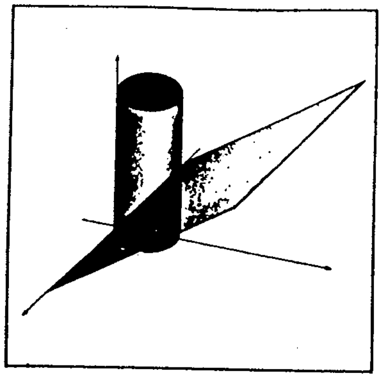


CHAPTER

T E N

Laplace Transforms

10.1 Introduction

The Laplace transform is a mathematical operation, like the Fourier transform, that replaces differentiation problems with algebraic ones, an essential simplification for ordinary and partial differential equations. Fourier transforms are associated with space variables; the Laplace transform is associated with time. In this section we give a brief review of the transform and its simple properties; the complex inversion integral is developed in Section 10.3, and the transform is applied to initial boundary value problems in Sections 10.2, 10.4, and 10.5.

The Laplace transform $\bar{f}(s)$ of a function $f(t)$ is defined by

$$\bar{f}(s) = \mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

provided the improper integral converges. When $f(t)$ is piecewise continuous on every finite interval $0 \le t \le T$, and $f(t)$ is of exponential order† α, its Laplace transform exists for $s > \alpha$.

† A function $f(t)$ is said to be of exponential order α, written $O(e^{\alpha t})$, if there exist constants T and M such that $|f(t)| < Me^{\alpha t}$ for all $t > T$. For example, e^{2t} is $O(e^{2t})$, $\sin t$ is $O(e^{0t})$, and t^n, n a nonnegative integer, is $O(e^{\varepsilon t})$ for arbitrarily small $\varepsilon > 0$.

When $\tilde{f}(s)$ is the Laplace transform of $f(t)$, we call $f(t)$ the *inverse Laplace transform of* $\tilde{f}(s)$ and write

$$f(t) = \mathscr{L}^{-1}\{\tilde{f}(s)\}. \tag{2}$$

The Laplace transforms contained in Table 10.1 are fundamental to applications of the transform to ordinary and partial differential equations; more extensive tables are contained in such references as *Tables of Integral Transforms*, Vol. 1, by Erdelyi, Magnus, Oberhettinger, and Tricomi (New York: McGraw-Hill, 1954). All but the last entry are straightforward applications of definition (1). The transform of $t^{-1/2}$ requires the improper integral of e^{-t^2} over the interval $0 \le t < \infty$, an integral that was evaluated in Exercise 24 of Section 7.2.

Table 10.1

$f(t)$	$\tilde{f}(s)$	$f(t)$	$\tilde{f}(s)$
t^n	$\frac{n!}{s^{n+1}}$	$\sinh at$	$\frac{a}{s^2 - a^2}$
e^{at}	$\frac{1}{s-a}$	$\cosh at$	$\frac{s}{s^2 - a^2}$
$\sin at$	$\frac{a}{s^2 + a^2}$	$t \sinh at$	$\frac{2as}{(s^2 - a^2)^2}$
$\cos at$	$\frac{s}{s^2 + a^2}$	$t \cosh at$	$\frac{s^2 + a^2}{(s^2 - a^2)^2}$
$t \sin at$	$\frac{2as}{(s^2 + a^2)^2}$	$\frac{1}{\sqrt{t}}$	$\sqrt{\frac{\pi}{s}}$
$t \cos at$	$\frac{s^2 - a^2}{(s^2 + a^2)^2}$		

Because the Laplace transform is an integral transform, $\tilde{f}(s)$ is unique for given $f(t)$, but there exist many functions $f(t)$ having the same transform $\tilde{f}(s)$. For example, the functions

$$f(t) = t^2 \quad \text{and} \quad g(t) = \begin{cases} 0 & t = 1 \\ t^2 & t \neq 1, 2, \\ 0 & t = 2 \end{cases}$$

which are identical except for their values at $t = 1$ and $t = 2$, both have the same transform $2/s^3$. What we are saying is that because the Laplace transform is not a one-to-one operation, the inverse transform $\mathscr{L}^{-1}\{\tilde{f}(s)\}$ in (2) cannot be a true inverse. In Section 10.3 we derive a formula for calculating inverse transforms, and this formula always yields a continuous function $f(t)$, if this is possible. In the event that this is not possible, the formula gives a piecewise continuous function whose value is the average of right and left limits at discontinuities, namely, $[f(t+) + f(t-)]/2$. This is reminiscent of equation (14a) in Chapter 4 for Fourier series and equation (6) in Chapter 7 for Fourier integrals. The importance, then, of this formula is that it defines $f(t) = \mathscr{L}^{-1}\{\tilde{f}(s)\}$ in a unique way. Other functions that have the same transform $\tilde{f}(s)$

differ from $f(t)$ only in their values at isolated points; they cannot differ from $f(t)$ over an entire interval $a \le t \le b$. In compliance with this anticipated formula, we adopt the procedure in this section and the next of always choosing a continuous function $\mathscr{L}^{-1}\{\tilde{f}(s)\}$ for given $\tilde{f}(s)$ or, when this is not possible, a piecewise continuous function.

The Laplace transform and its inverse are linear operators. Some of their simple properties are summarized below.

One of two shifting properties is

$$\mathscr{L}(e^{at}f(t)) = \tilde{f}(s-a), \tag{3a}$$

$$\mathscr{L}^{-1}\{\tilde{f}(s-a)\} = e^{at}f(t) \tag{3b}$$

(see Exercise 1). It states that multiplication by an exponential e^{at} in the time domain is equivalent to a translation in the s domain. For example, since $\mathscr{L}\{\cos 2t\} = s/(s^2+4)$, (3a) implies that

$$\mathscr{L}(e^{3t}\cos 2t) = \frac{s-3}{(s-3)^2+4}.$$

The other shifting property is

$$\mathscr{L}\{f(t-a)H(t-a)\} = e^{-as}\tilde{f}(s) \tag{4a}$$

and

$$\mathscr{L}^{-1}(e^{-as}\tilde{f}(s)) = f(t-a)H(t-a), \tag{4b}$$

where $H(t-a)$ is the Heaviside unit step function. It has value 0 when $t < a$ and value 1 when $t > a$. (See Exercise 2 for a proof of these properties.) These properties imply that multiplication by an exponential e^{-as} in the s domain is equivalent to a translation in the time domain. Graphs of $f(t)$ and $f(t-a)H(t-a)$ are shown in Figure 10.1.

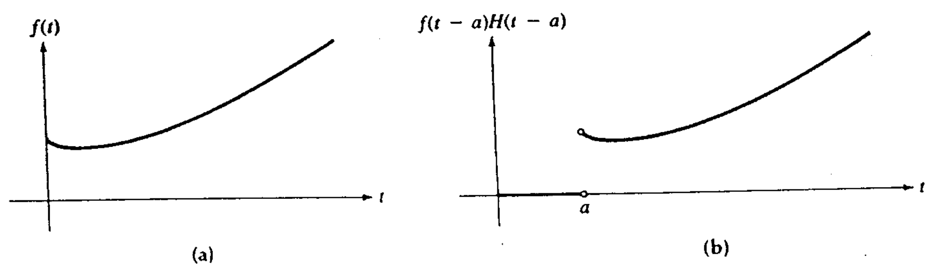


Figure 10.1

The following property is often called the *change of scale* property:

$$\mathscr{L}\{f(at)\} = a^{-1}\tilde{f}\left(\frac{s}{a}\right), \tag{5a}$$

$$\mathscr{L}^{-1}\{\tilde{f}(as)\} = a^{-1}f\left(\frac{t}{a}\right) \tag{5b}$$

(see Exercise 3). For instance, since $\mathscr{L}^{-1}\{1/(s^2+2)\} = (1/\sqrt{2})\sin\sqrt{2}\,t$, (5b) implies that

$$\mathscr{L}^{-1}\left(\frac{1}{4s^2+2}\right) = \mathscr{L}^{-1}\left(\frac{1}{(2s)^2+2}\right) = \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)\sin\left(\frac{\sqrt{2}\,t}{2}\right) = \frac{1}{2\sqrt{2}}\sin\left(\frac{t}{\sqrt{2}}\right).$$

When a function is periodic with period P, the improper integral in (1) may be replaced by an integral over $0 \le t \le P$:

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sP}} \int_0^P e^{-st} f(t)\,dt \tag{6}$$

(see Exercise 4).

The following theorem and its corollary eliminate much of the work when Laplace transforms are applied to initial boundary value problems.

Theorem 1

Suppose $f(t)$ is continuous with a piecewise continuous first derivative on every finite interval $0 \le t \le T$. If $f(t)$ is $O(e^{\alpha t})$, then $\mathscr{L}\{f'(t)\}$ exists for $s > \alpha$ and

$$\mathscr{L}\{f'(t)\} = s\tilde{f}(s) - f(0). \tag{7a}$$

Proof: If t_j, $j = 1, \ldots, n$ denote the discontinuities of $f'(t)$ in $0 \le t \le T$, then

$$\int_0^T e^{-st} f'(t)\,dt = \sum_{j=0}^{n} \int_{t_j}^{t_{j+1}} e^{-st} f'(t)\,dt,$$

where $t_0 = 0$ and $t_{n+1} = T$. Since $f'(t)$ is continuous on each subinterval, we may integrate by parts on each subinterval:

$$\int_0^T e^{-st} f'(t)\,dt = \sum_{j=0}^{n} \left(\{e^{-st} f(t)\}_{t_j}^{t_{j+1}} + s \int_{t_j}^{t_{j+1}} e^{-st} f(t)\,dt \right).$$

Because $f(t)$ is continuous, $f(t_j+) = f(t_j-)$, $j = 1, \ldots, n$, and therefore

$$\int_0^T e^{-st} f'(t)\,dt = -f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\,dt.$$

Thus,

$$\begin{aligned} \mathscr{L}\{f'(t)\} &= \int_0^\infty e^{-st} f'(t)\,dt = \lim_{T\to\infty} \int_0^T e^{-st} f'(t)\,dt \\ &= \lim_{T\to\infty} \left(-f(0) + e^{-sT} f(T) + s \int_0^T e^{-st} f(t)\,dt \right) \\ &= s\tilde{f}(s) - f(0) + \lim_{T\to\infty} e^{-sT} f(T), \end{aligned}$$

provided the limit on the right exists. Since $f(t)$ is $O(e^{\alpha t})$, there exist M and $\bar{T}$ such that for $t > \bar{T}$, $|f(t)| < Me^{\alpha t}$. Thus, for $T > \bar{T}$,

$$e^{-sT}|f(T)| < e^{-sT} M e^{\alpha T} = M e^{(\alpha - s)T},$$

which approaches zero as T approaches infinity (provided $s > \alpha$). Consequently, $\mathscr{L}\{f'(t)\} = s\tilde{f}(s) - f(0)$. ∎

This result is easily extended to second-order derivatives. The extension is stated in the following corollary and is verified in Exercise 5. For extensions when $f(t)$ is only piecewise continuous, see Exercise 35.

Corollary Suppose $f(t)$ and $f'(t)$ are continuous and $f''(t)$ is piecewise continuous on every finite interval $0 \le t \le T$. If $f(t)$ and $f'(t)$ are $O(e^{\alpha t})$, then $\mathscr{L}\{f''(t)\}$ exists for $s > \alpha$, and

$$\mathscr{L}\{f''(t)\} = s^2\tilde{f}(s) - sf(0) - f'(0). \tag{7b}$$

The following examples use these properties and at the same time indicate how Laplace transforms reduce ordinary differential equations to algebraic problems.

Example 1: Solve the differential equation

$$y'' - 2y' + y = 2e^t, \qquad y(0) = y'(0) = 0.$$

Solution: When we take Laplace transforms of both sides of the differential equation and use linearity of the operator,

$$\mathscr{L}\{y''\} - 2\mathscr{L}\{y'\} + \mathscr{L}\{y\} = 2\mathscr{L}\{e^t\}.$$

Properties (7a, b) yield

$$[s^2\tilde{y}(s) - sy(0) - y'(0)] - 2[s\tilde{y}(s) - y(0)] + \tilde{y}(s) = \frac{2}{s-1}.$$

We now use the initial conditions $y(0) = y'(0) = 0$,

$$s^2\tilde{y} - 2s\tilde{y} + \tilde{y} = \frac{2}{s-1}$$

and solve this equation for $\tilde{y}(s)$:

$$\tilde{y}(s) = \frac{2}{(s-1)^3}.$$

The required function $y(t)$ can now be obtained by taking the inverse transform of $\tilde{y}(s)$:

$$\begin{aligned} y(t) &= \mathscr{L}^{-1}\left(\frac{2}{(s-1)^3}\right) = 2\mathscr{L}^{-1}\left(\frac{1}{(s-1)^3}\right) \quad \text{(by linearity)} \\ &= 2e^t\mathscr{L}^{-1}\left(\frac{1}{s^3}\right) \quad \text{[by (3b)]} \\ &= 2e^t\left(\frac{t^2}{2}\right) \quad \text{(from Table 10.1)} \\ &= t^2e^t. \end{aligned}$$

■

Example 2: Solve the differential equation

$$y'' + 4y = 3\cos 2t, \qquad y(0) = 1, \qquad y'(0) = 0.$$

Solution: When we take the Laplace transforms of the differential equation and use the initial conditions,

$$[s^2\tilde{y} - s(1) - 0] + 4\tilde{y} = \frac{3s}{s^2+4}.$$

The solution of this equation for $\tilde{y}(s)$ is

$$\tilde{y}(s) = \frac{3s}{(s^2+4)^2} + \frac{s}{s^2+4},$$

and Table 10.1 gives

$$y(t) = 3\left(\frac{t}{4}\sin 2t\right) + \cos 2t.$$

■

When solving ordinary differential equations by means of Laplace transforms, considerable emphasis is placed on partial fraction decompositions of transform functions $\tilde{y}(s)$, and rightly so, because for ODEs, transform functions are often rational functions of s. Once the transform is decomposed into constituent fractions, and provided the decomposition is not too complicated, inverse transforms of individual terms can be located in tables. Unfortunately, transforms arising from PDEs are seldom rational functions, and there is therefore little point in our giving a detailed discussion of partial fractions.

It is often necessary in applications to find the inverse transform of the product of two functions $\tilde{f}(s)\tilde{g}(s)$ when inverse transforms of $\tilde{f}(s)$ and $\tilde{g}(s)$ are known. Recalling that convolutions were introduced for precisely the same problem associated with Fourier transforms, it should not be surprising that convolutions are defined for Laplace transforms. The convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f * g = \int_0^t f(u)g(t-u)\,du. \tag{8}$$

It has the same properties as convolution (32) in Chapter 7 [see equations (36) in Exercises 7.3], and its importance lies in the following theorem.

Theorem 2

If $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \le t \le T$, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\}, \qquad s > \alpha. \tag{9a}$$

Proof: If $\tilde{f}(s) = \mathscr{L}\{f(t)\}$ and $\tilde{g}(s) = \mathscr{L}\{g(t)\}$, then

$$\tilde{f}(s)\tilde{g}(s) = \int_0^\infty e^{-su}f(u)\,du \int_0^\infty e^{-s\tau}g(\tau)\,d\tau$$

$$= \int_0^\infty \int_0^\infty e^{-s(u+\tau)}f(u)g(\tau)\,d\tau\,du.$$

Suppose we change variables of integration in the inner integral with respect to τ by setting $t = u + \tau$. Then

$$\tilde{f}(s)\tilde{g}(s) = \int_0^\infty \int_u^\infty e^{-st}f(u)g(t-u)\,dt\,du.$$

Now, $g(t)$ is defined only for $t \geq 0$. If we set $g(t) = 0$ for $t < 0$, we may write

$$\tilde{f}(s)\tilde{g}(s) = \lim_{T \to \infty} \int_0^T \int_0^\infty e^{-st} f(u) g(t-u)\, dt\, du.$$

We would like to interchange orders of integration, but to do so requires that the inner integral converge uniformly with respect to u. To verify that this is indeed the case, we note that since $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \leq t \leq T$, there exists a constant M such that for all $t \geq 0$, $|f(t)| \leq Me^{\alpha t}$ and $|g(t)| \leq Me^{\alpha t}$. For each $u \geq 0$, we therefore have $|e^{-st}f(u)g(t-u)| < M^2 e^{-st} e^{\alpha u} e^{\alpha(t-u)} = M^2 e^{-t(s-\alpha)}$. Thus,

$$\left| \int_0^\infty e^{-st} f(u) g(t-u)\, dt \right| < M^2 \int_0^\infty e^{-t(s-\alpha)}\, dt = M^2 \left\{ \frac{e^{-t(s-\alpha)}}{\alpha - s} \right\}_0^\infty = \frac{M^2}{s-\alpha},$$

provided $s > \alpha$ and the improper integral is uniformly convergent with respect to u. The order of integration in the expression for $\tilde{f}(s)\tilde{g}(s)$ may therefore be interchanged, and we obtain

$$\begin{aligned} \tilde{f}(s)\tilde{g}(s) &= \lim_{T \to \infty} \int_0^\infty e^{-st} \int_0^T f(u) g(t-u)\, du\, dt \\ &= \lim_{T \to \infty} \left(\int_0^T e^{-st} \int_0^T f(u) g(t-u)\, du\, dt + \int_T^\infty e^{-st} \int_0^T f(u) g(t-u)\, du\, dt \right). \end{aligned}$$

Since
$$\begin{aligned} \left| \int_T^\infty e^{-st} \int_0^T f(u) g(t-u)\, du\, dt \right| &< \int_T^\infty \int_0^T M^2 e^{-t(s-\alpha)}\, du\, dt \\ &= M^2 T \left\{ \frac{e^{-t(s-\alpha)}}{\alpha - s} \right\}_T^\infty = \frac{M^2 T e^{-T(s-\alpha)}}{s-\alpha}, \end{aligned}$$

provided $s > \alpha$, it follows that

$$\lim_{T \to \infty} \int_T^\infty e^{-st} \int_0^T f(u) g(t-u)\, du\, dt = 0.$$

Further, due to the fact that $g(t-u) = 0$ for $u > t$, we may write, for $T > t$,

$$\int_0^T e^{-st} \int_0^T f(u) g(t-u)\, du\, dt = \int_0^T e^{-st} \int_0^t f(u) g(t-u)\, du\, dt = \int_0^T e^{-st} f * g\, dt.$$

Thus,
$$\tilde{f}(s)\tilde{g}(s) = \lim_{T \to \infty} \int_0^T e^{-st} f * g\, dt = \mathscr{L}\{f * g\}. \quad \blacksquare$$

More important in practice is the inverse of (9a).

Corollary

If $\mathscr{L}^{-1}\{\tilde{f}(s)\} = f(t)$ and $\mathscr{L}^{-1}\{\tilde{g}(s)\} = g(t)$, where $f(t)$ and $g(t)$ are $O(e^{\alpha t})$ and piecewise continuous on every finite interval, then

$$\mathscr{L}^{-1}\{\tilde{f}(s)\tilde{g}(s)\} = \int_0^t f(u) g(t-u)\, du. \tag{9b}$$

As an example to illustrate this corollary, consider finding $\mathcal{L}^{-1}\{2/[s^2(s^2+4)]\}$. Since $\mathcal{L}^{-1}\{2/(s^2+4)\} = \sin 2t$ and $\mathcal{L}^{-1}\{1/s^2\} = t$, we can state that the inverse transform of $2/[s^2(s^2+4)]$ is

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{2}{s^2(s^2+4)}\right) &= \int_0^t u \sin 2(t-u)\,du \\ &= \left\{\frac{u}{2}\cos 2(t-u) + \frac{1}{4}\sin 2(t-u)\right\}_0^t \\ &= \frac{t}{2} - \frac{1}{4}\sin 2t.\end{aligned}$$

Convolutions are particularly important in ODEs that contain unspecified forcing functions.

Example 3: Find the solution of the problem

$$y'' + 2y' - y = f(t), \qquad y(0) = A, \qquad y'(0) = B$$

for arbitrary constants A and B and an arbitrary function $f(t)$.

Solution: When we take Laplace transforms,

$$[s^2\tilde{y} - As - B] + 2[s\tilde{y} - A] - \tilde{y} = \tilde{f}(s),$$

and solve for $\tilde{y}$,

$$\tilde{y}(s) = \frac{\tilde{f}(s)}{s^2+2s-1} + \frac{As + B + 2A}{s^2+2s-1}.$$

To find the inverse transform of this function, we first note that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+2s-1}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2-2}\right) = e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2-2}\right) = \frac{1}{\sqrt{2}}e^{-t}\sinh\sqrt{2}\,t.$$

Convolution property (9b) on the first term of $\tilde{y}(s)$ now yields

$$\begin{aligned}y(t) &= \int_0^t f(u)\frac{1}{\sqrt{2}}e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + \mathcal{L}^{-1}\left(\frac{A(s+1)+(B+A)}{(s+1)^2-2}\right) \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + e^{-t}\mathcal{L}^{-1}\left(\frac{As+(B+A)}{s^2-2}\right) \\ &= \frac{1}{\sqrt{2}}\int_0^t f(u)e^{-(t-u)}\sinh\sqrt{2}(t-u)\,du + e^{-t}\left(A\cosh\sqrt{2}\,t + \frac{B+A}{\sqrt{2}}\sinh\sqrt{2}\,t\right).\end{aligned}$$

■

Exercises 10.1

1. (a) Verify shifting property (3).
   (b) Use (3a) and Table 10.1 to calculate Laplace transforms for the following:
   (i) $f(t) = t^3e^{-5t}$ (ii) $f(t) = e^{-t}\cos 2t + e^{3t}\sin 2t$
   (iii) $f(t) = e^{at}\cosh 4t - e^{-at}\sinh 4t$

(c) Use (3b) and Table 10.1 to calculate inverse Laplace transforms for the following:
(i) $\tilde{f}(s) = 1/(s^2 - 2s + 5)$ (ii) $\tilde{f}(s) = 1/\sqrt{s+3}$
(iii) $\tilde{f}(s) = s/(s^2 + 4s + 1)$

2. (a) Verify shifting property (4).
(b) Use (4a) and Table 10.1 to calculate Laplace transforms for the following:
(i) $f(t) = \begin{cases} 0 & 0 < t < 3 \\ t - 2 & t > 3 \end{cases}$ (ii) $f(t) = \begin{cases} 0 & 0 < t < a \\ 1 & t > a \end{cases}$
(iii) $f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & t > a \end{cases}$ (iv) $f(t) = \begin{cases} 0 & 0 < t < a \\ 1 & a < t < b \\ 0 & t > b \end{cases}$
(c) Use (4b) and Table 10.1 to calculate inverse Laplace transforms for the following:
(i) $\tilde{f}(s) = e^{-2s}/s^2$ (ii) $\tilde{f}(s) = e^{-3s}/(s^2 + 1)$
(iii) $\tilde{f}(s) = se^{-5s}/(s^2 - 2)$

3. (a) Verify change of scale property (5).
(b) Use (5a) and Table 10.1 to calculate Laplace transforms for the following:
(i) $f(t) = 4t^2 + \sinh 2t$ (ii) $f(t) = e^{4t}\cos 4t$
(c) Use (5b) and Table 10.1 to calculate inverse Laplace transforms for the following:
(i) $\tilde{f}(s) = s/(9s^2 + 2)$ (ii) $\tilde{f}(s) = \dfrac{1}{4s^2 - 6s - 5}$

4. (a) Verify equation (6).
(b) Find Laplace transforms for the following functions:
(i) $f(t) = t, \quad 0 < t < a, \quad f(t + a) = f(t)$
(ii) $f(t) = \begin{cases} 1 & 0 < t < a \\ -1 & a < t < 2a \end{cases}, \quad f(t + 2a) = f(t)$
(iii) $f(t) = |\sin at|$

5. Verify equation (7b).

In Exercises 6–9, use convolutions to find the inverse transform for the function.

6. $\tilde{f}(s) = \dfrac{1}{s(s+1)}$

7. $\tilde{f}(s) = \dfrac{1}{(s^2+1)(s^2+4)}$

8. $\tilde{f}(s) = \dfrac{s}{(s+4)(s^2-2)}$

9. $\tilde{f}(s) = \dfrac{s}{(s^2-4)(s^2-9)}$

In Exercises 10–15, find the Laplace transform of the function.

10. $f(t) = \begin{cases} 2t & 0 \le t \le 1 \\ t & t > 1 \end{cases}$

11. $f(t) = \begin{cases} t^2 & 0 \le t \le 1 \\ 2t & t > 1 \end{cases}$

12. $f(t) = \begin{cases} t & 0 < t < a \\ 2a - t & a < t < 2a \end{cases}, \quad f(t + 2a) = f(t)$

13. $f(t) = \begin{cases} 1 & 0 < t < a \\ 0 & a < t < 2a \end{cases}, \quad f(t + 2a) = f(t)$

14. $f(t) = \begin{cases} 0 & 0 < t < a \\ 1 & t > a \end{cases}$

15. $f(t) = \begin{cases} 0 & 0 < t < a \\ 1 & a < t < a + 1 \\ 0 & t > a + 1 \end{cases}$

In Exercises 16–25, find the inverse Laplace transform for the function.

16. $\tilde{f}(s) = \dfrac{s}{s^2 - 3s + 2}$

17. $\tilde{f}(s) = \dfrac{4s + 1}{(s^2 + s)(4s^2 - 1)}$

18. $\tilde{f}(s) = \dfrac{e^{-3s}}{s + 5}$

19. $\tilde{f}(s) = \dfrac{e^{-2s}}{s^2 + 3s + 2}$

20. $\tilde{f}(s) = \dfrac{1}{s^3 + 1}$

21. $\tilde{f}(s) = \dfrac{5s - 2}{3s^2 + 4s + 8}$

22. $\tilde{f}(s) = \dfrac{e^{-s}(1 - e^{-s})}{s(s^2 + 1)}$

23. $\tilde{f}(s) = \dfrac{s}{(s + 1)^5}$

24. $\tilde{f}(s) = \dfrac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)}$

25. $\tilde{f}(s) = \dfrac{s^2}{(s^2 - 4)^2}$

In Exercises 26–32, solve the differential equation.

26. $y'' + 2y' - y = e^t, \quad y(0) = 1, \quad y'(0) = 2$
27. $y'' + y = 2e^{-t}, \quad y(0) = y'(0) = 0$
28. $y'' + 2y' + y = t, \quad y(0) = 0, \quad y'(0) = 1$
29. $y''' - 3y'' + 3y' - y = t^2e^t, \quad y(0) = 1, \quad y'(0) = 0, \quad y''(0) = -2$
30. $y'' + 9y = \cos 2t, \quad y(0) = 1, \quad y(\pi/2) = -1$
31. $y''' - 3y'' + 3y' - y = t^2e^t$
32. $y'' - a^2y = f(t)$
33. Verify that the Laplace transform of a function $f(t)$ that is piecewise continuous on every finite interval $0 \le t \le T$ and is $O(e^{\alpha t})$ exists for $s > \alpha$.
34. (a) Prove that when n is a nonnegative integer, t^n is $O(e^{\varepsilon t})$ for every $\varepsilon > 0$.
    (b) Prove that when $f(t)$ is $O(e^{\alpha t})$, $t^n f(t)$ is $O[e^{(\alpha+\varepsilon)t}]$ for every $\varepsilon > 0$.
35. (a) Let $f(t)$ be $O(e^{\alpha t})$ and be continuous for $t \ge 0$ except for a finite discontinuity at $t = t_0 > 0$; and let $f'(t)$ be piecewise continuous on every finite interval $0 \le t \le T$. Show that

$$\mathscr{L}\{f'(t)\} = s\tilde{f}(s) - f(0) - e^{-st_0}[f(t_0+) - f(t_0-)].$$

    (b) What is the result in (a) if $t_0 = 0$?
36. Let $f(t)$ and $f'(t)$ be $O(e^{\alpha t})$, let $f'(t)$ be piecewise continuous on every finite interval $0 \le t \le T$, and let $f(t)$ have only a finite number of finite discontinuities for $t \ge 0$. Verify the "initial value theorem,"

$$\lim_{s\to\infty} s\tilde{f}(s) = \lim_{t\to 0^+} f(t).$$

Assume the result that

$$\lim_{s\to\infty} \tilde{f}(s) = 0$$

for functions that are piecewise continuous and of exponential order.

10.2 Laplace Transform Solutions for Problems on Unbounded Domains

In this section we illustrate the use of Laplace transforms on problems over unbounded domains. Such problems do not require the complex inversion formula of Section 10.3. We begin with a heat conduction problem on a semi-infinite interval.

xample 4: Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0, \tag{10a}$$

$$U(0,t) = U_0, \qquad t > 0, \tag{10b}$$

$$U(x,0) = 0, \qquad x > 0, \tag{10c}$$

for temperature in a semi-infinite rod that is initially at temperature 0°C. For time $t > 0$, its end at $x = 0$ is held at constant temperature U_0. (This problem was solved by Fourier sine transforms in Exercise 2 of Section 7.4.)

Solution: When we take Laplace transforms of PDE (10a) and use initial condition (10c), we obtain

$$s\tilde{U} = k\mathscr{L}\left(\frac{\partial^2 U}{\partial x^2}\right).$$

Since the integration with respect to t in the Laplace transform and the differentiation with respect to x are independent, we interchange the order of operations on the right:

$$s\tilde{U} = k\frac{\partial^2 \tilde{U}}{\partial x^2}.$$

Because only derivatives with respect to x remain, we replace the partial derivative with an ordinary derivative:

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = 0, \qquad x > 0. \tag{11a}$$

This ordinary differential equation is subject to the transform of (10b),

$$\tilde{U}(0,s) = \frac{U_0}{s}. \tag{11b}$$

For problems on finite domains, we have found it convenient to express general solutions of equations like (11a) in terms of hyperbolic functions. On infinite and semi-infinite intervals, it is advantageous to use the exponential formulation,

$$\tilde{U}(x,s) = Ae^{\sqrt{s/k}x} + Be^{-\sqrt{s/k}x}. \tag{12}$$

Because $U(x,t)$ must remain bounded as x becomes infinite, so also must $\tilde{U}(x,s)$. We must therefore set $A = 0$, in which case (11b) requires that $B = U_0/s$. Thus,

$$\tilde{U}(x,s) = \frac{U_0}{s}e^{-\sqrt{s/k}x}. \tag{13}$$

The inverse Laplace transform of this function is found in tables:

$$U(x,t) = U_0\mathscr{L}^{-1}\left(\frac{e^{-\sqrt{s/k}x}}{s}\right) = U_0\,\mathrm{erfc}\left(\frac{x}{2\sqrt{kt}}\right), \tag{14}$$

where $\mathrm{erfc}(x)$ is the complementary error function

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-u^2}\,du. \tag{15}$$

Notice that for any $x > 0$ and any $t > 0$, temperature $U(x,t)$ is positive. This indicates that the abrupt change in temperature at the end $x = 0$ from 0°C to U_0 is felt instantaneously at every point in the rod. In other words, energy is "transmitted" infinitely fast along the rod, a property of the heat equation that we mentioned in Section 5.6. ■

When $U(0,t)$ is a function of time in this example, say $U(0,t) = f_1(t)$, transform (13) is replaced by

$$\tilde{U}(x,s) = \tilde{f}_1(s)e^{-\sqrt{s/k}\,x}. \tag{16}$$

Because $\mathcal{L}^{-1}\{e^{-a\sqrt{s}}\} = [a/(2\sqrt{\pi t^3})]e^{-a^2/(4t)}$, it follows by convolution property (9b) that

$$U(x,t) = \int_0^t f_1(t-u)\frac{x}{2\sqrt{k\pi u^3}}e^{-x^2/(4ku)}\,du \tag{17a}$$

$$= \frac{x}{2\sqrt{k\pi}}\int_0^t u^{-3/2}f_1(t-u)e^{-x^2/(4ku)}\,du$$

or, alternatively, that

$$U(x,t) = \frac{x}{2\sqrt{k\pi}}\int_0^t (t-u)^{-3/2}f_1(u)e^{-x^2/[4k(t-u)]}\,du. \tag{17b}$$

In the next example we illustrate how a semi-infinite string falling under gravity reacts to one end being fixed.

Example 5: A semi-infinite string is supported from below so that it lies motionless on the x-axis. At time $t = 0$, the support is removed and gravity is permitted to act on the string. If the end $x = 0$ is fixed at the origin, find the displacement of the string.

Solution: The initial boundary value problem is

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + g, \qquad x > 0, \qquad t > 0, \tag{18a}$$

$$y(0,t) = 0, \qquad t > 0, \tag{18b}$$

$$y(x,0) = 0, \qquad x > 0, \tag{18c}$$

$$y_t(x,0) = 0, \qquad x > 0, \tag{18d}$$

where $g = -9.81$. When we apply the Laplace transform to the PDE and use the initial conditions,

$$s^2\tilde{y} = c^2\frac{d^2\tilde{y}}{dx^2} + \frac{g}{s}.$$

Thus, $\tilde{y}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{y}}{dx^2} - \frac{s^2}{c^2}\tilde{y} = -\frac{g}{c^2 s}, \qquad x > 0, \tag{19a}$$

subject to the transform of (18b),

$$\tilde{y}(0,s) = 0. \tag{19b}$$

The general solution of (19a) is

$$\bar{y}(x, s) = Ae^{sx/c} + Be^{-sx/c} + \frac{g}{s^3}.$$

For this function to remain bounded as $x \to \infty$, we must set $A = 0$, in which case boundary condition (19b) requires that $B = -g/s^3$. Hence,

$$\bar{y}(x, s) = \frac{g}{s^3}(1 - e^{-sx/c}). \tag{20}$$

The inverse transform of this function is

$$y(x, t) = \frac{gt^2}{2} - \frac{g}{2}\left(t - \frac{x}{c}\right)^2 H\left(t - \frac{x}{c}\right), \tag{21}$$

where $H(t - x/c)$ is the Heaviside unit step function. What this says is that a point x in the string falls freely under gravity for $0 < t < x/c$, after which it falls with constant velocity gx/c [since for $t > x/c$, $y(x, t) = -(g/2)(-2xt/c + x^2/c^2)$]. A picture of the string at any given time t_0 is shown in Figure 10.2. It is parabolic for $0 < x < ct_0$ and horizontal for $x > ct_0$. As t_0 increases, the parabolic portion lengthens and the horizontal section drops.

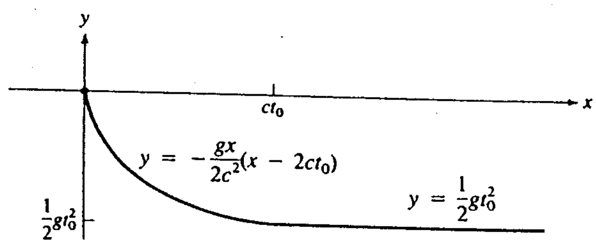


Figure 10.2

■

It is worthwhile noting that this problem cannot be solved by Fourier sine transforms because a constant function (g in this case) is not absolutely integrable on $0 \le x < \infty$.

Exercises 10.2

Part A—Heat Conduction

1. Solve Exercise 3 in Section 7.4.
2. Show that every solution $U(x, t)$ of the one-dimensional heat conduction equation

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x, t),$$

which at time $t = 0$ has value $U(x,0) = f(x)$, must have a Laplace transform of the form

$$\tilde{U}(x,s) = Ae^{\sqrt{s/k}x} + Be^{-\sqrt{s/k}x} - \sqrt{\frac{k}{s}}\int_0^x \left(\frac{f(u)}{k} + \frac{\tilde{g}(x,u)}{\kappa}\right)\sinh\sqrt{\frac{s}{k}}(x-u)\,du,$$

where A and B are independent of x. In Exercises 3–6 we use this result to solve various heat conduction problems on infinite and semi-infinite intervals.

3. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$
$$U(0,t) = f_1(t), \qquad t > 0,$$
$$U(x,0) = U_0 = \text{constant}, \qquad x > 0.$$

(b) Simplify the solution when $f_1(t) = \bar{U} = \text{constant}$. [See also Exercises 4(b) and (c) in Section 7.4.]

4. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$
$$\frac{\partial U(0,t)}{\partial x} = -\frac{f_1(t)}{\kappa}, \qquad t > 0,$$
$$U(x,0) = U_0 = \text{constant}, \qquad x > 0.$$

(b) Simplify the solution when $f_1(t) = Q_0 = \text{constant}$. [See also Exercises 5(b) and (c) in Section 7.4.]

5. (a) Use the result of Exercise 2 to solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad x > 0, \qquad t > 0,$$
$$-\kappa\frac{\partial U(0,t)}{\partial x} + \mu U(0,t) = \mu f_1(t), \qquad t > 0,$$
$$U(x,0) = 0, \qquad x > 0.$$

(b) Simplify the solution when $f_1(t) = U_m = \text{constant}$. [See also Exercise 6 in Section 7.4.]

6. (a) Use the result of Exercise 2, and the fact that the transform must remain bounded as $x \to \pm\infty$, to show that the transform of the function satisfying the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad -\infty < x < \infty, \qquad t > 0,$$
$$U(x,0) = U_0 H(x), \qquad -\infty < x < \infty,$$

must be of the form

$$\tilde{U}(x,s) = \begin{cases} Ae^{\sqrt{s/k}x} & x < 0 \\ Be^{-\sqrt{s/k}x} + \dfrac{U_0}{2s}(2 - e^{-\sqrt{s/k}x}) & x > 0 \end{cases}.$$

(b) By demanding that the expression for $\tilde{U}(x,s)$ and its first derivative with respect to x agree at $x = 0$, show that

$$\tilde{U}(x,s) = \frac{U_0}{2s}\begin{cases} e^{\sqrt{s/k}x} & x < 0 \\ 2 - e^{-\sqrt{s/k}x} & x \geq 0 \end{cases}.$$

(c) Find the inverse transform $U(x,t)$. [See also Case 2 for solution (47b) in Section 7.4.]

Part B—Vibrations

7. Show that every solution $y(x,t)$ of the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}$$

that also satisfies the initial conditions

$$y(x,0) = f(x), \qquad y_t(x,0) = g(x)$$

must have a Laplace transform of the form

$$\tilde{y}(x,s) = Ae^{sx/c} + Be^{-sx/c} - \frac{1}{cs}\int_0^x \left(sf(u) + g(u) + \frac{\tilde{F}(u,s)}{\rho}\right)\sinh\frac{s}{c}(x-u)\,du,$$

where A and B are independent of x. In Exercises 8 and 9 we use this result to solve vibration problems on semi-infinite intervals.

8. At time $t = 0$ a semi-infinite taut string lies motionless along the positive x-axis. If its left end is subjected to vertical motion described by $f_1(t)$ for $t > 0$, find its subsequent displacements. (See also Example 11 in Section 7.4.)

9. Solve Exercise 8 if $f_1(t)$ represents a force on the end $x = 0$ of the string; that is, replace the Dirichlet condition with the Neumann condition $\partial y(0,t)/\partial x = -\tau^{-1}f_1(t)$. (See also Exercise 8 in Section 7.4.)

10.3 The Complex Inversion Integral

Finding the inverse Laplace transform in Section 10.1 was a matter of organization and tables; we used properties (3b), (4b), (5b) (and partial fractions) to organize a given transform $\tilde{f}(s)$ into a form for which the inverse transform can be found in tables. In Section 10.2, for PDEs on infinite and semi-infinite intervals, tables and convolutions were once again prominent. For PDEs on finite domains, however, the situation is different; transform functions are so complicated that they can seldom be found in tables. What we need, then, is a direct method for inverting the Laplace transform. In this section we use the theory of functions of a complex variable to derive such a formula.

We first note that the results in equations (3)–(9) remain valid when s is complex; the complex derivation may be somewhat different from its real counterpart, but each result is valid when s is complex.

The following theorem shows that Laplace transforms are analytic functions of the complex variable s.

Theorem 3

If $f(t)$ is $O(e^{\alpha t})$ and piecewise continuous on every finite interval $0 \le t \le T$, the Laplace transform $\bar{f}(s) = \bar{f}(x + iy)$ of $f(t)$ is an analytic function of s in the half-plane $x > \alpha$.

Proof: If the real and imaginary parts of $\bar{f}(s)$ are denoted by $u(x, y)$ and $v(x, y)$,

$$\bar{f}(s) = u + iv = \int_0^\infty e^{-(x+iy)t} f(t)\, dt,$$

then
$$u(x, y) = \int_0^\infty e^{-xt} \cos yt\, f(t)\, dt, \qquad v(x, y) = \int_0^\infty -e^{-xt} \sin yt\, f(t)\, dt.$$

To verify the analyticity of $\bar{f}(s)$, we show that $u(x, y)$ and $v(x, y)$ have continuous first partial derivatives that satisfy the Cauchy-Riemann equations when $x > \alpha$. Now,

$$\left\{\begin{matrix} |e^{-xt} \cos yt\, f(t)| \\ |e^{-xt} \sin yt\, f(t)| \end{matrix}\right\} \le e^{-xt} |f(t)|,$$

and since $f(t)$ is $O(e^{\alpha t})$, there exist constants M and T such that for all $t > T$, $|f(t)| < Me^{\alpha t}$. Consequently, whenever $x \ge \alpha' > \alpha$ and $t > T$,

$$\left\{\begin{matrix} |e^{-xt} \cos yt\, f(t)| \\ |e^{-xt} \sin yt\, f(t)| \end{matrix}\right\} < e^{-xt} M e^{\alpha t} \le M e^{(\alpha - \alpha')t}$$

and
$$\left\{\begin{matrix} |u(x, y)| \\ |v(x, y)| \end{matrix}\right\} < \int_0^T e^{-xt} |f(t)|\, dt + \int_T^\infty M e^{(\alpha - \alpha')t}\, dt$$
$$\le \int_0^T e^{-\alpha' t} |f(t)|\, dt + M \left\{\frac{e^{(\alpha - \alpha')t}}{\alpha - \alpha'}\right\}_0^\infty$$
$$= \int_0^T e^{-\alpha' t} |f(t)|\, dt + \frac{M}{\alpha' - \alpha}.$$

Thus, the integrals representing u and v converge absolutely and uniformly with respect to x and y in the half-plane $x \ge \alpha' > \alpha$. Since $f(t)$ is piecewise continuous, u and v are continuous functions for $x \ge \alpha'$. Now,

$$\int_0^\infty \frac{\partial}{\partial x}(e^{-xt} \cos yt\, f(t))\, dt = \int_0^\infty -te^{-xt} \cos yt\, f(t)\, dt$$

and
$$\int_0^\infty \frac{\partial}{\partial y}(-e^{-xt} \sin yt\, f(t))\, dt = \int_0^\infty -te^{-xt} \cos yt\, f(t)\, dt.$$

Since $tf(t)$ is $O[e^{(\alpha+\epsilon)t}]$ for any $\epsilon > 0$ and is piecewise continuous on every finite interval $0 \le t \le T$, a similar argument to that above shows that this integral is absolutely and uniformly convergent with respect to x and y for $x \ge \alpha' > \alpha$. Because $\alpha' > \alpha$ is arbitrary, it follows that this integral converges to a continuous function that is equal to both $\partial u/\partial x$ and $\partial v/\partial y$ for $x > \alpha$. We have shown, then, that the first of the Cauchy-Riemann equations $\partial u/\partial x = \partial v/\partial y$ is satisfied for $x > \alpha$. In a similar way, we can show that $\partial u/\partial y = -\partial v/\partial x$, and therefore $\bar{f}(s)$ is analytic for $x > \alpha$. ∎

To obtain the complex inversion integral for $L^{-1}\{\tilde{f}(s)\}$, we use the extension of Cauchy's integral formula contained in the following theorem.

Theorem 4

Let $f(z)$ be a complex function analytic in a domain containing the half-plane $x \geq \gamma$ (Figure 10.3), and let $f(z)$ be $0(z^{-k})$† $(k > 0)$ as $|z| \to \infty$ in that half-plane. Then, if z_0 is any complex number with real part greater than γ,

$$f(z_0) = -\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \frac{f(z)}{z - z_0} dz. \tag{22}$$

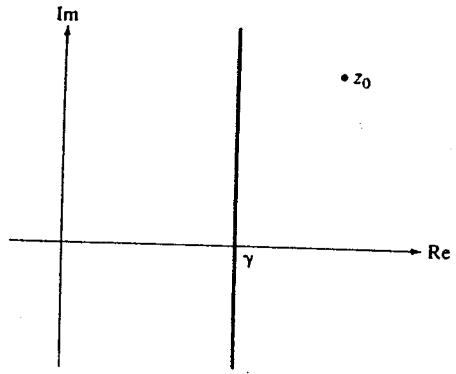


Figure 10.3

When a function $f(t)$ is $O(e^{\alpha t})$, we know that its transform $\tilde{f}(s)$ is analytic for $x > \alpha$ (see Theorem 3). It follows from (22) that when $\tilde{f}(s)$ is $O(s^{-k})$ in a half-plane $x \geq \gamma > \alpha$, we can write $\tilde{f}(s)$ in the form

$$\tilde{f}(s) = -\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} \frac{\tilde{f}(z)}{z - s} dz$$

for $x > \gamma$. If we formally take inverse transforms of both sides of this equation and interchange the order of integration and $\mathscr{L}^{-1}$, we obtain

$$f(t) = -\frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} - \tilde{f}(z)\mathscr{L}^{-1}\left(\frac{1}{s - z}\right) dz = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{zt}\tilde{f}(z)\, dz.$$

This expression,

$$\boxed{f(t) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st}\tilde{f}(s)\, ds,} \tag{23}$$

† A function $f(z)$ is said to be $O(z^{-k})$ as $|z| \to \infty$ if there exist constants M and r such that $|f(z)z^k| < M$ for $|z| > r$.

is called the *complex inversion integral* for the Laplace transformation. Although it is an integral in the complex plane along the line $x = \gamma$, it can be written as a complex combination of real improper integrals. But even for very simple functions $\tilde{f}(s)$, the integrations involved in this real form are usually very difficult (see Exercise 17). Fortunately, in Theorem 7 we prove that residues of $e^{st}\tilde{f}(s)$ may be used to evaluate the integral. First, however, we give conditions on a function $\tilde{f}(s)$ sufficient to guarantee that a function $f(t)$ exists whose transform is $\tilde{f}(s)$ and that $f(t)$ is given by this inversion integral.

Theorem 5

Let $\tilde{f}(s)$ be any function of the complex variable s that is analytic and $O(s^{-k})$, $k > 1$, for all $s = x + iy$ in a half-plane $x \geq \alpha$. Let also $\tilde{f}(x)$ be real when $x \geq \alpha$. Then the inversion integral of $\tilde{f}(s)$ along any line $x = \gamma (\gamma \geq \alpha)$ converges to a real-valued function $f(t)$ that is independent of γ and whose Laplace transform is $\tilde{f}(s)$ for $x > \alpha$. Furthermore, $f(t)$ is $O(e^{\alpha t})$, it is continuous, and $f(t) = 0$ for $t \leq 0$.

The conditions on $\tilde{f}(s)$ in this theorem are severe. They are not, for instance, satisfied by $\tilde{f}(s) = 1/s$, since this function is $O(s^{-1})$. By qualifying the function $f(t)$ instead of $\tilde{f}(s)$, it is possible to relax conditions on the inversion integral formula.

Theorem 6

If $\tilde{f}(s)$ is the Laplace transform of any function $f(t)$ of $O(e^{\alpha t})$, which is piecewise smooth on every finite interval $0 \leq t \leq T$, then the inversion integral of $\tilde{f}(s)$ along any line $x = \gamma > \alpha$ exists and represents $f(t)$. At any point of discontinuity of $f(t)$, the inversion integral represents $[f(t+) + f(t-)]/2$, and if $t = 0$, it represents $f(0+)/2$.

Proof: Define a function

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-\gamma t} f(t) & t > 0 \end{cases} \qquad \text{where } \gamma > \alpha.$$

Then $g(t)$ is piecewise smooth on every finite interval $0 \leq t \leq T$, and if T is such that $|f(t)| < Me^{\alpha t}$ for $t > T$,

$$\begin{aligned}\int_{-\infty}^{\infty} |g(t)|\,dt &= \int_0^{\infty} e^{-\gamma t}|f(t)|\,dt \leq \int_0^T e^{-\gamma t}|f(t)|\,dt + \int_T^{\infty} e^{-\gamma t}Me^{\alpha t}\,dt \\ &= \int_0^T e^{-\gamma t}|f(t)|\,dt + \frac{Me^{-(\gamma-\alpha)T}}{\gamma - \alpha};\end{aligned}$$

that is, $g(t)$ is absolutely integrable on $-\infty < t < \infty$. Consequently, $g(t)$ may be represented by Fourier's integral formula [see equation (6) in Chapter 7]:

$$\begin{aligned}\frac{g(t+) + g(t-)}{2} &= \int_0^{\infty}\left[\left(\frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\cos\lambda x\,dx\right)\cos\lambda t\right. \\ &\quad \left. + \left(\frac{1}{\pi}\int_{-\infty}^{\infty} g(x)\sin\lambda x\,dx\right)\sin\lambda t\right]d\lambda \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} g(x)\cos\lambda(t - x)\,dx\,d\lambda. \qquad (24)\end{aligned}$$

But because $\cos \lambda(x - t)$ and $\sin \lambda(x - t)$ are even and odd functions of λ, respectively, it follows that

$$\int_{-\infty}^{0} \int_{-\infty}^{\infty} g(x) \cos \lambda(t - x)\, dx\, d\lambda = \int_{0}^{\infty} \int_{-\infty}^{\infty} g(x) \cos \lambda(t - x)\, dx\, d\lambda$$

and

$$\int_{-\infty}^{0} \int_{-\infty}^{\infty} g(x) \sin \lambda(t - x)\, dx\, d\lambda = -\int_{0}^{\infty} \int_{-\infty}^{\infty} g(x) \sin \lambda(t - x)\, dx\, d\lambda.$$

This means that we can replace (24) with

$$\begin{aligned}\frac{g(t+) + g(t-)}{2} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) \cos \lambda(t - x)\, dx\, d\lambda \\ &\quad + \frac{i}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) \sin \lambda(t - x)\, dx\, d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x) e^{i\lambda(t-x)}\, dx\, d\lambda \\ &= \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\lambda t} \int_{-\infty}^{\infty} g(x) e^{-i\lambda x}\, dx\, d\lambda.\end{aligned}$$

Because $g(x)$ vanishes for $x < 0$ and is equal to $e^{-\gamma x} f(x)$ for $x > 0$, we may write

$$\begin{aligned}\frac{g(t+) + g(t-)}{2} &= \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\lambda t} \int_{0}^{\infty} e^{-(\gamma + i\lambda)x} f(x)\, dx\, d\lambda \\ &= \frac{1}{2\pi} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} e^{i\lambda t} \tilde{f}(\gamma + i\lambda)\, d\lambda.\end{aligned}$$

We now regard this integral as an integral along the line $x = \gamma$ in the complex plane by setting $s = \gamma + i\lambda$ and $ds = i\, d\lambda$. Multiplying both sides of the equation by $e^{\gamma t}$, we obtain

$$e^{\gamma t}\left(\frac{g(t+) + g(t-)}{2}\right) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \tilde{f}(s)\, ds.$$

But from the definition of $g(t)$, we see that when $t > 0$, $e^{\gamma t}[g(t+) + g(t-)]/2 = [f(t+) + f(t-)]/2$, and when $t = 0$, $e^{\gamma t}[g(t+) + g(t-)]/2 = f(0+)/2$. Thus,

$$\frac{f(t+) + f(t-)}{2} = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\gamma - i\beta}^{\gamma + i\beta} e^{st} \tilde{f}(s)\, ds. \qquad \blacksquare$$

One might argue that Theorem 6 is of little use in applications since we do not know $f(t)$; it is precisely $f(t)$ that is the unknown in the problem. For instance, suppose the Laplace transform was applied to an ODE (or a PDE) in $y(t)$ and some analysis was performed, leading to an expression for $\tilde{y}(s)$. It now remains to apply the inverse transform to find $y(t)$. But how then can we use Theorem 6? What we do is ignore the conditions of the theorem and simply apply the inversion integral to $\tilde{y}(s)$ to obtain a function $y(t)$, a function that we hope is both the inverse transform of $\tilde{y}(s)$ and a solution to our differential equation. To verify that this is indeed the case, we can proceed in two ways. First, we can take the Laplace transform of $y(t)$, and if we obtain $\tilde{y}(s)$, there is no question that $y(t)$ is the inverse transform of $\tilde{y}(s)$. Alternatively, we can set aside Laplace transforms completely and verify that $y(t)$ is a solution of the differential equation with which we began.

As we have already mentioned, the inversion integral is seldom used to find inverse transforms; it is circumvented with residues of the complex function $e^{st}\tilde{y}(s)$. The main argument of the method is contained in the following theorem, wherein $\tilde{f}(s)$ is assumed to satisfy conditions like those of Theorem 5 or Theorem 6.

Theorem 7

Let $\tilde{f}(s)$ be a function for which the inversion integral along a line $x = \gamma$ represents the inverse function $f(t)$, and let $\tilde{f}(s)$ be analytic except for isolated singularities $s_n (n = 1, \ldots)$ in the half-plane $x < \gamma$. Then the series of residues of $e^{st}\tilde{f}(s)$ at $s = s_n$ converges to $f(t)$ for each positive t,

$$f(t) = \text{sum of residues of } e^{st}\tilde{f}(s) \text{ at its singularities,}$$

provided a sequence C_n of contours can be found that satisfies the following properties:

(1) C_n consists of the straight line $x = \gamma$ from $\gamma - i\beta_n$ to $\gamma + i\beta_n$ and some curve Γ_n beginning at $\gamma + i\beta_n$, ending at $\gamma - i\beta_n$, and lying in $x \le \gamma$;

(2) C_n encloses $s_1, s_2, \ldots, s_n$;

(3) $\lim_{n\to\infty} \beta_n = \infty$; and

(4) $\lim_{n\to\infty} \int_{\Gamma_n} e^{st}\tilde{f}(s)\, ds = 0$ (Figure 10.4).

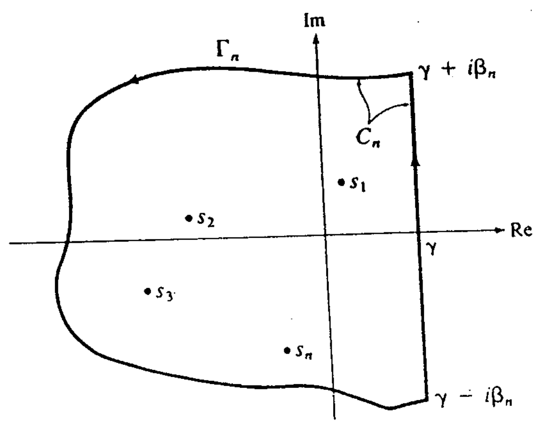


Figure 10.4

Proof: Since $e^{st}\tilde{f}(s)$ is analytic in C_n except at $s_1, \ldots, s_n$, the residue theorem states that

$$\begin{pmatrix}\text{Sum of residues of} \\ e^{st}\tilde{f}(s) \text{ at } s_1, \ldots, s_n\end{pmatrix} = \frac{1}{2\pi i} \oint_{C_n} e^{st}\tilde{f}(s)\, ds$$

$$= \frac{1}{2\pi i} \int_{\gamma - i\beta_n}^{\gamma + i\beta_n} e^{st}\tilde{f}(s)\, ds + \frac{1}{2\pi i} \int_{\Gamma_n} e^{st}\tilde{f}(s)\, ds.$$

When we take limits on n [and use conditions (3) and (4) in the theorem],

$$\begin{pmatrix}\text{Sum of residues of} \\ e^{st}\tilde{f}(s) \text{ at } s_1, \ldots, s_n, \ldots\end{pmatrix} = \frac{1}{2\pi i} \lim_{n\to\infty} \int_{\gamma - i\beta_n}^{\gamma + i\beta_n} e^{st}\tilde{f}(s)\,ds = f(t). \qquad \blacksquare$$

It is not essential, as condition (2) requires, that C_n contain precisely n of the singularities of $\tilde{f}(s)$. In fact, this could be very difficult to accomplish, depending on how the singularities are enumerated. What is essential is that as n increases, the C_n expand to enclose eventually all singularities of $\tilde{f}(s)$.

As a result of Theorem 7, finding the inverse transform of a function $\tilde{f}(s)$ is now a matter of calculating residues of the function $e^{st}\tilde{f}(s)$ at its singularities. When s_0 is a singularity of $e^{st}\tilde{f}(s)$, the residue at s_0 is defined as the coefficient of $(s - s_0)^{-1}$ in the Laurent expansion of $e^{st}\tilde{f}(s)$ about s_0. It can be found in one of two ways:

(1) Find the Laurent expansion of $e^{st}\tilde{f}(s)$ about s_0, or at least enough of it to identify the coefficient of $(s - s_0)^{-1}$.

(2) When it is known that s_0 is a pole of order m, the following formula yields the residue of $e^{st}\tilde{f}(s)$ at s_0:

$$\operatorname{Res}[e^{st}\tilde{f}(s), s_0] = \lim_{s\to s_0}\left[\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}\left((s - s_0)^m e^{st}\tilde{f}(s)\right)\right]. \tag{25}$$

Example 6: Use Theorem 5 to find inverse transforms when $\tilde{f}(s)$ is equal to (a) $\frac{1}{s^m}$, $m \geq 2$ an integer; (b) $1/(s^2 + 9)$; (c) $s^2/(s^2 + 1)^2$.

Solution: (a) The function $\tilde{f}(s) = 1/s^m$ has a pole of order m at $s = 0$, as does $e^{st}\tilde{f}(s)$. According to equation (25), the residue there is

$$\lim_{s\to 0}\frac{1}{(m-1)!}\frac{d^{m-1}}{ds^{m-1}}(s^m e^{st}\tilde{f}(s)) = \frac{1}{(m-1)!}\lim_{s\to 0}\frac{d^{m-1}}{ds^{m-1}}(e^{st}) = \frac{t^{m-1}}{(m-1)!}.$$

The contours Γ_n in Figure 10.5 clearly satisfy conditions (1) and (3) in Theorem 7. Furthermore, on Γ_n,

$$\left|\frac{e^{st}}{s^m}\right| = \left|\frac{e^{(x+iy)t}}{n^m e^{im\theta}}\right| = \frac{e^{xt}}{n^m} \leq \frac{e^t}{n^m}.$$

Thus,

$$\left|\int_{\Gamma_n}\frac{e^{st}}{s^m}\,ds\right| < \frac{2\pi n e^t}{n^m} = \frac{2\pi e^t}{n^{m-1}},^{\dagger}$$

and this expression approaches zero as $n \to \infty$. Consequently, the Γ_n satisfy condition (4) of Theorem 7, and by this theorem,

$$\mathscr{L}^{-1}\left\{\frac{1}{s^m}\right\} = \frac{t^{m-1}}{(m-1)!}.$$

† We have used the following result to arrive at this inequality. When $|f(z)| \leq M$ on a curve C of finite length L,

$$\left|\int_C f(z)\,dz\right| \leq ML.$$

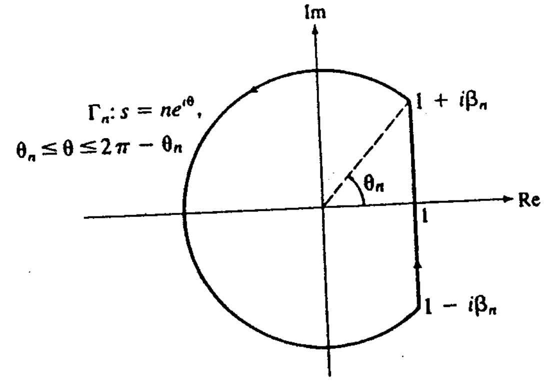


Figure 10.5

(b) The function $\tilde{f}(s) = 1/(s^2 + 9)$ has poles of order 1 at $s = \pm 3i$, as does $e^{st}\tilde{f}(s)$. The residue at $3i$ is

$$\operatorname{Res}[e^{st}\tilde{f}(s), 3i] = \lim_{s \to 3i} \frac{(s - 3i)e^{st}}{(s + 3i)(s - 3i)} = \frac{e^{3it}}{6i} = -\frac{i}{6}e^{3it}.$$

Similarly, $\operatorname{Res}[e^{st}\tilde{f}(s), -3i] = (i/6)e^{-3it}$. On the contour Γ_n ($n \ge 4$) in Figure 10.5,

$$\left|\frac{e^{st}}{s^2 + 9}\right| \le \frac{|e^{(x+iy)t}|}{|s|^2 - 9} = \frac{e^{xt}}{n^2 - 9} \le \frac{e^t}{n^2 - 9}.$$

Thus,

$$\left|\int_{\Gamma_n} \frac{e^{st}}{s^2 + 9}\, ds\right| < \frac{2\pi n e^t}{n^2 - 9},$$

and this expression approaches zero as $n \to \infty$. By Theorem 7, then,

$$\mathscr{L}^{-1}\left(\frac{1}{s^2 + 9}\right) = -\frac{i}{6}e^{3it} + \frac{i}{6}e^{-3it} = \frac{1}{3}\sin 3t.$$

(c) The function $\tilde{f}(s) = s^2/(s^2 + 1)^2$ has poles of order 2 at $s = \pm i$, as does $e^{st}\tilde{f}(s)$. The residue at i is

$$\begin{aligned}\operatorname{Res}[e^{st}\tilde{f}(s), i] &= \lim_{s \to i} \frac{d}{ds}\left(\frac{(s - i)^2 e^{st} s^2}{(s + i)^2 (s - i)^2}\right) \\ &= \lim_{s \to i}\left(\frac{(s + i)^2(2se^{st} + ts^2e^{st}) - s^2e^{st}(2)(s + i)}{(s + i)^4}\right) \\ &= \frac{1}{4}e^{it}(t - i).\end{aligned}$$

Similarly, $\operatorname{Res}[e^{st}\tilde{f}(s), -i] = (1/4)e^{-it}(t + i)$. On the contour Γ_n ($n \ge 2$) in Figure 10.5,

$$\left|\frac{s^2 e^{st}}{(s^2 + 1)^2}\right| \le \frac{|s|^2 e^{xt}}{(|s|^2 - 1)^2} \le \frac{n^2 e^t}{(n^2 - 1)^2}.$$

Thus,

$$\left|\int_{\Gamma_n} \frac{s^2 e^{st}}{(s^2 + 1)^2}\, ds\right| < \frac{2\pi n^3 e^t}{(n^2 - 1)^2},$$

and this expression approaches zero as $n \to \infty$. By Theorem 7, then,

$$\mathscr{L}^{-1}\left(\frac{s^2}{(s^2+1)^2}\right) = \frac{1}{4}e^{it}(t-i) + \frac{1}{4}e^{-it}(t+i) = \frac{t}{2}\cos t + \frac{1}{2}\sin t. \qquad \blacksquare$$

More complicated illustrations of Theorem 7 are contained in Example 7. This example is more typical of problems encountered in Section 10.4, where Laplace transforms are used to solve initial boundary value problems.

Example 7: Find inverse transforms for the following:

$$\text{(a)}\ \tilde{f}(s) = \frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}}; \qquad \text{(b)}\ \tilde{f}(s) = \frac{1}{s^3}(1-\cosh sx) + \frac{\sinh s \sinh sx}{s^3\cosh s}.$$

Solutions: (a) The function $\tilde{f}(s)$ has singularities at the zeros of sinh $\sqrt{s}$; that is, when $\sqrt{s} = n\pi i$ or $s = -n^2\pi^2$, $n \geq 0$ an integer. To determine the nature of the singularity at $s = 0$, we find the Laurent expansion of $\tilde{f}(s)$ about $s = 0$. We do this with expansions of the hyperbolic functions:

$$\tilde{f}(s) = \frac{1}{s}\left(\frac{\sqrt{s}\,x + \frac{1}{3!}(\sqrt{s}\,x)^3 + \cdots}{\sqrt{s} + \frac{1}{3!}(\sqrt{s})^3 + \cdots}\right) = \frac{1}{s}\left(x + \frac{s}{6}(x^3 - x) + \cdots\right).$$

Consequently, $\tilde{f}(s)$ has a pole of order 1 at $s = 0$, as does $e^{st}\tilde{f}(s)$. The following expansion shows that the residue of $e^{st}\tilde{f}(s)$ at this pole is x:

$$e^{st}\tilde{f}(s) = \left(1 + st + \frac{(st)^2}{2!} + \cdots\right)\left(\frac{1}{s}\right)\left(x + \frac{s}{6}(x^3 - x) + \cdots\right)$$
$$= \frac{1}{s}\left(x + \frac{s}{6}(6xt + x^3 - x) + \cdots\right).$$

Because the derivative of sinh $\sqrt{s}$ does not vanish at the remaining singularities $s = -n^2\pi^2$ $(n > 0)$, these are also poles of order 1, and the residues of $e^{st}\tilde{f}(s)$ at these poles are given by limit (25):

$$\lim_{s\to -n^2\pi^2}(s + n^2\pi^2)e^{st}\frac{\sinh\sqrt{s}\,x}{s\sinh\sqrt{s}} = e^{-n^2\pi^2 t}\frac{\sinh n\pi x i}{-n^2\pi^2}\lim_{s\to -n^2\pi^2}\frac{s + n^2\pi^2}{\sinh\sqrt{s}}.$$

L'Hôpital's rule can be used to evaluate this limit, which, combined with the facts that $\sinh i\theta = i\sin\theta$ and $\cosh i\theta = \cos\theta$, gives, for these residues,

$$-\frac{i}{n^2\pi^2}e^{-n^2\pi^2 t}\sin n\pi x \lim_{s\to -n^2\pi^2}\frac{1}{\frac{1}{2\sqrt{s}}\cosh\sqrt{s}}$$
$$= -\frac{2i}{n^2\pi^2}e^{-n^2\pi^2 t}\sin n\pi x\frac{n\pi i}{\cosh n\pi i}$$
$$= \frac{2}{n\pi}e^{-n^2\pi^2 t}\sin n\pi x\frac{1}{\cos n\pi} = \frac{2(-1)^n}{n\pi}e^{-n^2\pi^2 t}\sin n\pi x.$$

Thus, the sum of the residues of $e^{st}\tilde{f}(s)$ at its singularities is

$$f(t) = x + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2 t}\sin n\pi x.$$

Verification that this function is the inverse of $\tilde{f}(s)$ still requires the establishment of a sequence of contours satisfying the conditions of Theorem 7. We omit this part of the argument. Transforms of this type arise in heat conduction problems.

(b) This transform has singularities at $s = 0$ and $s = (2n - 1)\pi i/2$, n an integer (the zeros of $\cosh s$). The Laurent expansion of $\tilde{f}(s)$ about $s = 0$ can be found by expanding the hyperbolic functions in Maclaurin series:

$$\begin{aligned}\tilde{f}(s) &= \frac{1}{s^3}\left(1 - 1 - \frac{s^2x^2}{2!} - \frac{s^4x^4}{4!} - \cdots\right)\\ &\quad + \frac{1}{s^3}\left(\frac{s + \frac{s^3}{3!} + \frac{s^5}{5!} + \cdots}{1 + \frac{s^2}{2!} + \frac{s^4}{4!} + \cdots}\right)\left(sx + \frac{s^3x^3}{3!} + \cdots\right)\\ &= \left(-\frac{x^2}{2s} - \frac{x^4s}{24} - \cdots\right) + \left(s - \frac{s^3}{3} + \cdots\right)\left(\frac{x}{s^2} + \frac{x^3}{6} + \cdots\right)\\ &= \frac{x}{2s}(2 - x) + \frac{xs}{24}(-x^3 + 4x^2 - 8) + \cdots.\end{aligned}$$

Consequently, $\tilde{f}(s)$ has a pole of order 1 at $s = 0$, as does $e^{st}\tilde{f}(s)$. Multiplication of this series by the Maclaurin series for e^{st} gives

$$\begin{aligned}e^{st}\tilde{f}(s) &= \left(1 + st + \frac{(st)^2}{2!} + \cdots\right)\left(\frac{x}{2s}(2 - x) + \frac{xs}{24}(-x^3 + 4x^2 - 8) + \cdots\right)\\ &= \frac{x}{2s}(2 - x) + \frac{xt}{2}(2 - x) + \cdots,\end{aligned}$$

and therefore the residue of $e^{st}\tilde{f}(s)$ at $s = 0$ is $x(2 - x)/2$.

Because the derivative of $\cosh s$ does not vanish at $s = (2n - 1)\pi i/2$, these singularities are also poles of order 1, and the residues of $e^{st}\tilde{f}(s)$ at these poles are given by the limits

$$\begin{aligned}&\lim_{s\to(2n-1)\pi i/2}\left(s - \frac{(2n-1)\pi i}{2}\right)e^{st}\left(\frac{1}{s^3}(1 - \cosh sx) + \frac{\sinh s\sinh sx}{s^3\cosh s}\right)\\ &= \frac{e^{(2n-1)\pi t i/2}}{-(2n-1)^3\pi^3 i/8}\sinh\frac{(2n-1)\pi i}{2}\sinh\frac{(2n-1)\pi x i}{2}\lim_{s\to(2n-1)\pi i/2}\frac{s - (2n-1)\pi i/2}{\cosh s}\\ &= \frac{8e^{(2n-1)\pi t i/2}}{(2n-1)^3\pi^3 i}\sin\frac{(2n-1)\pi}{2}\sin\frac{(2n-1)\pi x}{2}\lim_{s\to(2n-1)\pi i/2}\frac{1}{\sinh s}\\ &\qquad\qquad\left(\text{using l'Hôpital's rule}\right)\end{aligned}$$

$$= \frac{8(-1)^{n+1}e^{(2n-1)\pi ti/2}}{(2n-1)^3\pi^3 i}\sin\frac{(2n-1)\pi x}{2}\,\frac{1}{\sinh\frac{(2n-1)\pi i}{2}}$$

$$= -\frac{8e^{(2n-1)\pi ti/2}}{(2n-1)^3\pi^3}\sin\frac{(2n-1)\pi x}{2}.$$

The sum of the residues of $e^{st}\tilde{f}(s)$ at its singularities is therefore

$$f(t) = \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=-\infty}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2}.$$

To simplify this expression, we separate it into two summations, one over positive n and the other over nonpositive n, and in the latter we set $m = 1 - n$:

$$\begin{aligned} f(t) &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &\quad - \frac{8}{\pi^3}\sum_{n=0}^{-\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &\quad - \frac{8}{\pi^3}\sum_{m=1}^{\infty}\frac{e^{[2(1-m)-1]\pi ti/2}}{[2(1-m)-1]^3}\sin\frac{[2(1-m)-1]\pi x}{2}. \end{aligned}$$

If we now replace m by n in the second summation and combine it with the first,

$$\begin{aligned} f(t) &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &\quad - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{-(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &= \frac{x}{2}(2-x) - \frac{8}{\pi^3}\sum_{n=1}^{\infty}\frac{e^{(2n-1)\pi ti/2} + e^{-(2n-1)\pi ti/2}}{(2n-1)^3}\sin\frac{(2n-1)\pi x}{2} \\ &= \frac{x}{2}(2-x) - \frac{16}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\frac{(2n-1)\pi t}{2}\sin\frac{(2n-1)\pi x}{2}. \end{aligned}$$

Once again, we omit verification of existence of a sequence of contours satisfying Theorem 7. Transforms of this type occur in vibration problems. ■

Exercises 10.3

In Exercises 1–16, use residues to find the inverse Laplace transform of the given function. In Exercises 1–5, verify the existence of contours satisfying the requirements of Theorem 7; in Exercises 6–16, neglect this verification.

1. $\tilde{f}(s) = s/(s-1)^3$
2. $\tilde{f}(s) = s/(s^2+4)^2$
3. $\tilde{f}(s) = \dfrac{1}{s^2(s+3)}$
4. $\tilde{f}(s) = \dfrac{s^2+2}{(s+1)^2(s-3)^3}$

5. $\tilde{f}(s) = \dfrac{s^2}{(s^2+1)(s^2+4)}$

6. $\tilde{f}(s) = \dfrac{s}{s^2-1}$

7. $\tilde{f}(s) = \dfrac{s^3}{(s^2-4)^3}$

8. $\tilde{f}(s) = \dfrac{1}{(s^2-2s+2)^2}$

9. $\tilde{f}(s) = \dfrac{s-1}{(s^2-2s+2)^2}$

10. $\tilde{f}(s) = \dfrac{s^2}{(s^2-2s+2)^2}$

11. $\tilde{f}(x,s) = \dfrac{1}{s}\left(x - \dfrac{\sinh\sqrt{s}\,x}{\sinh\sqrt{s}}\right)$

12. $\tilde{f}(x,u,s) = \dfrac{\sinh sx \sinh s(1-u)}{s\sinh s}$

13. $\tilde{f}(x,s) = \dfrac{2\sinh sx}{s^3\sinh s}(1-\cosh s) + \dfrac{2}{s^3}(\cosh sx - 1) + \dfrac{x}{s}(1-x)$

14. $\tilde{f}(x,s) = \dfrac{1}{s^3} + \dfrac{\cosh sx}{s^2\sinh s}$

15. $\tilde{f}(x,s) = \dfrac{\sinh sx}{(4s^2+\pi^2)\sinh s}$

16. $\tilde{f}(x,s) = \dfrac{\sinh sx}{(s^2+\pi^2)\sinh s}$

17. We have claimed that to use inversion integral (23) directly is usually impossible. Set up the complex combination of real improper integrals for (23) when $\tilde{f}(s) = 1/s^2$; that is, express (23) in the form

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2}\right\} = I_1 + iI_2,$$

where I_1 and I_2 are real, improper integrals. Use the line $\gamma = 1$.

10.4 Applications to Partial Differential Equations on Bounded Domains

Laplace transforms can be used to eliminate the time variable from initial boundary value problems. This reduces the PDE to an ODE or a PDE with one fewer variable. We illustrate with the following examples.

Example 8: Solve the heat conduction problem

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{26a}$$

$$U(0,t) = 0, \qquad t > 0, \tag{26b}$$

$$U(L,t) = 0, \qquad t > 0, \tag{26c}$$

$$U(x,0) = x, \qquad 0 < x < L. \tag{26d}$$

Solution: When we take Laplace transforms with respect to t on both sides of PDE (26a) and use property (7a),

$$s\tilde{U}(x,s) - x = k\frac{\partial^2 \tilde{U}}{\partial x^2}.$$

Thus, $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = -\frac{x}{k} \tag{27a}$$

subject to the transforms of (26b, c),

$$\tilde{U}(0,s) = 0, \tag{27b}$$

$$\tilde{U}(L,s) = 0. \tag{27c}$$

The general solution of ODE (27a) is

$$\tilde{U}(x,s) = C_1 \cosh\sqrt{\frac{s}{k}}x + C_2 \sinh\sqrt{\frac{s}{k}}x + \frac{x}{s},$$

and boundary conditions (27b, c) require that

$$0 = C_1, \qquad 0 = C_1 \cosh\sqrt{\frac{s}{k}}L + C_2 \sinh\sqrt{\frac{s}{k}}L + \frac{L}{s}.$$

From these,

$$\tilde{U}(x,s) = \frac{1}{s}\left(x - \frac{L\sinh\sqrt{s/k}\,x}{\sinh\sqrt{s/k}\,L}\right). \tag{28}$$

It remains now to find the inverse transform of $\tilde{U}(x,s)$.We do this by calculating the residues of $e^{st}\tilde{U}(x,s)$ at its singularities. To discover the nature of the singularity at $s = 0$, we expand $\tilde{U}(x,s)$ in a Laurent series around $s = 0$:

$$\begin{aligned}\tilde{U}(x,s) &= \frac{1}{s}\left(x - \frac{L[\sqrt{s/k}\,x + (\sqrt{s/k}\,x)^3/3! + \cdots]}{\sqrt{s/k}\,L + (\sqrt{s/k}\,L)^3/3! + \cdots}\right)\\ &= \frac{1}{s}\left(x - \frac{x + sx^3/(6k) + \cdots}{1 + sL^2/(6k) + \cdots}\right)\\ &= \frac{1}{s}\left(\frac{sx(L^2 - x^2)}{6k} + \cdots\right) = \frac{x(L^2 - x^2)}{6k} + \text{terms in } s, s^2, \ldots.\end{aligned}$$

It follows that $\tilde{U}(x,s)$ has a removable singularity at $s = 0$.

The remaining singularities of $\tilde{U}(x,s)$ occur at the zeros of $\sinh\sqrt{s/k}\,L$; that is, when $\sqrt{s/k}\,L = n\pi i$ or $s = -n^2\pi^2k/L^2$, n a positive integer. Because the derivative of $\sinh\sqrt{s/k}\,L$ does not vanish at $s = -n^2\pi^2k/L^2$, this function has zeros of order 1 at $s = -n^2\pi^2k/L^2$. It follows that $\tilde{U}(x,s)$ has poles of order 1 at these singularities, and, according to formula (25), the residue of $e^{st}\tilde{U}(x,s)$ at $s = -n^2\pi^2k/L^2$ is

$$\begin{aligned}&\lim_{s\to -n^2\pi^2k/L^2}\left(s + \frac{n^2\pi^2k}{L^2}\right)\frac{e^{st}}{s}\left(x - \frac{L\sinh\sqrt{s/k}\,x}{\sinh\sqrt{s/k}\,L}\right)\\ &\quad = -\frac{e^{-n^2\pi^2kt/L^2}}{-n^2\pi^2k/L^2}L\ \sinh\frac{n\pi xi}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{s + n^2\pi^2k/L^2}{\sinh\sqrt{s/k}\,L}.\end{aligned}$$

L'Hôpital's rule, together with the facts that $\sinh i\theta = i\sin\theta$ and $\cosh i\theta = \cos\theta$, yields

$$\frac{iL^3}{n^2\pi^2 k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\frac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}\,L}$$

$$=\frac{2iL^2}{n^2\pi^2 k}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\frac{1}{\frac{L}{n\pi ki}\cosh n\pi i}$$

$$=\frac{2L}{n\pi}(-1)^{n+1}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.$$

We sum these residues to find the inverse Laplace transform of $\tilde{U}(x,s)$:

$$U(x,t)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}. \tag{29}$$ ■

Before proceeding to further problems, some general comments are appropriate:

(1) In the above example, the Laplace transform was applied to the time variable t to eliminate the time derivative from the PDE and obtain an ODE in $\tilde{U}(x,s)$. The Laplace transform cannot be applied to the space variable x, because the range of x is only $0 \le x \le L$. It is the power of finite Fourier transforms to eliminate the space variable, not the Laplace transform. This is why Laplace transforms are applied to initial boundary value problems and not boundary value problems.

(2) The Laplace transform immediately incorporates the initial condition into the solution, and boundary conditions on $U(x,t)$ become boundary conditions for $\tilde{U}(x,s)$. Contrast this with finite Fourier transforms, which immediately incorporate boundary conditions and use the initial condition on $U(x,t)$ as an initial condition for $\tilde{U}(\lambda_n,t)$.

(3) Mathematically, the solution is not complete because the existence of a sequence of contours satisfying the properties of Theorem 7 has not been established, but we omit this part of the problem. We could circumvent this difficulty by now verifying that function (29) does indeed satisfy initial boundary value problem (26).

Problems with arbitrary initial conditions are more difficult to handle. This is illustrated in the next example.

Example 9: Solve the vibration problem

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2},\qquad 0<x<L,\qquad t>0, \tag{30a}$$

$$y(0,t)=0,\qquad t>0, \tag{30b}$$

$$y(L,t)=0,\qquad t>0, \tag{30c}$$

$$y(x,0) = f(x), \qquad 0 < x < L, \tag{30d}$$

$$y_t(x,0) = 0, \qquad 0 < x < L \tag{30e}$$

[see Exercise 10 in Section 3.2, with $g(x) = 0$].

Solution: When we take Laplace transforms of (30a) with respect to t and use initial conditions (30d, e) in property (7b),

$$s^2\tilde{y} - sf(x) = c^2\frac{\partial^2\tilde{y}}{\partial x^2}.$$

Thus, $\tilde{y}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{y}}{dx^2} - \frac{s^2}{c^2}\tilde{y} = -\frac{s}{c^2}f(x) \tag{31a}$$

subject to the transforms of (30b, c),

$$\tilde{y}(0,s) = 0, \qquad \tilde{y}(L,s) = 0. \tag{31b}$$

Variation of parameters (see Section 3.3) leads to the following form for the general solution of (31a):

$$\tilde{y}(x,s) = C_1\cosh\frac{sx}{c} + C_2\sinh\frac{sx}{c} - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du.$$

Boundary conditions (31b) on $\tilde{y}(x,s)$ require that

$$0 = C_1, \qquad 0 = C_1\cosh\frac{sL}{c} + C_2\sinh\frac{sL}{c} - \frac{1}{c}\int_0^L f(u)\sinh\frac{s}{c}(L-u)\,du,$$

from which

$$\tilde{y}(x,s) = \frac{\sinh\frac{sx}{c}}{c\sinh\frac{sL}{c}}\int_0^L f(u)\sinh\frac{s}{c}(L-u)\,du - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du$$

$$= \int_0^L f(u)\tilde{p}(x,u,s)\,du - \frac{1}{c}\int_0^x f(u)\sinh\frac{s}{c}(x-u)\,du, \tag{32a}$$

where

$$\tilde{p}(x,u,s) = \frac{\sinh\frac{sx}{c}\sinh\frac{s}{c}(L-u)}{c\sinh\frac{sL}{c}}. \tag{32b}$$

To obtain $y(x,t)$ by residues requires the singularities of $\tilde{y}(x,s)$. Provided $f(x)$ is piecewise continuous, integration with respect to u in (32a) and any differentiation with respect to s can be interchanged, and therefore the second integral in (32a) has no singularities. Singularities of the first integral are determined by those of $\tilde{p}(x,u,s)$. For the singularity at $s = 0$, we note that

$$\tilde{p}(x,u,s) = \frac{1}{c}\sinh\frac{s}{c}(L-u)\left(\frac{\sinh\frac{sx}{c}}{\sinh\frac{sL}{c}}\right)$$

$$= \frac{1}{c}\left(\frac{s}{c}(L-u) + \frac{s^3}{3!c^3}(L-u)^3 + \cdots\right)\left(\frac{\frac{sx}{c} + \frac{1}{3!}\left(\frac{sx}{c}\right)^3 + \cdots}{\frac{sL}{c} + \frac{1}{3!}\left(\frac{sL}{c}\right)^3 + \cdots}\right)$$

$$= \left(\frac{s}{c^2}(L-u) + \frac{s^3}{6c^4}(L-u)^3 + \cdots\right)\left(\frac{x + \frac{x^3s^2}{6c^2} + \cdots}{L + \frac{L^3s^2}{6c^2} + \cdots}\right)$$

$$= \frac{s}{c^2}(L-u)\frac{x}{L} + \text{terms in } s^2, s^3, \ldots,$$

and therefore $\tilde{p}(x,u,s)$ has a removable singularity at $s = 0$. The remaining singularities of $\tilde{p}(x,u,s)$ are $s = n\pi ci/L$, n a nonzero integer. Because the derivative of sinh (sL/c) does not vanish at $s = n\pi ci/L$, these singularities are poles of order 1. According to formula (25), the residue of $\tilde{p}(x,u,s)$ at $s = n\pi ci/L$ is

$$\lim_{s\to n\pi ci/L}\left(s - \frac{n\pi ci}{L}\right)\tilde{p}(x,u,s)$$

$$= \lim_{s\to n\pi ci/L}\left(s - \frac{n\pi ci}{L}\right)\frac{\sinh\frac{sx}{c}\sinh\frac{s}{c}(L-u)}{c\sinh\frac{sL}{c}}$$

$$= \sinh\frac{n\pi xi}{L}\sinh\frac{n\pi i(L-u)}{L}\lim_{s\to n\pi ci/L}\frac{s - \frac{n\pi ci}{L}}{c\sinh\frac{sL}{c}}$$

$$= -\sin\frac{n\pi x}{L}\sin\frac{n\pi}{L}(L-u)\lim_{s\to n\pi ci/L}\frac{1}{L\cosh\frac{sL}{c}} \quad \text{(by l'Hôpital's rule)}$$

$$= \frac{(-1)^n}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}\frac{1}{\cosh n\pi i}$$

$$= \frac{1}{L}\sin\frac{n\pi x}{L}\sin\frac{n\pi u}{L}.$$

The residue of e^{st} times the first integral in (32a) at $s = n\pi ci/L$ is now

$$\lim_{s \to n\pi ci/L} \left(s - \frac{n\pi ci}{L}\right) e^{st} \int_0^L f(u)\tilde{p}(x, u, s)\,du.$$

When we interchange the limit on s with the integration with respect to u, the residue becomes

$$\begin{aligned}
\int_0^L \lim_{s \to n\pi ci/L} &\left[e^{st}\left(s - \frac{n\pi ci}{L}\right) f(u)\tilde{p}(x, u, s)\right] du \\
&= \int_0^L e^{n\pi cti/L} f(u) \frac{1}{L} \sin\frac{n\pi x}{L} \sin\frac{n\pi u}{L}\,du \\
&= \frac{1}{L} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du.
\end{aligned}$$

The inverse transform of $\bar{y}(x, s)$ is the sum of all such residues:

$$y(x, t) = \frac{1}{L} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du. \tag{33}$$

To simplify this summation, we divide it into two parts,

$$\begin{aligned}
y(x, t) = &\frac{1}{L} \sum_{n=1}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du \\
&+ \frac{1}{L} \sum_{n=-\infty}^{-1} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du,
\end{aligned}$$

and replace n by $-n$ in the second term:

$$\begin{aligned}
y(x, t) = &\frac{1}{L} \sum_{n=1}^{\infty} e^{n\pi cti/L} \sin\frac{n\pi x}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du \\
&+ \frac{1}{L} \sum_{n=1}^{\infty} e^{-n\pi cti/L} \sin\left(\frac{-n\pi x}{L}\right) \int_0^L f(u) \sin\left(\frac{-n\pi u}{L}\right) du \\
= &\frac{1}{L} \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L} (e^{n\pi cti/L} + e^{-n\pi cti/L}) \int_0^L f(u) \sin\frac{n\pi u}{L}\,du \\
= &\frac{1}{L} \sum_{n=1}^{\infty} 2 \sin\frac{n\pi x}{L} \cos\frac{n\pi ct}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du \\
= &\sum_{n=1}^{\infty} a_n \cos\frac{n\pi ct}{L} \sin\frac{n\pi x}{L},
\end{aligned} \tag{34a}$$

where

$$a_n = \frac{2}{L} \int_0^L f(u) \sin\frac{n\pi u}{L}\,du. \tag{34b}$$

This is identical to the solution obtained by separation of variables in Exercise 10 of Section 3.2 when $g(x)$ is set equal to zero. ∎

Examples 8 and 9 were homogeneous problems. Convolutions can be used to handle problems with nonhomogeneities.

Example 10: Solve Example 8 if the end $x = 0$ of the rod has a prescribed temperature $f(t)$ and the initial temperature is zero throughout. Compare the solution with that obtained by eigenfunction expansions and finite Fourier transforms.

Solution: The initial boundary value problem in this case is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{35a}$$

$$U(0,t) = f(t), \qquad t > 0, \tag{35b}$$

$$U(L,t) = 0, \qquad t > 0, \tag{35c}$$

$$U(x,0) = 0, \qquad 0 < x < L. \tag{35d}$$

When the Laplace transform is applied to PDE (35a) and initial temperature (35d) is used, the transform function $\tilde{U}(x,s)$ must satisfy the ODE

$$\frac{d^2\tilde{U}}{dx^2} - \frac{s}{k}\tilde{U} = 0, \qquad 0 < x < L, \tag{36a}$$

$$\tilde{U}(0,s) = \tilde{f}(s), \tag{36b}$$

$$\tilde{U}(L,s) = 0. \tag{36c}$$

The solution of this system is

$$\tilde{U}(x,s) = \frac{\tilde{f}(s)\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}\,L}. \tag{37}$$

To find the inverse transform of this function, we first find the inverse of $\tilde{p}(x,s) = \sinh\sqrt{s/k}(L-x)/\sinh\sqrt{s/k}\,L$. This function has singularities when $\sqrt{s/k}\,L = n\pi i$ or $s = -n^2\pi^2k/L^2$, n a nonnegative integer. Expansion of $\tilde{p}(x,s)$ in a Laurent series around $s = 0$ immediately shows that $\tilde{p}(x,s)$ has a removable singularity at $s = 0$. The remaining singularities are poles of order 1, and the residue of $e^{st}\tilde{p}(x,s)$ at $s = -n^2\pi^2k/L^2$ is

$$\lim_{s\to -n^2\pi^2k/L^2}\left(s + \frac{n^2\pi^2k}{L^2}\right)e^{st}\frac{\sinh\sqrt{s/k}(L-x)}{\sinh\sqrt{s/k}\,L}$$

$$= e^{-n^2\pi^2kt/L^2}\sinh\frac{n\pi i(L-x)}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{s + n^2\pi^2k/L^2}{\sinh\sqrt{s/k}\,L}$$

$$= ie^{-n^2\pi^2kt/L^2}\sin\frac{n\pi(L-x)}{L}\lim_{s\to -n^2\pi^2k/L^2}\frac{1}{\dfrac{L}{2\sqrt{ks}}\cosh\sqrt{s/k}\,L}$$

$$= ie^{-n^2\pi^2kt/L^2}(-1)^{n+1}\sin\frac{n\pi x}{L}\,\frac{2nk\pi i}{L^2\cosh n\pi i}$$

$$= \frac{2nk\pi}{L^2}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.$$

Convolutions can now be used to invert $\tilde{U}(x,s)$ in (37):

$$U(x,t) = \mathscr{L}^{-1}[\tilde{f}(s)\tilde{p}(x,s)] = \int_0^t f(u)p(x,t-u)\,du$$
$$= \int_0^t f(u)\left(\frac{2k\pi}{L^2}\sum_{n=1}^{\infty} ne^{-n^2\pi^2k(t-u)/L^2}\sin\frac{n\pi x}{L}\right)du$$
$$= \frac{2k\pi}{L^2}\sum_{n=1}^{\infty} c_n(t)\sin\frac{n\pi x}{L}, \tag{38a}$$

where

$$c_n(t) = n\int_0^t f(u)e^{-n^2\pi^2k(t-u)/L^2}\,du. \tag{38b}$$

With eigenfunction expansions (from Section 3.3), the dependent variable is changed to $V(x,t) = U(x,t) - f(t)(1 - x/L)$, resulting in a problem with homogeneous boundary conditions for $V(x,t)$,

$$\frac{\partial V}{\partial t} = k\frac{\partial^2 V}{\partial x^2} - f'(t)\left(1 - \frac{x}{L}\right), \qquad 0 < x < L, \qquad t > 0,$$
$$V(0,t) = 0, \qquad t > 0,$$
$$V(L,t) = 0, \qquad t > 0,$$
$$V(x,0) = -f(0)\left(1 - \frac{x}{L}\right) = 0, \qquad 0 < x < L,$$

provided we assume that $f(0) = 0$. [The $f(0) \neq 0$ situation is discussed in Exercise 14.] An eigenfunction expansion

$$V(x,t) = \sum_{n=1}^{\infty} a_n(t)\sin\frac{n\pi x}{L}$$

leads to

$$a_n(t) = \frac{-2}{n\pi}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du,$$

and therefore

$$U(x,t) = f(t)\left(1 - \frac{x}{L}\right) - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du\right)\sin\frac{n\pi x}{L}. \tag{39}$$

That this is identical to (38) is verified by integrating (38b) by parts,

$$c_n(t) = n\left\{\frac{L^2}{n^2\pi^2k}f(u)e^{-n^2\pi^2k(t-u)/L^2}\right\}_0^t$$
$$- n\int_0^t \frac{L^2}{n^2\pi^2k}f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du$$
$$= \frac{L^2}{nk\pi^2}f(t) - \frac{L^2}{n\pi^2k}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du,$$

and substituting into (38a):

$$U(x,t) = \frac{2k\pi}{L^2}\sum_{n=1}^{\infty}\left(\frac{L^2}{nk\pi^2}f(t) - \frac{L^2}{nk\pi^2}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du\right)\sin\frac{n\pi x}{L}$$
$$= f(t)\sum_{n=1}^{\infty}\frac{2}{n\pi}\sin\frac{n\pi x}{L} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{1}{n}\int_0^t f'(u)e^{-n^2\pi^2k(t-u)/L^2}\,du\right)\sin\frac{n\pi x}{L}.$$

This is identical to (39) when we notice that the coefficients in the Fourier sine series of $1 - x/L$ are $2/(n\pi)$.

The finite Fourier transform

$$\tilde{f}(\lambda_n) = \int_0^L f(x)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}\,dx$$

applied to problem (35) gives the solution in form (38). ■

When we write solution (38) for problem (35) in the form

$$U(x,t) = \frac{2k\pi}{L^2}\sum_{n=1}^{\infty} b_n e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}, \tag{40a}$$

where

$$b_n = n\int_0^t f(u)e^{n^2\pi^2ku/L^2}\,du, \tag{40b}$$

we see that the exponentials in (40a) enhance convergence for large values of t. For instance, if the temperature of the left end is maintained at 100°C for $t > 0$, temperature function (40) reduces to

$$U(x,t) = \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}(1 - e^{-n^2\pi^2kt/L^2})\sin\frac{n\pi x}{L}, \tag{41}$$

which can also be expressed in the form

$$U(x,t) = 100\left(1 - \frac{x}{L}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}. \tag{42}$$

Suppose the rod is 1/5 m in length and is made from stainless steel with thermal diffusivity $k = 3.87 \times 10^{-6}$ m²/s. Consider finding the temperature at the midpoint $x = 1/10$ of the rod at the four times t = 2, 5, 30, and 100 min. Series (42) gives

$$U\left(\frac{1}{10}, 120\right) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.1145861n^2}\sin\frac{n\pi}{2}$$
$$= 0.10°\text{C};$$
$$U\left(\frac{1}{10}, 300\right) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-0.28646526n^2}\sin\frac{n\pi}{2}$$
$$= 3.80°\text{C};$$
$$U\left(\frac{1}{10}, 1800\right) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-1.7187915n^2}\sin\frac{n\pi}{2}$$
$$= 38.6°\text{C};$$

$$U\left(\frac{1}{10}, 6000\right) = 100\left(1 - \frac{1}{2}\right) - \frac{200}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}e^{-5.72930520n^2}\sin\frac{n\pi}{2}$$
$$= 49.8°\text{C}.$$

To obtain these temperatures, we required only four nonzero terms from the first series, three from the second, and one each from the third and fourth. This substantiates our claim that as t increases, fewer and fewer terms in series (42) are required for accurate calculations of temperature.

Laplace transforms can be used to give a completely different representation for the temperature in the rod when $f(t) = 100$. To find this representation, we return to expression (37) for the Laplace transform $\tilde{U}(x,s)$ of $U(x,t)$ and set $\tilde{f}(s) = 100/s$, the transform of $f(t) = 100$:

$$\tilde{U}(x,s) = \frac{100\sinh\sqrt{s/k}(L-x)}{s\sinh\sqrt{s/k}\,L} = \frac{100}{s}\frac{e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)}}{e^{\sqrt{s/k}L} - e^{-\sqrt{s/k}L}}$$
$$= \frac{100}{s}\frac{e^{-\sqrt{s/k}L}(e^{\sqrt{s/k}(L-x)} - e^{-\sqrt{s/k}(L-x)})}{1 - e^{-2\sqrt{s/k}L}}.$$

If we regard $1/(1 - e^{-2\sqrt{s/k}L})$ as the sum of an infinite geometric series with common ratio $e^{-2\sqrt{s/k}L}$, we may write

$$\tilde{U}(x,s) = \frac{100}{s}(e^{-\sqrt{s/k}x} - e^{-\sqrt{s/k}(2L-x)})\sum_{n=0}^{\infty}e^{-2n\sqrt{s/k}L}$$
$$= 100\sum_{n=0}^{\infty}\left(\frac{e^{-\sqrt{s/k}(2nL+x)}}{s} - \frac{e^{-\sqrt{s/k}[2(n+1)L-x]}}{s}\right). \tag{43}$$

Tables of Laplace transforms indicate that

$$\mathscr{L}^{-1}\left(\frac{e^{-a\sqrt{s}}}{s}\right) = \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right),$$

where $\operatorname{erfc}(x)$ is the complementary error function in equation (15). Hence, $U(x,t)$ may be expressed as a series of complementary error functions,

$$U(x,t) = 100\sum_{n=0}^{\infty}\left[\operatorname{erfc}\left(\frac{2nL+x}{2\sqrt{kt}}\right) - \operatorname{erfc}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right)\right]$$
$$= 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)L-x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{2nL+x}{2\sqrt{kt}}\right)\right], \tag{44}$$

where we have used the fact that $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$. This representation of $U(x,t)$ is valuable for small values of t [as opposed to (42), which converges rapidly for large t]. To understand this, consider temperature at the midpoint of the above stainless steel rod at $t = 300$ s:

$$U\left(\frac{1}{10}, 300\right) = 100\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{2(n+1)/5 - 1/10}{2\sqrt{3.87\times 10^{-6}(300)}}\right) - \operatorname{erf}\left(\frac{2n/5 + 1/10}{2\sqrt{3.87\times 10^{-6}(300)}}\right)\right].$$

For $n > 0$, all terms in this series essentially vanish, and

$$U\left(\frac{1}{10}, 300\right) = 100[\text{erf}(4.40) - \text{erf}(1.467)] = 3.80°\text{C}.$$

For $t = 1800$,

$$U\left(\frac{1}{10}, 1800\right) = 100 \sum_{n=0}^{\infty}\left[\text{erf}\left(\frac{2(n+1)/5 - 1/10}{2\sqrt{3.87 \times 10^{-6}(1800)}}\right) - \text{erf}\left(\frac{2n/5 + 1/10}{2\sqrt{3.87 \times 10^{-6}(1800)}}\right)\right].$$

Once again, only the $n = 0$ term is required; it yields

$$U\left(\frac{1}{10}, 1800\right) = 38.6°\text{C}.$$

Finally, for $t = 6000$,

$$U\left(\frac{1}{10}, 6000\right) = 100 \sum_{n=0}^{\infty}\left[\text{erf}\left(\frac{2(n+1)/5 - 1/10}{2\sqrt{3.87 \times 10^{-6}(6000)}}\right) - \text{erf}\left(\frac{2n/5 + 1/10}{2\sqrt{3.87 \times 10^{-6}(6000)}}\right)\right].$$

In this case, the $n = 0$ and $n = 1$ terms give

$$U\left(\frac{1}{10}, 6000\right) = 49.9°\text{C}.$$

For larger values of t, more and more terms of (44) are required.

The error function representation in (44) once again substantiates our claim in Section 5.6 that heat propagates with infinite speed. Because the error function is an increasing function of its argument, and the argument $(2nL + 2L - x)/(2\sqrt{kt})$ of the first error function in (44) is greater than the second argument, $(2nL + x)/(2\sqrt{kt})$, it follows that each term in (44) is positive. Since this is true for every x in $0 < x < L$ and every $t > 0$, the temperature at every point in the rod for every $t > 0$ is positive. This means that the effect of changing the temperature of the end $x = 0$ of the rod from 0°C to 100°C at time $t = 0$ is instantaneously felt at every point in the rod. The amount of heat transmitted to other parts of the rod may be minute, but nonetheless, heat is transmitted instantaneously to all parts of the rod.

Exercises 10.4

Use Laplace transforms to solve all problems in this set of exercises.

Part A—Heat Conduction

1. A homogeneous, isotropic rod with insulated sides has temperature $\sin m\pi x/L$ (m an integer) at time $t = 0$. For time $t > 0$, its ends ($x = 0$ and $x = L$) are held at temperature 0°C. Find a formula for temperature $U(x, t)$ in the rod for $0 < x < L$ and $t > 0$.
2. Solve Example 1 in Section 3.2 when the initial temperature is U_0 = constant.
3. Repeat Exercise 1 if the initial temperature is 10°C throughout.
4. Solve Exercise 8 in Section 3.3.

5. Repeat Exercise 4 if $g(x,t) = e^{-\alpha t}$. Assume that $\alpha \neq n^2\pi^2 k/L^2$ for any integer n.
6. (a) Repeat Exercise 5 if the initial temperature at time $t = 0$ is 10°C throughout.
   (b) Compare the solution with that obtained in Exercise 9 of Section 3.3.
7. Solve Exercise 2 in Section 3.2.
8. Solve Example 1 in Section 3.2 when the initial temperature is $f(x)$ (in place of x).
9. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For time $t > 0$, its left end, $x = 0$, is kept at 0°C and its right end, $x = L$, is kept at constant temperature U_L°C. Find two expressions for temperature in the rod, one in terms of exponentials in time and the other in terms of error functions.
10. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at constant temperature U_0°C throughout. For $t > 0$, its end $x = 0$ is insulated, and heat is added to the end $x = L$ at a constant rate Q W/m². Find the temperature in the rod for $0 < x < L$ and $t > 0$.
11. (a) A homogeneous, isotropic rod with insulated sides has, for time $t > 0$, its ends at $x = 0$ and $x = L$ kept at temperature zero. Initially its temperature is Ax, where A is constant. Show that temperature in the rod can be expressed in two ways:

$$U(x,t) = \frac{2AL}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}$$

and $$U(x,t) = A\left(x - L\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right) - \operatorname{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right)\right]\right).$$

   (b) Which of the two solutions do you expect to converge more quickly for small t? For large t?
   (c) Verify your conjecture in (b) by calculating the temperature at the midpoint of a stainless steel rod ($k = 3.87 \times 10^{-6}$) of length 1/5 m when $A = 500$ and
   (i) $t = 30$ s (ii) $t = 5$ min. (iii) $t = 100$ min.
12. A homogeneous, isotropic rod with insulated sides is initially ($t = 0$) at temperature 0°C throughout. For $t > 0$, its left end, $x = 0$, is kept at 0°C and heat is added to the end $x = L$ at a constant rate $Q > 0$ W/m². Find two series representations for $U(x,t)$, one in terms of error functions and one in terms of time exponentials.
13. Solve Exercise 13 in Section 6.2.
14. Show that the Laplace transform solution and the eigenfunction expansion solution to the problem in Example 10 are identical when $f(0) \neq 0$.
15. A homogeneous, isotropic rod with insulated sides has initial temperature distribution $U_L x/L$, $0 \le x \le L$ (U_L a constant). For time $t > 0$, its ends $x = 0$ and $x = L$ are held at temperatures 0°C and U_L°C, respectively. Find the temperature distribution in the rod for $t > 0$.
16. Repeat Exercise 15 if the initial temperature distribution is $f(x) = x$, $0 \le x \le L$, and the ends $x = 0$ and $x = L$ are held at constant temperatures U_0°C and 0°C, respectively, for $t > 0$.
17. Solve Exercise 5 in Section 3.3. (See also Exercise 8 in Section 6.2.)

Part B— Vibrations

18. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. If it is given an initial displacement (at time $t = 0$) of $f(x) = kx(L - x)$ (k a constant) and zero initial velocity, find its subsequent displacement.
19. Solve Exercise 8 in Section 3.2.

20. Repeat Exercise 18 for zero initial displacement and an unspecified initial velocity $g(x)$.
21. Solve Exercise 33(a) in Section 6.2.
22. Solve Exercise 23 in Section 6.2. Assume that $\omega \neq n\pi c/L$ for any integer n.
23. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. An external force (per unit x-length) $F = -ky$ $(k > 0)$ acts at each point on the string. Assuming an initial displacement $f(x)$ and a velocity $g(x)$, find subsequent displacements of the string.

For Exercises 24–29, solve Exercises 26–31 in Section 6.2.

30. Repeat Example 9 if gravity is taken into account. See also Exercise 37 in Section 6.2.
31. Solve Exercise 24 in Section 6.2.
32. Show that Laplace transforms lead to the solution in part (b) for the problem in Exercise 21 of Section 6.2.
33. (a) Find a series solution for displacements in the bar of Exercise 21 of Section 6.2 if the constant force per unit area F is replaced by an impulse force $F = F_0\delta(t)$. Use the fact that

$$\int_0^\infty f(t)\delta(t)\,dt = f(0+).$$

(b) Show that the displacement of the end $x = L$ is $cF_0/(AE)$ times the square wave function

$$M_{2L/c}(t) = \begin{cases} 1 & 0 < t < 2L/c \\ -1 & 2L/c < t < 4L/c \end{cases},$$

$$M_{2L/c}(t + 4L/c) = M_{2L/c}(t).$$

34. Solve Exercise 38 in Section 6.2.
35. A taut string of length L is initially at rest along the x-axis. For time $t > 0$, its ends are subjected to prescribed displacements

$$y(0, t) = f_1(t), \qquad y(L, t) = f_2(t).$$

Find its displacement for $0 < x < L$ and $t > 0$.

36. (a) Show that the Laplace transform of the displacement function $y(x, t)$ for the vibrations in Exercise 41 of Section 6.2 is

$$\tilde{y}(x, s) = \frac{F_0\omega c\sinh(sx/c)}{s(s^2 + \omega^2)[AE\cosh(sL/c) + mcs\sinh(sL/c)]}.$$

(b) Resonance occurs if either of the zeros $s = \pm i\omega$ of $s^2 + \omega^2$ coincides with a zero of

$$h(s) = AE\cosh\left(\frac{sL}{c}\right) + mcs\sinh\left(\frac{sL}{c}\right).$$

By expressing zeros of $h(s)$ in the form $s = c(\mu + i\lambda)$, show that

$$\tanh 2\mu L = \frac{-2AEmc^2\mu}{A^2E^2 + m^2c^4(\mu^2 + \lambda^2)}$$

and that therefore $\mu = 0$. Verify that resonance occurs if $\omega = c\lambda$ where λ is a root of the equation

$$\tan\lambda L = \frac{AE}{mc^2\lambda}.$$

37. Solve Example 3 in Section 3.2, but with an unspecified initial displacement $f(x)$. [*Hint:* Replace s by icq^2 in the ODE for $\bar{y}(x, s)$.]

38. (a) The top of the bar in Exercise 21 is attached to a spring with constant k. If $x = 0$ corresponds to the top end of the bar when the spring is unstretched, show that the Laplace transform of the displacement function for cross sections of the bar is

$$\bar{y}(x, s) = \frac{g}{s^3} - \frac{kgc\cosh[s(L-x)/c]}{s^3[AEs\sinh(sL/c) + kc\cosh(sL/c)]}.$$

(b) Verify that $\bar{y}(x, s)$ has a pole of order 1 at $s = 0$. What is the residue of $e^{st}\bar{y}(x, s)$ at $s = 0$?

(c) By setting $s = c(\mu + i\lambda)$ to obtain zeros of

$$h(s) = AEs\sinh\left(\frac{sL}{c}\right) + kc\cosh\left(\frac{sL}{c}\right),$$

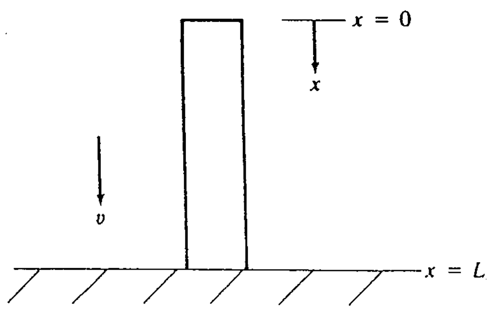
show that μ must be zero and that λ must satisfy

$$\tan\lambda L = \frac{k}{AE\lambda}.$$

(d) Find $y(x, t)$. (See also Exercise 34 in Section 6.2.)

39. (a) An unstrained elastic bar falls vertically under gravity with its axis in the vertical position (Figure 10.6). When its velocity is $v > 0$, it strikes a solid object and remains in contact with it thereafter. Show that the Laplace transform of displacements $y(x, t)$ of cross sections of the bar is

$$\bar{y}(x, s) = \left(\frac{v}{s^2} + \frac{g}{s^3}\right)\left(1 - \frac{\cosh(sx/c)}{\cosh(sL/c)}\right).$$



gure 10.6

(b) Use residues to find

$$y(x, t) = \frac{g(L^2 - x^2)}{2c^2} + \frac{8Lv}{\pi^2 c}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}$$
$$+ \frac{16L^2 g}{\pi^3 c^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{2L}\cos\frac{(2n-1)\pi x}{2L}.$$

(c) Verify that the second series in (b) may be expressed in the form

$$-\frac{g}{4c^2}(K(x + ct) + K(x - ct)),$$

where $K(x)$ is the even, odd-harmonic extension of $L^2 - x^2$, $0 \le x \le L$, to a function of period $4L$. (See Exercise 22 in Section 2.2 for the definition of an even, odd-harmonic function.)

(d) Verify that the first series in (b) may be expressed in the form

$$\frac{v}{2c}(M_L(x + ct) - M_L(x - ct)),$$

where $M_L(x)$ is the odd, odd-harmonic extension of x, $0 \le x \le L$, to a function of period $4L$. (See Exercise 21 in Section 2.2 for the definition of an odd, odd-harmonic function.)

(e) Find an expression for the force $F(t)$ due to the bar on the cross section at $x = L$. Sketch graphs of $F(t)$ when $v < 2Lg/c$ and $v > 2Lg/c$.

40. A bar 1/4 m long is falling as in Exercise 39 when it strikes an object squarely. Use the result of Exercise 39 to find a formula for the length of time of contact of the bar with the object. Use this formula to find the contact time for a steel bar with $\rho = 7.8 \times 10^3$ kg/m^3 and $E = 2.1 \times 10^{11}$ kg/m^2 when $v = 2$ m/s.

10.5 Laplace Transform Solutions to Problems in Polar, Cylindrical, and Spherical Coordinates

Laplace transforms can also be used to solve problems in polar, cylindrical, and spherical coordinates, but calculations are sometimes more difficult. We illustrate with the following examples.

Example 11: An infinitely long cylinder of radius r_2 is initially at temperature $f(r) = r_2^2 - r^2$, and for time $t > 0$, the boundary $r = r_2$ is insulated. Find the temperature in the cylinder for $t > 0$. (This problem was solved by separation of variables in Example 1 of Section 9.1.)

Solution: The initial boundary value problem for $U(r, t)$ is

$$\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r}\right), \qquad 0 < r < r_2, \qquad t > 0, \tag{45a}$$

$$\frac{\partial U(r_2, t)}{\partial r} = 0, \qquad t > 0, \tag{45b}$$

$$U(r, 0) = r_2^2 - r^2, \qquad 0 < r < r_2. \tag{45c}$$

When we take Laplace transforms of (45a) and use (45c),

$$s\tilde{U}(r, s) - (r_2^2 - r^2) = k\left(\frac{\partial^2 \tilde{U}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{U}}{\partial r}\right);$$

that is, $\tilde{U}(r, s)$ must satisfy the ODE

$$r\frac{d^2\tilde{U}}{dr^2} + \frac{d\tilde{U}}{dr} - \frac{sr}{k}\tilde{U} = \frac{r^3 - r_2^2 r}{k}, \qquad 0 < r < r_2 \tag{46a}$$

subject to the transform of boundary condition (45b),

$$\tilde{U}'(r_2, s) = 0. \tag{46b}$$

The change of independent variable $u = i\sqrt{s/k}r$ replaces the homogeneous equation

$$r\frac{d^2\tilde{U}}{dr^2} + \frac{d\tilde{U}}{dr} - \frac{sr}{k}\tilde{U} = 0 \tag{47}$$

with

$$u\frac{d^2\tilde{U}}{du^2} + \frac{d\tilde{U}}{du} + u\tilde{U} = 0. \tag{48}$$

This is Bessel's differential equation of order zero, with general solution

$$AJ_0(u) + BY_0(u).$$

Thus, the general solution of (47) is

$$AJ_0\left(i\sqrt{\frac{s}{k}}r\right) + BY_0\left(i\sqrt{\frac{s}{k}}r\right). \tag{49}$$

When the particular solution $-r^2/s + (r_2^2 s - 4k)/s^2$ of (46a) is added to (49), the general solution of (46a) is

$$\tilde{U}(r,s) = AJ_0\left(i\sqrt{\frac{s}{k}}r\right) + BY_0\left(i\sqrt{\frac{s}{k}}r\right) - \frac{4k}{s^2} + \frac{r_2^2 - r^2}{s}. \tag{50}$$

Because $U(r,t)$ must remain bounded as r approaches zero, so also must $\tilde{U}(r,s)$. This implies that B must vanish, in which case boundary condition (46b) requires that

$$i\sqrt{\frac{s}{k}}AJ_0'\left(i\sqrt{\frac{s}{k}}r_2\right) - \frac{2r_2}{s} = 0.$$

When this equation is solved for A and the result is substituted into (50),

$$\tilde{U}(r,s) = \frac{2r_2 J_0(i\sqrt{s/k}r)}{i\sqrt{s^3/k}\,J_0'(i\sqrt{s/k}r_2)} - \frac{4k}{s^2} + \frac{r_2^2 - r^2}{s}. \tag{51}$$

This function has singularities at $s = 0$ and values of s satisfying $J_0'(i\sqrt{s/k}r_2) = 0$. If we set $\sqrt{s/k}\,i = \lambda_n$, singularities occur for $s = -k\lambda_n^2$ where $J_0'(\lambda_n r_2) = 0$. Power series (15) in Section 8.3 can be used to expand $\tilde{U}(r,s)$ about $s = 0$:

$$\begin{aligned}\tilde{U}(r,s) &= \frac{2\sqrt{k}r_2}{is^{3/2}}\left(\frac{1 - \dfrac{(i\sqrt{s/k}r)^2}{4} + \dfrac{(i\sqrt{s/k}r)^4}{64} - \cdots}{-\dfrac{(i\sqrt{s/k}r_2)}{2} + \dfrac{(i\sqrt{s/k}r_2)^3}{16} - \cdots}\right) - \frac{4k}{s^2} + \frac{r_2^2 - r^2}{s}\\ &= \frac{2\sqrt{k}r_2}{is^{3/2}}\left[-\frac{2}{i\sqrt{s/k}r_2} - \frac{i\sqrt{s/k}}{r_2}\left(\frac{r_2^2}{4} - \frac{r^2}{2}\right) + \cdots\right] - \frac{4k}{s^2} + \frac{r_2^2 - r^2}{s}\\ &= \frac{r_2^2}{2s} + \cdots.\end{aligned}$$

When this result is multiplied by e^{st},

$$e^{st}\tilde{U}(r,s) = \left(1 + st + \frac{s^2t^2}{2} + \cdots\right)\left(\frac{r_2^2}{2s} + \cdots\right),$$

it is clear that the residue of $e^{st}\tilde{U}(r,s)$ at $s=0$ is $r_2^2/2$. Because the derivative of J_0' does not vanish at its zeros, the remaining singularities at $s=-k\lambda_n^2$ are poles of order 1, and the residues of $e^{st}\tilde{U}(r,s)$ at these poles are

$$\lim_{s\to -k\lambda_n^2}(s+k\lambda_n^2)e^{st}\left(\frac{2r_2J_0(i\sqrt{s/k}\,r)}{i\sqrt{s^3/k}\,J_0'(i\sqrt{s/k}\,r_2)}-\frac{4k}{s^2}+\frac{r_2^2-r^2}{s}\right)$$

$$=\frac{2r_2}{-k\lambda_n^3}e^{-k\lambda_n^2t}J_0(\lambda_n r)\lim_{s\to -k\lambda_n^2}\frac{s+k\lambda_n^2}{J_0'(i\sqrt{s/k}\,r_2)}$$

$$=\frac{-2r_2}{k\lambda_n^3}e^{-k\lambda_n^2t}J_0(\lambda_n r)\lim_{s\to -k\lambda_n^2}\frac{1}{\frac{ir_2}{2\sqrt{ks}}J_0''(i\sqrt{s/k}\,r_2)} \qquad \text{(by l'Hôpital's rule)}$$

$$=\frac{-4}{k\lambda_n^3}e^{-k\lambda_n^2t}J_0(\lambda_n r)\frac{1}{\frac{-1}{k\lambda_n}J_0''(\lambda_n r_2)}$$

$$=\frac{4}{\lambda_n^2J_0''(\lambda_n r_2)}e^{-k\lambda_n^2t}J_0(\lambda_n r).$$

But, because $J_0(\lambda_n r)$ satisfies equation (47) when $s=-k\lambda_n^2$,

$$r\frac{d^2J_0(\lambda_n r)}{dr^2}+\frac{dJ_0(\lambda_n r)}{dr}+\lambda_n^2 rJ_0(\lambda_n r)=0$$

or

$$\lambda_n^2 rJ_0''(\lambda_n r)+\lambda_n J_0'(\lambda_n r)+\lambda_n^2 rJ_0(\lambda_n r)=0.$$

When we set $r=r_2$ in this equation and note that $J_0'(\lambda_n r_2)=0$, we obtain

$$J_0''(\lambda_n r_2)=-J_0(\lambda_n r_2).$$

Residues of $e^{st}\tilde{U}(r,s)$ at $s=-k\lambda_n^2$ can therefore be expressed as

$$\frac{-4}{\lambda_n^2J_0(\lambda_n r_2)}e^{-k\lambda_n^2t}J_0(\lambda_n r).$$

The sum of the residues at $s=0$ and $s=-k\lambda_n^2$ yields the temperature function

$$U(r,t)=\frac{r_2^2}{2}-4\sum_{n=1}^{\infty}\frac{e^{-k\lambda_n^2t}J_0(\lambda_n r)}{\lambda_n^2J_0(\lambda_n r_2)}. \qquad \blacksquare \quad (52)$$

The following vibration problem has a nonhomogeneous boundary condition.

Example 12: A circular membrane of radius r_2 is initially at rest on the xy-plane. Find its displacement for time $t>0$ if its edge is forced to undergo periodic oscillations described by $A\sin\omega t$, A a constant.

Solution: The initial boundary value problem for displacements $z(r,t)$ of the membrane is

$$\frac{\partial^2 z}{\partial t^2}=c^2\left(\frac{\partial^2 z}{\partial r^2}+\frac{1}{r}\frac{\partial z}{\partial r}\right), \qquad 0<r<r_2, \qquad t>0, \qquad (53a)$$

$$z(r_2,t)=A\sin\omega t, \qquad t>0, \qquad (53b)$$

$$z(r,0) = 0, \qquad 0 < r < r_2, \tag{53c}$$
$$z_t(r,0) = 0, \qquad 0 < r < r_2. \tag{53d}$$

When we apply the Laplace transform to PDE (53a) and use initial conditions (53c, d),

$$s^2\tilde{z} = c^2\left(\frac{d^2\tilde{z}}{dr^2} + \frac{1}{r}\frac{d\tilde{z}}{dr}\right);$$

that is, $\tilde{z}(r,s)$ must satisfy

$$r\frac{d^2\tilde{z}}{dr^2} + \frac{d\tilde{z}}{dr} - \frac{s^2 r}{c^2}\tilde{z} = 0 \tag{54a}$$

subject to

$$\tilde{z}(r_2, s) = \frac{A\omega}{s^2 + \omega^2}. \tag{54b}$$

The change of independent variable $u = isr/c$ replaces this equation with

$$u\frac{d^2\tilde{z}}{du^2} + \frac{d\tilde{z}}{du} + u\tilde{z} = 0, \tag{55}$$

Bessel's differential equation of order zero. Since the general solution of (55) is $BJ_0(u) + DY_0(u)$, it follows that

$$\tilde{z}(r,s) = BJ_0\left(\frac{isr}{c}\right) + DY_0\left(\frac{isr}{c}\right). \tag{56}$$

Because $z(r,t)$ must remain bounded as r approaches zero, so also must $\tilde{z}(r,s)$. This implies that D must vanish, in which case boundary condition (54b) requires that

$$\frac{A\omega}{s^2 + \omega^2} = BJ_0\left(\frac{isr_2}{c}\right).$$

When this equation is solved for B and the result is substituted into (56),

$$\tilde{z}(r,s) = \frac{A\omega}{s^2 + \omega^2}\,\frac{J_0(isr/c)}{J_0(isr_2/c)}. \tag{57}$$

This function has singularities at $s = \pm i\omega$ and values of s satisfying $J_0(isr_2/c) = 0$. If we set $is/c = \lambda_n$, singularities occur for $s = -ic\lambda_n$ where $J_0(\lambda_n r_2) = 0$. (For every positive value of λ_n satisfying this equation, $\lambda_{-n} = -\lambda_n$ is also a solution.) Provided $\omega \neq c\lambda_n$ for any n, all singularities are poles of order 1. The residue of $e^{st}\tilde{z}(r,s)$ at $s = i\omega$ is

$$\begin{aligned}\lim_{s\to i\omega}(s - i\omega)e^{st}\tilde{z}(r,s) &= \lim_{s\to i\omega}(s - i\omega)\frac{A\omega e^{st}}{(s + i\omega)(s - i\omega)}\,\frac{J_0(isr/c)}{J_0(isr_2/c)}\\ &= \frac{A\omega e^{i\omega t}}{2i\omega}\,\frac{J_0(-\omega r/c)}{J_0(-\omega r_2/c)}\\ &= -\frac{i}{2}Ae^{i\omega t}\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)}.\end{aligned}$$

Similarly, the residue of $e^{st}\bar{z}(r,s)$ at $s = -i\omega$ is

$$\frac{i}{2}Ae^{-i\omega t}\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)}.$$

The residues of $e^{st}\bar{z}(r,s)$ at $s = -ic\lambda_n$ are

$$\lim_{s\to -ic\lambda_n}(s + ic\lambda_n)e^{st}\frac{A\omega}{s^2+\omega^2}\frac{J_0(isr/c)}{J_0(isr_2/c)}$$
$$= \frac{A\omega}{\omega^2 - c^2\lambda_n^2}e^{-ic\lambda_n t}J_0(\lambda_n r)\lim_{s\to -ic\lambda_n}\frac{s+ic\lambda_n}{J_0(isr_2/c)}$$
$$= \frac{A\omega}{\omega^2 - c^2\lambda_n^2}e^{-ic\lambda_n t}J_0(\lambda_n r)\lim_{s\to -ic\lambda_n}\frac{1}{\frac{ir_2}{c}J_0'(isr_2/c)} \quad \text{(by l'Hôpital's rule)}$$
$$= \frac{-iA\omega ce^{-ic\lambda_n t}}{r_2(\omega^2 - c^2\lambda_n^2)}J_0(\lambda_n r)\frac{1}{J_0'(\lambda_n r_2)} = \frac{iA\omega ce^{-ic\lambda_n t}}{r_2(\omega^2 - c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}.$$

The sum of the residues at $s = \pm i\omega$ and $s = -ic\lambda_n$ yields the displacement of the membrane,

$$z(r,t) = -\frac{i}{2}Ae^{i\omega t}\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)} + \frac{i}{2}Ae^{-i\omega t}\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)} + \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{iA\omega ce^{-ic\lambda_n t}}{r_2(\omega^2 - c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$$
$$= A\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)}\left(\frac{e^{i\omega t} - e^{-i\omega t}}{2i}\right) + \sum_{n=1}^{\infty}\frac{iA\omega ce^{-ic\lambda_n t}}{r_2(\omega^2 - c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$$
$$+ \sum_{n=-1}^{-\infty}\frac{iA\omega ce^{-ic\lambda_n t}}{r_2(\omega^2 - c^2\lambda_n^2)}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$$
$$= A\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)}\sin\omega t + \frac{iA\omega c}{r_2}\sum_{n=1}^{\infty}\frac{e^{-ic\lambda_n t}}{\omega^2 - c^2\lambda_n^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$$
$$+ \frac{iA\omega c}{r_2}\sum_{n=1}^{\infty}\frac{e^{-ic\lambda_{-n} t}}{\omega^2 - c^2(\lambda_{-n})^2}\frac{J_0(\lambda_{-n} r)}{J_1(\lambda_{-n} r_2)}.$$

Since $\lambda_{-n} = -\lambda_n$, and J_0 and J_1 are even and odd functions, respectively, it follows that

$$z(r,t) = A\sin\omega t\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)} - \frac{iA\omega c}{r_2}\sum_{n=1}^{\infty}\frac{e^{ic\lambda_n t} - e^{-ic\lambda_n t}}{\omega^2 - c^2\lambda_n^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$$
$$= A\sin\omega t\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)} + \frac{2A\omega c}{r_2}\sum_{n=1}^{\infty}\frac{1}{\omega^2 - c^2\lambda_n^2}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}\sin c\lambda_n t. \quad (58)$$

The solution of this problem, obtained by finite Fourier transforms in Exercise 16 of Section 9.2, is

$$z(r,t) = -\frac{2Ac}{r_2}\sum_{n=1}^{\infty}\frac{c\lambda_n\sin\omega t - \omega\sin c\lambda_n t}{(\omega^2 - c^2\lambda_n^2)J_1(\lambda_n r_2)}J_0(\lambda_n r).$$

The Laplace transform solution is preferable; it expresses part of the finite Fourier transform solution in closed form. ■

Exercises 10.5

Part A—Heat Conduction

1. Solve Exercise 1(b) in Section 9.1.
2. Solve Exercise 1(c) in Section 9.1.
3. Laplace transforms do not handle problems in polar coordinates efficiently when initial conditions contain unspecified functions. To illustrate this, find the Laplace transform of the PDE for Exercise 1(a) in Section 9.1. How difficult is it to solve the ODE in $\tilde{U}(r,s)$?
4. Solve Example 5 in Section 9.2.
5. **(a)** An infinitely long cylinder of radius r_2 is initially at temperature 0°C throughout. If the surface $r = r_2$ has variable temperature $f(t)$ for $t > 0$, find the temperature inside the cylinder.
   (b) Simplify the solution when $f(t) = \bar{U}$, a constant. Do you obtain the solution to Exercise 4?
6. Solve Exercise 2(b) in Section 9.2.
7. **(a)** A cylinder occupying the region $0 \le r \le r_2$, $0 \le z \le L$, is initially at constant temperature U_0°C throughout. What is the initial boundary value problem for temperature in the cylinder if its surface is held at 0°C for $t > 0$?
   (b) If a finite Fourier transform is used to remove the z-variable from the problem in $U(r, z, t)$, what is the initial boundary value problem for $\tilde{U}(r, \mu_m, t)$ (where $\mu_m = m\pi/L$ are eigenvalues associated with this transform)?
   (c) Show that when the Laplace transform is applied to the PDE in $\tilde{U}(r, \mu_m, t)$, the transform function $\tilde{\tilde{U}}(r, \mu_m, s)$ must satisfy

$$r\frac{d^2\tilde{\tilde{U}}}{dr^2} + \frac{d\tilde{\tilde{U}}}{dr} - r\left(\frac{s}{k} + \mu_m^2\right)\tilde{\tilde{U}} = -\frac{rU_0\tilde{1}}{k}, \qquad 0 < r < r_2,$$

$$\tilde{\tilde{U}}(r_2, \mu_m, s) = 0,$$

   where $\tilde{1} = \sqrt{2L}[1 + (-1)^{m+1}]/(m\pi)$ is the finite Fourier transform of the unity function.
   (d) Verify that the solution for $\tilde{\tilde{U}}(r, \mu_m, s)$ is

$$\tilde{\tilde{U}}(r, \mu_m, s) = \frac{U_0\tilde{1}}{s + k\mu_m^2}\left(1 - \frac{J_0(i\sqrt{\mu_m^2 + s/k}\,r)}{J_0(i\sqrt{\mu_m^2 + s/k}\,r_2)}\right).$$

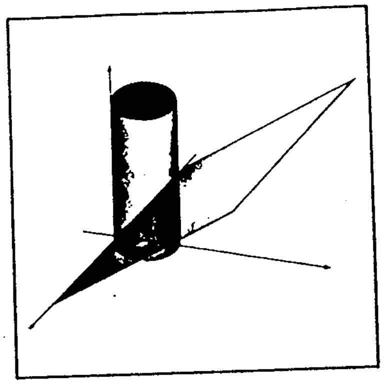
   (e) Prove that $\tilde{\tilde{U}}(r, \mu_m, s)$ has a removable singularity at $s = -k\mu_m^2$ and poles of order 1 at $s = -k(\lambda_n^2 + \mu_m^2)$ where $J_0(\lambda_n r_2) = 0$. Show that the residues of $e^{st}\tilde{\tilde{U}}(r, \mu_m, s)$ at these poles are

$$\frac{2U_0\tilde{1}}{r_2\lambda_n} e^{-k(\lambda_n^2 + \mu_m^2)t} \frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}.$$

   (f) Finally, invert the Laplace transform and the finite Fourier transform to find $U(r, z, t)$.

Part B—Vibrations

8. Solve Exercise 19 in Section 9.1.
9. Solve Exercise 20 in Section 9.1.
10. Solve Exercise 17 in Section 9.2 in the nonresonance case.



CHAPTER

ELEVEN

Green's Functions for Ordinary Differential Equations

11.1 Generalized Functions

To solve many physical problems, we create mathematical idealizations called "point" entities—point charges, point masses, point heat sources, and point forces, to name a few. For example, suppose a 1-N force is applied to the midpoint of a taut string (of negligible mass) as shown in Figure 11.1. The boundary value problem that describes static deflections of the string is

$$-\tau \frac{d^2 y}{dx^2} = F(x), \qquad 0 < x < L, \tag{1a}$$

$$y(0) = 0 = y(L), \tag{1b}$$

where τ is the constant tension in the string and $F(x)$ is the force per unit x-length on the string due to the applied force. Although it would seem to be a simple procedure to integrate the differential equation twice and apply the boundary conditions (for determination of constants of integration), integration of $F(x)$ presents a problem. If we use

$$F(x) = \begin{cases} 0 & 0 < x < L/2 \\ 1 & x = L/2 \\ 0 & L/2 < x < L \end{cases} \tag{2}$$

as the definition of $F(x)$, antidifferentiation gives

$$y(x) = \begin{cases} Ax + B & 0 < x < L/2 \\ Cx + D & L/2 < x < L \end{cases}.$$

(Recall from elementary calculus that we antidifferentiate only over an interval, not at a point; hence the absence of an antiderivative "at" $x = L/2$.) If we now apply boundary conditions (1b) and demand that $y(x)$ be continuous at $x = L/2$, we obtain

$$y(x) = \begin{cases} Ax & 0 \le x \le L/2 \\ -A(x - L) & L/2 \le x \le L \end{cases}.$$

But how do we calculate A? Certainly the size of the force (1 N here) and the tension τ in the string must determine A, but there seems to be no way to use this information. The problem must be representation (2) for a point force concentrated at $x = L/2$. Perhaps what we should do is distribute this force along the string, solve the problem, and then take a limit as the distributed force approaches a concentrated force. There is a multitude of ways that $F(x)$ might be defined, but clearly each must satisfy the condition

$$\int_0^L F(x)\,dx = 1. \tag{3}$$

Two possibilities, which are symmetric, are shown in Figure 11.2.

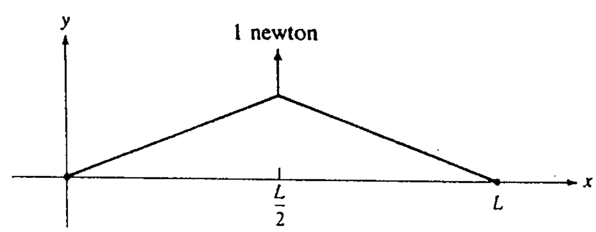


Figure 11.1

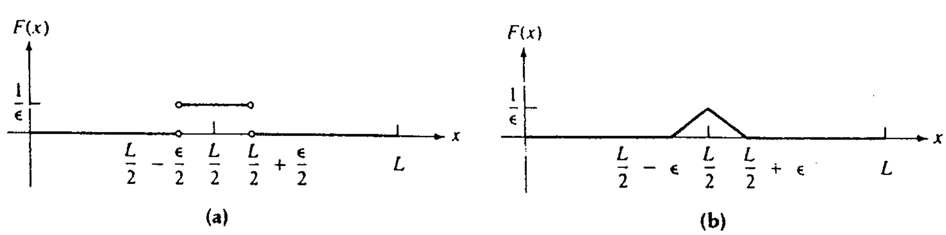


Figure 11.2

Suppose we solve the boundary value problem for $y(x)$ using the distribution in Figure 11.2(a). Then

$$\frac{d^2y}{dx^2} = -\frac{1}{\tau}\begin{cases} 0 & 0 < x < (L-\varepsilon)/2 \\ 1/\varepsilon & (L-\varepsilon)/2 < x < (L+\varepsilon)/2. \\ 0 & (L+\varepsilon)/2 < x < L \end{cases}$$

Integration leads to

$$y(x) = -\frac{1}{\tau}\begin{cases} Ax + B & 0 < x < (L-\varepsilon)/2 \\ x^2/(2\varepsilon) + Cx + D & (L-\varepsilon)/2 < x < (L+\varepsilon)/2. \\ Ex + F & (L+\varepsilon)/2 < x < L \end{cases}$$

If we apply boundary conditions (1b) and demand that $y(x)$ and $y'(x)$ be continuous at $x = (L-\varepsilon)/2$ and $x = (L+\varepsilon)/2$, we find that

$$y(x) = \frac{1}{\tau}\begin{cases} \dfrac{x}{2} & 0 \le x \le (L-\varepsilon)/2 \\ -\dfrac{x^2}{2\varepsilon} + \dfrac{Lx}{2\varepsilon} - \dfrac{1}{8\varepsilon}(L-\varepsilon)^2 & (L-\varepsilon)/2 \le x \le (L+\varepsilon)/2, \\ \dfrac{L-x}{2} & (L+\varepsilon)/2 \le x \le L \end{cases} \tag{4}$$

the graph of which is shown in Figure 11.3. To obtain the solution of (1) for a concentrated force, we now let ε approach zero. Geometrically, the parabolic section becomes smaller and smaller in width, and in the limit the two straight-line sections meet at $x = L/2$ (Figure 11.4). This implies that the displacement at $L/2$ is $L/(4\tau)$ and the displacement function for the unit point force in Figure 11.1 is that in Figure 11.4, defined algebraically by

$$y(x) = \begin{cases} x/(2\tau) & 0 \le x \le L/2 \\ (L-x)/(2\tau) & L/2 \le x \le L \end{cases} \tag{5}$$

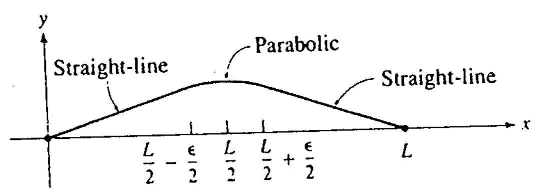


Figure 11.3

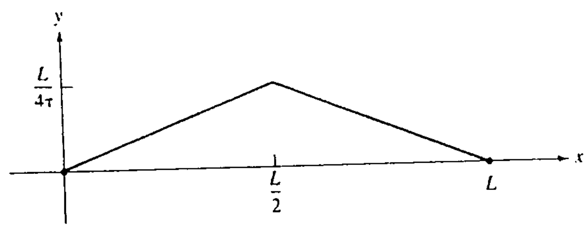


Figure 11.4

In Exercise 7, displacement $y(x)$ for the distributed load in Figure 11.2(b) is calculated. Although it is different from (4), its limit as ε approaches zero is once again (5).

We have attempted to illustrate with this one example that problems containing point sources can be solved with distributed sources and limits. This example and other physical situations in the exercises make it abundantly clear, however, that the method is extremely cumbersome. It is the purpose of this chapter and the next to develop representations for concentrated sources that are effective in solving ordinary and partial differential equations.

When we solve linear, second-order differential equations

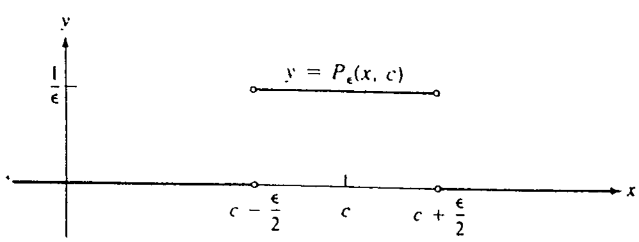
$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = f(x),$$

where $P(x)$, $Q(x)$, and $R(x)$ are continuous and $f(x)$ is piecewise continuous, the solution should be continuous and have a continuous first derivative. In fact, for the distributed load of Figure 11.2(a) we actually imposed these conditions at $x = (L \pm \varepsilon)/2$ to obtain displacement (4). But notice that limit function (5), shown in Figure 11.4, has a discontinuity in $y'(x)$ at $x = L/2$. In other words, when "point" sources influence second-order boundary value problems, we cannot expect solutions to have continuous first derivatives.

To begin our search for representations of concentrated sources, suppose that we have a time-independent one-dimensional problem along the x-axis (perhaps static deflections of a string, or steady-state heat conduction in a rod, or potential). We wish to define a function, which we denote by $\delta(x - c)$, to represent a unit point source at $x = c$. Based on the above example (where the unit force was distributed over an interval on the x-axis), it might seem reasonable to define $\delta(x - c)$ as the limit as $\varepsilon \to 0$ of the unit pulse function $P_\varepsilon(x, c)$ in Figure 11.5, that is, define

$$\delta(x - c) = \lim_{\varepsilon \to 0} P_\varepsilon(x, c). \tag{6}$$

Because the area under $P_\varepsilon(x, c)$ is unity for any $\varepsilon > 0$, this definition appears to preserve the "unit" character of the source. But, from the point of view of a function as a mapping from domain to range, definition (6) is unacceptable. It maps all values $x \neq c$ onto zero, and the value of $\delta(x - c)$ at $x = c$ is somehow "infinite." What we are saying is that $\delta(x - c)$ cannot be defined in a pointwise sense; functions that represent point sources require a completely new approach.



ure 11.5

To introduce this approach, recall that when $y_n(x)$ are normalized eigenfunctions of a Sturm-Liouville system on an interval $a \le x \le b$, and $f(x)$ is suitably behaved, the finite Fourier transform of $f(x)$ is

$$\tilde{f}(\lambda_n) = \int_a^b p(x) f(x) y_n(x)\, dx$$

[$p(x)$ is the weight function of the Sturm-Liouville system]. By this definition, each eigenfunction $y_n(x)$ associates with a function $f(x)$ its nth Fourier coefficient $\tilde{f}(\lambda_n)$,

$$f(x) \xrightarrow{y_n(x)} \tilde{f}(\lambda_n).$$

We have, then, an infinity of mappings $y_n(x)$. Each maps functions onto reals, and the real numbers are calculated by means of integrals. Such mappings are not restricted to eigenfunctions arising from Sturm-Liouville systems, however. We can associate such a mapping with any continuous function whatsoever. Indeed, if $g(x)$ is continuous on an interval $a \le x \le b$, we can associate an integral mapping with $g(x)$ according to

$$f(x) \xrightarrow{g(x)} \int_a^b f(x) g(x)\, dx;$$

that is, $g(x)$ is a *functional*, or *operator*, which maps functions $f(x)$ onto real numbers, and these numbers are defined by integrals. It is this view of an ordinary function as a functional or operator that we adopt to define $\delta(x - c)$. The "generalized" function[†] $\delta(x - c)$, called the (*Dirac*)[‡] *delta function*, is the functional that maps a function $f(x)$, continuous at $x = c$, onto its value at $x = c$,

$$f(x) \xrightarrow{\delta(x-c)} f(c).$$

For example,

$$x^2 + 2x - 3 \xrightarrow{\delta(x-2)} 5$$

and

$$(x + 1)^2 \cos x \xrightarrow{\delta(x)} 1.$$

In order that the delta functional have an integral representation, we write

$$f(x) \xrightarrow{\delta(x-c)} f(c) = \int_{-\infty}^{\infty} f(x)\delta(x - c)\, dx. \tag{7}$$

But because $\delta(x - c)$ cannot be regarded pointwise, the multiplication in this integral, and the integral itself, are symbolic. When we encounter an integral such as that in (7), we interpret it as the action of the functional $\delta(x - c)$ operating on $f(x)$ and immediately write $f(c)$. For example,

$$\int_{-\infty}^{\infty} \left(x^2 + \frac{2}{x - 1}\right) \delta(x)\, dx = -2$$

[†] A complete treatment of generalized functions can be found in M. J. Lighthill, *Introduction to Fourier Analysis and Generalized Functions* (Cambridge, England: Cambridge University Press, 1958).
[‡] After the mathematical physicist Paul Dirac.

and $$\int_{-\infty}^{\infty} \delta(x+2)\,dx = 1$$

[since the left side of the latter integral is interpreted as the delta function $\delta(x+2)$ operating on the function $f(x) \equiv 1$].

Because $\delta(x-c)$ picks out the value of a function at $x = c$, we write

$$\int_a^b f(x)\delta(x-c)\,dx = f(c) \qquad \textbf{(8a)}$$

as long as $a < c < b$; that is, the limits on the integral need not be $\pm\infty$. Furthermore, if $x = c$ is not between a and b, we set

$$\int_a^b f(x)\delta(x-c)\,dx = 0. \qquad \textbf{(8b)}$$

For instance,

$$\int_{-2}^{6} \sqrt{x+5}\,\delta(x)\,dx = \sqrt{5}$$

and $$\int_2^3 (x^2+2x-4)\delta(x+1)\,dx = 0.$$

From a functional point of view, it is not at all clear that the delta function $\delta(x-c)$ represents a point source at $x = c$. Our first evidence of this appears in the next section.

Exercises 11.1

In Exercises 1–6, evaluate the integral.

1. $\int_{-\infty}^{\infty} (x^2 - 2x + 4)\delta(x-1)\,dx$

2. $\int_{-8}^{3} \sin(3x+1)\delta(x)\,dx$

3. $\int_{-4}^{20} (e^x + x^2)\delta(x+3)\,dx$

4. $\int_{3}^{\infty} (x^2 + 1/x)\delta(x)\,dx$

5. $\int_{-\infty}^{\infty} (2x^2 + x^3 + 4)\delta(x-4)\,dx$

6. $\int_{-\infty}^{\infty} (1 + 4x - \cos x)\delta(x+10)\,dx$

7. Solve problem (1) when $F(x)$ is defined as in Figure 11.2(b), and sketch the displacement function. Show that the displacement of Figure 11.4 is obtained in the limit as $\varepsilon \to 0^+$.

8. Define your own distributed force function $F(x)$ [subject to condition (3)] and solve problem (1), taking limits as $F(x)$ approaches a point force. Do you obtain the result in Figure 11.4?

9. Calculate the displacement of a taut string (of negligible mass and length L) when two unit point masses are attached at distances $L/3$ from each end. Use distribution functions like that in Figure 11.2(a) for each mass.

10. A beam of length L and negligible weight is subjected to a unit load at its midpoint. If the left end of the beam ($x = 0$) is fixed horizontally and the right end ($x = L$) is free, use a distributed load like that of Figure 11.2(a) and limits as $\varepsilon \to 0^+$ to find the static deflection of the beam. Sketch the graph of the displacement function. Are $y'(x)$, $y''(x)$, and $y'''(x)$ continuous?

11. Find deflections of the beam in Exercise 10 if the point load is placed at the end $x = L$.
12. The displacement of a mass M from its equilibrium position at the end of a spring with constant k is described by the differential equation

$$M\frac{d^2y}{dt^2} + ky = F(t)$$

when viscous damping is negligible. In this exercise we determine the displacement $y(t)$ due to an instantaneous unit force $F(t)$ applied at time T,

$$F(t) = \begin{cases} 0 & 0 < t < T \\ 1 & t = T, \\ 0 & t > T \end{cases}$$

called a *unit impulse*. We do this by distributing the unit impulse in two ways.

(a) First, distribute $F(t)$ over a time interval of length ε around T according to

$$F_1(t) = \begin{cases} 0 & 0 < t < T - \varepsilon/2 \\ 1/\varepsilon & T - \varepsilon/2 < t < T + \varepsilon/2. \\ 0 & t > T + \varepsilon/2 \end{cases}$$

[Notice that the units of $F_1(t)$ are units of force per unit of time, so the total area "under" the $F_1(t)$ curve is unity.] Solve the differential equation with $F(t)$ replaced by $F_1(t)$ subject to the initial conditions $y(0) = y'(0) = 0$. Find and sketch the limit function as $\varepsilon \to 0^+$.

(b) Repeat (a) with the unit impulse $F(t)$ distributed over the time interval $T < t < T + \varepsilon$ according to

$$F_2(t) = \begin{cases} 0 & 0 < t < T \\ 1/\varepsilon & T < t < T + \varepsilon. \\ 0 & t > T + \varepsilon \end{cases}$$

13. Show that the same function as that in Exercise 12 is obtained if we assume that $y(t) = 0$ for $t < T$ and that for $t \geq T$, $y(t)$ satisfies

$$M\frac{d^2y}{dt^2} + ky = 0, \qquad t > T,$$

$$y(T) = 0, \qquad y'(T) = \frac{1}{M}.$$

Distributing point sources for multidimensional boundary value problems is more complex. The remaining exercises give examples.

14. A square membrane stretched tightly over the region $0 \leq x, y \leq L$ has its edges fixed on the xy-plane. Distribute a unit load at the midpoint of the membrane according to

$$F(x, y) = \begin{cases} -1/\varepsilon^2 & (L - \varepsilon)/2 < x, y < (L + \varepsilon)/2 \\ 0 & \text{otherwise} \end{cases}.$$

(a) Find the static deflection of the membrane due to this load by using the finite Fourier transform associated with the x-variable, or an eigenfunction expansion

$$z(x, y) = \sum_{n=1}^{\infty} a_n(y)\sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L}.$$

(b) Take the limit of the function $z(x, y)$ in (a) as $\varepsilon \to 0^+$ to find the static deflection of the membrane under a unit concentrated load at its center.

(c) Is the result in (b) defined at $(L/2, L/2)$?

15. Repeat Exercise 14 for a circular membrane of radius R. Distribute the unit load at the midpoint of the membrane according to

$$F(r, \theta) = \begin{cases} -1/(\pi\varepsilon^2) & 0 \le r < \varepsilon \\ 0 & \text{otherwise} \end{cases}.$$

11.2 Introductory Example

In this section we use a very simple example to illustrate the essential features of a Green's function. The example also helps justify our hopes that the delta function of Section 11.1 can be used to represent concentrated sources. The boundary value problem

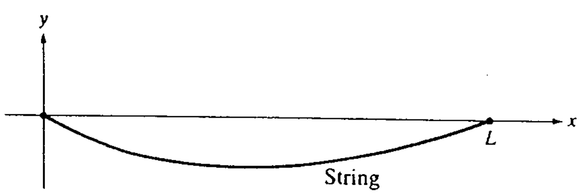
$$-\tau \frac{d^2y}{dx^2} = F(x), \tag{9a}$$

$$y(0) = y(L) = 0 \tag{9b}$$

describes static deflections of a taut string of negligible mass, tension τ, and length L due to a load $F(x)$ (Figure 11.6). We can solve this problem by using variation of parameters on the general solution $Ax + B$ of the associated homogeneous equation (see Section 3.3). Derivatives of $A(x)$ and $B(x)$ must satisfy

$$A'x + B' = 0,$$

$$A' = -\frac{F(x)}{\tau}.$$



ure 11.6

Solutions of these equations may be expressed as definite integrals

$$A(x) = \int_0^x -\tau^{-1}F(X)\,dX + C,$$

$$B(x) = \int_0^x \tau^{-1}XF(X)\,dX + D,$$

and hence
$$y(x) = x\left(\int_0^x -\tau^{-1}F(X)\,dX + C\right) + \int_0^x \tau^{-1}XF(X)\,dX + D$$
$$= \tau^{-1}\int_0^x (X - x)F(X)\,dX + Cx + D.$$

Boundary conditions (9b) require the constants C and D to satisfy
$$0 = y(0) = D,$$
$$0 = y(L) = \tau^{-1}\int_0^L (X - L)F(X)\,dX + CL + D,$$

and therefore
$$y(x) = \tau^{-1}\int_0^x (X - x)F(X)\,dX + (L\tau)^{-1}x\int_0^L (L - X)F(X)\,dX$$
$$= \tau^{-1}\int_0^x ((X - x) + L^{-1}x(L - X))F(X)\,dX + (L\tau)^{-1}x\int_x^L (L - X)F(X)\,dX$$
$$= (L\tau)^{-1}\int_0^x X(L - x)F(X)\,dX + (L\tau)^{-1}x\int_x^L (L - X)F(X)\,dX$$

or
$$y(x) = \int_0^L g(x;X)F(X)\,dX, \tag{10a}$$

where
$$g(x;X) = \begin{cases} \dfrac{X(L-x)}{L\tau} & 0 \le X \le x \\[2mm] \dfrac{x(L-X)}{L\tau} & x \le X \le L \end{cases}. \tag{10b}$$

The solution of problem (9) has therefore been expressed in integral form—the integral of the nonhomogeneity $F(x)$ multiplied by the function $g(x;X)$. The function $g(x;X)$ is called the Green's function for boundary value problem (9). It does not depend on $F(x)$; it depends only on the differential operator and the boundary conditions. Once $g(x;X)$ is known, the solution for any $F(x)$ can be represented in the form of a definite integral involving $g(x;X)$ and $F(x)$, and this integral representation clearly displays how the solution depends on $F(x)$. In addition, we shall see that when the boundary conditions are nonhomogeneous, representation of the solution in terms of the Green's function also indicates the nature of the dependence on these nonhomogeneities. Finally, it should be clear that formulation of the solution as a definite integral is a distinct advantage in numerical analysis.

The representation of $g(x;X)$ in (10b) regards X as the independent variable and x as a parameter. By interchanging the two expressions, we obtain a representation wherein X is the parameter and x is the independent variable:
$$g(x;X) = \begin{cases} \dfrac{x(L-X)}{L\tau} & 0 \le x \le X \\[2mm] \dfrac{X(L-x)}{L\tau} & X \le x \le L \end{cases}. \tag{10c}$$

With representation (10c), it is straightforward to illustrate three properties of this Green's function that are shared by all Green's functions. First,

$$g(x; X) \text{ is continuous for all } x \text{ (including } x = X). \tag{11a}$$

Second, the derivative of $g(x; X)$ with respect to x is continuous for all $x \neq X$, and

$$\lim_{x \to X^+} \frac{dg}{dx} - \lim_{x \to X^-} \frac{dg}{dx} = \left(\frac{-X}{L\tau}\right) - \left(\frac{L - X}{L\tau}\right) = -\frac{1}{\tau}. \tag{11b}$$

This jump is the reciprocal of the coefficient of d^2y/dx^2 in differential equation (9a). Finally, it is straightforward to check that at every $x \neq X$,

$$\begin{array}{l} g(x; X) \text{ satisfies the homogeneous version of the} \\ \text{differential equation from which it was derived.} \end{array} \tag{11c}$$

As we said, properties (11a–c) are shared by all Green's functions associated with ordinary differential equations. In fact, we shall use them to characterize Green's functions in Section 11.3.

In Section 11.1 we defined the delta function in hopes that it would represent a concentrated source. Let us see what happens if we set $F(x) = \delta(x - L/2)$ in (10):

$$\begin{aligned} y(x) &= \int_0^L g(x; X)\delta\left(X - \frac{L}{2}\right) dX \\ &= \int_0^x \frac{X(L - x)}{L\tau}\delta\left(X - \frac{L}{2}\right) dX + \int_x^L \frac{x(L - X)}{L\tau}\delta\left(X - \frac{L}{2}\right) dX. \end{aligned}$$

Since the first integral vanishes when $x < L/2$ and the second is zero when $x > L/2$, we separate the solution into two parts,

$$\begin{aligned} y(x) &= \begin{cases} \dfrac{x}{L\tau}\left(\dfrac{L}{2}\right) & 0 \le x < L/2 \\ \dfrac{1}{L\tau}\left(\dfrac{L}{2}\right)(L - x) & L/2 < x \le L \end{cases} \\ &= \begin{cases} \dfrac{x}{2\tau} & 0 \le x \le L/2 \\ \dfrac{1}{2\tau}(L - x) & L/2 \le x \le L \end{cases} \end{aligned}$$

(provided we demand continuity of the solution at $L/2$). But this is solution (5) to problem (1) for displacement due to a unit force concentrated at $x = L/2$. In other words, the delta function $\delta(x - L/2)$ appears to be a valid representation for a point force of magnitude unity at $x = L/2$.

Exercises 11.2

1. Consider the boundary value problem

$$\frac{d^2y}{dx^2} + y = F(x), \qquad 0 < x < L,$$

$$y(0) = 0 = y'(L).$$

(a) Use variation of parameters to show that the solution can be expressed in the form

$$y(x) = \int_0^L g(x; X)F(X)\,dX,$$

where $g(x; X)$ is the Green's function of the problem defined by

$$g(x; X) = \frac{-1}{\cos L}\begin{cases} \sin X \cos(L - x) & 0 \le X \le x \\ \sin x \cos(L - X) & x \le X \le L \end{cases}$$

(b) Show that $g(x; X)$ satisfies properties (11a–c).

11.3 Green's Functions

In this section we associate Green's functions with linear, second-order ordinary differential equations

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = f(x), \qquad \alpha < x < \beta. \tag{12}$$

Functions $P(x)$, $Q(x)$, and $R(x)$ are assumed continuous for $\alpha \le x \le \beta$, but no assumption is yet made on the behavior of $f(x)$. Provided $P(x)$ does not vanish on the interval $\alpha \le x \le \beta$, multiplication of (12) by $e^{\int(Q/P)\,dx}$ gives

$$\frac{d}{dx}\left(e^{\int(Q/P)\,dx}\frac{dy}{dx}\right) + \frac{R}{P}e^{\int(Q/P)\,dx}y = \frac{1}{P}e^{\int(Q/P)\,dx}f(x).$$

When we set $a(x) = e^{\int(Q/P)\,dx}$, $c(x) = RP^{-1}e^{\int(Q/P)\,dx}$, and $F(x) = P^{-1}e^{\int(Q/P)\,dx}f(x)$, the equation takes on a more pleasing appearance:

$$\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta. \tag{13a}$$

In other words, every linear, second-order differential equation for which $P(x) \ne 0$ can be expressed in form (13a), where $a(x) > 0$. This is called the *self-adjoint* form of the differential equation. We shall often find it convenient to denote the differential operator on the left side of (13a) by L, in which case the differential equation is expressed more compactly as

$$Ly = F(x), \qquad \alpha < x < \beta. \tag{13b}$$

To obtain a unique solution of (13), it is necessary to specify two boundary conditions. For the most part, we consider conditions of the form

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = m_1, \tag{14a}$$

$$B_2 y = l_2 y'(\beta) + h_2 y(\beta) = m_2, \tag{14b}$$

where l_1, l_2, h_1, h_2, m_1, and m_2 are given constants. They are called *unmixed* boundary conditions because one condition is at $x = \alpha$ and the other is at $x = \beta$. On occasion, however, we shall consider conditions of the form

$$y(\alpha) = y(\beta), \tag{15a}$$

$$y'(\alpha) = y'(\beta), \tag{15b}$$

called *periodic* boundary conditions. They arise only when $a(\alpha) = a(\beta)$, and they are always homogeneous. We have seen both types of conditions many times throughout the first ten chapters.

For the moment, we concentrate only on the operator L in (13), not on the differential equation or the boundary conditions. When $u(x)$ and $v(x)$ are continuously differentiable functions on $\alpha \leq x \leq \beta$ with piecewise continuous second derivatives, it is straightforward to show that

$$uLv - vLu = \frac{d}{dx} J(u, v) = \frac{d}{dx}(a(uv' - vu')). \tag{16}$$

This equation is known as *Lagrange's identity*; $J(u, v)$ is called the *conjunct* of u and v. The identity is valid at every point except discontinuities of the second derivatives of u and v. Because such discontinuities must be finite, (16) may be integrated between any two values of x in the interval $\alpha \leq x \leq \beta$:

$$\int_{x_1}^{x_2} (uLv - vLu)\,dx = \{J(u, v)\}_{x_1}^{x_2}. \tag{17}$$

This result is called *Green's formula on the interval* $x_1 \leq x \leq x_2$. When $x_1 = \alpha$ and $x_2 = \beta$, we obtain Green's formula on $\alpha \leq x \leq \beta$,

$$\boxed{\int_{\alpha}^{\beta} (uLv - vLu)\,dx = \{J(u, v)\}_{\alpha}^{\beta}.} \tag{18}$$

Identities (16)–(18) were based on the operator L in differential equation (13), but not on the differential equation itself; that is, $F(x)$ was not introduced. Nor were boundary conditions used in the derivation. In other words, (16)–(18) are properties of the operator L.

When $u(x)$ and $v(x)$ satisfy the homogeneous version of (13), it is obvious that their conjunct is constant. This result is sufficiently important that we state it in the form of a theorem.

Theorem 1

If $u(x)$ and $v(x)$ satisfy the homogeneous differential equation $Ly = 0$, then $J(u, v)$ is a constant (independent of x).

The constant value vanishes only if $u(x)$ and $v(x)$ are linearly dependent.

With these preliminaries out of the way, we are prepared to define Green's functions for boundary value problems of the form

$$Ly = \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta, \tag{19a}$$

$$B_1 y = m_1, \tag{19b}$$

$$B_2 y = m_2, \tag{19c}$$

where $a(x)$ is continuously differentiable and does not vanish for $\alpha \leq x \leq \beta$ and $c(x)$ is continuous therein. (If the boundary conditions are periodic, they are also homogeneous, $m_1 = m_2 = 0$.) Solutions of (19) are called classical when $F(x)$ is piecewise continuous. A solution $y(x)$ is *classical* if it is continuously differentiable, has a piecewise continuous second derivative, satisfies the boundary conditions (19b, c), and is such that Ly and $F(x)$ are identical at every point of continuity of $F(x)$. We mention this fact because Green's functions do not turn out to be classical solutions. The Green's function $g(x; X)$ for problem (19), if it exists, is defined as the solution of

$$Lg = \delta(x - X), \tag{20a}$$

$$B_1 g = 0, \tag{20b}$$

$$B_2 g = 0. \tag{20c}$$

It is the solution of the same problem, with two changes. The source function $F(x)$ is replaced by a concentrated unit source, and the boundary conditions are made homogeneous. Because $\delta(x - X)$ is not piecewise continuous, Green's function cannot be called a classical solution of (20). It is, however, an ordinary function (as opposed to a generalized function). This is established in Schwartz's theory of distributions, wherein it is also shown that solutions of differential equation (20a) have the following properties analogous to those in (11):

(1) $g(x; X)$ is continuous for $\alpha \leq x \leq \beta$; **(21a)**

(2) $dg(x; X)/dx$ is continuous except for a discontinuity at $x = X$ of magnitude $1/a(X)$; that is,

$$\lim_{x \to X^+} \frac{dg}{dx} - \lim_{x \to X^-} \frac{dg}{dx} = \frac{1}{a(X)}; \tag{21b}$$

(3) for all $x \neq X$,

$$Lg = 0. \tag{21c}$$

These properties, along with boundary conditions (20b, c), completely characterize Green's functions; in fact, we now use them to derive formulas for Green's functions.

Condition (21c) implies that $g(x; X)$ must be of the form

$$g(x; X) = \begin{cases} Eu(x) + Bv(x) & \alpha \leq x < X \\ Du(x) + Gv(x) & X < x \leq \beta' \end{cases} \tag{22}$$

where $u(x)$ and $v(x)$ are continuously differentiable solutions of $Ly = 0$. Inclusion of $x = \alpha$ and $x = \beta$ is a result of continuity condition (21a). Continuity at $x = X$ requires that

$$Eu(X) + Bv(X) = Du(X) + Gv(X),$$

and condition (21b) for the jump in dg/dx at $x = X$ implies that

$$Du'(X) + Gv'(X) - Eu'(X) - Bv'(X) = \frac{1}{a(X)}.$$

When these equations are solved for B and D in terms of E and G and substituted into (22), the result is

$$g(x; X) = \begin{cases} Eu(x) + Gv(x) - \dfrac{u(X)v(x)}{J(u, v)} & \alpha \leq x \leq X \\ Eu(x) + Gv(x) - \dfrac{v(X)u(x)}{J(u, v)} & X \leq x \leq \beta \end{cases}$$

The Heaviside unit step function can be used to combine these two expressions into one:

$$\begin{aligned} g(x; X) &= Eu(x) + Gv(x) - \frac{1}{J(u, v)}(u(X)v(x)H(X - x) + v(X)u(x)H(x - X)) \\ &= Eu(x) + Gv(x) - \frac{1}{J(u, v)}(u(X)v(x)[1 - H(x - X)] \\ &\quad + v(X)u(x)[1 - H(X - x)]) \\ &= \left(E - \frac{v(X)}{J(u, v)}\right)u(x) + \left(G - \frac{u(X)}{J(u, v)}\right)v(x) \\ &\quad + \frac{1}{J(u, v)}(u(x)v(X)H(X - x) + u(X)v(x)H(x - X)) \\ &= Au(x) + Cv(x) + \frac{1}{J(u, v)}(u(x)v(X)H(X - x) + u(X)v(x)H(x - X)). \end{aligned} \tag{23}$$

We understand that terms involving the step function are regarded in the limit sense $(x \to X)$ at $x = X$.

The remaining unknowns A and C are evaluated using boundary conditions (20b, c). They require that

$$0 = B_1 g = AB_1 u + CB_1 v + B_1 r, \tag{24a}$$

$$0 = B_2 g = AB_2 u + CB_2 v + B_2 r, \tag{24b}$$

where $r = J^{-1}[u(x)v(X)H(X-x) + u(X)v(x)H(x-X)]$. These are algebraic equations for A and C that have a unique solution provided

$$\begin{vmatrix} B_1 u & B_1 v \\ B_2 u & B_2 v \end{vmatrix} \neq 0. \tag{25}$$

Thus, when condition (25) is satisfied, $g(x;X)$ is defined by (23), where A and C are chosen so that $g(x;X)$ satisfies (20b, c).

We briefly examine here the significance of a vanishing determinant and deal with it more fully in Section 11.5. A vanishing determinant is equivalent to the existence of a constant $\lambda \neq 0$ such that

$$B_1 u = \lambda B_1 v \quad \text{and} \quad B_2 u = \lambda B_2 v \tag{26a}$$

or

$$B_1(u - \lambda v) = 0 \quad \text{and} \quad B_2(u - \lambda v) = 0. \tag{26b}$$

Since $u(x)$ and $v(x)$ are linearly independent, we can say that the determinant vanishes if and only if there is a nontrivial solution $u - \lambda v$ of the homogeneous boundary value problem

$$\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = 0, \qquad \alpha < x < \beta, \tag{27a}$$

$$B_1 y = 0, \tag{27b}$$

$$B_2 y = 0. \tag{27c}$$

We summarize these results in the following theorem.

Theorem 2

When homogeneous system (27) has only the trivial solution, the Green's function for problem (19) is uniquely given by

$$\boxed{g(x;X) = Au(x) + Cv(x) + \frac{1}{J(u,v)}(u(x)v(X)H(X-x) + u(X)v(x)H(x-X)),} \tag{28}$$

where $u(x)$ and $v(x)$ are linearly independent solutions of (27a) and A and C are chosen so that $g(x;X)$ satisfies (20b, c).

When the boundary conditions are unmixed, determination of $g(x;X)$ can be simplified further.

Corollary

When homogeneous system (27) has only the trivial solution and boundary conditions are unmixed, the Green's function for problem (19) is uniquely given by

$$\boxed{g(x;X) = \frac{1}{J(u,v)}(u(x)v(X)H(X-x) + u(X)v(x)H(x-X)),} \tag{29}$$

where $u(x)$ and $v(x)$ are linearly independent solutions of (27a) satisfying $B_1 u = 0$ and $B_2 v = 0$.

Proof: Certainly this function satisfies (20a) [since the function in (28) does]. In addition, when $x < X$, $g(x; X)$ reduces to $J^{-1}u(x)v(X)$, which, as a function of x, satisfies $B_1 g = 0$. Similarly, because $B_2 v = 0$, we must have $B_2 g = 0$. ∎

Once again, we point out that due to the step functions, expressions for $g(x; X)$ in (28) and (29) are not defined for $x = X$. However, continuity of $g(x; X)$ at $x = X$ implies that $g(x; X)$ must be given by either of the limits $\lim_{x \to X^+} g(x; X) = \lim_{x \to X^-} g(x; X)$, and we implicitly understand this when we write (28) and (29).

Notice that for unmixed boundary conditions, $g(x; X)$ is symmetric in x and X. That this is also true for periodic boundary conditions is verified in Theorem 5 of this section.

Example 1: Use formula (29) to find the Green's function for problem (9).

Solution: Solutions of $y'' = 0$ satisfying $y(0) = 0$ and $y(L) = 0$, respectively, are $u(x) = x$ and $v(x) = L - x$. With $J(u, v) = a(uv' - vu') = -\tau[(x)(-1) - (1)(L - x)] = L\tau$, (29) gives

$$g(x; X) = \frac{1}{L\tau}(x(L - X)H(X - x) + X(L - x)H(x - X)),$$

and this is (10c). ∎

Example 2: Find the Green's function for the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \qquad \alpha < x < \beta,$$

$$y(\alpha) = m_1, \qquad y'(\beta) = m_2.$$

Solution: Since solutions of $y'' + 4y = 0$ are of the form $A\sin 2(x + \phi)$ or $A\cos 2(x + \phi)$, solutions that satisfy $y(\alpha) = 0$ and $y'(\beta) = 0$, respectively, are $u(x) = \sin 2(x - \alpha)$ and $v(x) = \cos 2(\beta - x)$. With

$$\begin{aligned} J(u, v) &= uv' - vu' \\ &= 2\sin 2(x - \alpha)\sin 2(\beta - x) - 2\cos 2(x - \alpha)\cos 2(\beta - x) \\ &= -2\cos 2(\beta - \alpha), \end{aligned}$$

formula (29) gives

$$\begin{aligned} g(x; X) = \frac{1}{-2\cos 2(\beta - \alpha)}&(\sin 2(x - \alpha)\cos 2(\beta - X)H(X - x) \\ &+ \sin 2(X - \alpha)\cos 2(\beta - x)H(x - X)). \end{aligned}$$ ∎

Example 3: Find the Green's function for

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = F(x), \qquad 0 < x < \frac{\pi}{2},$$

$$y'(0) = 5, \qquad y\left(\frac{\pi}{2}\right) = 2.$$

Solution: Solutions of $y'' + 2y' + 10y = 0$ are always of the form $e^{-x}(A\sin 3x + B\cos 3x)$. Solutions that satisfy $y'(0) = 0$ and $y(\pi/2) = 0$, respectively, are

$$u(x) = e^{-x}(\sin 3x + 3\cos 3x) \quad \text{and} \quad v(x) = e^{-x}\cos 3x.$$

To find the conjunct of u and v, we express the differential equation in self-adjoint form by multiplying by e^{2x}:

$$e^{2x}\frac{d^2y}{dx^2} + 2e^{2x}\frac{dy}{dx} + 10e^{2x}y = e^{2x}F(x)$$

or

$$\frac{d}{dx}\left(e^{2x}\frac{dy}{dx}\right) + 10e^{2x}y = e^{2x}F(x).$$

With $a(x)$ identified as e^{2x},

$$\begin{aligned} J(u,v) &= e^{2x}(e^{-x}(\sin 3x + 3\cos 3x)(-e^{-x}\cos 3x - 3e^{-x}\sin 3x) \\ &\quad - e^{-x}\cos 3x(-10e^{-x}\sin 3x)) \\ &= -3, \end{aligned}$$

and therefore

$$\begin{aligned} g(x;X) = -\frac{1}{3}(&e^{-(x+X)}\cos 3X(\sin 3x + 3\cos 3x)H(X-x) \\ &+ e^{-(x+X)}\cos 3x(\sin 3X + 3\cos 3X)H(x-X)). \end{aligned}$$ ■

Example 4: Find the Green's function for the problem

$$\frac{d^2y}{dx^2} + y = F(x), \qquad 0 < x < 1,$$

$$y(0) - y(1) = 0, \qquad y'(0) - y'(1) = 0.$$

Solution: Since $u(x) = \sin x$ and $v(x) = \cos x$ are solutions of $y'' + y = 0$, we may take [according to (28)]

$$\begin{aligned} g(x;X) = A\sin x + C\cos x + &\frac{1}{J(\sin x, \cos x)}(\sin x\cos X\,H(X-x) \\ &+ \sin X\cos x\,H(x-X)), \end{aligned}$$

where $J(\sin x, \cos x) = \sin x(-\sin x) - \cos x(\cos x) = -1$. The boundary conditions must also be satisfied by $g(x;X)$, and therefore

$$C - A\sin 1 - C\cos 1 + \sin X\cos 1 = 0,$$

$$A - \cos X - A\cos 1 + C\sin 1 - \sin X\sin 1 = 0.$$

These can be solved for A and C:

$$A = \frac{\cos X - \cos(1+X)}{2(1-\cos 1)}, \qquad C = \frac{\sin X + \sin(1-X)}{2(1-\cos 1)}$$

and

$$\begin{aligned} g(x;X) = &\frac{1}{2(1-\cos 1)}(\sin x[\cos X - \cos(1+X)] + \cos x[\sin X + \sin(1-X)]) \\ &-\sin x\cos X H(X-x) - \sin X\cos x\,H(x-X). \end{aligned}$$ ■

The importance of Green's functions is contained in the following theorem.

orem 3

When $g(x;X)$ is the Green's function for the boundary value problem

$$Ly = \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta, \tag{30a}$$

$$B_1 y = 0, \tag{30b}$$

$$B_2 y = 0, \tag{30c}$$

the solution of the problem is

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX. \tag{31}$$

Notice that the boundary conditions are homogeneous; nonhomogeneous boundary conditions are discussed in Section 11.4.

Proof: The Green's function for (30a) satisfies equations (20). If we substitute (31) into (30a) and reverse orders of integration with respect to X and differentiations with respect to x,

$$\begin{aligned} Ly &= L\int_\alpha^\beta g(x;X)F(X)\,dX \\ &= \int_\alpha^\beta [Lg(x;X)]F(X)\,dX \\ &= \int_\alpha^\beta \delta(x-X)F(X)\,dX \quad \text{[by (20a)]} \\ &= F(x). \end{aligned}$$

Furthermore, because $g(x;X)$ satisfies (20b, c), $y(x)$ must satisfy (30b, c). ■

As a result of this theorem, once we know the Green's function for a boundary value problem, the solution for any source function $F(x)$ can be obtained by integration. Think of the integral as a superposition. Because the Green's function is the solution of problem (30) due to a unit point source at X, we interpret $g(x;X)F(X)\,dX$ as the effect due to that part $F(X)\,dX$ of the source over the interval dX of the x-axis, and the integral adds over all sources from $x = \alpha$ to $x = \beta$. Were the source composed of both a distributed portion $F(x)$ and n concentrated parts of magnitudes F_j at points x_j, the solution of (30) would be

$$\begin{aligned} y(x) &= \int_\alpha^\beta g(x;X)\left(F(X) + \sum_{j=1}^n F_j\delta(X-x_j)\right)dX \\ &= \int_\alpha^\beta g(x;X)F(X)\,dX + \sum_{j=1}^n F_j g(x;x_j). \end{aligned} \tag{32}$$

Before considering examples of Theorem 3 (in Section 11.4), we extend Green's formula to encompass delta functions. One of these extensions immediately implies that $g(x; X)$ is symmetric. Through the other, we see how Green's functions handle nonhomogeneous boundary conditions.

Theorem 4

Let L be the differential operator of problem (30). When $v(x; X)$ is a solution of $Lv = \delta(x - X)$ and $u(x)$ is continuously differentiable with a piecewise continuous second derivative,

$$\int_\alpha^\beta (uLv - vLu)\,dx = \{a(uv' - vu')\}_\alpha^\beta. \tag{33}$$

Proof: Suppose $u(x)$ has a discontinuity in its second derivative at a point $\bar{X} < X$. [Similar discussions can be made if $u(x)$ has more than one such point or if $\bar{X} > X$.] Then

$$\int_\alpha^\beta (uLv - vLu)\,dx = \int_\alpha^{\bar{X}} (uLv - vLu)\,dx + \int_{\bar{X}}^{X-\varepsilon} (uLv - vLu)\,dx$$
$$+ \int_{X-\varepsilon}^{X+\varepsilon} (uLv - vLu)\,dx + \int_{X+\varepsilon}^{\beta} (uLv - vLu)\,dx,$$

where $\varepsilon > 0$ is some small number. Green's formula (17) can be applied to the first, second, and fourth of these integrals since $Lv = 0$ therein [see condition (21c)]:

$$\int_\alpha^\beta (uLv - vLu)\,dx = \{a(uv' - vu')\}_\alpha^{\bar{X}} + \{a(uv' - vu')\}_{\bar{X}}^{X-\varepsilon}$$
$$+ \int_{X-\varepsilon}^{X+\varepsilon} (u\delta(x - X) - vLu)\,dx + \{a(uv' - vu')\}_{X+\varepsilon}^{\beta}.$$

Because a, u, u', v, and v' are all continuous at $\bar{X}$, terms in $\bar{X}$ vanish, and the expression on the right reduces to

$$\int_\alpha^\beta (uLv - vLu)\,dx = \{a(uv' - vu')\}_\alpha^\beta + a(X - \varepsilon)[u(X - \varepsilon)v'(X - \varepsilon; X)$$
$$- v(X - \varepsilon; X)u'(X - \varepsilon)] - a(X + \varepsilon)[u(X + \varepsilon)v'(X + \varepsilon; X)$$
$$- v(X + \varepsilon; X)u'(X + \varepsilon)] + u(X) - \int_{X-\varepsilon}^{X+\varepsilon} vLu\,dx.$$

We now take limits as $\varepsilon \to 0^+$. Since v, u, u', and u'' are continuous on $X - \varepsilon \le x \le X + \varepsilon$, the final integral vanishes in the limit, and the remaining terms give

$$\int_\alpha^\beta (uLv - vLu)\,dx = \{a(uv' - vu')\}_\alpha^\beta + a(X)[u(X)v'(X-; X) - v(X; X)u'(X)]$$
$$- a(X)[u(X)v'(X+; X) - v(X; X)u'(X)] + u(X)$$
$$= \{a(uv' - vu')\}_\alpha^\beta + a(X)u(X)[v'(X-; X) - v'(X+; X)] + u(X)$$
$$= \{a(uv' - vu')\}_\alpha^\beta$$

[because v satisfies condition (21b)]. ∎

A similar proof leads to the following extension of Green's formula.

Theorem 5

Let L be the differential operator of problem (30). When u and v satisfy $Lu = \delta(x - X)$ and $Lv = \delta(x - Y)$,

$$\int_\alpha^\beta (uLv - vLu)\,dx = \{a(uv' - vu')\}_\alpha^\beta. \tag{34}$$

In this case, the integral of $uLv - vLu$ over the interval $\alpha \le x \le \beta$ is subdivided into five integrals over the intervals

$$\alpha \le x \le X - \varepsilon, \quad X - \varepsilon \le x \le X + \varepsilon, \quad X + \varepsilon \le x \le Y - \varepsilon,$$
$$Y - \varepsilon \le x \le Y + \varepsilon, \quad Y + \varepsilon \le x \le \beta$$

(for $X < Y$), and Green's formula (17) is applied to the first, third, and fifth. Details are given in Exercise 20.

Formula (29) indicates that Green's functions for problems with unmixed boundary conditions are symmetric. That this is true for periodic boundary conditions as well is proved in the next theorem.

Theorem 6

When boundary conditions in problem (30) are unmixed or periodic, Green's function $g(x;X)$ is symmetric:

$$g(x;X) = g(X;x). \tag{35}$$

Proof: When we set $u = g(x;X)$ and $v = g(x;Y)$ in version (34) of Green's formula, the result is

$$\int_\alpha^\beta [g(x;X)Lg(x;Y) - g(x;Y)Lg(x;X)]\,dx$$
$$= \left\{a(x)\left(g(x;X)\frac{dg(x;Y)}{dx} - g(x;Y)\frac{dg(x;X)}{dx}\right)\right\}_\alpha^\beta.$$

It is straightforward to show that when $g(x;X)$ satisfies unmixed boundary conditions (14) or periodic conditions (15), the right side of this equation must vanish, and therefore

$$0 = \int_\alpha^\beta [g(x;X)\delta(x - Y) - g(x;Y)\delta(x - X)]\,dx = g(Y;X) - g(X;Y). \quad \blacksquare$$

It is interesting to interpret this symmetry physically, say, in string problem (9) of Section 11.2. Green's function $g(x;X)$ for this problem is the deflection of the string due to a unit force at position X. Symmetry of $g(x;X)$ means that the deflection at x due to a unit force at X is identical to the deflection at X due to a unit force at x. This is often referred to as *Maxwell's reciprocity* and is illustrated in Figure 11.7.

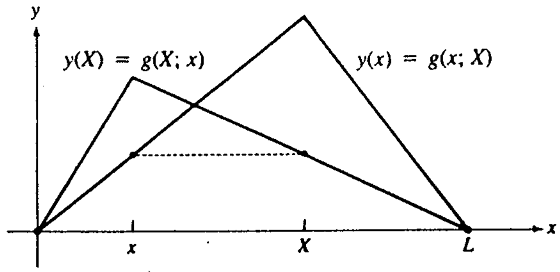


Figure 11.7

Exercises 11.3

In Exercises 1–5, write the differential equation in self-adjoint form.

1. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 3y = F(x)$

2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = F(x)$

3. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - (x+1)y = F(x)$

4. $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - (x+1)y = F(x)$

5. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} = F(x)$

In Exercises 6–14, find the Green's function for the boundary value problem.

6. $\dfrac{d^2y}{dx^2} = F(x), \quad 0 < x < 2, \quad y(0) = 0, \quad y'(2) = 0$

7. $\dfrac{d^2y}{dx^2} + y = F(x), \quad 0 < x < 1, \quad y(0) = 0, \quad y'(1) = 0$

8. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad 0 < x < \pi$ ($k > 0$ a constant, but not an integer), $y(0) = 0, \quad y(\pi) = 0$

9. $\dfrac{d^2y}{dx^2} = F(x), \quad 0 < x < 1, \quad y(0) = y'(0), \quad y'(1) = 0$

10. $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 4y = F(x), \quad 0 < x < 2, \quad y(0) = 0, \quad y'(2) = 0$

11. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 5y = F(x), \quad 0 < x < \pi/2, \quad y'(0) = 0, \quad y(\pi/2) = 0$

12. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 6y = F(x), \quad 1 < x < 2, \quad y(2) = y'(2), \quad y'(1) = 0$

13. $\dfrac{d^2y}{dx^2} + k^2y = F(x), \quad \alpha < x < \beta$ ($k > 0$ a constant), $y(\alpha) = y(\beta), \quad y'(\alpha) = y'(\beta)$. Would you place any restrictions on k?

14. $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = F(x), \quad 0 < \alpha < x < \beta, \quad y(\alpha) = 0, \quad y(\beta) = 0$

15. The boundary value problem for static deflections of a beam subjected to a distributed force

$F(x)$ is

$$EI\frac{d^4y}{dx^4} = F(x), \qquad 0 < x < L,$$

Boundary conditions at $x = 0$ and $x = L$,

where E and I are constants. The Green's function $g(x; X)$ for this fourth-order problem satisfies

$$EI\frac{d^4y}{dx^4} = \delta(x - X),$$

Homogeneous boundary conditions at $x = 0$ and $x = L$.

Thus, it is the solution of the problem due to a unit concentrated force at X (with homogeneous boundary conditions). Solutions of the differential equation are characterized by the following properties:

(i) $g(x; X)$, $dg(x; X)/dx$, and $d^2g(x; X)/dx^2$ are continuous for $0 \le x \le L$ except for a removable discontinuity at $x = X$.

(ii) $d^3g(x; X)/dx^3$ is continuous except for a discontinuity at $x = X$ of magnitude $(EI)^{-1}$; that is,

$$\lim_{x\to X^+}\frac{d^3g}{dx^3} - \lim_{x\to X^-}\frac{d^3g}{dx^3} = \frac{1}{EI}.$$

(iii) for any $x \ne X$,

$$EI\frac{d^4g(x; X)}{dx^4} = 0.$$

Use the characterization in (i), (ii), and (iii) to show that $g(x; X)$ can be expressed in the form

$$g(x; X) = \frac{1}{6EI}(x - X)^3H(x - X) + Ax^3 + Bx^2 + Cx + D,$$

where A, B, C, and D are constants. (The constants are evaluated using the homogeneous boundary conditions.)

In Exercises 16–19, use the result of Exercise 15 to find the Green's function for static deflections of a beam of length L ($0 \le x \le L$), where the boundary conditions are as given.

16. $y(0) = y'(0) = 0 = y''(L) = y'''(L)$ (cantilevered)

17. $y(0) = y''(0) = 0 = y(L) = y''(L)$ (simply supported at both ends)

18. $y(0) = y'(0) = 0 = y(L) = y'(L)$ (clamped at both ends)

19. $y(0) = y'(0) = 0 = y(L) = y''(L)$ (clamped at one end, simply supported at the other)

20. Prove Theorem 5.

21. When the boundary conditions in (19) are unmixed, it is sometimes advantageous to represent the Green's function of the problem in terms of orthonormal eigenfunctions of the corresponding Sturm-Liouville system:

$$\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + [c(x) + \lambda p(x)]y = 0, \qquad \alpha < x < \beta,$$

$$B_1y = 0, \qquad B_2y = 0.$$

[Notice that the weight function $p(x)$ is unspecified, but normally there is only one choice of $p(x)$ for which the differential equation gives rise to standard functions.] Show that when $y_n(x)$ are normalized eigenfunctions corresponding to eigenvalues λ_n, Green's function $g(x; X)$ can be expressed in the form

$$g(x;X) = \sum_{n=1}^{\infty} \frac{y_n(X)y_n(x)}{-\lambda_n}.$$

[*Hint:* Use Green's formula (33) with $u = y_n(x)$ and $v = g(x; X)$.]

22. Find an eigenfunction expansion for the Green's function of the boundary value problem

$$\frac{d^2y}{dx^2} = F(x), \qquad 0 < x < L,$$
$$y(0) = 0, \qquad y(L) = 0.$$

11.4 Solutions of Boundary Value Problems Using Green's Functions

In this section we show how easy it is to solve boundary value problems once the Green's function for the problem is known. Theorem 3 in Section 11.3 yields solutions to problems with homogeneous boundary conditions; we give two illustrative examples. Nonhomogeneous boundary conditions are handled either by superposition or by Green's formula.

Problems with Homogeneous Boundary Conditions

In Section 11.3 we defined the Green's function for the boundary value problem

$$Ly = \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta, \tag{36a}$$
$$B_1y = 0, \tag{36b}$$
$$B_2y = 0 \tag{36c}$$

as the solution of

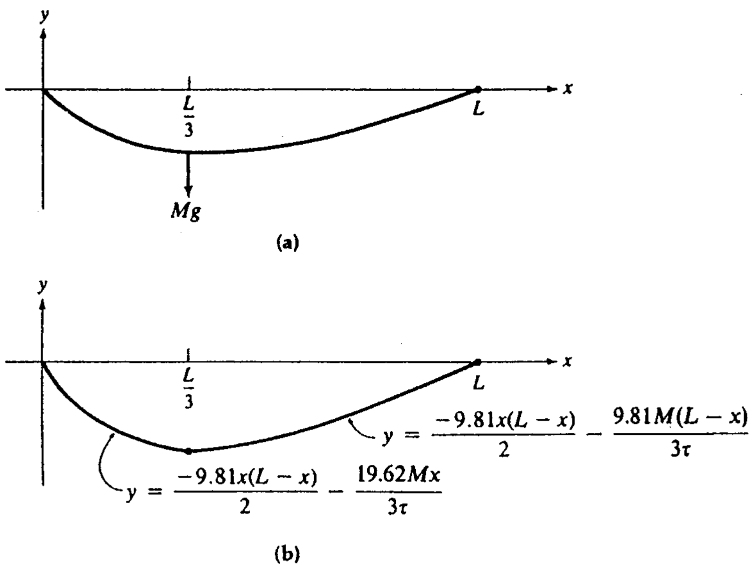
$$Lg = \delta(x - X), \tag{37a}$$
$$B_1g = 0, \tag{37b}$$
$$B_2g = 0. \tag{37c}$$

The solution of (36) is then given by the definite integral

$$y(x) = \int_\alpha^\beta g(x; X)F(X)\,dX \tag{38}$$

(see Theorem 3).

ımple 5: A taut string of length L has its ends fixed at $x = 0$ and $x = L$ on the x-axis. A concentrated mass of M kg is attached to the string at $x = L/3$ [Figure 11.8(a)]. Find the deflections in the string if gravity is also taken into account.



ure 11.8

olution: The boundary value problem for deflections in the string is

$$-\tau\frac{d^2y}{dx^2} = -9.81 - 9.81M\delta\left(x - \frac{L}{3}\right),$$
$$y(0) = 0 = y(L).$$

According to equation (10c) and Example 1, the Green's function for this problem is

$$g(x; X) = \frac{1}{L\tau}(x(L - X)H(X - x) + X(L - x)H(x - X)).$$

The solution is therefore defined by integral (38):

$$\begin{aligned}
y(x) &= \int_0^L g(x; X)\left[-9.81 - 9.81M\delta\left(X - \frac{L}{3}\right)\right]dX \\
&= -9.81\int_0^L g(x; X)\,dX - 9.81Mg\left(x; \frac{L}{3}\right) \\
&= \frac{-9.81}{L\tau}\int_0^x X(L - x)\,dX - \frac{9.81}{L\tau}\int_x^L x(L - X)\,dX \\
&\quad - \frac{9.81M}{L\tau}\left[x\left(L - \frac{L}{3}\right)H\left(\frac{L}{3} - x\right) + \left(\frac{L}{3}\right)(L - x)H\left(x - \frac{L}{3}\right)\right]
\end{aligned}$$

$$= \frac{-9.81}{L\tau}(L-x)\left(\frac{x^2}{2}\right) - \frac{9.81}{L\tau}\frac{x(L-x)^2}{2}$$
$$- \frac{9.81M}{L\tau}\left[\frac{2Lx}{3}H\left(\frac{L}{3}-x\right) + \left(\frac{L}{3}\right)(L-x)H\left(x-\frac{L}{3}\right)\right]$$
$$= \frac{-9.81x(L-x)}{2\tau} - \frac{9.81M}{3\tau}\begin{cases} 2x & 0 \le x \le L/3 \\ L-x & L/3 \le x \le L \end{cases}.$$

This is superposition of the displacement due to gravity (the first term) and that due to the concentrated load (the second term) [Figure 11.8(b)]. ■

Example 6: Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \qquad 0 < x < 3,$$
$$y(0) = 0 = y'(3)$$

when (a) $F(x) = 2x$ and (b) $F(x) = H(x-1) - H(x-2)$.

Solution: The Green's function for this problem can be obtained from Example 2 by setting $\alpha = 0$ and $\beta = 3$:

$$g(x;X) = \frac{-1}{2\cos 6}(\sin 2x \cos(6-2X)H(X-x) + \sin 2X \cos(6-2x)H(x-X)).$$

With source function $F(x)$, the solution of the boundary value problem is

$$y(x) = \int_0^3 g(x;X)F(X)\,dX.$$

(a) When $F(x) = 2x$,

$$y(x) = \int_0^3 2Xg(x;X)\,dX$$
$$= \frac{-1}{2\cos 6}\int_0^x 2X \sin 2X \cos(6-2x)\,dX$$
$$- \frac{1}{2\cos 6}\int_x^3 2X \sin 2x \cos(6-2X)\,dX$$
$$= \frac{-\cos(6-2x)}{\cos 6}\left\{\frac{-X\cos 2X}{2} + \frac{\sin 2X}{4}\right\}_0^x$$
$$- \frac{\sin 2x}{\cos 6}\left\{\frac{-X\sin(6-2X)}{2} + \frac{\cos(6-2X)}{4}\right\}_x^3$$
$$= \frac{x}{2} - \frac{\sin 2x}{4\cos 6}.$$

(This solution could also be derived very simply by finding the general solution of $y'' + 4y = 2x$ and using boundary conditions to evaluate arbitrary constants.)

(b) For $F(x) = H(x-1) - H(x-2)$, the solution is

$$y(x) = \int_0^3 [H(X-1) - H(X-2)]g(x;X)\,dX$$
$$= \int_1^2 g(x;X)\,dX.$$

When $x \le 1$,

$$y(x) = \int_1^2 \frac{-1}{2\cos 6} \sin 2x \cos(6-2X)\,dX$$
$$= \frac{\sin 2x}{2\cos 6}\left\{\frac{\sin(6-2X)}{2}\right\}_1^2$$
$$= \frac{\sin 2x(\sin 2 - \sin 4)}{4\cos 6};$$

when $1 < x < 2$,

$$y(x) = \int_1^x \frac{-1}{2\cos 6}\sin 2X\cos(6-2x)\,dX + \int_x^2 \frac{-1}{2\cos 6}\sin 2x\cos(6-2X)\,dX$$
$$= \frac{\cos(6-2x)}{2\cos 6}\left\{\frac{\cos 2X}{2}\right\}_1^x + \frac{\sin 2x}{2\cos 6}\left\{\frac{\sin(6-2X)}{2}\right\}_x^2$$
$$= \frac{1}{4} + \frac{1}{4\cos 6}[\sin 2x \sin 2 - \cos(6-2x)\cos 2];$$

and when $2 \le x < 3$,

$$y(x) = \int_1^2 \frac{-1}{2\cos 6}\sin 2X\cos(6-2x)\,dX = \frac{\cos(6-2x)}{2\cos 6}\left\{\frac{\cos 2X}{2}\right\}_1^2$$
$$= \frac{\cos(6-2x)(\cos 4 - \cos 2)}{4\cos 6}.$$

This solution is not so easily produced using methods from elementary differential equations. It requires integration of the differential equation on three separate intervals and matching of the solution and its first derivative at $x = 1$ and $x = 2$. ■

Problems with Nonhomogeneous Boundary Conditions

Suppose now that boundary conditions (36b, c) are not homogeneous, in which case problem (36) becomes

$$Ly = \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta, \tag{39a}$$

$$B_1 y = m_1, \tag{39b}$$

$$B_2 y = m_2. \tag{39c}$$

(Only nonhomogeneous unmixed boundary conditions are considered; periodic conditions are always homogeneous.) There are two ways to solve this problem; one is to use superposition, and the other is to use Green's formula. Both methods use Green's function for the associated problem with homogeneous boundary conditions:

$$Ly = \frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + c(x)y = F(x), \qquad \alpha < x < \beta, \tag{40a}$$

$$B_1 y = 0, \tag{40b}$$

$$B_2 y = 0. \tag{40c}$$

In the superposition method, we note that

$$y_1(x) = \int_\alpha^\beta g(x;X)F(X)\,dX,$$

where $g(x;X)$ is the associated Green's function, is a solution of (40). A solution of (39) will therefore be $y = y_1 + y_2$ if y_2 satisfies

$$Ly = 0, \qquad \alpha < x < \beta, \tag{41a}$$

$$B_1 y = m_1, \tag{41b}$$

$$B_2 y = m_2. \tag{41c}$$

In ODEs it is often quite straightforward to obtain $y_2(x)$—apply boundary conditions (41b, c) to a general solution of (41a). We illustrate with the following example.

Example 7: Solve the boundary value problem

$$-\tau\frac{d^2y}{dx^2} = F(x), \qquad 0 < x < L,$$

$$y(0) = m_1, \qquad y(L) = m_2.$$

Solution: In Section 11.2 we derived the solution

$$y_1(x) = \int_0^L g(x;X)F(X)\,dX = \frac{L-x}{L\tau}\int_0^x XF(X)\,dX + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX$$

for the associated problem with homogeneous boundary conditions. To this we must add the solution of

$$\frac{d^2y}{dx^2} = 0, \qquad y(0) = m_1, \qquad y(L) = m_2.$$

Since every solution of this differential equation must be of the form $y_2(x) = Ax + B$, to satisfy the boundary conditions we require that

$$m_1 = B, \qquad m_2 = AL + B.$$

Thus,

$$y_2(x) = (m_2 - m_1)\frac{x}{L} + m_1$$

and $$y(x) = y_1(x) + y_2(x) = (m_2 - m_1)\frac{x}{L} + m_1 + \frac{L-x}{L\tau}\int_0^x XF(X)\,dX + \frac{x}{L\tau}\int_x^L (L-X)F(X)\,dX.$$ ■

This superposition method works well for ODEs but fails to generalize to PDEs; it is not usually possible to produce general solutions of homogeneous PDEs and apply nonhomogeneous boundary conditions to determine arbitrary functions. An alternative approach, which does generalize to PDEs, is to use Green's formula (33). This method also illustrates how the solution depends on the nonhomogeneities in the boundary conditions.

If $y(x)$ is the required solution of (39) and $v(x)$ is the Green's function $g(x; X)$ for the problem, (33) becomes

$$\int_\alpha^\beta yLg(x;X)\,dx - \int_\alpha^\beta g(x;X)Ly\,dx = \left\{a(x)\left[y(x)\frac{\partial g(x;X)}{\partial x} - g(x;X)y'(x)\right]\right\}_\alpha^\beta.$$

Because $Ly = F(x)$ and $Lg(x;X) = \delta(x-X)$, we may write

$$\int_\alpha^\beta y(x)\delta(x-X)\,dx - \int_\alpha^\beta g(x;X)F(x)\,dx = \left\{a(x)\left[y(x)\frac{\partial g(x;X)}{\partial x} - g(x;X)y'(x)\right]\right\}_\alpha^\beta$$

or $$y(X) - \int_\alpha^\beta g(x;X)F(x)\,dx = a(\beta)\left(y(\beta)\frac{\partial g(\beta;X)}{\partial x} - g(\beta;X)y'(\beta)\right) - a(\alpha)\left(y(\alpha)\frac{\partial g(\alpha;X)}{\partial x} - g(\alpha;X)y'(\alpha)\right). \qquad (42)$$

If we now substitute from the boundary conditions

$$B_1 y = -l_1 y'(\alpha) + h_1 y(\alpha) = m_1, \qquad (43a)$$

$$B_2 y = l_2 y'(\beta) + h_2 y(\beta) = m_2, \qquad (43b)$$

$$\begin{aligned} y(X) - \int_\alpha^\beta g(x;X)F(x)\,dx &= a(\beta)\left[y(\beta)\frac{\partial g(\beta;X)}{\partial x} - g(\beta;X)\left(\frac{m_2}{l_2} - \frac{h_2}{l_2}y(\beta)\right)\right] \\ &\quad - a(\alpha)\left[y(\alpha)\frac{\partial g(\alpha;X)}{\partial x} - g(\alpha;X)\left(-\frac{m_1}{l_1} + \frac{h_1}{l_1}y(\alpha)\right)\right] \\ &= a(\beta)\left[-\frac{m_2}{l_2}g(\beta;X) + \frac{y(\beta)}{l_2}\left(l_2\frac{\partial g(\beta;X)}{\partial x} + h_2 g(\beta;X)\right)\right] \\ &\quad - a(\alpha)\left[\frac{m_1}{l_1}g(\alpha;X) - \frac{y(\alpha)}{l_1}\left(-l_1\frac{\partial g(\alpha;X)}{\partial x} + h_1 g(\alpha;X)\right)\right]. \end{aligned}$$

But $g(x;X)$ must satisfy homogeneous versions of (43); that is,

$$-l_1\frac{\partial g(\alpha;X)}{\partial x} + h_1 g(\alpha;X) = 0, \qquad (44a)$$

$$l_2\frac{\partial g(\beta;X)}{\partial x} + h_2 g(\beta;X) = 0. \qquad (44b)$$

Consequently,

$$y(X) = \int_\alpha^\beta g(x;X)F(x)\,dx - \frac{m_1}{l_1}a(\alpha)g(\alpha;X) - \frac{m_2}{l_2}a(\beta)g(\beta;X).$$

Finally, when we interchange x and X and use the fact that $g(x;X)$ is symmetric,

$$\boxed{y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX - \frac{m_1}{l_1}a(\alpha)g(x;\alpha) - \frac{m_2}{l_2}a(\beta)g(x;\beta).} \tag{45a}$$

When $l_1 = l_2 = 0$ (and we set $h_1 = h_2 = 1$), (42) yields the following replacement for (45a):

$$\boxed{y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX + m_2 a(\beta)\frac{\partial g(x;\beta)}{\partial X} - m_1 a(\alpha)\frac{\partial g(x;\alpha)}{\partial X}.} \tag{45b}$$

Both (45a) and (45b) clearly indicate the dependence of $y(x)$ on all three nonhomogeneities in problem (39). The integral term accounts for the nonhomogeneity $F(x)$ in the PDE, and the remaining terms contain contributions due to nonhomogeneities in the boundary conditions. With $F(x)$ piecewise continuous, the integral term in (45) is continuous in x. Furthermore, because $g(x;X)$ is continuous and $\partial g/\partial x$ has a discontinuity only when $x = X$, it follows that the additional terms in (45) due to the nonhomogeneities in the boundary conditions are also continuous. In other words, the representation of the solution to a boundary value problem in terms of its Green's function is always a continuous function.

Example 8: Solve the boundary value problem of Example 7.

Solution: The Green's function for this problem is

$$g(x;X) = (L\tau)^{-1}[x(L - X)H(X - x) + X(L - x)H(x - X)].$$

In Example 7 we used the direct method to find the particular solution satisfying the homogeneous differential equation and nonhomogeneous boundary conditions. Alternatively, according to equation (45b),

$$\begin{aligned} y(x) &= \int_0^L g(x;X)F(X)\,dX - \tau m_2\frac{\partial g(x;L)}{\partial X} + \tau m_1\frac{\partial g(x;0)}{\partial X} \\ &= \int_0^L g(x;X)F(X)\,dX - \frac{\tau m_2}{L\tau}(-xH(X - x) + (L - x)H(x - X))_{|X=L} \\ &\quad + \frac{\tau m_1}{L\tau}(-xH(X - x) + (L - x)H(x - X))_{|X=0} \end{aligned}$$

$$= \int_0^L g(x;X)F(X)\,dX + \frac{m_2}{L}x + \frac{m_1}{L}(L-x)$$

$$= \int_0^L g(x;X)F(X)\,dX + (m_2 - m_1)\frac{x}{L} + m_1. \qquad \blacksquare$$

Example 9: Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \qquad \alpha < x < \beta,$$

$$y(\alpha) = m_1, \qquad y'(\beta) = m_2.$$

Solution: According to Example 2, the Green's function for this problem is

$$g(x;X) = \frac{-1}{2\cos 2(\beta-\alpha)}(\sin 2(x-\alpha)\cos 2(\beta - X)H(X-x)$$
$$+ \sin 2(X-\alpha)\cos 2(\beta - x)H(x - X)).$$

To account for the nonhomogeneities m_1 and m_2 in the boundary conditions, we use the term in (45a) containing m_2 and the term in (45b) containing m_1:

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX - m_1\frac{\partial g(x;\alpha)}{\partial X} - m_2 g(x;\beta)$$

$$= \int_\alpha^\beta g(x;X)F(X)\,dX + \frac{m_1}{2\cos 2(\beta-\alpha)}$$
$$\times (2\sin 2(x-\alpha)\sin 2(\beta-\alpha)H(\alpha - x) + 2\cos 2(\beta - x)H(x-\alpha))$$
$$+ \frac{m_2}{2\cos 2(\beta-\alpha)}(\sin 2(x-\alpha)H(\beta - x) + \sin 2(\beta-\alpha)\cos 2(\beta-x)H(x-\beta))$$

$$= \int_\alpha^\beta g(x;X)F(X)\,dX + \frac{2m_1\cos 2(\beta-x) + m_2\sin 2(x-\alpha)}{2\cos 2(\beta-\alpha)}. \qquad \blacksquare$$

Exercises 11.4

Do the exercises in Part D first.

Part A—Heat Conduction

1. What is the Green's function for the boundary value problem for steady-state temperature in a rod from $x = 0$ to $x = L$ with constant thermal conductivity k and zero end temperatures?
2. Solve the boundary value problem

$$-\frac{d}{dx}\left(\kappa\frac{dU}{dx}\right) = F(x), \qquad \alpha < x < \beta,$$
$$U(\alpha) = 0 = U(\beta)$$

for steady-state temperature in a rod from $x = \alpha$ to $x = \beta$ with variable thermal conductivity $\kappa(x)$ and heat generation $F(x)$. Interpret the Green's function physically.

3. Two rods of lengths L_1 and L_2 and constant thermal conductivities κ_1 and κ_2 are joined end to end (the left end of L_1 at $x = 0$ and the right end of L_2 at $x = L_1 + L_2$). If the ends at $x = 0$ and $x = L_1 + L_2$ are kept at temperature zero, what is the Green's function for steady-state temperature in the rods?

Part B—Vibrations

In Exercises 4–9, the function $F(x)$ describes the appplied force on a massless string with constant tension τ stretched between two fixed points $x = 0$ and $x = L$. Find and sketch a graph of the displacement $y(x)$ in the string.

4. $F(x) = k < 0$ a constant

5. $F(x) = \begin{cases} -kx & 0 < x \le L/2 \\ k(x - L) & L/2 \le x < L \end{cases}$, $k < 0$ a constant

6. $F(x) = \begin{cases} 0 & 0 < x < L/4 \\ k & L/4 < x < 3L/4, \\ 0 & 3L/4 < x < L \end{cases}$ $k < 0$ a constant

7. $F(x)$ is due to two concentrated loads of magnitude $\bar{k}$ placed at $x = L/4$ and $x = 3L/4$.

8. $F(x)$ is due to the combination of the constant force k in Exercise 4 and the concentrated loads $\bar{k}$ in Exercise 7.

9. $F(x) = \begin{cases} k & 0 < x < L/4 \\ 0 & L/4 < x < 3L/4, \\ k & 3L/4 < x < L \end{cases}$ $k < 0$ a constant

10. Solve Exercise 13 in Section 1.3.

11. Solve Exercise 10 if a thin ring of mass m is attached halfway along the length of the bar.

12. Solve Exercise 10 if a mass M is attached to the lower end of the bar.

13. The bar in Exercise 12 is hung from a spring with constant k, and a thin ring of mass m is attached halfway along the length of the bar. Find displacements of its cross sections in the coordinate system shown in Figure 11.9.

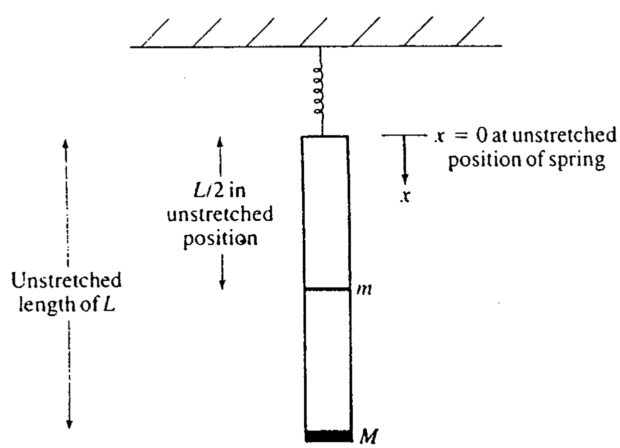


Figure 11.9

In Exercises 14–19, the function $F(x)$ describes the applied force on a beam of length L ($0 \le x \le L$), and the conditions represent boundary conditions at the ends of the beam. Use the Green's functions from Exercises 16–19 in Section 11.3 to find the static deflection of the beam. Sketch the deflection curve in Exercises 14–17.

14. $F(x)$ is due to a concentrated load of magnitude unity at $x = L/2$, and the weight of the beam is assumed negligible:

$$y(0) = y'(0) = 0 = y''(L) = y'''(L).$$

(See also Exercise 10 in Section 11.1.)

15. $F(x)$ is due to the load of Exercise 14 placed at $x = L$. (See also Exercise 11 in Section 11.1.)

16. $F(x)$ is due only to the weight per unit x-length w of a uniform beam:

$$y(0) = y''(0) = 0 = y(L) = y''(L).$$

17. $F(x)$ is due to a uniform weight per unit x-length w of a uniform beam and a concentrated load of magnitude k at $x = L/2$:

$$y(0) = y'(0) = 0 = y(L) = y'(L).$$

18. $$F(x) = \begin{cases} -w & 0 < x < L/4 \\ -(w + W) & L/4 < x < 3L/4, \\ -w & 3L/4 < x < L \end{cases} \qquad \begin{matrix} w = \text{constant} \\ W = \text{constant} \end{matrix},$$

$$y(0) = y'(0) = 0 = y''(L) = y'''(L)$$

19. $F(x)$ is due to a uniform weight per unit x-length W on $0 < x < L/2$ and a concentrated load of magnitude k at $x = L/4$. The weight of the beam itself is negligible:

$$y(0) = y'(0) = 0 = y(L) = y''(L).$$

Part D—General Results

In Exercises 20–27, find an integral representation for the solution of the boundary value problem.

20. $\dfrac{d^2y}{dx^2} = F(x)$, $1 < x < 2$, $y'(1) = m_1$, $y(2) = m_2$. What is the solution when $F(x) = xe^x$?

21. $\dfrac{d^2y}{dx^2} + y = F(x)$, $0 < x < 1$, $y(0) = m_1$, $y'(1) = m_2$. What is the solution when $F(x) = \cos x$?

22. $\dfrac{d^2y}{dx^2} + k^2y = F(x)$, $\alpha < x < \beta$ ($k > 0$ a constant), $y(\alpha) = 0$, $y'(\beta) = 1$. Is there any restriction on the value of k? What is the solution when $F(x) = 1$?

23. $\dfrac{d^2y}{dx^2} + k^2y = F(x)$, $\alpha < x < \beta$ ($k > 0$ a constant), $y(\alpha) = y(\beta)$, $y'(\alpha) = y'(\beta)$. (See Exercise 13 in Section 11.3 for the Green's function.) What is the solution when $F(x) = x$?

24. $(x + 1)\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = F(x)$, $0 < x < 1$, $y(0) = 0$, $y(1) = 0$. What is the solution when $F(x) = x$?

25. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 8y = F(x)$, $0 < x < \pi$, $y(0) = y(\pi)$, $y'(0) = 0$. What is the solution when $F(x) = e^{2x}$?

26. $\dfrac{d}{dx}\left(x\dfrac{dy}{dx}\right) = F(x), \quad 0 < \alpha < x < \beta, \quad y(\alpha) - y'(\alpha) = m_1, \quad y(\beta) = m_2.$

27. Show that the solution of nonhomogeneous problem (39) can be expressed in the form

$$y(x) = \int_\alpha^\beta g(x;X)F(X)\,dX + \frac{m_2}{B_2 y_1} y_1(x) + \frac{m_1}{B_1 y_2} y_2(x),$$

where $y_1(x)$ and $y_2(x)$ are nontrivial solutions of the associated homogeneous equation that satisfy $B_1(y_1) = 0$ and $B_2(y_2) = 0$, respectively.

11.5 Modified Green's Functions

When homogeneous problem (27) has nontrivial solutions, Green's function for the operator L and boundary conditions (27b, c) does not exist. Another way of saying the same thing is that Green's function does not exist when $\lambda = 0$ is an eigenvalue of the associated problem

$$\frac{d}{dx}\left(a(x)\frac{dy}{dx}\right) + [c(x) + \lambda p(x)]y = 0, \tag{46a}$$

$$B_1 y = 0, \tag{46b}$$

$$B_2 y = 0. \tag{46c}$$

A physical example to illustrate this is the boundary value problem

$$-\kappa\frac{d^2U}{dx^2} = F(x), \qquad 0 < x < L,$$

$$U'(0) = 0, \; U'(L) = 0$$

for steady-state heat conduction in a rod with insulated sides and ends. The associated homogeneous problem has nontrivial solutions $U =$ constant. Notice that if we integrate the differential equation from $x = 0$ to $x = L$,

$$\int_0^L F(x)\,dx = \int_0^L -\kappa\frac{d^2U}{dx^2}\,dx = \left\{-\kappa\frac{dU}{dx}\right\}_0^L = 0.$$

Thus, if there is to be a solution to this problem, $F(x)$ cannot be specified arbitrarily; it must satisfy the condition

$$\int_0^L F(x)\,dx = 0. \tag{47}$$

Physically this means that with insulated sides and ends, the only way a steady-state condition can prevail is if total internal heat generation is zero.

Since the delta function $\delta(x - X)$ does not satisfy this condition (a unit point source at $x = X$), there can be no solution to

$$-\kappa\frac{d^2g}{dx^2} = \delta(x - X),$$

$$g'(0;X) = 0, \qquad g'(L;X) = 0$$

for the associated Green's function $g(x;X)$.

The condition equivalent to (47) in more general problems with homogeneous boundary conditions is contained in the following theorem.

eorem 7

When a homogeneous boundary value problem

$$Ly = \frac{d}{dx}\left(a\frac{dy}{dx}\right) + cy = 0, \qquad \alpha < x < \beta, \tag{48a}$$

$$B_1 y = 0, \tag{48b}$$

$$B_2 y = 0 \tag{48c}$$

has nontrivial solutions $w(x)$, the nonhomogeneous problem

$$Ly = \frac{d}{dx}\left(a\frac{dy}{dx}\right) + cy = F(x), \tag{49a}$$

$$B_1 y = 0, \tag{49b}$$

$$B_2 y = 0 \tag{49c}$$

has a solution if and only if

$$\int_\alpha^\beta F(x)w(x)\,dx = 0 \tag{50}$$

for every such solution $w(x)$.

It is easy to establish the necessity of condition (50). If $y(x)$ is a solution of (49), then

$$\begin{aligned}\int_\alpha^\beta F(x)w(x)\,dx &= \int_\alpha^\beta (Ly)w(x)\,dx \\ &= \int_\alpha^\beta y(Lw)\,dx + \{a(wy' - yw')\}_\alpha^\beta \quad \text{[using Green's formula (18)]} \\ &= a(\beta)[w(\beta)y'(\beta) - y(\beta)w'(\beta)] - a(\alpha)[w(\alpha)y'(\alpha) - y(\alpha)w'(\alpha)],\end{aligned}$$

since $Lw = 0$. These terms both vanish when boundary conditions (49b, c) are unmixed, and they cancel when the boundary conditions are periodic.

When (48) has nontrivial solutions and consistency condition (50) is satisfied, the solution of (49) is not unique. If $y(x)$ is a solution, so also is $y(x) + Cw(x)$ for arbitrary C and $w(x)$ a solution of (48).

To solve (49) when condition (50) is satisfied, we introduce "modified" Green's functions. We do so because there can be no "ordinary" Green's function satisfying

$$Lg = \delta(x - X),$$

$$B_1 g = 0, \qquad B_2 g = 0,$$

since $\delta(x - X)$ does not satisfy (50). Two situations arise, depending on whether (48) has one or two linearly independent solutions. We consider first the case in which (48) has only one nontrivial solution that is unique to a multiplicative constant and may therefore be taken as normalized:

$$\int_\alpha^\beta [w(x)]^2\,dx = 1. \tag{51}$$

A *modified Green's function* associated with (49) is defined as a solution $\bar{g}(x; X)$ of

$$L\bar{g} = \delta(x - X) - w(x)w(X), \tag{52a}$$

$$B_1\bar{g} = 0, \tag{52b}$$

$$B_2\bar{g} = 0. \tag{52c}$$

Because the right side of equation (52a) satisfies consistency condition (50), Theorem 7 guarantees a solution $\bar{g}(x; X)$. But because the solution is not unique, $\bar{g}(x; X)$ may or may not be symmetric, depending on the method used in its construction. It is important to note, however, that because the differential equation for the ordinary Green's function is modified only by the term $w(x)w(X)$, the modified Green's function satisfies the same continuity properties as the ordinary Green's function. Indeed, we shall use these properties to find $\bar{g}(x; X)$.

Modified Green's functions can be used to solve problem (49), which has a solution provided $F(x)$ satisfies (50). Green's identity (33) with $u = y(x)$ and $v = \bar{g}(x; X)$ gives

$$\int_\alpha^\beta [yL\bar{g} - \bar{g}Ly]\,dx = \left\{a(x)\left(y(x)\frac{\partial \bar{g}(x; X)}{\partial x} - \bar{g}(x; X)y'(x)\right)\right\}_\alpha^\beta.$$

It is quickly shown that with either unmixed or periodic boundary conditions, the right side of this equation vanishes. Differential equations (49a) and (52a) then give

$$\begin{aligned} 0 &= \int_\alpha^\beta (y(x)[\delta(x - X) - w(x)w(X)] - \bar{g}(x; X)F(x))\,dx \\ &= y(X) - C_1 w(X) - \int_\alpha^\beta \bar{g}(x; X)F(x)\,dx, \end{aligned}$$

where $C_1 = \int_\alpha^\beta y(x)w(x)\,dx$. Thus,

$$y(X) = \int_\alpha^\beta \bar{g}(x; X)F(x)\,dx + C_1 w(X) \tag{53}$$

or, interchanging x and X,

$$y(x) = \int_\alpha^\beta \bar{g}(X; x)F(X)\,dX + C_1 w(x). \tag{54}$$

Because $y(x)$ is unique only to the addition of a term $Cw(x)$, we may drop the subscript in (54) and write

$$y(x) = \int_\alpha^\beta \bar{g}(X; x)F(X)\,dX + Cw(x), \tag{55}$$

where C is arbitrary.

Finally, when the construction of $\bar{g}(x; X)$ gives a symmetric function, we may write

$$y(x) = \int_\alpha^\beta \bar{g}(x; X)F(X)\,dX + Cw(x), \tag{56}$$

and the form of the solution, except for the $Cw(x)$ term, is identical to that for ordinary Green's functions. Exercise 11 describes a technique for calculating symmetric modified Green's functions from nonsymmetric ones.

Example 10: Solve the boundary value problem

$$\frac{d^2y}{dx^2} + 4y = F(x), \qquad 0 < x < \pi,$$

$$y(0) = 0 = y(\pi).$$

Solution: Solutions of the homogeneous differential equation $y'' + 4y = 0$ are of the form $y = A\cos 2x + B\sin 2x$. Since the function $\sin 2x$ satisfies both boundary conditions, the Green's function for this problem does not exist. We define a modified Green's function $\bar{g}(x; X)$ as the solution of

$$\frac{d^2\bar{g}}{dx^2} + 4\bar{g} = \delta(x - X) - \frac{2}{\pi}\sin 2x \sin 2X,$$

$$\bar{g}(0; X) = 0 = \bar{g}(\pi; X).$$

($2/\pi$ is the normalizing factor.) Because $\bar{g}(x; X)$ must satisfy property (21c), and a particular solution of $\bar{g}'' + 4\bar{g} = -(2/\pi)\sin 2x \sin 2X$ is $(2\pi)^{-1}x\sin 2X\cos 2x$, we take

$$\bar{g}(x; X) = \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x + B\cos 2x & 0 \le x < X \\ C\sin 2x + D\cos 2x & X < x \le \pi \end{cases}.$$

To determine A, B, C, and D, we apply boundary conditions $\bar{g}(0; X) = 0 = \bar{g}(\pi; X)$,

$$B = 0,$$

$$\frac{1}{2}\sin 2X + D = 0,$$

and continuity conditions (21a, b) at $x = X$,

$$A\sin 2X + B\cos 2X = C\sin 2X + D\cos 2X,$$

$$(2C\cos 2X - 2D\sin 2X) - (2A\cos 2X - 2B\sin 2X) = 1.$$

These four equations require that

$$B = 0, \qquad D = -\frac{1}{2}\sin 2X, \quad \text{and} \quad C = A + \frac{1}{2}\cos 2X,$$

where $A = A(X)$ is an arbitrary function of X. A modified Green's function is therefore

$$\begin{aligned} \bar{g}(x; X) &= \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x & 0 \le x \le X \\ \left(A + \frac{1}{2}\cos 2X\right)\sin 2x - \frac{1}{2}\sin 2X\cos 2x & X \le x \le \pi \end{cases} \\ &= \frac{x}{2\pi}\sin 2X\cos 2x + \begin{cases} A\sin 2x & 0 \le x \le X \\ A\sin 2x + \frac{1}{2}\sin 2(x - X) & X \le x \le \pi \end{cases} \\ &= \frac{x}{2\pi}\sin 2X\cos 2x + A\sin 2x + \frac{1}{2}\sin 2(x - X)H(x - X). \end{aligned}$$

Notice that the arbitrariness in $\bar{g}(x;X)$ is a constant $A(X)$ times $w(x)$, the solution of the homogeneous problem. Because $\bar{g}(x;X)$ is not symmetric, we use equation (55) to express the solution of the original boundary value problem in the form

$$\begin{aligned} y(x) &= \int_0^\pi \bar{g}(X;x)F(X)\,dX + C\sin 2x \\ &= \int_0^\pi \left(\frac{X}{2\pi}\sin 2x\cos 2X + A(x)\sin 2X + \frac{1}{2}\sin 2(X-x)H(X-x)\right)F(X)\,dX \\ &\quad + C\sin 2x \qquad (C \text{ a constant}) \\ &= C\sin 2x + \frac{\sin 2x}{2\pi}\int_0^\pi X\cos 2X\,F(X)\,dX + A(x)\int_0^\pi F(X)\sin 2X\,dX \\ &\quad + \frac{1}{2}\int_x^\pi \sin 2(X-x)\,F(X)\,dX. \end{aligned}$$

Since the first integral is a constant, the second term may be grouped with $C\sin 2x$. Furthermore, the second integral vanishes because of consistency condition (50). Thus, the final solution is

$$y(x) = C\sin 2x + \frac{1}{2}\int_x^\pi \sin 2(X-x)F(X)\,dX. \qquad \blacksquare$$

We have considered the situation in which the homogeneous problem (48) corresponding to (49) has a single nontrivial solution (unique to a multiplicative constant). The remaining possibility is that all solutions of (48a) satisfy boundary conditions (48b, c). In such a case, we can always find two orthonormal solutions $v(x)$ and $w(x)$ of $Ly = 0$. If $\psi(x)$ and $\phi(x)$ are linearly independent solutions, two orthonormal solutions are

$$v(x) = \psi(x)\left(\int_\alpha^\beta [\psi(x)]^2\,dx\right)^{-1/2}$$

and

$$w(x) = \left(\phi(x) - v(x)\int_\alpha^\beta \phi(x)v(x)\,dx\right)\left(\int_\alpha^\beta\left(\phi(x) - v(x)\int_\alpha^\beta \phi(x)v(x)\,dx\right)^2 dx\right)^{-1/2}.$$

[$\psi(x)$ is normalized to form $v(x)$. For $w(x)$, the component of $\phi(x)$ in the "direction" of $v(x)$ is removed, and the result is then normalized.] We define a modified Green's function $\bar{g}(x;X)$ associated with (49) as a solution of

$$L\bar{g} = \delta(x-X) - w(x)w(X) - v(x)v(X), \qquad (57a)$$

$$B_1\bar{g} = 0, \qquad (57b)$$

$$B_2\bar{g} = 0. \qquad (57c)$$

Because the right side of (57a) satisfies consistency condition (50), $\bar{g}(x;X)$ must indeed exist. Green's identity once again gives the solution of (49) as

$$y(x) = \int_{\alpha}^{\beta} \bar{g}(X; x)F(X)\,dX + Cw(x) + Dv(x), \tag{58}$$

where C and D are arbitrary constants.

Example 11: Solve the boundary value problem

$$\frac{d^2y}{dx^2} + y = F(x), \qquad 0 < x < 2\pi,$$

$$y(0) = y(2\pi), \qquad y'(0) = y'(2\pi).$$

Solution: The homogeneous problem has nontrivial solutions $\sin x$ and $\cos x$. Because these functions are orthogonal, a modified Green's function for this problem is defined by

$$\frac{d^2\bar{g}}{dx^2} + \bar{g} = \delta(x - X) - \pi^{-1}(\sin x \sin X + \cos x \cos X),$$

$$\bar{g}(0; X) = \bar{g}(2\pi; X), \qquad \frac{\partial \bar{g}(0; X)}{\partial x} = \frac{\partial \bar{g}(2\pi; X)}{\partial x}.$$

A solution of the differential equation is

$$\bar{g}(x; X) = \frac{x}{2\pi}\sin(X - x) + \begin{cases} A\sin x + B\cos x & 0 \le x < X \\ C\sin x + D\cos x & X < x \le 2\pi \end{cases}.$$

To determine A, B, C, and D, we first apply the boundary conditions

$$B = \sin X + D,$$

$$\frac{\sin X}{2\pi} + A = \frac{\sin X}{2\pi} - \cos X + C,$$

and then continuity conditions (21a, b) at $x = X$,

$$A\sin X + B\cos X = C\sin X + D\cos X,$$

$$C\cos X - D\sin X - A\cos X + B\sin X = 1.$$

These four conditions require that $A = C - \cos X$ and $B = D + \sin X$, where $C = C(X)$ and $D = D(X)$ are arbitrary functions of X. A modified Green's function is therefore

$$\bar{g}(x; X) = \frac{x}{2\pi}\sin(X - x) + C\sin x + D\cos x + \begin{cases} \sin X\cos x - \cos X\sin x & 0 \le x \le X \\ 0 & X \le x \le 2\pi \end{cases}$$

$$= C\sin x + D\cos x + \sin(X - x)\begin{cases} \dfrac{x}{2\pi} + 1 & 0 \le x \le X \\ \dfrac{x}{2\pi} & X \le x \le 2\pi \end{cases}$$

$$= C\sin x + D\cos x + \sin(X - x)\left(\frac{x}{2\pi} + H(X - x)\right).$$

According to (58), the solution of the boundary value problem is

$$
\begin{aligned}
y(x) &= \int_0^{2\pi} \bar{g}(X;x)F(X)\,dX + E\sin x + G\cos x \\
&= \int_0^{2\pi}\left[C\sin X + D\cos X + \sin(x-X)\left(\frac{X}{2\pi} + H(x-X)\right)\right]F(X)\,dX \\
&\quad + E\sin x + G\cos x \\
&= E\sin x + G\cos x + \int_0^{2\pi} \sin(x-X)\left(\frac{X}{2\pi} + H(x-X)\right)F(X)\,dX,
\end{aligned}
$$

since $F(x)$ must satisfy the consistency conditions

$$\int_0^{2\pi} F(x)\sin x\,dx = 0 = \int_0^{2\pi} F(x)\cos x\,dx. \qquad \blacksquare$$

When boundary conditions in (49b, c) are nonhomogeneous (and therefore unmixed), it is also necessary to introduce modified Green's functions into the results of Section 11.4. The following results are proved in Exercises 4 and 9.

When (48) has only one solution $w(x)$ (unique to a multiplicative constant), the solution of

$$Ly = \frac{d}{dx}\left(a\frac{dy}{dx}\right) + cy = F(x), \qquad \alpha < x < \beta, \tag{59a}$$

$$B_1 y = m_1, \tag{59b}$$

$$B_2 y = m_2 \tag{59c}$$

is

$$y(x) = \int_\alpha^\beta \bar{g}(X;x)F(X)\,dX + Cw(x) - \frac{m_1}{l_1}a(\alpha)\bar{g}(\alpha;x) - \frac{m_2}{l_2}a(\beta)\bar{g}(\beta;x) \tag{60a}$$

or, when $l_1 = l_2 = 0$,

$$y(x) = \int_\alpha^\beta \bar{g}(X;x)F(X)\,dX + Cw(x) + m_2 a(\beta)\frac{\partial\bar{g}(\beta;x)}{\partial X} - m_1 a(\alpha)\frac{\partial\bar{g}(\alpha;x)}{\partial X}. \tag{60b}$$

If (48) has two linearly independent solutions $v(x)$ and $w(x)$, the quantity $Cw(x)$ in equations (60) is replaced by $Cw(x) + Dv(x)$, and the solutions are otherwise the same.

In all cases, a solution of (59) exists if and only if $F(x)$, m_1, and m_2 satisfy the consistency condition

$$\int_\alpha^\beta F(x)w(x)\,dx = \frac{m_2}{l_2}a(\beta)w(\beta) + \frac{m_1}{l_1}a(\alpha)w(\alpha) \tag{61a}$$

or, when $l_1 = l_2 = 0$,

$$\int_\alpha^\beta F(x)w(x)\,dx = m_1 a(\alpha)w'(\alpha) - m_2 a(\beta)w'(\beta) \tag{61b}$$

for every solution $w(x)$ of the corresponding homogeneous problem.

Exercises 11.5

1. Solve the boundary value problem

$$-\kappa\frac{d^2U}{dx^2} = F(x), \qquad 0 < x < L,$$
$$U'(0) = 0 = U'(L)$$

when $F(x)$ satisfies consistency condition (47). Calculate the solution in closed form when $F(x) = \cos(\pi x/L)$.

2. Verify that the result in Example 11 gives the correct solution when $F(x) = \sin 2x$.

3. **(a)** Simplify the solution to Example 10 when $F(x) = \cos 2x$.

   (b) Use equation (60b) to find the solution when the boundary conditions are nonhomogeneous:

$$y(0) = m_1, \qquad y(\pi) = m_2.$$

   What condition must be imposed on m_1 and m_2?

4. Verify consistency conditions (61) for the nonhomogeneous problem (59).

5. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = F(x), \qquad 0 < x < L \qquad (k > 0 \text{ a constant}),$$
$$y(0) = 0 = y(L).$$

6. **(a)** Use the result of Exercise 5 to solve

$$\frac{d^2y}{dx^2} + \frac{9\pi^2}{L^2}y = F(x), \qquad 0 < x < L,$$
$$y(0) = m_1, \qquad y(L) = m_2.$$

   (b) Simplify the solution when $F(x) = x$. What is the consistency condition?

7. Solve the boundary value problem

$$\frac{d^2y}{dx^2} + k^2y = F(x), \qquad 0 < x < L \qquad (k > 0 \text{ a constant}),$$
$$y(0) = 0 = y'(L).$$

8. **(a)** Use the result of Exercise 7 to solve

$$\frac{d^2y}{dx^2} + \frac{25\pi^2}{4L^2}y = F(x), \qquad 0 < x < L,$$
$$y(0) = m_1, \qquad y'(L) = m_2.$$

   (b) Simplify the solution when $F(x) = x^2$. What is the consistency condition?

9. Verify the results in equations (60).

10. A modified Green's function for boundary value problem (59), when the corresponding homogeneous problem has only one solution $w(x)$ (unique to a multiplicative constant), is defined by boundary value problem (52). In this exercise we show that modified Green's functions can be defined in other ways. The homogeneous boundary value problem associated with the

heat conduction problem

$$-\kappa\frac{d^2U}{dx^2} = F(x), \qquad 0 < x < L,$$

$$U'(0) = m_1, \qquad U'(L) = m_2$$

has nontrivial solutions $y = \text{constant}$.

(a) Show that when a function $\bar{\bar{g}}(x; X)$ satisfies

$$-\kappa\frac{d^2\bar{\bar{g}}}{dx^2} = \delta(x - X),$$

$$\bar{\bar{g}}'(0; X) = \frac{1}{2\kappa}, \qquad \bar{\bar{g}}'(L; X) = \frac{-1}{2\kappa},$$

consistency condition (61a) for nonhomogeneous problems is satisfied.

(b) Use Green's formula (33) to show that $U(x)$ can be expressed in the form

$$U(x) = \int_0^L \bar{\bar{g}}(X; x)F(X)\,dX + \kappa[m_2\bar{\bar{g}}(L; x) - m_1\bar{\bar{g}}(0; x)] + C,$$

where C is an arbitrary constant. Find $\bar{\bar{g}}(x; X)$ and simplify this solution.

(c) Use the result in (b) to find the solution to the boundary value problem of Exercise 1 when $F(x) = \cos(\pi x/L)$.

11. (a) Show that there is only one modified Green's function $\bar{g}_s(x; X)$ satisfying (52) that is orthogonal to $w(x)$ and that this function is given by

$$\bar{g}_s(x; X) = \bar{g}(x; X) - w(x)\left(\int_\alpha^\beta \bar{g}(\xi; X)w(\xi)\,d\xi\right),$$

where $\bar{g}(x; X)$ is any modified Green's function whatsoever.

(b) Use Green's identity (34) with $u = \bar{g}_s(x; X)$ and $v = \bar{g}_s(x; Y)$ to show that $\bar{g}_s(x; X)$ is symmetric. Are there any other symmetric modified Green's functions?

12. Use Exercise 11 to find symmetric modified Green's functions for the problem in Exercise 1.

13. Use Exercise 11 to find symmetric modified Green's functions for the problem in Example 10.

14. (a) Show that there is only one modified Green's function $\bar{g}_s(x; X)$ satisfying (57) that is orthogonal to $w(x)$ and $v(x)$ and that this function is given by

$$\bar{g}_s(x; X) = \bar{g}(x; X) - w(x)\left(\int_\alpha^\beta \bar{g}(\xi; X)w(\xi)\,d\xi\right) - v(x)\left(\int_\alpha^\beta \bar{g}(\xi; X)v(\xi)\,d\xi\right),$$

where $\bar{g}(x; X)$ is any modified Green's function whatsoever.

(b) Use Green's identity (34) with $u = \bar{g}_s(x; X)$ and $v = \bar{g}_s(x; Y)$ to show that $\bar{g}_s(x; X)$ is symmetric. Are there any other symmetric modified Green's functions?

15. Use Exercise 14 to find symmetric modified Green's functions for the problem in Example 11.

11.6 Green's Functions for Initial Value Problems

When the conditions that accompany differential equation (19a) are of the form

$$y(\alpha) = 0, \qquad y'(\alpha) = 0, \tag{62}$$

they are called *initial conditions*, and the problem is known as an *initial value problem* rather than a boundary value problem. Because this situation arises most frequently when the independent variable is time t, we rewrite the initial value problem in the form

$$Ly = \frac{d}{dt}\left(a(t)\frac{dy}{dt}\right) + c(t)y = F(t), \qquad t > t_0, \tag{63a}$$

$$y(t_0) = m_1, \tag{63b}$$

$$y'(t_0) = m_2. \tag{63c}$$

Initial time t_0 is often chosen as $t_0 = 0$, but for the sake of generality we maintain arbitrary t_0.

It might seem natural to define the Green's function $g(t; T)$ for this problem as the function $g(t; T)$ satisfying

$$\frac{d}{dt}\left(a(t)\frac{dg}{dt}\right) + c(t)g = \delta(t - T), \tag{64a}$$

$$g(t_0; T) = 0 \qquad \frac{dg(t_0; T)}{dt} = 0. \tag{64b}$$

Unfortunately, this would lead to improper integral representations of solutions of (63), together with associated convergence problems. Instead, we define the Green's function $g(t; T)$ as what is called a *causal fundamental solution* of (63); it is the solution of

$$g(t; T) = 0, \qquad t_0 < t < T, \tag{65a}$$

$$Lg = \delta(t - T). \tag{65b}$$

Physically $g(t; T)$ is the reaction of the system described by (63) to a unit impulse at time T. Naturally, for time $t < T$, the system must be identically equal to zero [hence the requirement (65a)].

Provided $a(t)$ does not vanish for $t \geq t_0$, the solution of (65) exists and is unique. Furthermore, corresponding to properties (21), which characterize the Green's function for boundary value problem (19), the following conditions characterize the Green's function for initial value problem (63):

$$g(t; T) = 0, \qquad t_0 < t < T, \tag{66a}$$

$$Lg = \frac{d}{dt}\left(a(t)\frac{dg}{dt}\right) + c(t)g = 0, \qquad t > T, \tag{66b}$$

$$g(T+; T) = 0, \tag{66c}$$

$$\frac{dg(T+; T)}{dt} = \frac{1}{a(T)}. \tag{66d}$$

When $u(t)$ and $v(t)$ are linearly independent solutions of (66b), the function

$$g(t; T) = \frac{1}{J(u, v)}[u(T)v(t) - v(T)u(t)]H(t - T) \tag{67}$$

clearly satisfies (66) and must therefore be the Green's function for (63). This formula replaces (29) for boundary value problems, but notice that the condition that the associated homogeneous system have only the trivial solution is absent for initial value problems (it is always satisfied).

Example 12: What is the Green's function for the initial value problem

$$M\frac{d^2y}{dt^2} + ky = F(t), \qquad t > 0,$$
$$y(0) = m_1, \qquad y'(0) = m_2$$

for displacements of a mass M on the end of a spring with constant k?

Solution: Since $\sin\sqrt{k/M}\,t$ and $\cos\sqrt{k/M}\,t$ are solutions of $My'' + ky = 0$, the Green's function, according to (67), is

$$\begin{aligned} g(t;T) &= \frac{1}{J(\sin\sqrt{k/M}\,t, \cos\sqrt{k/M}\,t)}\left(\sin\sqrt{\frac{k}{M}}T\cos\sqrt{\frac{k}{M}}t\right. \\ &\qquad \left. -\cos\sqrt{\frac{k}{M}}T\sin\sqrt{\frac{k}{M}}t\right)H(t-T) \\ &= \frac{1}{-\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(T-t)\,H(t-T) \\ &= \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t-T)\,H(t-T). \end{aligned}$$ ■

The solution of an initial value problem can be expressed in terms of its Green's function. In particular, the solution of problem (63) is

$$y(t) = \int_{t_0}^{t} g(t;T)F(T)\,dT + a(t_0)\left(m_2 g(t;t_0) - m_1\frac{\partial g(t;t_0)}{\partial T}\right). \tag{68}$$

The integral term, which accounts for the nonhomogeneity in the differential equation, is interpreted as the superposition of incremental results. Because the Green's function $g(t;T)$ is the result at time t due to a unit impulse $\delta(t-T)$ at time T, $g(t;T)F(T)\,dT$ is the result at time t due to an incremental "force" $F(T)\,dT$ over dT. The integral then adds over all contributions, beginning at time t_0, to give the final result at time t. The last two terms in (68) account for nonhomogeneities in initial conditions (63b, c).

Example 13: What is the solution of the problem in Example 12?

Solution: According to (68), the solution is

$$\begin{aligned} y(t) &= \int_0^t \frac{1}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}(t-T)\,H(t-T)F(T)\,dT \\ &\quad + M\left(\frac{m_2}{\sqrt{kM}}\sin\sqrt{\frac{k}{M}}t + \frac{m_1}{\sqrt{kM}}\sqrt{\frac{k}{M}}\cos\sqrt{\frac{k}{M}}t\right) \\ &= \frac{1}{\sqrt{kM}}\int_0^t \sin\sqrt{\frac{k}{M}}(t-T)\,F(T)\,dT + \sqrt{\frac{M}{k}}m_2\sin\sqrt{\frac{k}{M}}t + m_1\cos\sqrt{\frac{k}{M}}t. \end{aligned}$$ ■

Exercises 11.6

1. A particle of mass M moves along the x-axis under the action of a force that is an explicit function $F(t)\,(t \geq 0)$ of time t only. Find an integral representation for its position as a function of time t if at time $t = 0$ it is moving with velocity v_0 at position x_0.
2. A mass M is suspended from a spring (with constant k). Vertical oscillations are initiated at time $t = 0$ by displacing M from its equilibrium position and giving it an initial speed. If motion takes place in a medium that causes a damping force proportional to velocity, and an external force $F(t)\,(t \geq 0)$ acts on M, find an integral representation for the position of M as a function of time t.
3. (a) Show that the solution of problem (63) can be expressed in the form

$$y(t) = \frac{1}{J(u, v)}\left(\int_{t_0}^{t} [u(T)v(t) - v(T)u(t)]F(T)\,dT + a(t_0)[m_1 v'(t_0) - m_2 v(t_0)]u(t) + a(t_0)[m_2 u(t_0) - m_1 u'(t_0)]v(t)\right),$$

where $u(t)$ and $v(t)$ are any two linearly independent solutions of $Ly = 0$.

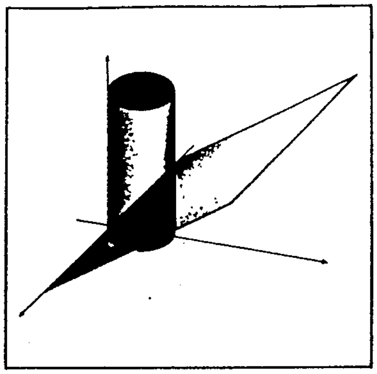
(b) Use the result in (a) to show that $y(t)$ can also be written in the form

$$y(t) = \frac{1}{a(t_0)}\int_{t_0}^{t} [u(T)v(t) - v(T)u(t)]F(T)\,dT + m_1 u(t) + m_2 v(t),$$

where $u(t)$ and $v(t)$ are solutions of $Ly = 0$ satisfying

$$u(t_0) = 1, \qquad u'(t_0) = 0; \qquad v(t_0) = 0, \qquad v'(t_0) = 1.$$

4. Use Exercise 3(b) to obtain the solution for Example 13.
5. Use Exercise 3(b) to solve Exercise 2.



C H A P T E R

T W E L V E

Green's Functions for Partial Differential Equations

12.1 Generalized Functions and Green's Identities

In this chapter we develop Green's functions for boundary value problems (and initial boundary value problems) associated with partial differential equations. Solutions to such problems can then be represented in terms of integrals of source functions and Green's functions. We begin by discussing multidimensional delta functions and Green's identities.

Two- and three-dimensional delta functions, like $\delta(x - c)$, are defined from a functional point of view. We discuss two-dimensional functions, but three-dimensional results are analogous. The generalized function $\delta(x - a, y - b)$ maps a function $f(x, y)$ continuous at (a, b) onto its value at (a, b); that is,

$$f(x, y) \xrightarrow{\delta(x-a,\, y-b)} f(a, b) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - a, y - b) f(x, y)\, dA. \tag{1}$$

Because successive applications of delta functions lead to

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x - a)\delta(y - b) f(x, y)\, dy\, dx = \int_{-\infty}^{\infty} \delta(x - a) f(x, b)\, dx = f(a, b), \tag{2}$$

it follows that

$$\delta(x-a, y-b) = \delta(x-a)\delta(y-b). \tag{3}$$

In other words, the two-dimensional delta function in Cartesian coordinates is the product of two one-dimensional delta functions. Corresponding to property (8) in Chapter 11, we take

$$\iint_R \delta(x-a, y-b) f(x,y)\,dA = \begin{cases} f(a,b) & (a,b) \text{ in } R \\ 0 & (a,b) \text{ not in } R \end{cases}. \tag{4}$$

Delta functions in curvilinear coordinates are defined analogously to those in Cartesian coordinates, but their expressions in terms of products of one-dimensional delta functions are complicated by formulas for area and volume elements in curvilinear coordinates. To illustrate, suppose that a point with Cartesian coordinates (x_0, y_0) has polar coordinates (r_0, θ_0). The delta function $\delta(r-r_0, \theta-\theta_0)$ in polar coordinates is that generalized function that assigns to a function $f(r,\theta)$, continuous at (r_0, θ_0), its value at (r_0, θ_0),

$$\iint_{R^2} \delta(r-r_0, \theta-\theta_0) f(r,\theta)\,dA = f(r_0, \theta_0), \tag{5a}$$

where R^2 refers to the xy-plane. But because $dA = r\,dr\,d\theta$, (5a) is expressible in the form

$$\int_{-\pi}^{\pi}\int_0^{\infty} \delta(r-r_0, \theta-\theta_0) f(r,\theta) r\,dr\,d\theta = f(r_0, \theta_0). \tag{5b}$$

Since

$$\int_{-\pi}^{\pi}\int_0^{\infty} \delta(r-r_0)\delta(\theta-\theta_0) f(r,\theta)\,dr\,d\theta = f(r_0, \theta_0), \tag{6}$$

it follows that $r\delta(r-r_0, \theta-\theta_0) = \delta(r-r_0)\delta(\theta-\theta_0)$, or

$$\delta(r-r_0, \theta-\theta_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0). \tag{7}$$

Since the delta function $\delta(x-x_0)\delta(y-y_0)$ and that in (7) pick out the value of a function at the same point, we may write

$$\delta(x-x_0)\delta(y-y_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0). \tag{8a}$$

Similarly, transformation laws from delta functions in Cartesian coordinates to those in cylindrical and spherical coordinates are

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \frac{1}{r}\delta(r-r_0)\delta(\theta-\theta_0)\delta(z-z_0) \tag{8b}$$

and

$$\delta(x-x_0)\delta(y-y_0)\delta(z-z_0) = \frac{1}{r^2\sin\phi}\delta(r-r_0)\delta(\theta-\theta_0)\delta(\phi-\phi_0). \tag{8c}$$

Many curvilinear coordinate systems, and in particular the above three, have *singular points*—points at which transformations between them and Cartesian coordinates fail to be one to one. In polar coordinates, the origin is singular, and in cylindrical and spherical coordinates, the z-axis is singular. Transformation laws (8) are not valid at singular points. To understand this, we first note that when the functional on the right side of (8a) operates on a function $f(r, \theta)$, it produces $f(r_0, \theta_0)$, the value of the function at (r_0, θ_0). But if $r_0 = 0$, the value of the function $f(r, \theta)$ does not depend on the value of θ; its value is completely dictated by setting $r = 0$. This means that when $r_0 = 0$, $\delta(\theta - \theta_0)$ on the right side of (8a) is redundant. To see how to remove this delta function, notice that if we write $F(0)$ for the value of $f(0, \theta)$, then

$$\int_0^\infty \delta(r) f(r, \theta)\, dr = F(0).$$

Integration of this result with respect to θ gives

$$\int_{-\pi}^{\pi} \int_0^\infty \delta(r) f(r, \theta)\, dr\, d\theta = \int_{-\pi}^{\pi} F(0)\, d\theta$$

or

$$\int_{-\pi}^{\pi} \int_0^\infty \frac{\delta(r)}{r} f(r, \theta) r\, dr\, d\theta = 2\pi F(0).$$

Thus,

$$\int_{-\pi}^{\pi} \int_0^\infty \frac{\delta(r)}{2\pi r} f(r, \theta) r\, dr\, d\theta = F(0).$$

But this equation implies that $\delta(r)/(2\pi r)$ must be the delta function at the origin, that is,

$$\delta(x)\delta(y) = \frac{\delta(r)}{2\pi r}. \tag{9}$$

A similar discussion in cylindrical coordinates shows that

$$\delta(x)\delta(y)\delta(z - z_0) = \frac{\delta(r)\delta(z - z_0)}{2\pi r}. \tag{10}$$

In spherical coordinates, we obtain

$$\delta(x)\delta(y)\delta(z - z_0) = \begin{cases} \dfrac{\delta(r - r_0)\delta(\phi)}{2\pi r^2 \sin\phi} & z_0 > 0 \\ \dfrac{\delta(r - r_0)\delta(\phi + \pi)}{2\pi r^2 \sin\phi} & z_0 < 0 \end{cases} \tag{11a}$$

and

$$\delta(x)\delta(y)\delta(z) = \frac{\delta(r)}{4\pi r^2}. \tag{11b}$$

Boundary value problems are associated with elliptic PDEs. We consider only two types in this chapter, those associated with the Helmholtz and Poisson's equations. The two-dimensional Helmholtz equation is

$$\nabla^2 u + k^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{12}$$

where A is some open region of the xy-plane (with a piecewise smooth boundary), and Poisson's equation is

$$\nabla^2 u = F(x, y), \qquad (x, y) \text{ in } A. \tag{13}$$

Green's (second) identity for both of these operators states that for functions $u(x, y)$ and $v(x, y)$ that have continuous first partial derivatives and piecewise continuous second partials in A,

$$\boxed{\iint_A (v\nabla^2 u - u\nabla^2 v)\,dA = \oint_{\beta(A)} (v\nabla u - u\nabla v)\cdot\hat{\mathbf{n}}\,ds,} \tag{14a}$$

where $\hat{\mathbf{n}}$ is the unit outward normal vector to the boundary $\beta(A)$ of A (see Appendix C). This identity is also valid when $u(x, y)$ and/or $v(x, y)$ satisfy the PDEs $\nabla^2 u + k^2 u = \delta(x - X, y - Y)$ or $\nabla^2 u = \delta(x - X, y - Y)$. These extensions parallel those in Theorems 4 and 5 in Chapter 11 for Green's formulas.

The three-dimensional version of Green's identity is

$$\boxed{\iiint_V (v\nabla^2 u - u\nabla^2 v)\,dV = \iint_{\beta(V)} (v\nabla u - u\nabla v)\cdot\hat{\mathbf{n}}\,dS,} \tag{14b}$$

where V is a volume in space with piecewise smooth boundary $\beta(V)$. It is also valid when either $u(x, y, z)$ or $v(x, y, z)$ satisfies PDE $\nabla^2 u + k^2 u = \delta(x - X, y - Y, z - Z)$ or PDE $\nabla^2 u = \delta(x - X, y - Y, z - Z)$.

12.2 Green's Functions for Dirichlet Boundary Value Problems

Dirichlet problems for the two-dimensional Helmholtz equation take the form

$$Lu = \nabla^2 u + k^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{15a}$$

$$u(x, y) = K(x, y), \qquad (x, y) \text{ on } \beta(A). \tag{15b}$$

For $k = 0$, we have the special case of Poisson's equation. When $F(x, y)$ has continuous first derivatives and piecewise continuous second derivatives in A, as does $K(x, y)$ on $\beta(A)$, problem (15) has a unique solution. A simplified example was discussed in Section 5.6 [see problem (74)]. In practical situations when $F(x, y)$ and $K(x, y)$ may not satisfy these conditions, verification of uniqueness is much more difficult, as is finding the solution by previous methods. Green's functions are an excellent alternative.

We define the Green's function $G(x, y; X, Y)$ for problem (15) as the solution of

$$LG = \nabla^2 G + k^2 G = \delta(x - X, y - Y), \qquad (x, y) \text{ in } A, \tag{16a}$$

$$G(x, y; X, Y) = 0, \qquad (x, y) \text{ on } \beta(A). \tag{16b}$$

It is the solution of (15) due to a unit point source at (X, Y) when boundary conditions are homogeneous. It is straightforward, then, to prove that the function

$$u(x, y) = \iint_A G(x, y; X, Y)F(X, Y)\,dA \tag{17}$$

satisfies (15a) (since integrations with respect to X and Y may be interchanged with differentiations with respect to x and y). Additional terms must be added to (17) in order to account for the nonhomogeneity $K(x, y)$ in boundary condition (15b). But clearly, $u(x, y) = 0$ on $\beta(A)$. In other words, when $G(x, y; X, Y)$ is the Green's function for (15), the function $u(x, y)$ in (17) satisfies

$$Lu = \nabla^2 u + k^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{18a}$$

$$u(x, y) = 0, \qquad (x, y) \text{ on } \beta(A). \tag{18b}$$

For boundary value problems associated with ODEs, we derived general formulas [equations (28) and (29) in Chapter 11] for Green's functions. This was possible because boundaries for ODEs consist of two points. For PDEs, boundaries consist of curves for two-dimensional problems and surfaces for three-dimensional problems. As a result, it is impossible to find formulas for Green's functions associated with multivariable boundary value problems. At best, we can hope to develop general techniques useful in large classes of problems. We illustrate some of these methods in this section. Before doing so, however, notice that if we substitute $u = G(x, y; X, Y)$ and $v = G(x, y; R, S)$ into Green's identity (14a),

$$\iint_A [G(x, y; R, S)\nabla^2 G(x, y; X, Y) - G(x, y; X, Y)\nabla^2 G(x, y; R, S)]\, dA = 0$$

[since $G(x, y; R, S)$ and $G(x, y; X, Y)$ satisfy boundary condition (16b)]. But because G is a solution of PDE (16a), we may write

$$\begin{aligned} 0 &= \iint_A \{G(x, y; R, S)[\delta(x - X, y - Y) - k^2 G(x, y; X, Y)] \\ &\quad - G(x, y; X, Y)[\delta(x - R, y - S) - k^2 G(x, y; R, S)]\}\, dA \\ &= G(X, Y; R, S) - G(R, S; X, Y). \end{aligned}$$

In other words, the Green's function is symmetric under an interchange of first and second variables with third and fourth:

$$G(x, y; X, Y) = G(X, Y; x, y). \tag{19}$$

This result is also valid when boundary condition (15b) is replaced by either a Neumann or a Robin condition.

We now illustrate four techniques for finding Green's functions.

Full Eigenfunction Expansion

In this method, the Green's function is expanded in terms of orthonormal eigenfunctions of the associated eigenvalue problem

$$Lu + \lambda^2 u = 0, \qquad (x, y) \text{ in } A, \tag{20a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A). \tag{20b}$$

We illustrate with the following example.

Example 1: Find the Green's function associated with the Dirichlet problem for the two-dimensional Laplacian on a rectangle $A: 0 \le x \le L, 0 \le y \le L'$.

Solution: Separation of variables on

$$\nabla^2 u + \lambda^2 u = 0, \qquad (x, y) \text{ in } A, \tag{21a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A), \tag{21b}$$

leads to normalized eigenfunctions

$$u_{mn}(x, y) = \frac{2}{\sqrt{LL'}} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'},$$

corresponding to eigenvalues $\lambda_{mn}^2 = (n\pi/L)^2 + (m\pi/L')^2$ (see Section 5.5). The eigenfunction expansion of $G(x, y; X, Y)$ in terms of these eigenfunctions is

$$G(x, y; X, Y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn} u_{mn}(x, y), \tag{22}$$

and this representation satisfies the boundary condition that G vanish on the edges of the rectangle. To calculate the coefficients c_{mn}, we substitute (22) into the PDE $\nabla^2 G = \delta(x - X, y - Y)$ for G and expand $\delta(x - X, y - Y)$ in terms of the $u_{mn}(x, y)$:

$$\begin{aligned}
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} c_{mn}\left(-\frac{n^2\pi^2}{L^2} - \frac{m^2\pi^2}{L'^2}\right) u_{mn}(x, y) \\
&= \delta(x - X, y - Y) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left(\int_0^L \int_0^{L'} \delta(x - X)\delta(y - Y) u_{mn}(x, y)\, dy\, dx\right) u_{mn}(x, y) \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} u_{mn}(X, Y) u_{mn}(x, y).
\end{aligned}$$

Consequently, $c_{mn} = u_{mn}(X, Y)/(-\lambda_{mn}^2)$, and

$$\begin{aligned}
G(x, y; X, Y) &= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{u_{mn}(X, Y)}{-\lambda_{mn}^2} u_{mn}(x, y) \\
&= \frac{-4}{LL'} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{\left(\frac{n\pi}{L}\right)^2 + \left(\frac{m\pi}{L'}\right)^2} \sin\frac{n\pi X}{L} \sin\frac{m\pi Y}{L'} \sin\frac{n\pi x}{L} \sin\frac{m\pi y}{L'}.
\end{aligned} \tag{23}$$

In Exercise 1 it is shown that (23) can also be obtained using Green's identity (14a). This avoids the interchange of the Laplacian and summation operations and the eigenfunction expansion of $\delta(x - X, y - Y)$. ∎

A general formula for full eigenfunction expansions can be found in Exercise 2, but such expansions are of limited calculational utility. First, they are possible only when the eigenvalue problem can be separated, and this requires that the boundary of A consist of coordinate curves (or coordinate surfaces, in three-dimensional problems). Second, in the case in which the full eigenfunction expansion is available, a partial eigenfunction expansion that converges more rapidly is also available.

Partial Eigenfunction Expansion

Like the full eigenfunction expansion, this method requires that region A be bounded by coordinate curves (or coordinate surfaces, in three-dimensional problems). It differs in that separation is considered on the homogeneous problem

$$Lu = 0, \qquad (x, y) \text{ in } A, \tag{24a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A), \tag{24b}$$

and is carried out until one variable remains. An eigenfunction expansion for the Green's function is then found in terms of normalized eigenfunctions already determined, with coefficients that are functions of the remaining variable. We illustrate once again with the problem in Example 1.

Example 2: Find a partial eigenfunction representation for the Green's function of Example 1.

***Solution*:** Separation of variables on

$$\nabla^2 u = 0, \qquad (x, y) \text{ in } A, \tag{25a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A), \tag{25b}$$

leads to normalized eigenfunctions $f_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. We expand $G(x, y; X, Y)$ in terms of these:

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} a_n(y) f_n(x). \tag{26}$$

In actual fact, coefficients $a_n(y)$ must also be functions of X and Y, but we shall understand this dependence implicitly rather than express it explicitly. To determine the $a_n(y)$, we substitute (26) into the PDE $\nabla^2 G = \delta(x - X, y - Y)$ for G and expand $\delta(x - X, y - Y)$ in terms of the $f_n(x)$:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{-n^2\pi^2}{L^2} a_n f_n(x) + \sum_{n=1}^{\infty} \frac{d^2 a_n}{dy^2} f_n(x) &= \delta(x - X, y - Y)\\ &= \sum_{n=1}^{\infty}\left(\int_0^L \delta(x - X, y - Y) f_n(x)\,dx\right) f_n(x)\\ &= \sum_{n=1}^{\infty} f_n(X)\delta(y - Y) f_n(x).\end{aligned}$$

This equation and the boundary conditions $G(x, 0; X, Y) = 0 = G(x, L'; X, Y)$ require the $a_n(y)$ to satisfy

$$\frac{d^2 a_n}{dy^2} - \frac{n^2\pi^2}{L^2} a_n = \delta(y - Y) f_n(X), \qquad 0 < y < L',$$

$$a_n(0) = 0, \qquad a_n(L') = 0.$$

We can solve this boundary value problem most easily by using our theory of Green's functions for ODEs. Since a solution of the homogeneous equation that satisfies the first boundary condition is $\sinh(n\pi y/L)$, and one that satisfies the second is $\sinh[n\pi(L' - y)/L]$, equation (29) in Chapter 11 gives

$$a_n(y) = \frac{1}{J}\left(\sinh\frac{n\pi y}{L}\sinh\frac{n\pi}{L}(L' - Y)H(Y - y) + \sinh\frac{n\pi Y}{L}\sinh\frac{n\pi}{L}(L' - y)H(y - Y)\right),$$

where J is the conjunct of $\sinh(n\pi y/L)$ and $\sinh[n\pi(L' - y)/L]$,

$$J = \frac{1}{f_n(X)}\left[\sinh\frac{n\pi y}{L}\left(\frac{-n\pi}{L}\right)\cosh\frac{n\pi}{L}(L' - y) - \left(\frac{n\pi}{L}\right)\cosh\frac{n\pi y}{L}\sinh\frac{n\pi}{L}(L' - y)\right]$$

$$= -\frac{n\pi\sinh(n\pi L'/L)}{\sqrt{2L}\sin(n\pi X/L)}.$$

Thus, an alternative expression to the double-series, full eigenfunction expansion for $G(x, y; X, Y)$ is the single-series, partial eigenfunction expansion

$$G(x, y; X, Y) = \sum_{n=1}^{\infty}\frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}}{n\pi\sinh\frac{n\pi L'}{L}}\left(\sinh\frac{n\pi y}{L}\sinh\frac{n\pi}{L}(L' - Y)H(Y - y)\right.$$

$$\left. + \sinh\frac{n\pi Y}{L}\sinh\frac{n\pi}{L}(L' - y)H(y - Y)\right)$$

$$= \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}\sinh\frac{n\pi}{L}(L' - Y)}{n\pi\sinh\frac{n\pi L'}{L}} & 0 \le y \le Y \\ \displaystyle\sum_{n=1}^{\infty}\frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\sinh\frac{n\pi Y}{L}\sinh\frac{n\pi}{L}(L' - y)}{n\pi\sinh\frac{n\pi L'}{L}} & Y \le y \le L' \end{cases}. \quad (27)$$

It is clear that we could find an equivalent solution by expanding G in a Fourier sine series in y. The result would be

$$G = \begin{cases} \displaystyle\sum_{n=1}^{\infty}\frac{-2\sin\frac{n\pi Y}{L'}\sin\frac{n\pi y}{L'}\sinh\frac{n\pi x}{L'}\sinh\frac{n\pi}{L'}(L - X)}{n\pi\sinh\frac{n\pi L}{L'}} & 0 \le x \le X \\ \displaystyle\sum_{n=1}^{\infty}\frac{-2\sin\frac{n\pi Y}{L'}\sin\frac{n\pi y}{L'}\sinh\frac{n\pi X}{L'}\sinh\frac{n\pi}{L'}(L - x)}{n\pi\sinh\frac{n\pi L}{L'}} & X \le x \le L \end{cases}. \quad (28)$$

A natural question to ask is, when should each of these expressions for $G(x, y; X, Y)$ be used? Since each is a Fourier series [(27) in x and (28) in y], rates of convergence of the series will depend on the relative magnitudes of coefficients. The coefficient of $\sin(n\pi x/L)$ in (27) for $y > Y$ is

$$\frac{-2\sin\dfrac{n\pi X}{L}\sinh\dfrac{n\pi Y}{L}\sinh\dfrac{n\pi}{L}(L'-y)}{n\pi\sinh\dfrac{n\pi L'}{L}},$$

and for large n we may drop the negative exponentials in the hyperbolic functions and approximate this quantity with

$$-\frac{e^{n\pi Y/L}e^{n\pi(L'-y)/L}}{n\pi e^{n\pi L'/L}}\sin\frac{n\pi X}{L} = \frac{-1}{n\pi}e^{n\pi(Y-y)/L}\sin\frac{n\pi X}{L}.$$

Similarly, when $y < Y$, the coefficient can, for large n, be approximated by

$$\frac{-1}{n\pi}e^{n\pi(y-Y)/L}\sin\frac{n\pi X}{L}.$$

Corresponding coefficients in (28) are approximated for large n by

$$\frac{-1}{n\pi}e^{-n\pi|X-x|/L'}\sin\frac{n\pi Y}{L'}.$$

It follows that to calculate $G(x, y; X, Y)$ at a value of x that is substantially different from X, it would be wise to use (28), and, conversely, when y is markedly different from Y, (27) would provide faster convergence.

In addition, when boundary integrals arise for the solution of Dirichlet's problem (15) [and this occurs for nonhomogeneous boundary conditions (15b)], it is advantageous to use (27) for integrations along $y = 0$ and $y = L'$ but use (28) along $x = 0$ and $x = L$. ■

Splitting Technique

Sometimes it is convenient to separate G into two parts, $G = U + g$, where U contains the singular part of G due to the delta function in (16a) and g guarantees that G satisfies the boundary conditions associated with L. This splitting technique permits consideration of the singular nature of the Green's function without the annoyance of boundary conditions. [The technique could have been used for ODEs, but it was unnecessary because formulas (28) and (29) in Chapter 11 were presented for Green's functions.] To be more specific, for the Green's function satisfying (16), we set $G = U + g$, where $U(x, y; X, Y)$ satisfies the PDE

$$LU = \delta(x - X, y - Y) \tag{29}$$

and g satisfies the boundary value problem

$$Lg = 0, \qquad (x, y) \text{ in } A, \tag{30a}$$

$$g = -U, \qquad (x, y) \text{ on } \beta(A). \tag{30b}$$

Because $U(x, y; X, Y)$ is not required to satisfy boundary conditions, it is often called the *free-space Green's function for the operator L*. Free-space Green's functions for the

Helmholtz, modified Helmholtz, and Laplace operators in two and three dimensions are listed in Table 12.1. Each is singular at the source point (X, Y).

Table 12.1 *Free-Space Green's Functions*

	∇^2 Laplacian	$\nabla^2 + k^2$ Helmholtz	$\nabla^2 - k^2$ Modified Helmholtz
xy-plane	$\frac{1}{2\pi}\ln\sqrt{(x-X)^2+(y-Y)^2}$	$\frac{1}{4}Y_0[k\sqrt{(x-X)^2+(y-Y)^2}]$	$-\frac{1}{2\pi}K_0[k\sqrt{(x-X)^2+(y-Y)^2}]$
xyz-space	$\frac{-1}{4\pi\sqrt{(x-X)^2+(y-Y)^2+(z-Z)^2}}$	$-\frac{e^{ikr}}{4\pi}, -\frac{e^{-ikr}}{4\pi}$	$-\frac{e^{kr}}{4\pi r}, -\frac{e^{-kr}}{4\pi r}$

We illustrate the splitting technique in the following example.

Example 3: Find the Green's function for the Dirichlet problem associated with Laplace's equation on a circle $0 \le r \le r_0$.

Solution: The Green's function associated with the Dirichlet problem for the Laplacian on a circle centered at the origin with radius r_0 satisfies

$$\nabla^2 G = \frac{\delta(r-R)\delta(\theta-\Theta)}{r}, \quad 0 < r < r_0, \quad -\pi < \theta \le \pi, \tag{31a}$$

$$G(r_0, \theta; R, \Theta) = 0, \quad -\pi < \theta \le \pi. \tag{31b}$$

The free-space Green's function for the two-dimensional Laplacian with singularity at (R, Θ) is

$$\begin{aligned} U(r,\theta;R,\Theta) &= \frac{1}{2\pi}\ln\sqrt{(r\cos\theta - R\cos\Theta)^2 + (r\sin\theta - R\sin\Theta)^2} \\ &= \frac{1}{4\pi}\ln[r^2 + R^2 - 2rR\cos(\theta-\Theta)] \end{aligned}$$

(see Table 12.1). When we split G into $G = U + g$, g must satisfy

$$\nabla^2 g = 0, \quad 0 < r < r_0, \quad -\pi < \theta \le \pi, \tag{32a}$$

$$g(r_0,\theta;R,\Theta) = -\frac{1}{4\pi}\ln[r_0^2 + R^2 - 2r_0R\cos(\theta-\Theta)], \quad -\pi < \theta \le \pi. \tag{32b}$$

Separation of variables on the PDE, together with boundedness at $r = 0$, leads to a solution of the form

$$g(r,\theta;R,\Theta) = \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(a_n r^n \frac{\cos n\theta}{\sqrt{\pi}} + b_n r^n \frac{\sin n\theta}{\sqrt{\pi}}\right)$$

[see equation (33a) in Section 5.3]. Boundary condition (32b) requires that

$$\begin{aligned} \frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(a_n r_0^n \frac{\cos n\theta}{\sqrt{\pi}} + b_n r_0^n \frac{\sin n\theta}{\sqrt{\pi}}\right) &= \frac{-1}{4\pi}\ln[r_0^2 + R^2 - 2r_0R\cos(\theta-\Theta)] \\ &= \frac{-1}{4\pi}\ln r_0^2 - \frac{1}{4\pi}\ln\left[1 + \left(\frac{R}{r_0}\right)^2 - 2\left(\frac{R}{r_0}\right)\cos(\theta-\Theta)\right]. \end{aligned}$$

With the result

$$\sum_{n=1}^{\infty} \frac{\alpha^n \cos n\phi}{n} = -\frac{1}{2}\ln(1 + \alpha^2 - 2\alpha\cos\phi) \qquad (|\alpha| < 1), \tag{33}$$

we may write

$$\begin{aligned}
&\frac{a_0}{\sqrt{2\pi}} + \sum_{n=1}^{\infty}\left(a_n r_0^n \frac{\cos n\theta}{\sqrt{\pi}} + b_n r_0^n \frac{\sin n\theta}{\sqrt{\pi}}\right) \\
&\quad = \frac{-1}{4\pi}\ln r_0^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/r_0)^n}{n}\cos n(\theta - \Theta) \\
&\quad = \frac{-1}{4\pi}\ln r_0^2 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(R/r_0)^n}{n}(\cos n\theta\cos n\Theta + \sin n\theta \sin n\Theta).
\end{aligned}$$

Comparison of coefficients requires that

$$\frac{a_0}{\sqrt{2\pi}} = \frac{-1}{4\pi}\ln r_0^2, \qquad \frac{a_n r_0^n}{\sqrt{\pi}} = \frac{(R/r_0)^n}{2\pi n}\cos n\Theta, \qquad \frac{b_n r_0^n}{\sqrt{\pi}} = \frac{(R/r_0)^n}{2\pi n}\sin n\Theta,$$

and therefore

$$\begin{aligned}
g(r,\theta;R,\Theta) &= \frac{-1}{2\pi}\ln r_0 + \sum_{n=1}^{\infty} r^n\left(\frac{(R/r_0)^n}{2\pi n r_0^n}\cos n\theta\cos n\Theta + \frac{(R/r_0)^n}{2\pi n r_0^n}\sin n\theta\sin n\Theta\right) \\
&= \frac{-1}{2\pi}\ln r_0 + \frac{1}{2\pi}\sum_{n=1}^{\infty}\frac{(rR/r_0^2)^n}{n}\cos n(\theta - \Theta).
\end{aligned}$$

But identity (33) permits evaluation of this series in closed form:

$$\begin{aligned}
g(r,\theta;R,\Theta) &= \frac{-1}{2\pi}\ln r_0 - \frac{1}{4\pi}\ln\left[1 + \left(\frac{rR}{r_0^2}\right)^2 - 2\left(\frac{rR}{r_0^2}\right)\cos(\theta - \Theta)\right] \\
&= \frac{1}{2\pi}\ln r_0 - \frac{1}{4\pi}\ln[r_0^4 + R^2r^2 - 2r_0^2 Rr\cos(\theta - \Theta)].
\end{aligned}$$

Finally,

$$\begin{aligned}
G(r,\theta;R,\Theta) = U + g = \frac{1}{4\pi}\ln[r^2 + R^2 - 2Rr\cos(\theta - \Theta)] + \frac{1}{2\pi}\ln r_0 \\
- \frac{1}{4\pi}\ln[r_0^4 + R^2r^2 - 2r_0^2Rr\cos(\theta - \Theta)].
\end{aligned} \tag{34}$$

This result is also obtained with a partial eigenfunction expansion in Exercise 13. ■

The splitting technique points out a distinct difference between Green's functions for one-dimensional problems and those for multidimensional problems. Green's function $g(x; X)$ for a one-dimensional boundary value problem (associated with a second-order ODE) is a continuous function of x (or can be made so) with a jump discontinuity in its first derivative. Green's functions for multidimensional boundary value problems can always be represented as the sum of a free-space Green's function U

and a regular part g, and, according to Table 12.1, free-space Green's functions are always singular at the source point. Thus, multivariable Green's functions always have discontinuities at source points.

Method of Images

The method of images is simply physical reasoning and intelligent guesswork in arriving at the function g in the splitting technique, and as such it works only on Laplace's equation with very simple geometries. When the Green's function G for a domain A is split into $U + g$, the free-space Green's function U can be regarded as the potential due to a unit point source interior to A. This source, by itself, induces an undesirable potential on $\beta(A)$. What is needed is a source distribution exterior to A whose potential g will cancel the effect of U on $\beta(A)$. (The fact that this distribution is exterior to A guarantees that $G = U + g$ satisfies $\nabla^2 G = \delta$ interior to A.)

We illustrate with the following three-dimensional problem.

Example 4: Find the Green's function associated with the three-dimensional Dirichlet problem in a sphere of radius r_0.

Solution: The Green's function satisfies

$$\nabla^2 G = \frac{\delta(r - R)\delta(\theta - \Theta)\delta(\phi - \Phi)}{r^2 \sin\phi}, \qquad 0 < r < r_0,$$

$$-\pi < \theta \le \pi, \qquad 0 < \phi < \pi, \tag{35a}$$

$$G(r_0, \theta, \phi; R, \Theta, \Phi) = 0, \qquad -\pi < \theta \le \pi, \qquad 0 < \phi < \pi. \tag{35b}$$

According to Table 12.1, the free-space Green's function with source point (X, Y, Z) is $-1/[4\pi\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}]$. When (R, Θ, Φ) are the spherical coordinates of (X, Y, Z), this function becomes

$$U(r, \theta, \phi; R, \Theta, \Phi) = \frac{-1}{4\pi\sqrt{r^2 + R^2 - 2Rr[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}.$$

What the method of images suggests is finding a source distribution exterior to the sphere, the potential g for which is such that $G = U + g$ vanishes on $r = r_0$. We might first consider whether a single source of magnitude q at a point (R^*, Θ^*, Φ^*) $(R^* > r_0)$ might suffice. Symmetry would suggest that such a source could eliminate U on $r = r_0$, which is symmetric around the line through the origin, and (R, Θ, Φ) (Figure 12.1) only if (R^*, Θ^*, Φ^*) were to lie on the line also. We assume, therefore, that $\Theta^* = \Theta$ and $\Phi^* = \Phi$, in which case the condition that $G = U + g$ vanish on $r = r_0$ is

$$0 = \frac{-1}{4\pi\sqrt{r_0^2 + R^2 - 2r_0R[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}$$
$$+ \frac{-q}{4\pi\sqrt{r_0^2 + R^{*2} - 2r_0R^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}}$$

or
$$-q\sqrt{r_0^2 + R^2 - 2r_0R[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}$$
$$= \sqrt{r_0^2 + R^{*2} - 2r_0R^*[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}.$$

Since this condition must be valid for all ϕ and θ, we set $\phi = 0$ and $\phi = \pi$:

$$-q\sqrt{r_0^2 + R^2 - 2r_0R\cos\Phi} = \sqrt{r_0^2 + R^{*2} - 2r_0R^*\cos\Phi},$$
$$-q\sqrt{r_0^2 + R^2 + 2r_0R\cos\Phi} = \sqrt{r_0^2 + R^{*2} + 2r_0R^*\cos\Phi}.$$

These two equations imply that $R^* = r_0^2/R$ and $q = -r_0/R$, and with these, $U + g$ vanishes identically on $r = r_0$. Thus, the Green's function for the Laplacian inside a sphere of radius r_0 is

$$G(r,\theta,\phi;R,\Theta,\Phi) = \frac{-1}{4\pi\sqrt{r^2+R^2-2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}$$
$$+\frac{r_0}{4\pi R\sqrt{r^2+\left(\frac{r_0^2}{R}\right)^2-2r\left(\frac{r_0^2}{R}\right)[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}$$
$$=\frac{-1}{4\pi\sqrt{r^2+R^2-2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}$$
$$+\frac{r_0}{4\pi\sqrt{R^2r^2+r_0^4-2r_0^2Rr[\cos\phi\cos\Phi+\sin\phi\sin\Phi\cos(\theta-\Theta)]}}. \quad \blacksquare \quad (36)$$

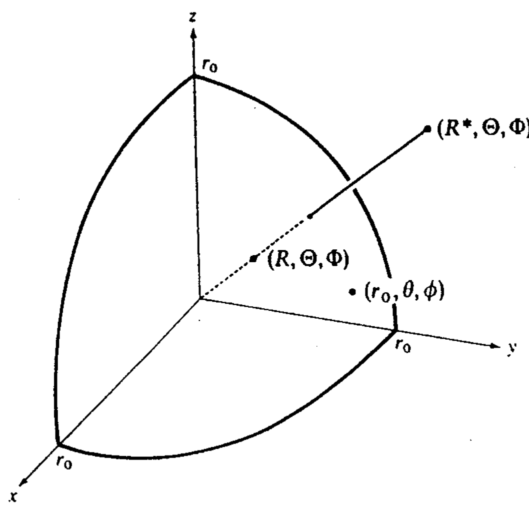


Figure 12.1

Exercises 12.2

1. Show that coefficients c_{mn} in (22) can be obtained by substituting $u = G(x,y;X,Y)$ and $v = u_{mn}(x,y)$ in Green's identity (14a).

2. Show that when $u_n(x, y)$ are orthonormal eigenfunctions of the eigenvalue problem

$$\nabla^2 u + \lambda^2 u = 0, \qquad (x, y) \text{ in } A, \tag{37a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A), \tag{37b}$$

associated with the Dirichlet problem

$$\nabla^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{38a}$$

$$u = K(x, y), \qquad (x, y) \text{ on } \beta(A), \tag{38b}$$

the full eigenfunction expansion for the Green's function is

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} \frac{u_n(X, Y)u_n(x, y)}{-\lambda_n^2}. \tag{39}$$

(This expansion should be compared with that in Exercise 21 of Section 11.3 for the Green's function of an ODE.)

In Exercises 3–8, use Exercise 2 (and its extension to three dimensions) to find full eigenfunction expansions for the Green's function associated with the Dirichlet problem for Poisson's equation on the given domain.

3. $0 \le r < r_0, \quad -\pi < \theta \le \pi$
4. $0 \le r < r_0, \quad 0 < \theta < \pi$
5. $0 \le r < r_0, \quad 0 < \theta < L$
6. $0 < x < L, \quad 0 < y < L', \quad 0 < z < L''$
7. $0 \le r < r_0, \quad -\pi < \theta \le \pi, \quad 0 < z < L$
8. $0 \le r < r_0, \quad -\pi < \theta \le \pi, \quad 0 \le \phi \le \pi$
9. Use the method of images and the result of Example 4 to find the Green's function for the Dirichlet problem associated with Poisson's equation in a hemisphere of radius r_0.
10. Use a "modified" method of images to find the Green's function for the Dirichlet problem associated with the two-dimensional Laplacian on a circle of radius r_0. Assume that g consists of a potential due to an exterior, negative unit point source plus a constant potential.
11. Use the result of Exercise 10 and the method of images to find the Green's function for the Dirichlet problem associated with Poisson's equation on a semicircle $0 < r < r_0$, $0 < \theta < \pi$. How does it compare with the representation in Exercise 4?
12. Use the method of images to find an expression for the Green's function of the Dirichlet problem for the Laplacian on the rectangle $0 < x < L$, $0 < y < L'$.
13. In this exercise we use a partial eigenfunction expansion to find Green's function (34) for problem (31).

(a) Show that the partial eigenfunction expansion for $G(r, \theta; R, \Theta)$ is

$$G(r, \theta; R, \Theta) = \frac{a_0(r)}{\sqrt{2\pi}} + \sum_{n=1}^{\infty} \left(a_n(r) \frac{\cos n\theta}{\sqrt{\pi}} + b_n(r) \frac{\sin n\theta}{\sqrt{\pi}} \right).$$

(b) Substitute the expansion in (a) into PDE (31a), and expand $\delta(r - R)\delta(\theta - \Theta)/r$ in a Fourier series to obtain the following boundary value problems for the coefficients:

$$\frac{d}{dr}\left(r \frac{da_0}{dr}\right) = \frac{\delta(r - R)}{\sqrt{2\pi}}, \qquad a_0(r_0) = 0;$$

$$\frac{d}{dr}\left(r \frac{da_n}{dr}\right) - \frac{n^2}{r} a_n = \delta(r - R) \frac{\cos n\theta}{\sqrt{\pi}}, \qquad a_n(r_0) = 0;$$

$$\frac{d}{dr}\left(r \frac{db_n}{dr}\right) - \frac{n^2}{r} b_n = \delta(r - R) \frac{\sin n\theta}{\sqrt{\pi}}, \qquad b_n(r_0) = 0.$$

(c) The systems in (b) are "singular" in the sense that there is only one boundary condition and the coefficient r in the derivative term vanishes at $r = 0$. As a result, equations (28) and (29) in Chapter 11 cannot be used to find a_n and b_n. Instead, use properties (21a–c) from Chapter 11 and the one boundary condition to show that

$$a_0(r) = \begin{cases} \dfrac{\ln(R/r_0)}{\sqrt{2\pi}} & 0 \le r \le R \\ \dfrac{\ln(r/r_0)}{\sqrt{2\pi}} & R \le r \le r_0 \end{cases},$$

$$a_n(r) = \begin{cases} \dfrac{\cos n\Theta}{2\sqrt{\pi n}}\left[\left(\dfrac{rR}{r_0^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right] & 0 \le r \le R \\ \dfrac{\cos n\Theta}{2\sqrt{\pi n}}\left[\left(\dfrac{rR}{r_0^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right] & R \le r \le r_0 \end{cases},$$

$$b_n(r) = \begin{cases} \dfrac{\sin n\Theta}{2\sqrt{\pi n}}\left[\left(\dfrac{rR}{r_0^2}\right)^n - \left(\dfrac{r}{R}\right)^n\right] & 0 \le r \le R \\ \dfrac{\sin n\Theta}{2\sqrt{\pi n}}\left[\left(\dfrac{rR}{r_0^2}\right)^n - \left(\dfrac{R}{r}\right)^n\right] & R \le r \le r_0 \end{cases}.$$

(d) Find $G(r, \theta; R, \Theta)$ and use (33) to reduce the function to the form in (34).

14. Use the technique of Exercise 13 to find a partial eigenfunction expansion for the Green's function of the Dirichlet problem for the Laplacian on the semicircle $0 < r < r_0$, $0 < \theta < \pi$. Show that it can be expressed in the form of Exercise 11.

15. Use the technique of Exercise 13 to find the partial eigenfunction expansion for the Green's function of Exercise 5.

16. Find a partial eigenfunction expansion for the Green's function of Exercise 6 using eigenfunctions in x and y.

17. Show that when $u_n(x, y)$ are orthonormal eigenfunctions of eigenvalue problem (20), the full eigenfunction expansion for the Green's function of the boundary value problem

$$\nabla^2 u + k^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{40a}$$

$$u = K(x, y), \qquad (x, y) \text{ on } \beta(A), \tag{40b}$$

is

$$G(x, y; X, Y) = \sum_{n=1}^{\infty} \frac{u_n(X, Y)u_n(x, y)}{k^2 - \lambda_n^2}, \tag{41}$$

provided $k \ne \lambda_n$ for any n. (The exceptional case is discussed in Exercise 8 of Section 12.3.)

In Exercises 18–24, use Exercise 17 to state Green's functions for problem (40) on the given domain. (See Example 1 and Exercises 3–8 for eigenpairs.)

18. $0 < x < L$, $0 < y < L'$

19. $0 \le r < r_0$, $-\pi < \theta \le \pi$

20. $0 \le r < r_0$, $0 < \theta < \pi$

21. $0 < r < r_0$, $0 < \theta < L$

22. $0 < x < L$, $0 < y < L'$, $0 < z < L''$

23. $0 \le r < r_0$, $-\pi < \theta \le \pi$, $0 < z < L$

24. $0 \le r < r_0$, $-\pi < \theta \le \pi$, $0 \le \phi \le \pi$

12.3 Solutions of Dirichlet Boundary Value Problems on Finite Regions

In this section we use Green's functions to solve Dirichlet boundary value problems associated with Poisson's equation on finite regions. Identical results for the Helmholtz equation are discussed in the exercises.

The Dirichlet boundary value problem for Poisson's equation in two dimensions is

$$\nabla^2 u = F(x,y), \qquad (x,y) \text{ in } A, \tag{42a}$$

$$u = K(x,y), \qquad (x,y) \text{ on } \beta(A), \tag{42b}$$

where A is a region with finite area. When $K(x,y) \equiv 0$, the solution is given by integral (17). The following theorem indicates that a line integral involving the normal derivative of $G(x,y;X,Y)$ incorporates nonzero $K(x,y)$.

Theorem 1

When $G(x,y;X,Y)$ is the Green's function for Dirichlet problem (42), the solution to (42) is

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N}\,ds, \tag{43}$$

where $\partial G/\partial N$ is the outward normal derivative of G with respect to the (X,Y) variables.

Proof: If in Green's identity (14a) we let $u = G(x,y;X,Y)$ and $v = u(x,y)$ be the solution of (42),

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}}\,ds.$$

Because $\nabla^2 u = F$ and $\nabla^2 G = \delta(x-X, y-Y)$ in A, and $u = K$ and $G = 0$ on $\beta(A)$,

$$\iint_A [u(x,y)\delta(x-X,y-Y) - G(x,y;X,Y)F(x,y)]\,dA$$

$$= \oint_{\beta(A)} K(x,y)\nabla G(x,y;X,Y)\cdot\hat{\mathbf{n}}\,ds$$

or
$$u(X,Y) = \iint_A G(x,y;X,Y)F(x,y)\,dy\,dx + \oint_{\beta(A)} K(x,y)\frac{\partial G(x,y;X,Y)}{\partial n}\,ds.$$

When we interchange (x,y) and (X,Y),

$$u(x,y) = \iint_A G(X,Y;x,y)F(X,Y)\,dY\,dX + \oint_{\beta(A)} K(X,Y)\frac{\partial G(X,Y;x,y)}{\partial N}\,ds$$

$$= \iint_A G(x,y;X,Y)F(X,Y)\,dY\,dX + \oint_{\beta(A)} K(X,Y)\frac{\partial G(x,y;X,Y)}{\partial N}\,ds$$

[because $G(x,y;X,Y)$ is symmetric.] ■

It is often helpful to interpret the integral terms in (43) physically. From an electrostatic point of view, problem (42) defines potential in a region A due to an area charge density determined by $F(x, y)$ and a boundary potential $K(x, y)$. (In actual fact, we are considering any cross section of a z-symmetric three-dimensional problem.) The area integral in (43) represents that part of the potential due to the interior charge, and the line integral is the boundary potential contribution. The Green's function $G(x, y; X, Y)$ is the potential at (x, y) due to a unit charge at (X, Y) when the boundary potential on $\beta(A)$ vanishes (which would be the case, say, for a grounded metallic surface). The double integral superposes over all elemental contributions $G(x, y; X, Y)F(X, Y)\,dY\,dX$ of internal charge.

From a heat conduction point of view, problem (42) describes steady-state temperature in a region A due to internal heat generation determined by $F(x, y)$ and boundary temperature $K(x, y)$. The area integral in (43) represents that part of the temperature due to internal sources. The Green's function is the temperature at (x, y) due to a unit source at (X, Y) when the boundary temperature is made to vanish. The line integral represents the effect of imposed boundary temperatures.

Finally, problem (42) also describes static deflections of a membrane stretched tautly over A. The double integral represents the effect due to applied forces [contained in $F(x, y)$], and the line integral determines the effect of boundary displacements.

We noted in Section 12.2 that $G(x, y; X, Y)$ is not continuous; it has a singularity when $(x, y) = (X, Y)$. The discontinuity cannot be too severe, however, since existence of the area integral in (43) (which integrates over the singularity) is guaranteed by Theorem 2. To illustrate this point, suppose A is the circle $r < r_0$ and $K = 0$ on $\beta(A)$. According to (43), the solution to problem (42) at any point (r, θ) in this case is

$$u(r, \theta) = \iint_A G(r, \theta; R, \Theta)F(R, \Theta)\,dA.$$

For simplicity, we consider the origin, in which case

$$u(0, \theta) = \iint_A G(0, \theta; R, \Theta)F(R, \Theta)\,dA.$$

Using equation (34) for $G(r, \theta; R, \Theta)$,

$$u(0, \theta) = \iint_A \frac{1}{2\pi}\ln\left(\frac{R}{r_0}\right)F(R, \Theta)\,dA,$$

and indeed we can see that $\ln(R/r_0)$ is singular at $R = 0$. However, the area element $dA = R\,dR\,d\Theta$ effectively removes this singularity, and

$$u(0, \theta) = \int_{-\pi}^{\pi}\int_0^{r_0} \frac{1}{2\pi}\ln\left(\frac{R}{r_0}\right)F(R, \Theta)R\,dR\,d\Theta$$

must converge. In particular, if $F(R, \Theta) \equiv 1$, integration by parts gives

$$u(0, \theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^{r_0} R\ln\left(\frac{R}{r_0}\right)dR\,d\Theta = -\frac{r_0^2}{4}.$$

For three-dimensional problems

$$\nabla^2 u = F(x, y, z), \qquad (x, y, z) \text{ in } V, \tag{44a}$$

$$u = K(x, y, z), \qquad (x, y, z) \text{ on } \beta(V), \tag{44b}$$

the solution is

$$u(x, y, z) = \iiint_V G(x, y, z; X, Y, Z)F(X, Y, Z)\,dV + \iint_{\beta(V)} K(X, Y, Z)\frac{\partial G(x, y, z; X, Y, Z)}{\partial N}\,dS. \tag{45}$$

We now consider some examples.

Example 5: Solve the boundary value problem

$$\nabla^2 u = F(r, \theta), \qquad 0 < r < r_0, \qquad -\pi < \theta \le \pi, \tag{46a}$$

$$u(r_0, \theta) = K(\theta), \qquad -\pi < \theta \le \pi. \tag{46b}$$

Solution: According to (43), the solution can be represented in the form

$$u(r, \theta) = \iint_A G(r, \theta; R, \Theta)F(R, \Theta)\,dA + \oint_{\beta(A)} K(\Theta)\frac{\partial G(r, \theta; r_0, \Theta)}{\partial R}\,ds,$$

where $G(r, \theta; R, \Theta)$ is the Green's function

$$G(r, \theta; R, \Theta) = \frac{1}{4\pi}\ln\left(r_0^2\frac{r^2 + R^2 - 2rR\cos(\theta - \Theta)}{r_0^4 + r^2R^2 - 2r_0^2rR\cos(\theta - \Theta)}\right)$$

[see equation (34)]. Now,

$$\begin{aligned}\frac{\partial G(r, \theta; r_0, \Theta)}{\partial R} &= \frac{1}{4\pi}\left(\frac{2R - 2r\cos(\theta - \Theta)}{r^2 + R^2 - 2rR\cos(\theta - \Theta)} - \frac{2r^2R - 2r_0^2r\cos(\theta - \Theta)}{r_0^4 + r^2R^2 - 2r_0^2Rr\cos(\theta - \Theta)}\right)_{|R = r_0}\\ &= \frac{1}{4\pi}\left(\frac{2r_0 - 2r\cos(\theta - \Theta)}{r^2 + r_0^2 - 2rr_0\cos(\theta - \Theta)} - \frac{2r^2r_0 - 2r_0^2r\cos(\theta - \Theta)}{r_0^4 + r^2r_0^2 - 2r_0^3r\cos(\theta - \Theta)}\right)\\ &= \frac{1}{2\pi r_0}\frac{r_0^2 - r^2}{r^2 + r_0^2 - 2r_0r\cos(\theta - \Theta)}.\end{aligned}$$

Thus,

$$\begin{aligned}u(r, \theta) &= \iint_A G(r, \theta; R, \Theta)F(R, \Theta)\,dA + \int_{-\pi}^{\pi} K(\Theta)\frac{r_0^2 - r^2}{2\pi r_0[r^2 + r_0^2 - 2r_0r\cos(\theta - \Theta)]}r_0\,d\Theta\\ &= \iint_A G(r, \theta; R, \Theta)F(R, \Theta)\,dA + \frac{r_0^2 - r^2}{2\pi}\int_{-\pi}^{\pi}\frac{K(\Theta)}{r^2 + r_0^2 - 2r_0r\cos(\theta - \Theta)}\,d\Theta. \qquad (47)\end{aligned}$$

When $F(r,\theta) \equiv 0$, the solution of Laplace's equation is

$$u(r,\theta) = \frac{r_0^2 - r^2}{2\pi} \int_{-\pi}^{\pi} \frac{K(\Theta)}{r^2 + r_0^2 - 2r_0 r\cos(\theta - \Theta)} d\Theta,$$

Poisson's integral formula for a circle [see equation (36) in Section 5.3]. ■

Example 6: Solve the following Dirichlet problem:

$$\nabla^2 u = F(x,y), \quad 0 < x < L, \quad 0 < y < L', \tag{48a}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{48b}$$

$$u(L,y) = 0, \quad 0 < y < L', \tag{48c}$$

$$u(x,L') = 0, \quad 0 < x < L, \tag{48d}$$

$$u(0,y) = g(y), \quad 0 < y < L'. \tag{48e}$$

Solution: The solution can be represented in the form

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA + \int_0^L -f(X)\frac{\partial G(x,y;X,0)}{\partial Y}dX + \int_{L'}^0 -g(Y)\frac{\partial G(x,y;0,Y)}{\partial X}(-dY),$$

where G is given by either (27) or (28). For the first line integral, we use (27) in the form

$$G(x,y;X,Y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\sinh\frac{n\pi Y}{L}\sinh\frac{n\pi(L'-y)}{L}}{n\pi\sinh\frac{n\pi L'}{L}} & 0 \le Y \le y \\ \displaystyle\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\sinh\frac{n\pi y}{L}\sinh\frac{n\pi(L'-y)}{L}}{n\pi\sinh\frac{n\pi L'}{L}} & y \le Y \le L' \end{cases}$$

to calculate

$$\frac{\partial G(x,y;X,0)}{\partial Y} = \sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\left(\frac{n\pi}{L}\right)\sinh\frac{n\pi(L'-y)}{L}}{n\pi\sinh\frac{n\pi L'}{L}}.$$

A similar calculation using (28) gives

$$\frac{\partial G(x,y;0,Y)}{\partial X} = \sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi Y}{L'}\sin\frac{n\pi y}{L'}\left(\frac{n\pi}{L'}\right)\sinh\frac{n\pi(L-x)}{L'}}{n\pi\sinh\frac{n\pi L}{L'}},$$

and therefore

$$u(x,y) = \iint_A G(x,y;X,Y)F(X,Y)\,dA$$
$$- \int_0^L f(X)\left(\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}\left(\frac{n\pi}{L}\right)\sinh\frac{n\pi}{L}(L'-y)}{n\pi\sinh\frac{n\pi L'}{L}}\right)dX$$
$$- \int_0^{L'} g(Y)\left(\sum_{n=1}^{\infty} \frac{-2\sin\frac{n\pi Y}{L'}\sin\frac{n\pi y}{L'}\left(\frac{n\pi}{L'}\right)\sinh\frac{n\pi}{L'}(L-x)}{n\pi\sinh\frac{n\pi L}{L'}}\right)dY$$
$$= \int_0^{L'}\int_0^L G(x,y;X,Y)F(X,Y)\,dX\,dY$$
$$+ \frac{2}{L}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi x}{L}\sinh\frac{n\pi(L'-y)}{L}}{\sinh\frac{n\pi L'}{L}} \int_0^L f(X)\sin\frac{n\pi X}{L}\,dX$$
$$+ \frac{2}{L'}\sum_{n=1}^{\infty} \frac{\sin\frac{n\pi y}{L'}\sinh\frac{n\pi(L-x)}{L'}}{\sinh\frac{n\pi L}{L'}} \int_0^{L'} g(Y)\sin\frac{n\pi Y}{L'}\,dY. \quad \blacksquare \quad (49)$$

Special cases of this problem that lead to solutions found in previous chapters are contained in Exercises 1 and 2.

Example 7: Solve the following Dirichlet problem on a sphere:

$$\nabla^2 u = F(r,\theta,\phi), \quad 0 < r < r_0, \quad -\pi < \theta \le \pi, \quad 0 < \phi < \pi, \quad (50a)$$
$$u(r_0,\theta,\phi) = K(\theta,\phi), \quad -\pi < \theta \le \pi, \quad 0 \le \phi \le \pi. \quad (50b)$$

Solution: The solution can be represented in the form

$$u(r,\theta,\phi) = \iiint_V G(r,\theta,\phi;R,\Theta,\Phi)F(R,\Theta,\Phi)\,dV + \iint_{\beta(V)} K(\Theta,\Phi)\frac{\partial G(r,\theta,\phi;r_0,\Theta,\Phi)}{\partial R}\,dS,$$

where the Green's function is contained in equation (36). Since $\partial G(r,\theta,\phi;r_0,\Theta,\Phi)/\partial R$ is equal to

$$\frac{1}{4\pi}\left(\frac{r_0 - r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[r^2 + r_0^2 - 2rr_0(\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right)$$
$$- \frac{r_0}{4\pi}\left(\frac{r_0 r^2 - r_0^2 r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta)]}{[r_0^2 r^2 + r_0^4 - 2r_0^3 r(\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta-\Theta))]^{3/2}}\right)$$

$$= \frac{1}{4\pi}\left(\frac{r_0 - r[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}{(r^2 + r_0^2 - 2rr_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)])^{3/2}}\right)$$
$$- \frac{r}{4\pi r_0}\left(\frac{r - r_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)]}{(r^2 + r_0^2 - 2rr_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)])^{3/2}}\right)$$
$$= \frac{r_0^2 - r^2}{4\pi r_0(r^2 + r_0^2 - 2rr_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)])^{3/2}},$$

we find that

$$u(r,\theta,\phi) = \iiint_V G(r,\theta,\phi;R,\Theta,\Phi)F(R,\Theta,\Phi)\,dV$$
$$+ \int_{-\pi}^{\pi}\int_0^{\pi} \frac{(r_0^2 - r^2)K(\Theta,\Phi)r_0^2\sin\Phi}{4\pi r_0(r^2 + r_0^2 - 2rr_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)])^{3/2}}\,d\Phi\,d\Theta$$
$$= \iiint_V G(r,\theta,\phi;R,\Theta,\Phi)F(R,\Theta,\Phi)\,dV + \frac{r_0^3 - r^2 r_0}{4\pi}$$
$$\times \int_{-\pi}^{\pi}\int_0^{\pi} \frac{K(\Theta,\Phi)\sin\Phi}{(r^2 + r_0^2 - 2rr_0[\cos\phi\cos\Phi + \sin\phi\sin\Phi\cos(\theta - \Theta)])^{3/2}}\,d\Phi\,d\Theta. \quad (51)$$

■

Exercises 12.3

1. Use the result of Example 6 to solve Exercise 18 in Section 3.2.
2. Use the result of Example 6 to solve Exercise 42 in Section 6.2.
3. Find an integral representation for the solution of the boundary value problem

$$\nabla^2 u = F(x,y), \quad 0 < x < L, \quad 0 < y < L',$$
$$u(x,0) = 0, \quad 0 < x < L,$$
$$u(L,y) = g(y), \quad 0 < y < L',$$
$$u(x,L') = f(x), \quad 0 < x < L,$$
$$u(0,y) = 0, \quad 0 < y < L'.$$

4. Find an integral representation for the solution of the following Dirichlet problem on a semicircle:

$$\nabla^2 u = F(r,\theta), \quad 0 < r < r_0, \quad 0 < \theta < \pi,$$
$$u(r_0,\theta) = f(\theta), \quad 0 < \theta < \pi,$$
$$u(r,0) = g_1(r), \quad 0 < r < r_0,$$
$$u(r,\pi) = g_2(r), \quad 0 < r < r_0.$$

(See Exercise 11 in Section 12.2 for the Green's function.)

In the remaining exercises we discuss results for the Dirichlet problem associated with the Helmholtz equation

$$(\nabla^2 + k^2)u = F(x,y), \qquad (x,y) \text{ in } A, \tag{52a}$$

$$u(x,y) = K(x,y), \qquad (x,y) \text{ on } \beta(A), \tag{52b}$$

where $k > 0$ is a constant.

5. Verify that (43) is the solution of problem (52) when $G(x,y;X,Y)$ is the Green's function satisfying

$$[\nabla^2 + k^2]G = \delta(x - X, y - Y), \qquad (x,y) \text{ in } A, \tag{53a}$$

$$G = 0, \qquad (x,y) \text{ on } \beta(A). \tag{53b}$$

6. What is the result corresponding to that in Exercise 5 for three-dimensional problems?

The homogeneous Dirichlet problem for the Laplacian

$$\nabla^2 u = 0, \qquad (x,y) \text{ in } A,$$

$$u = 0, \qquad (x,y) \text{ on } \beta(A),$$

has only the trivial solution. The homogeneous Dirichlet problem

$$[\nabla^2 + k^2]u = 0, \qquad (x,y) \text{ in } A,$$

$$u = 0, \qquad (x,y) \text{ on } \beta(A),$$

on the other hand, may have nontrivial solutions. In this case, it is necessary to introduce modified Green's functions. We illustrate this in Exercise 7 and discuss it in general in Exercise 8.

7. (a) Show that when A is the square $0 < x,\ y < L$, $w(x,y) = (2/L)\sin(\pi x/L)\sin(\pi y/L)$ is a (nontrivial) solution of

$$\nabla^2 u + \frac{2\pi^2}{L^2}u = 0, \qquad (x,y) \text{ in } A,$$

$$u = 0, \qquad (x,y) \text{ on } \beta(A).$$

(b) Prove that when the problem

$$\nabla^2 u + \frac{2\pi^2}{L^2}u = F(x,y), \qquad (x,y) \text{ in } A,$$

$$u = 0, \qquad (x,y) \text{ on } \beta(A),$$

has a solution $u(x,y)$, $F(x,y)$ satisfies

$$\int_0^L \int_0^L F(x,y)w(x,y)\,dy\,dx = 0.$$

[The converse is also valid; that is, when $F(x,y)$ satisfies this condition, the nonhomogeneous problem has a solution $u(x,y)$. It is not unique; $u(x,y) + Cw(x,y)$ is also a solution for any constant C.]

(c) Because the delta function does not satisfy the condition in (b), there can be no Green's function satisfying

$$\nabla^2 G + \frac{2\pi^2}{L^2} G = \delta(x - X, y - Y), \qquad (x, y) \text{ in } A,$$

$$G = 0, \qquad (x, y) \text{ on } \beta(A).$$

We therefore introduce a modified Green's function $\bar{G}(x, y; X, Y)$, satisfying

$$\left(\nabla^2 + \frac{2\pi^2}{L^2}\right)\bar{G} = \delta(x - X, y - Y) - w(x, y)w(X, Y), \qquad (x, y) \text{ in } A,$$

$$\bar{G} = 0, \qquad (x, y) \text{ on } \beta(A).$$

Show that the right side of the PDE for $\bar{G}$ satisfies the condition in (b).

(d) Find a partial eigenfunction expansion for $\bar{G}$ in terms of the normalized eigenfunctions $\sqrt{2/L}\sin(n\pi x/L)$.

(e) Find an integral representation for the solution of the boundary value problem in (b) in terms of $F(x, y)$ and $\bar{G}(x, y; X, Y)$.

8. (a) Show that when the homogeneous problem

$$[\nabla^2 + k^2]u = 0, \qquad (x, y) \text{ in } A, \tag{54a}$$

$$u = 0, \qquad (x, y) \text{ on } \beta(A), \tag{54b}$$

has nontrivial solutions $w(x, y)$, nonhomogeneous problem (52) has a solution only if $F(x, y)$ and $K(x, y)$ satisfy the condition that for every such solution $w(x, y)$,

$$\iint_A F(x, y)w(x, y)\,dA = -\oint_{\beta(A)} K(x, y)\frac{\partial w(x, y)}{\partial n}\,ds, \tag{55}$$

where $\partial w/\partial n$ is the derivative of w in the outwardly normal direction to $\beta(A)$. [The converse result is also valid; that is, when (55) is satisfied, (52) has a solution that is unique to an additive term $Cw(x, y)$, C an arbitrary constant.]

(b) Show that the solution of (52) can be expressed in the form

$$u(x, y) = \iint_A \bar{G}(X, Y; x, y)F(X, Y)\,dA + \oint_{\beta(A)} K(X, Y)\frac{\partial \bar{G}(X, Y; x, y)}{\partial N}\,ds + Cw(x, y), \tag{56}$$

where $\bar{G}(x, y; X, Y)$ is a modified Green's function satisfying

$$[\nabla^2 + k^2]\bar{G} = \delta(x - X, y - Y) - w(x, y)w(X, Y), \qquad (x, y) \text{ in } A, \tag{57a}$$

$$\bar{G} = 0, \qquad (x, y) \text{ on } \beta(A), \tag{57b}$$

and $w(x, y)$ is a normalized solution of (54).

12.4 Solutions of Neumann Boundary Value Problems on Finite Regions

The Neumann problem for Poisson's equation

$$\nabla^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{58a}$$

$$\frac{\partial u}{\partial n} = K(x, y), \qquad (x, y) \text{ on } \beta(A), \tag{58b}$$

is more difficult to handle than the Dirichlet problem because the corresponding homogeneous problem,

$$\nabla^2 u = 0, \qquad (x, y) \text{ in } A, \tag{59a}$$

$$\frac{\partial u}{\partial n} = 0, \qquad (x, y) \text{ on } \beta(A), \tag{59b}$$

always has nontrivial solutions $u = \text{constant}$. As a result, (58) does not have a unique solution; if $u(x, y)$ is a solution, so also is $u(x, y) + \text{constant}$. The following theorem shows that for there to be a solution of (58) at all, $F(x, y)$ and $K(x, y)$ must satisfy a consistency condition.

Theorem 2

Neumann problem (58) has a solution if and only if

$$\iint_A F(x, y)\,dA = \oint_{\beta(A)} K(x, y)\,ds. \tag{60}$$

When (58) is a steady-state heat conduction problem, condition (60) implies that heat generated within A must be compensated by heat crossing its boundary. This condition is the two-dimensional analog of condition (61a) in Chapter 11; its necessity is easily established with Green's theorem:

$$\oint_{\beta(A)} K(x, y)\,ds = \oint_{\beta(A)} \frac{\partial u}{\partial n}\,ds = \oint_{\beta(A)} \nabla u \cdot \hat{\mathbf{n}}\,ds$$
$$= \iint_A \nabla^2 u\,dA = \iint_A F(x, y)\,dA.$$

According to the following theorem, solutions of Neumann problems can be expressed in terms of modified Green's functions.

Theorem 3

When consistency condition (60) is satisfied, the solution of Neumann problem (58) is

$$u(x, y) = \iint_A N(x, y; X, Y)F(X, Y)\,dA - \oint_{\beta(A)} N(x, y; X, Y)K(X, Y)\,ds + C, \tag{61}$$

where C is an arbitrary constant and $N(x, y; X, Y)$ is the symmetric modified Green's function satisfying

$$\nabla^2 N = \delta(x - X, y - Y) - \frac{1}{\text{area}(A)}, \qquad (x, y) \text{ in } A, \tag{62a}$$

$$\frac{\partial N}{\partial n} = 0, \qquad (x, y) \text{ on } \beta(A). \tag{62b}$$

Proof: In Green's identity (14a) on A, we let $u = N(x, y; X, Y)$ and $v = u(x, y)$, the solution of (58):

$$\iint_A (u\nabla^2 N - N\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla N - N\nabla u) \cdot \hat{\mathbf{n}}\,ds.$$

Because $\nabla^2 u = F$ in A, $\nabla^2 N = \delta(x - X, y - Y) - 1/\text{area}(A)$, and $\partial u/\partial n = K$ and $\partial N/\partial n = 0$ on $\beta(A)$,

$$\iint_A \left(u(x, y)\left[\delta(x - X, y - Y) - \frac{1}{\text{area}(A)}\right] - N(x, y; X, Y)F(x, y)\right) dA$$
$$= \oint_{\beta(A)} -N(x, y; X, Y)K(x, y)\,ds$$

or

$$u(X, Y) = \iint_A N(x, y; X, Y)F(x, y)\,dA - \oint_{\beta(A)} N(x, y; X, Y)K(x, y)\,ds + \frac{C_1}{\text{area}(A)},$$

where $C_1 = \iint_A u(x, y)\,dA$. When we interchange (x, y) and (X, Y),

$$u(x, y) = \iint_A N(X, Y; x, y)F(X, Y)\,dA - \oint_{\beta(A)} N(X, Y; x, y)K(X, Y)\,ds + C \tag{63}$$
$$= \iint_A N(x, y; X, Y)F(X, Y)\,dA - \oint_{\beta(A)} N(x, y; X, Y)K(X, Y)\,ds + C,$$

where we have replaced $C_1/\text{area}(A)$ by C, since $u(x, y)$ is unique only to an additive constant. ■

If the modified Green's function $N(x, y; X, Y)$ is not symmetric, equation (63) must be used for the solution in place of (61).

Solutions to three-dimensional Neumann problems

$$\nabla^2 u = F(x, y, z), \qquad (x, y, z) \text{ in } V, \tag{64a}$$

$$\frac{\partial u}{\partial n} = K(x, y, z), \qquad (x, y, z) \text{ on } \beta(V), \tag{64b}$$

exist if and only if $F(x, y, z)$ and $K(x, y, z)$ satisfy

$$\iiint_V F(x, y, z)\,dV = \iint_{\beta(V)} K(x, y, z)\,dS. \tag{65}$$

When this condition is satisfied, the solution of (64) is

$$u(x,y,z) = \iiint_V N(x,y,z;X,Y,Z)F(X,Y,Z)\,dV - \iint_{\beta(V)} N(x,y,z;X,Y,Z)K(X,Y,Z)\,dS + C, \tag{66}$$

where C is an arbitrary constant and $N(x,y,z;X,Y,Z)$ is the symmetric modified Green's function satisfying

$$\nabla^2 N = \delta(x-X, y-Y, z-Z) - \frac{1}{\text{volume}(V)}, \qquad (x,y,z) \text{ in } V, \tag{67a}$$

$$\frac{\partial N}{\partial n} = 0, \qquad (x,y,z) \text{ on } \beta(V). \tag{67b}$$

When $N(x,y,z;X,Y,Z)$ is not symmetric in (x,y,z) and (X,Y,Z), (66) must be replaced by

$$u(x,y,z) = \iiint_V N(X,Y,Z;x,y,z)F(X,Y,Z)\,dV - \iint_{\beta(V)} N(X,Y,Z;x,y,z)K(X,Y,Z)\,dS + C. \tag{68}$$

As an example, we consider the following Neumann problem on a rectangle.

Example 8: Use a modified Green's function to solve the boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0, \qquad 0 < x < L, \qquad 0 < y < L',$$

$$\frac{\partial V(0,y)}{\partial x} = 0, \qquad 0 < y < L',$$

$$\frac{\partial V(L,y)}{\partial x} = 0, \qquad 0 < y < L',$$

$$\frac{\partial V(x,0)}{\partial y} = 0, \qquad 0 < x < L,$$

$$\frac{\partial V(x,L')}{\partial y} = f(x), \qquad 0 < x < L.$$

Solution: Modified Green's functions $N(x,y;X,Y)$ for this problem must satisfy

$$\frac{\partial^2 N}{\partial x^2} + \frac{\partial^2 N}{\partial y^2} = \delta(x-X, y-Y) - \frac{1}{LL'}, \qquad 0 < x < L, \qquad 0 < y < L',$$

$$N_x(0,y) = 0, \qquad 0 < y < L',$$

$$N_x(L,y) = 0, \qquad 0 < y < L',$$

$$N_y(x,0) = 0, \qquad 0 < x < L,$$

$$N_y(x,L') = 0, \qquad 0 < x < L.$$

Substitution of a partial eigenfunction expansion,

$$N(x, y; X, Y) = \sum_{n=0}^{\infty} a_n(y) f_n(x) = \frac{a_0(y)}{\sqrt{L}} + \sum_{n=1}^{\infty} a_n(y)\sqrt{\frac{2}{L}} \cos\frac{n\pi x}{L},$$

into the PDE for $N(x, y; X, Y)$ gives

$$\begin{aligned}\sum_{n=0}^{\infty} -\frac{n^2\pi^2}{L^2} a_n f_n(x) + \sum_{n=0}^{\infty} \frac{d^2 a_n}{dy^2} f_n(x) &= \delta(x - X, y - Y) - \frac{1}{LL'}\\ &= \sum_{n=0}^{\infty}\left(\int_0^L \left[\delta(x - X)\delta(y - Y) - \frac{1}{LL'}\right] f_n(x)\, dx\right) f_n(x)\\ &= \frac{1}{\sqrt{L}}\left(\delta(y - Y) - \frac{1}{L'}\right) f_0(x) + \sum_{n=1}^{\infty} f_n(X)\delta(y - Y) f_n(x).\end{aligned}$$

This equation, along with the boundary conditions $N_y(x, 0) = 0 = N_y(x, L')$, requires coefficients $a_n(y)$ to satisfy

$$\frac{d^2 a_0}{dy^2} = \frac{1}{\sqrt{L}}\left(\delta(y - Y) - \frac{1}{L'}\right), \qquad 0 < y < L',$$
$$a_0'(0) = a_0'(L') = 0,$$

and, for $n > 0$,

$$\frac{d^2 a_n}{dy^2} - \frac{n^2\pi^2}{L^2} a_n = f_n(X)\delta(y - Y), \qquad 0 < y < L',$$
$$a_n'(0) = a_n'(L') = 0.$$

Since the general solution of the differential equation $d^2a_n/dy^2 - (n^2\pi^2/L^2)a_n = 0$ is $A\cosh(n\pi y/L) + B\sinh(n\pi y/L)$, we take

$$a_n(y) = \begin{cases} A\cosh\dfrac{n\pi y}{L} + B\sinh\dfrac{n\pi y}{L} & 0 \le y < Y \\[2ex] C\cosh\dfrac{n\pi y}{L} + D\sinh\dfrac{n\pi y}{L} & Y < y \le L' \end{cases}$$

The boundary conditions require that

$$\frac{n\pi}{L} B = 0, \qquad C\sinh\frac{n\pi L'}{L} + D\cosh\frac{n\pi L'}{L} = 0,$$

and continuity conditions (21a, b) from Chapter 11 necessitate that

$$A\cosh\frac{n\pi Y}{L} + B\sinh\frac{n\pi Y}{L} = C\cosh\frac{n\pi Y}{L} + D\sinh\frac{n\pi Y}{L},$$
$$\left(C\sinh\frac{n\pi Y}{L} + D\cosh\frac{n\pi Y}{L}\right) - \left(A\sinh\frac{n\pi Y}{L} + B\cosh\frac{n\pi Y}{L}\right) = \frac{L}{n\pi} f_n(X).$$

These four equations can be solved for

$$A = \frac{-L\cosh\frac{n\pi}{L}(L' - Y)f_n(X)}{n\pi\sinh(n\pi L'/L)}, \qquad B = 0, \qquad C = \frac{-L\cosh\frac{n\pi L'}{L}\cosh\frac{n\pi Y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)},$$

$$D = \frac{L}{n\pi}\cosh\frac{n\pi Y}{L}f_n(X),$$

and hence

$$a_n(y) = \begin{cases} \dfrac{-L\cosh\frac{n\pi}{L}(L' - Y)\cosh\frac{n\pi y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)} & 0 \le y \le Y \\[2ex] \dfrac{-L\cosh\frac{n\pi L'}{L}\cosh\frac{n\pi Y}{L}\cosh\frac{n\pi y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)} + \dfrac{L\cosh\frac{n\pi Y}{L}\sinh\frac{n\pi y}{L}f_n(X)}{n\pi} & Y \le y \le L' \end{cases}$$

$$= \begin{cases} \dfrac{-L\cosh\frac{n\pi}{L}(L' - Y)\cosh\frac{n\pi y}{L}f_n(X)}{n\pi\sinh(n\pi L'/L)} & 0 \le y \le Y \\[2ex] \dfrac{-L\cosh\frac{n\pi Y}{L}\cosh\frac{n\pi}{L}(L' - y)f_n(X)}{n\pi\sinh(n\pi L'/L)} & Y \le y \le L' \end{cases}.$$

Because $-y^2/(2\sqrt{L}\,L')$ is a solution of $d^2a_0/dy^2 = -1/(\sqrt{L}\,L')$, we take

$$a_0(y) = \begin{cases} Ay + B - \dfrac{y^2}{2\sqrt{L}\,L'} & 0 \le y < Y \\[2ex] Dy + C - \dfrac{y^2}{2\sqrt{L}\,L'} & Y < y \le L' \end{cases}.$$

Boundary conditions $a_0'(0) = 0 = a_0'(L')$, and continuity conditions (21a, b) from Chapter 11, require that

$$A = 0, \qquad D - \frac{1}{\sqrt{L}} = 0,$$

$$AY + B = DY + C, \qquad D - A = \frac{1}{\sqrt{L}}.$$

These equations yield $A = 0$, $D = 1/\sqrt{L}$, and $B = Y/\sqrt{L} + C$, where C is arbitrary, and hence

$$a_0(y) = \begin{cases} \dfrac{Y}{\sqrt{L}} + C - \dfrac{y^2}{2\sqrt{L}\,L'} & 0 \le y \le Y \\[2ex] \dfrac{y}{\sqrt{L}} + C - \dfrac{y^2}{2\sqrt{L}\,L'} & Y \le y \le L' \end{cases}.$$

A modified Green's function is therefore

$$N(x,y;X,Y)=\begin{cases}\dfrac{Y}{L}+\dfrac{C}{\sqrt{L}}-\dfrac{y^2}{2LL'}-\displaystyle\sum_{n=1}^{\infty}\dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}\cosh\dfrac{n\pi(L'-Y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L} & 0\le y\le Y\\[2ex] \dfrac{y}{L}+\dfrac{C}{\sqrt{L}}-\dfrac{y^2}{2LL'}-\displaystyle\sum_{n=1}^{\infty}\dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L} & Y\le y\le L'\end{cases}$$

Because $N(x,y;X,Y)$ is not symmetric, we use (63) to express the solution of the original boundary value problem as a line integral along the edge $C':y=L'$,

$$\begin{aligned}V(x,y)&=-\int_{C'}N(X,Y;x,y)f(X)\,ds+D\\&=-\int_0^L N(X,L';x,y)f(X)\,dX+D,\end{aligned}$$

where D is an arbitrary constant, and

$$N(X,Y;x,y)=\begin{cases}\dfrac{y}{L}+\dfrac{C}{\sqrt{L}}-\dfrac{Y^2}{2LL'}-\displaystyle\sum_{n=1}^{\infty}\dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi Y}{L}\cosh\dfrac{n\pi(L'-y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L} & 0\le Y\le y\\[2ex] \dfrac{Y}{L}+\dfrac{C}{\sqrt{L}}-\dfrac{Y^2}{2LL'}-\displaystyle\sum_{n=1}^{\infty}\dfrac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}\cosh\dfrac{n\pi(L'-Y)}{L}}{n\pi\sinh(n\pi L'/L)}\cos\dfrac{n\pi x}{L} & y\le Y\le L'\end{cases}$$

When we use the latter of these expressions to evaluate $N(X,L';x,y)$ along C',

$$V(x,y)=-\int_0^L\left(\frac{L'}{L}+\frac{C}{\sqrt{L}}-\frac{L'}{2L}-\sum_{n=1}^{\infty}\frac{2\cos\dfrac{n\pi X}{L}\cosh\dfrac{n\pi y}{L}}{n\pi\sinh(n\pi L'/L)}\cos\frac{n\pi x}{L}\right)f(X)\,dX+D.$$

Since $f(x)$ must satisfy the consistency condition

$$\int_0^L f(x)\,dx=0,$$

this solution reduces to

$$V(x,y)=D+\sum_{n=1}^{\infty}a_n\cosh\frac{n\pi y}{L}\cos\frac{n\pi x}{L},$$

where
$$a_n=\frac{2}{n\pi\sinh(n\pi L'/L)}\int_0^L f(X)\cos\frac{n\pi X}{L}\,dX.$$

Had the nonhomogeneity been along either of the boundaries $x=0$ or $x=L$, or both, an eigenfunction expansion for $N(x,y;X,Y)$ in terms of functions $g_0(y)=1/\sqrt{L'}$ and $g_n(y)=\sqrt{2/L'}\cos(n\pi y/L')$ would have been used (see Exercise 1). ∎

Exercises 12.4

1. (a) Solve Example 8 when boundary conditions along $x = L$, $y = 0$, and $y = L'$ are homogeneous and that along $x = 0$ is $V_x(0, y) = f(y)$, $0 < y < L'$.
   (b) Find $V(x, y)$ when $f(y) = \delta(y - L'/4) - \delta(y - 3L'/4)$ and $V(0, L'/2) = 0$. What is the value of $V(x, y)$ at all points on the line $y = L'/2$?
2. What is the solution to Example 8 if the boundary condition along $y = 0$ is also nonhomogeneous, $V_y(x, 0) = g(x)$?
3. Verify that the steady-state heat conduction problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = -\frac{1}{\kappa}, \qquad 0 < x < L, \qquad 0 < y < L;$$

$$U_x(0, y) = \frac{L}{4\kappa}, \qquad 0 < y < L,$$

$$U_x(L, y) = \frac{-L}{4\kappa}, \qquad 0 < y < L,$$

$$U_y(x, 0) = \frac{L}{4\kappa}, \qquad 0 < x < L,$$

$$U_y(x, L) = \frac{-L}{4\kappa}, \qquad 0 < x < L,$$

satisfies consistency condition (60), and find its solution.

4. In this problem we develop a modified Green's function for the Neumann problem on a circle and solve the corresponding boundary value problem:

$$\nabla^2 u = F(r, \theta), \qquad 0 < r < r_0, \qquad -\pi < \theta \le \pi,$$

$$\frac{\partial u(r_0, \theta)}{\partial r} = K(\theta), \qquad -\pi < \theta < \pi.$$

   (a) What is the boundary value problem characterizing $N(r, \theta; R, \Theta)$ for this problem?
   (b) Using a partial eigenfunction expansion identical to that in Exercise 13(a) of Section 12.2, show that coefficient functions $a_0(r)$, $a_n(r)$, and $b_n(r)$ must satisfy

$$\frac{d^2 a_0}{dr^2} + \frac{1}{r}\frac{da_0}{dr} = \frac{\delta(r - R)}{\sqrt{2\pi}\, r} - \frac{\sqrt{2}}{\sqrt{\pi}\, r_0^2}, \qquad 0 < r < r_0,$$

$$a_0'(r_0) = 0;$$

$$\frac{d^2 a_n}{dr^2} + \frac{1}{r}\frac{da_n}{dr} - \frac{n^2}{r^2}a_n = \frac{\delta(r - R)\cos n\Theta}{\sqrt{\pi}\, r}, \qquad 0 < r < r_0,$$

$$a_n'(r_0) = 0;$$

$$\frac{d^2 b_n}{dr^2} + \frac{1}{r}\frac{db_n}{dr} - \frac{n^2}{r^2}b_n = \frac{\delta(r - R)\sin n\Theta}{\sqrt{\pi}\, r}, \qquad 0 < r < r_0,$$

$$b_n'(r_0) = 0.$$

(c) Solve the equations in (b) and hence show that

$$N(r,\bar{\theta};R,\Theta)=\begin{cases}\dfrac{A}{\sqrt{2\pi}}-\dfrac{r^2}{4\pi r_0^2}-\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\pi n}\left[\left(\frac{rR}{r_0^2}\right)^n+\left(\frac{r}{R}\right)^n\right]\cos n(\theta-\Theta), & 0\le r\le R\\ \dfrac{A}{\sqrt{2\pi}}+\dfrac{\ln(r/R)}{2\pi}-\dfrac{r^2}{4\pi r_0^2}-\displaystyle\sum_{n=1}^{\infty}\frac{1}{2\pi n}\left[\left(\frac{rR}{r_0^2}\right)^n+\left(\frac{R}{r}\right)^n\right]\cos n(\theta-\Theta), & R\le r\le r_0,\end{cases}$$

where A is independent of r and θ.

(d) Use (33) to simplify the modified Green's function to

$$N(r,\theta;R,\Theta)=\frac{A}{\sqrt{2\pi}}-\frac{r^2}{4\pi r_0^2}+\frac{1}{4\pi}\ln\left(\frac{[r^2+R^2-2rR\cos(\theta-\Theta)][r_0^4+r^2R^2-2rr_0^2R\cos(\theta-\Theta)]}{r_0^4R^2}\right).$$

(e) Find an integral representation for the solution of the boundary value problem in (a).

5. (a) To satisfy consistency condition (60), it is possible to change the boundary condition defining the modified Green's function instead of the PDE. Show that the function $\bar{N}(x,y;X,Y)$ defined by

$$\nabla^2\bar{N}=\delta(x-X,y-Y),\qquad (x,y)\text{ in }A,$$

$$\frac{\partial\bar{N}}{\partial n}=\frac{1}{L},\qquad (x,y)\text{ on }\beta(A),$$

where L is the length of $\beta(A)$, satisfies (60).

(b) Find the solution of problem (58) in terms of $\bar{N}(x,y;X,Y)$.

6. Use the modified Green's function of Exercise 5 to find the solution of the problem in Exercise 4.

7. What is the three-dimensional analog of Exercise 5?

8. (a) The Neumann problem for the Helmholtz equation is

$$(\nabla^2+k^2)u=F(x,y),\qquad (x,y)\text{ in }A, \tag{69a}$$

$$\frac{\partial u}{\partial n}=K(x,y),\qquad (x,y)\text{ on }\beta(A). \tag{69b}$$

The homogeneous system

$$(\nabla^2+k^2)u=0,\qquad (x,y)\text{ in }A, \tag{70a}$$

$$\frac{\partial u}{\partial n}=0,\qquad (x,y)\text{ on }\beta(A), \tag{70b}$$

has nontrivial solutions. (This is clear when $k=0$, since $u=$ constant is a solution, and it is also true when $k\neq 0$.) As a result, (69) does not have a unique solution; if $u(x,y)$ is a solution, then so also is $u(x,y)+Cw(x,y)$, where $w(x,y)$ is any solution of (70). In addition, $F(x,y)$ and $K(x,y)$ must satisfy a consistency condition for there to be a solution of (69) at all: problem

(69) has solutions if and only if

$$\iint_A w(x, y)F(x, y)\,dA = \oint_{\beta(A)} w(x, y)K(x, y)\,ds \tag{71}$$

for every solution $w(x, y)$ of (70). Prove the necessity of this condition.

(b) Show that when consistency condition (71) is satisfied, the solution of (69) is

$$u(x, y) = \iint_A N(x, y; X, Y)F(X, Y)\,dA - \oint_{\beta(A)} N(x, y; X, Y)K(X, Y)\,ds + Cw(x, y) \tag{72}$$

where $w(x, y)$ is the normalized solution of (70), C is an arbitrary constant, and $N(x, y; X, Y)$ is the symmetric modified Green's function satisfying

$$(\nabla^2 + k^2)N = \delta(x - X, y - Y) - w(x, y)w(X, Y), \qquad (x, y) \text{ in } A, \tag{73a}$$

$$\frac{\partial N}{\partial n} = 0, \qquad (x, y) \text{ on } \beta(A). \tag{73b}$$

9. State and prove the three-dimensional analog of Exercise 8.

12.5 Robin and Mixed Boundary Value Problems on Finite Regions

The Robin problem for Poisson's equation is

$$\nabla^2 u = F(x, y), \qquad (x, y) \text{ in } A, \tag{74a}$$

$$l\frac{\partial u}{\partial n} + hu = K(x, y), \qquad (x, y) \text{ on } \beta(A). \tag{74b}$$

Its solution can be represented in integral form in terms of the nonhomogeneities and the Green's function for the problem.

Theorem 4

The solution of problem (74) is

$$u(x, y) = \iint_A G(x, y; X, Y)F(X, Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x, y; X, Y)K(X, Y)\,ds, \tag{75}$$

where $G(x, y; X, Y)$ satisfies

$$\nabla^2 G = \delta(x - X, y - Y), \qquad (x, y) \text{ in } A, \tag{76a}$$

$$l\frac{\partial G}{\partial n} + hG = 0, \qquad (x, y) \text{ on } \beta(A). \tag{76b}$$

Proof: If in Green's identity (14a) on A we let $u = G(x, y; X, Y)$ and $v = u(x, y)$, the solution of (74),

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot\hat{\mathbf{n}}\,ds.$$

Because $\nabla^2 G = \delta(x - X, y - Y)$, $\nabla^2 u = F$, and $l\partial u/\partial n + hu = K$ and $l\partial G/\partial n + hG = 0$ on $\beta(A)$,

$$\iint_A [u(x, y)\delta(x - X, y - Y) - G(x, y; X, Y)F(x, y)]\,dA$$
$$= \oint_{\beta(A)} \left(\frac{u(x, y)}{l}[-hG(x, y; X, Y)] - \frac{G(x, y; X, Y)}{l}[K(x, y) - hu(x, y)]\right) ds$$

or
$$u(X, Y) = \iint_A G(x, y; X, Y)F(x, y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x, y; X, Y)K(x, y)\,ds.$$

When we interchange (x, y) and (X, Y),

$$u(x, y) = \iint_A G(X, Y; x, y)F(X, Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(X, Y; x, y)K(X, Y)\,ds$$
$$= \iint_A G(x, y; X, Y)F(X, Y)\,dA - \frac{1}{l}\oint_{\beta(A)} G(x, y; X, Y)K(X, Y)\,ds,$$

since $G(x, y; X, Y)$ must be symmetric (see Exercise 1). ∎

Because $G = -(l/h)\partial G/\partial n$ on $\beta(A)$, we may also express the solution in the form

$$u(x, y) = \iint_A G(x, y; X, Y)F(X, Y)\,dA + \frac{1}{h}\oint_{\beta(A)} \frac{\partial G(x, y; X, Y)}{\partial N} K(X, Y)\,ds, \tag{77}$$

where once again $\partial G/\partial N$ indicates the outward normal derivative of G with respect to the (X, Y) variables.

For three-dimensional problems

$$\nabla^2 u = F(x, y, z), \qquad (x, y, z) \text{ in } V, \tag{78a}$$

$$l\frac{\partial u}{\partial n} + hu = K(x, y, z), \qquad (x, y, z) \text{ on } \beta(V), \tag{78b}$$

the solution may be represented in either of the forms

$$u(x, y, z) = \iiint_V G(x, y, z; X, Y, Z)F(X, Y, Z)\,dV - \frac{1}{l}\iint_{\beta(V)} G(x, y, z; X, Y, Z)K(X, Y, Z)\,dS \tag{79a}$$

or

$$u(x, y, z) = \iiint_V G(x, y, z; X, Y, Z)F(X, Y, Z)\,dV + \frac{1}{h}\iint_{\beta(V)} \frac{\partial G(x, y, z; X, Y, Z)}{\partial N} K(X, Y, Z)\,dS. \tag{79b}$$

A boundary value problem is said to be *mixed* if all parts of the boundary are not subjected to the same type of condition. For instance, the unknown function may have to satisfy a Dirichlet condition on part of the boundary and a Neumann condition on the remainder.

Example 9: Solve the boundary value problem

$$\nabla^2 u = F(r,\theta), \qquad 0 < r < r_0, \qquad 0 < \theta < \pi,$$
$$u(r_0,\theta) = K_1(\theta), \qquad 0 < \theta < \pi,$$
$$\frac{\partial u(r,0)}{\partial \theta} = 0, \qquad 0 < r < r_0,$$
$$\frac{\partial u(r,\pi)}{\partial \theta} = 0, \qquad 0 < r < r_0.$$

Solution: The Green's function for this problem is

$$G(r,\theta;R,\Theta) = \frac{1}{4\pi}\ln\left(r_0^4\frac{[r^2+R^2-2Rr\cos(\theta+\Theta)][r^2+R^2-2Rr\cos(\theta-\Theta)]}{[R^2r^2+r_0^4-2r_0^2rR\cos(\theta+\Theta)][R^2r^2+r_0^4-2r_0^2rR\cos(\theta-\Theta)]}\right)$$

(see Exercise 2). To solve the boundary value problem, we apply identity (14a) to the semicircle with $u = G$ and $v = u(r,\theta)$, the solution of the problem:

$$\iint_A (u\nabla^2 G - G\nabla^2 u)\,dA = \oint_{\beta(A)} (u\nabla G - G\nabla u)\cdot\hat{n}\,ds.$$

With $\nabla^2 G = \delta(r-R,\theta-\Theta)/r$, $\nabla^2 u = F$, and the boundary conditions for G and u,

$$\iint_A\left(u(r,\theta)\frac{\delta(r-R,\theta-\Theta)}{r} - G(r,\theta;R,\Theta)F(r,\theta)\right)r\,dr\,d\theta$$
$$= \int_0^\pi K_1(\theta)\frac{\partial G(r_0,\theta;R,\Theta)}{\partial r}r_0\,d\theta$$

or

$$u(R,\Theta) = \int_0^\pi\int_0^{r_0} G(r,\theta;R,\Theta)F(r,\theta)r\,dr\,d\theta + \int_0^\pi r_0K_1(\theta)\frac{\partial G(r_0,\theta;R,\Theta)}{\partial r}\,d\theta.$$

When we interchange (r,θ) and (R,Θ) and note the symmetry in G,

$$u(r,\theta) = \int_0^\pi\int_0^{r_0} G(r,\theta;R,\Theta)F(R,\Theta)R\,dR\,d\Theta + \int_0^\pi r_0K_1(\Theta)\frac{\partial G(r,\theta;r_0,\Theta)}{\partial R}\,d\Theta.$$

■

Exercises 12.5

1. Verify that the Green's function for the Robin problem is symmetric.
2. Show that the Green's function for the boundary value problem of Example 9 is

$$\frac{1}{4\pi}\ln\left(r_0^4\frac{[r^2+R^2-2rR\cos(\theta+\Theta)][r^2+R^2-2rR\cos(\theta-\Theta)]}{[r^2R^2+r_0^4-2r_0^2rR\cos(\theta+\Theta)][r^2R^2+r_0^4-2r_0^2rR\cos(\theta-\Theta)]}\right).$$

3. Use a Green's function to find an integral representation for the solution of the boundary value problem

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=F(x,y),\qquad 0<x<L,\qquad 0<y<L,$$
$$u(0,y)=f(y),\qquad 0<y<L,$$
$$u(L,y)=0,\qquad 0<y<L,$$
$$u_y(x,0)=0,\qquad 0<x<L,$$
$$u(x,L)=0,\qquad 0<x<L.$$

4. Show that the solution of the Robin problem

$$(\nabla^2+k^2)u=F(x,y),\qquad (x,y)\text{ in }A,$$
$$l\frac{\partial u}{\partial n}+hu=K(x,y),\qquad (x,y)\text{ on }\beta(A)$$

is (75) when $G(x,y;X,Y)$ is the associated Green's function.

12.6 Green's Functions for Heat Conduction Problems

Green's functions can also be defined for initial boundary value problems; they encompass the character of Green's functions for boundary value problems and also the "causal" features of the initial value problems of Section 11.6.

The causal Green's function for the one-dimensional heat conduction problem

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{k}\frac{\partial U}{\partial t}-\frac{g(x,t)}{\kappa},\qquad 0<x<L,\qquad t>0,\tag{80a}$$

$$-l_1\frac{\partial U}{\partial x}+h_1U=f_1(t),\qquad x=0,\qquad t>0,\tag{80b}$$

$$l_2\frac{\partial U}{\partial x}+h_2U=f_2(t),\qquad x=L,\qquad t>0,\tag{80c}$$

$$U(x,0)=f(x),\qquad 0<x<L,\tag{80d}$$

is defined as the solution of the corresponding problem with homogeneous initial and boundary conditions when a unit of heat is inserted at position X and time T:

$$\frac{\partial^2 U}{\partial x^2}=\frac{1}{k}\frac{\partial U}{\partial t}-\frac{\delta(x-X)\delta(t-T)}{\kappa},\qquad 0<x<L,\qquad t>T,\tag{81a}$$

$$-l_1\frac{\partial U}{\partial x}+h_1U=0, \qquad x=0, \qquad t>T, \tag{81b}$$

$$l_2\frac{\partial U}{\partial x}+h_2U=0, \qquad x=L, \qquad t>T, \tag{81c}$$

$$U(x,t;X,T)=0, \qquad 0<x<L, \qquad t<T. \tag{81d}$$

For $t>T$, it can also be characterized as the solution of

$$\frac{\partial^2 G}{\partial x^2}=\frac{1}{k}\frac{\partial G}{\partial t}, \qquad 0<x<L, \qquad t>T, \tag{82a}$$

$$-l_1\frac{\partial G}{\partial x}+h_1G=0, \qquad x=0, \qquad t>T, \tag{82b}$$

$$l_2\frac{\partial G}{\partial x}+h_2G=0, \qquad x=L, \qquad t>T, \tag{82c}$$

$$G(x,T+;X,T)=\frac{k}{\kappa}\delta(x-X), \qquad 0<x<L; \tag{82d}$$

that is, the solution of (81) is $H(t-T)G(x,t;X,T)$ when $G(x,t;X,T)$ satisfies (82). What this means is that the effect of a unit heat source at position X and time T on a rod with zero temperature is equivalent to the effect of suddenly raising the temperature of the rod at point X to k/κ at time T. The causal Green's function for (80) is $H(t-T)G(x,t;X,T)$, where $G(x,t;X,T)$ satisfies (82). In essence, then, $G(x,t;X,T)$ is the causal Green's function for problem (80); we must simply remember to set it equal to zero for $t<T$. Because of this, we shall customarily call $G(x,t;X,T)$ the *causal Green's function*.

Example 10: Find the causal Green's function for problem (80) in the case that $l_1=0=h_2$.

Solution: Separation of variables on problem (82) with $l_1=h_2=0$ leads, for $t>T$, to a solution in the form

$$G(x,t;X,T)=\sum_{n=1}^{\infty}C_n e^{-(2n-1)^2\pi^2kt/(4L^2)}f_n(x),$$

where $f_n(x)=\sqrt{2/L}\sin[(2n-1)\pi x/(2L)]$. If $\delta(x-X)$ is given an eigenfunction expansion in terms of the $\{f_n(x)\}$, the initial condition requires that

$$\sum_{n=1}^{\infty}C_n e^{-(2n-1)^2\pi^2kT/(4L^2)}f_n(x)=\frac{k}{\kappa}\sum_{n=1}^{\infty}\left(\int_0^L\delta(x-X)f_n(x)\,dx\right)f_n(x)$$

$$=\frac{k}{\kappa}\sum_{n=1}^{\infty}f_n(X)f_n(x).$$

It follows, then, that

$$C_n e^{-(2n-1)^2\pi^2kT/(4L^2)}=\frac{k}{\kappa}f_n(X)$$

and

$$G(x,t;X,T) = \sum_{n=1}^{\infty} \frac{k}{\kappa} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} f_n(X) f_n(x)$$
$$= \frac{2k}{\kappa L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \sin\frac{(2n-1)\pi X}{2L} \sin\frac{(2n-1)\pi x}{2L}. \quad \blacksquare$$

The solution of problem (80) can be expressed in terms of the causal Green's function for the problem:

$$U(x,t) = \int_0^t \int_0^L G(x,t;X,T) g(X,T)\, dX\, dT + \frac{\kappa}{k} \int_0^L G(x,t;X,0) f(X)\, dX$$
$$+ \kappa \int_0^t \left(G(x,t;L,T) \frac{f_2(T)}{l_2} + G(x,t;0,T) \frac{f_1(T)}{l_1} \right) dT. \tag{83a}$$

The first term is the contribution of the internal heat source from $t = 0$ to present time, the second term is due to the initial temperature distribution in the rod, and the last integral represents the effects of heat transfer at the ends of the rod. Boundary conditions (82b, c) can be used to rewrite the last integral in the form

$$U(x,t) = \int_0^t \int_0^L G(x,t;X,T) g(X,T)\, dX\, dT + \frac{\kappa}{k} \int_0^L G(x,t;X,0) f(X)\, dX$$
$$+ \kappa \int_0^t \left(-\frac{\partial G(x,t;L,T)}{\partial X} \frac{f_2(T)}{h_2} + \frac{\partial G(x,t;0,T)}{\partial X} \frac{f_1(T)}{h_1} \right) dT \tag{83b}$$

(see Exercise 11). This form must be used when $l_1 = l_2 = 0$.

Example 11: Solve the heat conduction problem in Example 2 of Section 6.2.

Solution: The Green's function for this problem was obtained in the previous example. With $g(x,t) \equiv 0$, and $f_2(t)$ replaced by $-\kappa^{-1} f_2(t)$, we use (83a, b) to write

$$U(x,t) = \frac{\kappa}{k} \int_0^L G(x,t;X,0) f(X)\, dX$$
$$+ \kappa \int_0^t \left(-G(x,t;L,T) \frac{f_2(T)}{\kappa l_2} + \frac{\partial G(x,t;0,T)}{\partial X} \frac{f_1(T)}{h_1} \right) dT$$
$$= \frac{\kappa}{k} \int_0^L \left(\frac{2k}{\kappa L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 kt/(4L^2)} \sin\frac{(2n-1)\pi X}{2L} \sin\frac{(2n-1)\pi x}{2L} \right) f(X)\, dX$$
$$- \int_0^t \left(\frac{2k}{\kappa L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \sin\frac{(2n-1)\pi}{2} \sin\frac{(2n-1)\pi x}{2L} \right) f_2(T)\, dT$$
$$+ \kappa \int_0^t \left(\frac{2k}{\kappa L} \sum_{n=1}^{\infty} e^{-(2n-1)^2\pi^2 k(t-T)/(4L^2)} \left(\frac{(2n-1)\pi}{2L} \right) \sin\frac{(2n-1)\pi x}{2L} \right) f_1(T)\, dT.$$

When we interchange orders of summation and integration,

$$U(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(X)\sin\frac{(2n-1)\pi X}{2L}\,dX\right)e^{-(2n-1)^2\pi^2kt/(4L^2)}\sin\frac{(2n-1)\pi x}{2L}$$

$$+\frac{2k}{L}\sum_{n=1}^{\infty}\left(\int_0^t\left[\left(\frac{(2n-1)\pi}{2L}\right)f_1(T) + \frac{(-1)^n}{\kappa}f_2(T)\right]\right.$$

$$\left.\times\, e^{-(2n-1)^2\pi^2k(t-T)/(4L^2)}\,dT\right)\sin\frac{(2n-1)\pi x}{2L},$$

and this is solution (46) in Section 6.2. ∎

The causal Green's function for the two-dimensional heat conduction problem

$$\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{g(x,y,t)}{\kappa}, \qquad (x,y) \text{ in } A, \qquad t > 0, \tag{84a}$$

$$l\frac{\partial U}{\partial n} + hU = F(x,y,t), \qquad (x,y) \text{ on } \beta(A), \qquad t > 0, \tag{84b}$$

$$U(x,y,0) = f(x,y), \qquad (x,y) \text{ in } A, \tag{84c}$$

is defined as the solution of

$$\nabla^2 U = \frac{1}{k}\frac{\partial U}{\partial t} - \frac{\delta(x-X,y-Y)\delta(t-T)}{\kappa}, \qquad (x,y) \text{ in } A, \qquad t > T, \tag{85a}$$

$$l\frac{\partial U}{\partial n} + hU = 0, \qquad (x,y) \text{ on } \beta(A), \qquad t > T, \tag{85b}$$

$$U(x,y,t;X,Y,T) = 0, \qquad (x,y) \text{ in } A, \qquad t < T. \tag{85c}$$

It is also given by $H(t-T)G(x,y,t;X,Y,T)$, where $G(x,y,t;X,Y,T)$ satisfies

$$\nabla^2 G = \frac{1}{k}\frac{\partial G}{\partial t}, \qquad (x,y) \text{ in } A, \qquad t > T, \tag{86a}$$

$$l\frac{\partial G}{\partial n} + hG = 0, \qquad (x,y) \text{ on } \beta(A), \qquad t > T, \tag{86b}$$

$$G(x,y,T+;X,Y,T) = \frac{k}{\kappa}\delta(x-X,y-Y), \qquad (x,y) \text{ in } A. \tag{86c}$$

The solution of problem (84) can then be expressed in the form

$$U(x,y,t) = \int_0^t \iint_A G(x,y,t;X,Y,T)g(X,Y,T)\,dA\,dT$$

$$+\frac{\kappa}{k}\iint_A G(x,y,t;X,Y,0)f(X,Y)\,dA$$

$$+\frac{\kappa}{l}\int_0^t \oint_{\beta(A)} G(x,y,t;X,Y,T)F(X,Y,T)\,ds\,dT \tag{87a}$$

or
$$U(x,y,t) = \int_0^t \iint_A G(x,y,t;X,Y,T)g(X,Y,T)\,dA\,dT + \frac{\kappa}{k}\iint_A G(x,y,t;X,Y,0)f(X,Y)\,dA - \frac{\kappa}{h}\int_0^t \oint_{\beta(A)} F(X,Y,T)\frac{\partial G(x,y,t;X,Y,T)}{\partial N}\,ds\,dT. \quad \textbf{(87b)}$$

Exercises 12.6

In Exercises 1–4, find the causal Green's function for problem (80) when the values for l_1, l_2, h_1, and h_2 are as specified.

1. $l_1 = l_2 = 0,\quad h_1 = h_2 = 1$
2. $h_1 = h_2 = 0,\quad l_1 = l_2 = 1$
3. $l_2 = h_1 = 0,\quad l_1 = h_2 = 1$
4. $l_1 = 0,\quad h_1 = 1,\quad l_2h_2 \neq 0$

In Exercises 5–9, use formulas (83a, b) to solve the initial boundary value problem.

5. Exercise 8 in Section 3.3.
6. Exercise 9 in Section 3.3.
7. Exercise 1 in Section 6.2.
8. Exercise 7 in Section 6.2.
9. Exercise 15 in Section 6.2.
10. (a) What is the causal Green's function for problem (80)?
    (b) Use the representation in (a) to show that $G(x,t;X,T)$ satisfies the "reciprocity principle"
    $$G(x,t;X,T) = G(X,t;x,T).$$
    What does this mean physically?
    (c) Use the representation in (a) to show that $G(x,t;X,T)$ satisfies the "time-translation" property
    $$G(x,t-\bar{T};X,T) = G(x,t;X,T+\bar{T})$$
    (provided $t - T - \bar{T} > 0$). What does this mean physically?
11. Use the result in Exercise 10(b) to show that solution (83a) can be expressed in form (83b).

In Exercises 12–15, use formulas (87a, b) to solve the two-dimensional heat conduction problem.

12. Exercise 1 in Section 6.3.
13. Exercise 2(a) in Section 6.3.
14. Exercise 2(a) in Section 9.2.
15. Parts (a) and (b) of Exercise 3 in Section 9.2.
16. What are the three-dimensional analogs of equations (84)–(87)?

12.7 Green's Functions for the Wave Equation

The causal Green's function $G(x,t;X,T)$ for the one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{F(x,t)}{\rho c^2}, \qquad 0 < x < L, \qquad t > 0, \quad \textbf{(88a)}$$

$$-l_1\frac{\partial y}{\partial x} + h_1 y = f_1(t), \qquad x = 0, \qquad t > 0, \tag{88b}$$

$$l_2\frac{\partial y}{\partial x} + h_2 y = f_2(t), \qquad x = L, \qquad t > 0, \tag{88c}$$

$$y(x,0) = f(x), \qquad 0 < x < L, \tag{88d}$$

$$y_t(x,0) = g(x), \qquad 0 < x < L, \tag{88e}$$

is defined as the solution of

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} - \frac{\delta(x-X)\delta(t-T)}{\rho c^2}, \qquad 0 < x < L, \qquad t > T, \tag{89a}$$

$$-l_1\frac{\partial y}{\partial x} + h_1 y = 0, \qquad x = 0, \qquad t > T, \tag{89b}$$

$$l_2\frac{\partial y}{\partial x} + h_2 y = 0, \qquad x = L, \qquad t > T, \tag{89c}$$

$$y(x,t;X,T) = 0, \qquad 0 < x < L, \qquad t < T. \tag{89d}$$

It is also given by $H(t-T)G(x,t;X,T)$, where $G(x,t;X,T)$ satisfies

$$\frac{\partial^2 G}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}, \qquad 0 < x < L, \qquad t > T, \tag{90a}$$

$$-l_1\frac{\partial G}{\partial x} + h_1 G = 0, \qquad x = 0, \qquad t > T, \tag{90b}$$

$$l_2\frac{\partial G}{\partial x} + h_2 G = 0, \qquad x = L, \qquad t > T, \tag{90c}$$

$$G(x, T+; X, T) = 0, \qquad 0 < x < L, \tag{90d}$$

$$G_t(x, T+; X, T) = \frac{\delta(x-X)}{\rho}, \qquad 0 < x < L. \tag{90e}$$

In other words, the effect of an instantaneous unit force at position X and time T is equivalent to the effect of giving the point at X an instantaneous initial velocity $1/\rho$. Although the causal Green's function for (88) is $H(t-T)G(x,t;X,T)$, where $G(x,t;X,T)$ satisfies (90), we shall customarily call $G(x,t;X,T)$ itself the Green's function.

Problem (90) is easily solved by separation of variables.

Example 12: Find the causal Green's function for problem (88) when $l_1 = l_2 = 0$.

Solution: Separation of variables on (90a–d) leads, for $t > T$, to

$$G(x,t;X,T) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi c(t-T)}{L} f_n(x),$$

where $f_n(x) = \sqrt{2/L}\sin(n\pi x/L)$. If $\delta(x - X)$ is expanded in terms of the $\{f_n(x)\}$, initial condition (90e) requires that

$$\sum_{n=1}^{\infty} \frac{n\pi c}{L} A_n f_n(x) = \frac{1}{\rho}\sum_{n=1}^{\infty}\left(\int_0^L \delta(x - X) f_n(x)\,dx\right) f_n(x)$$
$$= \frac{1}{\rho}\sum_{n=1}^{\infty} f_n(X) f_n(x).$$

It follows, then, that

$$\frac{n\pi c}{L} A_n = \frac{1}{\rho} f_n(X)$$

and

$$G(x,t;X,T) = \sum_{n=1}^{\infty} \frac{L}{n\pi c\rho} f_n(X) \sin\frac{n\pi c(t-T)}{L} f_n(x)$$
$$= \frac{L}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c(t-T)}{L} f_n(X) f_n(x). \qquad \blacksquare$$

The solution of problem (88) can be expressed in terms of its Green's function:

$$y(x,t) = \int_0^t \int_0^L G(x,t;X,T)F(X,T)\,dX\,dT$$
$$+ \rho\int_0^L\left(g(X)G(x,t;X,0) - f(X)\frac{\partial G(x,t;X,0)}{\partial T}\right)dX$$
$$+ \rho c^2 \int_0^t\left(G(x,t;L,T)\frac{f_2(T)}{l_2} + G(x,t;0,T)\frac{f_1(T)}{l_1}\right)dT. \qquad \textbf{(91a)}$$

The first integral contains the effect of past external forces, and the second integral contains that of the initial displacement and velocity. The last integral is due to boundary disturbances. Boundary conditions (90b, c) can be used to rewrite the last integral in the form

$$y(x,t) = \int_0^t \int_0^L G(x,t;X,T)F(X,T)\,dX\,dT$$
$$+ \rho\int_0^L\left(g(X)G(x,t;X,0) - f(X)\frac{\partial G(x,t;X,0)}{\partial T}\right)dX$$
$$+ \rho c^2 \int_0^t\left(-\frac{f_2(T)}{h_2}\frac{\partial G(x,t;L,T)}{\partial X} + \frac{f_1(T)}{h_1}\frac{\partial G(x,t;0,T)}{\partial X}\right)dT. \qquad \textbf{(91b)}$$

This form must be used when $l_1 = l_2 = 0$.

Example 13: Solve the vibration problem of Example 3 in Section 6.2.

Solution: The Green's function for this problem was derived in Example 12. According to (91b), then,

$$y(x,t) = \rho c^2 \int_0^t -\frac{\partial G(x,t;L,T)}{\partial X} g(T)\,dT$$

$$= \rho c^2 \int_0^t \left(\frac{-L}{\rho \pi c} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi c(t-T)}{L} f_n'(L) f_n(x) \right) g(T)\, dT$$

$$= -\frac{Lc}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_0^t \sin \frac{n\pi c(t-T)}{L} g(T)\, dT \right) f_n'(L) f_n(x).$$

When $g(t) = A \sin \omega t$ and $\omega \neq n\pi c/L$ for any integer n,

$$\int_0^t \sin \frac{n\pi c(t-T)}{L} A \sin \omega T\, dT = \frac{AL^2}{n^2\pi^2c^2 - \omega^2 L^2} \left(\frac{n\pi c}{L} \sin \omega t - \omega \sin \frac{n\pi ct}{L} \right),$$

and therefore

$$y(x,t) = -\frac{Lc}{\pi} \sum_{n=1}^{\infty} \frac{AL^2}{n(n^2\pi^2c^2 - \omega^2L^2)} \left(\frac{n\pi c}{L} \sin \omega t - \omega \sin \frac{n\pi ct}{L} \right) \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L} \right) (-1)^n \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

$$= 2cA \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2\pi^2c^2 - \omega^2L^2} \left(\omega L \sin \frac{n\pi ct}{L} - n\pi c \sin \omega t \right) \sin \frac{n\pi x}{L}.$$

When $g(t) = A \sin(m\pi ct/L)$,

$$\int_0^t \sin \frac{n\pi c(t-T)}{L} g(T)\, dT = \begin{cases} \dfrac{AL}{\pi c(n^2 - m^2)} \left(n \sin \dfrac{m\pi ct}{L} - m \sin \dfrac{n\pi ct}{L} \right) & n \neq m \\[2ex] \dfrac{A}{2m\pi c} \left(L \sin \dfrac{m\pi ct}{L} - m\pi ct \cos \dfrac{m\pi ct}{L} \right) & n = m \end{cases}$$

and

$$y(x,t) = \frac{Lc}{-\pi} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{AL}{n\pi c(n^2 - m^2)} \left(n \sin \frac{m\pi ct}{L} - m \sin \frac{n\pi ct}{L} \right) f_n'(L) f_n(x)$$

$$- \frac{Lc}{\pi} \left[\frac{A}{2m^2\pi c} \left(L \sin \frac{m\pi ct}{L} - m\pi ct \cos \frac{m\pi ct}{L} \right) \right] f_m'(L) f_m(x)$$

$$= \frac{2A}{\pi} \sum_{\substack{n=1 \\ n \neq m}}^{\infty} \frac{(-1)^n}{n^2 - m^2} \left(m \sin \frac{n\pi ct}{L} - n \sin \frac{m\pi ct}{L} \right) \sin \frac{n\pi x}{L}$$

$$+ \frac{(-1)^m A}{m\pi L} \left(m\pi ct \cos \frac{m\pi ct}{L} - L \sin \frac{m\pi ct}{L} \right) \sin \frac{m\pi x}{L}. \quad ■$$

The causal Green's function for the two-dimensional vibration problem

$$\nabla^2 z = \frac{1}{c^2} \frac{\partial^2 z}{\partial t^2} - \frac{F(x,y,t)}{\rho c^2}, \qquad (x,y) \text{ in } A, \qquad t > 0, \tag{92a}$$

$$l \frac{\partial z}{\partial n} + hz = K(x,y,t), \qquad (x,y) \text{ on } \beta(A), \qquad t > 0, \tag{92b}$$

$$z(x,y,0) = f(x,y), \qquad (x,y) \text{ in } A, \tag{92c}$$

$$z_t(x,y,0) = g(x,y), \qquad (x,y) \text{ in } A, \tag{92d}$$

is defined as the solution of

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} - \frac{\delta(x-X, y-Y)\delta(t-T)}{\rho c^2}, \quad (x,y) \text{ in } A, \quad t > T, \tag{93a}$$

$$l\frac{\partial z}{\partial n} + hz = 0, \quad (x,y) \text{ on } \beta(A), \quad t > T, \tag{93b}$$

$$z(x,y,t;X,Y,T) = 0, \quad (x,y) \text{ in } A, \quad t < T. \tag{93c}$$

It is also given by $H(t-T)G(x,y,t;X,Y,T)$, where $G(x,y,t;X,Y,T)$ satisfies

$$\nabla^2 G = \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2}, \quad (x,y) \text{ in } A, \quad t > T, \tag{94a}$$

$$l\frac{\partial G}{\partial n} + hG = 0, \quad (x,y) \text{ on } \beta(A), \quad t > T, \tag{94b}$$

$$G(x,y,T+;X,Y,T) = 0, \quad (x,y) \text{ in } A, \tag{94c}$$

$$G_t(x,y,T+;X,Y,T) = \frac{\delta(x-X, y-Y)}{\rho}, \quad (x,y) \text{ in } A. \tag{94d}$$

The solution of (92) can then be expressed in the form

$$\begin{aligned} z(x,y,t) = &\int_0^t \iint_A G(x,y,t;X,Y,T)F(X,Y,T)\,dA\,dT \\ &+ \rho \iint_A \left(g(X,Y)G(x,y,t;X,Y,0) - f(X,Y)\frac{\partial G(x,y,t;X,Y,0)}{\partial T} \right) dA \\ &+ \frac{\rho c^2}{l} \int_0^t \oint_{\beta(A)} G(x,y,t;X,Y,T)K(X,Y,T)\,ds\,dT \end{aligned} \tag{95a}$$

or

$$\begin{aligned} z(x,y,t) = &\int_0^t \iint_A G(x,y,t;X,Y,T)F(X,Y,T)\,dA\,dT \\ &+ \rho \iint_A \left(g(X,Y)G(x,y,t;X,Y,0) - f(X,Y)\frac{\partial G(x,y,t;X,Y,0)}{\partial T} \right) dA \\ &- \frac{\rho c^2}{h} \int_0^t \oint_{\beta(A)} K(X,Y,T)\frac{\partial G(x,y,t;X,Y,T)}{\partial N}\,ds\,dT. \end{aligned} \tag{95b}$$

Exercises 12.7

In Exercises 1–3, find the causal Green's function for problem (88) when values for l_1, l_2, h_1, and h_2 are as specified.

1. $h_1 = h_2 = 0, l_1 = l_2 = 1$
2. $l_2 = h_1 = 0, l_1 = h_2 = 1$
3. $l_2 = h_1 = 1, l_1 = h_2 = 0$

In Exercises 4–6, use formulas (91a, b) to solve the initial boundary value problem.

4. Exercise 13 in Section 3.3 (see also Exercise 19 in Section 6.2).
5. Exercise 21(a) in Section 6.2.
6. Exercise 22 in Section 6.2.
7. A taut string initially at rest along the x-axis has its ends fixed at $x = 0$ and $x = L$.
   (a) Find displacements in the string for an arbitrary forcing function $F(x, t)$.
   (b) Simplify the solution in (a) when $F(x, t)$ is a time-independent, constant force F_0 concentrated at $x = x_0$.
   (c) Simplify the solution in (b) further if $x_0 = L/2$.
   (d) What is the solution in (b) if x_0 is a node of the mth normal mode of vibration of the string?
8. (a) What is the causal Green's function for problem (88)?
   (b) Use the representation in (a) to show that $G(x, t; X, T)$ satisfies the "reciprocity principle"

$$G(x, t; X, T) = G(X, t; x, T).$$

   What does this mean physically?
   (c) Use the representation in (a) to show that $G(x, t; X, T)$ satisfies the "time-translation" property

$$G(x, t; X, T) = G(x, t + \bar{T}; X, T + \bar{T})$$

   (provided $\bar{T} > 0$). What does this mean physically?

In Exercises 9 and 10, use formulas (95a, b) to solve the two-dimensional vibration problem.

9. Exercise 6 in Section 6.3.
10. Exercise 16 in Section 9.2.

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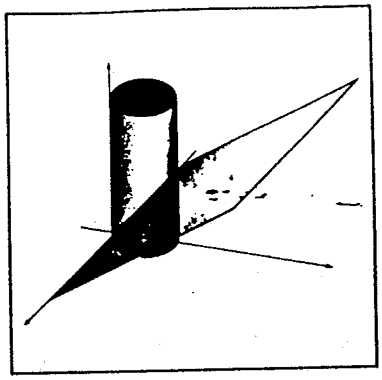


APPENDIX

A

Convergence of Fourier Series

In order to establish convergence of a Fourier series to the function that it represents, we require a few preliminary results on trigonometric integrals. These results are formulated so as to make them useful for Fourier integrals in Appendix B as well.

Result 1 (Riemann's Theorem)

If $f(x)$ is piecewise continuous on $a \leq x \leq b$, then

$$\lim_{\lambda \to \infty} \int_a^b f(x) \sin \lambda x \, dx = 0 = \lim_{\lambda \to \infty} \int_a^b f(x) \cos \lambda x \, dx. \tag{1}$$

Proof: The interval $a \leq x \leq b$ can be divided into a finite number of subintervals $p \leq x \leq q$ in each of which $f(x)$ is continuous even at the end points, provided we use the limits from the interior as values of $f(x)$ at the end points. The theorem then follows if we can show that

$$\lim_{\lambda \to \infty} \int_p^q f(x) \sin \lambda x \, dx = 0 = \lim_{\lambda \to \infty} \int_p^q f(x) \cos \lambda x \, dx$$

for continuous $f(x)$ on $p \leq x \leq q$. If we divide this interval into n equal parts by points $x_j =$

$p + (q - p)j/n$, $j = 0, \ldots, n$, then

$$\begin{aligned}\int_p^q f(x)\sin\lambda x\,dx &= \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}} f(x)\sin\lambda x\,dx\\ &= \sum_{j=0}^{n-1}\left(f(x_j)\int_{x_j}^{x_{j+1}}\sin\lambda x\,dx + \int_{x_j}^{x_{j+1}}[f(x)-f(x_j)]\sin\lambda x\,dx\right)\\ &= \sum_{j=0}^{n-1} f(x_j)\left(\frac{\cos\lambda x_j - \cos\lambda x_{j+1}}{\lambda}\right)\\ &\quad + \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}[f(x)-f(x_j)]\sin\lambda x\,dx.\end{aligned}$$

When we use the triangle inequality, $|a + b| \le |a| + |b|$, on each of these summations, and note that $|\sin\lambda x| \le 1$, we obtain

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| \le \sum_{j=0}^{n-1}|f(x_j)|\left|\frac{\cos\lambda x_j - \cos\lambda x_{j+1}}{\lambda}\right| + \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}|f(x) - f(x_j)|\,dx.$$

Clearly, $|\cos\lambda x_j - \cos\lambda x_{j+1}| \le |\cos\lambda x_j| + |\cos\lambda x_{j+1}| \le 2$, and if we denote the maximum value of $|f(x)|$ on $p \le x \le q$ by M, then

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| \le \frac{2Mn}{\lambda} + \sum_{j=0}^{n-1}\int_{x_j}^{x_{j+1}}|f(x) - f(x_j)|\,dx.$$

Because a continuous function $[f(x)]$ on a closed interval $[p \le x \le q]$ is uniformly continuous† thereon, we can state that corresponding to any number $\varepsilon > 0$, no matter how small, there exists an N large enough that when $n > N$ and $x_j \le x \le x_{j+1}$,

$$|f(x) - f(x_j)| < \frac{\varepsilon}{2(q-p)}.$$

For $n > N$, then,

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| \le \frac{2Mn}{\lambda} + \sum_{j=0}^{n-1}\frac{\varepsilon}{2(q-p)}(x_{j+1} - x_j) = \frac{2Mn}{\lambda} + \frac{\varepsilon}{2}.$$

Finally, if λ is chosen so large that $2Mn/\lambda < \varepsilon/2$, then

$$\left|\int_p^q f(x)\sin\lambda x\,dx\right| < \varepsilon;$$

† A function $f(x)$ is *uniformly continuous on an interval I* if given any $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever $|x_1 - x_2| < \delta$ and x_1 and x_2 are in I,

$$|f(x_1) - f(x_2)| < \varepsilon.$$

that is, λ can be chosen so large that the value of the integral can be made arbitrarily close to zero. This is tantamount to saying that

$$\lim_{\lambda \to \infty} \int_p^q f(x) \sin \lambda x \, dx = 0.$$

A similar proof yields the other limit. ■

When λ is set equal to $n\pi/L$, we obtain the following corollary to Result 1.

Corollary

If $f(x)$ is piecewise continuous on $0 \le x \le 2L$, then

$$\lim_{n \to \infty} \int_0^{2L} f(x) \cos \frac{n\pi x}{L} dx = 0 = \lim_{n \to \infty} \int_0^{2L} f(x) \sin \frac{n\pi x}{L} dx. \tag{2}$$

Result 2

If $f(x)$ is piecewise continuous on $0 \le x \le b$ and has a right derivative at $x = 0$, then

$$\lim_{\lambda \to \infty} \int_0^b f(x) \frac{\sin \lambda x}{x} dx = \frac{\pi}{2} f(0+). \tag{3}$$

Proof: We begin by expressing the integral in the form

$$\int_0^b f(x) \frac{\sin \lambda x}{x} dx = \int_0^b \left(\frac{f(x) - f(0+)}{x} \right) \sin \lambda x \, dx + f(0+) \int_0^b \frac{\sin \lambda x}{x} dx. \tag{4}$$

Now the function $[f(x) - f(0+)]/x$ is piecewise continuous on $0 \le x \le b$ [since $f(x)$ is, and provided we define the value at $x = 0$ by the limit that is the right derivative of $f(x)$ at $x = 0$]. Hence, by Riemann's theorem,

$$\lim_{\lambda \to \infty} \int_0^b \left(\frac{f(x) - f(0+)}{x} \right) \sin \lambda x \, dx = 0,$$

and the first integral on the right of (4) vanishes in the limit as $\lambda \to \infty$. Further, by the change of variable $u = \lambda x$ in the second integral, we find that

$$\lim_{\lambda \to \infty} \int_0^b \frac{\sin \lambda x}{x} dx = \lim_{\lambda \to \infty} \int_0^{b\lambda} \frac{\sin u}{u} du = \int_0^\infty \frac{\sin u}{u} du = \frac{\pi}{2}.^\dagger$$

Consequently, the limit of (4) as $\lambda \to \infty$ yields (3). ■

† This integral is quoted in many sources. See, for example, any edition of *Standard Mathematical Tables* by Chemical Rubber Publishing Company.

Result 3

If $f(x)$ is piecewise continuous on $a \le x \le b$, then at every x in $a < x < b$ at which $f(x)$ has a right and left derivative,

$$\lim_{\lambda\to\infty} \frac{1}{\pi}\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = \frac{f(x+)+f(x-)}{2}. \tag{5}$$

Proof: We begin by subdividing the interval of integration,

$$\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = \int_a^x f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt + \int_x^b f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt,$$

and make the changes of variables $u = x - t$ and $u = t - x$, respectively:

$$\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = \int_0^{x-a} f(x-u)\frac{\sin\lambda u}{u}\,du + \int_0^{b-x} f(x+u)\frac{\sin\lambda u}{u}\,du.$$

For fixed x, $f(x-u)$ is piecewise continuous in u on $0 \le u \le x-a$ and has a right derivative at $u = 0$ [namely, the negative of the left derivative of $f(x)$ at x]. It follows, then, from Result 2 that

$$\lim_{\lambda\to\infty}\int_0^{x-a} f(x-u)\frac{\sin\lambda u}{u}\,du = \frac{\pi}{2}f(x-).$$

A similar discussion yields

$$\lim_{\lambda\to\infty}\int_0^{b-x} f(x+u)\frac{\sin\lambda u}{u}\,du = \frac{\pi}{2}f(x+),$$

and these two facts give (5). ■

We are now prepared to prove Theorem 2 of Section 2.1.

Result 4

If $f(x)$ is piecewise continuous and of period $2L$, then at every x at which $f(x)$ has a right and left derivative, the Fourier series of $f(x)$ converges to $[f(x+)+f(x-)]/2$.

Proof: The nth partial sum of the Fourier series of $f(x)$ is

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos\frac{k\pi x}{L} + b_k\sin\frac{k\pi x}{L}\right).$$

Substitutions from definitions (12) in Section 2.1 for a_0, a_k, and b_k yield

$$\begin{aligned}S_n(x) &= \frac{1}{2L}\int_0^{2L} f(t)\,dt + \sum_{k=1}^{n}\left(\cos\frac{k\pi x}{L}\frac{1}{L}\int_0^{2L} f(t)\cos\frac{k\pi t}{L}\,dt\right.\\ &\qquad\left. + \sin\frac{k\pi x}{L}\frac{1}{L}\int_0^{2L} f(t)\sin\frac{k\pi t}{L}\,dt\right)\\ &= \frac{1}{L}\int_0^{2L}\left[\frac{1}{2}f(t) + \sum_{k=1}^{n} f(t)\left(\cos\frac{k\pi x}{L}\cos\frac{k\pi t}{L} + \sin\frac{k\pi x}{L}\sin\frac{k\pi t}{L}\right)\right]dt\end{aligned}$$

$$= \frac{1}{L}\int_0^{2L} f(t)\left(\frac{1}{2} + \sum_{k=1}^{n} \cos\frac{k\pi(x-t)}{L}\right) dt$$

$$= \frac{1}{L}\int_0^{2L} f(t)\,\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{2\sin\dfrac{\pi(x-t)}{2L}}\,dt.^{\dagger}$$

Since the integrand is of period $2L$, we may integrate over any interval of length $2L$. We choose an interval beginning at a, where $a < x < a + 2L$, and rearrange the integrand into the following form:

$$S_n(x) = \frac{1}{L}\int_a^{a+2L}\left(f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}\right)\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{x-t}\,dt.$$

In order to take limits as $n \to \infty$ and apply Result 3, we require piecewise continuity of

$$F(t) = f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}$$

on $a \le t \le a + 2L$ and existence of both of its one-sided derivatives at $t = x$ (x fixed). This will follow if

$$\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}$$

has these properties [since $f(t)$ has, by assumption]. Since $t = x$ is the only point in the interval $a \le t \le a + 2L$ at which the denominator of this function vanishes, it follows that it is indeed piecewise continuous thereon. Furthermore, it is easily shown that this function has a right and left derivative at $t = x$. By Result 3, then,

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty}\frac{1}{L}\int_a^{a+2L}\left(f(t)\frac{x-t}{2\sin\dfrac{\pi(x-t)}{2L}}\right)\frac{\sin\dfrac{(n+1/2)\pi(x-t)}{L}}{x-t}\,dt$$

$$= \frac{\pi}{L}\lim_{n\to\infty}\frac{1}{\pi}\int_a^{a+2L} F(t)\frac{\sin\dfrac{(2n+1)\pi(x-t)}{2L}}{x-t}\,dt = \frac{\pi}{2L}[F(x+) + F(x-)].$$

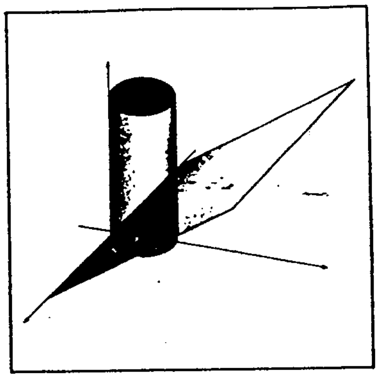
† We have used the identity

$$\frac{1}{2} + \sum_{k=1}^{n}\cos k\theta = \frac{\sin(n+1/2)\theta}{2\sin(\theta/2)}.$$

This formula can be established by expressing $\cos k\theta$ as a complex exponential $(e^{ik\theta} + e^{-ik\theta})/2$ and summing the two resulting geometric series. The identity is regarded in the limit sense at angles for which $\sin(\theta/2) = 0$.

Since $F(x+) = \lim_{t\to x^+} F(t) = f(x+)(L/\pi)$, and similarly for $F(x-)$, it follows that

$$\lim_{n\to\infty} S_n(x) = \frac{\pi}{2L}\left(\frac{L}{\pi}f(x+) + \frac{L}{\pi}f(x-)\right)$$
$$= \frac{f(x+) + f(x-)}{2}. \qquad \blacksquare$$



APPENDIX

B

Convergence of Fourier Integrals

In order to establish convergence of a Fourier integral to the function that it represents, we require some preliminary results on trigonometric integrals. They parallel and utilize analogous properties in Appendix A.

Result 1 (Riemann's Theorem)

If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx = 0 = \lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \cos \lambda x \, dx. \tag{1}$$

Proof: Since

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx = \lim_{\lambda \to \infty} \left(\lim_{\substack{r \to \infty \\ s \to \infty}} \int_{-s}^{r} f(x) \sin \lambda x \, dx \right),$$

and the limit on r and s is absolutely and uniformly convergent with respect to λ, limits may be reversed:

$$\lim_{\lambda \to \infty} \int_{-\infty}^{\infty} f(x) \sin \lambda x \, dx = \lim_{\substack{r \to \infty \\ s \to \infty}} \left(\lim_{\lambda \to \infty} \int_{-s}^{r} f(x) \sin \lambda x \, dx \right).$$

But Riemann's theorem for finite intervals (Result 1 in Appendix A) implies that the integral on the right vanishes for all r and s. ∎

Result 2

If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right and left derivative,

$$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} f(t) \frac{\sin \lambda(x-t)}{x-t} dt = \frac{f(x+) + f(x-)}{2}. \tag{2}$$

Proof: For each fixed x, the function

$$F(t) = f(t) \frac{\sin \lambda(x-t)}{x-t}$$

is piecewise continuous in t on every finite interval [provided we define $F(x)$ by the limit as t approaches x]. Further, since

$$|F(t)| = |\lambda||f(t)|\left|\frac{\sin \lambda(x-t)}{\lambda(x-t)}\right| \le |\lambda||f(t)|,$$

and $f(t)$ is absolutely integrable on $-\infty < t < \infty$, it follows that the improper integral

$$\int_{-\infty}^{\infty} F(t)\,dt = \int_{-\infty}^{\infty} f(t)\frac{\sin \lambda(x-t)}{x-t} dt$$

converges. If a and b are numbers such that $a < x < b$, then

$$\left|\int_{-\infty}^{\infty} F(t)\,dt - \pi\left(\frac{f(x+)+f(x-)}{2}\right)\right| \le \int_{-\infty}^{a} |F(t)|\,dt + \left|\int_{a}^{b} F(t)\,dt - \pi\left(\frac{f(x+)+f(x-)}{2}\right)\right| + \int_{b}^{\infty} |F(t)|\,dt.$$

Now,

$$\int_{-\infty}^{a} |F(t)|\,dt \le \int_{-\infty}^{a} \frac{|f(t)|}{|x-t|} dt \le \frac{1}{x-a}\int_{-\infty}^{a} |f(t)|\,dt.$$

Given any $\varepsilon > 0$, there exists $a(\varepsilon) < 0$, independent of λ, such that

$$\int_{-\infty}^{a} |F(t)|\,dt \le \frac{1}{x-a}\int_{-\infty}^{a} |f(t)|\,dt < \frac{\varepsilon}{3}.$$

Similarly, there exists $b(\varepsilon) > 0$ such that

$$\int_{b}^{\infty} |F(t)|\,dt < \frac{\varepsilon}{3}.$$

Since $f(t)$ is piecewise continuous on $a \le t \le b$ and has both one-sided derivatives at $t = x$, $a < x < b$, we have from Result 3 in Appendix A that

$$\lim_{\lambda \to \infty} \frac{1}{\pi} \int_{a}^{b} f(t) \frac{\sin \lambda(x-t)}{x-t} dt = \frac{f(x+) + f(x-)}{2};$$

that is, there exists $\lambda(\varepsilon)$ such that whenever $\lambda > \lambda(\varepsilon)$,

$$\left|\int_a^b f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt - \pi\left(\frac{f(x+)+f(x-)}{2}\right)\right| < \frac{\varepsilon}{3}.$$

Combining these three results, we have, for $\lambda > \lambda(\varepsilon)$,

$$\left|\int_{-\infty}^{\infty} F(t)\,dt - \pi\left(\frac{f(x+)+f(x-)}{2}\right)\right| < \varepsilon.$$

Since ε can be made arbitrarily small, it follows that

$$\lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\lambda(x-t)}{x-t}\,dt = \frac{f(x+)+f(x-)}{2}. \qquad \blacksquare$$

We can now establish Theorem 1 in Section 7.2.

Result 3 (Fourier Integral Theorem)

If $f(x)$ is piecewise continuous on every finite interval and absolutely integrable on $-\infty < x < \infty$, then at every x at which $f(x)$ has a right and left derivative,

$$\frac{f(x+)+f(x-)}{2} = \int_0^{\infty}[A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,d\lambda \tag{3a}$$

when $$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos\lambda x\,dx, \qquad B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin\lambda x\,dx. \tag{3b}$$

Proof: By Result 2, we may write

$$\frac{f(x+)+f(x-)}{2} = \lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\frac{\sin\alpha(x-t)}{x-t}\,dt.$$

Since $$\int_0^{\alpha}\cos\lambda(x-t)\,d\lambda = \left\{\frac{\sin\lambda(x-t)}{x-t}\right\}_0^{\alpha} = \frac{\sin\alpha(x-t)}{x-t},$$

it follows that

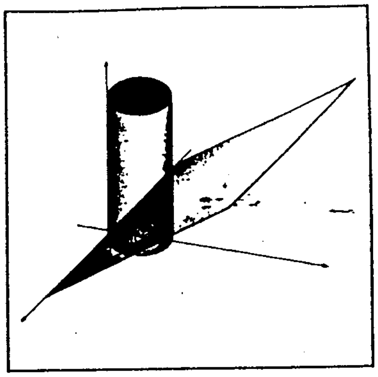
$$\begin{aligned}\frac{f(x+)+f(x-)}{2} &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\left(\int_0^{\alpha}\cos\lambda(x-t)\,d\lambda\right)dt\\ &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_{-\infty}^{\infty}\int_0^{\alpha} f(t)\cos\lambda(x-t)\,d\lambda\,dt.\end{aligned}$$

Since $$\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt$$

is uniformly convergent with respect to λ, we may interchange the order of integration and write

$$\begin{aligned}\frac{f(x+)+f(x-)}{2} &= \lim_{\alpha\to\infty}\frac{1}{\pi}\int_0^{\alpha}\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt\,d\lambda \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)\cos\lambda(x-t)\,dt\,d\lambda \\ &= \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(t)[\cos\lambda x\cos\lambda t+\sin\lambda x\sin\lambda t]\,dt\,d\lambda.\end{aligned}$$

This is the result in equation (3). ■



APPENDIX C

Vector Analysis

In this appendix we briefly mention the theorems from vector analysis that are used throughout the book.

When $f(x, y, z)$ is a scalar function with first partial derivatives in some region V of space, its gradient is a vector-valued function defined by

$$\operatorname{grad} f = \nabla f = \frac{\partial f}{\partial x}\hat{\mathbf{i}} + \frac{\partial f}{\partial y}\hat{\mathbf{j}} + \frac{\partial f}{\partial z}\hat{\mathbf{k}}. \tag{1}$$

This is a very important vector in applied mathematics, principally due to the properties stated in the following theorem.

Theorem 1

The directional derivative of a function $f(x, y, z)$ in any direction is the component of ∇f in that direction. Furthermore, $f(x, y, z)$ increases most rapidly in the direction ∇f, and its rate of change in this direction is $|\nabla f|$.

When $\mathbf{F}(x, y, z) = P(x, y, z)\hat{\mathbf{i}} + Q(x, y, z)\hat{\mathbf{j}} + R(x, y, z)\hat{\mathbf{k}}$ is a vector function with first partial derivatives, its divergence and curl are defined as

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}, \tag{2}$$

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\mathbf{i}} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\mathbf{j}} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{\mathbf{k}}. \quad (3)$$

The gradient, divergence, and curl are linear operators that satisfy the following identities:

$$\nabla(fg) = f\nabla g + g\nabla f, \quad (4a)$$
$$\nabla \cdot (f\mathbf{F}) = \nabla f \cdot \mathbf{F} + f\nabla \cdot \mathbf{F}, \quad (4b)$$
$$\nabla \times (f\mathbf{F}) = \nabla f \times \mathbf{F} + f(\nabla \times \mathbf{F}), \quad (4c)$$
$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}), \quad (4d)$$
$$\nabla \times (\nabla f) = \mathbf{0}, \quad (4e)$$
$$\nabla \cdot (\nabla \times \mathbf{F}) = 0, \quad (4f)$$

provided $f(x, y, z)$ and the components of vectors are sufficiently differentiable.

The line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$$

of a continuous vector function $\mathbf{F} = P\hat{\mathbf{i}} + Q\hat{\mathbf{j}} + R\hat{\mathbf{k}}$ along a smooth curve C can always be evaluated by substituting from parametric equations for C and evaluating the resulting definite integral. For example, the value of

$$\int_C y\,dx + x\,dy + z\,dz$$

along the curve C: $x = t^2$, $y = t + 1$, $z = 3t$, $0 \le t \le 1$, can be calculated with the definite integral

$$\int_0^1 (t + 1)(2t\,dt) + t^2\,dt + 3t(3\,dt) = \int_0^1 (3t^2 + 11t)\,dt = \frac{13}{2}.$$

In the event that a line integral is independent of path, and this occurs when $\mathbf{F}$ is the gradient of some scalar function $f(x, y, z)$, the value of the line integral is the difference in the values of $f(x, y, z)$ at terminal and initial points. The above line integral is independent of path, since $\nabla(xy + z^2/2) = y\hat{\mathbf{i}} + x\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, and therefore

$$\int_C y\,dx + x\,dy + z\,dz = \left\{xy + \frac{z^2}{2}\right\}_{(0,1,0)}^{(1,2,3)} = \frac{13}{2}.$$

The surface integral

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS$$

of a vector function $\mathbf{F}(x, y, z)$ over a smooth surface S with unit normal vector $\hat{\mathbf{n}}$ is usually evaluated by projecting the surface in a one-to-one fashion onto a coordinate plane, expressing $\mathbf{F} \cdot \hat{\mathbf{n}}$ and dS in terms of coordinates in this plane, and evaluating the resulting double integral. For example, when $\mathbf{F} = x^2y\hat{\mathbf{i}} + xz\hat{\mathbf{j}}$ and when $\hat{\mathbf{n}}$ is the upper normal to the surface $S: z = 4 - x^2 - y^2, z \geq 0$, it is appropriate to project S onto the xy-plane (Figure C.1). The unit upper normal to S is

$$\hat{\mathbf{n}} = \frac{\nabla(z - 4 + x^2 + y^2)}{|\nabla(z - 4 + x^2 + y^2)|} = \frac{(2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}}.$$

The relationship between a rectangular area $dy\,dx$ in the xy-plane and its projection dS on S is

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dy\,dx = \sqrt{1 + 4x^2 + 4y^2}\, dy\,dx.$$

Since S projects onto the circle $x^2 + y^2 \leq 4$, the value of the surface integral of $\mathbf{F}$ over S is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (x^2y, xz, 0) \cdot \frac{(2x, 2y, 1)}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2}\, dy\,dx \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2x^3y + 2xyz)\, dy\,dx \\ &= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [2x^3y + 2xy(4 - x^2 - y^2)]\, dy\,dx \\ &= 0.\end{aligned}$$

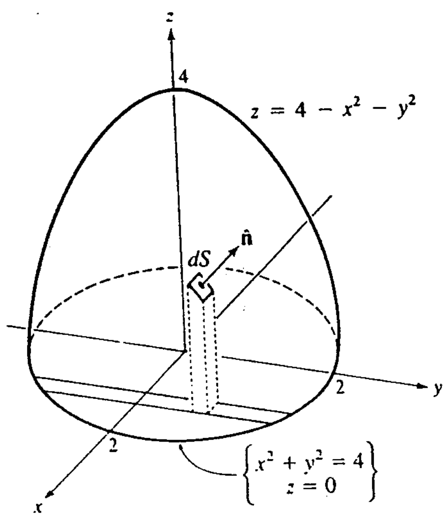


Figure C.1

When a surface does not project in a one-to-one fashion onto a coordinate plane (such would be the case, for example, if the surface were closed), it must be divided into subsurfaces that do project one-to-one. Alternatively, if the surface is indeed closed, the surface integral can be replaced by a triple integral over its interior. This is the result of the following theorem.

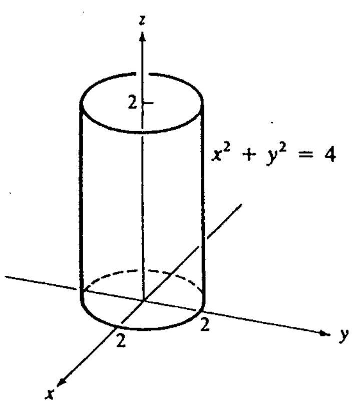
heorem 2 (Divergence Theorem)

Let S be a piecewise smooth surface enclosing a volume V. Let $\mathbf{F}(x, y, z)$ be a vector function whose components have continuous first partial derivatives inside and on S. If $\hat{\mathbf{n}}$ is the unit outward-pointing normal to S, then

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV. \tag{5}$$

For example, consider evaluating the surface integral of $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ over the surface S that encloses the volume described by $x^2 + y^2 \leq 4$, $0 \leq z \leq 2$ (Figure C.2). To do so by surface integrals would require that the top and bottom of the cylinder be projected onto the xy-plane and that the cylindrical side be divided into two parts, each of which projects one-to-one onto the xz-plane (or yz-plane). Alternatively, the divergence theorem yields

$$\oiint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V (1+1+1)\, dV = 3 \iiint_V dV = 3(\text{volume of } V) = 24\pi.$$



gure C.2

If we set $\mathbf{F} = u\nabla v$ in (5), where u and v are arbitrary functions of x, y, and z, and use identity (4b), we immediately obtain

$$\oiint_S (u\nabla v) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (u\nabla^2 v + \nabla u \cdot \nabla v)\, dV. \tag{6}$$

This result is called *Green's first identity*. When u and v are interchanged in (6) and the equations are subtracted, the result is called *Green's second identity*:

$$\oiint_S (u\nabla v - v\nabla u) \cdot \hat{\mathbf{n}}\, dS = \iiint_V (u\nabla^2 v - v\nabla^2 u)\, dV. \tag{7}$$

Stokes's theorem relates line integrals around closed curves to surface integrals over surfaces that have the curves as boundaries.

Theorem 3 (Stokes's Theorem)

Let C be a closed, piecewise smooth, non-self-intersecting curve, and let S be a piecewise smooth (orientable) surface with C as boundary (Figure C.3). Let $\mathbf{F}$ be a vector function whose components have continuous first partial derivatives in a region that contains S and C in its interior. Then,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{n}}\, dS, \tag{8}$$

where $\hat{\mathbf{n}}$ is the unit normal to S chosen in the following way. If when moving along C, the surface S is on the left side, then $\hat{\mathbf{n}}$ must be chosen as the unit normal on that side. On the other hand, if when moving along C, the surface is on the right, then $\hat{\mathbf{n}}$ must be chosen on the opposite side of S.

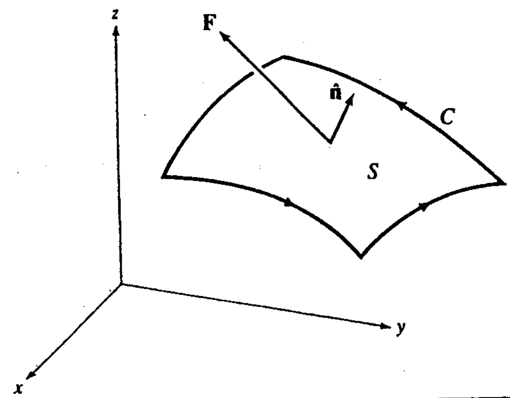


Figure C.3

For example, consider the line integral

$$\oint_C y^2\,dx + xy\,dy + xz\,dz,$$

where C is the curve of intersection of the surfaces $x^2 + y^2 = 2y$ and $y = z$, directed so that y increases when x is positive (Figure C.4). If we choose S as that part of the plane $y = z$ interior to C, then

$$\hat{\mathbf{n}} = \frac{\nabla(z-y)}{|\nabla(z-y)|} = \frac{(0,-1,1)}{\sqrt{2}}.$$

Since $\nabla \times \mathbf{F} = (0, -z, -y)$, it follows by Stokes's theorem that

$$\begin{aligned}\oint_C y^2\,dx + xy\,dy + xz\,dz &= \iint_S (0,-z,-y) \cdot \frac{(0,-1,1)}{\sqrt{2}}\,dS \\ &= \iint_S \frac{z-y}{\sqrt{2}}\,dS \\ &= 0,\end{aligned}$$

since $z = y$ at every point of S.

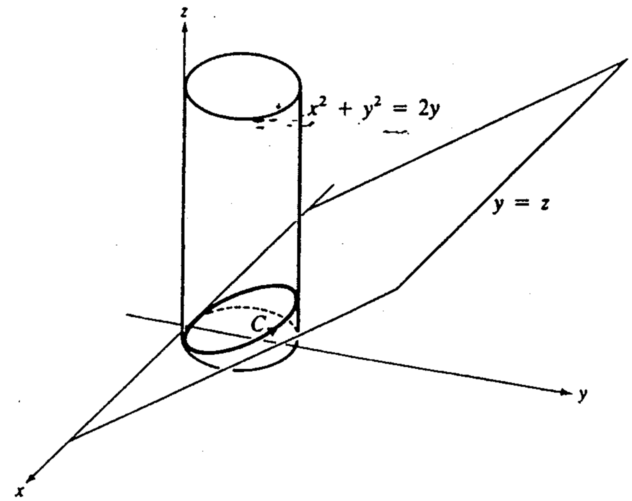


Figure C.4

When C is a curve in the xy-plane (directed counterclockwise) and S is chosen as that part A of the xy-plane interior to C, we obtain Green's theorem as a special case of Stokes's theorem (Figure C.5):

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C P(x, y)\, dx + Q(x, y)\, dy$$
$$= \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dy\, dx. \qquad (9)$$

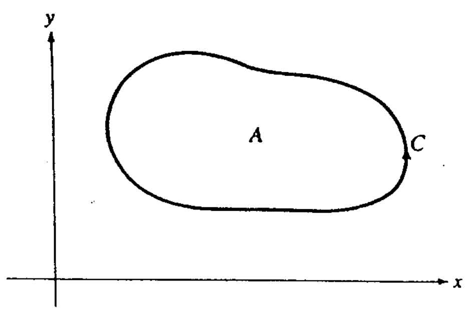


Figure C.5

The two-dimensional version of Green's first identity is obtained from (9) by setting $P = -u\partial v/\partial y$ and $Q = u\partial v/\partial x$, where u and v are functions of x and y:

$$\oint_C (u\nabla v) \cdot \hat{\mathbf{n}}\, ds = \iint_A (u\nabla^2 v + \nabla u \cdot \nabla v)\, dA. \qquad (10)$$

Interchanging u and v and subtracting gives the two-dimensional second identity,

$$\oint_C (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}\,ds = \iint_A (u\nabla^2 v - v\nabla^2 u)\,dA. \tag{11}$$

An alternative form of Green's theorem, which casts it as a two-dimensional version of the divergence theorem, is

$$\oint_C \mathbf{F}\cdot\hat{\mathbf{n}}\,ds = \iint_A \nabla\cdot\mathbf{F}\,dA, \tag{12}$$

where $\hat{\mathbf{n}}$ is the outward-pointing normal to C.

Answers to Selected Exercises

ection 1.1

(a) $V(0,y,z) = f_1(y,z),\ y > 0,\ z > 0;$
$V(L,y,z) = f_2(y,z),\ y > 0,\ z > 0;$
$V(x,0,z) = f_3(x,z),\ 0 < x < L,\ z > 0;$
$V(x,y,0) = f_4(x,y),\ 0 < x < L,\ y > 0$

(b) $\dfrac{-\partial V(0,y,z)}{\partial x} = f_1(y,z),\ y > 0,\ z > 0;$
$\dfrac{\partial V(L,y,z)}{\partial x} = f_2(y,z),\ y > 0,\ z > 0;$
$\dfrac{-\partial V(x,0,z)}{\partial y} = f_3(x,z),\ 0 < x < L,\ z > 0;$
$\dfrac{-\partial V(x,y,0)}{\partial z} = f_4(x,y),\ 0 < x < L,\ y > 0$

(c) $-l_1\dfrac{\partial V(0,y,z)}{\partial x} + h_1 V(0,y,z) = f_1(y,z),$
$y > 0,\ z > 0;$
$l_2\dfrac{\partial V(L,y,z)}{\partial x} + h_2 V(L,y,z) = f_2(y,z),$
$y > 0,\ z > 0;$
$-l_3\dfrac{\partial V(x,0,z)}{\partial y} + h_3 V(x,0,z) = f_3(x,z),$
$0 < x < L,\ z > 0;$
$-l_4\dfrac{\partial V(x,y,0)}{\partial z} + h_4 V(x,y,0) = f_4(x,y),$
$0 < x < L,\ y > 0$

(a) $V(r_0,\theta) = f_1(\theta),\ 0 < \theta < \pi;\ V(r,0) = f_2(r),$
$0 < r < r_0;\ V(r,\pi) = f_3(r),\ 0 < r < r_0$

(b) $\dfrac{\partial V(r_0,\theta)}{\partial r} = f_1(\theta),\ 0 < \theta < \pi;$
$-\dfrac{1}{r}\dfrac{\partial V(r,0)}{\partial \theta} = f_2(r),\ 0 < r < r_0;$
$\dfrac{1}{r}\dfrac{\partial V(r,\pi)}{\partial \theta} = f_3(r),\ 0 < r < r_0$

(c) $l_1\dfrac{\partial V(r_0,\theta)}{\partial r} + h_1 V(r_0,\theta) = f_1(\theta),\ 0 < \theta < \pi;$
$-\dfrac{l_2}{r}\dfrac{\partial V(r,0)}{\partial \theta} + h_2 V(r,0) = f_2(r),\ 0 < r < r_0;$
$\dfrac{l_3}{r}\dfrac{\partial V(r,\pi)}{\partial \theta} + h_3 V(r,\pi) = f_3(r),\ 0 < r < r_0$

7. (a) $V(r_0,\theta,\phi) = f_1(\theta,\phi),\ -\pi < \theta \le \pi,\ 0 \le \phi < \dfrac{\pi}{2};$
$V\left(r,\theta,\dfrac{\pi}{2}\right) = f_2(r,\theta),\ 0 \le r < r_0,\ -\pi < \theta \le \pi$

(b) $\dfrac{\partial V(r_0,\theta,\phi)}{\partial r} = f_1(\theta,\phi),\ -\pi < \theta \le \pi,\ 0 \le \phi < \dfrac{\pi}{2};$
$\dfrac{1}{r}\dfrac{\partial V(r,\theta,\pi/2)}{\partial \phi} = f_2(r,\theta),\ 0 < r < r_0,\ -\pi < \theta \le \pi$

(c) $l_1\dfrac{\partial V(r_0,\theta,\phi)}{\partial r} + h_1 V(r_0,\theta,\phi) = f_1(\theta,\phi),$
$-\pi < \theta \le \pi,\ 0 \le \phi < \dfrac{\pi}{2};$
$\dfrac{l_2}{r}\dfrac{\partial V(r,\theta,\pi/2)}{\partial \phi} + h_2 V\left(r,\theta,\dfrac{\pi}{2}\right) = f_2(r,\theta),$
$0 < r < r_0,\ -\pi < \theta \le \pi$

Section 1.2

3. $\dfrac{\partial U}{\partial t} = k\dfrac{\partial^2 U}{\partial x^2},\ 0 < x < L,\ t > 0;\ U_x(0,t) = 0,\ t > 0;$
$U(L,t) = 100,\ t > 0;\ U(x,0) = f(x),\ 0 < x < L$

5. $\dfrac{\partial U}{\partial t} = k\dfrac{\partial^2 U}{\partial x^2},\ 0 < x < L,\ t > 0,$
$-\kappa\dfrac{\partial U}{\partial x} + \mu_0 U = \mu_0 U_0,\ x = 0,\ t > 0,$
$\kappa\dfrac{\partial U}{\partial x} + \mu_L U = \mu_L U_L,\ x = L,\ t > 0;\ U(x,0) = f(x),$
$0 < x < L$

7. $\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, 0 < x < L, t > 0; -\kappa\frac{\partial U}{\partial x} = Q_0,$
$x = 0, t > 0; \kappa\frac{\partial U}{\partial x} = -\frac{Q_L(t)}{A}, x = L, t > 0;$
$U(x,0) = f(x), 0 < x < L$

10. $\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right) + \frac{kg(r)}{\kappa},$
$0 < r < r_0, -\pi < \theta \le \pi, t > 0; \kappa\frac{\partial U}{\partial r} + \mu U = 0,$
$r = r_0, -\pi < \theta \le \pi, t > 0; U(r,\theta,0) = f(r,\theta),$
$0 \le r < r_0, -\pi < \theta \le \pi$, where

$$g(r) = \begin{cases} 0 & 0 < r < r_1 \\ q & r_1 < r < r_2 \\ 0 & r_2 < r < r_0 \end{cases}$$

12. $\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial z^2}\right),$
$0 < r < r_0, -\pi < \theta \le \pi, 0 < z < L, t > 0;$
$U_z(r,\theta,0,t) = 0, 0 \le r < r_0, -\pi < \theta \le \pi, t > 0;$
$U_z(r,\theta,L,t) = 0, 0 \le r < r_0, -\pi < \theta \le \pi, t > 0;$
$U(r_0,\theta,z,t) = f_1(\theta,t), -\pi < \theta \le \pi, 0 < z < L,$
$t > 0; U(r,\theta,z,0) = f(r,\theta,z), 0 \le r < r_0,$
$-\pi < \theta \le \pi, 0 < z < L$

16. $\frac{\partial U}{\partial t} = k\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2}\right), 0 < r < r_0,$
$0 < \theta < \pi, t > 0; U_r(r_0,\theta,t) = 0, 0 < \theta < \pi, t > 0;$
$-\kappa r^{-1}U_\theta(r,0,t) = q, 0 < r < r_0, t > 0;$
$\kappa r^{-1}U_\theta(r,\pi,t) = q, 0 < r < r_0, t > 0;$
$U(r,\theta,0) = f(r,\theta), 0 \le r < r_0, 0 < \theta < \pi$

20. $\frac{(U_L - U_0)x}{L} + U_0$

21. $\frac{-q_0 x}{\kappa} + C$, C arbitrary, provided $q_L = -q_0$

22. (a) True; (b) Not necessarily true

26. (a) $\frac{-\kappa A(U_{out} - U_{in})}{L}$; (b) 660 W

27. (a) $\frac{-\kappa(U_{out} - U_{in})}{r\ln(r_{out}/r_{in})}$; (b) 2.36×10^4 W

33. (a) $\frac{U_a[\kappa + b\mu\ln(b/r)] + b\mu U_m\ln(r/a)}{\kappa + b\mu\ln(b/a)}$

34. $\frac{-I^2r^2}{4\kappa\pi^2a^4\sigma} + \frac{I^2}{4\pi^2a^2\sigma}\left(\frac{1}{\kappa} + \frac{2}{\kappa^*}\ln\left(\frac{b}{a}\right) + \frac{2}{b\mu^*}\right) + U_m,$
$0 \le r \le a; \frac{I^2}{2\kappa^*\pi^2a^2\sigma}\ln\left(\frac{b}{r}\right) + \frac{I^2}{2\pi^2a^2\sigma b\mu^*} + U_m,$
$a \le r < b$

Section 1.3

3. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} - \frac{\beta}{\rho}\frac{\partial y}{\partial t} + g, 0 < x < L, t > 0,$
$(g < 0, \beta > 0); -\frac{\tau\partial y}{\partial x} + k_1 y = 0, x = 0, t > 0;$
$\tau\frac{\partial y}{\partial x} + k_2 y = 0, x = L, t > 0; y(x,0) = f(x),$
$0 < x < L; y_t(x,0) = g(x), 0 < x < L$

6. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, 0 < x < L, t > 0; y(0,t) = 0, t > 0;$
$E\frac{\partial y(L,t)}{\partial x} = F, t > 0; y(x,0) = 0, 0 < x < L;$
$y_t(x,0) = 0, 0 < x < L$

9. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + g, 0 < x < L, t > 0\ (g < 0);$
$y(0,t) = y(L,t) = 0, t > 0; y(x,0) = y_t(x,0) = 0,$
$0 < x < L$
Static deflection: $y = \frac{gx(L-x)}{2c^2}$

11. $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + g, 0 < x < L, t > 0\ (g < 0);$
$y(0,t) = 0, t > 0; \tau\frac{\partial y(L,t)}{\partial x} = F_L, t > 0;$
$y(x,0) = y_t(x,0) = 0, 0 < x < L$
Static deflection: $y = \frac{gx(2L-x)}{2c^2} + F_L x/\tau$

13. $L + \frac{gL^2}{2c^2}$

14. $L + \frac{gL^2}{2c^2} + AL\rho g/k$; A = cross-sectional area of bar

Section 1.4

4. $\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{-\rho g}{\tau}, 0 < r < r_1, 0 < \theta < \alpha$
$(g < 0); z(r,0) = 0, 0 < r < r_1; z(r,\alpha) = 0,$
$0 < r < r_1; z(r_1,\theta) = f(\theta), 0 < \theta < \alpha$

5. $\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2}\right) + g - \frac{\beta}{\rho}\frac{\partial z}{\partial t},$
$0 < r < r_1, -\pi < \theta \le \pi, t > 0\ (g < 0, \beta > 0);$
$z(r_1,\theta,t) = 0, -\pi < \theta \le \pi, t > 0; z(r,\theta,0) = f(r,\theta),$
$0 \le r < r_1, -\pi < \theta \le \pi; z_t(r,\theta,0) = 0, 0 \le r < r_1,$
$-\pi < \theta \le \pi$

8. $z(r) = \frac{\rho g(r_2^2 - r^2)}{4\tau}$

9. (a) $z'' + r^{-1}z' = \dfrac{-f(r)}{\tau}, 0 < r < r_2; z(r_2) = 0$

(f) $\dfrac{k(9r_2r^2 - 4r^3 - 5r_2^3)}{36\tau}$

Section 1.5

1. $\dfrac{\partial^2 y}{\partial t^2} + c^2\dfrac{\partial^4 y}{\partial x^4} = g, 0 < x < L, t > 0 \left(c^2 = \dfrac{EI}{\rho}\right);$
$y(0,t) = y_x(0,t) = 0, t > 0; y_{xx}(L,t) = y_{xxx}(L,t) = 0,$
$t > 0; y(x,0) = f(x), 0 < x < L; y_t(x,0) = 0,$
$0 < x < L$

5. (a) $\dfrac{d^4y}{dx^4} = \dfrac{F}{EI}, 0 < x < L,$
$y(0) = y''(0) = 0 = y(L) = y''(L)$
$y(x) = \dfrac{Fx(x^3 - 2Lx^2 + L^3)}{24EI}$

(b) 9.2×10^{-7} m (c) 1.68×10^9 N/m

Section 1.6

1. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0, 0 < x < L, 0 < y < L'; V(0,y) = 0,$
$0 < y < L'; V(L,y) = 100, 0 < y < L'; V(x,0) = 0,$
$0 < x < L; V(x,L') = 100, 0 < x < L$

Section 1.8

2. Elliptic, $w_{vv} + w_{\eta\eta} = \left(\dfrac{1}{4}\right)\left[-3w_v - 6w_\eta + \left(\dfrac{\eta}{2} - v\right)w\right]$

4. Hyperbolic,
$w_{v\eta} = \left(\dfrac{w}{8}\right)[(3+2\sqrt{2})w_v + (3-2\sqrt{2})w_\eta]$

13. (b) $w_{v\eta} = \dfrac{-(w_v + w_\eta)}{6\eta + 6v}$ (c) $w_{vv} + w_{\eta\eta} = \dfrac{-w_v}{3v}$

(d) $u_{yy} = 0$

17. $w_{v\eta} = \dfrac{w_\eta}{v}$ 19. $v_{vv} + v_{\eta\eta} = \dfrac{45v}{64}$

20. $v_{v\eta} = \dfrac{v}{64}$ 21. $v_{vv} = -2v_\eta$

Section 2.1

2. $\dfrac{8L^2}{3} - 1 + \dfrac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\left(\dfrac{1}{n^2}\cos\dfrac{n\pi x}{L} - \dfrac{\pi}{n}\sin\dfrac{n\pi x}{L}\right)$

3. $\dfrac{2L^2}{3} - 1 + \dfrac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos\dfrac{n\pi x}{L}$

6. $\dfrac{3L}{4} + \dfrac{L}{\pi^2}\sum_{n=1}^{\infty}\left(\dfrac{3[1+(-1)^{n+1}]}{n^2}\cos\dfrac{n\pi x}{L} + \dfrac{\pi}{n}\sin\dfrac{n\pi x}{L}\right)$

9. $1 + \sin x - \cos 2x$ 11. $\dfrac{1}{2} + \dfrac{1}{2}\cos 4x$

14. $\dfrac{3}{\pi} - \dfrac{1}{2}\sin x - \dfrac{6}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2 - 1}\cos 2nx$

16. No 17. $\dfrac{L^2}{3} + \dfrac{2L^2}{\pi^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\dfrac{(-1)^n}{n^2}e^{-n\pi xi/L}$

18. $\dfrac{e^4 - 1}{2}\sum_{n=-\infty}^{\infty}\left(\dfrac{2 - n\pi i}{n^2\pi^2 + 4}\right)e^{-n\pi xi/2}$

Section 2.2

2. $\dfrac{\pi}{2} - \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos(2n-1)x$

4. $\dfrac{2L^2}{3} - 1 + \dfrac{8L^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^2}\cos\dfrac{n\pi x}{L}$

6. $\dfrac{2L}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}\sin\dfrac{n\pi x}{L}$ 9. $\dfrac{1}{2}\sin\dfrac{2\pi x}{L}$

10. $\dfrac{8L^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\sin\dfrac{(2n-1)\pi x}{L}$

11. $-\dfrac{L}{2} + \dfrac{4L}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}\cos\dfrac{(2n-1)\pi x}{L}$

13. $\dfrac{L^2}{6} - \dfrac{L^2}{\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\cos\dfrac{2n\pi x}{L}$

15. $\dfrac{1}{2} + \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{2n-1}\cos\dfrac{(2n-1)\pi x}{L}$

16. $\dfrac{2}{\pi} - \dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2 - 1}\cos 2nx$

19. (a) Yes; (b) Not necessarily

21. (b) Yes 22. (b) Yes

Section 2.3

2. (b) No

4. (a) Does not generally converge uniformly
(b) Does not generally converge uniformly
(c) Does converge uniformly

Section 3.1

1. Linear and homogeneous 2. Not linear

3. Not linear 4. Not linear

5. Linear and homogeneous

6. Linear and nonhomogeneous

7. Linear and nonhomogeneous
8. Not linear
9. Linear and homogeneous
10. Linear and homogeneous

Section 3.2

1. U_0

3. (a) $\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}e^{-(2n-1)^2\pi^2\kappa t/L^2}\sin\frac{(2n-1)\pi x}{L}$

(b) $\frac{4\kappa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2\kappa t/L^2}$; 0;

$\frac{-4\kappa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2\kappa t/L^2}$

(c) $\frac{4\kappa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1} = -\kappa, 0, \kappa; 0, 0, 0$

5. (a) $\frac{4\kappa}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{2n-1}e^{-(2n-1)^2\pi^2\kappa t/L^2}$; (b) $-\kappa$

7. $\frac{4L}{5\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

8. $\frac{8L^3}{\pi^4 c}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}\sin\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

10. $\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right)\sin\frac{n\pi x}{L}$, where

$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$

$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

13. $\frac{a_0+b_0 t}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right)\cos\frac{n\pi x}{L}$,

where $a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx,$

$b_n = \frac{2}{n\pi c}\int_0^L g(x)\cos\frac{n\pi x}{L}\,dx,\ b_0 = \frac{2}{L}\int_0^L g(x)\,dx$

16. $\frac{L^*-L}{2}-\frac{4(L^*-L)}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{L}$

$\times\cos\frac{(2n-1)\pi x}{L}$

17. $\frac{400}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)\sinh\frac{(2n-1)\pi L}{L'}}$

$\times\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$

20. $-\frac{400L'}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2\cosh\frac{(2n-1)\pi L}{L'}}$

$\times\sinh\frac{(2n-1)\pi(L-x)}{L'}\sin\frac{(2n-1)\pi y}{L'}$

21. $100x + B$; $100x - 50L$ 22. No

24. $\frac{4qL}{\pi^2\kappa}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\operatorname{sech}\frac{(2n-1)\pi L'}{L}$

$\times\sinh\frac{(2n-1)\pi(L'-y)}{L}\sin\frac{(2n-1)\pi x}{L}$

26. $\frac{-2kL}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{csch}\frac{n\pi L'}{L}\sinh\frac{n\pi(L'-y)}{L}\sin\frac{n\pi x}{L}$

$-\frac{2kL}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\operatorname{csch}\frac{n\pi L}{L'}\sinh\frac{n\pi(L-x)}{L'}\sin\frac{n\pi y}{L'}$

Section 3.3

3. $U_0+\frac{(U_L-U_0)x}{L}-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{U_0+(-1)^{n+1}U_L}{n}$

$\times e^{-n^2\pi^2\kappa t/L^2}\sin\frac{n\pi x}{L}+\frac{I^2x(L-x)}{2A^2\sigma\kappa}$

$+\frac{4}{\pi}\sum_{n=1}^{\infty}\left(\frac{20}{2n-1}-\frac{L^2I^2}{(2n-1)^3\pi^2A^2\sigma\kappa}\right)$

$\times e^{-(2n-1)^2\pi^2\kappa t/L^2}\sin\frac{(2n-1)\pi x}{L}$

6. $20+\frac{kI^2t}{A^2\sigma\kappa}$ 7. $20+\frac{k}{2A^2\sigma\kappa\alpha}(1-e^{-2\alpha t})$

9. $\frac{4}{\pi}\sum_{n=1}^{\infty}\Bigg(\frac{10}{2n-1}e^{-(2n-1)^2\pi^2\kappa t/L^2}$

$+\frac{kL^2}{\kappa(2n-1)[(2n-1)^2\pi^2k-\alpha L^2]}$

$\times\left[e^{-\alpha t}-e^{-(2n-1)^2\pi^2\kappa t/L^2}\right]\Bigg)\sin\frac{(2n-1)\pi x}{L}$

13. $\frac{kx(x-L)}{2\rho c^2}+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi ct}{L}+b_n\sin\frac{n\pi ct}{L}\right)$

$\times\sin\frac{n\pi x}{L},\ a_n=\frac{2}{L}\int_0^L\left(f(x)+\frac{kx(L-x)}{2\rho c^2}\right)$

$\times\sin\frac{n\pi x}{L}\,dx,\ b_n=\frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$

15. (a) No;

(c) $\frac{Fx}{E}+\frac{8LF}{\pi^2E}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}$

$\times\sin\frac{(2n-1)\pi x}{2L}\cos\frac{(2n-1)\pi ct}{2L}$

16. $\frac{(\rho+k)gx}{24EI}(x^3-2Lx^2+L^3)-\frac{4(\rho+k)gL^4}{EI\pi^5}$
$\times\sum_{n=1}^{\infty}\frac{1}{(2n-1)^5}\sin\frac{(2n-1)\pi x}{L}\cos\frac{(2n-1)^2\pi^2 ct}{L^2}$

21. $\frac{g(L^2-x^2)}{2\kappa}+\frac{16L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}$
$\times\left(\frac{g(-1)^n}{\kappa}+2(-1)^n+(2n-1)\pi\right)$
$\times\operatorname{sech}\frac{(2n-1)\pi L'}{2L}\cosh\frac{(2n-1)\pi(L'-y)}{2L}$
$\times\cos\frac{(2n-1)\pi x}{2L}$

Section 4.1

3. $\frac{n^2\pi^2}{16}, n\geq 0; \frac{1}{2}, \left(\frac{1}{\sqrt{2}}\right)\cos\left(\frac{n\pi x}{4}\right)$

5. $\frac{(2n-1)^2\pi^2}{4}, n>0; \sqrt{2}\cos\frac{(2n-1)\pi x}{2}$

7. $\frac{n^2\pi^2}{81}, n>0; \left(\frac{\sqrt{2}}{3}\right)\sin\frac{n\pi(x-1)}{9}$

8. $n^2\pi^2+\frac{1}{4}, n>0; \sqrt{2}\,e^{x/2}\sin(n\pi x)$

10. $\frac{n^2\pi^2}{L^2}, n\geq 0;$ 1, $A\cos\frac{n\pi x}{L}+B\sin\frac{n\pi x}{L}$

14. Sometimes

Section 4.2

2.

$\sin\lambda_n L$	$\cos\lambda_n L$
$\frac{(-1)^{n+1}\lambda_n\left(\frac{h_1}{l_1}+\frac{h_2}{l_2}\right)}{\left[\left(\lambda_n^2+\frac{h_1^2}{l_1^2}\right)\left(\lambda_n^2+\frac{h_2^2}{l_2^2}\right)\right]^{1/2}}$	$\frac{(-1)^{n+1}\left(\lambda_n^2-\frac{h_1h_2}{l_1l_2}\right)}{\left[\left(\lambda_n^2+\frac{h_1^2}{l_1^2}\right)\left(\lambda_n^2+\frac{h_2^2}{l_2^2}\right)\right]^{1/2}}$
$\frac{(-1)^{n+1}(h_1/l_1)}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$	$\frac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$
$\frac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$	$\frac{(-1)^{n}(h_1/l_1)}{\sqrt{\lambda_n^2+(h_1/l_1)^2}}$
$\frac{(-1)^{n+1}(h_2/l_2)}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$	$\frac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$
0	$(-1)^n$

(continued)

$\sin\lambda_n L$	$\cos\lambda_n L$
$\frac{(-1)^{n+1}\lambda_n}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$	$\frac{(-1)^{n}(h_2/l_2)}{\sqrt{\lambda_n^2+(h_2/l_2)^2}}$
$(-1)^{n+1}$	0
0	$(-1)^n$

7. $\frac{4L}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{\pi(-1)^{n+1}}{2n-1}-\frac{2}{(2n-1)^2}\right)\cos\frac{(2n-1)\pi x}{2L}$

10. $\frac{16L^2}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{\pi(-1)^{n+1}}{(2n-1)^2}-\frac{2}{(2n-1)^3}\right)\sin\frac{(2n-1)\pi x}{2L}$

11. 0, $1+\frac{n^2\pi^2}{L^2}(n>0); \sqrt{\frac{2}{e^{2L}-1}},$
$\frac{\sqrt{2L}e^{-x}}{\sqrt{n^2\pi^2+L^2}}\left(\frac{n\pi}{L}\cos\frac{n\pi x}{L}+\sin\frac{n\pi x}{L}\right)$

16. $\frac{(2n-1)^2\pi^2}{4(\ln b)^2}; \sqrt{\frac{2}{\ln b}}\cos\left[\frac{(2n-1)\pi\ln x}{2\ln b}\right]$

18. 9.84006, 39.3603, 88.5606, 157.441

20. (a) $\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}\cos\frac{(2n-1)\pi x}{L}$; (b) 0;
(c) 1, −1

22. No 24. Yes 25. Yes

Section 4.3

7. $X''+\lambda^2X=0, 0<x<L,$
$X(0)=0=X'(L)+200X(L);$
$Y''+\beta^2Y=0, 0<y<L',$
$Y'(0)=0=Y'(L')$

9. $X''+\lambda^2X=0, 0<x<L,$
$X(0)=0=X'(L)$
$Y(0)=0=Y(L')$
$Y''+\beta^2Y=0, 0<y<L',$

Section 5.2

2. $\sum_{n=1}^{\infty}c_ne^{-(2n-1)^2\pi^2kt/(4L^2)}\sqrt{\frac{2}{L}}\cos\frac{(2n-1)\pi x}{2L},$
$c_n=\sqrt{\frac{2}{L}}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx$

6. (b) $\frac{4L}{5\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\left((-1)^n+2\sin\frac{(2n-1)\pi}{4}\right)$
$\times\cos\frac{(2n-1)\pi ct}{2L}\sin\frac{(2n-1)\pi x}{2L}$

(c) $\dfrac{16L^3}{\pi^3 c}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\left(\dfrac{4}{(2n-1)\pi}+(-1)^n\right)$

$\times \sin\dfrac{(2n-1)\pi ct}{2L}\sin\dfrac{(2n-1)\pi x}{2L}$

7. (a) $\sum_{n=1}^{\infty}(a_n\cos\omega_n t+b_n\sin\omega_n t)\sqrt{\dfrac{2}{L}}\sin\dfrac{n\pi x}{L}$,

$\omega_n=\sqrt{\dfrac{n^2\pi^2c^2}{L^2}+\dfrac{k}{\rho}}$,

$a_n=\sqrt{\dfrac{2}{L}}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx$,

$b_n=\dfrac{\sqrt{2/L}}{\omega_n}\int_0^L g(x)\sin\dfrac{n\pi x}{L}\,dx$

Section 5.3

2. (b) $50+\dfrac{4L^2}{\pi^3}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3\sinh(2n-1)\pi}$

$\times\cosh\dfrac{(2n-1)\pi y}{L}\cos\dfrac{(2n-1)\pi x}{L}$

(c) No solution

3. (b) $f_1(x-L)+f_2$

6. (a) $\sum_{n=1}^{\infty}a_n r^n\sqrt{\dfrac{2}{\pi}}\sin n\theta$,

$a_n=\dfrac{1}{a^n}\int_0^{\pi}f(\theta)\sqrt{\dfrac{2}{\pi}}\sin n\theta\,d\theta$

(b) $\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{2n-1}\left(\dfrac{r}{a}\right)^{2n-1}\sin(2n-1)\theta$;

$\dfrac{4}{\pi}\operatorname{Tan}^{-1}\left(\dfrac{r}{a}\right)$

8. $\dfrac{2}{\pi}-\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{1}{4n^2-1}\left(\dfrac{r}{a}\right)^{2n}\cos 2n\theta$

17. (b) $\sum_{n=1}^{\infty}a_n r^{n/2}\dfrac{1}{\sqrt{\pi}}\sin\dfrac{n\theta}{2}$,

$a_n=R^{-n/2}\int_0^{2\pi}f(\theta)\dfrac{1}{\sqrt{\pi}}\sin\dfrac{n\theta}{2}\,d\theta$

(c) $\sqrt{\dfrac{r}{R}}\sin\left(\dfrac{\theta}{2}\right)$

20. $\dfrac{V_1+V_2}{2}+\pi^{-1}(V_1-V_2)\operatorname{Tan}^{-1}\left(\dfrac{2ar\sin\theta}{a^2-r^2}\right)$

Section 5.4

1. $\dfrac{2}{\sqrt{LL'}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{mn}e^{-(n^2/L^2+m^2/L'^2)\pi^2kt}\sin\dfrac{n\pi x}{L}\sin\dfrac{m\pi y}{L'}$,

$c_{mn}=\dfrac{2}{\sqrt{LL'}}\int_0^{L'}\int_0^{L}f(x,y)\sin\dfrac{n\pi x}{L}\sin\dfrac{m\pi y}{L'}\,dx\,dy$

5. $\dfrac{16U_0}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{e^{-k\pi^2[(2n-1)^2/L^2+(2m-1)^2/(4L''^2)]t}}{(2n-1)(2m-1)}$

$\times\sin\dfrac{(2n-1)\pi x}{L}\,\dfrac{\sin(2m-1)\pi z}{2L''}$

7. $\dfrac{4U_0}{\pi}\sum_{n=1}^{\infty}\dfrac{e^{-(2n-1)^2\pi^2kt/L^2}}{2n-1}\sin\dfrac{(2n-1)\pi x}{L}$

11. (b) $\dfrac{2L}{\pi^4}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+m}}{(2n-1)^2(2m-1)^2}$

$\times\cos\dfrac{c\pi\sqrt{(2n-1)^2+(2m-1)^2}\,t}{L}$

$\times\sin\dfrac{(2n-1)\pi x}{L}\sin\dfrac{(2m-1)\pi y}{L}$

17. $\dfrac{4L}{\pi^2\kappa}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2\sinh(2n-1)\pi}$

$\times\left(Q\cosh\dfrac{(2n-1)\pi(L-z)}{L}+q\cosh\dfrac{(2n-1)\pi x}{L}\right)$

$\times\sin\dfrac{(2n-1)\pi y}{L}$

Section 5.5

3. $W=$ constant

4. $\lambda_{mn}^2=\dfrac{(2n-1)^2\pi^2}{4L^2}+\dfrac{m^2\pi^2}{L'^2}$,

$W_{mn}=\dfrac{2}{\sqrt{LL'}}\sin\dfrac{(2n-1)\pi x}{2L}\sin\dfrac{m\pi y}{L'}$

6. $\lambda_{mn}^2=\dfrac{(2n-1)^2\pi^2}{4L^2}+\dfrac{(2m-1)^2\pi^2}{4L'^2}$,

$W_{mn}=\dfrac{2}{\sqrt{LL'}}\cos\dfrac{(2n-1)\pi x}{2L}\sin\dfrac{(2m-1)\pi y}{2L'}$

Section 6.1

1. $\dfrac{8\sqrt{2L^3}}{(2n-1)^2\pi^2}\left((-1)^{n+1}(L-1)-\dfrac{2L}{(2n-1)\pi}\right)$

6. $\dfrac{2\sqrt{2L}}{(2n-1)^2\pi^2-4L^2}[(-1)^{n+1}(2n-1)\pi\sin L-2L]$

7. $\tilde{f}(0)=\dfrac{e^L-1}{\sqrt{L}},\ \tilde{f}(\lambda_n)=\dfrac{-\sqrt{2L^3}}{n^2\pi^2+L^2}(1+(-1)^{n+1}e^L)$

9. $\tilde{f}(\lambda_2)=\sqrt{\dfrac{L}{8}},\ \tilde{f}(\lambda_n)=0$ for $n\neq 2$

11. $2x$ 12. $-3x^2$ 13. $2\sqrt{2}$ 14. $2x-1$

Section 6.2

1. $\dfrac{U_L x}{L} + \sum_{n=1}^{\infty} c_n e^{-n^2\pi^2 kt/L^2} \sin\dfrac{n\pi x}{L}$,

$c_n = \dfrac{2(-1)^n U_L}{n\pi} + \dfrac{2}{L}\int_0^L f(x)\sin\dfrac{n\pi x}{L}\,dx$

2. See the answer to Exercise 9 in Section 3.3.

5. $U_0 + \dfrac{8U_0}{\pi^2}\sum_{n=1}^{\infty}\dfrac{(-1)^n e^{-(2n-1)^2\pi^2 kt/(4L^2)}}{(2n-1)^2}$
$\times \sin\dfrac{(2n-1)\pi x}{2L}$

10. $\dfrac{2gL}{\kappa\pi^2}\sum_{n=1}^{\infty}\dfrac{1-e^{-n^2\pi^2 kt/L^2}}{n^2}\sin\dfrac{n\pi b}{L}\sin\dfrac{n\pi x}{L}$

14. $\dfrac{qx}{\kappa} + U_0 + \dfrac{8}{\pi^2\kappa}\sum_{n=1}^{\infty}\dfrac{(-1)^n(U_0\kappa + qL)}{(2n-1)^2}$
$\times e^{-(2n-1)^2\pi^2 kt/(4L^2)}\sin\dfrac{(2n-1)\pi x}{2L}$

15. (a) $U_0 + \dfrac{q(L-x)}{\kappa} - \dfrac{8qL}{\kappa\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}$
$\times e^{-(2n-1)^2\pi^2 kt/(4L^2)}\cos\dfrac{(2n-1)\pi x}{2L}$

(b) $U_0 + \dfrac{8qL}{\kappa\pi^2}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^2}$
$\times\left(e^{-(2n-1)^2\pi^2 k(t-t_0)/(4L^2)} - e^{-(2n-1)^2\pi^2 kt/(4L^2)}\right)$
$\times\cos\dfrac{(2n-1)\pi x}{2L}$

(c) U_0

23. $\dfrac{4F_0L^2}{\rho\pi}\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)[(2n-1)^2\pi^2c^2 - \omega^2L^2]}$
$\times\left(\sin\omega t - \dfrac{L\omega}{(2n-1)\pi c}\sin\dfrac{(2n-1)\pi ct}{L}\right)$
$\times\sin\dfrac{(2n-1)\pi x}{L}$ for the nonresonance case. When $\omega = m\pi c/L$, replace the mth term in the above series with $\dfrac{2F_0L}{m^2\pi^2\rho c}\left(\dfrac{L}{m\pi c}\sin\dfrac{m\pi ct}{L} - t\cos\dfrac{m\pi ct}{L}\right)\sin\dfrac{m\pi x}{L}$

25. $\dfrac{(2n-1)\pi c}{2L}$ 27. $\dfrac{(2n-1)\pi c}{2L}$ 29. $\dfrac{n\pi c}{L}$

31. $\omega = \dfrac{n\pi c}{L}$ or $\phi = \dfrac{n\pi c}{L}$; if $\omega = \phi$ and $A_0 = B_0$, then $\omega = \phi = \dfrac{(2n-1)\pi c}{L}$; if $\omega = \phi$ and $A_0 = -B_0$, then $\omega = \phi = \dfrac{2n\pi c}{L}$

38. $\dfrac{t^2}{2}\left(\dfrac{c^2F_0}{\tau L} + g\right) + M(x) - \dfrac{1}{2}[M(x+ct) + M(x-ct)]$, where $M(x)$ is the even, $2L$-periodic extension of $-\dfrac{F_0x^2}{2L\tau}$

42. (a) $\dfrac{\sigma x(L-x)}{2\varepsilon_0} - \dfrac{4\sigma L^2}{\varepsilon_0\pi^3}$
$\sum_{n=1}^{\infty}\dfrac{\sinh\dfrac{(2n-1)\pi y}{L} + \sinh\dfrac{(2n-1)\pi(L'-y)}{L}}{(2n-1)^3\sinh\dfrac{(2n-1)\pi L'}{L}}$
$\times\sin\dfrac{(2n-1)\pi x}{L}$

(b) $\dfrac{2L}{\varepsilon_0\pi^2}\sum_{n=1}^{\infty}\dfrac{\sigma_n}{n^2}\left(1 - \dfrac{\sinh\dfrac{n\pi y}{L} + \sinh\dfrac{n\pi(L'-y)}{L}}{\sinh\dfrac{n\pi L'}{L}}\right)$
$\times\sin\dfrac{n\pi x}{L}$, $\sigma_n = \int_0^L \sigma(x)\sin\dfrac{n\pi x}{L}\,dx$

(c) $\dfrac{2L^3}{\varepsilon_0\pi^3}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^3}\left(\dfrac{L'\sinh\dfrac{n\pi y}{L}}{\sinh\dfrac{n\pi L'}{L}} - y\right)\sin\dfrac{n\pi x}{L}$

Section 6.3

2. (a) $\dfrac{2}{L'\pi^3}\sum_{n=1}^{\infty}$
$\times\dfrac{\left(\pi^2L'n^2[U_1 + U_2(-1)^{n+1}] - L^2(\kappa_2^{-1}\phi_2 + \kappa_1^{-1}\phi_1)[1 + (-1)^{n+1}]\right)}{n^3}$
$\times(1 - e^{-n^2\pi^2 kt/L^2})\sin\dfrac{n\pi x}{L} + \dfrac{8L^2L'}{\pi^3}$
$\times\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\kappa_2^{-1}\phi_2(-1)^{m+1} - \kappa_1^{-1}\phi_1}{(2n-1)[(2n-1)^2L'^2 + m^2L^2]}$
$\times(1 - e^{-[(2n-1)^2/L^2 + m^2/L'^2]\pi^2 kt})\sin\dfrac{(2n-1)\pi x}{L}$
$\times\cos\dfrac{m\pi y}{L'}$

(b) $U_1 + \dfrac{(U_2 - U_1)}{L}x - \dfrac{2}{\pi}\sum_{n=1}^{\infty}\dfrac{U_1 + U_2(-1)^{n+1}}{n}$
$\times e^{-n^2\pi^2 kt/L^2}\sin\dfrac{n\pi x}{L}$

6. $\dfrac{16AL^2}{\rho\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{\cos\omega t-\cos\dfrac{c\pi\sqrt{(2n-1)^2+(2m-1)^2}\,t}{L}}{\left((2n-1)(2m-1)\times\{c^2\pi^2[(2n-1)^2+(2m-1)^2]-\omega^2L^2\}\right)}\times\sin\dfrac{(2n-1)\pi x}{L}\sin\dfrac{(2m-1)\pi y}{L}$

7. Same solution as for Exercise 6 except that the $m=1$ and $n=1$ term is replaced with

$$\frac{4\sqrt{2}\,AL}{\rho\pi^3c}t\sin\frac{\sqrt{2}\,\pi ct}{L}\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}$$

10. Same solution as for Exercise 6 except that the $(m,n)=(1,3)$ and $(m,n)=(3,1)$ terms are replaced with

$$\frac{8AL}{3\sqrt{10}\rho\pi^3c}t\sin\frac{\sqrt{10}\pi ct}{L}\times\left(\sin\frac{\pi x}{L}\sin\frac{3\pi y}{L}+\sin\frac{3\pi x}{L}\sin\frac{\pi y}{L}\right)$$

Section 7.2

2. $\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{e^{-k\lambda^2t}}{\lambda^3}(2\sin\lambda L-\lambda L(1+\cos\lambda L))\cos\lambda x\,d\lambda$

5. $\dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^L u(L-u)e^{-(u-x)^2/(4kt)}\,du$

6. $\dfrac{1}{2\sqrt{k\pi t}}\displaystyle\int_0^\infty f(u)\left(e^{-(u-x)^2/(4kt)}-e^{-(u+x)^2/(4kt)}\right)du$

11. $\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}g(u)\,du$, where $g(x)$ is extended as an odd function

13. (b) $\dfrac{2V_0}{\pi}\operatorname{Tan}^{-1}\left(\dfrac{\sin(\pi x/L)}{\sinh(\pi y/L)}\right)$

15. $\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{1}{\lambda}\cos\frac{\lambda(a+b-2x)}{2}\sin\frac{\lambda(b-a)}{2}\,d\lambda$

17. $\dfrac{4b}{\pi a^2}\displaystyle\int_0^\infty\frac{1}{\lambda^3}(\sin\lambda a-a\lambda\cos\lambda a)\cos\lambda x\,d\lambda$

19. $\dfrac{1}{\sqrt{k\pi}}\displaystyle\int_0^\infty e^{-\lambda^2/(4k)}\cos\lambda x\,d\lambda$

20. $\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{1}{\lambda}(\sin\lambda b-\sin\lambda a)\cos\lambda x\,d\lambda$;

$\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{1}{\lambda}(\cos\lambda a-\cos\lambda b)\sin\lambda x\,d\lambda$

Section 7.3

5. (a) $\mathscr{F}\{f^{(n)}(x)\}=(i\omega)^n\mathscr{F}\{f(x)\}$

10. (b) $\mathscr{F}_S\{\tilde{f}(x)\}=\left(\dfrac{\pi}{2}\right)f(\omega)$; $\mathscr{F}_C\{\tilde{f}(x)\}=\left(\dfrac{\pi}{2}\right)f(\omega)$

12. $\dfrac{2a}{\omega^2+a^2}$ 13. $\dfrac{n!}{(a+i\omega)^{n+1}}$

14. $\dfrac{2}{\omega}e^{-i\omega(a+b)/2}\sin\dfrac{\omega(b-a)}{2}$

16. $\dfrac{4b}{a\omega^2}\sin^2\left(\dfrac{\omega a}{2}\right)$ 17. $\dfrac{-4b}{a\omega^2}\cos a\omega+\dfrac{4b}{a^2\omega^3}\sin a\omega$

19. $\dfrac{1}{\omega}(\cos a\omega-\cos b\omega)$; $\dfrac{1}{\omega}(\sin b\omega-\sin a\omega)$

20. $\dfrac{4b}{a\omega^2}\sin\omega c\sin^2\left(\dfrac{a\omega}{2}\right)$; $\dfrac{4b}{a\omega^2}\cos\omega c\sin^2\left(\dfrac{a\omega}{2}\right)$

26. (a) (i) $\begin{cases}(x^2/2)e^{-8x} & x\ge 0\\ 0 & x<0\end{cases}$

(ii) $\begin{cases}(b/a)(x-a)[H(x-a)-1] & x>0\\ 0 & x<0\end{cases}$

(b) No

Section 7.4

1. (b) (i) $\dfrac{1}{2}\operatorname{erf}\left(\dfrac{x+a}{2\sqrt{kt}}\right)-\dfrac{1}{2}\operatorname{erf}\left(\dfrac{x-a}{2\sqrt{kt}}\right)$

(ii) $1-\dfrac{1}{2}\operatorname{erf}\left(\dfrac{x+a}{2\sqrt{kt}}\right)+\dfrac{1}{2}\operatorname{erf}\left(\dfrac{x-a}{2\sqrt{kt}}\right)$

2. (a) $\bar{U}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$

3. $\dfrac{Q_0}{\kappa}\left[2\sqrt{\dfrac{kt}{\pi}}e^{-x^2/(4kt)}-x\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)\right]$

4. (b) $U_0\operatorname{erf}\left(\dfrac{x}{2\sqrt{kt}}\right)$; (c) $\bar{U}\operatorname{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$

10. (a) (i) $\displaystyle\sum_{n=1}^{\infty}C_ne^{-n\pi x/L'}\sin\frac{n\pi y}{L'}$,

$C_n=\dfrac{2}{L'}\displaystyle\int_0^{L'}f(y)\sin\frac{n\pi y}{L'}\,dy$

(ii) $\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{\bar{g}(\omega)\sinh\omega y}{\sinh\omega L'}\sin\omega x\,d\omega$,

$\bar{g}(\omega)=\displaystyle\int_0^\infty g(x)\sin\omega x\,dx$

(iii) $\dfrac{2}{\pi}\displaystyle\int_0^\infty\frac{\bar{g}(\omega)\sinh\omega(L'-y)}{\sinh\omega L'}\sin\omega x\,d\omega$,

$\bar{g}(\omega)=\displaystyle\int_0^\infty g(x)\sin\omega x\,dx$

(iv) Sum of solutions in (i), (ii), and (iii)

12. (a) U_0;

(b) $C + \dfrac{4L'Q_0}{\pi^2\kappa}\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{(2n-1)^2}e^{-(2n-1)\pi x/L'}$ $\times \cos\dfrac{(2n-1)\pi y}{L'}$

(c) U_m

15. $\dfrac{V_L x}{L} + \dfrac{1}{\varepsilon}e^{-y}\left(\dfrac{\sin(L-x)+\sin x}{\sin L} - 1\right)$ $+\dfrac{2}{\pi}\sum_{n=1}^{\infty}\left(\dfrac{V_L(-1)^n}{n} - \dfrac{L^2[1+(-1)^{n+1}]}{n\varepsilon(n^2\pi^2 - L^2)}\right)e^{-n\pi y/L}$ $\times \sin\dfrac{n\pi x}{L}$

17. (d) $\dfrac{2}{\pi}\operatorname{Tan}^{-1}\left(\dfrac{x}{y}\right)$

18. (a) $\dfrac{y}{\pi}\displaystyle\int_{-\infty}^{\infty}\dfrac{f(u)}{(x-u)^2+y^2}\,du$

(b) $\dfrac{1}{2} + \dfrac{1}{\pi}\operatorname{Tan}^{-1}\left(\dfrac{x}{y}\right)$

Section 8.2

1. (a) 120 (b) 2.9812 (c) 7.3619
(d) −5.7386 (e) 0.6891 (f) −1.0276

Section 8.3

3. (a) 0.9604 (b) 0.6201 (c) 0.3688
(d) 0.0955 (e) 0.4448 (f) −0.2769
(g) 0.4333 (h) 0.1190

Section 8.4

2. (a) $2r_2^{\nu-1}\sum_{n=1}^{\infty}\dfrac{J_\nu(\lambda_{\nu n}r)}{\lambda_{\nu n}J_{\nu+1}(\lambda_{\nu n}r_2)}$

(b) $2\nu r_2^{\nu}\sum_{n=1}^{\infty}\dfrac{J_\nu(\lambda_{\nu n}r)}{[(\lambda_{\nu n}r_2)^2-\nu^2]J_\nu(\lambda_{\nu n}r_2)}$

3. $\dfrac{2}{r_2}\sum_{n=1}^{\infty}\dfrac{J_1(\lambda_n r_2)J_0(\lambda_n r)}{\lambda_n\left[1+\left(\dfrac{h_2}{\lambda_n l_2}\right)^2\right][J_0(\lambda_n r_2)]^2}$; when $l_2 = 0$,

$\dfrac{2}{r_2}\sum_{n=1}^{\infty}\dfrac{J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_2)}$; when $h_2 = 0$, $\dfrac{r_2}{\sqrt{2}}R_0(r)$

5. (b) $\dfrac{n^2\pi^2}{r_2^2}$; $\sqrt{\dfrac{2}{r_2}}\,r^{-1}\sin\left(\dfrac{n\pi r}{r_2}\right)$

6. (b) $\sqrt{3}\,r_2^{-3/2}$, $\dfrac{\sqrt{2}\sqrt{1+\lambda_n^2 r_2^2}}{\lambda_n r_2^{3/2} r}\sin\lambda_n r$

7. (b) $(rN)^{-1}\sin\lambda_n r$,
$2N^2 = r_2\left(1 + \dfrac{h_2 r_2/l_2 - 1}{\lambda_n^2 r_2^2 + (1 - h_2 r_2/l_2)^2}\right)$

8.

Condition at $r = r_2$	Eigenvalue Equation	NR_{mn}	$2N^2$
$h_2 l_2 \neq 0$	$0 = 2\lambda r_2 J'_{m+1/2}(\lambda r_2) + \left(\dfrac{2h_2r_2 - l_2}{l_2}\right)J_{m+1/2}(\lambda r_2)$	$\dfrac{J_{m+1/2}(\lambda_{mn}r)}{\sqrt{r}}$	$r_2^2\left[1 - \left(\dfrac{m+1/2}{\lambda_{mn}r_2}\right)^2 + \left(\dfrac{2h_2r_2/l_2 - 1}{2\lambda_{mn}r_2}\right)^2\right][J_{m+1/2}(\lambda_{mn}r_2)]^2$
$h_2 = 0$	$0 = 2\lambda r_2 J'_{m+1/2}(\lambda r_2) - J_{m+1/2}(\lambda r_2)$	$\dfrac{J_{m+1/2}(\lambda_{mn}r)}{\sqrt{r}}$	$r_2^2\left[1 - \left(\dfrac{m+1/2}{\lambda_{mn}r_2}\right)^2 + \left(\dfrac{1}{2\lambda_{mn}r_2}\right)^2\right][J_{m+1/2}(\lambda_{mn}r_2)]^2$
$l_2 = 0$	$0 = J_{m+1/2}(\lambda r_2)$	$\dfrac{J_{m+1/2}(\lambda_{mn}r)}{\sqrt{r}}$	$r_2^2[J'_{m+1/2}(\lambda_{mn}r_2)]^2 = r_2^2[J_{m+3/2}(\lambda_{mn}r_2)]^2$

Section 8.5

1. $1, x, \dfrac{3x^2-1}{2}, \dfrac{5x^3-3x}{2}, \dfrac{35x^4-30x^2+3}{8}$, $\dfrac{63x^5-70x^3+15x}{8}, \dfrac{231x^6-315x^4+105x^2-5}{16}$

10. (b) $\dfrac{(3x^2-1)Q_0}{2} - \dfrac{3x}{2}, \dfrac{(5x^3-3x)Q_0}{2} - \dfrac{5x^2}{2} + \dfrac{2}{3}$,
$\dfrac{(35x^4-30x^2+3)Q_0}{8} - \dfrac{35x^3}{8} + \dfrac{55x}{24}$

(c) $P_0Q_0 - \dfrac{3x}{2}, P_3Q_0 - \dfrac{5x^2}{2} + \dfrac{2}{3}, P_4Q_0 - \dfrac{35x^3}{8} + \dfrac{55x}{24}$

Section 8.6

1. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!(4n-1)}{2^{2n}n!(n-1)!} P_{2n-1}(\cos\phi)$

2. $\frac{\sqrt{2}}{5}\left(\frac{1}{\sqrt{2}}\right) + \frac{4\sqrt{10}}{35}\left(\sqrt{\frac{5}{2}}P_2(\cos\phi)\right) + \frac{8\sqrt{2}}{105}\left(\left(\frac{3}{\sqrt{2}}\right)P_4(\cos\phi)\right)$

3. $\frac{1}{4} + \frac{1}{2}\cos\phi + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!(4n+1)}{2^{2n+1}(n-1)!(n+1)!} P_{2n}(\cos\varphi)$

4. $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2n-2)!(4n+1)}{2^{2n}(n-1)!(n+1)!} P_{2n}(\cos\phi)$

5. $\lambda_n = 2n(2n-1),\ n \ge 1;\ \sqrt{4n-1}\,P_{2n-1}(\cos\phi)$

6. $\lambda_n = 2n(2n+1),\ n \ge 0;\ \sqrt{4n+1}\,P_{2n}(\cos\phi)$

Section 9.1

1. (b) $\frac{2U_0}{r_2}\sum_{n=1}^{\infty} \frac{e^{-k\lambda_n^2 t}J_0(\lambda_n r)}{\lambda_n J_1(\lambda_n r_2)}$

(c) $\frac{8}{r_2}\sum_{n=1}^{\infty}\frac{1}{\lambda_n^3}e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$

2. (b) $\frac{1}{\pi r_2^2}\int_{-\pi}^{\pi}\int_0^{r_2} rf(r)\,dr\,d\theta =$ average value of $f(r)$ over the circle $r \le r_2$

6. $\sqrt{2}\sum_{n=1}^{\infty} A_n e^{-k\lambda_n^2 t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n)}$, where

$$A_n = \frac{\sqrt{2}}{J_1(\lambda_n)}\int_0^1 rf(r)J_0(\lambda_n r)\,dr$$

8. $\frac{2}{r_2\sqrt{L}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}e^{-k[\lambda_n^2+(2m-1)^2\pi^2/(4L^2)]t}\frac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)} \times \cos\frac{(2m-1)\pi z}{2L},\ A_{mn} = \frac{2}{r_2\sqrt{L}J_1(\lambda_n r_2)} \times \int_0^L\int_0^{r_2} rf(r,z)J_0(\lambda_n r)\cos\frac{(2m-1)\pi z}{2L}\,dr\,dz$

9. (b) $\frac{2U_0 h_2 l_2}{r_2}\sum_{n=1}^{\infty}\frac{e^{-k\lambda_n^2 t}}{h_2^2+l_2^2\lambda_n^2}\frac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}$

10. (b) $\frac{2U_0}{\pi r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-n^2\pi^2 kt/r_2^2}\sin\frac{n\pi r}{r_2}$

12. (b) $\frac{2U_0\mu r_2}{\kappa r}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sqrt{\lambda_n^2 r_2^2+(1-\mu r_2/\kappa)^2}}{\lambda_n\left[\lambda_n^2 r_2^2+\frac{\mu r_2}{\kappa}\left(\frac{\mu r_2}{\kappa}-1\right)\right]} \times e^{-k\lambda_n^2 t}\sin\lambda_n r$

16. $\frac{4Lr_2^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}e^{-(2m-1)^2\pi^2 kt/(4L^2)}\cos\frac{(2m-1)\pi z}{2L} - \frac{32L}{\pi^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{e^{-k[\lambda_n^2+(2m-1)^2\pi^2/(4L^2)]t}}{(2m-1)^2\lambda_n^2} \times \frac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}\cos\frac{(2m-1)\pi z}{2L}$

17. (a) $\frac{\sqrt{2}}{r_2 J_1(\lambda_n r_2)}\int_0^{r_2} rf(r)J_0(\lambda_n r)\,dr$

19. $\frac{8}{r_2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^3 J_1(\lambda_n r_2)}\cos c\lambda_n t$

20. $\frac{2v_0}{cr_2}\sum_{n=1}^{\infty}\frac{J_0(\lambda_n r)}{\lambda_n^2 J_1(\lambda_n r_2)}\sin c\lambda_n t$

24. (b) No; (c) $\frac{J_0(kr)}{J_0(kr_2)}$

27. (b) $\frac{2U_0\mu\kappa}{r_2}\sum_{n=1}^{\infty}\frac{1}{(\mu^2+\lambda_n^2\kappa^2)\sinh\lambda_n L} \times \sinh\lambda_n(L-z)\frac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}$

29. $V_0\left(\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(4n-1)(2n-2)!}{2^{2n}n!(n-1)!} \times \left(\frac{r}{r_2}\right)^{2n-1}P_{2n-1}(\cos\phi)\right)$

30. $\frac{V_0+V_1}{2} - (V_0-V_1)\sum_{n=1}^{\infty}\frac{(-1)^n(4n-1)(2n-2)!}{2^{2n}n!(n-1)!} \times \left(\frac{r}{r_2}\right)^{2n-1}P_{2n-1}(\cos\phi)$

33. (d) $\frac{4U_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\frac{I_0[(2n-1)\pi r/L]}{I_0[(2n-1)\pi r_2/L]}\sin\frac{(2n-1)\pi z}{L}$

34. (c) $A_0 + \frac{2}{L}\sum_{n=1}^{\infty}A_n I_0\left(\frac{n\pi r}{L}\right)\cos\frac{n\pi z}{L}$, where

$$A_0 = \frac{1}{L}\int_0^L f(z)\,dz,$$

$$A_n = \frac{1}{I_0(n\pi r_2/L)}\int_0^L f(z)\cos\frac{n\pi z}{L}\,dz$$

(d) U_0

35. (c) $\frac{Q}{4\pi\varepsilon_0 a}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(\frac{r}{a}\right)^{2n}P_{2n}(\cos\phi),\ r < a;$

$\frac{Q}{4\pi\varepsilon_0 r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\phi),\ r > a$

36. (c) $\dfrac{Q}{2\pi\varepsilon_0 a}\left(1 - \dfrac{r}{a}\cos\phi + \sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}(2n-2)!}{2^{2n-1}n!(n-1)!}\right.$
$\left.\times\left(\dfrac{r}{a}\right)^{2n}P_{2n}(\cos\phi)\right),\ r < a;\ \dfrac{Q}{4\pi\varepsilon_0 a}$
$\times\sum_{n=0}^{\infty}\dfrac{(-1)^n(2n)!}{2^{2n}n!(n+1)!}\left(\dfrac{a}{r}\right)^{2n+1}P_{2n}(\cos\phi),$
$r > a$

Section 9.2

2. (b) $\dfrac{Q}{4\kappa r_2}(2r^2 - r_2^2 + 8kt) - \dfrac{2Q}{\kappa r_2}\sum_{n=1}^{\infty}\dfrac{e^{-k\lambda_n^2 t}}{\lambda_n^2}\dfrac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}$

3. (b) $\dfrac{2g}{r_2\kappa}\sum_{n=1}^{\infty}\dfrac{1 - e^{-k\lambda_n^2 t}}{\lambda_n^3}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$

(c) $\dfrac{g}{4\kappa}(r_2^2 - r^2) - \dfrac{2g}{r_2\kappa}\sum_{n=1}^{\infty}\dfrac{e^{-k\lambda_n^2 t}}{\lambda_n^3}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)}$

4. (b) $\dfrac{kgt}{\kappa}$

5. (b) $\dfrac{2\mu}{r_2}\sum_{n=1}^{\infty}\left(\dfrac{g + \kappa U_m\lambda_n^2}{\mu^2 + \kappa^2\lambda_n^2}\right)\left(\dfrac{1 - e^{-k\lambda_n^2 t}}{\lambda_n^2}\right)\dfrac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}$

(c) $U_m + \dfrac{g}{4}\left(\dfrac{r_2^2}{\kappa} + \dfrac{2r_2}{\mu} - \dfrac{r^2}{\kappa}\right)$
$-\dfrac{2\mu}{r_2}\sum_{n=1}^{\infty}\left(\dfrac{g + \kappa U_m\lambda_n^2}{\mu^2 + \kappa^2\lambda_n^2}\right)\dfrac{e^{-k\lambda_n^2 t}}{\lambda_n^2}\dfrac{J_0(\lambda_n r)}{J_0(\lambda_n r_2)}$

6. (b) $\dfrac{g}{6\kappa}(r_2^2 - r^2) + \dfrac{2r_2^3 g}{\pi^3\kappa r}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n^3}e^{-n^2\pi^2 kt/r_2^2}\sin\dfrac{n\pi r}{r_2}$

(c) $f_1\left(1 + \dfrac{2r_2}{\pi r}\sum_{n=1}^{\infty}\dfrac{(-1)^n}{n}e^{-n^2\pi^2 kt/r_2^2}\sin\dfrac{n\pi r}{r_2}\right)$

7. (b) $\dfrac{3kf_1 t}{\kappa r_2} + \dfrac{f_1}{10\kappa r_2}(5r^2 - 3r_2^2) + \dfrac{2f_1}{\kappa r_2 r}$
$\times\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\lambda_n^3}\sqrt{1 + \lambda_n^2 r_2^2}$
$e^{-k\lambda_n^2 t}\sin\lambda_n r$

10. $-\dfrac{2\sin\theta}{r_2}\sum_{n=1}^{\infty}\dfrac{1 - e^{-k\lambda_n^2 t}}{\lambda_n}\dfrac{J_1(\lambda_n r)}{J_0(\lambda_n r_2)}$

16. $\dfrac{2Ac}{r_2}\sum_{n=1}^{\infty}\dfrac{c\lambda_n\sin\omega t - \omega\sin c\lambda_n t}{c^2\lambda_n^2 - \omega^2}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)},$
$\omega \neq c\lambda_m;\ \dfrac{A(-c\lambda_m t\cos c\lambda_m t + \sin c\lambda_m t)J_0(\lambda_m r)}{r_2\lambda_m J_1(\lambda_m r_2)}$
$-\dfrac{2A}{r_2}\sum_{\substack{n=1\\ n\neq m}}^{\infty}\dfrac{\lambda_n\sin c\lambda_m t - \lambda_m\sin c\lambda_n t}{\lambda_m^2 - \lambda_n^2}\dfrac{J_0(\lambda_n r)}{J_1(\lambda_n r_2)},$
$\omega = c\lambda_m$

20. For $\beta \neq \dfrac{\pi}{2}, \pi, \dfrac{3\pi}{2}$,
$\dfrac{4\sigma\beta^2}{\pi\varepsilon}\sum_{n=1}^{\infty}\dfrac{r^2}{(2n-1)[(2n-1)^2\pi^2 - 4\beta^2]}$
$\times\left[1 - \left(\dfrac{r}{r_0}\right)^{(2n-1)\pi/\beta - 2}\right]\sin\dfrac{(2n-1)\pi\theta}{\beta};$

when $\beta = \dfrac{\pi}{2}$,
$\dfrac{\sigma r^2}{\varepsilon\pi}\ln\left(\dfrac{r_0}{r}\right)\sin 2\theta$
$+\dfrac{\sigma}{\pi\varepsilon}\sum_{n=2}^{\infty}\dfrac{r^2}{(2n-1)[(2n-1)^2 - 1]}$
$\times\left[1 - \left(\dfrac{r}{r_0}\right)^{4n-4}\right]\sin 2(2n-1)\theta;$

when $\beta = \pi$,
$\dfrac{4\sigma}{\pi\varepsilon}\sum_{n=1}^{\infty}\dfrac{r^2}{(2n-1)[(2n-1)^2 - 4]}\left[1 - \left(\dfrac{r}{r_0}\right)^{2n-3}\right]$
$\times\sin(2n-1)\theta;$

when $\beta = \dfrac{3\pi}{2}$,
$\dfrac{9\sigma r^2}{8\varepsilon\pi}\left[\left(\dfrac{r_0}{r}\right)^{4/3} - 1\right]\sin\dfrac{2\theta}{3} - \dfrac{\sigma r^2}{3\pi\varepsilon}\ln\left(\dfrac{r}{r_0}\right)$
$\times\sin 2\theta + \dfrac{9\sigma}{\pi\varepsilon}\sum_{n=3}^{\infty}\dfrac{r^2}{(2n-1)[(2n-1)^2 - 9]}$
$\times\left[1 - \left(\dfrac{r}{r_0}\right)^{4(n-2)/3}\right]\sin\dfrac{2(2n-1)\theta}{3}$

23. (a) $gr_2\beta = 2q + 2\beta Q$

(b) $\dfrac{A}{\sqrt{\beta}} - \dfrac{gr^2}{4\kappa} + \dfrac{gr\cos(\beta - \theta)}{\kappa\sin\beta} + \dfrac{2q\beta^2 r_2}{\kappa\pi}$
$\times\sum_{n=1}^{\infty}\dfrac{(r/r_2)^{n\pi/\beta}}{n(n^2\pi^2 - \beta^2)}\cos\dfrac{n\pi\theta}{\beta}$

Section 9.3

1. $\dfrac{ag}{\kappa}\displaystyle\int_0^{\infty}\dfrac{1}{\lambda^2}(1 - e^{-\kappa\lambda^2 t})J_1(\lambda a)J_0(\lambda r)\,d\lambda$

2. $\bar{U} - \dfrac{4\bar{U}}{\alpha}\sum_{n=1}^{\infty}\left(\displaystyle\int_0^{\infty}\dfrac{1}{\lambda}e^{-k\lambda^2 t}J_{(2n-1)\pi/\alpha}(\lambda r)\,d\lambda\right)$
$\times\sin\dfrac{(2n-1)\pi\theta}{\alpha}$

5. $\dfrac{aQ}{\kappa}\displaystyle\int_0^{\infty}\dfrac{1}{\lambda}e^{-\lambda z}J_1(\lambda a)J_0(\lambda r)\,d\lambda$

6. $a\bar{U}\displaystyle\int_0^{\infty}e^{-\lambda z}J_1(\lambda a)J_0(\lambda r)\,d\lambda$

Section 10.1

1. (b) (i) $\dfrac{6}{(s+5)^4}$; (ii) $\dfrac{s+1}{s^2+2s+5}+\dfrac{2}{s^2-6s+13}$;
   (c) (i) $\left(\dfrac{1}{2}\right)e^t \sin 2t$; (ii) $\left(\dfrac{1}{\sqrt{\pi t}}\right)e^{-3t}$
2. (b) (i) $\dfrac{e^{-3s}(s+1)}{s^2}$; (ii) $\dfrac{e^{-as}}{s}$; (iii) $\dfrac{1-e^{-as}}{s}$;
   (c) (i) $\begin{cases} 0 & 0<t<2 \\ t-2 & t>2 \end{cases}$;
   (ii) $\begin{cases} 0 & 0<t<3 \\ \sin(t-3) & t>3 \end{cases}$
3. (b) (i) $\dfrac{8}{s^3}+\dfrac{2}{s^2-4}$; (ii) $\dfrac{s-4}{s^2-8s+32}$;
   (c) (i) $\left(\dfrac{1}{9}\right)\cos\left(\dfrac{\sqrt{2}\,t}{3}\right)$; (ii) $\left(\dfrac{1}{\sqrt{29}}\right)e^{3t/4}\sinh\left(\dfrac{\sqrt{29}\,t}{4}\right)$
4. (b) (i) $\dfrac{1-e^{-as}(1+as)}{s^2(1-e^{-as})}$; (ii) $\dfrac{1-e^{-as}}{s(1+e^{-as})}$
6. $1-e^{-t}$
8. $\left(\dfrac{2}{7}\right)\cosh\sqrt{2}\,t-\left(\dfrac{\sqrt{2}}{14}\right)\sinh\sqrt{2}\,t-\dfrac{2}{7}e^{-4t}$
10. $\dfrac{2}{s^2}-\dfrac{e^{-s}(s+1)}{s^2}$ 12. $\dfrac{1-e^{-as}}{s^2(1+e^{-as})}$ 14. $\dfrac{e^{-as}}{s}$
16. $2e^{2t}-e^t$ 17. $e^{-t}+e^{t/2}-e^{-t/2}-1$
19. $0,\ t<2;\ e^{-(t-2)}-e^{-2(t-2)},\ t>2$
20. $\dfrac{e^{-t}}{3}+\left(\dfrac{e^{t/2}}{3}\right)\left[\sqrt{3}\sin\left(\dfrac{\sqrt{3}\,t}{2}\right)-\cos\left(\dfrac{\sqrt{3}\,t}{2}\right)\right]$
22. $0,\ t<1;\ 1-\cos(t-1),\ 1<t<2;$
    $\cos(t-2)-\cos(t-1),\ t>2$
23. $\dfrac{e^{-t}(4t^3-t^4)}{24}$ 25. $\dfrac{\sinh 2t+2t\cosh 2t}{4}$
26. $\left(\dfrac{1}{2}\right)e^t+\left(\dfrac{1}{2}\right)e^{-t}\left[\cosh\sqrt{2}\,t+\left(\dfrac{4}{\sqrt{2}}\right)\sinh\sqrt{2}\,t\right]$
27. $e^{-t}-\cos t+\sin t$ 28. $2(1+t)e^{-t}+t-2$
30. $\dfrac{\cos 2t+4\cos 3t+4\sin 3t}{5}$
32. $\dfrac{1}{a}\displaystyle\int_0^t f(u)\sinh a(t-u)\,du+A\cosh at+B\sinh at$
35. (b) $s\tilde{f}(s)-f(0+)$

Section 10.2

3. (b) $U_0+(\bar{U}-U_0)\,\mathrm{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)$
4. (b) $U_0+\dfrac{Q_0}{\kappa}\left[2\sqrt{\dfrac{kt}{\pi}}\,e^{-x^2/(4kt)}-x\,\mathrm{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)\right]$
5. (b) $U_m\left[\mathrm{erfc}\left(\dfrac{x}{2\sqrt{kt}}\right)-e^{\mu x/\kappa+k\mu^2 t/\kappa^2}\right.$
   $\left.\times\,\mathrm{erfc}\left(\dfrac{x}{2\sqrt{kt}}+\dfrac{\mu\sqrt{kt}}{\kappa}\right)\right]$
6. (c) $\left(\dfrac{U_0}{2}\right)\left[1+\mathrm{erf}\left(\dfrac{x}{2\sqrt{kt}}\right)\right]$
8. $f_1\left(t-\dfrac{x}{c}\right)H\left(t-\dfrac{x}{c}\right)$

Section 10.3

1. $\dfrac{t(t+2)e^t}{2}$ 3. $\dfrac{3t-1+e^{-3t}}{9}$
5. $\dfrac{2\sin 2t-\sin t}{3}$ 7. $\dfrac{2t^2\cosh 2t+3t\sinh 2t}{16}$
9. $\left(\dfrac{1}{2}\right)te^t\sin t$
11. $\dfrac{2}{\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{n}e^{-n^2\pi^2 t}\sin n\pi x$
13. $\dfrac{8}{\pi^3}\displaystyle\sum_{n=1}^{\infty}\dfrac{1}{(2n-1)^3}\cos(2n-1)\pi t\sin(2n-1)\pi x$
15. $\dfrac{1}{2\pi}\sin\dfrac{\pi t}{2}\sin\dfrac{\pi x}{2}+\dfrac{2}{\pi^2}\displaystyle\sum_{n=1}^{\infty}\dfrac{(-1)^n}{4n^2-1}\sin n\pi t\sin n\pi x$
16. $\dfrac{1}{2\pi^2}[\sin\pi t(\sin\pi x-2\pi x\cos\pi x)-2\pi t\cos\pi t\sin\pi x]$
    $+\dfrac{2}{\pi^2}\displaystyle\sum_{n=2}^{\infty}\dfrac{(-1)^n}{n^2-1}\sin n\pi t\sin n\pi x$

Section 10.4

1. $e^{-m^2\pi^2 kt/L^2}\sin\dfrac{m\pi x}{L}$ 2. U_0
5. $\dfrac{ke^{-\alpha t}}{\kappa\alpha}\left(-1+\dfrac{\cos\sqrt{\alpha/k}\,(L/2-x)}{\cos\sqrt{\alpha/k}\,L/2}\right)$
   $+\dfrac{4kL^2}{\kappa\pi}\displaystyle\sum_{n=1}^{\infty}\dfrac{e^{-(2n-1)^2\pi^2 kt/L^2}}{(2n-1)[\alpha L^2-(2n-1)^2\pi^2 k]}$
   $\times\sin\dfrac{(2n-1)\pi x}{L}$
6. (a) Add the following series to the solution of Exercise 5:

$$\frac{40}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}e^{-(2n-1)^2\pi^2 kt/L^2}\sin\frac{(2n-1)\pi x}{L}$$

9. $U_L\left(\frac{x}{L}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}\right)$;
$U_L\sum_{n=0}^{\infty}\left[\operatorname{erf}\left(\frac{(2n+1)L+x}{2\sqrt{kt}}\right)-\operatorname{erf}\left(\frac{(2n+1)L-x}{2\sqrt{kt}}\right)\right]$

16. $U_0\left(1-\frac{x}{L}\right)-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{U_0+(-1)^nL}{n}e^{-n^2\pi^2kt/L^2}\times\sin\frac{n\pi x}{L}$

18. $\frac{8kL^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^3}\cos\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

22. $\frac{F_0}{\rho\omega^2\sin\frac{\omega L}{c}}\left(\sin\frac{\omega x}{c}+\sin\frac{\omega(L-x)}{c}-\sin\frac{\omega L}{c}\right)\sin\omega t$
$+\frac{4F_0\omega L^3}{\rho c\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2[\omega^2L^2-(2n-1)^2\pi^2c^2]}\times\sin\frac{(2n-1)\pi ct}{L}\sin\frac{(2n-1)\pi x}{L}$

31. $\frac{A\sin(\omega x/c)\sin\omega t}{\sin(\omega L/c)}+2A\omega Lc\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\pi^2c^2-\omega^2L^2}\times\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$ for the nonresonance case.
When $\omega=\frac{m\pi c}{L}$, $\frac{A(-1)^m}{2m\pi L}\left(2m\pi ct\sin\frac{m\pi x}{L}\cos\frac{m\pi ct}{L}-L\sin\frac{m\pi x}{L}\sin\frac{m\pi ct}{L}+2m\pi x\cos\frac{m\pi x}{L}\sin\frac{m\pi ct}{L}\right)$
$+\frac{2Am}{\pi}\sum_{\substack{n=1\\n\neq m}}^{\infty}\frac{(-1)^n}{n^2-m^2}\sin\frac{n\pi ct}{L}\sin\frac{n\pi x}{L}$

37. $\sum_{n=1}^{\infty}C_n\cos\frac{n^2\pi^2ct}{L^2}\sin\frac{n\pi x}{L}$, $C_n=\frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$

38. (b) $\frac{g}{2c^2}\left(\frac{2AEL}{k}+2Lx-x^2\right)$

40. 9.6×10^{-5} s

Section 10.5

7. (f) $\frac{8U_0}{r_2\pi}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2m-1)\lambda_n}e^{-k[\lambda_n^2+(2m-1)^2\pi^2/L^2]t}\times\frac{J_0(\lambda_nr)}{J_1(\lambda_nr_2)}\sin\frac{(2m-1)\pi z}{L}$

10. $\frac{A}{\rho\omega^2}\left(\frac{J_0(\omega r/c)}{J_0(\omega r_2/c)}-1\right)-\frac{2A\omega}{\rho cr_2}\sum_{n=1}^{\infty}\frac{\sin c\lambda_nt}{\lambda_n^2(c^2\lambda_n^2-\omega^2)}\times\frac{J_0(\lambda_nr)}{J_1(\lambda_nr_2)}$

Section 11.1

1. 3 **2.** sin 1 **3.** $9+e^{-3}$ **4.** 0 **5.** 100
6. $-39-\cos 10$
9. $\frac{-x}{\tau}, 0\le x\le\frac{L}{3}$; $\frac{-L}{3\tau}, \frac{L}{3}\le x\le\frac{2L}{3}$; $\frac{x-L}{\tau}, \frac{2L}{3}\le x\le L$
11. $\frac{x^3-3Lx^2}{6EI}$
12. $0, 0\le t\le T$; $\left(\frac{1}{\sqrt{kM}}\right)\sin\sqrt{\frac{k}{M}}(t-T), t\ge T$

Section 11.3

1. $\frac{d}{dx}\left(x\frac{dy}{dx}\right)+3y=F(x)$
2. $\frac{d}{dx}\left(e^x\frac{dy}{dx}\right)-2e^xy=e^xF(x)$
5. $\frac{d}{dx}\left(e^{4x}\frac{dy}{dx}\right)=e^{4x}F(x)$
6. $XH(x-X)-xH(X-x)$
8. $-k^{-1}\csc k\pi[\sin kx\sin k(\pi-X)H(X-x)+\sin kX\sin k(\pi-x)H(x-X)]$
10. $[5(1+4e^{10})]^{-1}[(e^{-x}-e^{4x})\times(4e^{10-X}+e^{4X})H(X-x)+(e^{-X}-e^{4X})\times(4e^{10-x}+e^{4x})H(x-X)]$
11. $\left(\frac{1}{4}\right)e^{-(x+X)}[(2\cos 2x+\sin 2x)\sin 2XH(X-x)+(2\cos 2X+\sin 2X)\sin 2xH(x-X)]$
13. $\{2k[1-\cos k(\beta-\alpha)]\}^{-1}\{[\sin k(\beta-\alpha-X+x)+\sin k(X-x)]H(X-x)+[\sin k(\beta-\alpha-x+X)+\sin k(x-X)]H(x-X)\}$
16. $(6EI)^{-1}[(x-X)^3H(x-X)-x^3+3Xx^2]$
18. $(6EI)^{-1}\left[(x-X)^3H(x-X)+\frac{x^3(3L^2X^2-L^4-2LX^3)}{L^4}+\frac{3x^2(X^3-2LX^2+L^2X)}{L^2}\right]$
22. $-\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}$

Section 11.4

1. $(\kappa L)^{-1}[x(L-X)H(X-x)+X(L-x)H(x-X)]$

4. $\dfrac{kx(L-x)}{2\tau}$

6. $\dfrac{k}{32\tau}\begin{cases} 8xL & 0 \le x \le L/4 \\ -L^2+16Lx-16x^2 & L/4 \le x \le 3L/4 \\ 8L(L-x & 3L/4 \le x \le L \end{cases}$

8. $\dfrac{1}{4\tau}\begin{cases} 2kx(L-x)-4\bar{k}x & 0 \le x \le L/4 \\ 2kx(L-x)-\bar{k}L & L/4 \le x \le 3L/4 \\ 2kx(L-x)-4\bar{k}(L-x) & 3L/4 \le x \le L \end{cases}$

11. $\dfrac{L(2\rho c^2+\rho gL+mg)}{2\rho c^2}$

14. $\dfrac{1}{48EI}\begin{cases} 8x^3-12Lx^2 & 0 \le x \le L/2 \\ L^3-6L^2x & L/2 \le x \le L \end{cases}$

16. $\dfrac{wx(L-x)(x^2-Lx-L^2)}{24EI}$

18. $\dfrac{w(4Lx^3-6L^2x^2-x^4)}{24EI}+\dfrac{W}{24EI}$

$\times\begin{cases} Lx^2(2x-3L) & 0 \le x \le L/4 \\ -x^4+3Lx^3-(27L^2x^2/8) \\ \quad +(L^3x/16)-(L^4/256) & L/4 \le x \le 3L/4 \\ (5L^4/16)-(13L^3x/8) & 3L/4 \le x \le L \end{cases}$

21. $\dfrac{x}{2}\sin x-\dfrac{\sin x}{2\cos 1}(\sin 1+\cos 1)$

$+\dfrac{m_2\sin x+m_1\cos(1-x)}{\cos 1}$

23. $\dfrac{x}{k^2}+\dfrac{(\beta-\alpha)[\cos k(x-\alpha)-\cos k(\beta-x)]}{2k^2[1-\cos k(\beta-\alpha)]}$

25. $\left(\dfrac{1}{4}\right)e^{2x}(1-\cos 2x)$

Section 11.5

1. $E+\dfrac{L^2}{\kappa\pi^2}\cos\dfrac{\pi x}{L}$

3. (a) $\left(D+\dfrac{x}{4}\right)\sin 2x$;

(b) $\left(D+\dfrac{x}{4}\right)\sin 2x+m_1\cos 2x$

5. $D\sin\dfrac{n\pi x}{L}+\dfrac{L}{n\pi}\displaystyle\int_x^L F(X)\sin\dfrac{n\pi(X-x)}{L}\,dX$

when $k=n\pi/L$

7. $D\sin\dfrac{(2n-1)\pi x}{2L}+\dfrac{2L}{(2n-1)\pi}\displaystyle\int_x^L F(X)$

$\times\sin\dfrac{(2n-1)\pi(X-x)}{2L}\,dX$

when $k=(2n-1)\pi/(2L)$

Section 11.6

1. $\dfrac{1}{M}\displaystyle\int_0^t(t-T)F(T)\,dT+x_0+v_0t$

2. $\dfrac{1}{M\omega}\displaystyle\int_0^t e^{-\beta(t-T)/(2M)}\sin\omega(t-T)F(T)\,dT$

$+e^{-\beta t/(2M)}\left[x_0\cos\omega t+\left(\dfrac{v_0}{\omega}+\dfrac{\beta x_0}{2M\omega}\right)\sin\omega t\right]$

Section 12.2

4. $\dfrac{-4}{\pi r_0^2}\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\dfrac{J_m(\lambda_{mn}r)J_m(\lambda_{mn}R)\sin m\Theta\sin m\Theta}{\lambda_{mn}^2[J_{m+1}(\lambda_{mn}r_0)]^2}$

6. $\dfrac{-8}{LL'L''}\displaystyle\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}$

$\sin\dfrac{n\pi x}{L}\sin\dfrac{n\pi X}{L}\sin\dfrac{j\pi y}{L'}\sin\dfrac{j\pi Y}{L'}$

$\times\dfrac{\times\sin\dfrac{m\pi z}{L''}\sin\dfrac{m\pi Z}{L''}}{\dfrac{n^2\pi^2}{L^2}+\dfrac{j^2\pi^2}{L'^2}+\dfrac{m^2\pi^2}{L''^2}}$

11. $\dfrac{1}{4\pi}\ln\left(\dfrac{[r_0^4+r^2R^2-2r_0^2rR\cos(\theta+\Theta)]\times[R^2+r^2-2rR\cos(\theta-\Theta)]}{[r_0^4+r^2R^2-2r_0^2rR\cos(\theta-\Theta)]\times[R^2+r^2-2rR\cos(\theta+\Theta)]}\right)$

12. $\displaystyle\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\dfrac{(-1)^{n+m+1}}{4\pi}$

$\times\ln[(x-x_n)^2+(y-y_m)^2]$,

$x_n=\dfrac{L}{2}+nL+(-1)^n\left(x-\dfrac{L}{2}\right)$,

$y_m=\dfrac{L'}{2}+mL'+(-1)^m\left(Y-\dfrac{L'}{2}\right)$

15. $\dfrac{1}{4L}\ln\left[\dfrac{\left(r_0^4+r^2R^2-2r_0^2rR\cos\dfrac{\pi(\theta+\Theta)}{L}\right)\times\left(R^2+r^2-2rR\cos\dfrac{\pi(\theta-\Theta)}{L}\right)}{\left(r_0^4+r^2R^2-2r_0^2rR\cos\dfrac{\pi(\theta-\Theta)}{L}\right)\times\left(R^2+r^2-2rR\cos\dfrac{\pi(\theta+\Theta)}{L}\right)}\right]$

18. $$\frac{4}{LL'}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2-\dfrac{n^2\pi^2}{L^2}-\dfrac{m^2\pi^2}{L'^2}}\times\sin\frac{n\pi X}{L}\sin\frac{m\pi Y}{L'}\sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L'}$$

20. $$\frac{4}{\pi r_0^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{k^2-\lambda_{mn}^2}\times\frac{J_m(\lambda_{mn}R)J_m(\lambda_{mn}r)\sin m\Theta\sin m\theta}{[J_{m+1}(\lambda_{mn}r_0)]^2}$$

22. $$\frac{8}{LL'L''}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\times\frac{1}{k^2-\dfrac{n^2\pi^2}{L^2}-\dfrac{j^2\pi^2}{L'^2}-\dfrac{m^2\pi^2}{L''^2}}\times\sin\frac{n\pi X}{L}\sin\frac{j\pi Y}{L'}\sin\frac{m\pi Z}{L''}\times\sin\frac{n\pi x}{L}\sin\frac{j\pi y}{L'}\sin\frac{m\pi z}{L''}$$

Section 12.3

3. $$\int_0^{L'}\int_0^{L}G(x,y;X,Y)F(X,Y)\,dX\,dY+\frac{2}{L'}\sum_{n=1}^{\infty}B_n\operatorname{csch}\frac{n\pi L}{L'}\sinh\frac{n\pi x}{L'}\sin\frac{n\pi y}{L'}+\frac{2}{L}\sum_{n=1}^{\infty}A_n\operatorname{csch}\frac{n\pi L'}{L}\sinh\frac{n\pi y}{L}\sin\frac{n\pi x}{L},$$
$$A_n=\int_0^L f(X)\sin\frac{n\pi X}{L}\,dX,$$
$$B_n=\int_0^{L'} g(Y)\sin\frac{n\pi Y}{L'}\,dY$$

Section 12.4

1. (a) $$C-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{A_n}{n\sinh(n\pi L/L')}\cosh\frac{n\pi(L-x)}{L'}\times\cos\frac{n\pi y}{L'},\quad A_n=\int_0^{L'}f(Y)\cos\frac{n\pi Y}{L'}\,dY;$$
(b) $$-\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\cos(n\pi/4)-\cos(3n\pi/4)}{n\sinh(n\pi L/L')}\times\cosh\frac{n\pi(L-x)}{L'}\cos\frac{n\pi y}{L'};\ 0$$

3. $$C+\frac{L(x+y)-(x^2+y^2)}{4\kappa}$$

4. (e) $$C+\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{r_0}RF(R,\Theta)\times\ln\left(\frac{[r^2+R^2-2rR\cos(\theta-\Theta)]\times[r_0^4+r^2R^2-2r_0^2rR\cos(\theta-\Theta)]}{r_0^4r^2}\right)\times dR\,d\Theta-\frac{r_0}{2\pi}\int_{-\pi}^{\pi}K(\Theta)\times\ln\left(\frac{r^2+r_0^2-2r_0r\cos(\theta-\Theta)}{r_0r}\right)d\Theta$$

Section 12.6

1. $$\frac{2k}{\kappa L}\sum_{n=1}^{\infty}e^{-n^2\pi^2k(t-T)/L^2}\sin\frac{n\pi X}{L}\sin\frac{n\pi x}{L}$$

3. $$\frac{2k}{\kappa L}\sum_{n=1}^{\infty}e^{-(2n-1)^2\pi^2k(t-T)/(4L^2)}\times\cos\frac{(2n-1)\pi X}{2L}\cos\frac{(2n-1)\pi x}{2L}$$

Section 12.7

1. $$\frac{1}{\rho}\left(\frac{t-T}{L}+\frac{2}{\pi c}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi c(t-T)}{L}\times\cos\frac{n\pi X}{L}\cos\frac{n\pi x}{L}\right)$$

3. $$\frac{4}{\rho\pi c}\sum_{n=1}^{\infty}\frac{1}{2n-1}\sin\frac{(2n-1)\pi c(t-T)}{2L}\times\sin\frac{(2n-1)\pi X}{2L}\sin\frac{(2n-1)\pi x}{L}$$

7. (b) $$\frac{2F_0L}{\rho\pi^2c^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left(1-\cos\frac{n\pi ct}{L}\right)\sin\frac{n\pi x_0}{2L}\sin\frac{n\pi x}{L}$$
(c) $$\frac{2F_0L}{\rho\pi^2c^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\left(1-\cos\frac{(2n-1)\pi ct}{L}\right)\times\sin\frac{(2n-1)\pi x}{L}$$
(d) Same answer as for (b), but with the $n=m$ term absent

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