointwise convergence of Fourier series for a piecewise smooth function to the value of the function at a point of continuity of the function, and to average values of right- and left-hand limits at a point of discontinuity. Because the eigenfunctions in Theorem 2 of Section 2.1 are periodic, convergence is also assured at the end points of the interval $0 \le x \le 2L$. This is not the case in Theorem 2 above. Eigenfunctions are not generally periodic, and convergence at $x = a$ and $x = b$ is not guaranteed. It should be clear, however, that when $l_1 = 0$ [in which case $y_n(a) = 0$] convergence of (14a) at $x = a$ can be expected only if $f(a) = 0$ also. A similar statement can be made at $x = b$.

When a regular Sturm-Liouville system satisfies the conditions of this theorem as well as the conditions that $q(x) \ge 0$, $a \le x \le b$, and $l_1 h_1 \ge 0$, $l_2 h_2 \ge 0$, it is said to be a *proper Sturm-Liouville system.* For such a system we shall take l_1, l_2, h_1, and h_2 al nonnegative, in which case we can prove the following corollary.

Corollary

All eigenvalues of a proper Sturm-Liouville system are nonnegative. Furthermore, zero is a eigenvalue of a proper Sturm-Liouville system only when $q(x) \equiv 0$ and $h_1 = h_2 = 0$.

Proof: Let λ and $y(\lambda, x)$ be an eigenpair of a regular Sturm-Liouville system. Multiplication of (3a) b $y(\lambda, x)$ and integration from $x = a$ to $x = b$ gives

$$\lambda \int_a^b p(x)y^2(\lambda, x)\,dx = \int_a^b q(x)y^2(\lambda, x)\,dx - \int_a^b y(\lambda, x)[r(x)y'(\lambda, x)]'\,dx$$
$$= \int_a^b q(x)y^2(\lambda, x)\,dx - \{r(x)y(\lambda, x)y'(\lambda, x)\}_a^b$$
$$+ \int_a^b r(x)[y'(\lambda, x)]^2\,dx.$$

When we solve boundary conditions (3b, c) for $y'(\lambda, b)$ and $y'(\lambda, a)$ and substitute into the secor term on the right, we obtain

$$\lambda \int_a^b p(x)y^2(\lambda, x)\,dx = \int_a^b q(x)y^2(\lambda, x)\,dx + \int_a^b r(x)[y'(\lambda, x)]^2\,dx$$
$$+ \frac{h_2}{l_2} r(b)y^2(\lambda, b) + \frac{h_1}{l_1} r(a)y^2(\lambda, a).$$

When the Sturm-Liouville system is proper, every term on the right is nonnegative, as is t integral on the left, and therefore $\lambda \geq 0$. (If either $l_1 = 0$ or $l_2 = 0$, the corresponding terms in t above equation are absent and the result is the same.)

Furthermore, if $\lambda = 0$ is an eigenvalue, then each of the four terms on the right side of t above equation must vanish separately. The first requires that $q(x) \equiv 0$ and the second th $y'(\lambda, x) = 0$. But the fact that $y(\lambda, x)$ is constant implies that the last two terms can vanish only $h_1 = h_2 = 0$.

Since eigenvalues of a proper Sturm-Liouville system must be nonnegative, ` may replace λ by λ^2 in differential equation (3a) whenever it is convenient to do :

$$\frac{d}{dx}\left(r(x)\frac{dy}{dx}\right) + [\lambda^2 p(x) - q(x)]y = 0, \qquad a < x < b.$$

This often has the advantage of eliminating unnecessary square roots in calculatio:

Example 4: Expand the function $f(x) = L - x$ in terms of normalized eigenfunctions of the Sturı Liouville system of Example 1.

Solution: According to Example 1, eigenfunctions of this system are $\sin[(2n-1)\pi x/(2L$ Because

$$\left\|\sin\frac{(2n-1)\pi x}{2L}\right\|^2 = \int_0^L \left(\sin\frac{(2n-1)\pi x}{2L}\right)^2 dx = \frac{L}{2},$$

normalized eigenfunctions are $X_n(x) = \sqrt{2/L}\sin[(2n-1)\pi x/(2L)]$. The generalized Fourier series for $f(x) = L - x$ in terms of these eigenfunctions is

$$L - x = \sum_{n=1}^{\infty} c_n X_n(x),$$

where $$c_n = \int_0^L (L-x)\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}\,dx = \frac{2\sqrt{2}L^{3/2}}{\pi^2}\left(\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right).$$

Thus, $$L - x = \frac{2\sqrt{2}L^{3/2}}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\pi}{2n-1} + \frac{2(-1)^n}{(2n-1)^2}\right)\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}.$$

Theorem 2 guarantees convergence of the series to $L - x$ for $0 < x < L$. It obviously does not converge to $L - x$ at $x = 0$, but it does converge to $L - x$ at $x = L$. This follows from the facts that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1} = \frac{\pi}{4} \quad \text{and} \quad \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}. \qquad \blacksquare$$

In Chapter 3, when separation of variables was applied to (initial) boundary value problems, all boundary conditions in a given problem were either of Dirichlet type or of Neumann type. These led to Fourier sine and cosine series, series that we now know are eigenfunction expansions in terms of eigenfunctions of Sturm-Liouville systems (1) and (2). We did not consider problems with Robin conditions, nor did we mix Dirichlet and Neumann conditions. That would have led to series expansions for which we would have had no backup theory. With our results on Sturm-Liouville systems, we are now well prepared to tackle these expansions. A proper Sturm-Liouville system that arises repeatedly in our discussions is

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0, \qquad 0 < x < L, \tag{15a}$$

$$-l_1 X' + h_1 X = 0, \qquad x = 0, \tag{15b}$$

$$l_2 X' + h_2 X = 0, \qquad x = L. \tag{15c}$$

[Systems (1) and (2) are special cases of (15) when $l_1 = l_2 = 0$ and $h_1 = h_2 = 0$, respectively. Examples 1 and 4 contain the special case of $l_1 = h_2 = 0$ and $l_2 = h_1 = 1$.] We consider here the most general case, in which $h_1h_2l_1l_2 \neq 0$; special cases in which one or two of h_1, h_2, l_1, and l_2 vanish are tabulated later. In the general case when $h_1h_2l_1l_2 \neq 0$, we could divide (15b) by either l_1 or h_1. This would lead to a boundary condition with only one arbitrary constant (h_1/l_1 or l_1/h_1). Likewise, we could divide (15c) by l_2 or h_2 and express this boundary condition in terms of the ratio h_2/l_2 or the ratio l_2/h_2. However, when this is done, it is not quite so transparent how to specialize the results we obtain here in the cases in which one or two of h_1, h_2, l_1, and l_2 vanish. For this reason, we prefer to leave (15b, c) in their present forms.

We are justified in representing the eigenvalues of system (15) by λ^2 rather than λ, because all eigenvalues of a proper Sturm-Liouville system are nonnegative. The general solution of differential equation (15a) is

$$X(\lambda, x) = A\cos\lambda x + B\sin\lambda x, \tag{16}$$

and when we impose boundary conditions (15b, c),

$$-l_1\lambda B + h_1 A = 0, \tag{17a}$$

$$l_2(-A\lambda\sin\lambda L + B\lambda\cos\lambda L) + h_2(A\cos\lambda L + B\sin\lambda L) = 0. \tag{17b}$$

We solve (17a) for $B = h_1 A/(l_1\lambda)$ and substitute into (17b). After rearrangement, we obtain

$$\tan\lambda L = \frac{\lambda\left(\frac{h_1}{l_1} + \frac{h_2}{l_2}\right)}{\lambda^2 - \frac{h_1 h_2}{l_1 l_2}}, \tag{18}$$

the equation that must be satisfied by λ. We denote by λ_n $(n = 1, \ldots)$ the eigenvalues of this transcendental equation, although, in fact, *λ_n^2 are the eigenvalues of the Sturm-Liouville system*. Corresponding to these eigenvalues are the orthonormal eigenfunctions

$$X_n(x) = X(\lambda_n, x) = \frac{1}{N}\left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right), \tag{19a}$$

where

$$N^2 = \int_0^L \left(\cos\lambda_n x + \frac{h_1}{\lambda_n l_1}\sin\lambda_n x\right)^2 dx. \tag{19b}$$

In Exercise 1, integration is shown to lead to

$$2N^2 = \left[1 + \left(\frac{h_1}{\lambda_n l_1}\right)^2\right]\left[L + \frac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}\right] + \frac{h_1/l_1}{\lambda_n^2}. \tag{19c}$$

Of the nine possible combinations of boundary conditions at $x = 0$ and $x = L$, we have considered only one, the most general in which none of h_1, h_2, l_1, and l_2 vanishes. Results for the remaining eight cases can be obtained from (18) and (19), or by similar analyses; they are tabulated in Table 4.1.

Each eigenvalue equation in Table 4.1 is unchanged if λ is replaced by $-\lambda$, so that for every positive solution λ of the equation, $-\lambda$ is also a solution. Since NX_n is invariant (up to a sign change) by the substitution of $-\lambda_n$ for λ_n, it is necessary only to consider the nonnegative solutions of the eigenvalue equations. This agrees with the fact that the eigenvalues of the Sturm-Liouville system are λ_n^2 and that there cannot be two linearly independent eigenfunctions corresponding to the same eigenvalue. Table 4.1 gives the eigenvalues explicitly in only four of the nine cases. The eigenvalues in the remaining five cases are illustrated geometrically below and on the following pages.

If $h_1 h_2 l_1 l_2 \neq 0$, eigenvalues are illustrated graphically in Figure 4.1 as points of intersection of the curves

$$y = \tan\lambda L, \qquad y = \frac{\lambda(h_1/l_1 + h_2/l_2)}{\lambda^2 - [(h_1 h_2)/(l_1 l_2)]}.$$

It might appear that $\lambda = 0$ is an eigenvalue in this case. However, the corollary to Theorem 2 indicates that zero is an eigenvalue only when $h_1 = h_2 = 0$. This can also be verified using conditions (17), which led to the eigenvalue equation (see Exercise 3).

Table 4.1 *Eigenpairs for the Sturm-Liouville System $X'' + \lambda^2 X = 0$, $0 < x < L$, $-l_1 X'(0) + h_1 X(0) = 0$, $l_2 X'(L) + h_2 X(L) = 0$.*

Condition at $x = 0$	Condition at $x = L$	Eigenvalue Equation	NX_n	$2N^2$
$h_1 l_1 \neq 0$	$h_2 l_2 \neq 0$	$\tan \lambda L = \dfrac{\lambda\left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\lambda^2 - \dfrac{h_1 h_2}{l_1 l_2}}$	$\cos \lambda_n x + \dfrac{h_1}{\lambda_n l_1} \sin \lambda_n x$	$\dfrac{h_1/l_1}{\lambda_n^2}\left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] \times \left[L + \dfrac{h_2/l_2}{\lambda_n^2 + \left(\dfrac{h_2}{l_2}\right)^2}\right]$
$h_1 l_1 \neq 0$	$h_2 = 0$ $(l_2 = 1)$	$\tan \lambda L = \dfrac{h_1}{\lambda l_1}$	$\dfrac{\cos \lambda_n (L - x)}{\cos \lambda_n L}$	$L\left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] + \dfrac{h_1/l_1}{\lambda_n^2}$
$h_1 l_1 \neq 0$	$l_2 = 0$ $(h_2 = 1)$	$\cot \lambda L = -\dfrac{h_1}{\lambda l_1}$	$\dfrac{\sin \lambda_n (L - x)}{\sin \lambda_n L}$	$L\left[1 + \left(\dfrac{h_1}{\lambda_n l_1}\right)^2\right] + \dfrac{h_1/l_1}{\lambda_n^2}$
$h_1 = 0$ $(l_1 = 1)$	$h_2 l_2 \neq 0$	$\tan \lambda L = \dfrac{h_2}{\lambda l_2}$	$\cos \lambda_n x$	$L + \dfrac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}$
$h_1 = 0$ $(l_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\sin \lambda L = 0$ $\lambda_n = \dfrac{n\pi}{L}$, $n = 0, 1, 2, \ldots$	$\cos \lambda_n x$	$L\ (n \neq 0)$ $2L\ (n = 0)$
$h_1 = 0$ $(l_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\cos \lambda L = 0$ $\lambda_n = \left(\dfrac{2n-1}{2}\right)\dfrac{\pi}{L}$, $n = 1, 2, \ldots$	$\cos \lambda_n x$	L
$l_1 = 0$ $(h_1 = 1)$	$h_2 l_2 \neq 0$	$\cot \lambda L = -\dfrac{h_2}{\lambda l_2}$	$\sin \lambda_n x$	$L + \dfrac{h_2/l_2}{\lambda_n^2 + (h_2/l_2)^2}$
$l_1 = 0$ $(h_1 = 1)$	$h_2 = 0$ $(l_2 = 1)$	$\cos \lambda L = 0$ $\lambda_n = \left(\dfrac{2n-1}{2}\right)\dfrac{\pi}{L}$, $n = 1, 2, \ldots$	$\sin \lambda_n x$	L
$l_1 = 0$ $(h_1 = 1)$	$l_2 = 0$ $(h_2 = 1)$	$\sin \lambda L = 0$ $\lambda_n = \dfrac{n\pi}{L}$, $n = 1, 2, \ldots$	$\sin \lambda_n x$	L

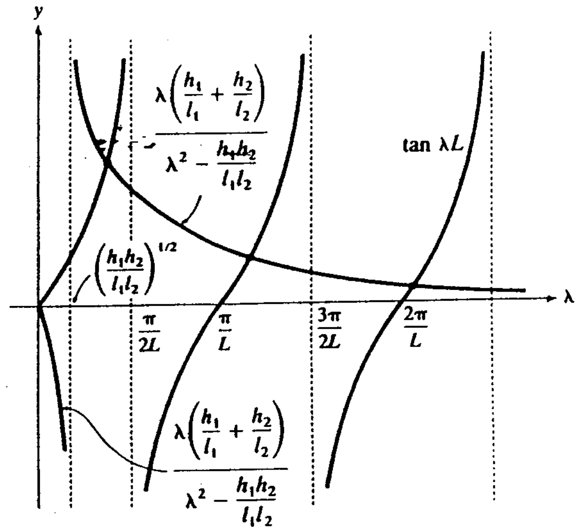


Figure 4.1

If $h_1l_1 \neq 0$ and $h_2 = 0$ (in which case we set $l_2 = 1$), eigenvalues are illustrated graphically (Figure 4.2) as points of intersection of the curves

$$y = \tan \lambda L \quad \text{and} \quad y = \frac{h_1}{\lambda l_1}.$$

A similar situation arises when $h_2l_2 \neq 0$ and $h_1 = 0$.

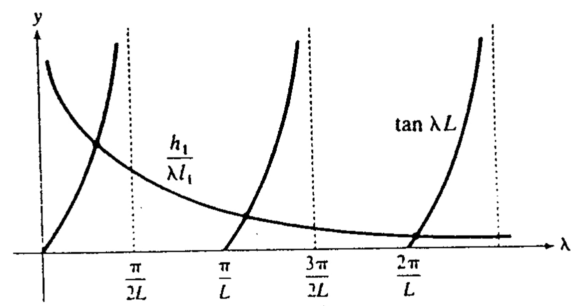


Figure 4.2

If $h_1l_1 \neq 0$ and $l_2 = 0$ (in which case we set $h_2 = 1$), eigenvalues are illustrated graphically (Figure 4.3) as points of intersection of

$$y = \cot \lambda L \quad \text{and} \quad y = -\frac{h_1}{\lambda l_1}.$$

A similar situation arises when $h_2l_2 \neq 0$ and $l_1 = 0$.

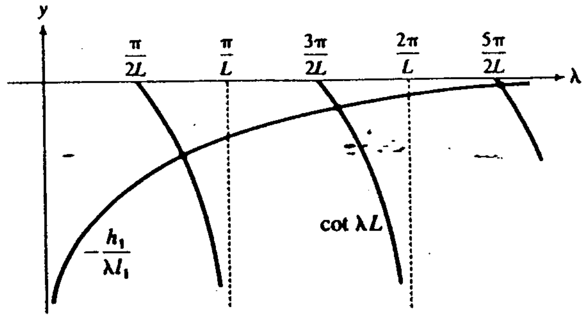


Figure 4.3

Theorem 2 states that when a function $f(x)$ is piecewise smooth on the interval $0 \le x \le L$, we may write for $0 < x < L$

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x) \tag{20a}$$

where

$$c_n = \int_0^L f(x) X_n(x)\, dx. \tag{20b}$$

Example 5: Expand the function $f(x) = 2x - 1$, $0 \le x \le 4$ in terms of the orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \qquad 0 < x < 4,$$
$$X'(0) = 0 = X(4).$$

Solution: When we set $L = 4$ in line 6 of Table 4.1, normalized eigenfunctions of the Sturm-Liouville system are

$$X_n(x) = \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \qquad n = 1, 2, \ldots.$$

For $0 < x < 4$, we may write

$$2x - 1 = \sum_{n=1}^{\infty} c_n X_n(x),$$

where

$$\begin{aligned} c_n &= \int_0^4 (2x-1) X_n(x)\, dx \\ &= \frac{1}{\sqrt{2}} \left\{ \frac{8(2x-1)}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{8} + \frac{128}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi x}{8} \right\}_0^4 \\ &= \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2}. \end{aligned}$$

Thus,

$$\begin{aligned} 2x - 1 &= \sum_{n=1}^{\infty} \frac{-8[16 + 7(-1)^n(2n-1)\pi]}{\sqrt{2}(2n-1)^2\pi^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8} \\ &= -\frac{4\sqrt{2}}{\pi^2} \sum_{n=1}^{\infty} \frac{16 + 7(-1)^n(2n-1)\pi}{(2n-1)^2} \frac{1}{\sqrt{2}} \cos \frac{(2n-1)\pi x}{8}, \end{aligned}$$

$$0 < x < 4.$$

■

Periodic Sturm-Liouville systems do not come under the purview of Theorem 2. In particular, this theorem does not guarantee expansions in terms of normalized eigenfunctions of periodic Sturm-Liouville systems. For instance, eigenvalues for the periodic Sturm-Liouville system of Example 2 are $\lambda_n = n^2\pi^2/L^2$ $(n = 0, 1, \ldots)$, with corresponding eigenfunctions

$$\lambda_0 \leftrightarrow 1, \qquad \lambda_n \leftrightarrow \sin\frac{n\pi x}{L}, \qquad \cos\frac{n\pi x}{L} \qquad (n > 0).$$

Normalized eigenfunctions are

$$\lambda_0 \leftrightarrow \frac{1}{\sqrt{2L}}, \qquad \lambda_n \leftrightarrow \frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}, \qquad \frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} \qquad (n > 0).$$

Theorem 2 does not ensure the expansion of a function $f(x)$ in terms of these eigenfunctions, but our theory of ordinary Fourier series does. These are precisely the basis functions for ordinary Fourier series, except for normalizing factors, so we may write

$$f(x) = \frac{a_0}{\sqrt{2L}} + \sum_{n=1}^{\infty}\left(a_n\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L} + b_n\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right), \tag{21a}$$

where

$$a_0 = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{2L}}\right)dx, \qquad a_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\cos\frac{n\pi x}{L}\right)dx, \tag{21b}$$

$$b_n = \int_{-L}^{L} f(x)\left(\frac{1}{\sqrt{L}}\sin\frac{n\pi x}{L}\right)dx. \tag{21c}$$

Exercises 4.2

1. Obtain expression (19c) for $2N^2$ by direct integration of (19b). *Hint:* Show that

$$\sin\lambda_n L = \frac{(-1)^{n+1}\lambda_n\left(\dfrac{h_1}{l_1} + \dfrac{h_2}{l_2}\right)}{\left[\left(\lambda_n^2 + \dfrac{h_1^2}{l_1^2}\right)\left(\lambda_n^2 + \dfrac{h_2^2}{l_2^2}\right)\right]^{1/2}}$$

and

$$\cos\lambda_n L = \frac{(-1)^{n+1}\left(\lambda_n^2 - \dfrac{h_1 h_2}{l_1 l_2}\right)}{\left[\left(\lambda_n^2 + \dfrac{h_1^2}{l_1^2}\right)\left(\lambda_n^2 + \dfrac{h_2^2}{l_2^2}\right)\right]^{1/2}}.$$

2. For each Sturm-Liouville system in Table 4.1, find expressions for $\sin\lambda_n L$ and $\cos\lambda_n L$ that involve only h_1, h_2, l_1, l_2, and/or λ_n. These should be tabulated and attached to Table 4.1 for future reference.

3. Use equations (17) to verify that $\lambda = 0$ is an eigenvalue of Sturm-Liouville system (15) only when $h_1 = h_2 = 0$.

In Exercises 4–9, express the function $f(x) = x$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system.

4. $X'' + \lambda^2 X = 0, \quad X(0) = X(L) = 0$
5. $X'' + \lambda^2 X = 0, \quad X'(0) = X'(L) = 0$
6. $X'' + \lambda^2 X = 0, \quad X(0) = X'(L) = 0$
7. $X'' + \lambda^2 X = 0, \quad X'0) = X(L) = 0$
8. $X'' + \lambda^2 X = 0, \quad X'(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$
9. $X'' + \lambda^2 X = 0, \quad X(0) = 0, \quad l_2 X'(L) + h_2 X(L) = 0$
10. Express the function $f(x) = x^2$, $0 \le x \le L$, in terms of orthonormal eigenfunctions of the Sturm-Liouville system

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L,$$
$$X(0) = 0 = X'(L).$$

In Exercises 11–13, find eigenvalues and orthonormal eigenfunctions of the proper Sturm-Liouville system.

11. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L, \quad y'(0) = 0 = y'(L)$
12. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L\ (\beta \neq 0 \text{ a given constant}), \quad y(0) = 0 = y(L)$
13. $\dfrac{d^2y}{dx^2} + \beta\dfrac{dy}{dx} + \lambda^2 y = 0, \quad 0 < x < L\ (\beta \neq 0 \text{ a given constant}), \quad y'(0) = 0 = y'(L)$
14. (a) Find eigenvalues and (nonnormalized) eigenfunctions for the proper Sturm-Liouville system

$$y'' + \lambda^2 y = 0, \qquad -L < x < L,$$
$$y'(-L) = 0 = y'(L).$$

   (b) Show that the eigenfunctions in (a) can be expressed in the compact form $\cos[n\pi(x + L)/(2L)]$, $n = 0, 1, 2, \ldots$.
   (c) Normalize the eigenfunctions.
15. (a) Show that the transformation $x = e^z$ replaces the Sturm-Liouville system

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2 y = 0, \qquad 1 < x < L,$$
$$y(1) = 0 = y(L)$$

   in $y(x)$ with the system

$$\frac{d^2y}{dz^2} + \lambda^2 y = 0, \qquad 0 < z < \ln L,$$
$$y(0) = 0 = y(\ln L)$$

   in $y(z)$. If we use Table 4.1, what are the normalized eigenfunctions $y_n(z)$? Replace z by $\ln x$ to obtain eigenfunctions $y_n(x)$.
   (b) Repeat (a) with the transformation $x = L^z$.

16. Find eigenvalues and orthonormal eigenfunctions of the Sturm-Liouville system

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \frac{\lambda^2}{x}y = 0, \qquad 1 < x < b,$$

$$y(b) = 0 = y'(1).$$

In Exercises 17–19, find approximations for the four smallest eigenvalues of the Sturm-Liouville system.

17. $X'' + \lambda^2 X = 0,\ \ 0 < x < 1,\ \ -200X'(0) + 400{,}000X(0) = 0,\ \ X'(1) = 0$

18. $X'' + \lambda^2 X = 0,\ \ 0 < x < 1,\ \ X(0) = 0,\ \ 150X'(1) + 100{,}000X(1) = 0$

19. $X'' + \lambda^2 X = 0,\ \ 0 < x < 1,\ \ -200X'(0) + 400X(0) = 0,\ \ 200X'(1) + 100X(1) = 0$

20. (a) Expand the function

$$f(x) = \begin{cases} 1 & 0 < x < L/2 \\ -1 & L/2 < x < L \end{cases}$$

in terms of the normalized eigenfunctions of Sturm-Liouville system (2).

(b) What does the series converge to at $x = L/2$? Is this to be expected?

(c) What does the series converge to at $x = 0$ and $x = L$? Are these to be expected?

21. Repeat Exercise 20 with the eigenfunctions of Sturm-Liouville system (1).

22. In Exercise 11 of Section 4.1, we suggested two ways of interpreting the 4 in the differential equation. Does it make a difference as far as eigenfunction expansions are concerned?

23. The initial boundary value problem for transverse vibrations $y(x, t)$ of a beam simply supported at one end ($x = L$) and horizontally built in at the other end ($x = 0$) when gravity is negligible compared with internal forces is

$$\frac{\partial^2 y}{\partial t^2} + c^2\frac{\partial^4 y}{\partial x^4} = 0, \qquad 0 < x < L, \qquad t > 0,$$

$$y(0, t) = y_x(0, t) = 0, \qquad t > 0,$$

$$y(L, t) = y_{xx}(L, t) = 0, \qquad t > 0,$$

$$y(x, 0) = f(x) \qquad 0 < x < L,$$

$$y_t(x, 0) = g(x), \qquad 0 < x < L.$$

(a) Show that by assuming that $y(x, t) = X(x)T(t)$, eigenfunctions

$$X_n(x) = \frac{1}{\cos\lambda_n L}\sin\lambda_n(L - x) - \frac{1}{\cosh\lambda_n L}\sinh\lambda_n(L - x)$$

are obtained, where eigenvalues λ_n must satisfy

$$\tan\lambda L = \tanh\lambda L.$$

(b) Prove that these eigenfunctions are orthogonal on the interval $0 \le x \le L$ with respect to the weight function $p(x) = 1$. [*Hint:* Use the differential equation defining $X_n(x)$ and a construction like that in Theorem 1.]

24. Does the Sturm-Liouville system in line 6 of Table 4.1 give rise to the expansion in Exercise 2: of Section 2.2 for even and odd-harmonic functions?

25. Does the Sturm-Liouville system in line 8 of Table 4.1 give rise to the expansion in Exercise 21 of Section 2.2 for odd and odd-harmonic functions?
26. Show that the Sturm-Liouville system

$$\frac{d^2X}{dx^2} + \lambda X = 0, \qquad 0 < x < L,$$
$$X'(0) = 0,$$
$$l_2X'(L) - h_2X(L) = 0, \qquad (l_2 > 0, h_2 > 0)$$

has exactly one negative eigenvalue. What is the corresponding eigenfunction?

4.3 Further Properties of Sturm-Liouville Systems

When $f(x)$ is a piecewise smooth function, its Fourier series converges to $[f(x+) + f(x-)]/2$. When $f(x)$ is continuous, the Fourier series converges absolutely and uniformly (see Theorem 10 in Section 2.3). The counterpart of the latter result for generalized Fourier series is contained in the following theorem.

Theorem 3

Suppose that $f(x)$ is continuous and $f'(x)$ is piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system on $a \le x \le b$, then the generalized Fourier series (*14*) for $f(x)$ converges uniformly to $f(x)$ for $a \le x \le b$.

For example, when the function $x(L-x)$ is expanded in terms of the eigenfunctions of Sturm-Liouville system (1), the result is

$$x(L-x) = \frac{2\sqrt{2}L^{5/2}}{\pi^3}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n^3}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$
$$= \frac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{L}, \qquad 0 \le x \le L.$$

Because $x(L-x)$ satisfies conditions (1b, c), convergence is uniform. This could also be verified with the Weierstrass M-test (see Section 2.3).

Exercise 4 in Section 2.3 contains special cases of this result.

Expansions of functions as generalized Fourier series are very different from power series expansions. A function can be represented in a Taylor series on an interval only if the function and all of its derivatives are continuous throughout the interval, and even these conditions may not be sufficient to guarantee convergence of the series to the function. Eigenfunction expansions, however, are valid even though a function and its first derivative may each possess a finite number of finite discontinuities.

On the other hand, whereas Taylor series expansions may be differentiated term by term inside the interval of convergence of the series, such may not be the case for generalized Fourier series. The following result is analogous to Theorem 3 in Section 2.1.

Theorem 4

Suppose that $f(x)$ is continuous and $f'(x)$ and $f''(x)$ are piecewise continuous on $a \le x \le b$. If $f(x)$ satisfies the boundary conditions of a proper Sturm-Liouville system, then for any x in $a < x < b$, series (14) may be differentiated term by term with the resulting series converging to $[f'(x+) + f'(x-)]/2$.

We now prove the Sturm comparison theorem, a result that has implications when we study singular Sturm-Liouville systems in Chapter 8.

Theorem 5 (Sturm Comparison Theorem)

Let $r(x)$ be a function that is positive on the interval $a < x < b$ and has a continuous first derivative for $a \le x \le b$. Suppose that $s_1(x)$ and $s_2(x)$ are continuous functions for $a < x < b$ such that $s_2(x) > s_1(x)$ thereon. If $y_1(x)$ and $y_2(x)$ satisfy

$$\frac{d}{dx}\left(r(x)\frac{dy_1}{dx}\right) + s_1(x)y_1 = 0 \tag{22a}$$

and

$$\frac{d}{dx}\left(r(x)\frac{dy_2}{dx}\right) + s_2(x)y_2 = 0 \tag{22b}$$

there is at least one zero of $y_2(x)$ between every consecutive pair of zeros of $y_1(x)$ in $a < x < b$.

Proof: Let α and β be any two consecutive zeros of $y_1(x)$ in $a < x < b$, and suppose that $y_2(x)$ has no zero between α and β. We assume, without loss in generality, that $y_1(x) > 0$ and $y_2(x) > 0$ on $\alpha < x < \beta$. [If this were not true, we would work with $-y_1(x)$ and $-y_2(x)$.] When (22a) and (22b) are multiplied by y_2 and y_1, respectively, and the results are subtracted,

$$0 = y_1\left[\frac{d}{dx}\left(r\frac{dy_2}{dx}\right) + s_2y_2\right] - y_2\left[\frac{d}{dx}\left(r\frac{dy_1}{dx}\right) + s_1y_1\right].$$

Integration of this equation from α to β gives

$$\begin{aligned}\int_\alpha^\beta (s_2 - s_1)y_1y_2\,dx &= \int_\alpha^\beta [(ry_1')'y_2 - (ry_2')'y_1]\,dx \\ &= \int_\alpha^\beta (ry_1'y_2 - ry_2'y_1)'\,dx \\ &= \{r(y_2y_1' - y_1y_2')\}_\alpha^\beta \\ &= r(\beta)[y_2(\beta)y_1'(\beta) - y_1(\beta)y_2'(\beta)] - r(\alpha)[y_2(\alpha)y_1'(\alpha) - y_1(\alpha)y_2'(\alpha)] \\ &= r(\beta)y_2(\beta)y_1'(\beta) - r(\alpha)y_2(\alpha)y_1'(\alpha),\end{aligned}$$

since $y_1(\alpha) = y_1(\beta) = 0$. Because $y_1(x) > 0$ for $\alpha < x < \beta$, it follows that $y_1'(\alpha) \ge 0$ and $y_1'(\beta) \le 0$. Furthermore, because $r(\alpha)$, $r(\beta)$, $y_2(\alpha)$, and $y_2(\beta)$ are all positive, we must have

$$r(\beta)y_2(\beta)y_1'(\beta) - r(\alpha)y_2(\alpha)y_1'(\alpha) \le 0.$$

But this contradicts the fact that

$$\int_{\alpha}^{\beta} [s_2(x) - s_1(x)] y_2(x) y_1(x)\, dx > 0,$$

since $s_2 > s_1$ on $\alpha \leq x \leq \beta$. Consequently, $y_2(x)$ must have a zero between α and β. ■

To see the implication of this theorem in Sturm-Liouville theory, we set $s_1(x) = \lambda_1 p(x) - q(x)$ and $s_2(x) = \lambda_2 p(x) - q(x)$, where $\lambda_2 > \lambda_1$ are eigenvalues of system (3). It then follows that between every pair of zeros of the eigenfunction $y_1(x)$ corresponding to λ_1, there is at least one zero of the eigenfunction $y_2(x)$ associated with λ_2. Figure 4.4 illustrates the situation for eigenfunction $X_3(x)$ and $X_4(x)$ of Sturm-Liouville system (1).

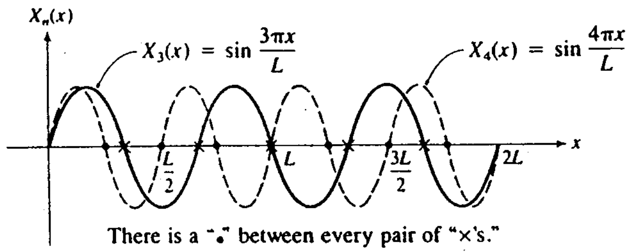


Figure 4.4 There is a "•" between every pair of "x's."

As a final consideration in this chapter, we show that the Sturm-Liouville systems in Table 4.1 arise when separation of variables is applied to (initial) boundary value problems having second-order PDEs expressed in Cartesian coordinates. To illustrate this, we apply separation of variables to the rather general second-order PDE

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \tag{23}$$

where p, q, and s are constants and t is time. We consider this PDE because it includes as special cases many of those in Chapter 1. In particular,

(1) if $V = V(\mathbf{r}, t)$, $p = s = 0$, and $q = k^{-1}$, then (23) is the one-, two-, or three-dimensional heat conduction equation;

(2) if $V = V(\mathbf{r}, t)$, and $p = \rho/\tau$ (or ρ/E), then (23) is the one-, two-, or three-dimensional wave equation;

(3) if $V = V(\mathbf{r})$, $p = q = s = 0$, then (23) is the one-, two-, or three-dimensional Laplace equation.

Thus, the results obtained here are valid for heat conduction, vibration, and potential problems.

When (23) is to be solved in some finite region, boundary conditions and possibly initial conditions are associated with the PDE. If this region is a rectangular parallelopiped (box) in space, Cartesian coordinates can be chosen to specify the region in the form $0 \leq x \leq L, 0 \leq y \leq L', 0 \leq z \leq L''$ (Figure 4.5). Boundary conditions must then be specified on the six faces $x = 0$, $y = 0$, $z = 0$, $x = L$, $y = L'$, and $z = L''$. Suppose, for example, that the following homogeneous Dirichlet, Neumann, and Robin conditions accompany (23):

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV,$$

$$0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{24a}$$

$$V = 0, \quad x = 0, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{24b}$$

$$\frac{\partial V}{\partial x} = 0, \quad x = L, \quad 0 < y < L', \quad 0 < z < L'', \quad t > 0, \tag{24c}$$

$$-l_3\frac{\partial V}{\partial y} + h_3 V = 0, \quad y = 0, \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{24d}$$

$$V = 0, \quad y = L', \quad 0 < x < L, \quad 0 < z < L'', \quad t > 0, \tag{24e}$$

$$\frac{\partial V}{\partial z} = 0, \quad z = 0, \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{24f}$$

$$l_6\frac{\partial V}{\partial z} + h_6 V = 0, \quad z = L'', \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{24g}$$

Initial conditions, if applicable. (24h)

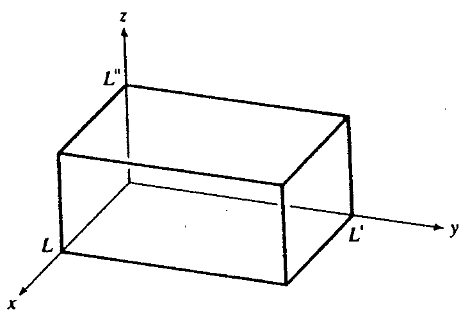


Figure 4.5

If we assume that a function with variables separated, $V(x, y, z, t) = X(x)Y(y)Z(z)T(t)$, satisfies (24a),

$$X''YZT + XY''ZT + XYZ''T = pXYZT'' + qXYZT' + sXYZT.$$

Division by $XYZT$ gives

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = \frac{pT'' + qT' + sT}{T}$$

or

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}.$$

The separation principle (see Section 3.1) implies that each side of this equation must be equal to a constant, say α:

$$-\frac{X''}{X} = \alpha = \frac{Y''}{Y} + \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T}. \tag{25}$$

Thus $X(x)$ must satisfy the ODE $X'' + \alpha X = 0$, $0 < x < L$. When the separated function is substituted into boundary conditions (24b, c), there results

$$X(0)Y(y)Z(z)T(t) = 0, \qquad X'(L)Y(y)Z(z)T(t) = 0.$$

From these, $X(0) = 0 = X'(L)$, and hence $X(x)$ must satisfy

$$X'' + \alpha X = 0, \qquad 0 < x < L, \tag{26a}$$

$$X(0) = 0 = X'(L). \tag{26b}$$

This is proper Sturm-Liouville system (15) with $l_1 = h_2 = 0$ and $h_1 = l_2 = 1$. When we set $\alpha = \lambda^2$, eigenvalues λ_n^2 and orthonormal eigenfunctions $X_n(x)$ are then as given in line 8 of Table 4.1:

$$\lambda_n^2 = \frac{(2n-1)^2\pi^2}{4L^2}, \qquad X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}.$$

Further separation of (25) gives

$$-\frac{Y''}{Y} = \beta = \frac{Z''}{Z} - \frac{pT'' + qT' + sT}{T} - \lambda_n^2. \tag{27}$$

where β is a constant. Boundary conditions (24d, e) imply that $Y(y)$ must satisfy

$$Y'' + \beta Y = 0, \qquad 0 < y < L', \tag{28a}$$

$$-l_3 Y'(0) + h_3 Y(0) = 0, \tag{28b}$$

$$Y(L') = 0. \tag{28c}$$

This is Sturm-Liouville system (15) with y's replacing x's, with h_3, l_3, and L replacing h_1, l_1, and L, and with $l_2 = 0, h_2 = 1$. When we set $\beta = \mu^2$, the eigenvalue equation and orthonormal eigenfunctions are as found in line 3 of Table 4.1:

$$\cot \mu L' = -\frac{h_3}{\mu l_3}, \qquad N Y_m(y) = \frac{1}{\sin \mu_m L'}\sin \mu_m(L' - y),$$

where
$$2N^2 = L'\left[1 + \left(\frac{h_3}{\mu_m l_3}\right)^2\right] + \frac{h_3/l_3}{\mu_m^2}.$$

Continued separation of (27) yields

$$-\frac{Z''}{Z} = \gamma = -\frac{pT'' + qT' + sT}{T} - \lambda_n^2 - \mu_m^2, \tag{29}$$

where γ is a constant. When this is combined with boundary conditions (24f, g), $Z(z)$ must satisfy the Sturm-Liouville system

$$Z'' + \gamma Z = 0, \qquad 0 < z < L'', \tag{30a}$$

$$Z'(0) = 0, \tag{30b}$$

$$l_6 Z'(L'') + h_6 Z(L'') = 0. \tag{30c}$$

With changes in notation, this is the Sturm-Liouville system in line 4 of Table 4.1. Eigenvalues $\gamma = \nu^2$ are defined by

$$\tan \nu L'' = \frac{h_6}{\nu l_6},$$

with orthonormal eigenfunctions $N^{-1} \cos \nu_j z$ where

$$2N^2 = L'' + \frac{h_6/l_6}{\nu_j^2 + (h_6/l_6)^2}.$$

The time-dependent part $T(t)$ of $V(x, y, z, t)$ is obtained by solving the ODE

$$pT'' + qT' + sT = -(\lambda_n^2 + \mu_m^2 + \nu_j^2)T. \tag{31}$$

In summary, separation of variables on (initial) boundary value problem (24) has led to the Sturm-Liouville systems in lines 3, 4, and 8 of Table 4.1. Other choices for boundary conditions led to the remaining five Sturm-Liouville systems in Table 4.1 (see Exercises 7–9).

Exercises 4.3

1. Theorem 4 indicates that generalized Fourier series from proper Sturm-Liouville systems may be differentiated term by term when $f(x)$ is continuous, $f'(x)$ and $f''(x)$ are piecewise continuous, and $f(x)$ satisfies the boundary conditions of the system. We illustrate with two examples.
   (a) Find the eigenfunction expansion for

   $$f(x) = \begin{cases} x & 0 \le x \le L/2 \\ L - x & L/2 \le x \le L \end{cases}$$

   in terms of the normalized eigenfunctions of Sturm-Liouville system (1). Show graphically that $f(x)$ is continuous, and $f'(x)$ and $f''(x)$ are piecewise continuous, on $0 \le x \le L$. Since $f(0) = f(L) = 0$, Theorem 4 guarantees that term-by-term differentiation of the eigenfunction expansion for $f(x)$ yields a series that converges to $[f'(x+) + f'(x-)]/2$ for $0 < x < L$. Verify that this is indeed true, but do so without using Theorem 4.
   (b) Find the eigenfunction expansion for $g(x) = 1, 0 \le x \le L$, in terms of the eigenfunctions of (1). Show that term-by-term differentiation of this series gives a series that converges only for $x = L/2$. Which of the conditions in Theorem 4 are violated by $g(x)$?
2. Prove Bessel's inequality for eigenfunction expansions: If $f(x)$ is a piecewise continuous function on $a \le x \le b$, and $y_n(x)$ are the eigenfunctions of a proper Sturm-Liouville system, the generalized Fourier coefficients of $f(x)$ satisfy the inequality

   $$\sum_{n=1}^{\infty} c_n^2 \le \int_a^b p(x)[f(x)]^2\,dx.$$

   This result is extended to an equality in Exercise 3.
3. Parseval's theorem states that when $f(x)$ is a piecewise continuous function on $a \le x \le b$, and $y_n(x)$ are the eigenfunctions of a proper Sturm-Liouville system, the generalized Fourie

coefficients of $f(x)$ satisfy the equality

$$\sum_{n=1}^{\infty} c_n^2 = \int_a^b p(x)[f(x)]^2\,dx.$$

Verify this in the case that $f(x)$ is continuous, has a piecewise continuous first derivative, and satisfies the boundary conditions of the Sturm-Liouville system.

4. In Exercise 6 of Section 2.3, it was shown that the partial sums of the Fourier series of a function $f(x)$ are the best trigonometric approximation of the function in the mean square sense. In this exercise we show that generalized Fourier series are also best approximations in the mean square sense. A piecewise continuous function $f(x)$ is to be approximated by a sum of the form

$$S_n(x) = \sum_{k=1}^{n} \alpha_k y_k(x),$$

where α_k are constants and $y_k(x)$ are normalized eigenfunctions of Sturm-Liouville system (3). One measure of the accuracy of this approximation is the quantity

$$E_n = \int_a^b p(x)[f(x) - S_n(x)]^2\,dx,$$

called the mean square error with respect to the weight function $p(x)$.

(a) Show that E_n can be expressed in the form

$$E_n = \int_a^b p(x)[f(x)]^2\,dx + \sum_{k=1}^{n} \alpha_k^2 - 2\sum_{k=1}^{n} \alpha_k c_k,$$

where c_k are the generalized Fourier coefficients of $f(x)$.

(b) Regarding E_n as a function of n variables $\alpha_1, \ldots, \alpha_n$, show that E_n is minimized when the α_k are chosen as c_k.

5. The generalized Fourier series of $f(x)$ in terms of the eigenfunctions of Sturm-Liouville system (3) is said to converge in the mean to $f(x)$ on $a \le x \le b$ if

$$\lim_{n\to\infty} \int_a^b p(x)[S_n(x) - f(x)]^2\,dx = 0,$$

where $S_n(x)$ is the nth partial sum

$$S_n(x) = \sum_{k=1}^{n} c_k y_k(x)$$

and

$$c_k = \int_a^b p(x) f(x) y_k(x)\,dx.$$

Use the results of Exercises 3 and 4 to show that this is indeed the case when $f(x)$ is continuous and $f'(x)$ is piecewise continuous.

6. (a) Show that an eigenvalue λ_n of a regular Sturm-Liouville system can be expressed in terms of its corresponding normalized eigenfunction $y_n(x)$ according to

$$\lambda_n = \int_a^b \left(r(x)[y_n'(x)]^2 + q(x)[y_n(x)]^2\right)dx - \{r(x)y_n(x)y_n'(x)\}_a^b.$$

(b) What form does the expression in (a) take when both boundary conditions are Dirichlet? When both are Neumann?

In Exercises 7–9, determine all Sturm-Liouville systems that result when separation of variables is used to solve the problem. Do not solve the problem; simply find the Sturm-Liouville systems. Find eigenvalues (or eigenvalue equations) for each Sturm-Liouville system and orthonormal eigenfunctions. Give a physical interpretation of each problem.

7. $\dfrac{\partial^2 U}{\partial x^2} + \dfrac{\partial^2 U}{\partial y^2} = k\dfrac{\partial U}{\partial t}$, $0 < x < L$, $0 < y < L'$, $t > 0$; $U(0, y, t) = 0$, $0 < y < L'$, $t > 0$; $\partial U(L, y, t)/\partial x + 200U(L, y, t) = 0$, $0 < y < L'$, $t > 0$; $\partial U(x, 0, t)/\partial y = 0$, $0 < x < L$, $t > 0$; $\partial U(x, L', t)/\partial y = 0$, $0 < x < L$, $t > 0$; $U(x, y, 0) = f(x, y)$, $0 < x < L$, $0 < y < L'$

8. $\dfrac{\partial^2 y}{\partial t^2} = c^2\dfrac{\partial^2 y}{\partial x^2} - \beta\dfrac{\partial y}{\partial t}$, $0 < x < L$, $t > 0$; $-\tau\partial y(0, t)/\partial x + ky(0, t) = 0$, $t > 0$; $y(L, t) = 0$, $t > 0$; $y(x, 0) = f(x)$, $0 < x < L$; $y_t(x, 0) = 0$, $0 < x < L$

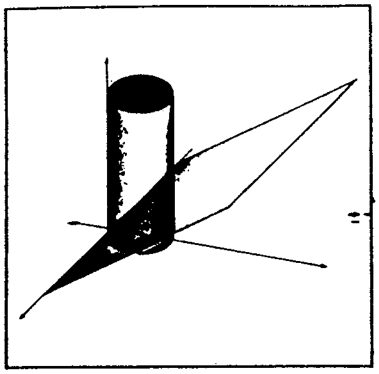
9. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} = 0$, $0 < x < L$, $0 < y < L'$, $0 < z < L''$; $V(0, y, z) = 0$, $0 < y < L'$, $0 < z < L''$; $\partial V(L, y, z)/\partial x = 0$, $0 < y < L'$, $0 < z < L''$; $V(x, 0, z) = 0$, $0 < x < L$, $0 < z < L''$; $V(x, L', z) = 0$, $0 < x < L$, $0 < z < L''$; $V(x, y, 0) = f(x, y)$, $0 < x < L$, $0 < y < L'$; $V(x, y, L'') = 0$, $0 < x < L$, $0 < y < L'$

10. (a) Show that the homogeneous PDE

$$a(x, y)\frac{\partial^2 u}{\partial x^2} + b(x, y)\frac{\partial^2 u}{\partial x \partial y} + c(x, y)\frac{\partial^2 u}{\partial y^2} + d(x, y)\frac{\partial u}{\partial x} + e(x, y)\frac{\partial u}{\partial y} + f(x, y)u = 0$$

is separable if $a(x, y) = a(x)$, $b(x, y) = \text{constant}$, $c(x, y) = c(y)$, $d(x, y) = d(x)$, $e(x, y) = e(y)$, and $f(x, y) = f_1(x) + f_2(y)$.

(b) Are the conditions in (a) necessary for separability?



C H A P T E R

F I V E

Solution of Homogeneous Problems by Separation of Variables

5.1 Introduction

In Chapter 1 we developed boundary value and initial boundary value problems to describe physical phenomena such as heat conduction, vibrations, and electrostatic potentials. In Chapter 2 we introduced ordinary Fourier series, which we then used in Chapter 3, in conjunction with separation of variables, to solve very simple problems. These straightforward examples led to consideration of Sturm-Liouville systems in Chapter 4. We are now ready to apply these results in more complex homogeneous problems. In Chapter 6 we introduce finite Fourier transforms to solve nonhomogeneous problems.

A great variety of homogeneous problems could be considered—heat conduction, vibration, or potential; one-, two-, or three-dimensional; time dependent or steady-state. Because we cannot hope to consider all of these problems, we select a few straightforward examples to illustrate the technique; this puts us in a position to consider quite general PDEs, such as

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \tag{1}$$

where p, q, and s are constants. We pointed out in Section 4.3 that this PDE contains many of the PDEs in Chapter 1 [see equation (23) in Chapter 4]. It follows that initial boundary value problems associated with PDE (1) contain as special cases many of the (initial) boundary value problems of Chapter 1. In fact, when we solve (1) subject to Robin boundary conditions, we obtain general formulas that may be specialized to give solutions to many problems. We begin in Sections 5.2 and 5.3 with problems in two independent variables. In Section 5.4 we generalize to problems in higher dimensions.

5.2 Homogeneous Initial Boundary Value Problems in Two Variables

We begin this section by using separation of variables to solve two initial boundary value problems, one in heat conduction and the other in vibrations. What we learn from these examples will prepare us for separation of variables in more general problems. The heat conduction problem is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{2a}$$

$$U_x(0,t) = 0, \qquad t > 0, \tag{2b}$$

$$\kappa\frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = 0, \qquad t > 0, \tag{2c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{2d}$$

Physically described is a rod of uniform cross section and insulated sides that at time $t = 0$ has temperature $f(x)$ (Figure 5.1). For time $t > 0$, the end $x = 0$ is also insulated. and heat is exchanged at the other end with an environment at temperature 0°C. The problem is said to be homogeneous because PDE (2a) and boundary conditions (2b, c are homogeneous.

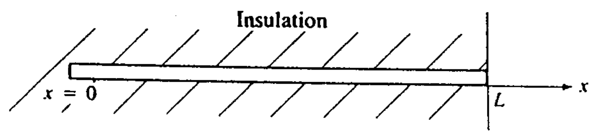


Figure 5.1

If we assume that a function $U(x,t)$, separated in the form $U(x,t) = X(x)T(t)$ satisfies PDE (2a), then

$$XT' = kX''T,$$

or

$$\frac{X''}{X} = \frac{T'}{kT} = \alpha = \text{constant}.$$

When this is combined with boundary conditions (2b, c), $X(x)$ must satisfy the system

$$X'' - \alpha X = 0, \qquad 0 < x < L, \tag{3}$$

$$X'(0) = 0, \tag{3}$$

$$\kappa X'(L) + \mu X(L) = 0 \tag{3}$$

and $T(t)$ must satisfy the ODE

$$T' - \alpha k T = 0, \qquad t > 0. \tag{4}$$

System (3) is a special case of proper Sturm-Liouville system (15) in Section 4.2. Since eigenvalues $(-\alpha)$ must be positive, we set $-\alpha = \lambda^2$, in which case line 4 in Table 4.1 defines eigenvalues as solutions of the equation

$$\tan \lambda L = \frac{\mu}{\kappa\lambda}$$

and orthonormal eigenfunctions as $X_n(x) = N^{-1} \cos \lambda_n x$, where the normalizing factor N^{-1} is given by

$$2N^2 = L + \frac{\mu/\kappa}{\lambda_n^2 + (\mu/\kappa)^2}.$$

For these eigenvalues, the general solution of (4) is $T(t) = ce^{-k\lambda_n^2 t}$, where c is an arbitrary constant. It follows that separated functions

$$ce^{-k\lambda_n^2 t} X_n(x)$$

for any constant c and any eigenvalue λ_n satisfy PDE (2a) and boundary conditions (2b, c). To satisfy initial condition (2d), we superpose separated functions (the PDE and boundary conditions being linear and homogeneous) and take

$$U(x, t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x), \tag{5}$$

where the c_n are constants. Condition (2d) now implies that

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \qquad 0 < x < L. \tag{6}$$

But this equation states that the c_n are the Fourier coefficients in the generalized Fourier series of $f(x)$ in terms of $X_n(x)$ and are defined according to equation (20b) in Section 4.2 by

$$c_n = \int_0^L f(x) X_n(x)\, dx = \frac{1}{N} \int_0^L f(x) \cos \lambda_n x\, dx. \tag{7a}$$

The final formal solution of problem (2) is therefore

$$U(x, t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} N^{-1} \cos \lambda_n x. \tag{7b}$$

where the c_n are defined in (7a). To see how the boundary conditions affect temperature in the rod, we consider a specific initial temperature distribution. Suppose, for example, that the rod is 1 m long and that $f(x) = 100(1 - x)$. Furthermore, suppose that the conductivity κ and diffusivity k of the material in the rod are 48 W/mK and 12×10^{-6} m^2/s and that the heat transfer coefficient at $x = L$ is $\mu = 96$ W/m^{2}K. With these physical attributes, eigenvalues are defined by

$$\tan \lambda = \frac{2}{\lambda}$$

and normalizing factors are

$$2N^2 = 1 + \frac{2}{\lambda_n^2 + 4}.$$

Coefficients c_n are given by (7a):

$$c_n = \frac{1}{N}\int_0^1 100(1 - x)\cos \lambda_n x \, dx = \frac{100}{N\lambda_n^2}(1 - \cos \lambda_n).$$

Thus,

$$U(x,t) = \sum_{n=1}^{\infty} \frac{100}{N\lambda_n^2}(1 - \cos \lambda_n)e^{-12\times 10^{-6}\lambda_n^2 t}N^{-1}\cos \lambda_n x$$

$$= \sum_{n=1}^{\infty} \frac{200(\lambda_n^2 + 4)(1 - \cos \lambda_n)}{\lambda_n^2(\lambda_n^2 + 6)} e^{-12\times 10^{-6}\lambda_n^2 t}\cos \lambda_n x.$$

When this series is approximated by its first four terms, sketches for various values of t are as shown in Figure 5.2. (The four smallest positive solutions of $\tan \lambda = 2/\lambda$ are 1.076874, 3.643597, 6.578334, and 9.629560.)

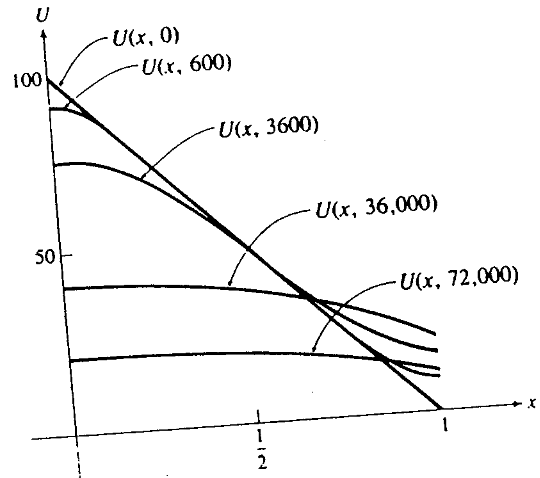


Figure 5.2

In Figure 5.3 we show temperature in the rod for the same times when boundary condition (2c) is replaced by $U(L,t) = 0$. What this means is that the heat transfer coefficient μ in (2c) has become very large and there is essentially no resistance to heat transfer across the boundary $x = L$. The solution in this case is

$$U(x,t) = \frac{800}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-12\times 10^{-6}(2n-1)^2\pi^2 t/4}\cos\frac{(2n-1)\pi x}{2}.$$

Ultimately, the solution approaches the situation in which temperature in the rod is identically zero, as it does in problem (2), but it does so more quickly.

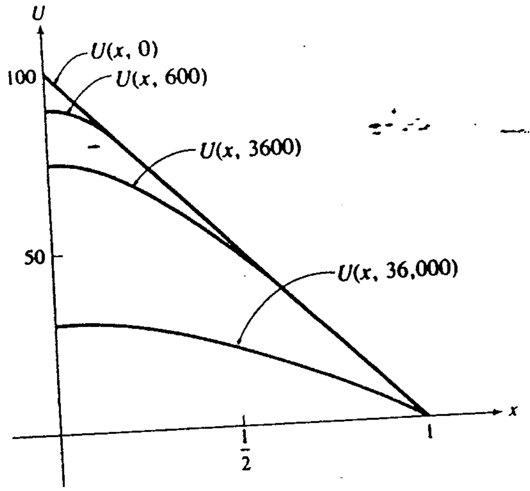


Figure 5.3

Our second illustrative example is concerned with displacements of the taut string in Figure 5.4. The end at $x = L$ is fixed on the x-axis, while the end at $x = 0$ is looped around a vertical support and can move thereon without friction. If the position of the string is initially parabolic, $x(L - x)$, and it is motionless, subsequent displacements are described by the homogeneous initial boundary value problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{8a}$$

$$y_x(0, t) = 0, \qquad t > 0, \tag{8b}$$

$$y(L, t) = 0, \qquad t > 0, \tag{8c}$$

$$y(x, 0) = x(L - x), \qquad 0 < x < L, \tag{8d}$$

$$y_t(x, 0) = 0, \qquad 0 < x < L. \tag{8e}$$

Figure 5.4

When a separated function $y(x, t) = X(x)T(t)$ is substituted into PDE (8a), boundary conditions (8b, c), and initial condition (8e), a Sturm-Liouville system in $X(x)$ results

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L, \tag{9a}$$

$$X'(0) = 0 = X(L), \tag{9b}$$

and an ODE in $T(t)$,

$$T'' + c^2\lambda^2 T = 0, \qquad t > 0, \tag{10a}$$

$$T'(0) = 0. \tag{10b}$$

According to line 6 in Table 4.1, eigenvalues of the Sturm-Liouville system are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$ $(n \geq 1)$, with orthonormal eigenfunctions $X_n(x) = \sqrt{2/L}\cos\lambda_n x$. For these eigenvalues, the solution of (10) is $T(t) = A\cos c\lambda_n t$, where A is an arbitrary constant. We have shown, therefore, that separated functions $A\cos c\lambda_n t\, X_n(x)$ for any constant A and any eigenvalues λ_n satisfy PDE (8a), boundary conditions (8b, c), and initial condition (8e). To satisfy initial condition (8d), we superpose separated functions and take

$$y(x,t) = \sum_{n=1}^{\infty} A_n \cos c\lambda_n t\, X_n(x), \tag{11}$$

where the A_n are constants. Condition (8d) now requires that

$$x(L-x) = \sum_{n=1}^{\infty} A_n X_n(x), \qquad 0 < x < L. \tag{12}$$

Consequently, the A_n are Fourier coefficients in the generalized Fourier series of $x(L-x)$; that is,

$$A_n = \int_0^L x(L-x)X_n(x)\,dx = \int_0^L x(L-x)\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{2L}\,dx$$

$$= \frac{16\sqrt{2}\,L^{5/2}(-1)^{n+1}}{(2n-1)^3\pi^3} - \frac{4\sqrt{2}\,L^{5/2}}{(2n-1)^2\pi^2}.$$

When these are substituted into (11), the formal solution is

$$y(x,t) = \frac{-8L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(2n-1)\pi + 4(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}. \tag{13}$$

Each term in (13) is called a *normal mode of vibration* of the string. The first term, let us denote it by

$$H_1(x,t) = \frac{-8L^2(\pi-4)}{\pi^3}\cos\frac{\pi ct}{2L}\cos\frac{\pi x}{2L} = 0.22L^2\cos\frac{\pi ct}{2L}\cos\frac{\pi x}{2L},$$

is called the *fundamental mode* or *first harmonic*. As a separated function, $H_1(x,t)$ satisfies (8a, b, c, e); at time $t = 0$, it reduces to $0.22L^2\cos[\pi x/(2L)]$. In other words, $H_1(x,t)$ describes displacements of a string identical to that in problem (8), except that the initial displacement is $0.22L^2\cos[\pi x/(2L)]$ instead of $x(L-x)$. Positions of this string for various values of t are shown in Figure 5.5. The string vibrates back and forth between the enveloping curves $\pm 0.22L^2\cos[\pi x/(2L)]$, always maintaining the shape of a cosine.

The second harmonic is the second term in (13),

$$H_2(x,t) = -0.13L^2\cos\frac{3\pi ct}{2L}\cos\frac{3\pi x}{2L};$$

it represents displacements of the same string were the initial displacement $-0.13L^2\cos[3\pi x/(2L)]$. Positions of this string for various values of t are shown in Figure 5.6. The point at $x = L/3$ in the string remains motionless; it is called a *node* of $H_2(x, t)$.

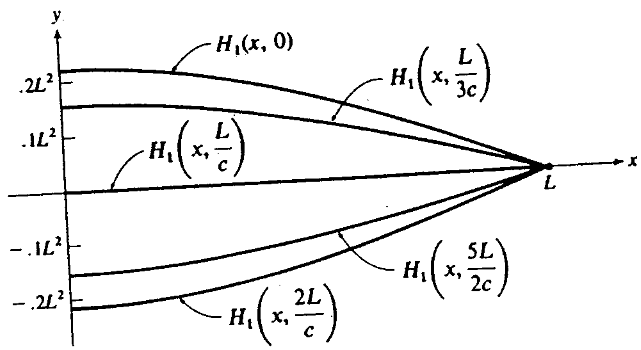


Figure 5.5

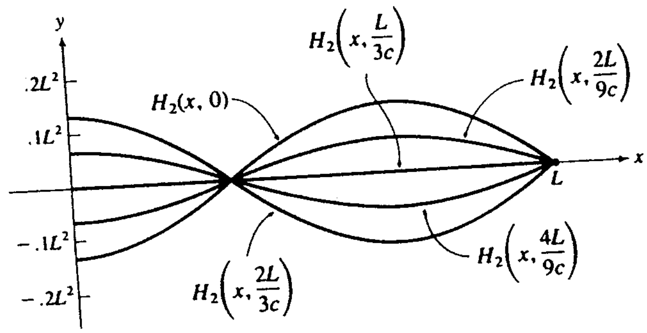


Figure 5.6

The third harmonic,

$$H_3(x, t) = -0.024L^2 \cos\frac{5\pi ct}{2L}\cos\frac{5\pi x}{2L},$$

is shown in Figure 5.7. It has two nodes, one at $x = L/5$ and the other at $x = 3L/5$.

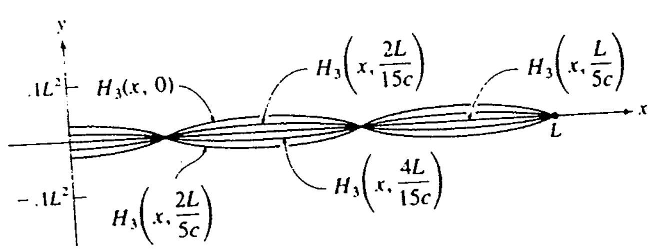


Figure 5.7

Solution (13) of problem (8) is the sum of all its harmonics. Because A_n decreases rapidly with increasing n, lower harmonics are more significant than higher ones.

We are now in a position to consider the general homogeneous initial boundary value problem

$$\frac{\partial^2 V}{\partial x^2} = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV, \qquad 0 < x < L, \qquad t > 0, \tag{14a}$$

$$-l_1\frac{\partial V}{\partial x} + h_1 V = 0, \qquad x = 0, \qquad t > 0, \tag{14b}$$

$$l_2\frac{\partial V}{\partial x} + h_2 V = 0, \qquad x = L, \qquad t > 0, \tag{14c}$$

$$V(x,0) = f(x), \qquad 0 < x < L, \tag{14d}$$

$$V_t(x,0) = g(x), \qquad 0 < x < L. \tag{14e}$$

It is said to be homogeneous because the PDE and boundary conditions are homogeneous. Problem (14) includes as special cases the following problems from Chapter 1:

(1) If $V(x,t) = U(x,t)$, $p = s = 0$, and $q = k^{-1}$, then (14) is the one-dimensional heat conduction problem with no internal heat generation but with heat transfer at ends $x = 0$ and $x = L$ into or from media at temperature zero. In this case, initial condition (14e) would be absent.

(2) If $V(x,t) = y(x,t)$, $p = \rho\tau^{-1}$ (or ρE^{-1}), $q = \beta\tau^{-1}$, and $s = k\tau^{-1}$, then (14) is the one-dimensional vibration problem with a damping force proportional to velocity and a restoring force proportional to displacement.

When a function separated in the form $V(x,t) = X(x)T(t)$ is substituted into (14a),

$$X''T = pXT'' + qXT' + sXT$$

or

$$\frac{X''}{X} = \frac{pT'' + qT' + sT}{T} = \alpha = \text{constant independent of } x \text{ and } t.$$

This separation, together with boundary conditions (14b, c), leads to a Sturm-Liouville system in $X(x)$,

$$X'' - \alpha X = 0, \qquad 0 < x < L, \tag{15a}$$

$$-l_1 X' + h_1 X = 0, \qquad x = 0, \tag{15b}$$

$$l_2 X' + h_2 X = 0, \qquad x = L, \tag{15c}$$

and an ODE in $T(t)$,

$$pT'' + qT' + (s - \alpha)T = 0, \qquad t > 0. \tag{16}$$

System (15) is precisely Sturm-Liouville system (15) in Chapter 4. When we set $\alpha = -\lambda$ ($\lambda \geq 0$) (since eigenvalues of a proper Sturm-Liouville system must be nonnegative eigenvalues λ_n and orthonormal eigenfunctions $X_n(x) = X(\lambda_n, x)$ are as listed in Table 4.1.

When $p = 0$, ODE (16) has general solution

$$T(t) = ce^{-(s+\lambda_n^2)t/q}, \tag{17}$$

where c is a constant. We have shown, therefore, that separated functions

$$V(x, t) = X(x)T(t) = ce^{-(s+\lambda_n^2)t/q}X_n(x),$$

for any constant c, and any eigenvalue λ_n are solutions of PDE (14a) and boundary conditions (14b, c). There is but one initial condition when $p = 0$, namely (14d), and to satisfy it, we superpose separated functions (the PDE and boundary conditions being linear and homogeneous) and take

$$V(x, t) = \sum_{n=1}^{\infty} c_n X_n(x) e^{-(s+\lambda_n^2)t/q}, \tag{18}$$

where the c_n are constants. Initial condition (14d) now implies that the c_n must satisfy

$$f(x) = \sum_{n=1}^{\infty} c_n X_n(x), \qquad 0 < x < L. \tag{19}$$

The constants c_n are therefore Fourier coefficients in the generalized Fourier series of $f(x)$,

$$c_n = \int_0^L f(x)X_n(x)\,dx. \tag{20}$$

The formal solution of (14) for $p = 0$ is therefore (18) with the c_n defined by (20).

When $p \neq 0$, ODE (16) has general solution

$$T(t) = c\phi_1(t) + d\phi_2(t), \tag{21}$$

where $\phi_1(t)$ and $\phi_2(t)$ are independent solutions of (16) and c and d are arbitrary constants. In this case, separated functions

$$V(x, t) = X(x)T(t) = X_n(x)\{c\phi_1(t) + d\phi_2(t)\},$$

for any constants c and d and any eigenvalue λ_n, are solutions of PDE (14a) and boundary conditions (14b, c). To satisfy the initial conditions, we superpose separated functions and take

$$V(x, t) = \sum_{n=1}^{\infty} X_n(x)\{c_n\phi_1(t) + d_n\phi_2(t)\}, \tag{22}$$

where c_n and d_n are constants. Initial conditions (14d, e) now imply that the c_n and d_n must satisfy

$$f(x) = \sum_{n=1}^{\infty} X_n(x)\{c_n\phi_1(0) + d_n\phi_2(0)\}, \qquad 0 < x < L \tag{23a}$$

and

$$g(x) = \sum_{n=1}^{\infty} X_n(x)\{c_n\phi_1'(0) + d_n\phi_2'(0)\}, \qquad 0 < x < L. \tag{23b}$$

If we multiply (23a) by $\phi_2'(0)$, multiply (23b) by $\phi_2(0)$, and subtract,

$$\phi_2'(0)f(x) - \phi_2(0)g(x) = \sum_{n=1}^{\infty} c_n\{\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)\}X_n(x). \tag{24}$$

This equation implies that $c_n\{\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)\}$ must be the Fourier coefficients in the generalized Fourier series of $\phi_2'(0)f(x) - \phi_2(0)g(x)$ in terms of $X_n(x)$ and are therefore defined by equation (20b) of Chapter 4:

$$c_n\{\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)\} = \int_0^L \{\phi_2'(0)f(x) - \phi_2(0)g(x)\}X_n(x)\,dx. \tag{25}$$

[Equation (25) can also be obtained by multiplying (24) by $X_m(x)$ and integrating with respect to x from $x = 0$ to $x = L$.] Thus,

$$c_n = \frac{1}{\phi_1(0)\phi_2'(0) - \phi_1'(0)\phi_2(0)}\int_0^L \{\phi_2'(0)f(x) - \phi_2(0)g(x)\}X_n(x)\,dx. \tag{26}$$

Similarly, it can be shown that

$$d_n = \frac{1}{\phi_1'(0)\phi_2(0) - \phi_1(0)\phi_2'(0)}\int_0^L \{\phi_1'(0)f(x) - \phi_1(0)g(x)\}X_n(x)\,dx. \tag{27}$$

The formal solution of (14) for $p \neq 0$ is therefore (22), where c_n and d_n are defined by (26) and (27).

We have demonstrated that separation of variables can be used to solve initial boundary value problems of form (14) and therefore, as special cases, problems (1) and (2) following (14). In fact, (18) and (22) represent formulas for solutions of many of these problems. For example, to solve heat conduction problem (19) in Section 3.2, we could set $p = s = h_1 = h_2 = 0$, $l_1 = l_2 = 1$, and $q = k^{-1}$ in (14), delete initial condition (14e), and set $f(x) = x$. According to (18), the solution is

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t}X_n(x),$$

where

$$c_n = \int_0^L f(x)X_n(x)\,dx.$$

Eigenpairs are found in line 5 of Table 4.1:

$$\lambda_0 = 0 \leftrightarrow X_0(x) = \frac{1}{\sqrt{L}}, \qquad \lambda_n = \frac{n\pi}{L} \leftrightarrow \sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}.$$

With these,

$$c_0 = \int_0^L x\frac{1}{\sqrt{L}}\,dx = \frac{L^{3/2}}{2}, \qquad c_n = \int_0^L x\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L}\,dx = \frac{\sqrt{2}\,L^{3/2}[(-1)^n - 1]}{n^2\pi^2},$$

and therefore

$$\begin{aligned} U(x,t) &= \frac{L^{3/2}}{2}\left(\frac{1}{\sqrt{L}}\right) + \sum_{n=1}^{\infty}\frac{\sqrt{2}\,L^{3/2}[(-1)^n - 1]}{n^2\pi^2}e^{-n^2\pi^2kt/L^2}\sqrt{\frac{2}{L}}\cos\frac{n\pi x}{L} \\ &= \frac{L}{2} - \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}e^{-(2n-1)^2\pi^2kt/L^2}\cos\frac{(2n-1)\pi x}{L}. \end{aligned}$$

This is solution (24) of problem (19) in Section 3.2.

Exercises for Section 5.2

We are not in the habit of recommending that you use results such as (18) and (22) as formulas. Formulas are fine for those who have mastered fundamentals and are now looking for shortcuts in solving large classes of problems. We prefer to regard our analysis of equation (14) as an illustration of the fact that any problem of this form can be solved by separation of variables. The procedure leading from problem (14) to either solution (18) or solution (22) should be used as a guideline for solving other problems—separate variables, obtain the appropriate Sturm-Liouville system, solve the system (perhaps by quoting Table 4.1), solve the ODE in $T(t)$, superpose separated functions, and apply the nonhomogeneous initial condition(s).

Exercises 5.2

Part A—Heat Conduction

1. (a) A homogeneous, isotropic rod with insulated sides has temperature $f(x) = L - x$, $0 \le x \le L$, at time $t = 0$. If, for time $t > 0$, the end $x = 0$ is insulated and the end $x = L$ is held at temperature 0°C, find the temperature in the rod.
   (b) Find an expression (in series form) for the amount of heat leaving the end $x = L$ of the rod as a function of time t.
   (c) Sketch a graph of the function in (b) if $\kappa = 48$ W/mK, $k = 12 \times 10^{-6}$ m²/s, and $L = 1$ m.
2. What is the solution to Exercise 1(a) for an arbitrary initial temperature $f(x)$?
3. Let $U(x, t)$ denote temperature in the thin-wire problem (see Exercise 31 in Section 1.2) of a thin wire of length L lying along the x-axis. When the temperature of the surrounding medium is zero and there is no heat generation, $U(x, t)$ must satisfy the PDE

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - hU, \qquad 0 < x < L, \qquad t > 0,$$

   where $h > 0$ is a constant.
   (a) If the ends of the wire are insulated and the initial temperature distribution is denoted by some function $f(x)$, find and solve the initial boundary value problem for $U(x, t)$.
   (b) Compare the solution in (a) with that obtained when the lateral sides are also insulated.
4. Exercise 3 suggests the following result. The general homogeneous thin-wire problem (see Exercise 31 in Section 1.2) is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - hU, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \qquad x = L, \qquad t > 0,$$

$$U(x, 0) = f(x), \qquad 0 < x < L.$$

   (Homogeneity requires an environmental temperature identically zero. Nonzero environmental temperatures and other nonhomogeneities are considered in the exercises in Section 6.2.) Show that the solution of this problem is always e^{-ht} times that of the corresponding problem when no heat transfer takes place over the surface of the wire.

Part B—Vibrations

5. (a) A taut string is given an initial displacement (at time $t = 0$) of $f(x)$, $0 \le x \le L$, and initial velocity $g(x)$, $0 \le x \le L$. If the ends $x = 0$ and $x = L$ of the string are fixed on the x-axis, find displacements of points in the string thereafter.
   (b) As functions of time, what are the amplitudes of the first, second, and third harmonics? Sketch graphs of these harmonics for various fixed values of t. Are frequencies of higher harmonics integer multiples of the frequency of the fundamental mode?
   (c) The nodes of a normal mode of vibration are those points that remain motionless for that mode. What are the nodes for the first three harmonics?
6. (a) A taut string is given an initial displacement (at time $t = 0$) of $f(x)$, $0 \le x \le L$, and initial velocity $g(x)$, $0 \le x \le L$. The end $x = 0$ is fixed on the x-axis, while the end $x = L$ is looped around a vertical support and can move thereon without friction. Find displacements in the string for $0 < x \le L$ and $t > 0$.
   (b) Specialize the result in (a) when

$$f(x) = \begin{cases} x/10 & 0 \le x \le L/2 \\ (L - x)/10 & L/2 \le x \le L \end{cases}, \qquad g(x) \equiv 0.$$

   (c) Repeat (b) for $f(x) \equiv 0$ and $g(x) = x(L - x)$.
7. (a) Repeat Exercise 5(a) if an external force (per unit x-length) $F = -ky$ ($k > 0$) acts at each point in the string.
   (b) Compare the normal modes of vibration with those in Exercise 5.
8. Repeat Exercise 5(a) if an external force (per unit x-length) $F = -\beta \partial y/\partial t$ $(0 < \beta < 2\pi\rho c/L)$ acts at every point on the string.
9. A taut string is given a displacement bx, b a constant, $0 \le x \le L$, and zero initial velocity. The end $x = 0$ is fixed on the x-axis, and the right end moves vertically but is restrained by a spring (constant k) that is unstretched on the x-axis (Figure 5.8).

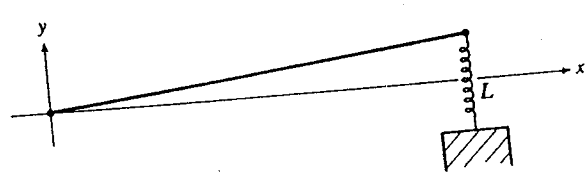


Figure 5.8

   (a) Show that subsequent displacements of points on the string can be expressed in the form

$$y(x, t) = \frac{2b(kL + \tau)}{\tau} \sum_{n=1}^{\infty} \frac{(\tau^2\lambda_n^2 + k^2)\sin \lambda_n L}{\lambda_n^2[L(\tau^2\lambda_n^2 + k^2) + k\tau]} \cos c\lambda_n t \sin \lambda_n x,$$

   where λ_n are the positive solutions of the equation

$$\cot \lambda L = \frac{-k}{\tau\lambda},$$

   τ is the constant tension in the string, and $c^2 = \tau/\rho$, where ρ is the constant density of the string.

(b) Reduce the expression in (a) to

$$y(x, t) = 2b(kL + \tau) \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sqrt{k^2 + \tau^2\lambda_n^2}}{\lambda_n[L(k^2 + \tau^2\lambda_n^2) + k\tau]} \cos c\lambda_n t \sin \lambda_n x.$$

10. A bar of uniform cross section and length L lies along the x-axis. Its left end is fixed at $x = 0$, and its right end is attached to a spring with constant k that is unstretched when the bar is unstrained (Figure 5.9). If at time $t = 0$ the bar is pulled to the right so that cross sections are displaced according to $f(x) = x/100$, then released from rest at this position, find subsequent displacements of cross sections.

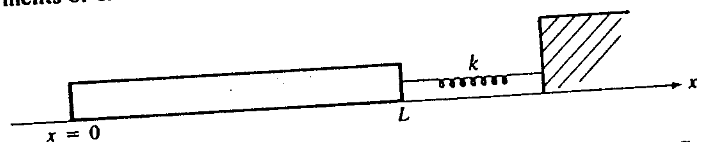


Figure 5.9

Equations (14a–e) describe displacements of a taut string when $p \neq 0$. Separation leads to a solution in form (22) with coefficients c_n and d_n given by (26) and (27). The normal modes of this solution are

$$H_n(x, t) = X_n(x)[c_n\phi_1(t) + d_n\phi_2(t)],$$

where $X_n(x)$ are the eigenfunctions in Table 4.1. Nodes of $H_n(x, t)$ are points that remain motionless for all t. They are the zeros of $X_n(x)$. In Exercises 11–16, we show that the number of nodes of the nth mode is exactly $n - 1$ (except when both ends of the string are looped around vertical supports and move freely without friction).

11. Show that when both ends are fixed on the x-axis, the distance between successive nodes is L/n, and hence there are $n - 1$ equally spaced nodes between $x = 0$ and $x = L$.

12. Show that when the end $x = 0$ is fixed on the x-axis and the end $x = L$ is looped around a vertical support and moves without friction thereon (a free end), there are $n - 1$ nodes between $x = 0$ and $x = L$. A similar result holds when the left end is free and the right end is fixed.

13. Verify that when both ends are free, the nth mode has n nodes.

14. **(a)** Verify that when the end $x = 0$ is fixed on the x-axis and the end $x = L$ satisfies a homogeneous Robin condition, nodes of the nth mode occur for $x_m = m\pi/\lambda_n$, $m > 0$ an integer.

(b) Use Figure 4.3 to establish that eigenvalues λ_n satisfy

$$\frac{(n-1)\pi}{L} < \lambda_n < \frac{n\pi}{L}.$$

Use this to verify the existence of exactly $n - 1$ nodes. A similar result holds when the right end is fixed and the left end satisfies a homogeneous Robin condition.

15. **(a)** Verify that when end $x = 0$ is free and end $x = L$ satisfies a homogeneous Robin condition, nodes of the nth mode occur for $x_m = (2m - 1)\pi/(2\lambda_n)$, $m > 0$ an integer.

(b) Establish the inequality

$$\frac{(n-1)\pi}{L} < \lambda_n < \frac{(2n-1)\pi}{2L}$$

for this case, and use it to verify that there are exactly $n - 1$ nodes. A similar result holds when the right end is free and the left end satisfies a homogeneous Robin condition.

16. The final case is when both ends of the string satisfy homogeneous Robin conditions, in which case $X_n(x)$ is given in line 1 of Table 4.1.
    (a) Show that zeros of $X_n(x)$ occur for
$$x_m = \frac{m\pi}{\lambda_n} - \phi_n, \qquad m \text{ an integer,}$$
    where $\phi_n = \lambda_n^{-1}\,\mathrm{Tan}^{-1}(\lambda_n l_1/h_1)$.
    (b) Establish the inequality in Exercise 15(b) and the fact that the difference between successive nodes is π/λ_n.
    (c) Use the results in (b) to verify that there are exactly $n - 1$ nodes between $x = 0$ and $x = L$.

5.3 Homogeneous Boundary Value Problems in Two Variables

The Helmholtz equation on a rectangle $0 \le x \le L, 0 \le y \le L'$ takes the form

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} - sV = 0, \qquad 0 < x < L, \qquad 0 < y < L' \tag{28a}$$

where s is some given constant. When $s = 0$, we obtain the extremely important special case of Laplace's equation. A boundary value problem accompanying either of these equations is said to be homogeneous if the boundary conditions on a pair of parallel sides are homogeneous. For example, the following conditions on $x = 0$ and $x = L$ yield a homogeneous problem:

$$V(0, y) = 0, \qquad 0 < y < L', \tag{28b}$$
$$\frac{\partial V(L, y)}{\partial x} = 0, \qquad 0 < y < L', \tag{28c}$$
$$V(x, 0) = f(x), \qquad 0 < x < L, \tag{28d}$$
$$V(x, L') = g(x), \qquad 0 < x < L. \tag{28e}$$

No real difficulty is encountered in the solution of problem (28) if (28b, c) are not homogeneous, if say,

$$V(0, y) = h(y), \qquad 0 < y < L', \tag{28f}$$
$$\frac{\partial V(L, y)}{\partial x} = k(y), \qquad 0 < y < L'. \tag{28g}$$

We simply use superposition to write $V(x, y) = V_1(x, y) + V_2(x, y)$, where V_1 and V_2 both satisfy PDE (28a) and the following boundary conditions:

$$\begin{aligned}
V_1(0, y) &= 0, & 0 < y < L', \qquad V_2(0, y) &= h(y), & 0 < y < L',\\
\frac{\partial V_1(L, y)}{\partial x} &= 0, & 0 < y < L', \qquad \frac{\partial V_2(L, y)}{\partial x} &= k(y), & 0 < y < L',\\
V_1(x, 0) &= f(x), & 0 < x < L, \qquad V_2(x, 0) &= 0, & 0 < x < L,\\
V_1(x, L') &= g(x), & 0 < x < L; \qquad V_2(x, L') &= 0, & 0 < x < L.
\end{aligned}$$

In other words, the nonhomogeneous boundary value problem (28a, d, e, f, g) can be divided into two homogeneous problems. It follows, then, that separation of variables as illustrated here in problem (28a–e) is typical for all boundary value problems on rectangles (provided the PDE is homogeneous).

Substitution of a separated function $V(x, y) = X(x)Y(y)$ into (28a, b, c) leads to a Sturm-Liouville system in $X(x)$,

$$X'' + \lambda^2 X = 0, \qquad 0 < x < L,$$
$$X(0) = 0 = X'(L),$$

and an ODE in $Y(y)$,

$$Y'' - (s + \lambda^2)Y = 0, \qquad 0 < y < L'.$$

Eigenpairs of the Sturm-Liouville system are $\lambda_n^2 = (2n-1)^2\pi^2/(4L^2)$ and $X_n(x) = \sqrt{2/L}\sin\lambda_n x$. Corresponding solutions for $Y(y)$ are $Y(y) = A\cosh\sqrt{s + \lambda_n^2}\,y + B\sinh\sqrt{s + \lambda_n^2}\,y$. We superpose separated functions and take

$$V(x, y) = \sum_{n=1}^{\infty} [A_n \cosh\sqrt{s + \lambda_n^2}\,y + B_n \sinh\sqrt{s + \lambda_n^2}\,y]X_n(x). \tag{29a}$$

Boundary conditions (28d, e) require that

$$f(x) = \sum_{n=1}^{\infty} A_n X_n(x), \qquad 0 < x < L$$

and $g(x) = \sum_{n=1}^{\infty} [A_n \cosh\sqrt{s + \lambda_n^2}\,L' + B_n \sinh\sqrt{s + \lambda_n^2}\,L']X_n(x), \qquad 0 < x < L.$

These imply that

$$A_n = \int_0^L f(x)X_n(x)\,dx \tag{29b}$$

and

$$A_n \cosh\sqrt{s + \lambda_n^2}\,L' + B_n \sinh\sqrt{s + \lambda_n^2}\,L' = \int_0^L g(x)X_n(x)\,dx \tag{29c}$$

or

$$B_n = \frac{1}{\sinh\sqrt{s + \lambda_n^2}\,L'}\left(\int_0^L g(x)X_n(x)\,dx - A_n \cosh\sqrt{s + \lambda_n^2}\,L'\right). \tag{29d}$$

The formal solution of (28a–e) is therefore

$$V(x, y) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} [A_n \cosh\sqrt{s + \lambda_n^2}\,y + B_n \sinh\sqrt{s + \lambda_n^2}\,y]\sin\lambda_n x, \tag{30}$$

where A_n and B_n are calculated in (29b, d).

As a specific example, suppose $s = 0$, so that (28a) becomes Laplace's equation, and suppose that $f(x) = 0$ and $g(x) = x$. One possible interpretation of problem (28) would be that for determining steady-state temperature in a rectangle in which sides $x = 0$ and $y = 0$ are held at temperature 0°C, side $x = L$ is insulated, and $y = L'$ has temperature x. The solution to this problem is

$$V(x, y) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} B_n \sinh\lambda_n y \sin\frac{(2n-1)\pi x}{2L},$$

where $B_n = \dfrac{1}{\sinh\lambda_n L'}\displaystyle\int_0^L x\sqrt{\frac{2}{L}}\sin\frac{(2n-1)\pi x}{2L}\,dx = \frac{4\sqrt{2}\,L^{3/2}(-1)^{n+1}}{(2n-1)^2\pi^2\sinh\lambda_n L'}.$

Thus,

$$V(x,y) = \sqrt{\frac{2}{L}} \sum_{n=1}^{\infty} \frac{4\sqrt{2}\, L^{3/2}(-1)^{n+1}}{(2n-1)^2\pi^2 \sinh \lambda_n L'} \sinh \lambda_n y \sin \frac{(2n-1)\pi x}{2L}$$

$$= \frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2 \sinh[(2n-1)\pi L'/(2L)]} \sinh \frac{(2n-1)\pi y}{2L} \sin \frac{(2n-1)\pi x}{2L}.$$

We now consider Laplace's equation in a circle of radius a with a Dirichlet boundary condition (Figure 5.10):

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0, \qquad 0 < r < a, \qquad -\pi < \theta \le \pi, \tag{31a}$$

$$V(a,\theta) = f(\theta), \qquad -\pi < \theta \le \pi. \tag{31b}$$

The solution of this problem describes a number of physical phenomena. It represents (axially symmetric) electrostatic potential in a source-free cylinder $r \le a$, with potential prescribed on the surface of the cylinder $r = a$ as $f(\theta)$. Also described is steady-state temperature in a thin circular plate, insulated at top and bottom, with circumferential temperature $f(\theta)$. Finally, $V(r,\theta)$ represents static deflections of a circular membrane subjected to no external forces but with edge deflections $f(\theta)$.

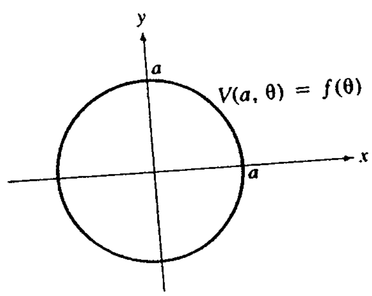


Figure 5.10

When we substitute a separated function $V(r,\theta) = R(r)\Theta(\theta)$ into (31a) and multiply by $r^2/V(r,\theta)$, separation results:

$$\frac{r^2 R''}{R} + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \text{constant}.$$

Thus, $R(r)$ and $\Theta(\theta)$ must satisfy the ODEs

$$r^2R'' + rR' - \lambda R = 0, \qquad \Theta'' + \lambda\Theta = 0.$$

Now, $V(r,\theta)$ must be 2π-periodic in θ, as must its first derivative with respect to θ; that is,

$$V(r, \theta + 2\pi) = V(r,\theta),$$

$$\frac{\partial V(r, \theta + 2\pi)}{\partial \theta} = \frac{\partial V(r,\theta)}{\partial \theta}.$$

These imply that $\Theta(\theta)$ and $\Theta'(\theta)$ must also be periodic. It follows that $\Theta(\theta)$ must satisfy the periodic Sturm-Liouville system

$$\Theta'' + \lambda\Theta = 0, \qquad -\pi < \theta < \pi,$$
$$\Theta(-\pi) = \Theta(\pi),$$
$$\Theta'(-\pi) = \Theta'(\pi).$$

According to Example 2 in Section 4.1, eigenvalues of this system are $\lambda_n = n^2$ $(n \geq 0)$, with a single eigenfunction, $1/\sqrt{2\pi}$, corresponding to $\lambda_0 = 0$ and a pair of eigenfunctions, $(1/\sqrt{\pi})\cos n\theta$ and $(1/\sqrt{\pi})\sin n\theta$ corresponding to $\lambda_n = n^2$ $(n > 0)$.

The differential equation in $R(r)$ is a Cauchy-Euler equation, which can be solved (in the case in which $n > 0$) by setting $R(r) = r^m$, m an unknown constant. This results in the general solution

$$R(r) = \begin{cases} A + B\ln r & n = 0 \\ Ar^n + Br^{-n} & n \geq 1 \end{cases}. \tag{32}$$

For these solutions to remain bounded near $r = 0$, we must set $B = 0$. Separated functions have now been determined to be $A/\sqrt{2\pi}$, corresponding to $\lambda_0 = 0$, and $(Ar^n/\sqrt{\pi})\cos n\theta$ and $(Ar^n/\sqrt{\pi})\sin n\theta$, corresponding to $\lambda_n = n^2$ $(n > 0)$. To satisfy boundary condition (31b), we superpose separated functions and take

$$V(r, \theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} r^n\left(A_n\frac{\cos n\theta}{\sqrt{\pi}} + B_n\frac{\sin n\theta}{\sqrt{\pi}}\right). \tag{33a}$$

The boundary condition requires that

$$f(\theta) = \frac{A_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} a^n\left(A_n\frac{\cos n\theta}{\sqrt{\pi}} + B_n\frac{\sin n\theta}{\sqrt{\pi}}\right), \qquad -\pi < \theta \leq \pi,$$

from which

$$A_0 = \int_{-\pi}^{\pi} f(\theta)\frac{1}{\sqrt{2\pi}}\,d\theta, \qquad A_n = \frac{1}{a^n}\int_{-\pi}^{\pi} f(\theta)\frac{\cos n\theta}{\sqrt{\pi}}\,d\theta,$$
$$B_n = \frac{1}{a^n}\int_{-\pi}^{\pi} f(\theta)\frac{\sin n\theta}{\sqrt{\pi}}\,d\theta \tag{33b}$$

[see equations (21) in Section 4.2, with $L = \pi$ and x replaced by θ]. The formal solution of problem (31) is now complete; it is (33a) with coefficients defined in (33b). An integral expression for the solution can be obtained by substituting coefficients A_n and B_n into (33a). In order to keep variable θ in (33a) distinct from the variable of integration in (33b), we replace θ by u in (33b):

$$\begin{aligned} V(r, \theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(u)\,du \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{\pi}\left(\frac{r}{a}\right)^n\left(\cos n\theta\int_{-\pi}^{\pi} f(u)\cos nu\,du + \sin n\theta\int_{-\pi}^{\pi} f(u)\sin nu\,du\right) \\ &= \frac{1}{\pi}\left(\frac{1}{2}\int_{-\pi}^{\pi} f(u)\,du + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\int_{-\pi}^{\pi} f(u)\cos n(\theta - u)\,du\right). \end{aligned} \tag{34}$$

If we interchange the order of integration and summation,

$$V(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - u)\right] f(u)\,du.$$

The series can be summed in closed form by noting that $\cos n(\theta - u)$ is the real part of a complex exponential, $\cos n(\theta - u) = \mathrm{Re}(e^{in(\theta - u)}]$,

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - u) = \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \mathrm{Re}[e^{in(\theta - u)}] = \mathrm{Re}\left[\sum_{n=1}^{\infty}\left(\frac{r}{a}e^{i(\theta - u)}\right)^n\right].$$

Since the right side is a geometric series with common ratio $(r/a)e^{i(\theta - u)}$, converging therefore when $r < a$, we may write

$$\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \cos n(\theta - u) = \mathrm{Re}\left(\frac{(r/a)e^{i(\theta - u)}}{1 - (r/a)e^{i(\theta - u)}}\right) = \mathrm{Re}\left(\frac{r[\cos(\theta - u) + i\sin(\theta - u)]}{a - r[\cos(\theta - u) + i\sin(\theta - u)]}\right)$$
$$= \frac{ar\cos(\theta - u) - r^2}{a^2 + r^2 - 2ar\cos(\theta - u)}. \tag{35}$$

Consequently,

$$V(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2} + \frac{ar\cos(\theta - u) - r^2}{a^2 + r^2 - 2ar\cos(\theta - u)}\right) f(u)\,du$$
$$= \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{a^2 + r^2 - 2ar\cos(\theta - u)}\,du. \tag{36}$$

This result is called *Poisson's integral formula*. It expresses the solution to Laplace's equation inside the circle $r \le a$ in terms of its values on the circle. Immediate consequences of the Poisson integral formula are the following two results.

Theorem 1

When $V(r,\theta)$ is the solution to Dirichlet's problem for Laplace's equation in a circle $r \le a$, the value $V(0,\theta)$ at the center of the circle is the average of its values on $r = a$.

Proof: According to (36), the value of $V(r,\theta)$ at $r = 0$ is

$$V(0,\theta) = \frac{a^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(u)}{a^2}\,du = \frac{1}{2\pi a}\int_{-\pi}^{\pi} f(\theta)\,a\,d\theta,$$

the average value of $f(\theta)$ on $r = a$. ■

Corollary

When $V(r,\theta)$ is the solution to Dirichlet's problem for Laplace's equation in a circle $r \le a$, the average value of $V(r,\theta)$ around every circle centered at $r = 0$ is $V(0,\theta)$.

Exercises 5.3

1. (a) Solve Exercise 17 from Section 3.2.
   (b) Find an approximate value for the potential at the center of the plate if the plate is square.

Exercises for Section 5.3

2. (a) Find the steady-state temperature $U(x, y)$ inside a plate $0 \le x, y \le L$ if the sides $x = 0, y = 0$, and $x = L$ are all insulated and the boundary condition on $y = L$ is $\partial U(x, L)/\partial y = f(x)$. Can $f(x)$ be specified arbitrarily?
   (b) What is the solution when $f(x) = (L - 2x)/2$ and the temperature at the center of the plate is 50°C?
   (c) What is the solution when $f(x) = x(L - x)$?
3. (a) Find the steady-state temperature $U(x, y)$ inside a rectangular plate $0 \le x \le L, 0 \le y \le L'$ if the sides $y = 0$ and $y = L'$ are insulated; the temperature along $x = L$ is prescribed by the function $f_2(y)$, $0 < y < L'$; and the boundary condition along $x = 0$ is $\partial U(0, y)/\partial x = f_1(y)$, $0 < y < L'$.
   (b) Simplify the solution in (a) when $f_1(y)$ and $f_2(y)$ are constants.
4. A membrane is stretched tightly over the rectangle $0 \le x \le L, 0 \le y \le L'$. Its edges are given deflections that are described by the following boundary conditions:

$$\begin{aligned} z(0, y) &= f_1(y), & 0 < y < L', \\ z(L, y) &= f_2(y), & 0 < y < L', \\ z(x, 0) &= g_1(x), & 0 < x < L, \\ z(x, L') &= g_2(x), & 0 < x < L. \end{aligned}$$

   Find static deflections of the membrane when all external forces are negligible compared with tensions in the membrane.
5. Solve Laplace's equation in the rectangle $0 \le x \le L$, $0 \le y \le L'$ subject to the following boundary conditions:

$$\begin{aligned} V(0, y) &= f_1(y), & 0 < y < L', \\ V_x(L, y) &= f_2(y), & 0 < y < L', \\ V_y(x, 0) &= 0, & 0 < x < L, \\ V(x, L') &= g(x), & 0 < x < L. \end{aligned}$$

6. (a) Solve Laplace's equation in a semicircle $r \le a, 0 \le \theta \le \pi$ when the unknown function is zero on the diameter and $f(\theta)$ on the semicircle.
   (b) Simplify the solution when $f(\theta) \equiv 1$. Evaluate this solution along the y-axis.
7. (a) Along the circle $r = a$, a solution $V(r, \theta)$ of Laplace's equation must take on the value 1 for $0 < \theta < \pi$ and 0 for $-\pi < \theta < 0$. Show that the series solution for $V(r, \theta)$ is

$$V(r, \theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(r/a)^{2n-1}}{2n - 1} \sin(2n - 1)\theta.$$

   A closed-form solution of this problem is found in Exercise 19.
   (b) What is the value of $V(r, \theta)$ along the x-axis?
8. Find the steady-state temperature inside the quarter-circle $r \le a, 0 \le \theta \le \pi/2$ if its straight edges are insulated and the temperature along the curved edge is $\sin \theta$.
9. (a) Solve boundary value problem (31) when boundary condition (31b) is of Neumann type:

$$\frac{\partial V(a, \theta)}{\partial r} = f(\theta), \qquad -\pi < \theta \le \pi.$$

(b) Show that the solution can be expressed in the form

$$V(r,\theta) = C - \frac{a}{2\pi}\int_{-\pi}^{\pi} f(u)\ln[a^2 + r^2 - 2ar\cos(\theta - u)]\,du,$$

where C is an arbitrary constant. This result is called Dini's integral.

10. (a) Solve boundary value problem (31) when boundary condition (31b) is of Robin type:

$$l\frac{\partial V}{\partial r} + hV = f(\theta), \qquad r = a, \qquad -\pi < \theta \le \pi.$$

11. What is the solution to boundary value problem (31) in the exterior region $r > a$ if
   (a) $V(r,\theta)$ is required to be bounded at infinity [i.e., $V(r,\theta)$ must be bounded for large r]?
   (b) $V(r,\theta)$ must vanish at infinity?

12. Solve the boundary value problem of Exercise 11 when the boundary condition is of Neumann type:

$$-\frac{\partial V(a,\theta)}{\partial r} = f(\theta), \qquad -\pi < \theta \le \pi.$$

13. Solve the boundary value problem of Exercise 11 when the boundary condition is of Robin type:

$$-l\frac{\partial V}{\partial r} + hV = f(\theta), \qquad r = a, \qquad -\pi < \theta \le \pi.$$

14. Solve Laplace's equation inside a circular annulus $a < r < R$ with Dirichlet boundary conditions

$$V(a,\theta) = f_1(\theta), \qquad V(R,\theta) = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

15. Solve Exercise 14 when the boundary conditions are Neumann:

$$-\frac{\partial V(a,\theta)}{\partial r} = f_1(\theta), \qquad \frac{\partial V(R,\theta)}{\partial r} = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

16. Solve Exercise 14 when the boundary conditions are Robin:

$$-l_1\frac{\partial V(a,\theta)}{\partial r} + h_1 V(a,\theta) = f_1(\theta), \qquad -\pi < \theta \le \pi,$$

$$l_2\frac{\partial V(R,\theta)}{\partial r} + h_2 V(R,\theta) = f_2(\theta), \qquad -\pi < \theta \le \pi.$$

17. (a) A circular membrane of radius R is in a steady-state position with radial lines $\theta = 0$ and $\theta = \alpha$ clamped on the xy-plane. If the displacement of the edge $r = R$ is $f(\theta)$ for $0 < \theta < \alpha$, find the displacement in the sector $0 < \theta < \alpha$.
   (b) Take the limit of your answer in (a) as $\alpha \to 2\pi$. What does this function represent physically?
   (c) What is the answer in (b) if $f(\theta) = \sin(\theta/2)$?

When $f(\theta)$ in the boundary condition for Dirichlet problem (31) is piecewise constant, Poisson's integral can be evaluated analytically. We illustrate this in Exercises 18–20.

18. Show that

$$\int \frac{1}{a^2 + r^2 - 2ar\cos(\theta - u)}\,du = \frac{-2}{a^2 - r^2}\,\mathrm{Tan}^{-1}\left[\frac{a+r}{a-r}\tan\left(\frac{\theta - u}{2}\right)\right] + C,$$

provided $u \ne \theta \pm \pi$.

19. (a) When

$$f(\theta) = \begin{cases} 0 & -\pi < \theta < 0 \\ 1 & 0 < \theta < \pi \end{cases},$$

use the result of Exercise 18 to obtain the following solution for problem (31):

$$V(r,\theta) = \begin{cases} \frac{1}{\pi}\operatorname{Tan}^{-1}\left[\frac{a+r}{a-r}\tan\left(\frac{\theta}{2}\right)\right] + \frac{1}{\pi}\operatorname{Tan}^{-1}\left[\frac{a+r}{a-r}\cot\left(\frac{\theta}{2}\right)\right] & 0 < \theta < \pi \\ 1 + \frac{1}{\pi}\operatorname{Tan}^{-1}\left[\frac{a+r}{a-r}\tan\left(\frac{\theta}{2}\right)\right] + \frac{1}{\pi}\operatorname{Tan}^{-1}\left[\frac{a+r}{a-r}\cot\left(\frac{\theta}{2}\right)\right] & -\pi < \theta \le 0. \end{cases}$$

(b) For $\theta = 0$ and $\theta = \pi$, the solution in (a) must be regarded in the sense of limits as $\theta \to 0^+$ and $\theta \to \pi^-$. What are $V(r,0)$ and $V(r,\pi)$?

(c) Use trigonometry to combine the description for $V(r,\theta)$ in (a) into the single expression

$$V(r,\theta) = \frac{1}{2} + \frac{1}{\pi}\operatorname{Tan}^{-1}\left(\frac{2ar\sin\theta}{a^2 - r^2}\right).$$

(d) Solve the expression in (c) for r in terms of V and θ, and use the result to plot equipotential curves for $V = 1/8, 1/4, 3/8, 5/8, 3/4$, and $7/8$.

20. Use the result of Exercise 19 to solve problem (31) when

$$f(\theta) = \begin{cases} V_2 & -\pi < \theta < 0 \\ V_1 & 0 < \theta < \pi \end{cases}.$$

21. Find expressions similar to those in Exercise 19(a) when the boundary condition is

$$f(\theta) = \begin{cases} 0, & -\pi < \theta < 0 \\ 1, & 0 < \theta < \pi/2. \\ 0, & \pi/2 < \theta \le \pi \end{cases}$$

5.4 Homogeneous Problems in Three and Four Variables (Cartesian Coordinates Only)

In this section we extend the technique of separation of variables to homogeneous problems in two and three space variables, but confine our discussions to rectangles $0 \le x \le L, 0 \le y \le L'$ in the xy-plane and boxes $0 \le x \le L, 0 \le y \le L', 0 \le z \le L''$ in space. In other words, boundaries of the region under consideration must be coordinate curves $x =$ constant and $y =$ constant in the xy-plane and coordinate surfaces $x =$ constant, $y =$ constant, and $z =$ constant in space. This is an inherent restriction on the method of separation of variables for any problem whatsoever, be it initial boundary value or boundary value; be it two- or three-dimensional; be it in Cartesian, polar, cylindrical, or spherical coordinates. Separation of variables requires a region bounded by coordinate curves or surfaces; then and only then will separation of variables lead to Sturm-Liouville systems in space variables.

First consider the homogeneous initial boundary value problem

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0, \tag{37a}$$

$$z(0, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{37b}$$

$$z(L, y, t) = 0, \quad 0 < y < L', \quad t > 0, \tag{37c}$$

$$z(x, 0, t) = 0, \quad 0 < x < L, \quad t > 0, \tag{37d}$$

$$z(x, L', t) = 0, \quad 0 < x < L, \quad t > 0, \tag{37e}$$

$$z(x, y, 0) = f(x, y), \quad 0 < x < L, \quad 0 < y < L', \tag{37f}$$

$$z_t(x, y, 0) = 0, \quad 0 < x < L, \quad 0 < y < L'. \tag{37g}$$

Physically described are the vertical oscillations of a rectangular membrane that is released from rest at time $t = 0$ with displacement described by $f(x, y)$. Its edges are fixed on the xy-plane for all time, and no external forces act on the membrane.

If a function separated in the form $z(x, y, t) = X(x)Y(y)T(t)$ is substituted into (37a), the x-dependence can be separated from the y- and t-dependence:

$$\frac{X''}{X} = -\frac{Y''}{Y} + \frac{T''}{c^2 T} = \alpha = \text{constant independent of } x, y, \text{ and } t.$$

When this is combined with boundary conditions (37b, c), the Sturm-Liouville system

$$X'' - \alpha X = 0, \quad 0 < x < L, \tag{38a}$$

$$X(0) = X(L) = 0 \tag{38b}$$

is obtained. Since eigenvalues of a proper Sturm-Liouville system must be nonnegative, we set $\alpha = -\lambda^2$ $(\lambda \geq 0)$, in which case eigenvalues are $\lambda_n^2 = n^2\pi^2/L^2$, with normalized eigenfunctions $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$.

We continue to separate the equation in $Y(y)$ and $T(t)$:

$$\frac{Y''}{Y} = \frac{T''}{c^2 T} + \lambda_n^2 = \beta = \text{constant independent of } y \text{ and } t.$$

Combine this with boundary conditions (38d, e), and the system

$$Y'' - \beta Y = 0, \quad 0 < y < L', \tag{39a}$$

$$Y(0) = Y(L') = 0 \tag{39b}$$

results. With $\beta = -\mu^2$, the eigenvalues of this proper Sturm-Liouville system are $\mu_m^2 = m^2\pi^2/L'^2$, with orthonormal eigenfunctions $Y_m(y) = \sqrt{2/L'}\sin(m\pi y/L')$.

The ordinary differential equation

$$T'' + c^2(\lambda_n^2 + \mu_m^2)T = 0$$

has general solution $A\cos c\sqrt{\lambda_n^2 + \mu_m^2}\,t + B\sin c\sqrt{\lambda_n^2 + \mu_m^2}\,t$. But initial condition (37g) requires that $B = 0$, and therefore

$$T(t) = A\cos c\sqrt{\lambda_n^2 + \mu_m^2}\,t.$$

We have determined that separated functions

$$X_n(x)Y_m(y)T(t) = A\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\cos c\sqrt{\lambda_n^2 + \mu_m^2}\,t,$$

for any positive integers m and n and any constant A, satisfy PDE (37a), boundary conditions (37b–e), and initial condition (37g). Since these conditions are all linear and homogeneous, we superpose separated functions in an attempt to satisfy the initial displacement condition,

$$z(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\cos c\sqrt{\lambda_n^2 + \mu_m^2}\,t, \tag{40a}$$

where A_{mn} are constants. Condition (37f) requires that

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}, \qquad 0 < x < L, \qquad 0 < y < L'.$$

If we multiply this equation by $\sqrt{2/L}\sin(k\pi x/L)$, integrate with respect to x from $x = 0$ to $x = L$, and interchange orders of summation and integration on the right, orthogonality of the eigenfunctions $\sqrt{2/L}\sin(n\pi x/L)$ leads to

$$\int_0^L f(x, y)\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\,dx = \sum_{m=1}^{\infty} A_{mk}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}, \qquad 0 < y < L'.$$

Multiplication by $\sqrt{2/L'}\sin(j\pi y/L')$ and integration with respect to y from $y = 0$ to $y = L'$ gives, similarly,

$$\int_0^{L'}\left(\int_0^L f(x, y)\sqrt{\frac{2}{L}}\sin\frac{k\pi x}{L}\,dx\right)\sqrt{\frac{2}{L'}}\sin\frac{j\pi y}{L'}\,dy = A_{jk}.$$

Thus, coefficients A_{mn} in (40a) are given by

$$A_{mn} = \int_0^{L'}\int_0^L f(x, y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\,dx\,dy, \tag{40b}$$

and the formal solution is complete.

As a special case, suppose $f(x, y) = xy(L - x)(L' - y)$ so that cross sections of the initial displacement parallel to the xz- and yz-planes are parabolic. Integration by parts in (40b) yields

$$A_{mn} = \frac{8(LL')^{5/2}[1 + (-1)^{n+1}][1 + (-1)^{m+1}]}{n^3m^3\pi^6},$$

and hence

$$z(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{8(LL')^{5/2}[1 + (-1)^{n+1}][1 + (-1)^{m+1}]}{n^3m^3\pi^6}\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$
$$\times\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\cos c\sqrt{\frac{n^2\pi^2}{L^2} + \frac{m^2\pi^2}{L'^2}}\,t.$$

Since terms are nonzero only when both m and n are odd integers, we may write

$$z(x,y,t) = \frac{64(LL')^2}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3(2m-1)^3} \sin\frac{(2n-1)\pi x}{L} \times \sin\frac{(2m-1)\pi y}{L'} \cos \pi c \sqrt{\frac{(2n-1)^2}{L^2} + \frac{(2m-1)^2}{L'^2}}\, t. \tag{41}$$

The terms in this series are called the *normal modes of vibration* for the membrane (similar to the normal modes of a vibrating string in Section 5.2). The first term corresponding to $n = 1$ and $m = 1$,

$$H_{1,1}(x,y,t) = \frac{64(LL')^2}{\pi^6} \sin\frac{\pi x}{L} \sin\frac{\pi y}{L'} \cos \pi c \sqrt{\frac{1}{L^2} + \frac{1}{L'^2}}\, t,$$

is called the *fundamental mode of vibration*. It represents displacements of a membrane identical to that in problem (37), except that the initial displacement is $[64(LL')^2/\pi^6]\sin(\pi x/L)\sin(\pi y/L')$. For such an initial displacement, the membrane oscillates back and forth between the enveloping surfaces $\pm[64(LL')^2/\pi^6]\sin(\pi x/L)$ $\sin(\pi y/L')$; the shape of the membrane is always the same, the cosine factor describing the time dependence of the oscillations.

The $n = 1$ and $m = 2$ term in (41) is

$$H_{2,1}(x,y,t) = \frac{64(LL')^2}{27\pi^6} \sin\frac{\pi x}{L} \sin\frac{3\pi y}{L'} \cos \pi c \sqrt{\frac{1}{L^2} + \frac{9}{L'^2}}\, t.$$

It represents vibrations of the same membrane with an initial displacement $[64(LL')^2/(27\pi^6)]\sin(\pi x/L)\sin(3\pi y/L')$. The membrane oscillates back and forth between this surface and its negative. The lines $y = L'/3$ and $y = 2L'/3$, which always remain motionless, are called *nodal curves* for this mode of vibration.

The mode

$$H_{1,2}(x,y,t) = \frac{64(LL')^2}{27\pi^6} \sin\frac{3\pi x}{L} \sin\frac{\pi y}{L'} \cos \pi c \sqrt{\frac{9}{L^2} + \frac{1}{L'^2}}\, t$$

is similar with nodal curves $x = L/3$ and $x = 2L/3$.

Solution (41) is the sum of an infinity of modes of vibration, the modes of lower orders contributing more significantly than higher-order ones.

We now consider a three-dimensional boundary value problem,

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \quad 0 < z < L'', \tag{42a}$$

$$\frac{\partial U(0,y,z)}{\partial x} = 0, \quad 0 < y < L', \quad 0 < z < L'', \tag{42b}$$

$$U(L,y,z) = 0, \quad 0 < y < L', \quad 0 < z < L'', \tag{42c}$$

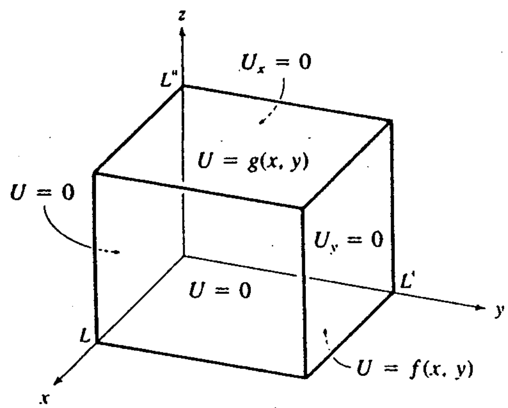
$$U(x,0,z) = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{42d}$$

$$\frac{\partial U(x,L',z)}{\partial y} = 0, \quad 0 < x < L, \quad 0 < z < L'', \tag{42e}$$

$$U(x, y, 0) = f(x, y), \quad 0 < x < L, \quad 0 < y < L', \tag{42f}$$

$$U(x, y, L'') = g(x, y), \quad 0 < x < L, \quad 0 < y < L'. \tag{42g}$$

The problem describes steady-state temperature $U(x, y, z)$ in the box of Figure 5.11, where two faces ($x = 0$ and $y = L'$) are insulated, two faces ($x = L$ and $y = 0$) are held at temperature zero, and the remaining faces have prescribed nonzero temperatures $f(x, y)$ and $g(x, y)$. The problem is said to be homogeneous because the PDE is homogeneous, and all boundary conditions are homogeneous except those on a single pair of opposite faces.



gure 5.11

When a function with variables separated, $U(x, y, z) = X(x)Y(y)Z(z)$, is substituted into (42a), separation gives

$$-\frac{X''}{X} = \frac{Y''}{Y} + \frac{Z''}{Z} = \lambda^2 = \text{constant independent of } x, y, \text{ and } z.$$

Combined with boundary conditions (42b, c), this yields

$$X'' + \lambda^2 X = 0, \quad 0 < x < L,$$
$$X'(0) = X(L) = 0.$$

Eigenvalues of this Sturm-Liouville system are $\lambda_n^2 = (2n - 1)^2\pi^2/(4L^2)$, with eigenfunctions $X_n(x) = \sqrt{2/L}\cos \lambda_n x$ (see Table 4.1).

Further separation of the equation in Y and Z leads to

$$-\frac{Y''}{Y} = \frac{Z''}{Z} - \lambda_n^2 = \mu^2 = \text{constant independent of } y \text{ and } z.$$

This equation, along with boundary conditions (42d, e), gives

$$Y'' + \mu^2 Y = 0, \quad 0 < y < L',$$
$$Y(0) = Y'(L') = 0.$$

Eigenpairs of this Sturm-Liouville system are $\mu_m^2 = (2m-1)^2\pi^2/(4L'^2)$ and $Y_m(y) = \sqrt{2/L'} \sin \mu_m y$.

Finally, the ODE

$$Z'' - (\lambda_n^2 + \mu_m^2)Z = 0$$

has general solution

$$Z(z) = A\cosh\sqrt{\lambda_n^2 + \mu_m^2}\, z + B\sinh\sqrt{\lambda_n^2 + \mu_m^2}\, z.$$

We have now determined that separated functions

$$X_n(x)Y_m(y)Z(z) = X_n(x)Y_m(y)[A\cosh\sqrt{\lambda_n^2 + \mu_m^2}\, z + B\sinh\sqrt{\lambda_n^2 + \mu_m^2}\, z]$$

for positive integers n and m and arbitrary constants A and B satisfy PDE (42a) and boundary conditions (42b–e). To accommodate boundary conditions (42f, g), we superpose separated functions and take

$$U(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_n(x)Y_m(y)[A_{mn}\cosh\sqrt{\lambda_n^2 + \mu_m^2}\, z + B_{mn}\sinh\sqrt{\lambda_n^2 + \mu_m^2}\, z], \quad \textbf{(43a)}$$

in which case (42f, g) require that

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}X_n(x)Y_m(y), \qquad 0 < x < L, \qquad 0 < y < L'$$

and

$$g(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_n(x)Y_m(y)[A_{mn}\cosh\sqrt{\lambda_n^2 + \mu_m^2}\, L'' + B_{mn}\sinh\sqrt{\lambda_n^2 + \mu_m^2}\, L''],$$
$$0 < x < L, \qquad 0 < y < L'.$$

Successive multiplications of these equations by eigenfunctions in x and y and integrations with respect to x and y lead to the following expressions for A_{mn} and B_{mn}:

$$A_{mn} = \int_0^{L'}\int_0^{L} f(x,y)X_n(x)Y_m(y)\,dx\,dy \quad \textbf{(43b)}$$

and

$$B_{mn} = \frac{1}{\sinh\sqrt{\lambda_n^2 + \mu_m^2}\, L''} \times \left[\int_0^{L'}\int_0^{L} g(x,y)X_n(x)Y_m(y)\,dx\,dy - A_{mn}\cosh\sqrt{\lambda_n^2 + \mu_m^2}\, L''\right]. \quad \textbf{(43c)}$$

In Section 5.5, solutions like (40) and (43) are approached from a different point of view.

Exercises 5.4

Part A—Heat Condition

1. A thin rectangle occupying the region $0 \le x \le L$, $0 \le y \le L'$ has its top and bottom faces insulated. At time $t = 0$, its temperature is described by the function $f(x, y)$. Find its temperature for $t > 0$ if all four edges, $x = 0$, $y = 0$, $x = L$, and $y = L'$, are maintained at 0 C.

2. Repeat Exercise 1 if edges $x = 0$ and $y = L'$ are insulated.
3. (a) Repeat Exercise 1 if edges $y = 0$ and $y = L'$ are insulated.
   (b) Simplify the solution if the initial temperature is a function only of x.
4. Repeat Exercise 1 if heat is transferred to an environment at temperature 0°C along the edge $x = L$ (according to Newton's law of cooling).
5. A block of metal occupies the region $0 \le x \le L$, $0 \le y \le L'$, $0 \le z \le L''$. The surfaces $y = 0$, $y = L'$, and $z = L''$ are insulated, and faces $x = 0$, $x = L$, and $z = 0$ are held at temperature 0°C. If the temperature of the block is initially a constant U_0 throughout, find the temperature in the block thereafter.
6. Repeat Exercise 5 if heat is transferred to the surrounding medium, at temperature zero, according to Newton's law of cooling on the face $z = L''$.
7. Repeat Exercise 5 if face $z = 0$ is insulated.
8. Repeat Exercise 5 if face $y = L'$ is held at temperature 0°C.
9. In this exercise we prove a result for homogeneous heat conduction problems in two or three space variables that uses solutions of one-dimensional problems provided the initial temperature distribution is the product of one-dimensional functions. In particular, show that the solution of the two-dimensional problem

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \quad x = 0, \quad 0 < y < L', \quad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \quad x = L, \quad 0 < y < L', \quad t > 0,$$

$$-l_3\frac{\partial U}{\partial y} + h_3 U = 0, \quad y = 0, \quad 0 < x < L, \quad t > 0,$$

$$l_4\frac{\partial U}{\partial y} + h_4 U = 0, \quad y = L', \quad 0 < x < L, \quad t > 0,$$

$$U(x, y, 0) = f(x)g(y), \quad 0 < x < L, \quad 0 < y < L'$$

is the product of the solutions of the one-dimensional problems

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \quad 0 < x < L, \quad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \quad x = 0, \quad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \quad x = L, \quad t > 0,$$

$$U(x, 0) = f(x), \quad 0 < x < L;$$

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial y^2}, \quad 0 < y < L', \quad t > 0,$$

$$-l_3\frac{\partial U}{\partial y} + h_3 U = 0, \quad y = 0, \quad t > 0,$$

$$l_4\frac{\partial U}{\partial y} + h_4 U = 0, \quad y = L', \quad t > 0,$$

$$U(y, 0) = g(y), \quad 0 < y < L'.$$

This result is easily extended to heat conduction problems in x, y, z, and t. In addition, it can sometimes be generalized to other coordinate systems (see Exercise 15 in Section 9.1).

10. (a) Use the result of Exercise 9 in this section, together with those of Exercise 1 in Section 5.2 and

Example 1 in Section 3.2, to solve the following heat conduction problem:

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right), \quad 0 < x < L, \quad 0 < y < L', \quad t > 0,$$

$$U_x(0, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$

$$U_x(L, y, t) = 0, \quad 0 < y < L', \quad t > 0,$$

$$U_y(x, 0, t) = 0, \quad 0 < x < L, \quad t > 0,$$

$$U(x, L', t) = 0, \quad 0 < x < L, \quad t > 0,$$

$$U(x, y, 0) = x(L' - y), \quad 0 < x < L, \quad 0 < y < L'.$$

(b) Solve the problem in (a) by separation of variables. Are the solutions identical?

Part B—Vibrations

11. **(a)** A membrane is stretched tightly over the square $0 \le x, y \le L$. If all four edges are clamped on the xy-plane and the membrane is released from rest at an initial displacement $f(x, y)$, find its subsequent displacements.

(b) Simplify the solution when

$$f(x, y) = \frac{(L - 2|x - L/2|)(L - 2|y - L/2|)}{32L}.$$

12. **(a)** A membrane is stretched tightly over the rectangle $0 \le x \le L, 0 \le y \le L'$. Edges $x = 0$ and $x = L$ are clamped on the xy-plane, but $y = 0$ and $y = L'$ are free to move vertically. If the membrane is released from rest at time $t = 0$ from a position described by $f(x, y)$, determine subsequent displacements of the membrane.

(b) Simplify the solution when $f(x, y) = (L - 2|x - L/2|)/(32L)$.

13. Equation (40) describes displacements of a rectangular membrane with edges fixed on the xy-plane when oscillations are initiated by releasing the membrane from rest at a prescribed displacement. Find nodal curves for the mode $2A_{mn}/\sqrt{LL'}\sin(n\pi x/L)\sin(m\pi y/L') \times \cos c\sqrt{n^2/L^2 + m^2/L'^2}\,\pi t$.

14. Is there a result analogous to that in Exercise 9 for the vibration problem of displacements in a membrane?

Part C—Potential, Steady-State Heat Conduction

15. Find the potential inside the rectangular parallelopiped $0 \le x \le L, 0 \le y \le L', 0 \le z \le L''$ if faces $x = 0$, $y = 0$, $x = L$, and $y = L$ are all held at potential zero while faces $z = 0$ and $z = L''$ are maintained at potentials $f(x, y)$ and $g(x, y)$, respectively.

16. Repeat Exercise 15 if faces $x = 0$ and $x = L$ are held at potentials $h(y, z)$ and $k(y, z)$, the other four faces remaining unchanged.

17. Find the steady-state temperature distribution inside a cube $0 \le x \le L, 0 \le y \le L, 0 \le z \le L$ if faces $x = 0$ and $z = L$ are insulated, faces $y = 0$ and $y = L$ are held at temperature zero, and heat is added to faces $x = L$ and $z = 0$ at constant rates q and Q W/m^2, respectively.

5.5 The Multidimensional Eigenvalue Problem

In Section 5.4 we demonstrated that successively separating off Cartesian variables in homogeneous problems leads to the Sturm-Liouville systems of Section 4.2. When the problem is an initial boundary value one, as opposed to a boundary value problem, there remains an ODE for the time dependence of the unknown function. An alternative procedure is first to separate off the time dependence, leaving what is called the multidimensional eigenvalue problem. To illustrate, suppose that the unknown function V in the homogeneous PDE

$$\nabla^2 V = p\frac{\partial^2 V}{\partial t^2} + q\frac{\partial V}{\partial t} + sV \tag{44}$$

is separated into a spatial part, which we designate by W, and a time-dependent part, $T(t)$, $V = WT(t)$. (We have purposely not expressed W as a function of coordinates because what we are about to do is independent of the particular choice of coordinate system.) When this product representation for V is substituted into (44), the time dependence contained in T may be separated from the spatial dependence in W:

$$\frac{\nabla^2 W}{W} = \frac{pT'' + qT' + sT}{T} = -k = \text{constant independent of all variables.}$$

It follows that $T(t)$ must satisfy the ODE

$$pT'' + qT' + (s + k)T = 0$$

and W must satisfy the Helmholtz equation

$$\nabla^2 W + kW = 0.$$

When PDE (44) is accompanied by homogeneous boundary conditions on V, these become homogeneous boundary conditions for W. If we set $k = \lambda^2$, the problem

$$\nabla^2 W + \lambda^2 W = 0, \tag{45a}$$

$$\text{Homogeneous boundary conditions} \tag{45b}$$

is called the *multidimensional eigenvalue problem*. For certain eigenvalues λ^2, there exist nontrivial solutions of (45) called *eigenfunctions*. Properties of eigenvalues and eigenfunctions of this eigenvalue problem parallel those of Sturm-Liouville systems in Chapter 4, but important differences do exist. We consider one example here and give general discussions and further examples in the exercises.

When boundary conditions (45b) are of Dirichlet type on the edges of a rectangle $0 \le x \le L, 0 \le y \le L'$, (45) takes the form

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \lambda^2 W = 0, \qquad 0 < x < L, \qquad 0 < y < L', \tag{46a}$$

$$W(0, y) = 0, \qquad 0 < y < L', \tag{46b}$$

$$W(L, y) = 0, \qquad 0 < y < L', \tag{46c}$$

$$W(x, 0) = 0, \qquad 0 < x < L, \tag{46d}$$

$$W(x, L') = 0, \qquad 0 < x < L. \tag{46e}$$

To solve this problem, we separate $W(x, y) = X(x)Y(y)$. This results in the SL-systems

$$X'' + \mu^2 X = 0, \quad 0 < x < L, \qquad Y'' + (\lambda^2 - \mu^2)Y = 0, \quad 0 < y < L',$$
$$X(0) = X(L) = 0; \qquad Y(0) = Y(L') = 0,$$

solutions of which are

$$X_n(x) = \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} \quad \text{corresponding to eigenvalues } \mu_n^2 = n^2\pi^2/L^2,$$

and $Y_m(y) = \sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}$ corresponding to eigenvalues $\lambda^2 - \mu_n^2 = m^2\pi^2/L'^2$.

In other words, eigenvalues of (46) are $\lambda_{mn}^2 = n^2\pi^2/L^2 + m^2\pi^2/L'^2$, with corresponding eigenfunctions

$$W_{mn}(x, y) = \frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}. \tag{47}$$

It is straightforward to show that these functions are orthonormal on the rectangle with respect to the weight function $p(x, y) = 1$; that is,

$$\int_0^L \int_0^{L'} W_{mn}(x, y)W_{kl}(x, y)\,dy\,dx = \begin{cases} 1 & \text{if } m = k \text{ and } n = l \\ 0 & \text{otherwise} \end{cases} \tag{48}$$

Furthermore, suppose we are given a function $f(x, y)$ that is, along with its first partial derivatives, piecewise continuous on the rectangle $0 \le x \le L, 0 \le y \le L'$. For fixed y, $f(x, y)$ and $\partial f(x, y)/\partial x$ are piecewise continuous functions of x, and we may therefore express $f(x, y)$ in terms of $X_n(x)$; that is, the eigenfunction expansion of $f(x, y)$ as a function of x is

$$\frac{f(x+, y) + f(x-, y)}{2} = \sum_{n=1}^{\infty} d_n(y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}, \tag{49a}$$

where the functions $d_n(y)$ are defined by

$$d_n(y) = \int_0^L f(x, y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx. \tag{49b}$$

Equations (49) are valid provided $f(x, y)$ is continuous in y at the chosen value of y. When this is not the case, these equations must be replaced by appropriate limiting expressions. Because $d_n(y)$ is itself piecewise continuous, with a piecewise continuous first derivative, it may be expanded in terms of $Y_m(y)$:

$$\frac{d_n(y+) + d_n(y-)}{2} = \sum_{m=1}^{\infty} c_{mn}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'} \tag{50a}$$

where

$$c_{mn} = \int_0^{L'} d_n(y)\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\,dy. \tag{50b}$$

We combine these expressions to write

$$\left(\frac{f(x+,y+)+f(x-,y+)}{2}\right)+\left(\frac{f(x+,y-)+f(x-,y-)}{2}\right)$$
$$=\sum_{n=1}^{\infty} d_n(y+)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}+\sum_{n=1}^{\infty} d_n(y-)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$
$$=\sum_{n=1}^{\infty}[d_n(y+)+d_n(y-)]\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$
$$=\sum_{n=1}^{\infty}\left(2\sum_{m=1}^{\infty} c_{mn}\sqrt{\frac{2}{L'}}\sin\frac{m\pi y}{L'}\right)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}$$
$$=2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} c_{mn}\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}.$$

In other words, the function $f(x, y)$ has been expanded in terms of the orthonormal eigenfunctions of eigenvalue problem (46),

$$\frac{f(x+,y+)+f(x+,y-)+f(x-,y+)+f(x-,y-)}{4}$$
$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}W_{mn}(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{mn}\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}, \qquad (51a)$$

where
$$C_{mn}=\int_0^L\int_0^{L'} f(x,y)W_{mn}(x,y)\,dy\,dx$$
$$=\int_0^L\int_0^{L'} f(x,y)\frac{2}{\sqrt{LL'}}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}\,dy\,dx, \qquad (51b)$$

and this result is valid for $0 < x < L, 0 < y < L'$.

We have illustrated with this example that for the multidimensional eigenvalue problem we should expect multisubscripted eigenvalues, orthogonal eigenfunctions, and multidimensional eigenfunction expansions. This is illustrated further in the exercises.

When solving homogeneous initial boundary value problems by separation of variables, there is always the choice of separating off the time dependence first or last. The solution will ultimately be the same for either approach, but the steps differ in arriving at this solution. Let us illustrate with the heat conduction problem

$$\frac{\partial U}{\partial t}=k\left(\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}\right),\quad 0<x<L,\quad 0<y<L',\quad t>0, \qquad (52a)$$
$$U(0,y,t)=0,\quad 0<y<L',\quad t>0, \qquad (52b)$$
$$U(L,y,t)=0,\quad 0<y<L',\quad t>0, \qquad (52c)$$
$$U(x,0,t)=0,\quad 0<x<L,\quad t>0, \qquad (52d)$$
$$U(x,L',t)=0,\quad 0<x<L,\quad t>0, \qquad (52e)$$
$$U(x,y,0)=f(x,y),\quad 0<x<L,\quad 0<y<L'. \qquad (52f)$$

If the x- and y-dependences of a separated function $U(x, y, t) = X(x)Y(y)T(t)$ are separated off first (as was done in Section 5.4), Sturm-Liouville systems in $X(x)$ and $Y(y)$ are obtained:

$$X'' + \lambda^2 X = 0, \quad 0 < x < L, \qquad Y'' + \mu^2 Y = 0, \quad 0 < y < L',$$
$$X(0) = 0 = X(L); \qquad Y(0) = 0 = Y(L').$$

Eigenpairs of these systems are

$$\lambda_n^2 = \frac{n^2\pi^2}{L^2}, \qquad X_n(x) = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L},$$
$$\mu_m^2 = \frac{m^2\pi^2}{L'^2}, \qquad Y_m(y) = \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'}.$$

What remains is an ODE in $T(t)$, namely,

$$T' + k(\lambda_n^2 + \mu_m^2)T = 0, \qquad t > 0,$$

with general solution

$$T(t) = Ae^{-k(\lambda_n^2 + \mu_m^2)t}.$$

To satisfy the initial condition, separated functions are superposed in the form

$$U(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-k(\lambda_n^2 + \mu_m^2)t} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'}, \tag{53a}$$

and the initial temperature $f(x, y)$ at $t = 0$ then requires that

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'}, \qquad 0 < x < L, \qquad 0 < y < L'. \tag{54a}$$

To find expressions for the A_{mn}, we multiply successively by $\sqrt{2/L}\sin(k\pi x/L)$ and $\sqrt{2/L'}\sin(j\pi y/L')$, integrate with respect to x and y, and use orthogonality. The result is

$$A_{mn} = \int_0^{L'} \int_0^L f(x, y) \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \sqrt{\frac{2}{L'}} \sin\frac{m\pi y}{L'} \, dx\, dy. \tag{53b}$$

Alternatively, if time is the first variable separated off by setting $U(x, y, t) = W(x, y)T(t)$, the ODE

$$T' + k\lambda^2 T = 0$$

is obtained along with eigenvalue problem (46). With the eigenpairs $\lambda_{mn}^2 = n^2\pi^2/L^2 + m^2\pi^2/L'^2$ and $W_{mn}(x, y) = (2/\sqrt{LL'})\sin(n\pi x/L)\sin(m\pi y/L')$, the solution for $T(t)$ is

$$T(t) = Ae^{-k\lambda_{mn}^2 t}.$$

Superposition of separated functions gives

$$U(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} e^{-k\lambda_{mn}^2 t} W_{mn}(x, y), \tag{55a}$$

and the initial condition (52f) requires that

$$f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} A_{mn} W_{mn}(x, y), \qquad 0 < x < L, \qquad 0 < y < L'. \tag{54b}$$

But then, A_{mn} are the Fourier coefficients in the eigenfunction expansion of $f(x, y)$ in terms of the $W_{mn}(x, y)$,

$$A_{mn} = \int_0^{L'} \int_0^{L} f(x, y) W_{mn}(x, y)\, dx\, dy. \tag{55b}$$

Solutions (53) and (55) are identical; it is only the way in which we regard equations (54a, b) that differs in our arriving at the solution.

Exercises 5.5

In Exercises 1–3 we prove some general results concerning eigenvalue problem (45) in the xy-plane. Results in three space variables are analogous.

1. Prove the following result corresponding to Theorem 1 in Chapter 4. All eigenvalues of the multidimensional eigenvalue problem

$$\nabla^2 W + \lambda^2 W = 0, \qquad (x, y) \text{ in } A, \tag{56a}$$

$$l\frac{\partial W}{\partial n} + hW = 0, \qquad (x, y) \text{ on } \beta(A), \qquad h > 0, \qquad l > 0, \tag{56b}$$

are real, and eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the unit weight function.

2. Use eigenvalue problem (46) (with $L' = 2L$) to illustrate that a multidimensional eigenvalue problem can have linearly independent eigenfunctions corresponding to the same eigenvalue. (Contrast this with Exercise 12 in Section 4.1 for Sturm-Liouville systems.)
3. Show that all eigenvalues of (56) are nonnegative and that $\lambda = 0$ is an eigenvalue only when the boundary condition is Neumann. In this case, what is the eigenfunction corresponding to $\lambda = 0$?

In Exercises 4–8, find eigenvalues and orthonormal eigenfunctions of eigenproblem (45) on the rectangle A: $0 \le x \le L$, $0 \le y \le L'$ for the given boundary conditions.

4. $W(0, y) = 0$, $0 < y < L'$; $W_x(L, y) = 0$, $0 < y < L'$; $W(x, 0) = 0$, $0 < x < L$; $W(x, L') = 0$, $0 < x < L$
5. $W(0, y) = 0$, $0 < y < L'$; $W(L, y) = 0$, $0 < y < L'$; $\partial W(x, 0)/\partial y = 0$, $0 < x < L$; $\partial W(x, L')/\partial y = 0$, $0 < x < L$
6. $\partial W(0, y)/\partial x = 0$, $0 < y < L'$; $W(L, y) = 0$, $0 < y < L'$; $W(x, 0) = 0$, $0 < x < L$; $\partial W(x, L')/\partial y = 0$, $0 < x < L$
7. $W(0, y) = 0$, $0 < y < L'$; $W(L, y) = 0$, $0 < y < L'$; $\partial W(x, 0)/\partial y = 0$, $0 < x < L$; $l\partial W(x, L')/\partial y + hW(x, L') = 0$, $0 < x < L$
8. $-l_1 \partial W/\partial x + h_1 W = 0$, $x = 0$, $0 < y < L'$; $l_2 \partial W/\partial x + h_2 W = 0$, $x = L$, $0 < y < L'$; $-l_3 \partial W/\partial y + h_3 W = 0$, $y = 0$, $0 < x < L$; $l_4 \partial W/\partial y + h_4 W = 0$, $y = L'$, $0 < x < L$

In Exercises 9–11, use the multidimensional eigenvalue problem approach to solve the initial boundary value problem.

9. Exercise 11(a) in Section 5.4.
10. Exercise 12(a) in Section 5.4.
11. Exercise 5 in Section 5.4.

5.6 Properties of Parabolic Partial Differential Equations

We now return to a difficulty posed in Chapter 3. In what sense are the series obtained in Chapters 3 and 5 "solutions" of their respective problems? In arriving at each series solution, we superposed an infinity of functions satisfying a linear, homogeneous PDE and linear, homogeneous boundary and/or initial conditions. Because of the questionable validity of this step (superposition principle 1 in Section 3.1 endorses only finite linear combinations), we have called each series a formal solution. It is now incumbent on us to verify that each formal solution is indeed a valid solution of its (initial) boundary value problem. Unfortunately, it is not possible to prove general results that encompass all problems solved by means of separation of variables and generalized Fourier series; on the other hand, the situation is not so bad that every problem is its own special case. Techniques exist that verify formal solutions for large classes of problems. In this section and Sections 5.7 and 5.8, we illustrate techniques that work when separation of variables leads to the Sturm-Liouville systems in Table 4.1. At the same time, we take the opportunity to develop properties of solutions of parabolic, hyperbolic, and elliptic PDEs. Time-dependent heat conduction problems are manifested in parabolic equations; vibrations invariably involve hyperbolic equations; and potential problems give rise to elliptic equations.

We choose to illustrate the situation for parabolic PDEs with the heat conduction problem in equation (2) of Section 5.2:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{57a}$$

$$\frac{\partial U(0,t)}{\partial x} = 0, \qquad t > 0, \tag{57b}$$

$$\kappa\frac{\partial U(L,t)}{\partial x} + \mu U(L,t) = 0, \qquad t > 0, \tag{57c}$$

$$U(x,0) = f(x), \qquad 0 < x < L. \tag{57d}$$

The formal solution of this problem is

$$U(x,t) = \sum_{n=1}^{\infty} c_n e^{-k\lambda_n^2 t} X_n(x) \tag{58a}$$

where

$$c_n = \int_0^L f(x) X_n(x)\, dx. \tag{58b}$$

Eigenfunctions are $X_n(x) = N^{-1}\cos\lambda_n x$, where normalizing factors are $2N^2 = L + (\mu/\kappa)/[\lambda_n^2 + (\mu/\kappa)^2]$, and eigenvalues are defined by the equation $\tan\lambda L = \mu/(\kappa\lambda)$.

We shall show by direct substitution that the function $U(x,t)$ defined by series (58) does indeed satisfy PDE (57a), its boundary conditions (57b, c), and its initial condition (57d).

When coefficients c_n are calculated according to (58b), the series $\sum_{n=1}^{\infty} c_n X_n(x)$ converges to $f(x)$ for $0 < x < L$ [provided $f(x)$ is piecewise smooth for $0 \le x \le L$]. Since this series is $U(x,0)$, it follows that initial condition (57d) is satisfied if $f(x)$ is piecewise smooth on $0 \le x \le L$, provided at any point of discontinuity of $f(x)$, $f(x)$ is defined by $f(x) = [f(x+) + f(x-)]/2$.

To verify (57a–c) is not quite so simple. We first show that series (58a) converges for all $0 \le x \le L$ and $t > 0$ and can be differentiated with respect to either x or t. Because eigenfunctions $X_n(x)$ are uniformly bounded (see Theorem 2 of Chapter 4), there exists a constant M such that for all $n \ge 1$ and $0 \le x \le L$, $|X_n(x)| \le N^{-1} \le M$. Further, since $f(x)$ is piecewise continuous on $0 \le x \le L$, it is also bounded thereon: $|f(x)| \le K$. These two results imply that the coefficients c_n defined by (58b) are bounded by

$$|c_n| \le \int_0^L |f(x)||X_n(x)|\,dx \le KML. \tag{59}$$

It follows that for any x in $0 \le x \le L$, and any $t \ge t_0 > 0$.

$$\sum_{n=1}^{\infty} |c_n X_n(x) e^{-k\lambda_n^2 t}| \le KM^2L \sum_{n=1}^{\infty} (e^{-kt_0})^{\lambda_n^2}.$$

Figure 4.2 indicates that the nth eigenvalue $\lambda_n \ge (n-1)\pi/L$. Combine this with the fact that $e^{-kt_0} < 1$, and we may write, for $0 \le x \le L$ and $t \ge t_0 > 0$,

$$\begin{aligned}\sum_{n=1}^{\infty} |c_n X_n(x) e^{-k\lambda_n^2 t}| &\le KM^2L \sum_{n=1}^{\infty} (e^{-kt_0})^{(n-1)^2\pi^2/L^2} \\ &\le KM^2L \sum_{n=1}^{\infty} [(e^{-kt_0})^{\pi^2/L^2}]^{n-1} = KM^2L \sum_{n=1}^{\infty} r^{n-1},\end{aligned} \tag{60}$$

and the geometric series on the right converges, since $r = e^{-kt_0\pi^2/L^2} < 1$. According to the Weierstrass M-test (Theorem 4 in Section 2.3), series (58a) converges absolutely and uniformly with respect to x and t for $0 \le x \le L$ and $t \ge t_0 > 0$. Because $t_0 > 0$ is arbitrary, it also follows that series (58a) converges absolutely for $0 \le x \le L$ and $t > 0$.

Term-by-term differentiation of series (58a) with respect to t gives

$$\sum_{n=1}^{\infty} -k\lambda_n^2 c_n X_n(x) e^{-k\lambda_n^2 t}. \tag{61}$$

Since $\lambda_n \le n\pi/L$ (see, once again, Figure 4.2), it follows that for all $0 \le x \le L$ and $t \ge t_0 > 0$,

$$\sum_{n=1}^{\infty} |-k\lambda_n^2 c_n X_n(x) e^{-k\lambda_n^2 t}| \le \frac{kKM^2\pi^2}{L} \sum_{n=1}^{\infty} n^2 r^{n-1}. \tag{62}$$

Because the series $\sum_{n=1}^{\infty} n^2 r^{n-1}$ converges, we conclude that series (61) converges absolutely and uniformly with respect to x and t for $0 \le x \le L$ and $t \ge t_0 > 0$. As a result, series (61) represents $\partial U/\partial t$ for $0 \le x \le L$ and $t \ge t_0 > 0$ (Theorem 8 in Section 2.3). But, once again, the fact that t_0 is arbitrary implies that we may write

$$\frac{\partial U}{\partial t} = \sum_{n=1}^{\infty} -k\lambda_n^2 c_n X_n(x) e^{-k\lambda_n^2 t} \tag{63}$$

for $0 \le x \le L$ and $t > 0$.

Term-by-term differentiation of series (58a) with respect to x gives

$$\sum_{n=1}^{\infty} c_n X_n'(x) e^{-k\lambda_n^2 t} = \sum_{n=1}^{\infty} c_n(-\lambda_n) N^{-1} \sin \lambda_n x\, e^{-k\lambda_n^2 t}. \tag{64}$$

$0 \le t \le T$, where $\varepsilon > 0$ is a very small number. Because U satisfies (72a), we can say that for $0 < x < L$ and $0 < t < T$,

$$\frac{\partial V}{\partial t} - k\frac{\partial^2 V}{\partial x^2} = \frac{\partial U}{\partial t} - k\left(\frac{\partial^2 U}{\partial x^2} + 2\varepsilon\right) = -2k\varepsilon < 0. \tag{73}$$

Assuming that $U(x,t)$ is continuous, so also is $V(x,t)$, and therefore $V(x,t)$ must take on a maximum in the closed rectangle $\bar{A}$ of Figure 5.12. This value must occur either on the edge of the rectangle or at an interior point (x^*, t^*). In the latter case, $V(x,t)$ must necessarily have a relative maximum at (x^*, t^*), and therefore $\partial V/\partial t = \partial V/\partial x = 0$ and $\partial^2 V/\partial x^2 \le 0$ at (x^*, t^*). But then $\partial V/\partial t - k\partial^2 V/\partial x^2 \ge 0$ at (x^*, t^*), contradicting (73). Hence, the maximum value of V must occur on the boundary of $\bar{A}$. It cannot occur along $t = T$, for, in this case, $\partial V/\partial t \ge 0$ at the point and $\partial^2 V/\partial x^2$ would still be nonpositive. Once again, (73) would be violated. Consequently, the maximum value of V on $\bar{A}$ must occur on one of the three boundaries $t = 0$, $x = 0$, or $x = L$. Since $U \le U_M$ on these three lines, it follows that $V \le U_M + \varepsilon L^2$ on these lines and therefore in $\bar{A}$. But because $U(x,t) \le V(x,t)$, we can state that, in $\bar{A}$, $U(x,t) \le U_M + \varepsilon L^2$. Since ε can be made arbitrarily small, it follows that U_M must be the maximum value of U for $0 \le x \le L$ and $0 \le t \le T$.

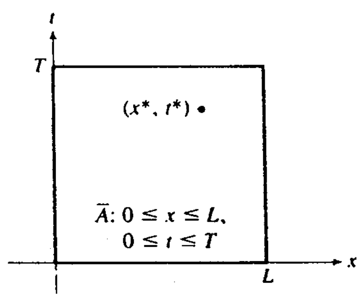


Figure 5.12

When this result is applied to $-U$, the *minimum principle* is obtained—at no point in the rod during the time interval $0 \le t \le T$ can the temperature ever be less than the minimum of the initial temperature of the rod and that found (or applied) at the ends of the rod up to time T.

We mention one final property of heat conduction problems, which, unfortunately, is not demonstrable with the series solutions of Chapters 3 and 5. [It is illustrated for infinite rods in Case 2 of solution (47b) in Section 7.4 and for finite rods in solution (44) of Section 10.4.] When heat is added to any part of an object, its effect is instantaneously felt throughout the whole object. For instance, suppose that the initial temperature $f(x)$ of the rod in problem (57) is identically equal to zero, and at $t = 0$ a small amount of heat is added to either end of the rod or over some cross section of the rod. Instantaneously, the temperature of every point of the rod rises. The increase may be extremely small, but, nonetheless, every point in the rod has a positive temperature for arbitrarily small $t > 0$, and this is true for arbitrarily large L. In other words, heat has been propagated infinitely fast from the source point to all other points in the rod. This apparent paradox is a result of the macroscopic derivation of the heat equation in Section 1.2. On a microscopic level, it would be necessary to take into account the moment of inertia of the molecules transmitting heat, and this would lead to a finite speed for propagation of heat.

Exercises 5.6

1. (a) What is the formal series solution of the one-dimensional heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$

$$-l_1\frac{\partial U}{\partial x} + h_1 U = 0, \qquad x = 0, \qquad t > 0,$$

$$l_2\frac{\partial U}{\partial x} + h_2 U = 0, \qquad x = L, \qquad t > 0,$$

$$U(x,0) = f(x), \qquad 0 < x < L?$$

   (b) Use a technique similar to verification of formal solution (58) for problem (57) to verify that the formal solution in (a) satisfies the four equations in (a) when $f(x)$ is piecewise smooth on $0 \le x \le L$.
   (c) Assuming further that $f(x)$ is continuous on $0 \le x \le L$, show that there is one and only one solution of the problem in (a) that also satisfies continuity conditions (57e, f).
   (d) Verify that the formal solution in (a) satisfies (57e, f) when $f(x)$ satisfies the boundary conditions of the associated Sturm-Liouville system.
2. Use Green's first identity (see Appendix C) to verify that there cannot be more than one solution to problem (71).
3. Repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Robin type.
4. Can you repeat Exercise 2 if the boundary condition on $\beta(V)$ is of Neumann type?
5. In this exercise we prove three-dimensional maximum and minimum principles. Let $U(x, y, z, t)$ be the continuous solution of the homogeneous three-dimensional heat conduction equation in some open region V,

$$\frac{\partial U}{\partial t} = k\nabla^2 U, \qquad (x, y, z) \text{ in } V, \qquad t > 0,$$

   which also satisfies the initial condition

$$U(x, y, z, 0) = f(x, y, z), \qquad (x, y, z) \text{ in } \bar{V},$$

   where $\bar{V}$ is the closed region consisting of V and its boundary $\beta(V)$. Let U_M be the maximum value of $f(x, y, z)$ and the value of U on $\beta(V)$ for $0 \le t \le T$, T some given time.
   (a) Define a function

$$W(x, y, z, t) = U(x, y, z, t) + \varepsilon(x^2 + y^2 + z^2),$$

where $\varepsilon > 0$ is a very small number. Show that

$$\frac{\partial W}{\partial t} - k\nabla^2 W < 0$$

for (x, y, z) in V and $0 < t < T$, and use this fact to verify that W cannot have a relative maximum for a point (x, y, z) in V and a time $0 < t < T$.

(b) Prove the maximum principle that $U(x, y, z, t) \le U_M$ for (x, y, z) in $\bar{V}$ and $0 \le t \le T$.

(c) What is the minimum principle for this situation?

5.7 Properties of Elliptic Partial Differential Equations

Verifications that formal solutions of boundary value problems do indeed satisfy the elliptic PDEs and boundary conditions from which they were derived are similar to those for parabolic (heat) problems. We illustrate with the following Dirichlet problem for Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \quad 0 < x < L, \quad 0 < y < L', \tag{74a}$$

$$V(0, y) = 0, \quad 0 < y < L', \tag{74b}$$

$$V(L, y) = 0, \quad 0 < y < L', \tag{74c}$$

$$V(x, 0) = 0, \quad 0 < x < L, \tag{74d}$$

$$V(x, L') = f(x), \quad 0 < x < L. \tag{74e}$$

Separation leads to the formal solution

$$V(x, y) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi y}{L} X_n(x), \tag{75a}$$

where

$$A_n = \frac{1}{\sinh(n\pi L'/L)} \int_0^L f(x) X_n(x)\, dx \tag{75b}$$

and $X_n(x) = \sqrt{2/L}\sin(n\pi x/L)$.

Theorem 2 of Chapter 4 guarantees that boundary condition (74e) is satisfied when $f(x)$ is piecewise smooth on $0 \le x \le L$ [provided $f(x)$ is defined as the average of right- and left-hand limits at any point of discontinuity]. Boundary conditions (74b–d) are clearly satisfied by (75a). To verify that $V(x, y)$ as defined by (75a) satisfies PDE (74a), we first note that when $f(x)$ is piecewise continuous, it is necessarily bounded [$|f(x)| \le K$)]. Combine this with the fact that $|X_n(x)| \le \sqrt{2/L}$, and we obtain

$$\begin{aligned} |A_n| &\le \frac{1}{|\sinh(n\pi L'/L)|} \int_0^L |f(x)||X_n(x)|\, dx \\ &\le \frac{K\sqrt{2/L}(L)}{\sinh(n\pi L'/L)} = \frac{\sqrt{2L}\,K}{\sinh(n\pi L'/L)}. \end{aligned} \tag{76}$$

With this result, we may write, for any x in $0 \le x \le L$ and any y in $0 \le y \le y_0 < L'$,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|A_n \sinh\frac{n\pi y}{L} X_n(x)\right| &\le \sum_{n=1}^{\infty} \frac{\sqrt{2L}K}{\sinh(n\pi L'/L)}\sinh\frac{n\pi y}{L}\sqrt{\frac{2}{L}}\\ &= 2K\sum_{n=1}^{\infty}\frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)} \le 2K\sum_{n=1}^{\infty} e^{-n\pi(L'-y)/L}\\ &\le 2K\sum_{n=1}^{\infty} e^{-n\pi(L'-y_0)/L} = 2K\sum_{n=1}^{\infty}(e^{-\pi(L'-y_0)/L})^n\\ &= 2K\sum_{n=1}^{\infty} r^n, \end{aligned} \tag{77}$$

a convergent geometric series since $r = e^{-\pi(L'-y_0)/L} < 1$. Consequently, according to the Weierstrass M-test, series (75a) converges absolutely and uniformly with respect to x and y for $0 \le x \le L$ and $0 \le y \le y_0 < L'$. Because y_0 is arbitrary, series (75a) converges absolutely for $0 \le x \le L$ and $0 \le y < L'$. In addition, series (75a) represents a continuous function for $0 \le x \le L$ and $0 \le y < L'$. Thus, even though $f(x)$ may have discontinuities, the solution of Laplace's equation must be a continuous function. In other words, Laplace's equation smooths out discontinuities in boundary data.

Term-by-term differentiation of series (75a) with respect to x gives

$$\sum_{n=1}^{\infty} A_n \sinh\frac{n\pi y}{L} X_n'(x), \tag{78}$$

where $X_n'(x) = (n\pi/L)\sqrt{2/L}\cos(n\pi x/L)$. It follows that, for $0 \le x \le L$ and $0 \le y \le y_0 < L'$,

$$\sum_{n=1}^{\infty}\left|A_n \sinh\frac{n\pi y}{L} X_n'(x)\right| \le \frac{2K\pi}{L}\sum_{n=1}^{\infty} nr^n. \tag{79}$$

Because $\sum_{n=1}^{\infty} nr^n$ converges, series (78) converges absolutely and uniformly. Thus, series (75a) may be differentiated term by term to yield, for $0 \le x \le L$ and $0 \le y < L'$,

$$\frac{\partial V}{\partial x} = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi y}{L} X_n'(x). \tag{80}$$

Similarly, for $0 \le x \le L$ and $0 \le y < L'$,

$$\frac{\partial^2 V}{\partial x^2} = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi y}{L} X_n''(x). \tag{81}$$

Term-by-term differentiation of (75a) with respect to y gives

$$\frac{\pi}{L}\sum_{n=1}^{\infty} nA_n \cosh\frac{n\pi y}{L} X_n(x). \tag{82}$$

Using inequality (76) and the fact that $|X_n(x)| \le \sqrt{2/L}$, we may write

$$\frac{\pi}{L}\sum_{n=1}^{\infty}\left|nA_n \cosh\frac{n\pi y}{L} X_n(x)\right| \le \frac{2K\pi}{L}\sum_{n=1}^{\infty}\frac{n\cosh(n\pi y/L)}{\sinh(n\pi L'/L)}. \tag{83}$$

Now N can always be chosen sufficiently large that $\sinh(n\pi L'/L) \geq (1/4)e^{n\pi L'/L}$, whenever $n \geq N$. For such N,

$$\frac{\pi}{L}\sum_{n=N}^{\infty}\left|nA_n \cosh\frac{n\pi y}{L}X_n(x)\right| \leq \frac{2K\pi}{L}\sum_{n=N}^{\infty}\frac{ne^{n\pi y/L}}{(1/4)e^{n\pi L'/L}}$$
$$= \frac{8K\pi}{L}\sum_{n=N}^{\infty} ne^{-n\pi(L'-y)/L} \leq \frac{8K\pi}{L}\sum_{n=N}^{\infty} ne^{-n\pi(L'-y_0)/L}$$
$$= \frac{8K\pi}{L}\sum_{n=N}^{\infty} nr^n, \tag{84}$$

where $r = e^{-\pi(L'-y_0)/L}$, provided also that $0 \leq x \leq L$ and $0 \leq y \leq y_0 < L'$. Since the series $\sum_{n=1}^{\infty} nr^n$ converges, it follows that series (82) converges uniformly and absolutely for $0 \leq x \leq L$ and $0 \leq y \leq y_0 < L'$. Thus, series (75a) may be differentiated term by term with respect to y to yield, for $0 \leq x \leq L$ and $0 \leq y < L'$,

$$\frac{\partial V}{\partial y} = \frac{\pi}{L}\sum_{n=1}^{\infty} nA_n \cosh\frac{n\pi y}{L}X_n(x). \tag{85}$$

For the same values of x and y, we also obtain

$$\frac{\partial^2 V}{\partial y^2} = \frac{\pi^2}{L^2}\sum_{n=1}^{\infty} n^2A_n \sinh\frac{n\pi y}{L}X_n(x). \tag{86}$$

Because $X_n''(x) = (-n^2\pi^2/L^2)X_n(x)$, expressions (81) and (86) clearly indicate that $V(x, y)$ satisfies Laplace's equation (74a). We have shown, therefore, that series solution (75) satisfies problem (74).

In order to guarantee a unique solution of (74), continuity conditions must also accompany the problem. We show that when $f(x)$ is a continuous function with a continuous first derivative $f'(x)$ and a piecewise continuous second derivative $f''(x)$, for which $f(0) = f(L) = 0$, appropriate conditions are

$$V, \quad \frac{\partial V}{\partial x}, \text{ and } \frac{\partial V}{\partial y} \quad \text{continuous for } 0 \leq x \leq L \text{ and } 0 \leq y \leq L'; \tag{74f}$$

second partial derivatives of $V(x, y)$ continuous for

$$0 < x < L, 0 < y < L'. \tag{74g}$$

Suppose, to the contrary, that there exist two solutions, $V_1(x, y)$ and $V_2(x, y)$, satisfying (74). The difference $V(x, y) = V_1(x, y) - V_2(x, y)$ must also satisfy (74), but with (74e) replaced by the homogeneous condition $V(x, L') = 0, 0 < x < L$. If we multiply (74a) by $V(x, y)$, integrate over the rectangle R: $0 < x < L, 0 < y < L'$, and use Green's first identity (Appendix C), we obtain

$$0 = \iint_R V\nabla^2 V\, dA = \oint_{\beta(R)} V\frac{\partial V}{\partial n}ds - \iint_R |\nabla V|^2\, dA, \tag{87}$$

where $\partial V/\partial n$ is the directional derivative of V outwardly normal to $\beta(R)$. Since $V \equiv 0$ on $\beta(R)$,

$$0 = -\iint_R |\nabla V|^2\, dA.$$

But this result requires that $\nabla V \equiv 0$ in R, and therefore $V(x, y)$ must be constant in R. Because V is constant in R, vanishes on $\beta(R)$, and is continuous for $0 \le x \le L$, $0 \le y \le L'$, it follows that $V(x, y) \equiv 0$. In other words, conditions (74f, g) guarantee a unique solution of problem (74).

Once again, we point out that Laplace's equation, like the heat equation, smooths out discontinuities. Even when the boundary data function $f(x)$ has discontinuities in its second derivative, (74g) demands that second derivatives of $V(x, y)$ be continuous for $0 < x < L$ and $0 < y < L'$.

We now establish that solution (75) of problem (74a–e) also satisfies conditions (74f, g). The facts that series (81) and (86) converge uniformly for $0 \le x \le L$ and $0 \le y \le y_0 < L'$ and y_0 is arbitrary imply that $\partial^2 V/\partial x^2$ and $\partial^2 V/\partial y^2$ are continuous for $0 \le x \le L$ and $0 \le y < L'$. To verify (74f), we use Theorem 5 in Section 2.3. First, note that with continuity of $f(x)$ and $f(0) = f(L) = 0$, the Fourier series of $f(x)$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L'}{L} X_n(x), \tag{88}$$

converges uniformly to $f(x)$ on $0 \le x \le L$ (see Theorem 3 in Section 4.3). Series (75a) can be obtained from series (88) by multiplying the nth term of (88) by

$$Y_n(y) = \frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)}.$$

These functions are uniformly bounded for $0 \le y \le L'$. For fixed y in $0 \le y \le L'$, the derivative of $Y_n(y)$ as a function of a continuous variable n is

$$\frac{\partial Y_n}{\partial n} = \frac{(\pi y/L)\sinh(n\pi L'/L)\cosh(n\pi y/L) - (\pi L'/L)\sinh(n\pi y/L)\cosh(n\pi L'/L)}{\sinh^2(n\pi L'/L)}.$$

Thus,

$$\begin{aligned}
\frac{L}{\pi}\sinh^2\left(\frac{n\pi L'}{L}\right)\frac{\partial Y_n}{\partial n} &= y\sinh\frac{n\pi L'}{L}\cosh\frac{n\pi y}{L} - L'\sinh\frac{n\pi y}{L}\cosh\frac{n\pi L'}{L} \\
&= \frac{y}{2}\left(\sinh\frac{n\pi(L'+y)}{L} + \sinh\frac{n\pi(L'-y)}{L}\right) - \frac{L'}{2}\left(\sinh\frac{n\pi(y+L')}{L} + \sinh\frac{n\pi(y-L')}{L}\right) \\
&= \frac{L'+y}{2}\sinh\frac{n\pi(L'-y)}{L} - \frac{L'-y}{2}\sinh\frac{n\pi(L'+y)}{L} \\
&= \frac{L'+y}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{n\pi(L'-y)}{L}\right)^{2m+1} - \frac{L'-y}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}\left(\frac{n\pi(L'+y)}{L}\right)^{2m+1} \\
&= \frac{(L'+y)(L'-y)}{2}\sum_{m=0}^{\infty}\frac{1}{(2m+1)!}((L'-y)^{2m} - (L'+y)^{2m})\left(\frac{n\pi}{L}\right)^{2m+1},
\end{aligned}$$

which is clearly nonpositive. Thus, for each fixed y in $0 \le y \le L'$, the sequence $\{Y_n(y)\}$ is nonincreasing, and by Theorem 5 in Section 2.3, series (75a) converges uniformly for $0 \le x \le L$ and $0 \le y \le L'$. This series therefore defines a continuous function $V(x, y)$ on $0 \le x \le L, 0 \le y \le L'$.

Because $f'(x)$ is continuous [and $f''(x)$ is piecewise continuous], the Fourier (cosine) series

$$f'(x) = \sum_{n=1}^{\infty} A_n \sinh\frac{n\pi L'}{L} X_n'(x) = \frac{\sqrt{2}\,\pi}{L^{3/2}} \sum_{n=1}^{\infty} nA_n \sinh\frac{n\pi L'}{L} \cos\frac{n\pi x}{L}$$

converges uniformly to $f'(x)$ for $0 \le x \le L$ [see Exercise 4(c) in Section 2.3]. Since series (80) for $\partial V/\partial x$ can be obtained from this series by multiplying the nth term by $Y_n(y)$, it follows that series (80) converges uniformly to $\partial V/\partial x$ for $0 \le x \le L$ and $0 \le y \le L'$ and that $\partial V/\partial x$ is continuous thereon.

Finally, we must show that $\partial V/\partial y$ as defined by series (85) is continuous. Because the above series for $f'(x)$ is uniformly convergent for $0 \le x \le L$, it follows (by setting $x = 0$) that the series

$$\sum_{n=1}^{\infty} \left| nA_n \sinh\frac{n\pi L'}{L} \right|$$

is convergent. Consequently, the series

$$\sum_{n=1}^{\infty} nA_n \sinh\frac{n\pi L'}{L} X_n(x)$$

converges absolutely and uniformly for $0 \le x \le L$. Series (85) for $\partial V/\partial y$ can be obtained from this series by multiplying the nth term by

$$Z_n(y) = \frac{\cosh(n\pi y/L)}{\sinh(n\pi L'/L)}.$$

These functions are uniformly bounded for $0 \le y \le L'$, and, furthermore,

$$[Z_n(y)]^2 = \frac{\cosh^2(n\pi y/L)}{\sinh^2(n\pi L'/L)} = \frac{1}{\sinh^2(n\pi L'/L)} + \left(\frac{\sinh(n\pi y/L)}{\sinh(n\pi L'/L)}\right)^2$$
$$= \frac{1}{\sinh^2(n\pi L'/L)} + [Y_n(y)]^2.$$

For fixed y in $0 \le y \le L'$, the sequence $\{Y_n(y)\}$ is nonincreasing, as is the sequence $\{1/\sinh^2(n\pi L'/L)\}$. Consequently, the same can be said for $\{Z_n(y)\}$, and it follows by Theorem 5 in Section 2.3 that series (85) converges uniformly for $0 \le x \le L$ and $0 \le y \le L'$. Thus, $\partial V/\partial y$ must be continuous thereon, and this completes the proof that solution (75) satisfies conditions (74f, g).

The method used to verify that problem (74a–g) has a unique solution is applicable to much more general problems. Consider, for example, the three-dimensional boundary value problem

$$\nabla^2 U = F(x, y, z), \qquad (x, y, z) \text{ in } V, \tag{89a}$$

$$l\frac{\partial U}{\partial n} + hU = f(x, y, z), \qquad (x, y, z) \text{ on } \beta(V), \tag{89b}$$

U and its first derivatives continuous in $\bar{V}$, (89c)

Second derivatives of U continuous in V, (89d) where $\bar{V}$ is the closed region consisting of V and its boundary and $l \geq 0$ and $h \geq 0$ are constants. In Exercise 2 it is shown that when $h \neq 0$, there cannot be more than one solution of this problem, and when $h = 0$, the solution is unique to an additive constant (i.e., if U is a solution, then all solutions are of the form $U + C$, C = constant). Uniqueness also results when different parts of $\beta(V)$ are subjected to different types of boundary conditions. For U not to be unique, the boundary condition must be Neumann on all of $\beta(V)$.

Maximum and minimum principles for elliptic problems are important theoretically and practically. We verify three-dimensional principles here. The maximum principle for Poisson's equation is as follows:

If $U(x, y, z)$ is a continuous solution of (89a), and $F(x, y, z) \geq 0$ in V, then at no point in V can the value of $U(x, y, z)$ exceed the maximum value of U on $\beta(V)$.

To prove this result, we let U_M be the maximum value of U on $\beta(V)$ and define a function $W(x, y, z) = U(x, y, z) + \varepsilon(x^2 + y^2 + z^2)$ in $\bar{V}$, where $\varepsilon > 0$ is a very small number. Because U satisfies (89a), we can say that in V,

$$\nabla^2 W = \nabla^2 U + 6\varepsilon = F(x, y, z) + 6\varepsilon > 0. \tag{90}$$

Because W is continuous in $\bar{V}$, it must attain an absolute maximum therein. Suppose this maximum occurs at a point (x^*, y^*, z^*) in the interior V (which therefore must be a relative maximum). It follows, then, that

$$\frac{\partial W}{\partial x} = \frac{\partial W}{\partial y} = \frac{\partial W}{\partial z} = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial x^2} \leq 0, \qquad \frac{\partial^2 W}{\partial y^2} \leq 0, \qquad \frac{\partial^2 W}{\partial z^2} \leq 0,$$

all at (x^*, y^*, z^*). Because the last three inequalities contradict (90), the maximum of W must occur on $\beta(V)$.

Since $U \leq U_M$ on $\beta(V)$, $W \leq U_M + \varepsilon R^2$ on $\beta(V)$, where R is the radius of a sphere centered at the origin that contains V (such a sphere must exist when V is bounded). Since the maximum value of W must occur on $\beta(V)$, we can state further that $W \leq U_M + \varepsilon R^2$ for all (x, y, z) in $\bar{V}$. But because $U(x, y, z) \leq W(x, y, z)$ in $\bar{V}$, it follows that in $\bar{V}$, $U(x, y, z) \leq U_M + \varepsilon R^2$. Since ε can be made arbitrarily small, we conclude that $U(x, y, z) \leq U_M$ in $\bar{V}$, and the proof is complete.

When $U(x, y, z)$ is a solution of Laplace's equation, the above maximum principle still holds. In addition, the principle may also be applied to $-U$, resulting in a minimum principle. In other words, we have the following maximum-minimum principle for Laplace's equation:

If a continuous solution of Laplace's equation $\nabla^2 U = 0$ in V satisfies the condition that $U_m \leq U \leq U_M$ on $\beta(V)$, then $U_m \leq U \leq U_M$ in V also.

This principle provides an alternative, and very simple, proof for uniqueness of solutions to problem (89) when the boundary condition is Dirichlet. If U_1 and U_2 are solutions of Poisson's equation (89a) and a Dirichlet condition $U = f(x, y, z)$ on $\beta(V)$, then $U = U_1 - U_2$ is a solution of Laplace's equation $\nabla^2 U = 0$ subject to $U = 0$ on $\beta(V)$. But, according to the maximum-minimum principle for Laplace's equation, U must then be identically equal to zero in V; that is, $U_1 \equiv U_2$.

difficulty, we use d'Alembert's representation of (92),

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} c_n \sqrt{\frac{2}{L}}\left(\sin\frac{n\pi(x+ct)}{L} + \sin\frac{n\pi(x-ct)}{L}\right) \tag{93}$$
$$= \frac{1}{2}[f(x+ct) + f(x-ct)].$$

For this solution to define $y(x,t)$ for $0 \le x \le L$ and $t \ge 0$, $f(x)$ is extended as an odd, $2L$-periodic function. This extension immediately implies that (93) satisfies boundary conditions (91b, c) and initial condition (91e). Initial condition (91d) is clearly satisfied. With continuity of $f''(x)$, it is a straightforward application of chain rules to verify (91a).

We now show that problem (91) has a unique solution when $y(x,t)$ is also required to satisfy the condition that

$$\begin{gathered}y(x,t) \text{ and its first and second partial derivatives}\\ \text{be continuous for } 0 \le x \le L \text{ and } t \ge 0.\end{gathered} \tag{91f}$$

Suppose, to the contrary, that $y_1(x,t)$ and $y_2(x,t)$ are two solutions of (91a–f). Their difference, $y(x,t) = y_1(x,t) - y_2(x,t)$, must then satisfy (91a, b, c, e, f), but (91d) is replaced by the homogeneous initial condition $y(x,0) = 0$, $0 < x < L$. If we multiply (91a) by $\partial y/\partial t$ and integrate with respect to x from $x = 0$ to $x = L$,

$$\int_0^L \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial t}\,dx = \int_0^L c^2 \frac{\partial^2 y}{\partial x^2}\frac{\partial y}{\partial t}\,dx, \qquad t > 0.$$

Integration by parts on the right gives

$$\frac{1}{2}\int_0^L \frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 dx = c^2\left\{\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right\}_0^L - c^2\int_0^L \frac{\partial y}{\partial x}\frac{\partial^2 y}{\partial x\,\partial t}\,dx$$
$$= c_2\left\{\frac{\partial y}{\partial t}\frac{\partial y}{\partial x}\right\}_0^L - c^2\int_0^L \frac{1}{2}\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2 dx, \qquad t > 0. \tag{94}$$

Because the ends of the string are fixed on the x-axis, it follows that $\partial y(0,t)/\partial t = \partial y(L,t)/\partial t = 0$, and therefore (94) reduces to

$$0 = \frac{1}{2}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 + c^2\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2\right]dx, \qquad t > 0. \tag{95}$$

When this equation is antidifferentiated with respect to time, the result is

$$\frac{1}{2}\int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right]dx = K, \qquad t > 0, \tag{96}$$

where K is a constant. To evaluate K, we take the limit of each term in this equation as $t \to 0^+$. Because $\partial y/\partial t$ and $\partial y/\partial x$ are assumed continuous [condition (91f)],

$$\lim_{t\to 0^+} \frac{\partial y(x,t)}{\partial t} = \frac{\partial y(x,0)}{\partial t} = 0, \qquad 0 < x < L$$

[initial condition (91e)]. Furthermore, because $y(x,0) = y_1(x,0) - y_2(x,0) = 0$, we find that

$$\lim_{t\to 0^+} \frac{\partial y(x,t)}{\partial x} = \frac{\partial y(x,0)}{\partial x} = 0, \qquad 0 < x < L.$$

With these results, limits as $t \to 0^+$ in equation (96) show that $K = 0$, and, therefore, for $t \geq 0$ we may write

$$\int_0^L \left[\left(\frac{\partial y}{\partial t}\right)^2 + c^2\left(\frac{\partial y}{\partial x}\right)^2\right] dx = 0. \tag{97}$$

Since each term in this equation is continuous and nonnegative, it follows that each must vanish separately; that is, we must have $\partial y/\partial x = \partial y/\partial t = 0$ for $0 \leq x \leq L$, $t \geq 0$. These imply that $y(x,t)$ is constant for $0 \leq x \leq L$ and $t \geq 0$, and this constant must be zero since $y(x,0) = 0$. Thus, $y(x,t) \equiv 0$, and the solution of (91) is unique.

That (93) satisfies continuity condition (91f) is an immediate consequence of the assumption that $f''(x)$ is continuous for $0 \leq x \leq L$.

In Section 5.6 we saw that discontinuities in the initial temperature function were smoothed out by the heat equation. Likewise, discontinuities in boundary data were smoothed out by Laplace's equation. This is not the case for hyperbolic equations; a distinguishing property of hyperbolic equations is that discontinuities in initial data are propagated by the solution. We have already seen this with the discontinuity in $f'(x)$ for $f(x)$ in Figure 1.30(a). The discontinuity in $f'(x)$ is propagated in both directions along the string at speed c; it is not smoothed out. For a small time t (before the disturbance reaches the ends of the string), the discontinuity is found at positions $x = L/2 \pm ct$, that is, at points given by $x \pm ct = L/2$. But these are equations of characteristic curves for the one-dimensional wave equation (see Example 6 in Section 1.8). We have illustrated, therefore, that discontinuities in derivatives of initial data are propagated along characteristic curves of hyperbolic equations. These characteristics are shown in Figure 5.13. At time $t = L/(2c)$, the discontinuities reach the ends of the string for the first time, whereupon they are reflected to travel once again along the string. By drawing a horizontal line, say $t = t_0$, to intersect the broken lines in this figure, we obtain the positions of the discontinuities at time t_0. Intersections with a vertical line $x = x_0$ give the times at which the discontinuities pass through the point x_0 on the string.

The formal solution of problem (91) when $f(x)$ is as shown in Figure 1.30(a) is still defined by (92) or, more compactly, by (93). It is not, however, a function that satisfies (91a) for all $0 < x < L$ and $t > 0$. It satisfies (91a) at all points (x,t) in Figure 5.13 that are not on the characteristics $x = L/2 \pm ct$ and their reflections.

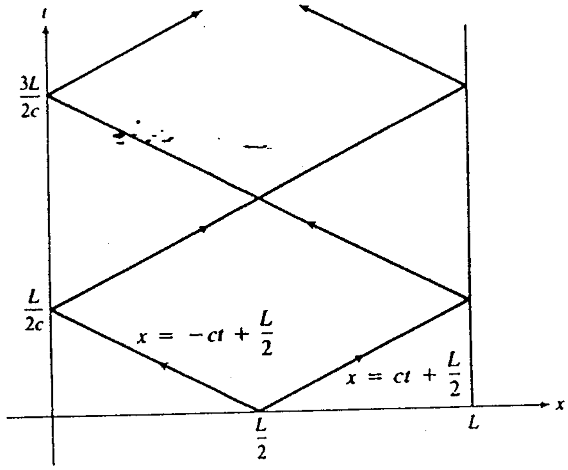


Figure 5.13

Exercises 5.8

1. (a) What is the formal series solution of the vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$

$$y_x(0,t) = 0, \qquad t > 0,$$

$$y_x(L,t) = 0, \qquad t > 0,$$

$$y(x,0) = f(x), \qquad 0 < x < L,$$

$$y_t(x,0) = 0, \qquad 0 < x < L?$$

Express this solution in closed form.

(b) Verify that the formal solution in (a) satisfies the five equations in (a) when $f(x)$, $f'(x)$, and $f''(x)$ are continuous on $0 \le x \le L$ and $f'(0) = f'(L) = 0$.

(c) Show that there is a unique solution to the problem in (a) that also satisfies continuity condition (91f).

(d) Verify that the formal solution in (a) satisfies (91f).

2. (a) What is the formal series solution of vibration problem (91) if initial conditions (91d, e) are replaced by

$$y(x,0) = 0, \qquad y_t(x,0) = g(x), \qquad 0 < x < L?$$

Express the formal solution in closed form when $g(x)$ and $g'(x)$ are continuous for $0 \le x \le L$ and $g(0) = g(L) = 0$.

(b) Verify that the formal solution in (a) satisfies (91a–c) and the initial conditions in (a).

(c) Show that there is a unique solution to the problem in (a) that also satisfies continuity condition (91f).

(d) Verify that the formal solution in (a) satisfies (91f).