APPLIED PARTIAL DIFFERENTIAL EQUATIONS

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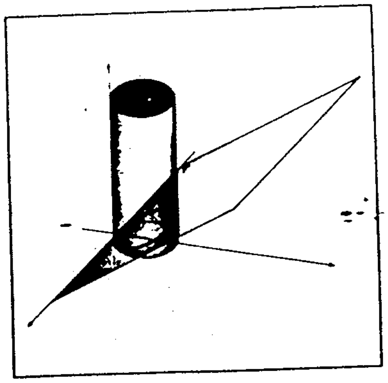

Preface

This text evolved, as have so many others, from notes used to teach partial differential equations to advanced undergraduate mathematics students and graduate engineering students. Major emphasis is placed on techniques for solving partial differential equations found in physics and engineering, but discussions on existence and uniqueness of solutions are also included. Every opportunity is taken to show that there may be more than one way to solve a particular problem and to discuss the advantages of each solution relative to the others. In addition, physical interpretations of mathematical solutions are stressed whenever possible.

Section 1.1 introduces partial differential equations and describes how initial boundary value problems are associated with such equations. To distinguish between physical assumptions leading to the various models of heat conduction, vibration, potential problems, and so forth, and the mathematical techniques used to solve these problems, models are developed in Sections 1.2–1.6, with no attempt at solutions. At this stage, the reader concentrates only on how mathematics describes physical phenomena. Once these ideas are firmly entrenched, it is then reasonable to proceed to various solution techniques. It has been our experience that confusion often arises when new mathematical techniques are prematurely applied to unfamiliar problems.

One of the most fundamental classical techniques for solving partial differentia equations is that of separation of variables, which leads, in the simplest of examples, t trigonometric Fourier series. Chapter 2 develops the theory of Fourier series to th point where it is easily accessible to separation of variables in Chapter 3. Eigenfunctio expansions are used to handle nonhomogeneities in this chapter. The examples i Chapter 3 also suggest the possibility of expansions other than trigonometric Fourie series, and these are discussed in detail through Sturm-Liouville systems in Chapter 4

The reader can proceed in a variety of ways through Chapters 5–9. One obviou way is to follow the order of topics as presented. This begins in Chapter 5, wit separation of variables on homogeneous problems that are more difficult than thos encountered in Chapter 3. In this chapter we also illustrate how to verify serie solutions of initial boundary value problems, and we discuss distinguishing propertie of parabolic, elliptic, and hyperbolic partial differential equations. In Chapter 6, finit Fourier transforms are presented as an alternative to eigenfunction expansions fo nonhomogeneous problems. In Chapter 7 we discuss homogeneous and nonhomc geneous problems on unbounded domains using separation of variables, Fourie integrals, and Fourier transforms. Chapters 8 and 9 essentially repeat material i Chapters 4, 5, 6, and 7, but in polar, cylindrical, and spherical coordinates.

For those who prefer to study bounded domain problems in polar, cylindrical, an spherical coordinates before considering problems on unbounded domains, we sugge one of three reorderings of sections in Chapters 5–9:

Chapter 8	Chapter 5	Chapter 5
Chapter 5 and Section 9.1	Chapter 6	Chapter 8
Chapter 6 and Section 9.2	Chapter 8	Section 9.1
Chapter 7 and Section 9.3	Sections 9.1 and 9.2	Chapter 6 and Section 9
	Chapter 7 and Section 9.3	Chapter 7 and Section 9

To work through most sections of the book, students require a first course ordinary differential equations and an introduction to advanced calculus. Sectio 10.3–10.5, which deal with Laplace transform solutions of initial boundary valu problems, assume a working knowledge of complex variable theory. This chapter ca also be adapted to the above schemes. Sections 10.1–10.4 can be covered at any tin after Chapter 5. Section 10.5 requires material from Chapter 8.

Green's functions for ordinary and partial differential equations are discussed Chapters 11 and 12. Green's functions for ordinary differential equations can l studied at any time. Chapter 12 utilizes separation techniques from Chapter 5 ar Section 9.1.

We are of the opinion that exercises are of the utmost importance to a student learning. There must be straightforward problems to reinforce fundamentals and mo difficult problems to challenge enterprising students. We have attempted to provic more than enough of each type. Problems in each set of exercises are graded from eas to difficult, and answers to selected exercises are provided at the back of the boo Exercise sets in 16 sections (3.2, 3.3; 5.2, 5.3, 5.4; 6.2, 6.3; 7.2, 7.4; 9.1, 9.2, 9.3; 10.2, 10. 10.5; 11.4) stress applications. They have been divided into four parts:

Part A—Heat Conduction

Part B—Vibrations

Part C—Potential, Steady-State Heat Conduction, Static Deflections of Membranes

Part D—General Results

Students interested in heat conduction should concentrate on problems from Part A. Students interested in mechanical vibrations will find problems in Part B particularly appropriate. All students can profit from problems in Part C, since every problem therein, although stated in terms of one of the three applications, is easily interpretable in terms of the other two. We recommend the exercises in Part D to all students.

A student supplement containing solutions to many of the exercises is available from the author.

The author wishes to acknowledge the students who provided initial motivation for writing this book and the students who suffered through its many revisions. Appreciation is also expressed to the reviewers, who made many valuable suggestions:

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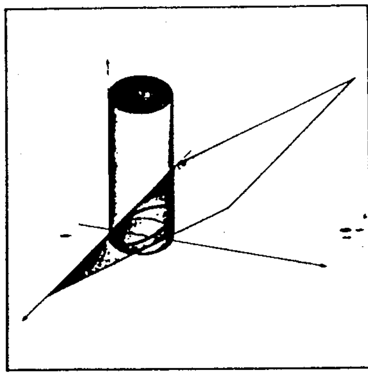
Finally, many thanks to the staff of PWS-KENT for their cooperation throughout the duration of this project.

D.W.T.

In memory of my father. Not many words passed between us, but we knew.



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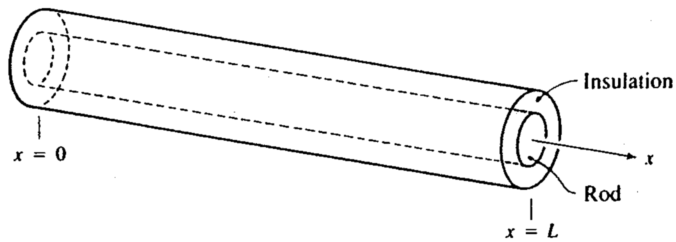


Figure 1.1

Other PDEs we shall consider include the one-dimensional wave equation for displacement $y(x, t)$ of vibrating strings,

$$\frac{\partial^2 y}{\partial t^2} = \frac{\tau}{\rho} \frac{\partial^2 y}{\partial x^2} \tag{2}$$

(Section 1.3); the three-dimensional Poisson's equation for potential $V(x, y, z)$,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = F(x, y, z) \tag{3}$$

(Section 1.6); and the beam-vibration equation for displacement $y(x, t)$,

$$\frac{w}{g} \frac{\partial^2 y}{\partial t^2} + EI \frac{\partial^4 y}{\partial x^4} = F(x, t) \tag{4}$$

(Section 1.5).

The *order* of a PDE is the highest-order partial derivative contained therein. Equations (1a), (2), and (3) are therefore second order, and equation (4) is fourth order.

Since partial derivatives of multivariable functions are ordinary derivatives with respect to one variable (the others being held constant), it might seem that the study of partial differential equations should be an easy extension of the theory for ordinary differential equations (ODEs). Such is not the case. Partial differential equations and ordinary differential equations are approached in fundamentally different ways. To understand why, recall that in your studies of ODEs it was customary to solve a certain class of equations and thereafter to deal with applications involving equations in this class. For example, a general solution of the second-order linear ODE

$$p \frac{d^2 y}{dt^2} + q \frac{dy}{dt} + ry = 0$$

is $y(t) = Ay_1(t) + By_2(t)$, where A and B are arbitrary constants and $y_1(t)$ and $y_2(t)$ are any two linearly independent solutions of the equation. Once $y_1(t)$ and $y_2(t)$ are known, every solution of the equation is of the form $Ay_1(t) + By_2(t)$ for some A and B. When such an equation is found in an application, say a vibrating mass–spring system or an LCR circuit, it is accompanied by two initial conditions that the solution $y(t)$ must satisfy. These conditions determine the values for A and B. What we are saying is that in applications, ODEs are often solved by first finding general solutions and then using subsidiary conditions to determine arbitrary constants.

It is very unusual to approach PDEs in this way, principally because arbitrary constants in general solutions of ODEs are replaced by arbitrary functions in PDEs, and determination of these arbitrary functions using subsidiary conditions is usually impossible. In other words, general solutions of PDEs are of limited utility in solving applied PDEs. [The one major exception is wave equation (2), and this particular situation is discussed in Section 1.7.] In general, then, it is necessary to consider a PDE and any extra conditions that accompany the equation simultaneously. We must proceed directly to a solution of the PDE and subsidiary conditions, as opposed to PDE first and subsidiary conditions later.

Subsidiary conditions that accompany PDEs are called initial or boundary conditions. For example, it is clear that the temperature function $U(x,t)$ for the rod in Figure 1.1 must also satisfy the *boundary conditions*

$$U(0,t) = 100, \tag{1b}$$

$$U(L,t) = 100, \tag{1c}$$

since the ends of the rod, $x = 0$ and $x = L$, are held at temperature 100°C. In addition, $U(x,t)$ must satisfy the *initial condition*

$$U(x,0) = 10, \tag{1d}$$

since its temperature at time $t = 0$ was 10°C throughout.

Partial differential equation (1a), boundary conditions (1b, c), and initial condition (1d) constitute the complete *initial boundary value problem* for temperature in the rod. It is more precise to describe the problem as follows:

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{5a}$$

$$U(0,t) = 100, \qquad t > 0, \tag{5b}$$

$$U(L,t) = 100, \qquad t > 0, \tag{5c}$$

$$U(x,0) = 10, \qquad 0 < x < L. \tag{5d}$$

All that we have done is affix intervals on which conditions (1) must be satisfied, but, perhaps unexpectedly, these intervals are all open. To see why this is the case, consider first PDE (5a). Physically, $U(x,t)$ is a function of one space variable x and the time variable t, but mathematically, it is simply a function of two independent variables x and t. It must satisfy PDE (5a) in some region of the xt-plane, and we take this region to be described by the inequalities $0 < x < L$ and $t > 0$ (Figure 1.2). By keeping these intervals open, we avoid discussing the PDE on the boundary of the region. Otherwise

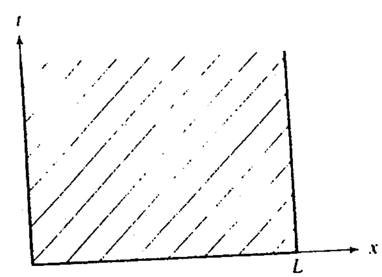


Figure 1.2

it would be necessary to consider one-sided derivatives with respect to x along $x = 0$ and $x = L$, one-sided derivatives with respect to t along $t = 0$, and both types of one-sided derivatives at $(0, 0)$ and $(L, 0)$. We take as a general principle that partial differential equations are always considered on open† regions.

Replacement of $t > 0$ and $0 < x < L$ in (5b–d) with $t \geq 0$ and $0 \leq x \leq L$ would lead to contradictions. Conditions (5b, c) would then require $U(x, t)$ to have values $U(0, 0) = U(L, 0) = 100$, whereas (5d) would demand that $U(0, 0) = U(L, 0) = 10$. By imposing boundary and initial conditions on open intervals, we eliminate such mathematical contradictions. Realize, however, that although (5) contains no mathematical contradictions, it is physically impossible to change the temperature of the ends of the rod instantaneously from 10°C to 100°C, and yet (5) does demand this. We must therefore anticipate some type of anomaly in the solution to (5) near positions $x = 0$ and $x = L$ at times close to $t = 0$.

It is not always necessary to use open intervals for boundary and initial conditions. If the initial temperature in the rod were not constant but varied with x according to, say, $f(x) = 400x(L - x) + 100$, it would not be necessary to heat the ends of the rod suddenly to 100°C at time $t = 0$; they would already be at that temperature, since $f(0) = f(L) = 100$. It would be necessary only to maintain them at 100°C thereafter. In this case, it would be quite acceptable to replace the open intervals in (5b–d) with

$$U(0, t) = 100, \qquad t \geq 0,$$
$$U(L, t) = 100, \qquad t \geq 0,$$
$$U(x, 0) = 400x(L - x) + 100, \qquad 0 \leq x \leq L.$$

It will be our practice to state initial and boundary conditions on open intervals even when closed intervals are acceptable.

Example 1: The ends of a violin string of length L are fixed on the x-axis at positions $x = 0$ and $x = L$. When the middle of the string is elevated to the position in Figure 1.3 and then released from rest (at time $t = 0$), subsequent displacements of particles of the string must satisfy PDE (2), where τ is the tension in the string and ρ is its linear density. What are the boundary and initial conditions for $y(x, t)$?

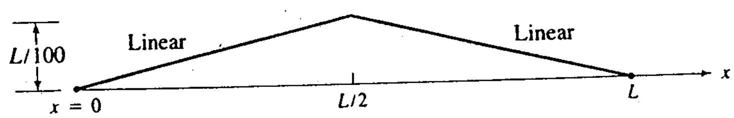


Figure 1.3

Solution: Since the ends of the string are fixed on the x-axis, boundary conditions are

$$y(0, t) = 0, \qquad t > 0$$

at $x = 0$ and

$$y(L, t) = 0, \qquad t > 0$$

at $x = L$.

† A region of the xy-plane is said to be open if about every point in the region there can be drawn a circle such that its interior contains only points of the region. A region in space is open if about every point in the region there can be drawn a sphere such that its interior contains only points of the region.

Because the string has the position shown in Figure 1.3 at time $t = 0$, $y(x, t)$ must satisfy the initial condition

$$y(x,0) = \begin{cases} x/50 & 0 < x < L/2 \\ (L-x)/50 & L/2 < x < L \end{cases}.$$

In addition, the fact that the string is released from rest indicates that its velocity at time $t = 0$ is equal to zero. Since velocity is the time rate of change of displacement, the second initial condition is

$$\frac{\partial y(x,0)}{\partial t} = 0, \qquad 0 < x < L.$$

There would be no conflict in replacing each of the open intervals in these four conditions with closed intervals. ■

In problem (5), boundary conditions (5b, c) specify the temperature of the rod at its ends, $x = 0$ and $x = L$. Likewise, in Example 1, the boundary conditions specify the displacement of the string at its ends. These are examples of what are called *Dirichlet* boundary conditions. A Dirichlet boundary condition specifies the value of the unknown function on a physical boundary. As another example, consider the two-dimensional version of Poisson's equation (3),

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x, y), \qquad (x, y) \text{ in } R, \tag{6a}$$

for the region R in Figure 1.4. [$F(x, y)$ is a given function.] In compliance with our previous remarks, R is the open region consisting of all points interior to the bounding curve $\beta(R)$ but not including $\beta(R)$ itself. A Dirichlet boundary condition specifies the value for $V(x, y)$ on $\beta(R)$:

$$V(x, y) = G(x, y), \qquad (x, y) \text{ on } \beta(R), \tag{6b}$$

$G(x, y)$ some given function. Poisson's equation (6a) together with boundary condition (6b) is called a *boundary value problem*.

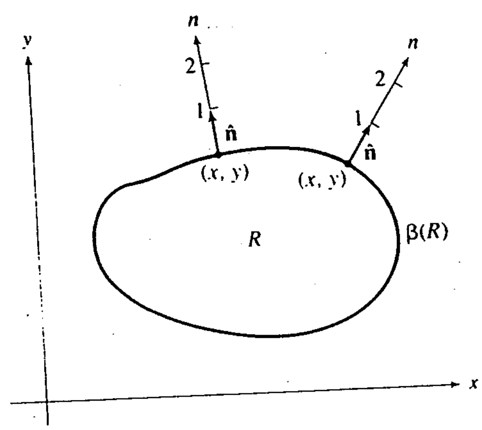


Figure 1.4

Two other types of boundary conditions arise frequently in applications—Neumann and Robin. A *Neumann* boundary condition for equation (6a) specifies the rate of change of $V(x, y)$ at points on $\beta(R)$ in a direction outwardly normal (perpendicular) to $\beta(R)$. We express this in the form

$$\frac{\partial V}{\partial n} = G(x, y), \qquad (x, y) \text{ on } \beta(R), \tag{7a}$$

where n is understood to be a measure of distance at (x, y) in a direction perpendicular to $\beta(R)$ (Figure 1.4). Because $\partial V/\partial n$ is in reality the directional derivative of V along the outward normal to $\beta(R)$, (7a) may be expressed in the equivalent form

$$\nabla V \cdot \hat{\mathbf{n}} = G(x, y), \qquad (x, y) \text{ on } \beta(R), \tag{7b}$$

where ∇V is the gradient of V at (x, y) and $\hat{\mathbf{n}}$† is the unit outward normal vector to $\beta(R)$ at (x, y).

A *Robin* boundary condition is a linear combination of a Dirichlet and a Neumann condition. For equation (6a), it takes the form

$$l\frac{\partial V}{\partial n} + hV = G(x, y), \qquad (x, y) \text{ on } \beta(R), \tag{8a}$$

where l and h are nonzero constants. What is important is not the individual values of l and h but their ratio, l/h or h/l; division of (8a) by l or h leads to boundary conditions

$$\frac{\partial V}{\partial n} + \left(\frac{h}{l}\right)V = \frac{G(x, y)}{l}, \qquad (x, y) \text{ on } \beta(R) \tag{8b}$$

and

$$\left(\frac{l}{h}\right)\frac{\partial V}{\partial n} + V = \frac{G(x, y)}{h}, \qquad (x, y) \text{ on } \beta(R), \tag{8c}$$

both of which are equivalent to (8a). The advantage of (8a), however, is that solutions of problems with Dirichlet and Neumann boundary conditions are easily obtained from those with Robin conditions by specifying $l = 0$, $h = 1$ and $h = 0$, $l = 1$, respectively. Boundary conditions (6b), (7), and (8) are said to be *homogeneous* if $G(x, y) \equiv 0$; otherwise, they are said to be *nonhomogeneous*. Physical interpretations of Neumann and Robin boundary conditions are discussed in Sections 1.2–1.7.

Example 2: What form do Robin boundary conditions take for the heat conduction problem described by equations (5a–d)?

Solution: At the end $x = L$ of the rod, the outward normal is in the positive x-direction. Consequently, at $x = L$, $\partial U/\partial n = \partial U/\partial x$, and the general Robin boundary condition there is

$$l_2\frac{\partial U(L, t)}{\partial x} + h_2 U(L, t) = G_2(t), \qquad t > 0.$$

† A "^" over a vector indicates that the vector is of unit length.

Because the outward normal at $x = 0$ is in the negative x-direction, it follows that $\partial U(0,t)/\partial n = -\partial U(0,t)/\partial x$, and a Robin boundary condition there takes the form

$$-l_1 \frac{\partial U(0,t)}{\partial x} + h_1 U(0,t) = G_1(t), \qquad t > 0.$$

■

In order that an (initial) boundary value problem adequately represent a physical situation, its solution should have certain properties. First, there should be a solution to the (initial) boundary value problem. Second, this solution should be unique; that is, the problem should not have more than one solution. For example, if problem (5) had more than one solution, how could it possibly be an accurate description of the temperature in the rod? Solutions should also have one further property, which we explain through problem (6). The solution of this problem depends on the functions $F(x, y)$ and $G(x, y)$. In practice, these quantities may not be known exactly; they may, for instance, be obtained from physical measurements. It would be reasonable to expect that small changes in either $F(x, y)$ or $G(x, y)$ should not appreciably affect $V(x, y)$. These three conditions lead to what is called a "well-posed" problem. An (initial) boundary value problem is said to be *well posed* if

(1) it has a solution;

(2) the solution is unique;

(3) the solution depends continuously on source terms and initial and boundary data (i.e., small changes in source terms and initial and boundary data produce small changes in the solution).

All stable physical situations should be modeled by well-posed problems.

In this book we discuss only existence and uniqueness of solutions; continuous dependence of solutions on source terms and subsidiary data is beyond our scope. Existence of solutions can be approached in two ways. One might be interested in knowing whether a particular initial boundary value problem has a solution but might not be at all interested in what the solution is. This is "existence" in its purest sense. Our approach is more pragmatic. We discuss different ways to solve (initial) boundary value problems, and if one of these methods succeeds in giving a solution to a problem, then clearly "existence" of a solution has been established. It is important to know that a problem has only one solution, however, since then, and only then, may we conclude that once *a* solution has been found, it must be *the* solution to the problem. Uniqueness is discussed in Sections 5.6–5.8.

In Sections 1.2–1.6 we derive partial differential equations that arise in physics and engineering. Each section is self-contained and may therefore be read independently of the others. This means that readers interested in heat conduction could study Section 1.2 and omit Section 1.3–1.6 without fear of missing any general ideas concerning PDEs. Likewise, readers interested in mechanical vibrations could omit Section 1.2 and 1.6 and concentrate on Sections 1.3–1.5.

Arising in many of these applications is the "Laplacian" of a function. The Laplacian of a function $V(x, y)$ of Cartesian coordinates x and y is defined as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} \qquad \text{(9a)}$$

and, if $V(x, y, z)$ is a function of three variables, as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \tag{9b}$$

When a function is expressed in polar, cylindrical, or spherical coordinates, its Laplacian is more complicated to calculate. We list the formulas here, leaving verification to Exercises 9 and 10. In polar coordinates (r, θ) (Figure 1.5),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}; \tag{10a}$$

in cylindrical coordinates (r, θ, z),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}; \tag{10b}$$

and in spherical coordinates (r, θ, ϕ) (Figure 1.6),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial V}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}. \tag{10c}$$

The PDE obtained by setting the Laplacian of a function equal to zero,

$$\nabla^2 V = 0, \tag{11}$$

is called *Laplace's equation.*

Figure 1.5

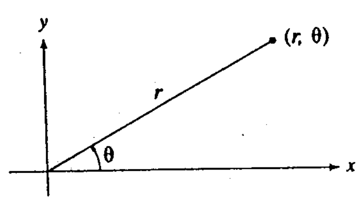
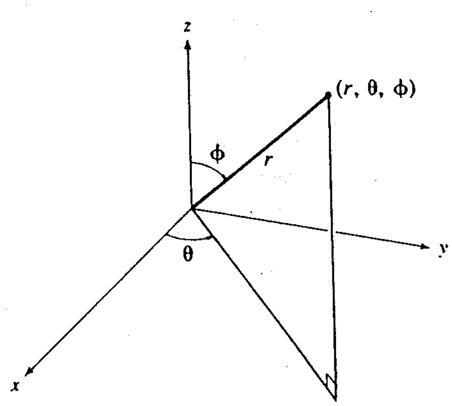


Figure 1.6



Exercises 1.1

On the regions in Exercises 1–7 what form do Dirichlet, Neumann, and Robin boundary conditions take for the PDE?

1. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = F(x, y), \quad 0 < x < L, \quad 0 < y < L'$

2. $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} + \dfrac{\partial^2 V}{\partial z^2} + V = F(x, y, z), \quad 0 < x < L, \quad y > 0, \quad z > 0$

3. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = F(r, \theta), \quad 0 < r < r_0, \quad -\pi < \theta \leq \pi, \quad (r, \theta)$ polar coordinates

4. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} = F(r, \theta), \quad 0 < r < r_0, \quad 0 < \theta < \pi$

5. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 V}{\partial \theta^2} + \dfrac{\partial^2 V}{\partial z^2} = F(r, \theta, z), \quad 0 < r < r_0, \quad -\pi < \theta \leq \pi, \quad z > 0, \quad (r, \theta, z)$ cylindrical coordinates

6. $\dfrac{\partial^2 V}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial V}{\partial r} + \dfrac{1}{r^2 \sin\phi}\dfrac{\partial}{\partial \phi}\left(\sin\phi \dfrac{\partial V}{\partial \phi}\right) + \dfrac{1}{r^2 \sin^2\phi}\dfrac{\partial^2 V}{\partial \theta^2} = F(r, \theta, \phi), \quad 0 < r < r_0,$ $-\pi < \theta \leq \pi, \quad 0 < \phi < \pi, \quad (r, \theta, \phi)$ spherical coordinates

7. Use the same PDE as in Exercise 6, but on the region

$$0 < r < r_0, \qquad -\pi < \theta \leq \pi, \qquad 0 < \phi < \frac{\pi}{2}.$$

8. When a boundary value problem (but *not* an initial boundary value problem) has a Neumann boundary condition on all parts of its boundary, it must satisfy a consistency condition. In this exercise, we derive this condition for two- and three-dimensional problems.

   (a) Consider the two-dimensional boundary value problem consisting of Poisson's equation (6a) and Neumann boundary condition (7a). Use Green's theorem in the plane to show that

$$\oint_{\beta(R)} G(x, y)\, ds = \iint_R F(x, y)\, dA. \qquad \textbf{(12a)}$$

   (Green's theorem is stated in Appendix C.) The left side of this equation is the line integral of $G(x, y)$ around the bounding curve $\beta(R)$, and the right side is the double integral of $F(x, y)$ over R. Thus, the "source term" $F(x, y)$ in (6a) and the boundary data $G(x, y)$ in (7a) cannot be specified independently; they must satisfy consistency condition (12a). Physical interpretations of this condition will be given later (see, for example, Exercise 23 in Section 1.2).

   (b) Show that the analog of (12a) for the three-dimensional boundary value problem

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = F(x, y, z), \qquad (x, y, z) \text{ in } V,$$

$$\frac{\partial V}{\partial n} = G(x, y, z), \qquad (x, y, z) \text{ on } \beta(V)$$

   is

$$\iint_{\beta(V)} G(x, y, z)\, dS = \iiint_V F(x, y, z)\, dV. \qquad \textbf{(12b)}$$

   (You will need the divergence theorem from Appendix C.)

9. In this exercise we verify expression (10a) for the Laplacian in polar coordinates. Formula (10b) is then obvious.
   (a) Verify that when a function $V(x, y)$ is expressed in polar coordinates r and θ, its Cartesian derivatives $\partial V/\partial x$ and $\partial V/\partial y$ may be calculated according to
$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial x}, \qquad \frac{\partial V}{\partial y} = \frac{\partial V}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial V}{\partial \theta}\frac{\partial \theta}{\partial y}.$$
   (b) Obtain formulas for $\partial r/\partial x$, $\partial r/\partial y$, $\partial \theta/\partial x$, and $\partial \theta/\partial y$ from the relations $x = r\cos\theta$ and $y = r\sin\theta$ between polar and Cartesian coordinates, and use them to show that
$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial \theta}, \qquad \frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial \theta}.$$
   (c) Use the results in (b) to calculate the following expressions for second partial derivatives of V with respect to x and y:
$$\frac{\partial^2 V}{\partial x^2} = \cos^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} - \frac{2\sin\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial \theta},$$
$$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial \theta^2} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial V}{\partial \theta} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\,\partial \theta}.$$
   (d) Finally, add the results in (c) to obtain (10a).

10. Use the technique of Exercise 9 to obtain (10c).

1.2 Heat Conduction

In this section we develop the mathematics necessary to describe conductive heat flow in various physical media—rods, plates, and three-dimensional bodies. We could begin with one-dimensional flow, such as that in the rod of Figure 1.1, and generalize later to plates and volumes. Alternatively, we could begin with three-dimensional heat flow and specialize later to plates and rods. We find the latter approach more satisfactory; it does not require special physical apparatus to ensure heat flow in only one or two directions. Furthermore, the mathematical and physical quantities that describe heat flow have units that are more natural in a three-dimensional setting.

When we consider temperature at various points in some body (say the human body), seldom is it constant; temperature normally varies from point to point and changes with time. Experience has shown that when temperature does vary, heat flows by conduction. Heat can flow by other means as well, namely by convection and by radiation. Heat received by the earth from the sun is due to *radiation*. We do not consider heat transfer by radiation in this book. The engine of a car illustrates the difference between convective and conductive heat flow. In order to keep the engine cool, water carries heat from the engine to the radiator through hoses; it is the motion of the water that transfers heat from engine to radiator. This is called *convective* heat transfer. Heat will also pass through the walls of the engine to be dissipated into the air. The process by which heat is moved from molecule to molecule in the engine wall is called heat transfer by *conduction*; it is due to vibrations of molecules, the vibrations increasing with higher and higher temperatures. In this book we discuss only heat transfer by conduction. To describe conductive heat flow in a medium, and ultimately obtain a PDE that determines temperature in the medium, we introduce the heat flux vector:

The heat flux vector q(r, *t*) is a vector function of position r and time *t*. Its direction corresponds to the direction of heat flow at position r and time *t*, and its magnitude is equal to the amount of heat per unit time crossing unit area normal to the direction of q.

This vector, which has units of watts per square meter (W/m^2), is defined at every point in a conducting medium except possibly at sources or sinks of heat (Figure 1.7).

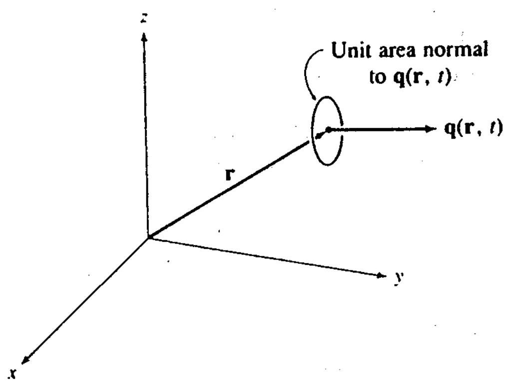


Figure 1.7

A medium is said to be *isotropic* if, when any point within it is heated, heat spreads out equally in all directions. In other words, isotropic media have no preferred directions for heat flow. It has been shown experimentally that in an isotropic medium, heat flows in the direction in which temperature decreases most rapidly, and the amount of heat flowing in that direction is proportional to the rate of change of temperature in that direction. This is called *Fourier's law of heat conduction*. Mathematically, if $U(\mathbf{r}, t)$ is the temperature distribution in the medium, then its gradient, ∇U, points in the direction in which the function U increases most rapidly and $|\nabla U|$ is the maximum rate of increase. Consequently, Fourier's law of heat conduction in an isotropic medium can be stated vectorially as

$$\mathbf{q}(\mathbf{r}, t) = -\kappa \nabla U; \tag{13}$$

where $\kappa > 0$ is the "constant" of proportionality called the *thermal conductivity* of the medium. It has units of watts per meter per degree Kelvin or Celsius (W/mK). In general, thermal conductivity may depend both on the temperature of and the position in the medium. If, however, the range of temperature is "limited" (and we shall consider only this case), the variation of κ with temperature is negligible and κ becomes a

Table 1.1

Thermal Properties of Some Materials

Material	Density (kg/m^3)	Specific Heat (Ws/kgK)	Thermal Conductivity (W/mK) at 273 K	Thermal Diffusivity (m^2/s)
Copper	8950	381	390	114×10^{-6}
Mild steel	7884	460	45	12.4×10^{-6}
Pyrex glass	2413	837	1.18	0.584×10^{-6}
Water	1000	1000	0.600	0.600×10^{-6}
Asbestos	579	1047	0.15	0.247×10^{-6}

function of position only, $\kappa = \kappa(\mathbf{r})$. The medium is said to be *homogeneous* if κ is independent of position, in which case κ becomes a numerical constant. Rough values for thermal conductivities of various homogeneous materials are given in Table 1.1. The larger the value of κ, the more readily the material conducts heat. Other thermal properties are also included.

To obtain a PDE governing temperature in a medium, we consider an imaginary surface S bounding a portion of the medium of volume V (Figure 1.8). Heat is added to (or removed from) V in two ways—across S and by internal heat sources or sinks. When $g(\mathbf{r}, t)$ is the amount of heat generated (or removed) per unit time per unit volume at position $\mathbf{r}$ and time t, the total heat generation per unit time within V is expressed by the triple integral

$$\iiint_V g(\mathbf{r}, t)\, dV. \tag{14}$$

The amount of heat flowing into V through S per unit time is given by the surface integral on the left side of the equation

$$\iint_S \mathbf{q} \cdot (-\hat{\mathbf{n}})\, dS = \iint_S \kappa \nabla U \cdot \hat{\mathbf{n}}\, dS, \tag{15}$$

where $\hat{\mathbf{n}}$ is the unit outward-pointing normal to S. Equation (13) has been used to obtain the integral on the right. The total heat represented by (14) and (15) changes the temperature of points (x, y, z) in V by an amount $\partial U/\partial t$ in unit time. The heat requirement for this change is

$$\iiint_V \frac{\partial U}{\partial t} s\rho\, dV, \tag{16}$$

where ρ and s are the density and specific heat of the medium. (Specific heat is the amount of heat required to produce unit temperature change in unit mass.) Energy balance requires that (16) be equal to (14) plus (15):

$$\iiint_V \frac{\partial U}{\partial t} s\rho\, dV = \iiint_V g(\mathbf{r}, t)\, dV + \iint_S \kappa \nabla U \cdot \hat{\mathbf{n}}\, dS, \tag{17}$$

and when the divergence theorem (see Appendix C) is applied to the surface integral, the result is

$$\iiint_V \left(\rho s \frac{\partial U}{\partial t} - g(\mathbf{r}, t) - \nabla \cdot (\kappa \nabla U)\right) dV = 0. \tag{18}$$

For this integral to vanish for an arbitrary volume V, in particular for an arbitrarily small volume, the integrand must vanish at each point of V; that is, U must satisfy the PDE

$$\rho s \frac{\partial U}{\partial t} - g(\mathbf{r}, t) - \nabla \cdot (\kappa \nabla U) = 0. \tag{19}$$

In actual fact, this conclusion is correct only when we know that the integrand in (18) is a continuous function throughout V. When this is not the case, (19) may not be valid at every point of V. It will, however, be true in each subregion of V in which the integrand is continuous. Since (18) must be valid even when its integrand is discontinuous, it is a more general statement of energy balance than (19).

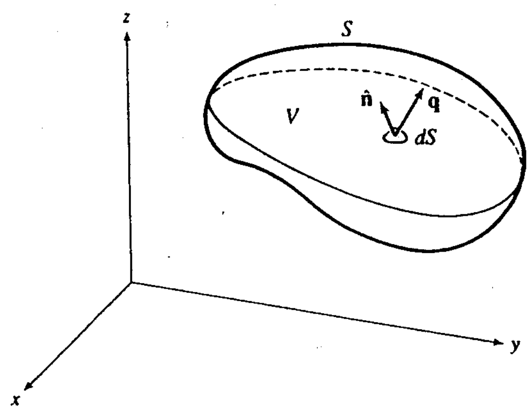


Figure 1.8

Equation (19) is the PDE for heat conduction in an isotropic medium. If the medium is also homogeneous, we define $k = \kappa/(s\rho)$ as the *thermal diffusivity* of the medium, in which case (19) reduces to

$$\boxed{\frac{\partial U}{\partial t} = k\left(\nabla^2 U + \frac{g(\mathbf{r}, t)}{\kappa}\right).} \tag{20}$$

The units of k are meters squared per second (m^2/s); typical values are given in Table 1.1.

Accompanying the PDE of heat conduction in any given problem will be initial and/or boundary conditions. An initial condition describes the temperature throughout the extent of the medium, R, at some initial time (usually $t = 0$):

$$U(\mathbf{r}, 0) = f(\mathbf{r}), \qquad \mathbf{r} \text{ in } R, \tag{21}$$

$f(\mathbf{r})$ some given function of position.

The three types of boundary conditions that we consider are those introduced in Section 1.1—Dirichlet, Neumann, and Robin. A Dirichlet condition prescribes temperature on the boundary $\beta(R)$ of R:

$$U = F(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0, \tag{22}$$

$F(\mathbf{r}, t)$ a given function.

Sometimes in applications we know that a certain amount of heat is being conducted across $\beta(R)$; that is, we know that the heat flux vector $\mathbf{q}$ on $\beta(R)$ is normal to $\beta(R)$, and its magnitude is specified (Figure 1.9). Suppose in this situation that we represent $\mathbf{q}$ on $\beta(R)$ by $\mathbf{q} = q(\mathbf{r}, t)\hat{\mathbf{n}}$, where $q(\mathbf{r}, t)$ is the component of $\mathbf{q}$ in direction $\hat{\mathbf{n}}$ (q is negative when heat is added to R and positive when heat is extracted). Fourier's law (13) on $\beta(R)$ yields

$$q\hat{\mathbf{n}} = -\kappa \nabla U, \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0, \tag{23}$$

and scalar products with $\hat{\mathbf{n}}$ give

$$\frac{\partial U}{\partial n} = -\frac{q(\mathbf{r}, t)}{\kappa}, \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0. \tag{24}$$

In other words, specification of heat flow across $\beta(R)$ leads to a Neumann boundary condition. In particular, if a bounding surface is insulated, the heat flux vector thereon vanishes and consequently that surface satisfies a homogeneous Neumann boundary condition

$$\frac{\partial U}{\partial n} = 0, \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0. \tag{25}$$

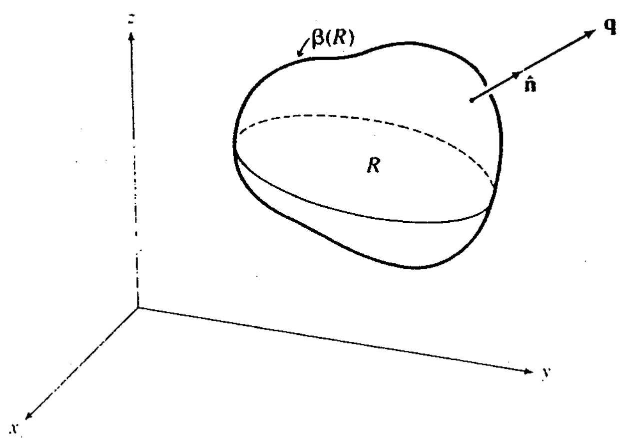


Figure 1.9

A Robin boundary condition is a linear combination of a Dirichlet and a Neumann condition:

$$l\frac{\partial U}{\partial n} + hU = F(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0. \tag{26}$$

Dirichlet and Neumann boundary conditions are obtained by setting l and h, respectively, equal to zero. To show that Robin boundary conditions are physically realistic, suppose the conducting material transfers heat to or from a surrounding medium according to Newton's law of cooling (heat transfer proportional to temperature difference). Then

$$-\kappa\frac{\partial U}{\partial n} = \mu(U - U_m), \qquad \mu > 0, \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0, \tag{27a}$$

where U_m is the temperature of the surrounding medium, or

$$\kappa\frac{\partial U}{\partial n} + \mu U = \mu U_m = F(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0. \tag{27b}$$

The constant of proportionality μ is called the *surface heat transfer coefficient*. This is clearly a Robin condition. Homogeneous Robin conditions

$$\kappa\frac{\partial U}{\partial n} + \mu U = 0, \qquad \mathbf{r} \text{ on } \beta(R), \qquad t > 0 \tag{28}$$

describe heat transfer according to Newton's law of cooling to media at temperature zero.

The initial boundary value problem of heat conduction in a homogeneous, isotropic medium can thus be stated as

$$\frac{\partial U}{\partial t} = k\left[\nabla^2 U + \frac{g(\mathbf{r}, t)}{\kappa}\right], \qquad \mathbf{r} \text{ in } R, \qquad t > 0, \tag{29a}$$

Boundary conditions, if applicable, (29b)

Initial condition $U(\mathbf{r}, 0) = f(\mathbf{r})$, $\mathbf{r}$ in R, if applicable. (29c)

If boundary conditions (29b) and heat sources $g(\mathbf{r}, t)$ in (29a) are independent of time, there may exist solutions of (29a, b) that are also independent of time. Such solutions are called *steady-state* solutions; they satisfy

$$\nabla^2 U = -\frac{g(\mathbf{r})}{\kappa}, \qquad \mathbf{r} \text{ in } R, \tag{30a}$$

Boundary conditions, if applicable. (30b)

For example, suppose a conducting sphere of radius a (Figure 1.10) has at time $t = 0$ some temperature distribution $f(r, \theta, \phi)$, where r, θ, and ϕ are the spherical coordinates shown in Figure 1.6. If the sphere is suddenly packed on the outside with perfect insulation, and no heat generation occurs within the sphere, the temperature distribution $U(r, \theta, \phi, t)$ thereafter must satisfy the initial boundary value problem

$$\frac{\partial U}{\partial t} = k\nabla^2 U, \qquad 0 < r < a, \qquad 0 < \phi < \pi, \qquad -\pi < \theta \leq \pi, \quad t > 0, \tag{31a}$$

$$\frac{\partial U(a, \theta, \phi, t)}{\partial r} = 0, \qquad 0 \leq \phi \leq \pi, \quad -\pi < \theta \leq \pi, \qquad t > 0, \tag{31b}$$

$$U(r, \theta, \phi, 0) = f(r, \theta, \phi), \quad 0 \leq r < a, \qquad 0 \leq \phi \leq \pi, \qquad -\pi < \theta \leq \pi. \tag{31c}$$

Steady-state solutions $U(r, \theta, \phi)$ for this problem, if there are any, must satisfy

$$\nabla^2 U = 0, \qquad 0 < r < a, \qquad 0 < \phi < \pi, \qquad -\pi < \theta \le \pi, \tag{32a}$$

$$\frac{\partial U(a, \theta, \phi)}{\partial r} = 0, \qquad 0 \le \phi \le \pi, \qquad \pi < \theta \le \pi. \tag{32b}$$

Obviously, a solution of (32) is $U = C$, C any constant whatsoever. Thus, constant functions are steady-state solutions for problem (31). We can realize the physical significance of steady-state solutions and determine a useful value for C if we return to initial boundary value problem (31). Physically it is clear that because no heat can enter or leave the sphere, heat will eventually redistribute itself until the temperature at every point in the sphere becomes the same constant value. [In Section 9.1 we prove that the value of this constant is the average value $\bar{U}$ of $f(r, \theta, \phi)$ over the sphere.] In other words, the useful steady-state solution will be $U = \bar{U}$. Later we shall see that the solution of (31) contains two parts. One is the steady-state (time-independent) part $U = \bar{U}$; the other is a transient (time-dependent) part that describes the transition from initial temperature $f(r, \theta, \phi)$ to final temperature $\bar{U}$.

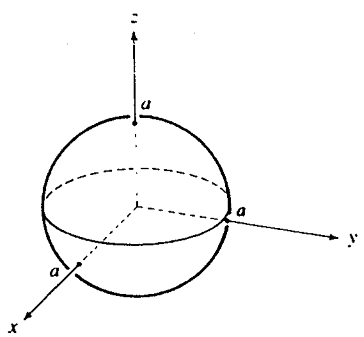


Figure 1.10

When $g(\mathbf{r})$ in Poisson's equation (30a) is identically zero (i.e., no internal heat generation occurs within R), the PDE reduces to Laplace's equation. Problem (30) then reads

$$\nabla^2 U = 0, \qquad \mathbf{r} \text{ in } R, \tag{33a}$$

$$\text{Boundary conditions, if applicable.} \tag{33b}$$

Problems (30) and (33) are called boundary value problems rather than initial boundary value problems, since no initial conditions are present.

Example 3: Formulate the initial boundary value problem for the temperature in a cylindrical rod with insulated sides and with flat ends at $x = 0$ and $x = L$. The end at $x = 0$ is kept at temperature 60°C; the end at $x = L$ is insulated; and at time $t = 0$ the temperature distribution throughout the rod is $f(x)$, $0 \le x \le L$. Assume no internal heat generation. Are there steady-state solutions for this problem?

Solution: Notwithstanding the fact that the rod is three-dimensional, we note that because all cross sections are identical, the sides are insulated, and the initial temperature distribution is a function of x alone, heat flows only in the x-direction. In other words, the heat conduction problem is one-dimensional, namely

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0,$$

$$U(0,t) = 60, \qquad t > 0,$$

$$\frac{\partial U(L,t)}{\partial x} = 0, \qquad t > 0,$$

$$U(x,0) = f(x), \qquad 0 < x < L.$$

Steady-state solutions $\psi(x)$ for this problem must satisfy

$$\frac{d^2\psi}{dx^2} = 0, \qquad 0 < x < L,$$

$$\psi(0) = 60, \qquad \psi'(L) = 0.$$

The general solution of this ODE is $\psi(x) = Ax + B$, and the boundary conditions require that

$$60 = B, \qquad 0 = A;$$

that is, $\psi(x) = 60$. After a very long time, the temperature in the rod will become 60°C throughout. ∎

Exercises 1.2

1. (a) A cylindrical, homogeneous, isotropic rod has flat ends at $x = 0$ and $x = L$ and insulated sides. Initially the temperature distribution in the rod is a function of x only, and heat generation at points x in the rod takes place uniformly over the cross section at x. Apply an energy balance to a segment of the rod from a fixed point $x = a$ to an arbitrary value x to show that the PDE governing temperature $U(x,t)$ in the rod is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} + \frac{k}{\kappa}g(x,t),$$

where $g(x,t)$ is the amount of heat per unit volume per unit time generated at position x and time t.

(b) What form do Robin boundary conditions take at $x = 0$ and $x = L$?

In Exercises 2–19, set up, but do not solve, an (initial) boundary value problem for the required temperature. Assume that the medium is isotropic and homogeneous.

2. A cylindrical rod has flat ends at $x = 0$ and $x = L$ and insulated sides. At time $t = 0$ its temperature is a function $f(x)$, $0 \leq x \leq L$, of x only. If both ends are kept at 100°C for $t > 0$, formulate the initial boundary value problem for temperature $U(x,t)$ in the rod for $0 < x < L$ and $t > 0$.
3. Repeat Exercise 2 except that the end at $x = 0$ is insulated.
4. Repeat Exercise 2 except that the temperature at end $x = L$ is changed from 0°C to 100°C at a constant rate over a period of T seconds and maintained at 100°C thereafter.

5. Repeat Exercise 2 except that heat is transferred according to Newton's law of cooling from the ends $x = 0$ and $x = L$ into media at temperatures U_0 and U_L, respectively.
6. Repeat Exercise 2 except that both ends are insulated and at each point in the rod heat is generated at a rate $g(x,t)$ per unit volume per unit time. What is $g(x,t)$ if heat generation is q calories per cubic centimeter per minute over that part of the rod between $x = L/4$ and $x = 3L/4$ and is zero otherwise?
7. Repeat Exercise 2 except that heat is added to the end $x = 0$ at a constant rate $Q_0 > 0$ W/m^2 uniformly over the end and is removed at a variable rate $Q_L(t) > 0$ W at $x = L$ uniformly over the end.
8. The top and bottom of a horizontal rectangular plate $0 \le x \le L$, $0 \le y \le L'$ are insulated. At time $t = 0$ its temperature is a function $f(x,y)$ of x and y only. If the edges $x = 0$ and $y = L'$ are kept at 50°C for $t > 0$ and the edges $y = 0$ and $x = L$ are insulated, formulate the initial boundary value problem for temperature $U(x,y,t)$ in the plate for $0 < x < L$, $0 < y < L'$, and $t > 0$.
9. Repeat Exercise 8 except that along $y = 0$ heat is transferred according to Newton's law of cooling into a medium with temperature $f_1(t)$, and heat is generated at a rate e^{at} joules per cubic meter per second at every point in the plate for the first T seconds.
10. The top and bottom of a horizontal circular plate $0 \le r \le r_0$, $-\pi < \theta \le \pi$ are insulated. At time $t = 0$ its temperature is a function $f(r,\theta)$ of polar coordinates r and θ only. For $t > 0$, heat is transferred along its edge according to Newton's law of cooling into a medium at temperature zero, and heat is generated at constant rate q W/m^3 inside the ring $0 < r_1 < r < r_2 < r_0$. Formulate the initial boundary value problem for temperature in the plate.
11. A right circular cylinder of length L and radius r_0 has its axis along the z-axis with flat faces in the planes $z = 0$ and $z = L$. At time $t = 0$ its temperature is a function $f(r,\theta)$ of r and θ only. For $t > 0$, faces $z = 0$ and $z = L$ are insulated, and $r = r_0$ is kept at temperature $f_1(\theta,t)$. Formulate the initial boundary value problem for temperature in the cylinder.
12. Repeat Exercise 11 except that $f(r,\theta)$ is replaced by $f(r,\theta,z)$.
13. Repeat Exercise 11 except that the ends $z = 0$ and $z = L$ are kept at 100°C for $t > 0$ and the cylindrical side is insulated.
14. Repeat Exercise 11 except that heat is transferred according to Newton's law of cooling from the top and cylindrical faces into air at temperature 20°C. Initially, temperature is a function $f(r)$ of r only.
15. Repeat Exercise 11 except that the initial temperature is a function $f(r)$ of r only and $r = r_0$ is kept at temperature $f_1(t)$.
16. The top and bottom of a horizontal semicircular plate $0 \le r \le r_0$, $0 \le \theta \le \pi$ are insulated. At time $t = 0$, its temperature is $f(r,\theta)$. For $t > 0$, the curved edge of the plate is insulated, but along the straight edge, heat is added at a constant rate $q > 0$ W/m^2. Formulate the initial boundary value problem for temperature in the plate.
17. Repeat Exercise 16 except that along $r = r_0$, heat is extracted at a constant rate $q > 0$ W/m^2 and along the straight edge, heat is exchanged according to Newton's law of cooling with an environment at constant temperature U_0.
18. A sphere of radius r_0 has an initial temperature ($t = 0$) of 100°C. If, for $t > 0$, it transfers heat according to Newton's law of cooling to an environment at constant temperature 10°C, what is the initial boundary value problem for temperature in the sphere?
19. A hemisphere of radius r_0 has its flat face in the xy-plane. The curved face of the sphere is insulated. If the heat flux vector on the face $z = 0$ is $\mathbf{q} = f(r,\theta)\hat{\mathbf{k}}$, formulate the boundary value

problem for steady-state temperature in the hemisphere. Can $f(r, \theta)$ be arbitrarily specified? [See Exercise 8(b) in Section 1.1.]

20. A homogeneous, isotropic rod with insulated sides has its ends $x = 0$ and $x = L$ held at temperatures U_0 and U_L, respectively. If no heat is generated in the rod, can there be a steady-state temperature distribution in the rod?

21. Heat is added (or removed) at the ends $x = 0$ and $x = L$ of a homogeneous, isotropic rod with insulated sides at constant rates q_0 and q_L, respectively. Can there be a steady-state temperature distribution in the rod?

22. Discuss each of the following statements for temperature in a homogeneous, isotropic rod with insulated sides:

(a) If temperature at points in the rod changes in time, heat must flow in the rod.

(b) If heat flows in the rod, temperature at points in the rod must change in time.

23. (a) Suppose there is a steady-state temperature distribution in a region R of the xy-plane that satisfies Poisson's equation $\nabla^2 U = -g(x, y)/\kappa$. Suppose further that the boundary condition on the boundary $\beta(R)$ of R is of Neumann type, $\partial U/\partial n = f(x, y)$ for (x, y) on $\beta(R)$. Use the result of Exercise 8 in Section 1.1 (or Green's theorem) to show that $f(x, y)$ and $g(x, y)$ must satisfy the consistency condition

$$\oint_{\beta(R)} f(x, y)\, ds = \iint_R -\frac{g(x, y)}{\kappa}\, dA.$$

What is the physical significance of this requirement?

(b) What is the three-dimensional analog of the result in (a)?

24. In Exercise 1 we developed the one-dimensional heat conduction equation based on energy balance for a small segment of the rod. In this exercise we use the PDE to discuss energy balance for the entire rod. Multiply the PDE in Exercise 1 by $A\kappa/k$ (A is the cross-sectional area of the rod), integrate with respect to x over the length $0 \le x \le L$ of the rod, and integrate with respect to t from $t = 0$ to an arbitrary value of t, to obtain the following result:

$$\int_0^L A\rho s U(x, t)\, dx - \int_0^L A\rho s U(x, 0)\, dx$$
$$= \int_0^t A\kappa \frac{\partial U(L, t)}{\partial x}\, dt - \int_0^t A\kappa \frac{\partial U(0, t)}{\partial x}\, dt + \int_0^t \int_0^L A g(x, t)\, dx\, dt.$$

Interpret each term in this equation physically, and hence deduce that the equation is a statement of energy balance for the rod.

25. Repeat Exercise 24 to obtain an energy balance for a volume V using PDE (20).

26. (a) The inside temperature of a flat wall is a constant U_{in} °C and the outside temperature is a constant U_{out} C. If the wall is considered as part of an infinite slab that is in a steady-state temperature situation, find an expression for the amount of heat lost through an area A of the wall per unit time. Is this expression inversely proportional to the thickness of the wall?

(b) Evaluate the result in (a) if A is 15 m², the thickness of the wall is 10 cm, the thermal conductivity of the material in the wall is 0.11 W/mK, $U_{out} = -20$°C, and $U_{in} = 20$°C.

27. (a) Steam is passed through a pipe with inner radius r_{in} and outer radius r_{out}. The temperature of the inner wall is a constant U_{in} C and that on the outer wall is a constant U_{out} °C. If the pipe is considered part of an infinitely long pipe that is in a steady-state temperature situation, find an expression for the amount of heat per unit area per unit time flowing radially outward.

(b) How much heat (per second) is lost at the outer surface of the pipe in a section 2 m long if $r_{in} = 3.75$ cm, $r_{out} = 5.0$ cm, $U_{in} = 205°$C, $U_{out} = 195°$C, and $\kappa = 54$ W/mK?

(c) Illustrate that the same amount of heat is transferred through the inner wall of the section. Must this be the case?

28. A homogeneous, isotropic rod with insulated sides has temperature $\sin(n\pi x/L)$, n a positive integer, at time $t = 0$. For time $t > 0$ its ends at $x = 0$ and $x = L$ are held at temperature 0°C.

(a) Find the initial boundary value problem for temperature in the rod and verify that a solution is

$$U(x, t) = e^{-n^2\pi^2kt/L^2}\sin\frac{n\pi x}{L}.$$

(b) Find the rate of heat flow across cross sections of the rod at $x = 0$, $x = L/2$, and $x = L$ by calculating

$$\lim_{x\to 0^+} q(x, t), \qquad q\left(\frac{L}{2}, t\right), \qquad \lim_{x\to L^-} q(x, t).$$

(c) Calculate limits of the heat flows in (b) as $t \to 0^+$ and $t \to \infty$.

29. (a) When two media with different thermal conductivities κ_1 and κ_2 are brought into intimate contact, heat flows from the hotter to the cooler medium. Assuming that heat transfer follows Newton's law of cooling, show that the following boundary conditions must be satisfied by the temperatures in the media at the interface:

$$-\kappa_1\frac{\partial U(0-)}{\partial n} = \mu[U(0-) - U(0+)],$$

$$\kappa_2\frac{\partial U(0+)}{\partial n} = \mu[U(0+) - U(0-)], \qquad -\kappa_1\frac{\partial U(0-)}{\partial n} = -\kappa_2\frac{\partial U(0+)}{\partial n},$$

where n is a coordinate perpendicular to the interface with positive direction from medium 1 into medium 2. Are these conditions independent?

(b) What do these conditions become in the event that μ is so high that there is essentially no resistance to heat flow across the interface?

30. (a) A homogeneous, isotropic sphere of radius R is heated uniformly from heat sources within at the rate of Q watts per cubic meter. Heat is transferred to a surrounding medium at constant temperature U_m according to Newton's law of cooling until a steady-state situation is achieved. Find the steady-state temperature distribution in the sphere.

(b) What is the initial boundary value problem for temperature in the sphere for $t > 0$ if the heat sources are turned off at time $t = 0$ and the steady-state situation has been achieved?

31. A thin wire of uniform cross section radiates heat from its sides (not ends) at a rate per unit area per unit time that is proportional to the difference between the temperature of the wire on its surface and that of its surroundings. It follows that variations in temperature should occur over cross sections of the wire. In many applications, these variations are sufficiently small that they may be considered negligible. In such a case, temperature at points in the wire is a function of time t and only one space variable along the wire, which we take as x, $U = U(x, t)$. Temperature problems of this type are called *thin-wire problems*. By considering heat flow into, and out of, the segment of the wire from a fixed point $x = a$ to an arbitrary x, show that the PDE for thin-wire problems is

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2} - h(U - U_m) + \frac{k}{\kappa}g(x,t),$$

where $h > 0$ is a constant and U_m is the temperature of the medium surrounding the wire.

32. Heat generation within a rod can be effected by passing an electric current along the length of the rod. Show that when the current is I,

$$g(x,t) = \frac{I^2}{A^2\sigma},$$

where σ is the electrical conductivity of the material of the rod and A is its cross-sectional area.

33. A cylindrical pipe of inner and outer radii a and b is sufficiently long that end effects may be neglected. The temperature of the inner wall is a constant U_a, and heat is transferred at the outer wall to a medium at constant temperature $U_m < U_a$ with surface heat transfer coefficient μ.
    (a) Find U as a function of r when the steady-state situation has been achieved.
    (b) Show that the amount of heat flowing radially through a unit length of the pipe at any radius $a < r < b$ is

$$\frac{2\pi\mu\kappa b(U_a - U_m)}{\kappa + \mu b\ln(b/a)}.$$

34. A long, straight wire of circular cross section has thermal conductivity κ and carries a current I. Surrounding the wire is insulation with thermal conductivity κ^*, $b - a$ units thick. If r is a radial coordinate measured from the center of the wire, the wire occupies the region $0 < r < a$, and the insulation, $a < r < b$. Heat transfer takes place at $r = b$ into a medium at constant temperature U_m with surface heat transfer coefficent μ^*. Find the steady-state temperature $U(r)$ in the wire and insulation under the assumption that $U(r)$ must be continuous at $r = a$. [*Hint:* See Exercise 32 for $g(r)$ and Exercise 29 for the additional boundary condition at the wire-insulation interface.

35. Repeat Exercise 34 except that continuity of $U(r)$ at $r = a$ is replaced by the condition that heat transfer from the wire to the insulation occurs according to Newton's law of cooling with surface heat transfer coefficient μ.

1.3 Transverse Vibrations of Strings; Longitudinal and Angular Vibrations of Bars

In this section we discuss three vibration problems that all give rise to the same mathematical representation.

Transverse Vibrations of Strings

A perfectly flexible string (such as, perhaps, a violin string) is stretched tightly between two fixed points $x = 0$ and $x = L$ on the x-axis (Figure 1.11). Suppose the string is somehow set into motion in the xy-plane (possibly by pulling vertically on the midpoint of the string and then releasing it). Our objective is to study the subsequent motion of the string. When the string is very taut and displacements are small, horizontal displacements of particles of the string are negligible compared with vertical displacements; that is, displacements may be taken as purely transverse, representable in the form $y(x, t)$.

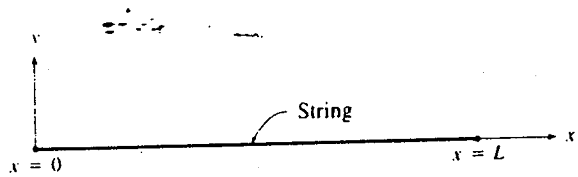


Figure 1.11

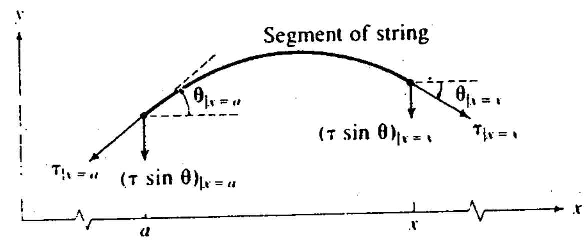


Figure 1.12

To find a PDE for $y(x, t)$, we analyze the forces on a segment of the string from a fixed position $x = a$ to an arbitrary position x (Figure 1.12). We denote by $\tau(x, t)$ the magnitude of the tension in the string at position x and time t. Because the string is perfectly flexible, tension in the string is always in the tangential direction of the string. This means that the y-component of the resulting force due to tension at the ends of the segment is $(\tau \sin\theta)_{|x=x} - (\tau \sin\theta)_{|x=a}$. We group all other forces acting on the segment into one function by letting $F(x, t)$ be the y-component of the sum of all external forces acting on the string per unit length in the x-direction. The total of all external forces acting on the segment then has y-component

$$\int_a^x F(\zeta, t)\, d\zeta.$$

Newton's second law states that the time rate of change of the momentum of the segment of the string must be equal to the resultant force thereon:

$$\frac{\partial}{\partial t}\left(\int_a^x \frac{\partial y(\zeta, t)}{\partial t}\rho(\zeta, t)\sqrt{1 + \left(\frac{\partial y(\zeta, t)}{\partial x}\right)^2}\, d\zeta\right)$$
$$= (\tau \sin\theta)_{|x=x} - (\tau \sin\theta)_{|x=a} + \int_a^x F(\zeta, t)\, d\zeta, \quad (34)$$

where $\rho(x, t)$ is the density of the string (mass per unit length). The quantity $\sqrt{1 + [\partial y(\zeta, t)/\partial x]^2}\, d\zeta$ is the length of string that projects onto a length $d\zeta$ along the x-axis. Multiplication by $\rho(\zeta, t)\partial y(\zeta, t)/\partial t$ gives the momentum of this infinitesimal length of string, and integration yields the momentum of that segment of the string from $x = a$ to an arbitrary position x. If we differentiate this equation with respect to x, we obtain

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial y}{\partial t}\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2}\right) = \frac{\partial}{\partial x}(\tau \sin\theta) + F(x,t). \tag{35}$$

When vibrations of the string are such that the slope of the displaced string, $\partial y/\partial x$, is very much less than unity (and this is the only case that we consider), the radical may be dropped from the equation and $\sin\theta$ approximated by $\tan\theta = \partial y/\partial x$. The resulting PDE for $y(x, t)$ is

$$\frac{\partial}{\partial t}\left(\rho \frac{\partial y}{\partial t}\right) = \frac{\partial}{\partial x}\left(\tau \frac{\partial y}{\partial x}\right) + F(x,t). \tag{36}$$

For most applications, both the density of and the tension in the string may be taken as constant, in which case (36) reduces to

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F}{\rho}, \qquad c^2 = \frac{\tau}{\rho}.} \tag{37}$$

This is the mathematical model for small transverse vibrations of a taut string; it is called the *one-dimensional wave equation*. In its derivation we have assumed that the slope of the string at every point is always very much less than 1 and that tension and density are constant.

When the only external force acting on the string is gravity, $F(x, t)$ takes the form

$$F = \rho g, \qquad g < 0. \tag{38}$$

Other possibilities include a damping force proportional to velocity,

$$F = -\beta \frac{\partial y}{\partial t}, \qquad \beta > 0; \tag{39}$$

and a restoring force proportional to displacement,

$$F = -ky, \qquad k > 0. \tag{40}$$

Accompanying the wave equation will be initial and/or boundary conditions. Initial conditions describe the displacement and velocity of the string at some initial time (usually $t = 0$):

$$y(x,0) = f(x), \qquad x \text{ in } I, \tag{41a}$$

$$\frac{\partial y(x,0)}{\partial t} = y_t(x,0)^{\dagger} = g(x), \qquad x \text{ in } I, \tag{41b}$$

where I is the interval over which the string is stretched. In Figure 1.11, I is $0 < x < L$, but other intervals are also possible. Interval I also dictates the number of boundary conditions. There are three possibilities, depending upon whether the string is of finite

† Subscripts are often used to denote partial derivatives. In (41b), y_t denotes $\partial y/\partial t$. In a similar way, we may use the notation y_{tt} in place of $\partial^2 y/\partial t^2$.

length, of "semi-infinite" length, or of "infinite" length. If the string is of finite length, the interval I is customarily taken as $0 < x < L$ and two boundary conditions result, one at each end. The string is said to be of semi-infinite length, or the problem is semi-infinite, if the string has only one end that satisfies some prescribed condition. The interval I in this case is always chosen as $0 < x < \infty$, and the one boundary condition is at $x = 0$. The string is said to be of infinite length, or the problem is infinite, if the string has no ends. In this case interval I becomes $-\infty < x < \infty$ and there are no boundary conditions.

It might be argued that there is no such thing as a semi-infinitely long or infinitely long string, and we must agree. There are, however, situations in which the model of a semi-infinite or infinite string is definitely advantageous. For example, suppose a fairly long string (with ends at $x = 0$ and $x = L$) is initially at rest along the x-axis. Suddenly, something disturbs the string at its midpoint, $x = L/2$ (perhaps it is struck by an object). The effect of this disturbance travels along the string in both directions toward $x = 0$ and $x = L$. Before the disturbance reaches $x = 0$ and $x = L$, the string reacts exactly as if it had no ends whatsoever. If we are interested only in these initial disturbances, and consideration of the "infinite" problem provides straightforward explanations, it is an advantage to analyze the "infinite" problem rather than the finite one.

We consider only three types of boundary conditions at an end of the string—Dirichlet, Neumann, and Robin. When the string has an end at $x = 0$, a Dirichlet boundary condition takes the form

$$y(0, t) = f_1(t), \qquad t > 0. \tag{42a}$$

It states that the end $x = 0$ of the string is caused by some external mechanism to perform the vertical motion described by $f_1(t)$. Similarly, if the string has an end at $x = L$, a Dirichlet condition

$$y(L, t) = f_2(t), \qquad t > 0 \tag{42b}$$

indicates that this end has a vertical displacement described by $f_2(t)$. For the string in Figure 1.11, $f_1(t) = f_2(t) = 0$.

Instead of prescribing the motion of the end $x = 0$ of the string, suppose that this end is attached to a mass m (Figure 1.13) and, furthermore, that motion of the mass is restricted to be vertical by a containing tube. The vertical component of the tension of the string acting on m at $x = 0$ is $\tau(0, t) \sin\theta$, which for small slopes can be approximated by

$$\tau(0, t) \sin\theta \approx \tau(0, t) \tan\theta = \tau(0, t)\frac{\partial y(0, t)}{\partial x}. \tag{43}$$

Consequently, when Newton's second law is applied to the motion of m,

$$m\frac{\partial^2 y(0, t)}{\partial t^2} = \tau(0, t)\frac{\partial y(0, t)}{\partial x} + f_1(t), \qquad t > 0, \tag{44}$$

where $f_1(t)$ represents the y-component of all other forces acting on m.

If m is sufficiently small that it may be taken as negligible (for instance, as with a very light loop around a vertical rod), this equation takes the form

$$\frac{\partial y(0,t)}{\partial x} = -\frac{1}{\tau(0,t)} f_1(t), \qquad t > 0, \tag{45}$$

a Neumann boundary condition. In particular, if the massless end of the string is free to slide vertically with no forces acting on it except tension in the string, it satisfies a homogeneous Neumann condition

$$\frac{\partial y(0,t)}{\partial x} = 0. \tag{46}$$

What this equation says is that when the end of a taut string is free of external forces, the slope of the string there will always be zero.

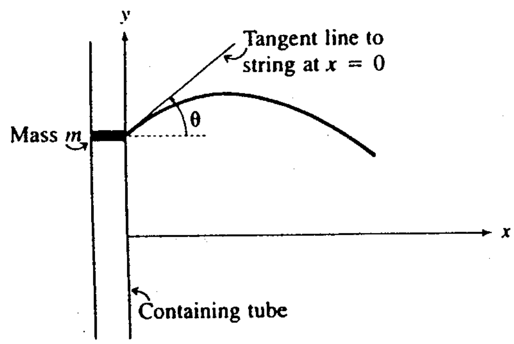


Figure 1.13

Similarly, if the string has a massless end at $x = L$ that is subjected to a vertical force with component $f_2(t)$, the boundary condition there is once again Neumann:

$$\frac{\partial y(L,t)}{\partial x} = \frac{1}{\tau(L,t)} f_2(t), \qquad t > 0. \tag{47}$$

What we have shown, then, is that Neumann boundary conditions result when the ends of the string, taken as massless, move vertically under the influence of forces that are specified as functions of time.

Robin boundary conditions, which are linear combinations of Dirichlet and Neumann conditions, arise when the ends of the string are attached to springs that are unstretched on the x-axis (Figure 1.14). When this is the case at $x = 0$, equation (44) becomes

$$m\frac{\partial^2 y(0,t)}{\partial t^2} = \tau(0,t)\frac{\partial y(0,t)}{\partial x} - ky(0,t) + f_1(t), \tag{48}$$

where $f_1(t)$ now represents all external forces acting on m other than the spring and tension in the string. For a massless end ($m = 0$) and constant tension τ, (48) takes the form

$$-\tau\frac{\partial y}{\partial x} + ky = f_1(t), \qquad x = 0, \qquad t > 0. \tag{49a}$$

Similarly, attaching the end $x = L$ to a spring gives the Robin condition

$$\tau\frac{\partial y}{\partial x} + ky = f_2(t), \qquad x = L, \qquad t > 0. \tag{49b}$$

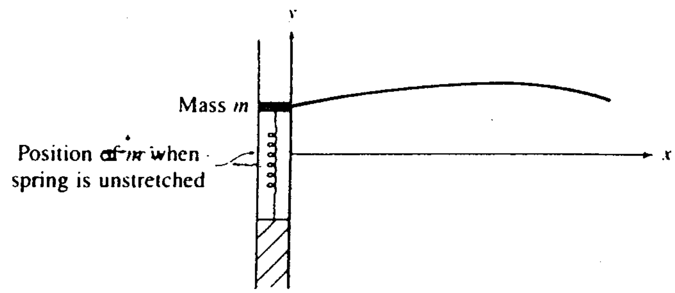


Figure 1.14

The initial boundary value problem for the vibrating string consists of the one-dimensional wave equation together with two initial conditions and/or zero, one, or two boundary conditions:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad x \text{ in } I, \qquad t > 0, \tag{50a}$$

Boundary conditions, if applicable, **(50b)**

$$y(x,0) = f(x), \qquad x \text{ in } I, \text{ if applicable}, \tag{50c}$$

$$y_t(x,0) = g(x), \qquad x \text{ in } I, \text{ if applicable}. \tag{50d}$$

When the boundary conditions and external force F are independent of time, there may exist solutions of (50a, b) that are also independent of time. Such solutions, called *static deflections*, satisfy the boundary value problem

$$\frac{d^2 y}{dx^2} = -\frac{F(x)}{\tau}, \qquad x \text{ in } I, \tag{51a}$$

Boundary conditions. **(51b)**

No vibrations occur; the string remains in static equilibrium under the forces present. We shall see that the solution of (50) divides into two parts: the static deflection part plus a second part that represents vibrations about the static solution.

Example 4: Formulate the initial boundary value problem for transverse vibrations of a string stretched tightly along the x-axis between $x = 0$ and $x = L$. The end $x = 0$ is free to move without friction along a vertical support, and the end $x = L$ is fixed on the x-axis. Initially, the string is released from rest at a position described by the function $f(x)$, $0 \le x \le L$. Take gravity into account. Are there static deflections for this problem?

Solution: The initial boundary value problem for displacements $y(x,t)$ of points in the string is

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - 9.81, \qquad 0 < x < L, \qquad t > 0,$$

$$\frac{\partial y(0,t)}{\partial x} = 0, \qquad t > 0,$$

$$y(L,t) = 0, \qquad t > 0,$$
$$y(x,0) = f(x), \qquad 0 < x < L,$$
$$\frac{\partial y(x,0)}{\partial t} = 0, \qquad 0 < x < L.$$

The PDE is a result of equations (37) and (38), and the boundary condition at $x = 0$ is equation (46). Static deflections must satisfy

$$0 = c^2 \frac{d^2 y}{dx^2} - 9.81, \qquad 0 < x < L,$$
$$y'(0) = 0, \qquad y(L) = 0,$$

the solution of which is

$$y(x) = \frac{9.81}{2c^2}(x^2 - L^2)$$

(Figure 1.15). This is the position that the string would occupy were it to hang motionless under gravity. Notice, in particular, that the parabola has zero slope at its free end, $x = 0$.

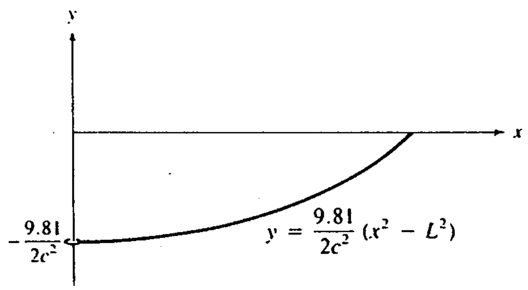


Figure 1.15

■

It is a standard example in ODEs to find the shape of a string that hangs between two points under the influence of gravity. The solution, called a *catenary*, is a hyperbolic cosine function, not a parabola as derived in Example 4. The difference lies in the assumptions leading to the ODEs describing the two situations. In Example 4, it is assumed that tension τ in the string is constant, and this leads to the differential equation $d^2y/dx^2 = 9.81\rho/\tau$ for static deflections. For the catenary problem, the string is not sufficiently taut that tension is constant. This leads to the differential equation $d^2y/dx^2 = (9.81\rho/\tau)\sqrt{1 + (dy/dx)^2}$, where τ is tension at only the lowest point in the string.

Longitudinal Vibrations of Bars

In Figure 1.16 we show a circular bar of natural length L lying along the x-axis. Suppose that the end $x = 0$ is clamped at that position and the end $x = L$ is struck with a hammer. This will set up longitudinal vibrations in the bar. We show that the one-dimensional wave equation, which describes transverse vibrations of a taut string, also describes these longitudinal vibrations of the bar. Although we have drawn the bar in a horizontal position, it could equally well be vertical. We denote by x the positions of cross sections of the bar when the bar is in an unstrained state, and we denote by $y(x, t)$ the positions of cross sections relative to their unstrained positions (Figure 1.17). It is assumed that cross sections remain plane during vibrations.

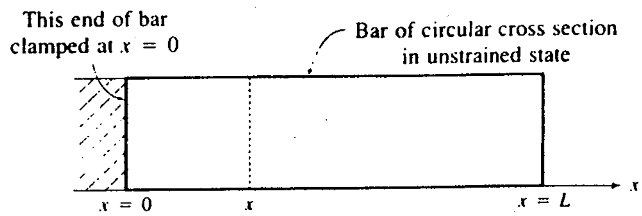


Figure 1.16

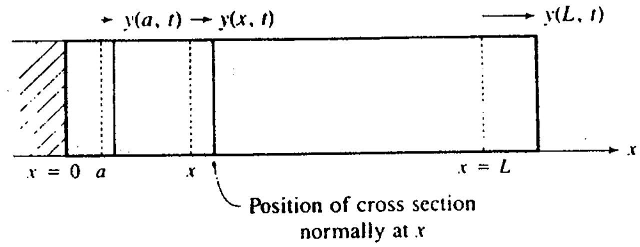


Figure 1.17

Consider the segment of the bar that in an unstrained state occupies the region between $x = a$ (a some fixed number) and an arbitrary position x. At time t, this segment is stretched an amount $y(x,t) - y(a,t)$. Hooke's law states that the force exerted across the segment due to this extension (or compression) is given by

$$AE\left(\frac{y(x,t) - y(a,t)}{x - a}\right), \tag{52}$$

where A is the cross-sectional area of the bar and E is Young's modulus of elasticity of the material in tension and compression. It follows (by limits as $x \to a$) that the internal force exerted on the face at $x = a$ by that part of the bar to its right at time t has component

$$AE\frac{\partial y(a,t)}{\partial x}. \tag{53}$$

[The internal force on the face at $x = a$ due to that part of the bar to its left has component $-AE\partial y(a,t)/\partial x$.]

We now apply Newton's second law to the motion of the above segment of the bar:

$$AE\frac{\partial y(x,t)}{\partial x} - AE\frac{\partial y(a,t)}{\partial x} + \int_a^x F(\zeta,t)A\,d\zeta = \frac{\partial}{\partial t}\left(\int_a^x \frac{\partial y(\zeta,t)}{\partial t}\rho(\zeta,t)A\,d\zeta\right), \tag{54}$$

where $\rho(x,t)$ is the density of the bar (mass per unit volume) and $F(x,t)$ is the x-component of all external forces acting on the bar per unit volume. It is assumed that these external forces are constant over each cross section of the bar. Differentiation of this equation with respect to x and division by A give

$$E\frac{\partial^2 y}{\partial x^2} + F(x,t) = \frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right). \tag{55}$$

In most applications, ρ can be taken as constant, in which case (55) reduces to the one-dimensional wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{F(x,t)}{\rho}, \qquad c^2 = \frac{E}{\rho}.} \tag{56}$$

Initial conditions that accompany PDE (56) describe the displacement and velocity of cross sections of the bar at some initial time, usually $t = 0$ [see equations (41a, b)]. Boundary conditions must also be specified. When the bar is of finite length ($0 < x < L$), two boundary conditions occur, one at each end. If the bar is of semi-infinite length ($0 < x < \infty$), only the end $x = 0$ satisfies a boundary condition; and when the bar is of infinite length, no boundary conditions are present. Dirichlet boundary conditions are of form (42a, b); they specify displacements $y(0,t)$ and $y(L,t)$ of the ends of the bar. Neumann boundary conditions result when longitudinal forces that are prescribed functions of time are applied to the faces of the bar. To see this, note that the force exerted on the face $x = 0$ by the bar (to the right) is $AE\partial y(0,t)/\partial x$. Consequently, if the end $x = 0$ of the bar is subjected to an external force with x-component $f_1(t)$, then taking the face as massless, Newton's second law for the face gives

$$AE\frac{\partial y(0,t)}{\partial x} + f_1(t) = 0 \tag{57a}$$

or

$$\frac{\partial y(0,t)}{\partial x} = -\frac{1}{AE}f_1(t), \qquad t > 0, \tag{57b}$$

a Neumann condition. Similarly, if the bar has an end at $x = L$ with external force $f_2(t)$, the Neumann boundary condition there is

$$\frac{\partial y(L,t)}{\partial x} = \frac{1}{AE}f_2(t), \qquad t > 0. \tag{58}$$

Homogeneous Neumann boundary conditions describe free ends.

Were we to attach the end of the bar at $x = 0$ to the origin by a spring (of constant $k > 0$) so that the spring were unstretched when the end is at $x = 0$, (57a) would be replaced by

$$AE\frac{\partial y(0,t)}{\partial x} - ky(0,t) = 0, \qquad t > 0$$

or

$$-AE\frac{\partial y(0,t)}{\partial x} + ky(0,t) = 0, \qquad t > 0. \tag{59a}$$

This is a homogeneous Robin condition. Similarly, when end $x = L$ is attached to a spring, the resulting boundary condition is the homogeneous Robin condition

$$AE\frac{\partial y(L,t)}{\partial x} + ky(L,t) = 0, \qquad t > 0. \tag{59b}$$

The initial boundary value problem for longitudinal displacements in the bar consists of the one-dimensional wave equation (56) together with two initial conditions and zero, one, or two boundary conditions, a problem identical to that for string vibrations.

Angular Vibrations of Bars

Angular vibrations of a bar also give rise to the above mathematical problem. Let x denote distance from some fixed reference point to cross sections of a cylindrical elastic bar (Figure 1.18). At time t, the angular displacement of the section labeled x from its torque-free position is denoted by $y(x,t)$, where it is assumed that in each cross section, lines that are radial in the bar before torque is applied remain straight after the bar is twisted. At this time, the segment of the bar between a and x has its right face twisted relative to its left face by an amount $y(x,t) - y(a,t)$. The torque exerted across the element is then

$$IE\left(\frac{y(x,t) - y(a,t)}{x - a}\right), \tag{60}$$

where I is the moment of inertia of the cross-sectional area about the axis of the bar and E is Young's modulus of elasticity of the material in shear. It follows (by limits) that the internal torque exerted on the face at $x = a$ by that part of the bar to its right at time t is

$$IE\frac{\partial y(a,t)}{\partial x}. \tag{61}$$

[The internal torque on the face at $x = a$ due to that part of the bar to its left is $-IE\partial y(a,t)/\partial x$.]

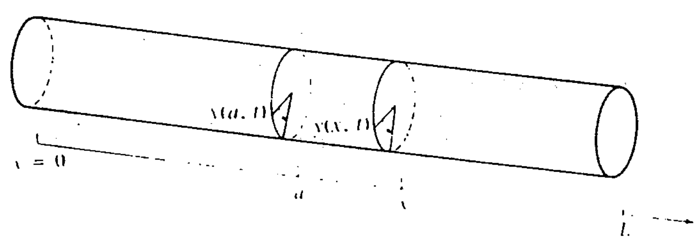


Figure 1.18

If, in addition, an external torque per unit length $\tau(x,t)$ acts, and $\rho(x,t)$ is the density (mass per unit volume) of the bar, then the PDE for angular vibrations of th bar can be obtained from Newton's second law applied to the element between a and x:

$$IE\frac{\partial y(x,t)}{\partial x} - IE\frac{\partial y(a,t)}{\partial x} + \int_a^x \tau(\zeta,t)\,d\zeta = \frac{\partial}{\partial t}\left(\int_a^x I\rho(\zeta,t)\frac{\partial y(\zeta,t)}{\partial t}\,d\zeta\right). \tag{62}$$

Differentiation of this equation with respect to x and division by I give

$$E\frac{\partial^2 y}{\partial x^2} + \frac{\tau(x,t)}{I} = \frac{\partial}{\partial t}\left(\rho\frac{\partial y}{\partial t}\right). \tag{63}$$

When ρ is constant, (63) reduces to the one-dimensional wave equation

$$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} + \frac{\tau(x,t)}{\rho I}, \qquad c^2 = \frac{E}{\rho}.} \tag{64}$$

Accompanying this PDE will be two initial conditions and/or zero, one, or two boundary conditions.

Exercises 1.3

In Exercises 1–12, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that density of and tension in the string are constant (or that Young's modulus and density are constant in the bar).

1. A taut string has its ends fixed at $x = 0$ and $x = L$ on the x-axis. It is given an initial displacement at $t = 0$ of $f(x)$, $0 \le x \le L$ and initial velocity $g(x)$, $0 \le x \le L$. Formulate the initial boundary value problem for displacement $y(x, t)$ of the string for $0 < x < L$ and $t > 0$.
2. Repeat Exercise 1 except that the end at $x = L$ is free to slide without friction along a vertical support.
3. Repeat Exercise 1 where oscillations take place in a medium that creates a damping force proportional to velocity and the ends of the string are elastically connected to the x-axis. Furthermore, do not neglect the weight of the string.
4. Repeat Exercise 1 except that a vertical force $F(t) = \cos \omega t$, $t > 0$, acts on the end $x = 0$ of the string and the string is initially at rest along the x-axis.
5. Repeat Exercise 3 except that the force $F(t)$ in Exercise 4 also acts on the end $x = 0$.
6. A horizontal cylindrical bar is originally at rest and unstrained along the x-axis between $x = 0$ and $x = L$. For time $t > 0$, the left end is fixed and the right end is subjected to a constant elongating force per unit area F parallel to the bar. Formulate the initial boundary value problem for displacements $y(x, t)$ of cross sections of the bar.
7. A bar of unstrained length L is clamped along its length, turned to the vertical position, and hung from its end $x = 0$. At time $t = 0$, the clamp is removed and gravity is therefore permitted to act on the bar. Formulate an initial boundary value problem for displacements $y(x, t)$ of cross sections of the bar.
8. Repeat Exercise 7 except that the top of the bar is attached to a spring with constant k. Let $x = 0$ correspond to the top of the bar when the spring is in the unstretched position at $t = 0$.
9. The ends of a taut string are fixed at $x = 0$ and $x = L$ on the x-axis. The string is initially at rest

along the x-axis, then is allowed to drop under its own weight. Formulate an initial boundary value problem for displacements of the string. What are the static deflections for this string?

10. Repeat Exercise 9 except that motion takes place in a medium that creates a damping force proportional to velocity.
11. Repeat Exercise 9 except that the end of the string at $x = L$ is looped around a smooth vertical support and a constant vertical force F_L acts on this loop. What are the static deflections for the string?
12. An unstrained elastic bar falls vertically under gravity with its axis vertical. When its velocity is v (which we take as time $t = 0$), it strikes a solid object and remains in contact with it thereafter. Formulate an initial boundary value problem for displacements of cross sections of the bar.
13. A cylindrical bar has unstrained length L. If it is hung vertically from one end so that no oscillations occur, what is its length?
14. The bar in Exercise 13 is hung from a spring with constant $k > 0$. How far below $x = 0$ (the position of the lower end of the spring in the unstretched position) will the lower end of the bar lie?
15. Verify that Robin and Neumann conditions at $x = L$ take the forms (49b) and (47) for massless ends.
16. The end $x = 0$ of a horizontal bar of length L is kept fixed, and the other end has a mass m attached to it. The mass m is then subjected to a horizontal periodic force $F = F_0 \sin \omega t$. If the bar is initially unstrained and at rest, set up the initial boundary value problem for longitudinal displacements in the bar.
17. The one-dimensional wave equation (37) for vibrations of a taut string was derived by applying Newton's second law to a segment of the string. In this exercise, we use the PDE to discuss energy balance for the entire string (assumed finite in length).
    (a) Multiply the PDE by $\partial y/\partial t$, integrate the result with respect to x over the length of the string $0 \le x \le L$, and use integration by parts to obtain
    $$\frac{1}{2}\int_0^L \left[\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial t}\right)^2 + c^2\frac{\partial}{\partial t}\left(\frac{\partial y}{\partial x}\right)^2\right]dx = c^2\left\{\frac{\partial y}{\partial x}\frac{\partial y}{\partial t}\right\}_0^L + \int_0^L \frac{F(x,t)}{\rho}\frac{\partial y}{\partial t}\,dx.$$
    (b) Integrate the result in (a) with respect to time from $t = 0$ to an arbitrary t to show that
    $$\begin{aligned}\int_0^L \frac{\rho}{2}\left(\frac{\partial y}{\partial t}\right)^2 dx + \int_0^L \frac{\tau}{2}\left(\frac{\partial y}{\partial x}\right)^2 dx = &\int_0^L \frac{\rho}{2}\left(\frac{\partial y(x,0)}{\partial t}\right)^2 dx \\ &+ \int_0^L \frac{\tau}{2}\left(\frac{\partial y(x,0)}{\partial x}\right)^2 dx \\ &+ \int_0^t \left(\tau\frac{\partial y(L,t)}{\partial x}\right)\frac{\partial y(L,t)}{\partial t}\,dt \\ &+ \int_0^t \left(-\tau\frac{\partial y(0,t)}{\partial x}\right)\frac{\partial y(0,t)}{\partial t}\,dt \\ &+ \int_0^t\int_0^L F(x,t)\frac{\partial y}{\partial t}\,dx\,dt.\end{aligned}$$
    Interpret each of these terms physically and thereby conclude that the equation is a statement of work-energy balance. It is often called the "energy equation" for the string.

18. Show that when the cross-sectional area of the bar in Figure 1.16 varies with position, equation (56) is replaced by

$$\frac{\partial^2 y}{\partial t^2} = \frac{c^2}{A(x)} \frac{\partial}{\partial x}\left(A(x)\frac{\partial y}{\partial x}\right) + \frac{F(x,t)}{\rho}, \qquad c^2 = \frac{E}{\rho},$$

provided expression (53) still gives forces across-cross sections of the bar.

19. A bar of unstrained length L is clamped at end $x = 0$. For time $t < 0$, it is at rest, subjected to a force with x-component F distributed uniformly over the other end. If the force is removed at time $t = 0$, formulate the initial boundary value problem for subsequent displacements in the bar.

20. In this exercise we derive the PDE for small vibrations of a suspended heavy cable. Consider a heavy cable of uniform density ρ (mass/length) and length L suspended vertically from one end. Take the origin of coordinates at the position of equilibrium of the lower end of the cable and the positive x-axis along the cable. Denote by $y(x,t)$ small horizontal deflections of points in the cable from equilibrium.

(a) Apply Newton's second law to a segment of the cable to obtain the PDE for small deflections

$$\frac{\partial^2 y}{\partial t^2} = -g\frac{\partial}{\partial x}\left(x\frac{\partial y}{\partial x}\right) + \frac{F}{\rho},$$

where $g < 0$ is the acceleration due to gravity and F is the y-component of all external horizontal forces per unit length in the x-direction.

(b) What boundary condition must $y(x,t)$ satisfy?

1.4 Transverse Vibrations of Membranes

In this section we study vibrations of perfectly flexible membranes stretched over regions of the xy-plane (Figure 1.19). When the membrane is very taut and displacements are small, the horizontal components of these displacements are negligible compared with vertical components; that is, displacements may be taken as purely transverse, representable in the form $z(x, y, t)$.

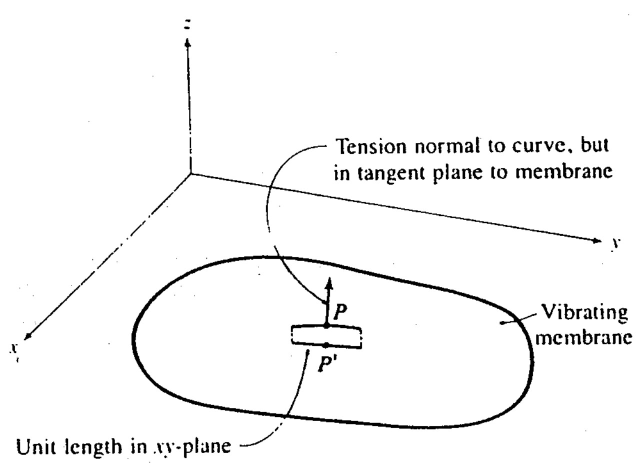


Figure 1.19

In discussing transverse vibrations of strings, tension played an integral role. No less important is the tension in a membrane. Suppose a line of unit length is drawn in any direction at a point P' in the xy-plane and projected onto a curve on the membrane (Figure 1.19). The material on one side of the curve exerts a force on the material on the other side, the force acting normal to the curve and in the tangent plane of the surface at P. This force is called the tension τ of the membrane.

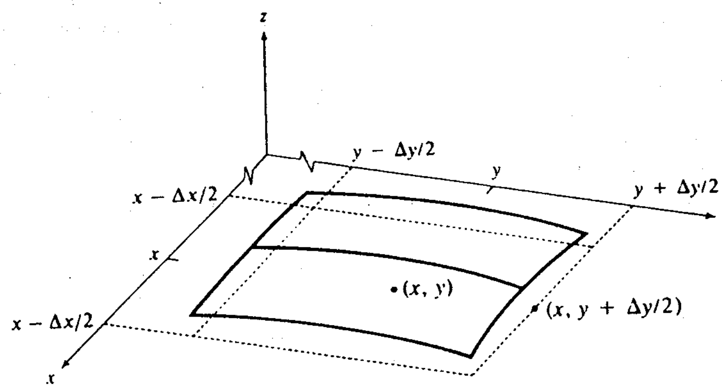


Figure 1.20

To obtain a PDE for displacements $z(x, y, t)$ of the membrane, we examine forces acting on an element of the membrane that projects onto a small rectangle in the xy-plane (Figure 1.20). The vertical component of the tension force on the element is obtained by taking vertical components of the tensions on the boundaries. The tension at the point on the membrane corresponding to the point $(x, y + \Delta y/2)$ in the xy-plane acts in the tangential direction of the curve

$$x = x \quad \text{(fixed)}, \qquad y = y, \qquad z = z(x, y, t),$$

namely,

$$\left(0, 1, \frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \tag{65}$$

A unit vector in this direction is

$$\frac{(0, 1, \partial z/\partial y)}{\sqrt{1 + (\partial z/\partial y)^2}}_{|(x, y+\Delta y/2, t)} \tag{66}$$

When vibrations of the membrane are such that $\partial z/\partial y$ is very much less than unity (and we consider only this case), the denominator in (66) may be approximated by 1, and (66) is replaced by (65). The vertical component of the tension force acting along that part of the boundary containing the point $(x, y + \Delta y/2, z)$ may therefore be approximated by

$$\tau_{|(x, y+\Delta y/2, t)}\left(0, 1, \frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \Delta x \cdot \hat{\mathbf{k}} = \left(\tau \frac{\partial z}{\partial y}\right)_{|(x, y+\Delta y/2, t)} \Delta x. \tag{67}$$

A similar analysis may be made on the remaining three boundaries, resulting in a total vertical force on the element (due to tension) of approximately

$$\left[\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}\right]\Delta x + \left[\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}\right]\Delta y. \tag{68}$$

When Newton's second law (force equals time rate of change of momentum) is applied to this element of the membrane, the result is

$$\left[\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}\right]\Delta x + \left[\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}\right]\Delta y + F\,\Delta x\,\Delta y = \frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\Delta x\,\Delta y\right), \tag{69}$$

where ρ is the density of the membrane (mass per unit area) and F is the sum of all vertical external forces on the membrane per unit area in the xy-plane. If we divide both sides of this equation by $\Delta x\,\Delta y$ and take limits as Δx and Δy approach zero,

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right) = \lim_{\Delta y\to 0}\frac{\left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y+\Delta y/2,\,t)} - \left(\tau\frac{\partial z}{\partial y}\right)_{|(x,\,y-\Delta y/2,\,t)}}{\Delta y} + \lim_{\Delta x\to 0}\frac{\left(\tau\frac{\partial z}{\partial x}\right)_{|(x+\Delta x/2,\,y,\,t)} - \left(\tau\frac{\partial z}{\partial x}\right)_{|(x-\Delta x/2,\,y,\,t)}}{\Delta x} + F$$

or

$$\frac{\partial}{\partial t}\left(\rho\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial y}\left(\tau\frac{\partial z}{\partial y}\right) + \frac{\partial}{\partial x}\left(\tau\frac{\partial z}{\partial x}\right) + F. \tag{70}$$

For most applications, both the density of and the tension in the membrane may be taken as constant, in which case (70) reduces to

$$\boxed{\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F}{\rho}, \qquad c^2 = \frac{\tau}{\rho}.} \tag{71}$$

This is the PDE for transverse vibrations of the membrane, called the *two-dimensional wave equation.*

For an external force due only to gravity,

$$F = \rho g, \qquad g < 0; \tag{72}$$

for a damping force proportional to velocity,

$$F = -\beta \frac{\partial z}{\partial t}, \qquad \beta > 0; \tag{73}$$

and for a restoring force proportional to displacement,

$$F = -kz, \qquad k > 0. \tag{74}$$

Initial conditions that accompany (71) describe the displacement and velocity of the membrane at some initial time (usually $t = 0$):

$$z(x, y, 0) = f(x, y), \qquad (x, y) \text{ in } R, \tag{75a}$$

$$\frac{\partial z(x, y, 0)}{\partial t} = g(x, y), \qquad (x, y) \text{ in } R, \tag{75b}$$

where R is the region in the xy-plane onto which the membrane projects. A Dirichlet boundary condition for (71) prescribes the value of $z(x, y, t)$ on the boundary $\beta(R)$ of R,

$$z(x, y, t) = f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \tag{76}$$

$f(x, y, t)$ some given function.

Suppose instead that the edge of the membrane can move vertically and that it is subjected to an external vertical force $f(x, y, t)$ per unit length. The edge is also acted on by the tension in the membrane, and the magnitude of the z-component of the tension acting across a unit length along $\beta(R)$ is $|\tau \partial z/\partial n|$, where n is a coordinate measuring distance in the xy-plane normal to $\beta(R)$ (Figure 1.21). Consequently, if we take the edge of the membrane as massless, Newton's second law for vertical components of forces on an element ds of $\beta(R)$ gives

$$-\left(\tau \frac{\partial z}{\partial n}\right)_{|\beta(R)} ds + f(x, y, t)\, ds = 0 \tag{77a}$$

or

$$\frac{\partial z}{\partial n} = \frac{1}{\tau} f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0. \tag{77b}$$

This is a nonhomogeneous Neumann boundary condition. When the only force acting on the edge of the membrane is that due to tension, $z(x, y, t)$ must satisfy a homogeneous Neumann condition,

$$\frac{\partial z}{\partial n} = 0, \qquad (x, y) \text{ on } \beta(R), \qquad t > 0. \tag{78}$$

Another possibility is to have the edge of the membrane elastically attached to the xy-plane in such a way that the restoring force per unit length along $\beta(R)$ is proportional to displacement. Then, according to (77a),

$$-\left(\tau \frac{\partial z}{\partial n}\right) ds + [-kz + f(x, y, t)]\, ds = 0, \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \tag{79a}$$

where $k > 0$, and $f(x, y, t)$ now represents all external forces acting on $\beta(R)$ other than tension and the restoring force. Equation (79a) can be written in the equivalent form

$$\tau\frac{\partial z}{\partial n} + kz = f(x, y, t), \qquad (x, y) \text{ on } \beta(R), \qquad t > 0, \qquad \textbf{(79b)}$$

a Robin condition.

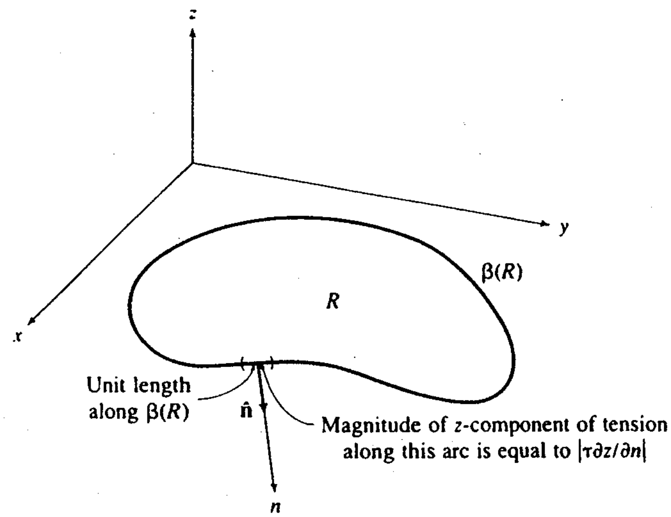


Figure 1.21

The initial boundary value problem for displacements in the membrane is

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{F(x, y, t)}{\rho}, \qquad (x, y) \text{ in } R, \qquad t > 0, \qquad \textbf{(80a)}$$

$$\text{Boundary conditions}, \qquad \textbf{(80b)}$$

$$z(x, y, 0) = f(x, y), \qquad (x, y) \text{ in } R, \qquad \textbf{(80c)}$$

$$z_t(x, y, 0) = g(x, y), \qquad (x, y) \text{ in } R. \qquad \textbf{(80d)}$$

If boundary conditions (80b) and external force $F(x, y, t)$ are independent of time, there may exist solutions of (80a, b) that are also independent of time. Such solutions, called *static deflections*, satisfy Poisson's equation

$$\nabla^2 z = -\frac{F(x, y)}{\tau} \qquad \textbf{(81a)}$$

and Boundary conditions. **(81b)**

If, in addition, no external forces are present, the PDE reduces to Laplace's equation

$$\nabla^2 z = 0. \qquad \textbf{(82)}$$

An important technique in solving the two-dimensional wave equation is the method of separation of variables, a method we shall deal with at length in Section 3.2. In this method it is assumed that displacement can be separated into a function of x and y multiplied by a function of time t, $z(x, y, t) = u(x, y)T(t)$. Substitution of this into equation (80a) when $F \equiv 0$ gives

$$u(x, y)\frac{d^2T}{dt^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}T(t) + \frac{\partial^2 u}{\partial y^2}T(t)\right)$$

or

$$\frac{1}{T}\frac{d^2T}{dt^2} = \frac{c^2}{u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right). \tag{83}$$

Because the left side of this equation is a function of only t and the right side is a function of x and y, it follows that each must be equal to a constant, say $-k$. Then $u(x, y)$ must satisfy

$$\frac{c^2}{u}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = -k$$

or

$$\boxed{\nabla^2 u + \frac{k}{c^2}u = 0.} \tag{84}$$

This equation is called the two-dimensional *Helmholtz equation*. In the present context, it is also called the *reduced wave equation*. In essence, it describes the amplitude of the oscillations of each point in the membrane; $T(t)$ contains the time dependence of the vibrations. Boundary conditions for the wave equation will yield boundary conditions for the Helmholtz equation.

Exercises 1.4

In Exercises 1–7, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that density of and tension in the membrane are constant.

1. A vibrating circular membrane of radius r_1 is given an initial displacement $f(r, \theta)$ and zero initial velocity. If its edge $r = r_1$ is fixed on the xy-plane, formulate an initial boundary value problem for subsequent displacements of the membrane. Assume that no external forces act on the membrane.
2. Repeat Exercise 1 except that $f(r, \theta)$ is replaced by $f(r)$.
3. A circular membrane of radius r_1 is in a static position with radial lines $\theta = 0$ and $\theta = \alpha$ clamped on the xy-plane. If the displacement of the edge $r = r_1$ is $f(\theta)$ for $0 \le \theta \le \alpha$, formulate the boundary value problem for displacement in the sector $0 < \theta < \alpha$. Would there be any restriction on $f(\theta)$?
4. Repeat Exercise 3 except that gravity acts on the membrane.
5. Repeat Exercise 1 except that gravity, as well as a damping force proportional to velocity, acts on the membrane.
6. A rectangular membrane is initially ($t = 0$) at rest over the region $0 \le x \le L$, $0 \le y \le L'$ in the xy-plane. For time $t > 0$, a periodic force per unit area $\cos \omega t$ acts at all points in the membrane.

If the edge of the membrane is fixed on the xy-plane, formulate an initial boundary value problem for displacements of the membrane.

7. Repeat Exercise 6 except that the boundaries $x = 0$ and $x = L$ are elastically connected to the xy-plane and the boundaries $y = 0$ and $y = L'$ are forced to exhibit motion described by $f_1(x, t)$ and $f_2(x, t)$, respectively.
8. A circular membrane of radius r_2 has its edge $r = r_2$ fixed on the xy-plane. If gravity and tension are the only forces acting on the membrane, what are the static deflections of points of the membrane?
9. In this exercise we replace gravity in Exercise 8 with an arbitrary (but continuous) function $f(r)$; that is, assume that the only forces acting on the membrane are tension and a force per unit area with y-component $f(r)$.
   (a) What is the boundary value problem for static deflections $z(r)$ of the membrane?
   (b) Show that $z'(r)$ must be of the form
   $$z'(r) = \frac{-1}{r\tau}\int rf(r)\,dr.$$
   (c) Express the antiderivative in (b) as a definite integral
   $$z'(r) = \frac{-1}{r\tau}\int_0^r uf(u)\,du$$
   and integrate once more to find $z(r)$ in the form
   $$z(r) = \frac{1}{\tau}\left(\int_0^{r_2}\int_0^v \frac{u}{v} f(u)\,du\,dv - \int_0^r\int_0^v \frac{u}{v} f(u)\,du\,dv\right).$$
   (d) Interchange orders of integration to obtain
   $$z(r) = \frac{1}{\tau}\left(\int_0^{r_2} uf(u)\ln\left(\frac{r_2}{u}\right)du - \int_0^r uf(u)\ln\left(\frac{r}{u}\right)du\right).$$
   (e) Verify that the result in (d) yields the solution to Exercise 8 when $f(r) = \rho g$.
   (f) Find deflections when $f(r) = k(r - r_2)$, $k > 0$ a constant.
10. The two-dimensional wave equation (71) was derived by applying Newton's second law to a segment of the membrane. In this exercise we use the PDE to discuss energy balance for the entire membrane.
   (a) Multiply (71) by $\partial z/\partial t$ and integrate over the region R in the xy-plane onto which the membrane projects to show that
   $$\iint_R \frac{1}{2}\frac{\partial}{\partial z}\left(\frac{\partial z}{\partial t}\right)^2 dA = c^2\iint_R \frac{\partial z}{\partial t}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right)dA + \iint_R \frac{F(x, y, t)}{\rho}\frac{\partial z}{\partial t}\,dA.$$
   (b) Verify that for z a function of x, y, and t,
   $$\frac{\partial z}{\partial t}\nabla^2 z = \nabla\cdot\left(\frac{\partial z}{\partial t}\nabla z\right) - \frac{1}{2}\frac{\partial}{\partial t}|\nabla z|^2,$$
   and use this identity together with Green's theorem to rewrite the result in (a) in the form
   $$\frac{1}{2}\iint_R\left[\frac{\partial}{\partial t}\left(\frac{\partial z}{\partial t}\right)^2 + c^2\frac{\partial}{\partial t}|\nabla z|^2\right]dA = c^2\oint_{\beta(R)}\frac{\partial z}{\partial t}\frac{\partial z}{\partial n}\,ds + \iint_R \frac{F(x, y, t)}{\rho}\frac{\partial z}{\partial t}\,dA.$$

(c) Integrate the result in (b) with respect to time from $t = 0$ to an arbitrary t to obtain

$$\iint_R \left[\frac{\rho}{2}\left(\frac{\partial z}{\partial t}\right)^2 + \frac{\tau}{2}|\nabla z|^2\right] dA = \iint_R \left[\frac{\rho}{2}\left(\frac{\partial z(x,y,0)}{\partial t}\right)^2 + \frac{\tau}{2}|\nabla z(x,y,0)|^2\right] dA$$
$$+ \int_0^t \oint_{\beta(R)} \left(\tau\frac{\partial z}{\partial n}\right)\frac{\partial z}{\partial t}\,ds\,dt + \int_0^t \iint_R F(x,y,t)\frac{\partial z}{\partial t}\,dA\,dt.$$

Interpret each term in this result physically, and hence obtain a physical interpretation of the equation as a whole. It is often called the "energy equation" for the membrane.

1.5 Transverse Vibrations of Beams

In this section we study vertical oscillations of horizontal beams (Figure 1.22). It is assumed that the beam is symmetric about the xy-plane and that all cross sections (which would be plane in the absence of any loading) remain plane during vibrations. Displacements are then described by the position $y(x, t)$ of the neutral axis.

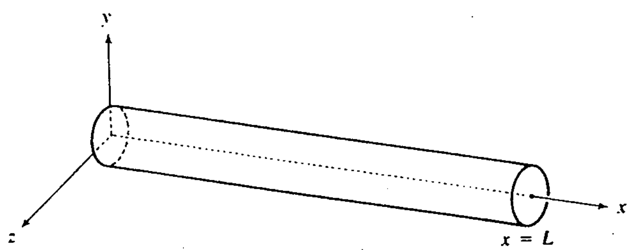


Figure 1.22

Stretches and compressions in various parts of the beam lead to internal forces and moments. It has been shown experimentally that the bending moment $M(x, t)$ on the right face of the cross section of the beam at position x due to the rest of the beam to its right is related to the signed curvature $\kappa(x, t)$ of the neutral axis by the equation

$$M = EI\kappa, \tag{85}$$

where $E = E(x) > 0$ is Young's modulus of elasticity (depending on the material in the beam) and $I = I(x)$ is the moment of inertia of the cross section of the beam (Figure 1.23). It is shown in elementary calculus that

$$\kappa = \frac{\partial^2 y/\partial x^2}{\left[1 + \left(\frac{\partial y}{\partial x}\right)^2\right]^{3/2}}, \tag{86}$$

but if we assume that vibrations produce only small slopes, then $\partial y/\partial x \ll 1$, and we may take

$$\kappa = \frac{\partial^2 y}{\partial x^2}. \tag{87}$$

Consequently, for vibrations producing small slopes, bending moments are related to curvature by

$$M(x,t) = EI\frac{\partial^2 y}{\partial x^2}. \tag{88}$$

Since $\partial^2 y/\partial x^2$ is positive when the beam is concave upward (as in Figure 1.23), it follows that M must be positive on the right face for the direction shown. The moment on the left face of the same cross section due to the material in the beam on its left is therefore $-M(x,t) = -EI\partial^2 y/\partial x^2$.

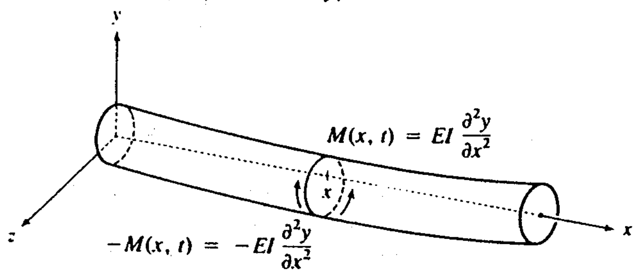


Figure 1.23

Shear forces also act on any cross section. We denote by $Q(x,t)$ the y-component of the shear force acting on the right face of the cross section at position x due to that part of the beam to its right. Then $-Q(x,t)$ is the shear force acting on the left face. Shear and bending moments are related. To see how, we apply Newton's second law for rotational motion of a segment of the beam from a fixed value $x = a$ to an arbitrary value x (Figure 1.24); the sum of all moments of all forces on the segment must equal the moment of inertia multiplied by the angular acceleration. Since motion about the line $x = \bar{x}$ through the center of mass of the segment is strictly translational, moments about $x = \bar{x}$ yield

$$0 = M(x,t) - M(a,t) + (x - \bar{x})Q(x,t) - (a - \bar{x})Q(a,t) + \int_a^x (\zeta - \bar{x})F(\zeta,t)\,d\zeta, \tag{89}$$

where $F(x,t)$ is the sum of all vertical forces on the beam per unit x-length. Differentiation of this equation with respect to x gives

$$0 = \frac{\partial M(x,t)}{\partial x} + (x - \bar{x})\frac{\partial Q(x,t)}{\partial x} + \left(1 - \frac{\partial \bar{x}}{\partial x}\right)Q(x,t) + Q(a,t)\frac{\partial \bar{x}}{\partial x} + \int_a^x -\frac{\partial \bar{x}}{\partial x}F(\zeta,t)\,d\zeta + (x - \bar{x})F(x,t). \tag{90}$$

If we now take limits as x approaches a, $\bar{x}$ approaches a also, and

$$0 = \frac{\partial M(a,t)}{\partial x} + Q(a,t). \tag{91}$$

Because a is arbitrary, we obtain the following relationship between shear and bending moments:

$$Q(x,t) = -\frac{\partial M(x,t)}{\partial x}. \tag{92}$$

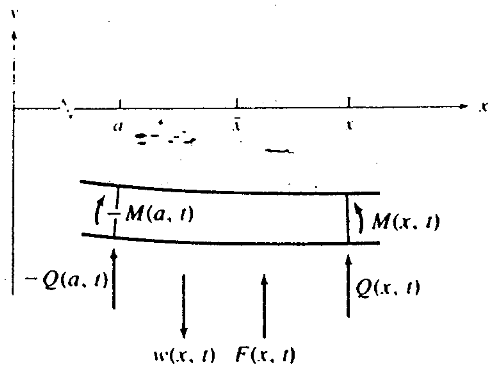


Figure 1.24

Vibrations of the beam are determined by the interactions of the internal bending moments and shear forces with the exterior loading $w(x,t)$ per unit x-length (including the weight of the beam) and all external forces $F(x,t)$ per unit x-length (including loading). The function $w(x,t)$ is the y-component of the loading and as such is negative, while $F(x,t)$, the y-component of all external forces, may be positive, negative, or zero. To describe these interactions, we apply Newton's second law to the vertical translational motion of the segment of the beam in Figure 1.24:

$$\frac{\partial}{\partial t}\left(\int_a^x \frac{\partial y(\zeta,t)}{\partial t}\frac{w}{g}\,d\zeta\right) = \int_a^x F(\zeta,t)\,d\zeta + Q(x,t) - Q(a,t), \qquad (g<0). \tag{93}$$

The integral on the left is the momentum of the segment—$w\,d\zeta/g$ is the mass of an element of the beam of length $d\zeta$ along the x-axis, and multiplication by velocity $\partial y(\zeta,t)/\partial t$ gives momentum. The integral on the right is the sum of all external forces on the segment, and $Q(x,t)$ and $Q(a,t)$ are the shear forces on the faces at x and a, respectively. Differentiation of this equation with respect to x gives

$$\frac{\partial}{\partial t}\left(\frac{w}{g}\frac{\partial y}{\partial t}\right) = F(x,t) + \frac{\partial Q}{\partial x}. \tag{94}$$

Substitutions for $\partial Q/\partial x$ and M from equations (92) and (88) yield the PDE satisfied by transverse vibrations of the beam:

$$\frac{\partial}{\partial t}\left(\frac{w}{g}\frac{\partial y}{\partial t}\right) + \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right) = F(x,t). \tag{95}$$

When E and I are independent of x and $w(x,t)$ is independent of t, the PDE can be written in the simplified form

$$\boxed{\frac{w}{EIg}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = \frac{F}{EI}.} \tag{96}$$

In many applications, the internal forces in the beam are so large that the effect of F is negligible. In such cases, (96) may be replaced by the "homogeneous"† equation

† A general definition of what it means for a PDE to be homogeneous is given in Section 3.1.

$$\frac{w}{EIg}\frac{\partial^2 y}{\partial t^2} + \frac{\partial^4 y}{\partial x^4} = 0. \tag{97}$$

This is illustrated in Exercise 5, where it is shown that when $F(x)$ is due only to the weight of the beam itself, static deflections are very small.

Accompanying (96) or (97) will be two initial conditions that describe the displacement and velocity of the beam at some initial time (usually $t = 0$):

$$y(x,0) = f(x), \qquad 0 < x < L, \tag{98a}$$

$$y_t(x,0) = g(x), \qquad 0 < x < L. \tag{98b}$$

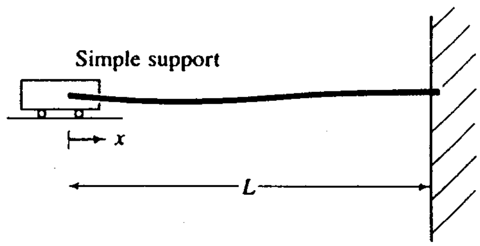


Figure 1.25

Various types of boundary conditions may exist at each end of the beam. If the end $x = 0$ is "simply supported" (Figure 1.25), displacement and curvature (moment) there are both zero:

$$y(0,t) = 0, \qquad \frac{\partial^2 y(0,t)}{\partial x^2} = 0. \tag{99}$$

If this end is "built in" horizontally (Figure 1.26), displacement and slope vanish:

$$y(0,t) = 0, \qquad \frac{\partial y(0,t)}{\partial x} = 0. \tag{100}$$

Finally, if this end is "free" (Figure 1.27), curvature and shear are both zero:

$$\frac{\partial^2 y(0,t)}{\partial x^2} = 0, \qquad \frac{\partial^3 y(0,t)}{\partial x^3} = 0. \tag{101}$$

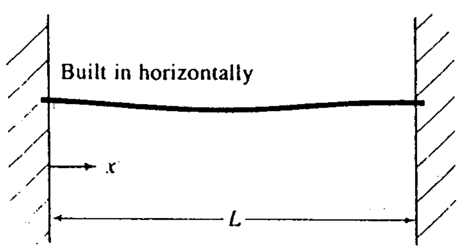


Figure 1.26

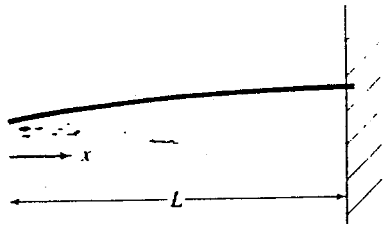


Figure 1.27

Similar conditions exist at the end $x = L$.

Exercises 1.5

In Exercises 1–4, set up, but do not solve, an (initial) boundary value problem for the required displacement. Assume that Young's modulus E and the moment of inertia I of the cross section of the beam are both constant.

1. A horizontal beam of length L has flat ends at $x = 0$ and $x = L$. At time $t = 0$, it is at rest but its neutral axis is deflected according to the function $f(x)$, $0 \le x \le L$. It is then released from this position. The left end of the beam is built in horizontally, and the right end is free.
2. Repeat Exercise 1 except that both ends are simply supported on the x-axis.
3. Repeat Exercise 1 except that a mass m is distributed uniformly along the beam and both ends are built in horizontally.
4. A beam of length L is clamped horizontally at $x = 0$ and is cantilevered (not supported) at $x = L$. For time $t < 0$, it is deflected, but motionless, under a downward force of magnitude F at $x = L$ and its own weight. At time $t = 0$, this force is removed. [*Hint:* In the static situation, the boundary conditions at $x = L$ are $y''(L) = 0$ and $y'''(L) = F/(EI)$.]
5. In this exercise we illustrate that for small external forces, the nonhomogeneous term $[F/(EI)]$ in equation (96) may be neglected.
   (a) What is the boundary value problem for static deflections of a beam of length L, simply supported at both ends? Solve this problem when the external force is constant.
   (b) Suppose now that F is due only to the weight of the beam itself. Find the maximum deflection of the beam using the following data:
   $$E = 2.1 \times 10^{11}\ \mathrm{N/m^2}, \qquad \rho = 7.85 \times 10^{3}\ \mathrm{kg/m^3}, \qquad L = 5\ \mathrm{m},$$
   $$I = 6.5\ \mathrm{kg \cdot m^2}, \qquad \text{Cross-sectional area} = 0.02\ \mathrm{m^2}.$$
   (c) What constant force (per unit x-length) over the beam would create a maximum deflection of 1 cm? How large is this compared with the weight per unit length of the beam?
6. Show that when the ends of the beam in Exercise 5 are clamped horizontally, the maximum deflection is only one-fifth that for the simply supported beam.

1.6 Electrostatic Potential

When two positive point charges Q and q are r units apart in free space, Coulomb's law states that each repels the other with a force whose magnitude is

$$F = \frac{qQ}{4\pi\varepsilon_0 r^2}, \tag{102}$$

where ε_0 is the permittivity of free space. The force on unit charge q due to Q is called the electric field intensity

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^3}\mathbf{r}, \tag{103}$$

where $\mathbf{r}$ is the vector from Q to $q = 1$ (Figure 1.28). It is straightforward to show that the curl and divergence of this vector field vanish:

$$\nabla \times \mathbf{E} = \mathbf{0}, \tag{104a}$$

$$\nabla \cdot \mathbf{E} = 0. \tag{104b}$$

A vanishing curl implies the existence (in a suitably defined domain not containing Q) of a potential function V satisfying

$$\mathbf{E} = -\nabla V. \tag{105}$$

Combine this with (104b), and we find that V must satisfy Laplace's equation

$$\nabla^2 V = 0. \tag{106}$$

For such a simple charge distribution, it is easily shown [from (105)] that to an additive constant,

$$V = \frac{Q}{4\pi\varepsilon_0 r}. \tag{107}$$

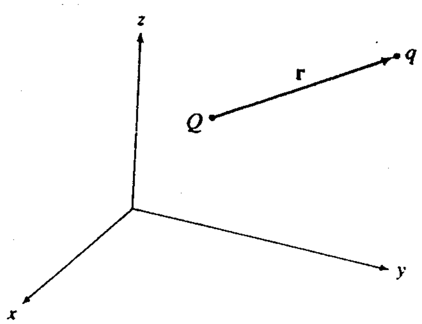


Figure 1.28

When Q is replaced by a distribution of charge with density σ in some region of free space (or other medium), determination of a potential function is more complex. In this case we appeal to Maxwell's equations, which govern all electromagnetic fields. In a static situation, Maxwell's equations still require the electric field intensity $\mathbf{E}$ to satisfy

(104a), in which case the potential function V associated with the field is once again defined by (105). Unfortunately, however, we do not know **E** (as we did for the point charge) and therefore cannot solve (105) for V. To find an equation determining V that does not contain **E**, we use another of Maxwell's equations that requires the electric displacement **D** to satisfy

$$\nabla \cdot \mathbf{D} = \sigma \tag{108}$$

at each point in the medium. When the medium is isotropic with constant permittivity ε, then **D** and **E** are related by

$$\mathbf{D} = \varepsilon \mathbf{E}, \tag{109}$$

and hence **E** must satisfy

$$\nabla \cdot \mathbf{E} = \frac{\sigma}{\varepsilon}. \tag{110}$$

Between equations (105) and (110) we may eliminate **E**, the result being Poisson's equation

$$\boxed{\nabla^2 V = -\frac{\sigma}{\varepsilon}.} \tag{111}$$

In other words, the electrostatic potential function V associated with an electrostatic field **E** must satisfy (111) at every point interior to the charge distribution. At points outside the charge distribution, σ vanishes and V satisfies Laplace's equation (106).

Equations (106) and (111) are not, by themselves, sufficient to determine V. It is necessary to specify boundary conditions as well. A Dirichlet boundary condition specifies $V(x, y, z)$ on the bounding surface $\beta(R)$ of the medium:

$$V(x, y, z) = f(x, y, z), \qquad (x, y, z) \text{ on } \beta(R), \tag{112}$$

$f(x, y, z)$ a given function. A Neumann boundary condition prescribes the directional derivative of $V(x, y, z)$ normal to the bounding surface:

$$\frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}} = f(x, y, z), \qquad (x, y, z) \text{ on } \beta(R), \tag{113}$$

where $\hat{\mathbf{n}}$ is the unit outward normal to $\beta(R)$. Since $\nabla V = -\mathbf{E}$, it follows that specification of the electrostatic force on a bounding surface yields a Neumann boundary condition. If a bounding surface is free of electrostatic forces, it satisfies a homogeneous Neumann boundary condition.

A Robin boundary condition is a linear combination of a Dirichlet and a Neumann condition:

$$l\frac{\partial V}{\partial n} + hV = f(x, y, z), \qquad (x, y, z) \text{ on } \beta(R). \tag{114}$$

Dirichlet and Neumann boundary conditions are obtained by setting l and h equal to zero, respectively.

Exercises 1.6

In Exercises 1 and 2, set up, but do not solve, a boundary value problem for the required potential.

1. Region R in space is bounded by the planes $x = 0$, $y = 0$, $x = L$, and $y = L'$. If the planes $y = 0$ and $x = 0$ are held at zero potential, whereas $x = L$ and $y = L'$ are maintained at a potential of 100, what is the boundary value problem for potential in R?
2. Repeat Exercise 1 except that a uniform charge (with density σ) is spread over the volume $L/4 \le x \le 3L/4$, $L'/4 \le y \le 3L'/4$.
3. A region R of space has a subregion $\bar{R}$ occupied by charge with density $\sigma(x, y, z)$ coulombs per cubic meter, assumed continuous (Figure 1.29). Consider the function $V(x, y, z)$ defined by

$$V(x, y, z) = \iiint_{\bar{R}} \frac{\sigma(X, Y, Z)}{4\pi\varepsilon_0\sqrt{(x - X)^2 + (y - Y)^2 + (z - Z)^2}}\, dZ\, dY\, dX.$$

Coordinates (X, Y, Z) identify points in $\bar{R}$.

(a) When (x, y, z) is in R but not in $\bar{R}$, $V(x, y, z)$ is clearly well defined. By using spherical coordinates originating at (x, y, z) for integration variables, show that when (x, y, z) is in $\bar{R}$, the improper integral converges. In other words, $V(x, y, z)$ is well defined throughout all R.

(b) By interchanging the order of differentiations with respect to x, y, and z and integrations with respect to X, Y, and Z, show that when (x, y, z) is in R, but not in $\bar{R}$, $V(x, y, z)$ satisfies Laplace's equation (106).

To prove that $V(x, y, z)$ satisfies Poisson's equation (111) when (x, y, z) is in $\bar{R}$ requires the theory of "generalized functions." Parts of this theory are introduced in Chapters 11 and 12, but the development is not carried far enough to permit verification of the integral as a solution to Poisson's equation. This is not really a problem, however, because the integral representation of $V(x, y, z)$ is of limited utility anyway. Seldom can the integral be evaluated in closed form. In addition, the integral does not take into account any boundary conditions that may be present, and there is no obvious way to modify the integral in order to encompass boundary conditions.

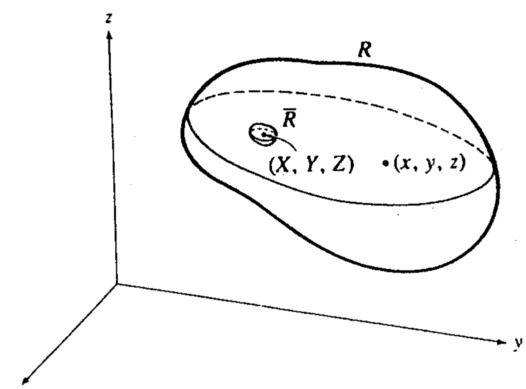


Figure 1.29

1.7 General Solutions of Partial Differential Equations

When boundary and/or initial conditions accompany an ODE, we often find a general solution and then use the subsidiary conditions to determine the arbitrary constants.

This procedure seldom works for PDEs. Arbitrary constants in ODEs are replaced by arbitrary functions in PDEs, and to use initial and/or boundary conditions to determine these functions is usually impossible. We give one very simple example to illustrate the direction the analysis might take in using a general solution for a PDE to solve an initial boundary value problem. The one-dimensional vibration problem

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}, \qquad 0 < x < L, \qquad t > 0, \tag{115a}$$

$$y(0, t) = 0, \qquad t > 0, \tag{115b}$$

$$y(L, t) = 0, \qquad t > 0, \tag{115c}$$

$$y(x, 0) = f(x), \qquad 0 < x < L, \tag{115d}$$

$$y_t(x, 0) = g(x), \qquad 0 < x < L, \tag{115e}$$

describes free oscillations of a taut string with fixed ends. For consistency, we assume that the initial displacement and velocity functions satisfy $f(0) = g(0) = f(L) = g(L) = 0$. By changing independent variables according to $v = x + ct$ and $\eta = x - ct$ and denoting $y[x(v, \eta), t(v, \eta)]$ by $w(v, \eta)$, the wave equation (115a) is replaced by

$$\frac{\partial^2 w}{\partial v \partial \eta} = 0 \tag{116}$$

(see Exercise 1 for details). The general solution of this PDE is

$$w(v, \eta) = F(v) + G(\eta), \tag{117}$$

where F and G are arbitrary but continuous functions with continuous first derivatives. As a result, the general solution of (115a) is

$$y(x, t) = F(x + ct) + G(x - ct). \tag{118}$$

It now remains to determine the exact form of these functions. Application of initial conditions (115d) and (115e) requires that

$$f(x) = F(x) + G(x), \qquad 0 < x < L,$$
$$g(x) = cF'(x) - cG'(x), \qquad 0 < x < L.$$

When the first of these is differentiated with respect to x and combined with the second,

$$F'(x) = \frac{1}{2c}[cf'(x) + g(x)]$$

and therefore

$$F(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\zeta)\, d\zeta + D,$$

$$G(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\zeta)\, d\zeta - D$$

(D an arbitrary constant). When x is replaced by $x + ct$ in $F(x)$ and by $x - ct$ in $G(x)$, we obtain

$$\begin{aligned} y(x,t) &= \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(\zeta)\,d\zeta + D \\ &\quad + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(\zeta)\,d\zeta - D \\ &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta. \end{aligned} \tag{119}$$

At first sight, (119) would seem to be the complete solution of (115), but this would be very strange, since boundary conditions (115b, c) have not been used, and the solution of an initial boundary value problem cannot be independent of its boundary conditions. The reason that (119) is not a complete solution is that f and g are defined only for $0 \le x \le L$, and yet for (119) to represent $y(x,t)$ for all x and t, these functions must be defined for all real numbers. The boundary conditions will show us how to extend the domains of f and g beyond the interval $0 \le x \le L$. Boundary condition (115b) demands that

$$0 = \frac{1}{2}[f(ct) + f(-ct)] + \frac{1}{2c}\int_{-ct}^{ct} g(\zeta)\,d\zeta,$$

and this equation is satisfied if we separately set

$$f(ct) + f(-ct) = 0, \qquad \int_{-ct}^{ct} g(\zeta)\,d\zeta = 0.$$

These imply that $f(x)$ and $g(x)$ must be extended from their original domain of definition $0 \le x \le L$ as odd functions. The functions are now defined for $-L \le x \le L$. Finally, boundary condition (115c) at $x = L$ is satisfied if we choose

$$0 = f(L+ct) + f(L-ct), \qquad 0 = \int_{L-ct}^{L+ct} g(\zeta)\,d\zeta.$$

These imply that the odd extensions of f and g must also be $2L$-periodic, and this completes definitions of f and g for all real arguments.

The function in (119) can now be used to calculate the position of the string for any x in $0 < x < L$ and any time $t > 0$; it is called *d'Alembert's solution* of initial boundary value problem (115).

As was stated earlier, this is a particularly simple example, and analyses of this type are not usually possible. For this reason, it is unusual to solve initial boundary value problems by finding a general solution for the PDE and attempting to use initial and/or boundary conditions to determine the arbitrary functions. More direct methods must be devised.

Notwithstanding the fact that general solutions of PDEs are seldom of use in solving initial boundary value problems, d'Alembert's solution (119) of (115) provides considerable insight into the behavior of vibrating strings that are free of external forces. Consider first a taut string that at time $t = 0$ is released from rest $[g(x) \equiv 0]$ from the position in Figure 1.30(a) $[f(x) = 0$ for $|x - L/2| \geq L/16]$. This is not a particularly realistic initial displacement in view of the assumptions in Section 1.3 that displacements and slopes must be small. But because our discussion is independent of $f(x)$, we have purposely exaggerated the initial shape in order that our graphical representations be unmistakable. According to d'Alembert's solution (119), subsequent displacements of the string are defined by

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)], \tag{120}$$

and it is quite simple to obtain a pictorial history of the string using this function. For any given time t, the graph of $f(x + ct)/2$ is one-half that of $f(x)$ translated ct units to the left; $f(x - ct)/2$ is one-half of $f(x)$ shifted ct units to the right. The position of the string at this particular time is the sum of these two graphs. We have shown this procedure for the times $t = L/(64c)$, $L/(32c)$, $3L/(64c)$, $L/(16c)$, and $L/(8c)$ in Figures 1.30(b), (c), (d), (e), and (f), respectively. The dotted curves represent $f(x + ct)/2$, the dashed curves $f(x - ct)/2$, and the solid curves $y(x, t)$.

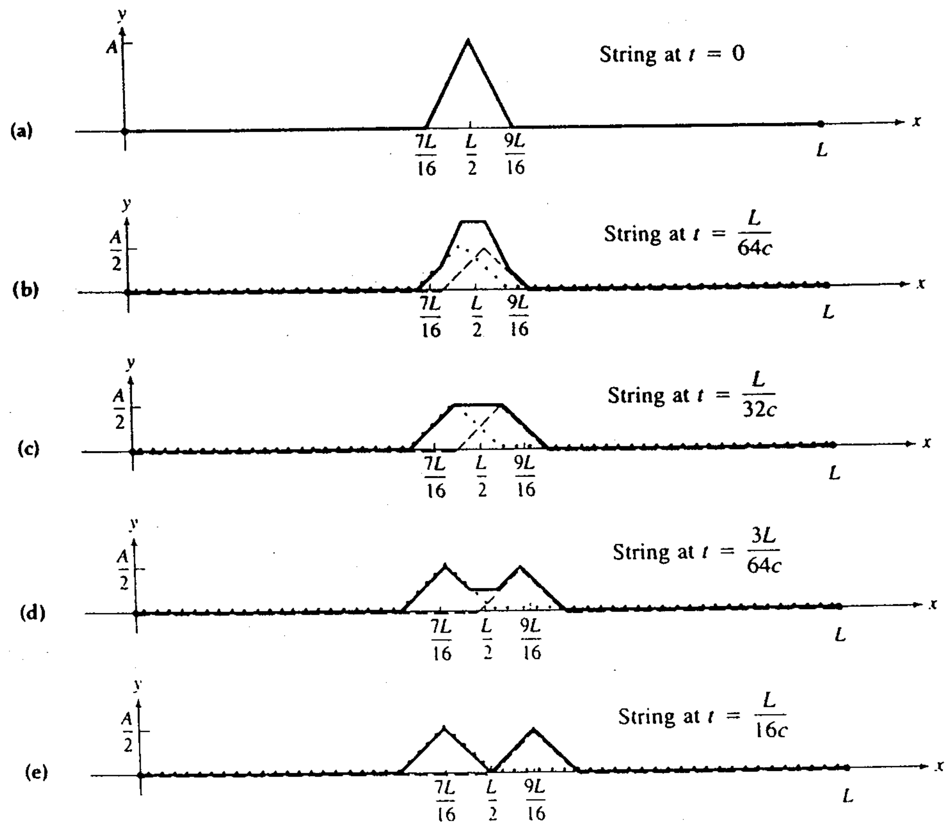


Figure 1.30

(Continued)

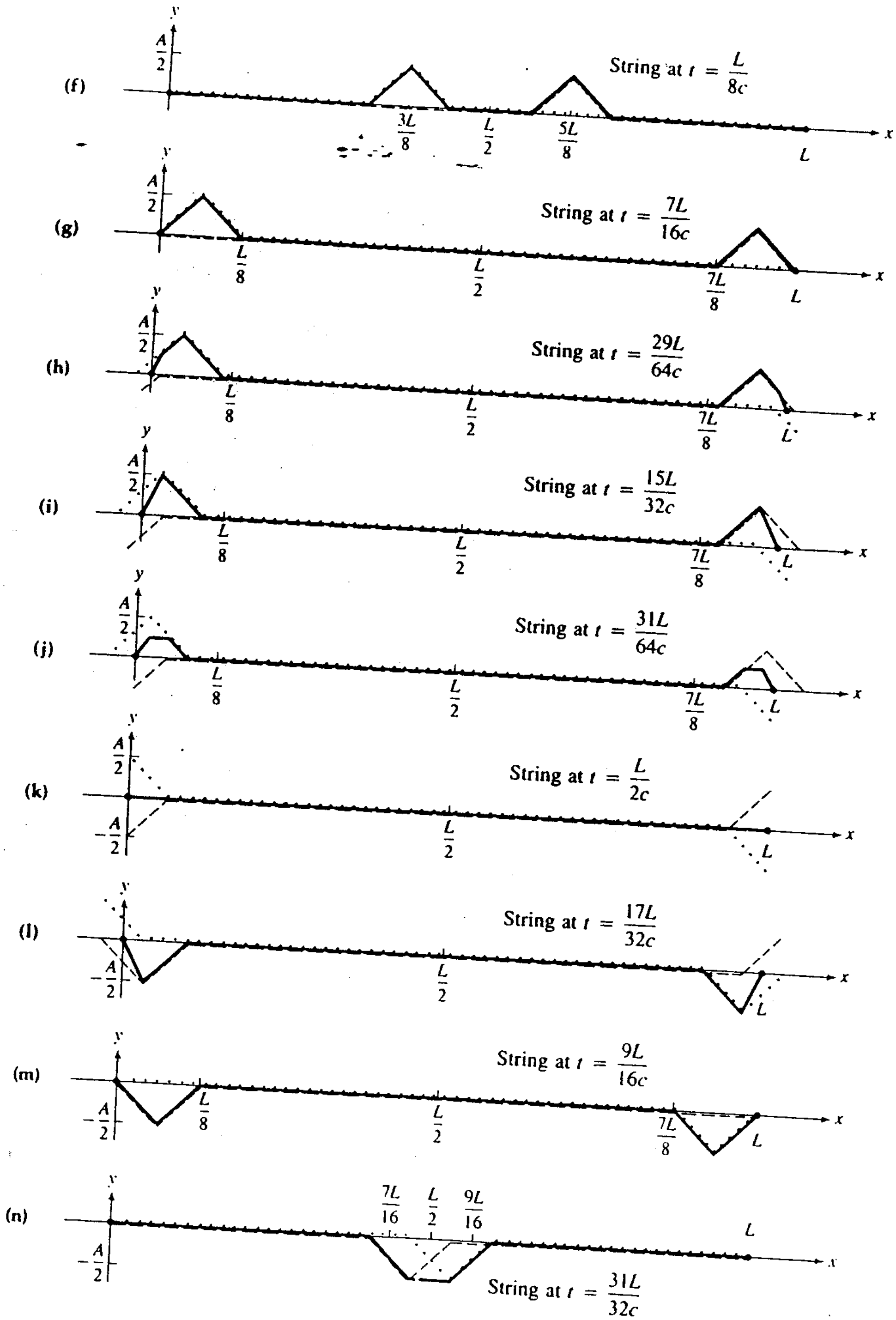


Figure 1.30 (Continued)

(Continued)

(o)

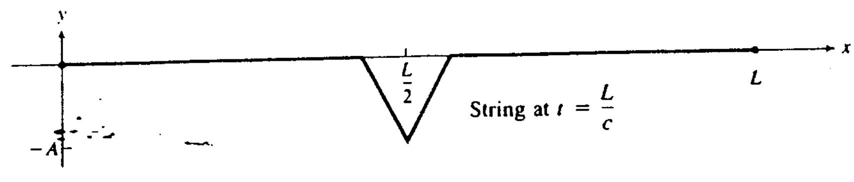


Figure 1.30 (*Continued*)

In Figures 1.30(g), (h), (i), (j), and (k), where we have continued this graphical construction at times $t = 7L/(16c)$, $29L/(64c)$, $15L/(32c)$, $31L/(64c)$, and $L/(2c)$, respectively, we have used the fact that $f(x)$ must be extended as an odd, $2L$-periodic function. At $t = L/(2c)$, the string is completely horizontal. Figures 1.30(l), (m), (n), and (o) show the string at four additional times. This procedure clearly yields the position of the string at any required time.

What is most interesting is that these graphs suggest the following physical description for the motion of the string. Figures 1.30(a)–(g) indicate that the initial deflection $f(x)$ in the string divides into two parts, each equal to one-half of $f(x)$, one traveling to the left with velocity $-c$ and the other traveling to the right with velocity c. Figures 1.30(h)–(o) suggest that when these disturbances reach the fixed ends of the string at time $7L/(16c)$, they are reflected there with a reversal of sign. The reflected disturbance then combines with the original disturbance to yield the total deflection. Reflected disturbances then travel toward one another at speed c, eventually combining at time $t = L/c$ to give a disturbance identical to that in Figure 1.30(a), but with a reversal in sign.

For times $t > L/c$, the disturbances separate again, travel to the ends of the string, are reflected there, and recombine at $t = 2L/c$ to yield the initial position in Figure 1.30(a).

For times $t > 2L/c$, the two disturbances continue to travel back and forth along the string, interfering constructively near the center of the string and destructively at the ends.

All of these things happen very quickly. For instance, if the tension in a 1-m string with density $\rho = 2.0$ g/m is 50 N, then $2L/c = 0.0126$. Thus, the initial displacement separates into two parts, and these two disturbances travel twice the length of the string and recombine to give the initial displacement in 0.0126 s. In other words, all of this happens $1/0.0126 = 79$ times each second, too fast for the human eye, but not for sophisticated cameras.

Example 5: Find the position of the string described by (120) at time $t = 1023L/(32c)$ when $f(x)$ is as shown in Figure 1.30(a).

Solution: In each time interval of length $2L/c$ after $t = 0$, the initial disturbance separates into two parts; each part travels to an end of the string and is reflected, then travels to the other end of the string and is reflected, and the parts then recombine to form $f(x)$ once again. Since $1023L/(32c) = 15(2L/c) + 63L/(32c)$, the position of the string at time $t = 1023L/(32c)$ is identical to that at $t = 63L/(32c)$. But this is 63/64 of the time for a complete cycle; that is, the two waves will be in the positions shown in Figure 1.31(a). These are combined to give the position of the string in Figure 1.31(b).

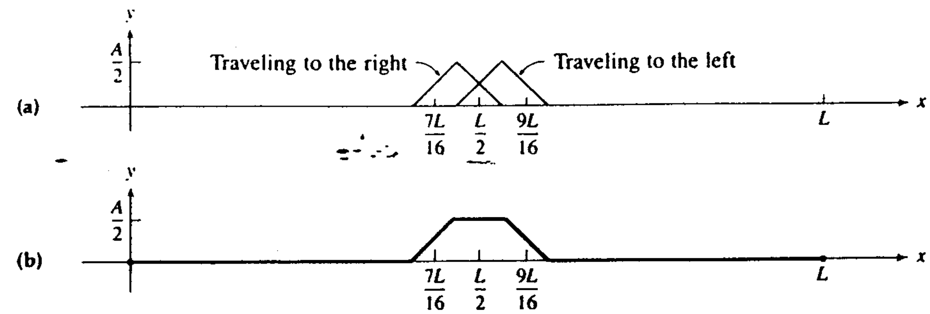


Figure 1.31

An alternative procedure is to write (120) at $t = 1023L/(32c)$ in the form

$$y\left(x, \frac{1023L}{32c}\right) = \frac{1}{2}\left[f\left(x + \frac{1023L}{32}\right) + f\left(x - \frac{1023L}{32}\right)\right]$$
$$= \frac{1}{2}\left[f\left(x + 16(2L) - \frac{L}{32}\right) + f\left(x - 16(2L) + \frac{L}{32}\right)\right]$$
$$= \frac{1}{2}\left[f\left(x - \frac{L}{32}\right) + f\left(x + \frac{L}{32}\right)\right],$$

since $f(x)$ is $2L$-periodic $[f(x + 2L) = f(x)]$. These functions are shown in Figure 1.31(a) and added in 1.31(b). ■

The above discussion and example have illustrated that the motion of a string with initial displacement $f(x)$ as shown in Figure 1.30(a) and zero initial velocity is easily described. For more complicated functions $f(x)$, such as in Figure 1.32, the principles are the same; the only difference is that reflections at the ends of the string begin immediately. Examples of this are given in Exercises 3 and 4.

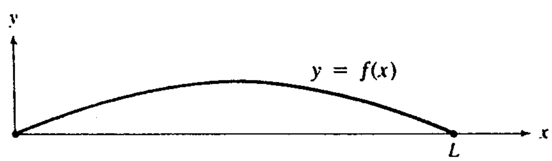


Figure 1.32

These ideas may also be extended to the situation in which the string is given a nonzero initial velocity $g(x)$, but no initial displacement, $f(x) \equiv 0$. In this case, (119) yields

$$y(x, t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\zeta)\,d\zeta \qquad \textbf{(121a)}$$

as the displacement of the string at position x and time t. Suppose, for example, that

$$g(x) = \begin{cases} 0 & 0 \le x < 7L/16 \\ k & 7L/16 < x < 9L/16, \\ 0 & 9L/16 < x \le L \end{cases}$$

where $k > 0$ is a constant (Figure 1.33). (It is not obviously so, but such an initial velocity can be achieved by striking that part of the string $7L/16 < x < 9L/16$ with a hammer.)

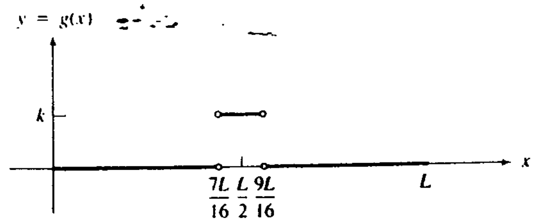


Figure 1.33

If we denote by $G(x)$ the antiderivative

$$G(x) = \frac{1}{2c}\int_0^x g(\zeta)\,d\zeta,$$

$y(x, t)$ can be expressed in the form

$$y(x, t) = G(x + ct) - G(x - ct), \tag{121b}$$

where, because $g(x)$ is extended as an odd, $2L$-periodic function [Figure 1.34(a)], the graph of $G(x)$ is as shown in Figure 1.34(b). The position of the string at any given time can now be obtained by the (destructive) combination of the left-traveling wave $G(x + ct)$ and the right-traveling wave $G(x - ct)$. Results are shown for various times in Figure 1.35.

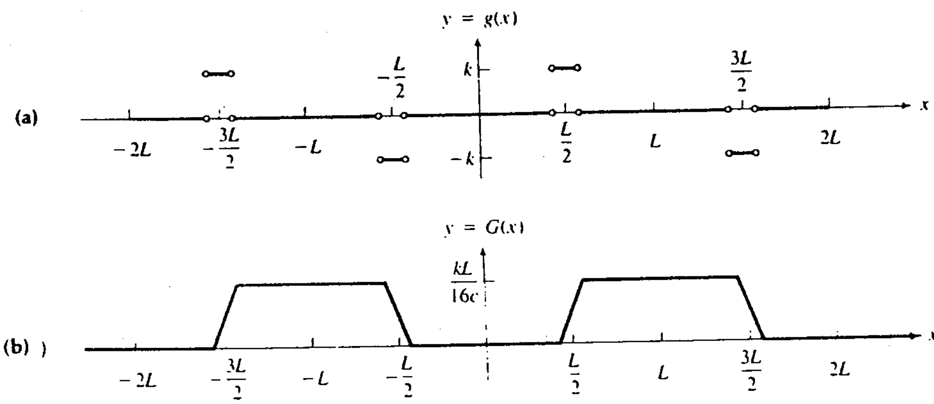


Figure 1.34

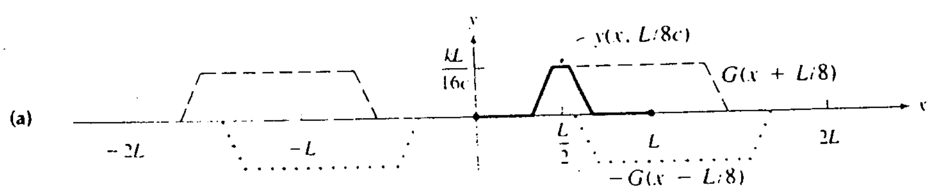


Figure 1.35

(*Continued*)

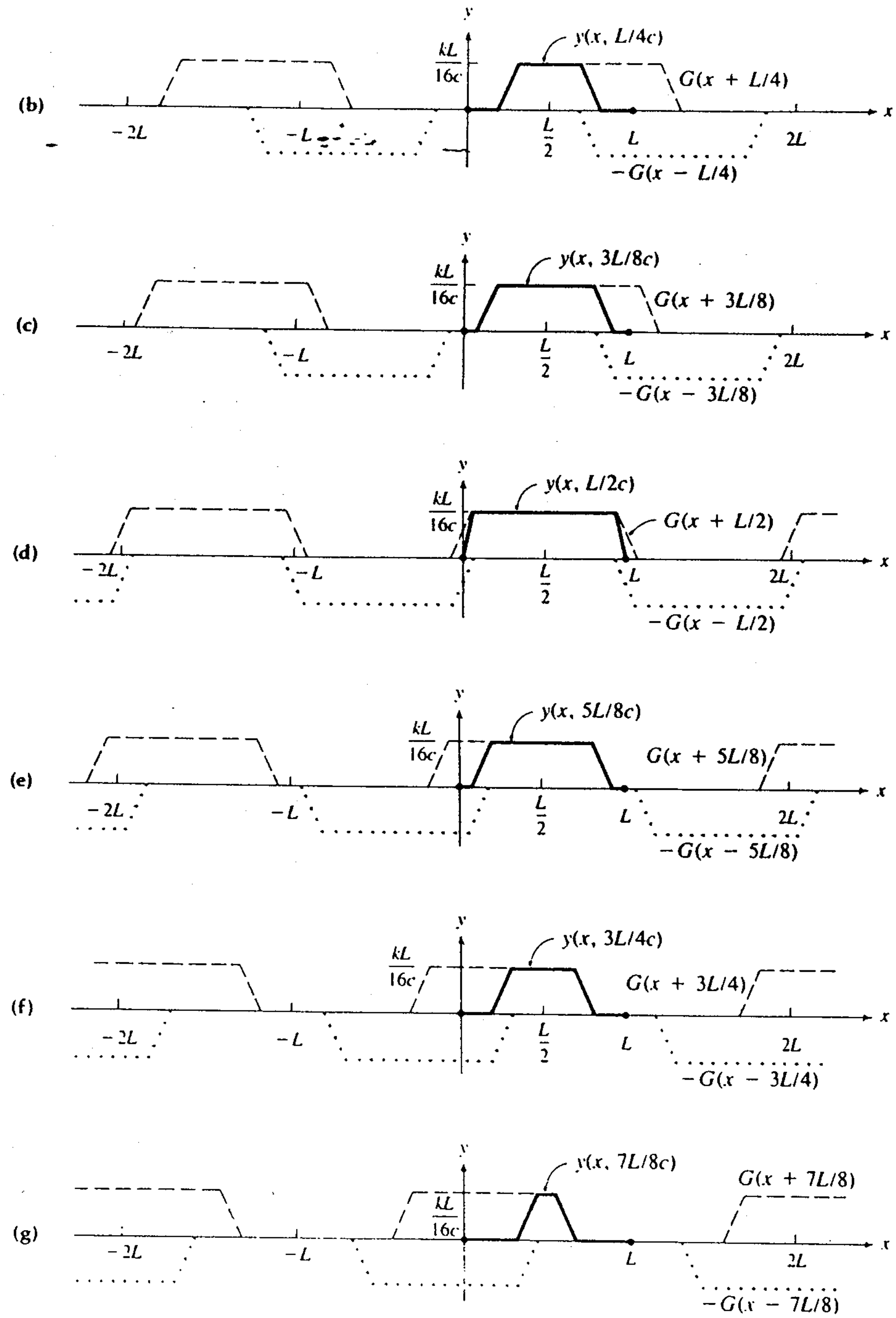


Figure 1.35 (*Continued*)

(*Continued*)

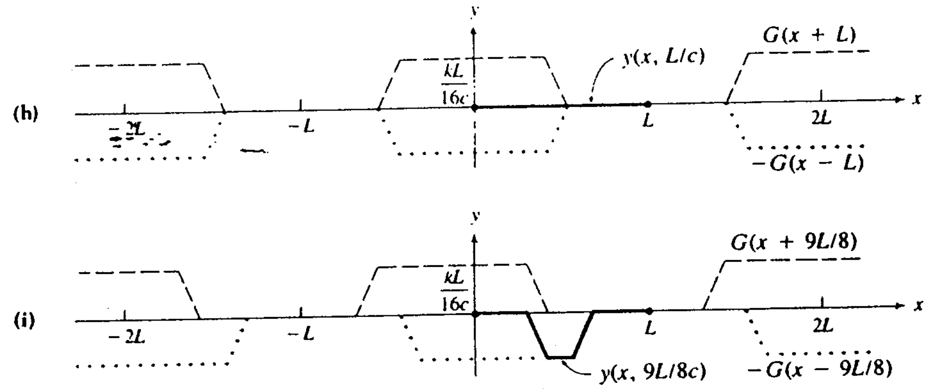


Figure 1.35 (*Continued*)

When a string has both an initial displacement $f(x)$ and an initial velocity $g(x)$, graphical techniques may still be used to determine the solution of (115). We express $y(x,t)$ in the form $y(x,t) = u(x,t) + v(x,t)$, where $u(x,t)$ and $v(x,t)$ satisfy the problems

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} &\qquad &\frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial x^2} \\ &u(0,t) = 0 & &v(0,t) = 0 \\ &u(L,t) = 0 & &v(L,t) = 0 \\ &u(x,0) = f(x) & &v(x,0) = 0 \\ &u_t(x,0) = 0 & &v_t(x,0) = g(x). \end{aligned}$$

Exercises 1.7

1. Show that the transformation of independent variables $v = x + ct$ and $\eta = x - ct$ replaces the wave equation (115a) with (116).
2. Determine the position of the string in Figure 1.30(a) when (a) $t = L/c$; (b) $t = 21L/(8c)$.
3. Use the graphical techniques of this section to determine the displacements of a string with zero initial velocity and initial displacement

$$f(x) = \begin{cases} x/8 & 0 \le x \le L/2 \\ (L - x)/8 & L/2 \le x \le L \end{cases}$$

at the following times:

(a) $t = L/(8c)$ (b) $t = L/(4c)$ (c) $t = 3L/(8c)$ (d) $t = L/(2c)$
(e) $t = 5L/(8c)$ (f) $t = 3L/(4c)$ (g) $t = 7L/(8c)$ (h) $t = L/c$

4. Repeat Exercise 3 with $f(x) = \sin(2\pi x/L)$, $0 \le x \le L$.
5. Use the graphical techniques of this section to determine the displacements of a string with zero initial displacement and initial velocity

$$g(x) = \begin{cases} 0 & 0 \le x < L/4 \\ 1 & L/4 < x < 3L/4 \\ 0 & 3L/4 < x \le L \end{cases}$$

for the times in Exercise 3.

6. Repeat Exercise 5 with

$$g(x) = \begin{cases} 0 & 0 \le x < L/8 \\ 1 & L/8 < x < 3L/8 \\ 0 & 3L/8 < x < 5L/8. \\ 1 & 5L/8 < x < 7L/8 \\ 0 & 7L/8 < x \le L \end{cases}$$

1.8 Classification of Second-Order Partial Differential Equations

The material in this section is not essential at this point in our discussions. It can be considered at any time, since it is neither a prerequisite for subsequent discussions nor dependent on them. We include it here because it acts somewhat as a justification for the approach that we take in the remainder of the book. We intend solving the initial boundary value problems in Sections 1.2–1.6 using the techniques of separation of variables; Fourier transforms, both finite and infinite; Laplace transforms; and Green's functions. Second-order PDEs play a prominent role in these problems; the only application we have seen so far that gives rise to a PDE that is not second order is that for beam vibrations. What we illustrate here is that all linear second-order PDEs (we define this term shortly) are basically of three types. These types correspond generally to Poisson's equation, the wave equation, and the heat conduction equation. Consequently, once we learn how to apply the above techniques to these three equations, we have essentially learned how to handle all second-order linear equations.

For purposes of classification, it is not necessary to restrict consideration to linear equations. The classification is also valid for equations that are linear only in their second derivatives. Such equations, in two independent variables, are of the form

$$a(x, y)u_{xx} + b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u, u_x, u_y), \tag{122}$$

where f is any function of its arguments whatsoever. It is assumed that a, b, and c have continuous first partial derivatives in some domain D and that these coefficient functions do not all vanish simultaneously. We classify such PDEs into one of three types—elliptic, parabolic, or hyperbolic—and each type of PDE displays characteristics quite distinct from the others. This classification is stated as follows:

Partial differential equation (122) is said to be hyperbolic at a point (x, y) if

$$b^2 - 4ac > 0; \tag{123a}$$

parabolic at a point (x, y) if

$$b^2 - 4ac = 0; \tag{123b}$$

and elliptic at a point (x, y) if

$$b^2 - 4ac < 0. \tag{123c}$$

This classification is a pointwise one so that a PDE may change its type from point to point. The one-dimensional wave equation (37) is hyperbolic at all points; the one-dimensional heat conduction equation in Example 3 of Section 1.2 is parabolic; and Poisson's equation (6a) is elliptic.

We shall show that by means of a change of independent variables

$$v = v(x, y), \qquad \eta = \eta(x, y), \tag{124}$$

PDE (122) can be transformed into simpler forms. We require that functions $v(x, y)$ and $\eta(x, y)$ have continuous second partial derivatives in D and that the Jacobian

$$\frac{\partial(v, \eta)}{\partial(x, y)} \neq 0 \tag{125}$$

in order that the original variables x and y be retrievable:

$$x = x(v, \eta), \qquad y = y(v, \eta). \tag{126}$$

When we replace x and y by v and η, we denote the dependent variable by $w(v, \eta) = u[x(v, \eta), y(v, \eta)]$. It follows, then, that $u(x, y) = w[v(x, y), \eta(x, y)]$, and chain rules for partial derivatives permit us to express derivatives of $u(x, y)$ with respect to x and y in terms of derivatives of w with respect to v and η:

$$u_x = w_v v_x + w_\eta \eta_x, \qquad u_y = w_v v_y + w_\eta \eta_y$$

and

$$\begin{aligned} u_{xx} &= (w_{vv} v_x + w_{v\eta} \eta_x) v_x + w_v v_{xx} + (w_{\eta v} v_x + w_{\eta\eta} \eta_x)\eta_x + w_\eta \eta_{xx}, \\ u_{xy} &= (w_{vv} v_y + w_{v\eta} \eta_y) v_x + w_v v_{xy} + (w_{\eta v} v_y + w_{\eta\eta} \eta_y)\eta_x + w_\eta \eta_{xy}, \\ u_{yy} &= (w_{vv} v_y + w_{v\eta} \eta_y) v_y + w_v v_{yy} + (w_{\eta v} v_y + w_{\eta\eta} \eta_y)\eta_y + w_\eta \eta_{yy}. \end{aligned}$$

The PDE in w as a function of v and η equivalent to (122) is therefore

$$\begin{aligned} &(av_x^2 + bv_x v_y + cv_y^2)w_{vv} + [2av_x\eta_x + b(v_x\eta_y + v_y\eta_x) + 2cv_y\eta_y]w_{v\eta} \\ &\quad + (a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2)w_{\eta\eta} + (av_{xx} + bv_{xy} + cv_{yy})w_v \\ &\quad + (a\eta_{xx} + b\eta_{xy} + c\eta_{yy})w_\eta = f[x(v, \eta), y(v, \eta), w, w_v v_x + w_\eta \eta_x, w_v v_y + w_\eta \eta_y] \end{aligned}$$

or

$$\alpha w_{vv} + \beta w_{v\eta} + \gamma w_{\eta\eta} = F(v, \eta, w, w_v, w_\eta), \tag{127a}$$

where

$$\begin{aligned} \alpha &= av_x^2 + bv_x v_y + cv_y^2, \\ \beta &= 2av_x\eta_x + b(v_x\eta_y + v_y\eta_x) + 2cv_y\eta_y, \\ \gamma &= a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 \end{aligned} \tag{127b}$$

and

$$\begin{aligned} F(v, \eta, w, w_v, w_\eta) = {} & f[x(v, \eta), y(v, \eta), w, w_v v_x + w_\eta \eta_x, w_v v_y + w_\eta \eta_y] \\ & - (av_{xx} + bv_{xy} + cv_{yy})w_v \\ & - (a\eta_{xx} + b\eta_{xy} + c\eta_{yy})w_\eta. \end{aligned} \tag{127c}$$

It is a simple exercise to show that

$$\beta^2 - 4\alpha\gamma = (b^2 - 4ac)\left(\frac{\partial(v, \eta)}{\partial(x, y)}\right)^2, \tag{128}$$

a result that proves that our classification of PDEs is invariant under a real transformation of independent variables.

We now suppose that PDE (122) is of the same type at every point (x, y) in D and show that hyperbolic PDEs can be transformed into the *canonical* form

$$w_{\nu\eta} = F(\nu, \eta, w, w_\nu, w_\eta); \tag{129a}$$

parabolic PDEs can be transformed into the canonical form

$$w_{\nu\nu} = F(\nu, \eta, w, w_\nu, w_\eta); \tag{129b}$$

and elliptic PDEs can be transformed into the form

$$w_{\nu\nu} + w_{\eta\eta} = F(\nu, \eta, w, w_\nu, w_\eta). \tag{129c}$$

Hyperbolic Equations

For hyperbolic PDEs, we claim the existence of a transformation (124) that reduces the PDE to canonical form (129a). This is possible if functions $\nu(x, y)$ and $\eta(x, y)$ can be found to satisfy

$$0 = \alpha = a\nu_x^2 + b\nu_x\nu_y + c\nu_y^2, \tag{130a}$$

$$0 = \gamma = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2 \tag{130b}$$

or

$$0 = a\left(\frac{\nu_x}{\nu_y}\right)^2 + b\frac{\nu_x}{\nu_y} + c, \tag{131a}$$

$$0 = a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\frac{\eta_x}{\eta_y} + c; \tag{131b}$$

that is, the ratios ν_x/ν_y and η_x/η_y must satisfy the same equation,

$$a\lambda^2 + b\lambda + c = 0. \tag{132}$$

Since $b^2 - 4ac > 0$, there are two distinct solutions, $\lambda_1 = \lambda_1(x, y)$ and $\lambda_2 = \lambda_2(x, y)$, of this quadratic. Consequently, when functions $\nu(x, y)$ and $\eta(x, y)$ satisfy the first-order PDEs

$$\nu_x = \lambda_1(x, y)\nu_y, \qquad \eta_x = \lambda_2(x, y)\eta_y, \tag{133}$$

the PDE in w as a function of ν and η is reduced to the form

$$\beta w_{\nu\eta} = F(\nu, \eta, w, w_\nu, w_\eta). \tag{134}$$

Since $\beta^2 - 4\alpha\gamma = \beta^2 = (b^2 - 4ac)[\partial(\nu, \eta)/\partial(x, y)]^2 \neq 0$, we may divide by β and obtain the canonical form for a hyperbolic PDE.

Because of the form of PDEs (133), solutions can be obtained with ODEs. Indeed, suppose the ordinary differential equation

$$\frac{dy}{dx} = -\lambda_1(x, y) \tag{135}$$

has a solution defined implicitly by

$$\nu(x, y) = C_1. \tag{136}$$

Then each curve in this one-parameter family has slope defined by

$$v_x + v_y \frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{v_x}{v_y}. \tag{137}$$

Consequently, when (135) is solved in form (136), function $v(x, y)$ satisfies the PDE

$$\frac{v_x}{v_y} = \lambda_1(x, y). \tag{138}$$

The curves defined implicitly by (136) are called *characteristic curves* for the hyperbolic PDE; they are determined by the coefficients a, b, and c in the equation.

Similarly, the ODE

$$\frac{dy}{dx} = -\lambda_2(x, y) \tag{139}$$

defines a one-parameter family of curves

$$\eta(x, y) = C_2, \tag{140}$$

also called characteristic curves, and $\eta(x, y)$ is a solution of $\eta_x = \lambda_2(x, y)\eta_y$.

Each of the families $v(x, y) = C_1$ and $\eta(x, y) = C_2$ forms a covering of the domain of the xy-plane in which the PDE is hyperbolic (Figure 1.36). Furthermore, at no point can the particular curves from each family share a common tangent (else $\lambda_1 = \lambda_2$ at that point).

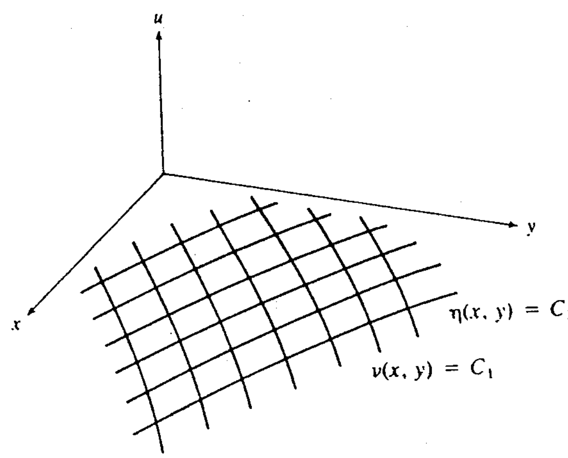


Figure 1.36

Under the transformation $v = v(x, y)$, $\eta = \eta(x, y)$, regarded as a mapping from the xy-plane to the $v\eta$-plane, curves along which v and η are constant in the xy-plane become coordinate lines in the $v\eta$-plane. Since these are precisely the characteristic curves, we conclude that when a hyperbolic PDE is in canonical form, coordinate lines are characteristic curves for the PDE. In other words, characteristic curves of a hyperbolic PDE are those curves to which the PDE must be referred as coordinate curves in order that it take on canonical form.

Example 6: Show that the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} - \frac{\tau}{\rho}\frac{\partial^2 y}{\partial x^2} = \frac{1}{\rho}F(x, t, y, y_x, y_t)$$

is hyperbolic, and find an equivalent equation in canonical form.

Solution: Since $b^2 - 4ac = 4\tau/\rho > 0$, the PDE is hyperbolic. Characteristic curves can be found by solving the ODE

$$\frac{dx}{dt} = -\lambda(x, t),$$

where $\lambda(x, t)$ satisfies $\lambda^2 - \tau/\rho = 0$. From the equations

$$\frac{dx}{dt} = -\lambda_1 = -\sqrt{\frac{\tau}{\rho}} \quad \text{and} \quad \frac{dx}{dt} = -\lambda_2 = \sqrt{\frac{\tau}{\rho}},$$

we obtain the characteristic curves

$$x = -\sqrt{\frac{\tau}{\rho}}\,t + C_1, \qquad x = \sqrt{\frac{\tau}{\rho}}\,t + C_2.$$

It follows, then, that the transformation

$$\nu = x + \sqrt{\frac{\tau}{\rho}}\,t, \qquad \eta = x - \sqrt{\frac{\tau}{\rho}}\,t$$

will reduce the wave equation to canonical form in $w(\nu, \eta)$:

$$\frac{\partial^2 w}{\partial \nu \partial \eta} = \frac{-1}{4\tau}F\left(x(\nu,\eta), t(\nu,\eta), w, w_\nu + w_\eta, \sqrt{\frac{\tau}{\rho}}w_\nu - \sqrt{\frac{\tau}{\rho}}w_\eta\right). \quad \blacksquare$$

Notice that when $F \equiv 0$ in this example, the canonical form for the one-dimensional wave equation $y_{tt} - (\tau/\rho)y_{xx} = 0$ is

$$\frac{\partial^2 w}{\partial \nu \partial \eta} = 0.$$

This is precisely equation (116) of Section 1.7, but now we see the origin of the transformation $\nu = x + ct$ and $\eta = x - ct$.

If $\psi(\nu)$ and $\phi(\eta)$ are any two (twice continuously differentiable) functions of the canonical variables ν and η, then

$$\psi_x = \psi'(\nu)\nu_x = \lambda_1(x, y)\nu_y\psi'(\nu) = \lambda_1(x, y)\psi_y,$$
$$\phi_x = \phi'(\eta)\eta_x = \lambda_2(x, y)\eta_y\phi'(\eta) = \lambda_2(x, y)\phi_y.$$

Thus, $\psi[\nu(x, y)]$ and $\phi[\eta(x, y)]$ also satisfy (133), and it follows that any transformation of the form

$$\psi = \psi[\nu(x, y)], \qquad \phi = \phi[\eta(x, y)] \tag{141}$$

also reduces the PDE to canonical form (in ψ and ϕ).

Finally, notice that if we set $r = v + \eta$, $s = v - \eta$, and $f(r,s) = w[v(r,s), \eta(r,s)]$, then

$$w_v = f_r r_v + f_s s_v = f_r + f_s$$

and

$$w_{v\eta} = f_{rr} r_\eta + f_{rs} s_\eta + f_{sr} r_\eta + f_{ss} s_\eta$$
$$= f_{rr} - f_{rs} + f_{sr} - f_{ss} = f_{rr} - f_{ss}.$$

Consequently, the PDE in $f(r,s)$ corresponding to (129a) is

$$f_{rr} - f_{ss} = F[v(r,s), \eta(r,s), f, f_r + f_s, f_r - f_s]; \qquad (142)$$

this is sometimes used as a canonical form for hyperbolic equations.

Parabolic Equations

Parabolic equations can be transformed into canonical form (129b) by (124) if functions $v(x, y)$ and $\eta(x, y)$ can be found to satisfy

$$0 = \beta = 2av_x\eta_x + b(v_x\eta_y + v_y\eta_x) + 2cv_y\eta_y, \qquad (143a)$$
$$0 = \gamma = a\eta_x^2 + b\eta_x\eta_y + c\eta_y^2. \qquad (143b)$$

The second equation can be written in the form

$$0 = a\left(\frac{\eta_x}{\eta_y}\right)^2 + b\left(\frac{\eta_x}{\eta_y}\right) + c \qquad (144)$$

so that η_x/η_y must satisfy

$$0 = a\lambda^2 + b\lambda + c. \qquad (145)$$

Since $b^2 - 4ac = 0$, there is exactly one solution $\lambda = \lambda(x, y)$ of this quadratic, and $\eta(x, y)$ must therefore satisfy the first-order PDE

$$\eta_x = \lambda(x, y)\eta_y. \qquad (146)$$

When $\eta(x, y)$ is so defined, $\gamma = 0$ and, from (128),

$$0 = (b^2 - 4ac)\left(\frac{\partial(v, \eta)}{\partial(x, y)}\right)^2 = \beta^2 - 4\alpha\gamma = \beta^2.$$

Thus, β must also vanish, and PDE (127a) in the parabolic case reduces to

$$\alpha w_{vv} = F(v, \eta, w, w_v, w_\eta). \qquad (147)$$

Since $\alpha \neq 0$ (why?), we may divide to obtain the canonical form (129b) for a parabolic PDE.

We may obtain $\eta(x, y)$ by writing the solutions of the ODE

$$\frac{dy}{dx} = -\lambda(x, y) \qquad (148)$$

in the form

$$\eta(x, y) = C. \qquad (149)$$

The curves in this one-parameter family are called characteristic curves for the parabolic PDE. Parabolic PDEs therefore have only one family of characteristic curves. Notice that no mention of v has been made throughout the discussion. It follows that the canonical form for parabolic PDEs is obtained for arbitrary $v(x, y)$, except that $v(x, y)$ must be chosen to yield a nonvanishing Jacobian (125).

Example 7: Is the one-dimensional heat conduction equation

$$k\frac{\partial^2 U}{\partial x^2} = \frac{\partial U}{\partial t} - \frac{k}{\kappa}g(x, t)$$

parabolic?

Solution: The equation is already in canonical form for a parabolic PDE. ■

Example 8: Show that the PDE

$$x^2u_{xx} - 2xyu_{xy} + y^2u_{yy} = x^2 + u_y$$

is parabolic, and find an equivalent PDE in canonical form.

Solution: Because $b^2 - 4ac = (-2xy)^2 - 4(x^2)(y^2) = 0$, the PDE is everywhere parabolic. Characteristic curves can be found by solving

$$\frac{dy}{dx} = -\lambda(x, y)$$

where $\lambda(x, y)$ satisfies

$$0 = x^2\lambda^2 - 2xy\lambda + y^2 = (x\lambda - y)^2.$$

Consequently, we solve

$$\frac{dy}{dx} = -\frac{y}{x},$$

the solution of which can be written in the form $xy = C$. We choose therefore $\eta(x, y) = xy$, and $v(x, y)$ is arbitrary except that the Jacobian $\partial(v, \eta)/\partial(x, y) \neq 0$. If we choose $v(x, y) = y$, then

$$\frac{\partial(v, \eta)}{\partial(x, y)} = \begin{vmatrix} 0 & 1 \\ y & x \end{vmatrix} = -y \neq 0 \quad \text{(except along the } x\text{-axis).}$$

Instead of using (127) to write the PDE in $w(v, \eta)$ equivalent to the original equation in $u(x, y)$, let us perform the transformation. To do this, we require the following partial derivatives:

$$u_x = w_v v_x + w_\eta \eta_x = y w_\eta, \qquad u_y = w_v v_y + w_\eta \eta_y = w_v + x w_\eta,$$
$$u_{xx} = y(w_{\eta v} v_x + w_{\eta\eta}\eta_x) = y^2 w_{\eta\eta},$$
$$u_{xy} = w_\eta + y(w_{\eta v} v_y + w_{\eta\eta}\eta_y) = w_\eta + y w_{\eta v} + xy w_{\eta\eta},$$
$$u_{yy} = w_{vv} v_y + w_{v\eta}\eta_y + x(w_{\eta v} v_y + w_{\eta\eta}\eta_y) = w_{vv} + 2x w_{v\eta} + x^2 w_{\eta\eta}.$$

Substitution of these into the PDE for $u(x, y)$ along with $x = \eta/v$ and $y = v$ gives

$$\frac{\eta^2}{v^2}v^2 w_{\eta\eta} - 2\frac{\eta}{v}v\left(w_\eta + vw_{\eta v} + \frac{\eta}{v}vw_{\eta\eta}\right) + v^2\left(w_{vv} + 2\frac{\eta}{v}w_{\eta v} + \frac{\eta^2}{v^2}w_{\eta\eta}\right)$$

$$= \frac{\eta^2}{v^2} + w_v + \frac{\eta}{v}w_\eta.$$

Thus, the PDE equivalent to the given equation is

$$w_{vv} = \frac{1}{v^4}[\eta^2 + v^2 w_v + \eta v(1 + 2v)w_\eta],$$

valid in any domain not containing points on the x-axis (for which $v = 0$). ■

Elliptic Equations

Transformation (124) reduces an elliptic PDE to canonical form (129c) if functions $v(x, y)$ and $\eta(x, y)$ can be found to satisfy

$$0 = 2av_x\eta_x + b(v_x\eta_y + v_y\eta_x) + 2cv_y\eta_y, \tag{150a}$$

$$0 = a(v_x^2 - \eta_x^2) + b(v_xv_y - \eta_x\eta_y) + c(v_y^2 - \eta_y^2). \tag{150b}$$

For hyperbolic PDEs, $v(x, y)$ and $\eta(x, y)$ satisfied first-order PDEs that were separated one from the other. Similarly, $\eta(x, y)$ in the parabolic case satisfied a first-order equation that was independent of $v(x, y)$. Unfortunately, equations (150) for $v(x, y)$ and $\eta(x, y)$ are mixed; both unknowns appear in both equations. In an attempt to separate them, we multiply the first by the complex number i and add to the second to give

$$a(v_x + i\eta_x)^2 + b(v_x + i\eta_x)(v_y + i\eta_y) + c(v_y + i\eta_y)^2 = 0.$$

This equation can be solved for two possible values of the ratio

$$\frac{v_x + i\eta_x}{v_y + i\eta_y} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} \tag{151}$$

(since $b^2 - 4ac$ is known to be negative). Real and imaginary parts of equation (151) give

$$v_x = \frac{-bv_y - (\pm)\eta_y\sqrt{4ac - b^2}}{2a}, \qquad \eta_x = \frac{-b\eta_y \pm v_y\sqrt{4ac - b^2}}{2a} \tag{152}$$

or

$$2av_x + bv_y = -(\pm)\eta_y\sqrt{4ac - b^2}, \qquad \pm v_y\sqrt{4ac - b^2} = 2a\eta_x + b\eta_y. \tag{153}$$

These are linear equations in v_x and v_y that have the following solutions in terms of the partial derivatives η_x and η_y:

$$v_x = -\frac{2c\eta_y + b\eta_x}{\pm\sqrt{4ac - b^2}}, \qquad v_y = \frac{2a\eta_x + b\eta_y}{\pm\sqrt{4ac - b^2}}. \tag{154}$$

These equations (called the Beltrami equations) are equivalent to (150), but they still form a mixed set of equations in the sense that v and η appear in both. A second-order

PDE for $\eta(x, y)$ is evidently

$$\frac{\partial}{\partial x}\left(\frac{2a\eta_x + b\eta_y}{\sqrt{4ac - b^2}}\right) = \frac{\partial}{\partial y}\left(-\frac{2c\eta_y + b\eta_x}{\sqrt{4ac - b^2}}\right). \tag{155}$$

If this equation is solved for $\eta(x, y)$ and then used to determine $v(x, y)$, the original PDE in u is transformed to the form

$$\alpha w_{vv} + \alpha w_{\eta\eta} = F(v, \eta, w, w_v, w_\eta). \tag{156}$$

Since $0 < (b^2 - 4ac)[\partial(v, \eta)/\partial(x, y)]^2 = \beta^2 - 4\alpha\gamma = -4\alpha^2$, it follows that $\alpha \neq 0$, and the elliptic PDE can be obtained in canonical form (129c).

The only difficulty with this procedure is that in general, PDE (155) for $\eta(x, y)$ may not be significantly easier to solve than the original PDE in $u(x, y)$. Instead, notice that the form of equation (151) suggests that we define a complex function $\phi(x, y)$ of two real variables x and y,

$$\phi(x, y) = v(x, y) + i\eta(x, y), \tag{157}$$

in which case $v(x, y)$ and $\eta(x, y)$ can be retrieved as the real and imaginary parts of $\phi(x, y)$. It is clear that $\phi(x, y)$ must satisfy one of the equations

$$\frac{\phi_x}{\phi_y} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a}. \tag{158}$$

To solve either one of these complex PDEs for $\phi(x, y)$, we employ the same technique used for hyperbolic and parabolic equations: we consider the ordinary differential equation

$$\frac{dy}{dx} = \frac{b + i\sqrt{4ac - b^2}}{2a} \tag{159}$$

[or $dy/dx = (b - i\sqrt{4ac - b^2})/(2a)$] for y as a function of x. Because the right side is complex, we must (temporarily) regard x and y as complex variables. If we obtain a solution in the form

$$\phi(x, y) = C, \tag{160}$$

then

$$\frac{dy}{dx} = -\frac{\phi_x}{\phi_y} = \frac{b + i\sqrt{4ac - b^2}}{2a}, \tag{161}$$

clearly indicating that $\phi(x, y)$ is the required function. Real and imaginary parts of ϕ (once again regarding x and y as real) give $v(x, y)$ and $\eta(x, y)$.

Because x and y in (159) are considered complex, elliptic PDEs do not have real characteristic curves.

Example 9: Find regions in which

$$u_{xx} + x^2 u_{yy} = y u_y$$

is elliptic, and find an equivalent PDE in canonical form.

Solution: Since $b^2 - 4ac = -4x^2$, the PDE is elliptic in any region that does not contain points on the y-axis. To find a transformation that will reduce the PDE to canonical form, we set

$$\frac{dy}{dx} = -\lambda(x, y),$$

where $\lambda(x, y)$ is one of the complex solutions of $\lambda^2 + x^2 = 0$. If we choose $\lambda = -ix$, then

$$\frac{dy}{dx} = ix$$

and $y = ix^2/2 + C$. The transformation functions v and η are the real and imaginary parts of $y - ix^2/2$,

$$v(x, y) = y, \qquad \eta(x, y) = \frac{-x^2}{2}.$$

With this transformation,

$$u_x = w_v v_x + w_\eta \eta_x = -x w_\eta, \qquad u_y = w_v v_y + w_\eta \eta_y = w_v,$$

$$u_{xx} = -w_\eta - x(w_{v\eta} v_x + w_{\eta\eta} \eta_x) = -w_\eta + x^2 w_{\eta\eta},$$

$$u_{yy} = w_{vv} v_y + w_{v\eta} \eta_y = w_{vv}.$$

Substitution of these into the original PDE gives

$$w_{vv} + w_{\eta\eta} = \frac{-1}{2\eta}(w_\eta + v w_v).$$

Had we chosen to set $dy/dx = -ix$, the transformation would have been $v(x, y) = y$, $\eta(x, y) = x^2/2$, and the equivalent PDE would have been

$$w_{vv} + w_{\eta\eta} = \frac{1}{2\eta}(w_\eta + v w_v).$$ ■

To summarize our results, all second-order PDEs in two independent variables that are linear in their second derivatives can be classified as hyperbolic, parabolic, or elliptic. The one-dimensional wave equation is hyperbolic, the one-dimensional heat equation is parabolic, and the two-dimensional Poisson's equation is elliptic. We can therefore discover properties of all second-order PDEs in two independent variables that are linear in second derivatives by analyzing vibrating strings, heat conduction in rods, and two-dimensional electrostatic problems. Each type of equation has properties distinct from the others; properties of hyperbolic equations differ from those of parabolic equations, and these in turn differ from those of elliptic equations. For instance, in Section 1.7 we saw that a disturbance (more generally, information) is transmitted by the wave equation (a hyperbolic equation) at finite speed. Information (in the form of heat) is transmitted infinitely fast by the heat equation (see Section 5.6). Elliptic equations represent static or steady-state situations. Other properties of hyperbolic, parabolic, and elliptic equations are discussed throughout the book, particularly in Sections 5.6–5.8. Problems associated with these equations are even characterized differently; all three are accompanied by boundary conditions, but the wave equation has two initial conditions, the heat equation has one, and Poisson's equation has none.

Second-order PDEs in more than two independent variables can also be classified into types, including parabolic, elliptic, and hyperbolic. However, it is not usually possible to reduce such equations to simple canonical forms. One instance in which a canonical form is possible is for PDEs with constant coefficients. We shall not discuss the classification and canonical forms here, but we should point out that in this classification, the three-dimensional Laplace equation is elliptic, the multidimensional wave equation is hyperbolic, and the multidimensional heat equation is parabolic.

Exercises 1.8

In Exercises 1–4, classify the PDE as hyperbolic, parabolic, or elliptic and find an equivalent PDE in canonical form.

1. $u_{xx} + 2u_{xy} + u_{yy} = u_x - xu_y$
2. $u_{xx} + 2u_{xy} + 5u_{yy} = 3u_x - yu$
3. $3u_{xx} + 10u_{xy} + 3u_{yy} = 0$
4. $u_{xx} + 6u_{xy} + u_{yy} = 4uu_x$

In Exercises 5–11, determine where the PDE is hyperbolic, parabolic, and elliptic. Illustrate each region graphically in the xy-plane.

5. $u_{xx} + 2yu_{xy} + 5u_{yy} = 15x + 2y$
6. $x^2u_{xx} + 4yu_{yy} = u$
7. $x^2yu_{xx} + xyu_{xy} - y^2u_{yy} = 0$
8. $xyu_{xx} - xu_{xy} + u_{yy} = uu_x + 3$
9. $(\sin x)u_{xx} + (2\cos x)u_{xy} + (\sin x)u_{yy} = 0$
10. $(x \ln y)u_{xx} + 4u_{yy} = u_x - 3xyu$
11. $u_{xx} + xu_{xy} + yu_{yy} = 0$
12. Find a PDE in canonical form equivalent to the PDE in Example 8 that is valid in regions not containing points on the y-axis.
13. (a) Show that the Tricomi PDE

$$yu_{xx} + u_{yy} = 0$$

is hyperbolic when $y < 0$, parabolic when $y = 0$, and elliptic when $y > 0$.

(b) Find an equivalent PDE in canonical form when $y < 0$.

(c) Find an equivalent PDE in canonical form when $y > 0$.

(d) Find an equivalent PDE in canonical form when $y = 0$.

14. Find regions in which

$$x^2u_{xx} + 4u_{yy} = u$$

is hyperbolic, parabolic, and elliptic. In each such region, find an equivalent PDE in canonical form.

15. Show that

$$y^2u_{xx} - 2xyu_{xy} + x^2u_{yy} = 0$$

is everywhere parabolic. Find an equivalent PDE in canonical form valid in regions not containing points on the x-axis.

16. Show that

$$u_{xx} + x^2 u_{xy} - \left(\frac{x^2}{2} + \frac{1}{4}\right) u_{yy} = 0$$

is hyperbolic in the entire xy-plane. Find its characteristic curves and illustrate them geometrically.

17. Show that the PDE

$$x u_{xy} + y u_{yy} = 0$$

is hyperbolic when $x \neq 0$. Find an equivalent PDE in canonical form.

18. (a) A second-order PDE in two independent variables x and y is said to be linear if it is of the form

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g(x, y),$$

where a, b, c, d, e, and f are functions of x and y only. Show that when these coefficients are constants, the canonical forms for hyperbolic, parabolic, and elliptic equations remain linear with constant coefficients:

$$w_{\nu\eta} + p w_\nu + q w_\eta + r w = g \quad \text{(hyperbolic)},$$
$$w_{\nu\nu} + p w_\nu + q w_\eta + r w = g \quad \text{(parabolic)},$$
$$w_{\nu\nu} + w_{\eta\eta} + p w_\nu + q w_\eta + r w = g \quad \text{(elliptic)}.$$

(b) Prove that in the case of a hyperbolic equation, the first-derivative terms w_ν and w_η can be eliminated by a change of dependent variable

$$v(\nu, \eta) = e^{\varepsilon\nu + \rho\eta} w(\nu, \eta)$$

for appropriate constants ε and ρ.

(c) Verify that the transformation in (b) can be used to eliminate first derivatives for elliptic equations also.

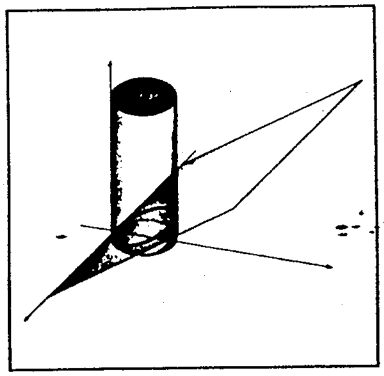
(d) Show that the transformation in (b) will eliminate w_ν and w for a parabolic equation when $q \neq 0$ and will eliminate w_ν and w_η when $q = 0$.

In Exercises 19–21, use the results of Exercise 18 to find simplified canonical representations for the PDE.

19. $u_{xx} + 2u_{xy} + 5u_{yy} = 3u_x$

20. $u_{xx} + 6u_{xy} + u_{yy} = 4u_x$

21. $u_{xx} + 2u_{xy} + u_{yy} = u_x - u_y$



CHAPTER

TWO

Fourier Series

2.1 Fourier Series

Power series play an integral part in real (and complex) analysis. Given a function $f(x)$ and a point $x = a$, it is investigated to what extent $f(x)$ can be expressed in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(x - a)^n.$$

Perhaps one of the most important uses of such series (and one that we require in Chapter 8) is the solution of linear ODEs with variable coefficients. In this chapter we introduce a new type of series called a *Fourier series*; such series are indispensable to the study of PDEs. Fourier series are used in a theoretical way to determine properties of solutions of PDEs and in a practical way to find explicit representations of solutions. Some of the terminology associated with Fourier series is borrowed from ordinary vectors; in addition, many of the ideas in Fourier series have their origin in vector analysis. A quick review of pertinent ideas from vector analysis will therefore facilitate later comparisons and help to solidify underlying concepts in the new theory.

The Cartesian components of a vector $\mathbf{v}$ in space are three scalars v_x, v_y, and v_z such that $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$. Formulas for these components are

$$v_x = \mathbf{v} \cdot \hat{\mathbf{i}}, \qquad v_y = \mathbf{v} \cdot \hat{\mathbf{j}}, \qquad v_z = \mathbf{v} \cdot \hat{\mathbf{k}}. \tag{1}$$

These expressions are very simple, and the reason for this is that the basis vectors $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ are orthonormal; that is, they are mutually orthogonal (or perpendicular) and have length 1. Given different basis vectors, say $\mathbf{e}_1 = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{e}_2 = \hat{\mathbf{i}} - \hat{\mathbf{j}}$, and $\mathbf{e}_3 = 3\hat{\mathbf{k}}$, which remain orthogonal, it is still possible to express $\mathbf{v}$ in terms of the $\mathbf{e}_j$,

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3. \tag{2}$$

However, because the $\mathbf{e}_j$ do not have length 1, component formulas (1) must be replaced by somewhat more complicated expressions. Scalar products of (2) with $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ give

$$v_1 = \frac{\mathbf{v}\cdot\mathbf{e}_1}{|\mathbf{e}_1|^2}, \qquad v_2 = \frac{\mathbf{v}\cdot\mathbf{e}_2}{|\mathbf{e}_2|^2}, \qquad v_3 = \frac{\mathbf{v}\cdot\mathbf{e}_3}{|\mathbf{e}_3|^2}. \tag{3}$$

Were the $\mathbf{e}_j$ not orthogonal, expressions for components would be even more complicated, but we have no need for such generality here.

Thus, when an orthogonal basis is used for vectors, equations (3) yield components, and when the basis is orthonormal, the simpler expressions (1) prevail.

We now generalize these ideas to functions. When two functions $f(x)$ and $g(x)$ are defined on the interval $a \le x \le b$, their *scalar product* with respect to a weight function $w(x)$ is defined as

$$\int_a^b w(x)f(x)g(x)\,dx. \tag{4}$$

This definition is much like the definition of the scalar product of ordinary vectors, $\mathbf{u}\cdot\mathbf{v} = u_xv_x + u_yv_y + u_zv_z$, provided we think of a function as having an infinite number of components, its values at the points in the interval $a \le x \le b$. Corresponding components of $f(x)$ and $g(x)$ are then multiplied together and added in (4). The weight function in scalar products (1) and (3) is unity; definition (4) is more general; it permits a variable weight function $w(x)$. Corresponding to the test for orthogonality of nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, namely $\mathbf{u}\cdot\mathbf{v} = 0$, we make the following definition for orthogonality of functions.

Definition

Two nonzero functions $f(x)$ and $g(x)$ are said to be orthogonal on the interval $a \le x \le b$ with respect to the weight function $w(x)$ if their scalar product vanishes:

$$\int_a^b w(x)f(x)g(x)dx = 0. \tag{5}$$

A sequence of functions $\{f_n(x)\} = f_1(x), f_2(x), \ldots$ is said to be orthogonal on $a \le x \le b$ with respect to $w(x)$ if every pair of functions is orthogonal:

$$\int_a^b w(x)f_n(x)f_m(x)\,dx = 0, \quad \text{when } n \ne m. \tag{6}$$

For example, since

$$\int_0^{2\pi} \sin nx \sin mx\,dx = \int_0^{2\pi} \frac{1}{2}(\cos(n-m)x - \cos(n+m)x)\,dx$$
$$= \frac{1}{2}\left\{\frac{\sin(n-m)x}{n-m} - \frac{\sin(n+m)x}{n+m}\right\}_0^{2\pi} = 0,$$

the sequence of functions $\{\sin nx\}$ is orthogonal on the interval $0 \leq x \leq 2\pi$ with respect to the weight function $w(x) \equiv 1$. The sequence is also orthogonal with the same weight function on the interval $0 \leq x \leq \pi$.

By analogy with geometric vectors, where $|\mathbf{v}|^2 = \mathbf{v} \cdot \mathbf{v}$, we regard the scalar product of a function $f(x)$ with itself as the square of its length; that is, we define the *length* $\|f(x)\|$ of a function on the interval $a \leq x \leq b$ with respect to the weight function $w(x)$ as

$$\|f(x)\| = \sqrt{\int_a^b w(x)[f(x)]^2\,dx}. \tag{7}$$

A sequence of functions $\{f_n(x)\}$ is said to be *orthonormal* on $a \leq x \leq b$ with respect to the weight function $w(x)$ if

$$\int_a^b w(x) f_n(x) f_m(x)\,dx = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}. \tag{8}$$

Condition (8) therefore requires the functions $f_n(x)$ to be mutually orthogonal and of unit length.

Any orthogonal sequence can be made orthonormal simply by dividing each function by its length; that is, when $\{f_n(x)\}$ is orthogonal, $\{f_n(x)/\|f_n(x)\|\}$ is orthonormal. For example, since

$$\int_0^{\pi} (\sin nx)^2\,dx = \frac{\pi}{2},$$

the sequence $\{\sqrt{2/\pi}\sin nx\}$ is orthonormal with respect to the weight function $w(x) \equiv 1$ on $0 \leq x \leq \pi$.

With these preliminaries out of the way, we are now ready to consider Fourier series. In the theory of Fourier series, it is investigated to what extent a function $f(x)$ can be represented in an infinite series of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right), \tag{9}$$

where a_n and b_n are constants. The 2 in the first term of this series is included simply as a matter of convenience. (The formula for a_n, $n > 0$, will then include a_0 as well.)

Because $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ have period $2L/n$, it follows that any function $f(x)$ expressible in form (9) must necessarily be of period $2L$ (or of a period that evenly

[*] The notation $\{F(x)\}_a^b$ to represent $F(b) - F(a)$ is useful in evaluating definite integrals.

divides $2L$). That many $2L$-periodic functions can be expressed in this form is to a large extent attributable to the fact that the sine and cosine functions satisfy the following theorem.

Theorem 1

The set of functions $\{1, \cos(n\pi x/L), \sin(n\pi x/L)\}$ is orthogonal over the interval $0 \le x \le 2L$ with respect to the weight function $w(x) = 1$. Furthermore,

$$\int_0^{2L} 1^2\,dx = 2L; \qquad \int_0^{2L}\left(\cos\frac{n\pi x}{L}\right)^2 dx = \int_0^{2L}\left(\sin\frac{n\pi x}{L}\right)^2 dx = L. \tag{10}$$

(See Exercise 15 for a proof of this result.)

It follows that the functions $\{1/\sqrt{2L}, (1/\sqrt{L})\cos(n\pi x/L), (1/\sqrt{L})\sin(n\pi x/L)\}$ are orthonormal with respect to the weight function $w(x) = 1$ on the interval $0 \le x \le 2L$.

Suppose we neglect for the moment questions of convergence and formally set

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right). \tag{11}$$

Just as v_x, v_y, and v_z are the components of $\mathbf{v}$ with respect to the basis vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$, and $\hat{\mathbf{k}}$ in $\mathbf{v} = v_x\hat{\mathbf{i}} + v_y\hat{\mathbf{j}} + v_z\hat{\mathbf{k}}$, we regard the coefficients $a_0/2$, a_n, and b_n as components of $f(x)$ with respect to the basis functions 1, $\cos(n\pi x/L)$, and $\sin(n\pi x/L)$. If we integrate both sides of (11) from $x = 0$ to $x = 2L$, and formally interchange the order of integration and summation on the right, we obtain

$$\int_0^{2L} f(x)\,dx = \frac{a_0}{2}(2L).$$

Thus,

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)\,dx; \tag{12a}$$

that is, if (11) is to hold, the constant term $a_0/2$ must be the average value of $f(x)$ over the interval $0 \le x \le 2L$. When we multiply both sides of equation (11) by $\cos(k\pi x/L)$, integrate from $x = 0$ to $x = 2L$, and once again interchange the order of integration and summation,

$$\int_0^{2L} f(x)\cos\frac{k\pi x}{L}\,dx = \int_0^{2L}\frac{a_0}{2}\cos\frac{k\pi x}{L}\,dx + \sum_{n=1}^{\infty}\left(\int_0^{2L} a_n\cos\frac{n\pi x}{L}\cos\frac{k\pi x}{L}\,dx + \int_0^{2L} b_n\sin\frac{n\pi x}{L}\cos\frac{k\pi x}{L}\,dx\right)$$

$$= a_k(L) \quad \text{(by the orthogonality of Theorem 1).}$$

Thus,

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad n > 0. \tag{12b}$$

Similarly,

$$b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{12c}$$

We have found, therefore, that *if* $f(x)$ can be represented in form (11), and *if* the series is suitably convergent, coefficients a_n and b_n must be calculated according to (12). What we must answer is the converse question: If a_n and b_n are calculated according to (12), does series (11) converge to $f(x)$? Does it converge pointwise, uniformly, or in any other sense? When a_0, a_n, and b_n are calculated according to (12), the right side of (11) is called the *Fourier series* of $f(x)$. Numbers a_0, a_n, and b_n are called the *Fourier coefficients* of $f(x)$; they are, as we have already suggested, components of $f(x)$ with respect to the basis functions 1, $\cos(n\pi x/L)$, and $\sin(n\pi x/L)$.

Theorem 2, which follows shortly, guarantees that series (11) essentially converges to $f(x)$ when $f(x)$ is piecewise continuous and has a piecewise continuous first derivative. A function $f(x)$ is *piecewise continuous* on an interval $a \leq x \leq b$ if the interval can be divided into a finite number of subintervals inside each of which $f(x)$ is continuous and has finite limits as x approaches either end point of the subinterval from the interior. A $2L$-periodic function is said to be piecewise continuous if it is piecewise continuous on the interval $0 \leq x \leq 2L$. Figure 2.1(a) illustrates a $2L$-periodic function that is piecewise continuous; its discontinuities at $x = c$ and $x = d$ are finite. Because the discontinuity at $x = c$ in Figure 2.1(b) is not finite, this function is not piecewise continuous.

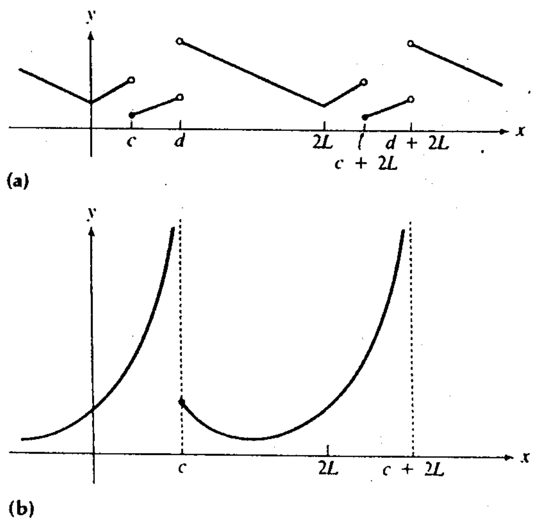
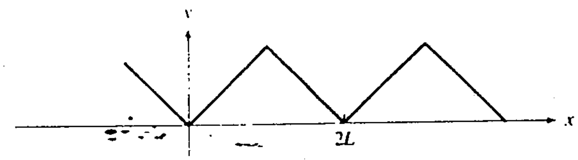


Figure 2.1

A function $f(x)$ is said to be *piecewise smooth* on an interval $a \leq x \leq b$ if $f(x)$ and $f'(x)$ are both piecewise continuous therein. A $2L$-periodic function is piecewise smooth if it is piecewise smooth on $0 \leq x \leq 2L$. The periodic functions in Figure 2.2 are both continuous: that in Figure 2.2(a) is piecewise smooth; that in Figure 2.2(b) is not. The $2L$-periodic function in Figure 2.3 is piecewise smooth.

Theorem 2

The Fourier series of a periodic, piecewise continuous function $f(x)$ converges to $[f(x+) + f(x-)]/2$ at any point at which $f(x)$ has both a left and right derivative.



(a)

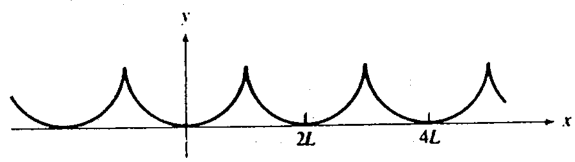


Figure 2.2 (b)

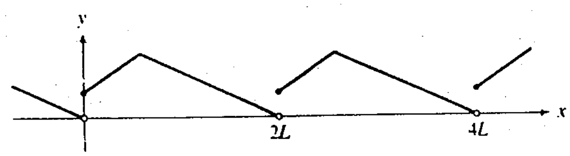


Figure 2.3

By $f(x+)$ we mean the right-hand limit of $f(x)$ at x, $\lim_{\varepsilon\to0^+} f(x+\varepsilon)$. Similarly, $f(x-) = \lim_{\varepsilon\to0^+} f(x-\varepsilon)$. The proof of this theorem is very lengthy; it requires verification of a number of preliminary results that, although interesting in their own right, detract from the flow of our discussion. We have therefore included the proof as Appendix A following Chapter 12.

Since functions that are piecewise smooth must have right and left derivatives at all points, we may state the following corollary to Theorem 2.

Corollary

When $f(x)$ is a periodic, piecewise smooth function, its Fourier series converges to $[f(x+)+f(x-)]/2$.

For such functions, we therefore write

$$\frac{f(x+)+f(x-)}{2} = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right), \tag{13a}$$

where

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{13b}$$

There is nothing sacrosanct about the limits $x = 0$ and $x = 2L$ on the integrals in (13b); all that is required is integration over one full period of length $2L$. In other words, expressions (13b) could be replaced by

$$a_n = \frac{1}{L}\int_c^{c+2L} f(x)\cos\frac{n\pi x}{L}\,dx, \qquad b_n = \frac{1}{L}\int_c^{c+2L} f(x)\sin\frac{n\pi x}{L}\,dx, \tag{13c}$$

where c is any number whatsoever.

If we make the agreement that at any point of discontinuity, $f(x)$ shall be defined (or redefined if necessary) as the average of its right- and left-hand limits, (13a) may be replaced by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right). \tag{14}$$

For example, the Fourier series of the function $f(x)$ in Figure 2.1(a) converges to the function in Figure 2.4; $f(x)$ must be defined as the average of its right- and left-hand limits at $x = d + 2nL$ and redefined as the average of its right- and left-hand limits at $x = c + 2nL$.

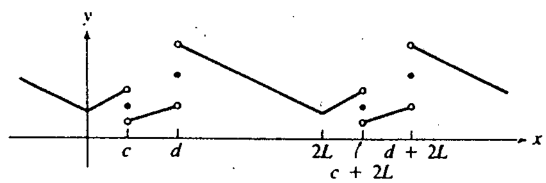


Figure 2.4

Example 1: Find the Fourier series of the function $f(x)$ that is equal to x for $0 < x < 2L$ and is $2L$-periodic.

Solution: According to (13b), the Fourier coefficients are

$$a_0 = \frac{1}{L}\int_0^{2L} x\,dx = \frac{1}{L}\left\{\frac{x^2}{2}\right\}_0^{2L} = 2L;$$

$$a_n = \frac{1}{L}\int_0^{2L} x\cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left\{\frac{Lx}{n\pi}\sin\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\cos\frac{n\pi x}{L}\right\}_0^{2L} = 0, \qquad n > 0;$$

$$b_n = \frac{1}{L}\int_0^{2L} x\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{L}\left\{-\frac{Lx}{n\pi}\cos\frac{n\pi x}{L} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi x}{L}\right\}_0^{2L} = -\frac{2L}{n\pi}, \qquad n > 0.$$

We may therefore write

$$f(x) = L + \sum_{n=1}^{\infty} -\frac{2L}{n\pi}\sin\frac{n\pi x}{L} = L\left(1 - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\right),$$

provided we define $f(x)$ as L at its points of discontinuity $x = 2nL$ (Figure 2.5).

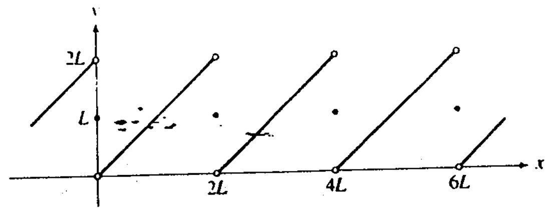


Figure 2.5

Example 2: Find the Fourier series of the function $f(x)$ that is equal to x^2 for $-L \le x \le L$ and is of period $2L$.

Solution: In this example, it is more convenient to integrate over the interval $-L \le x \le L$. In other words, we use (13c) with $c = -L$ to calculate the Fourier coefficients:

$$a_0 = \frac{1}{L}\int_{-L}^{L} x^2\,dx = \frac{1}{L}\left\{\frac{x^3}{3}\right\}_{-L}^{L} = \frac{2L^2}{3};$$

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} x^2 \cos\frac{n\pi x}{L}\,dx \\ &= \frac{1}{L}\left\{\frac{Lx^2}{n\pi}\sin\frac{n\pi x}{L} + \frac{2L^2 x}{n^2\pi^2}\cos\frac{n\pi x}{L} - \frac{2L^3}{n^3\pi^3}\sin\frac{n\pi x}{L}\right\}_{-L}^{L} \\ &= \frac{4L^2(-1)^n}{n^2\pi^2}, \qquad n > 0; \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{L}\int_{-L}^{L} x^2 \sin\frac{n\pi x}{L}\,dx \\ &= \frac{1}{L}\left\{\frac{-Lx^2}{n\pi}\cos\frac{n\pi x}{L} + \frac{2L^2 x}{n^2\pi^2}\sin\frac{n\pi x}{L} + \frac{2L^3}{n^3\pi^3}\cos\frac{n\pi x}{L}\right\}_{-L}^{L} \\ &= 0, \qquad n > 0. \end{aligned}$$

Because $f(x)$ is continuous for all x (Figure 2.6), we may write

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2(-1)^n}{n^2\pi^2}\cos\frac{n\pi x}{L} = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\frac{n\pi x}{L}.$$

This Fourier series can be used to find the sum of the series $\sum_{n=1}^{\infty} 1/n^2$. When we set $x = L$, and note that $f(L) = L^2$,

$$\begin{aligned} L^2 &= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi \\ &= \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}. \end{aligned}$$

This equation can be solved for

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4L^2}\left(L^2 - \frac{L^2}{3}\right) = \frac{\pi^2}{6}.$$

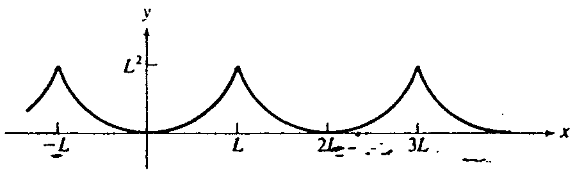


Figure 2.6

Example 3: Find the Fourier series for the 2π-periodic function $f(x)$ in Figure 2.7.

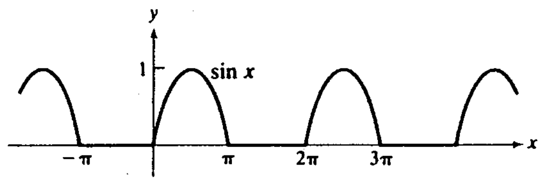


Figure 2.7

Solution: With $L = \pi$ in (13b),

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x\,dx = \frac{1}{\pi}\{-\cos x\}_0^{\pi} = \frac{2}{\pi};$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{1}{\pi}\begin{cases} \left\{\dfrac{1}{2}\sin^2 x\right\}_0^{\pi} & n = 1 \\ \left\{\dfrac{\cos(n-1)x}{2(n-1)} - \dfrac{\cos(n+1)x}{2(n+1)}\right\}_0^{\pi} & n > 1 \end{cases}$$

$$= \begin{cases} 0 & n = 1 \\ -\dfrac{[1 + (-1)^n]}{\pi(n^2 - 1)} & n > 1 \end{cases};$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} \sin x \sin nx\,dx$$

$$= \frac{1}{\pi}\begin{cases} \left\{\dfrac{x}{2} - \dfrac{\sin 2x}{4}\right\}_0^{\pi} & n = 1 \\ \left\{\dfrac{\sin(n-1)x}{2(n-1)} - \dfrac{\sin(n+1)x}{2(n+1)}\right\}_0^{\pi} & n > 1 \end{cases}$$

$$= \begin{cases} \dfrac{1}{2} & n = 1 \\ 0 & n > 1 \end{cases}.$$

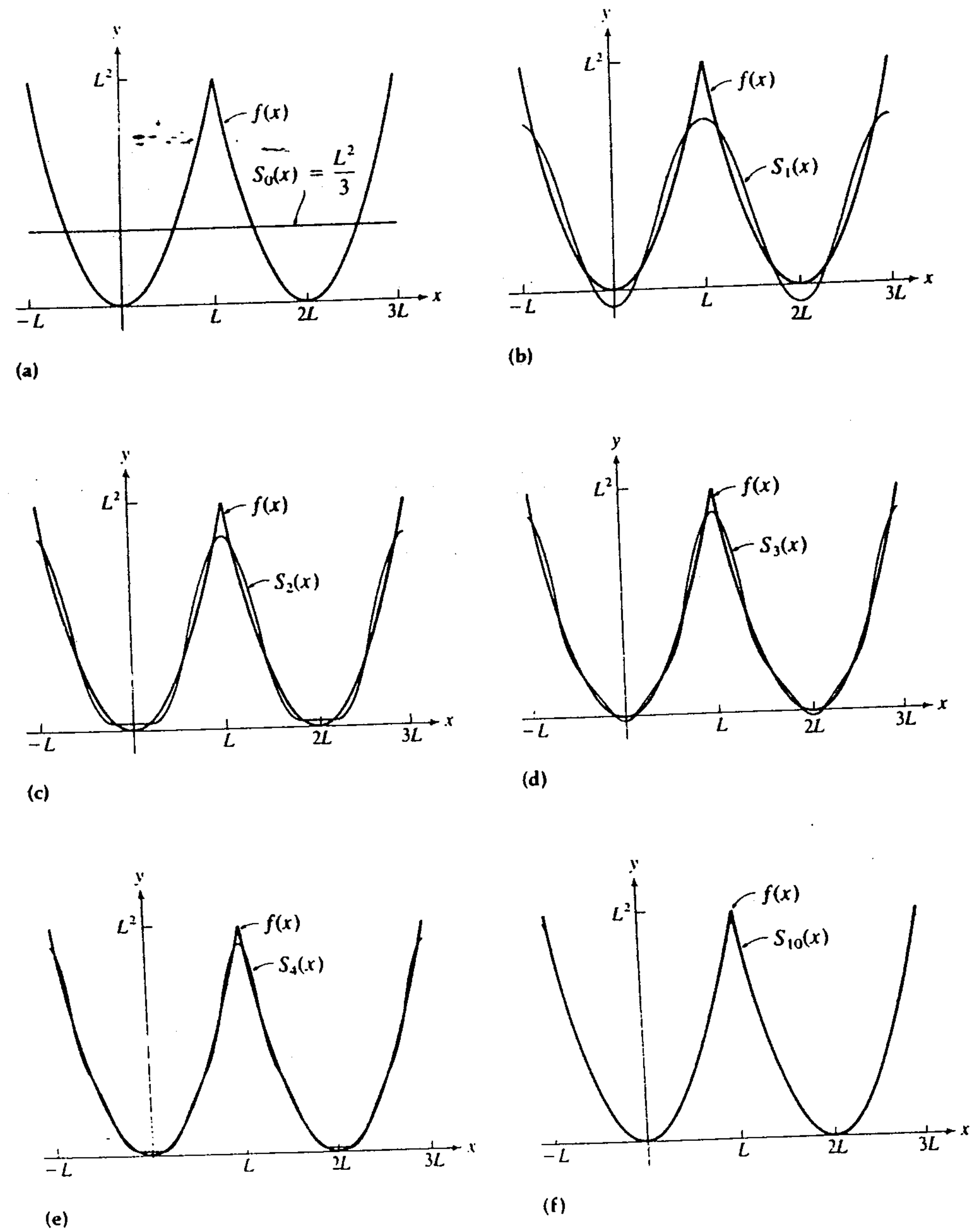
Because $f(x)$ is continuous for all x, we may write

$$f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} -\frac{[1 + (-1)^n]}{\pi(n^2 - 1)}\cos nx + \frac{1}{2}\sin x.$$

Terms in the series vanish when n is odd. To display only the even terms, we replace n by $2n$:

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.$$

■


y
L²
f(x)
S₀(x) = L²/3
−L
L
2L
3L
x
(a)
y
L²
f(x)
S₁(x)
−L
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3L
x
(b)
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L²
f(x)
S₂(x)
−L
L
2L
3L
x
(c)
y
L²
f(x)
S₃(x)
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(d)
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2L
3L
x
(e)
y
L²
f(x)
S₁₀(x)
−L
L
2L
3L
x
(f)

Figure 2.8

When we write (14), we mean that the sequence of partial sums $\{S_n(x)\}$ of the series on the right converges to $f(x)$ for all x; that is, were we to plot the functions in the sequence

$$S_0(x) = \frac{a_0}{2},$$

$$S_1(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L}\right),$$

$$S_2(x) = \frac{a_0}{2} + \left(a_1 \cos\frac{\pi x}{L} + b_1 \sin\frac{\pi x}{L}\right) + \left(a_2 \cos\frac{2\pi x}{L} + b_2 \sin\frac{2\pi x}{L}\right),$$

and so forth, their graphs should resemble more and more closely that of $f(x)$. Figure 2.8 illustrates this fact with the partial sums $S_0(x)$, $S_1(x)$, $S_2(x)$, $S_3(x)$, $S_4(x)$, and $S_{10}(x)$ for the function $f(x)$ in Example 2. Graphs are plotted only for $-L \le x \le 3L$; they are extended periodically outside this interval.

Figure 2.9 illustrates the same partial sums for the function $f(x)$ in Example 1, but convergence in this case is much slower. This is easily explained by the fact that the Fourier coefficients in Example 2 have a factor n^2 in the denominator, whereas in Example 1 the factor is only n. Figure 2.9 also indicates a property of all Fourier series at points of discontinuity of the function $f(x)$. On either side of the discontinuity, the partial sums eventually overshoot $f(x)$, and this overshoot does not diminish in size as more and more terms of the Fourier series are included. This is known as the Gibbs phenomenon; it states that for large n, $S_n(x)$ overshoots the curve at a discontinuity by about 9% of the size of the jump in the function.

Entire books have been written on Fourier series and their properties; some of these properties are developed in Section 2.3. For most of our discussions on partial differential equations, we require the basic ideas of pointwise convergence (contained in Theorem 2) and the ability to differentiate Fourier series. The following theorem indicates conditions that permit Fourier series to be differentiated term by term.

Theorem 3

If $f(x)$ is a continuous function of period $2L$ with piecewise continuous derivatives $f'(x)$ and $f''(x)$, the Fourier series of $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right),$$

can be differentiated term by term to yield

$$f'(x) = \frac{f'(x+) + f'(x-)}{2} = \frac{\pi}{L}\sum_{n=1}^{\infty} n\left(-a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right). \tag{15}$$

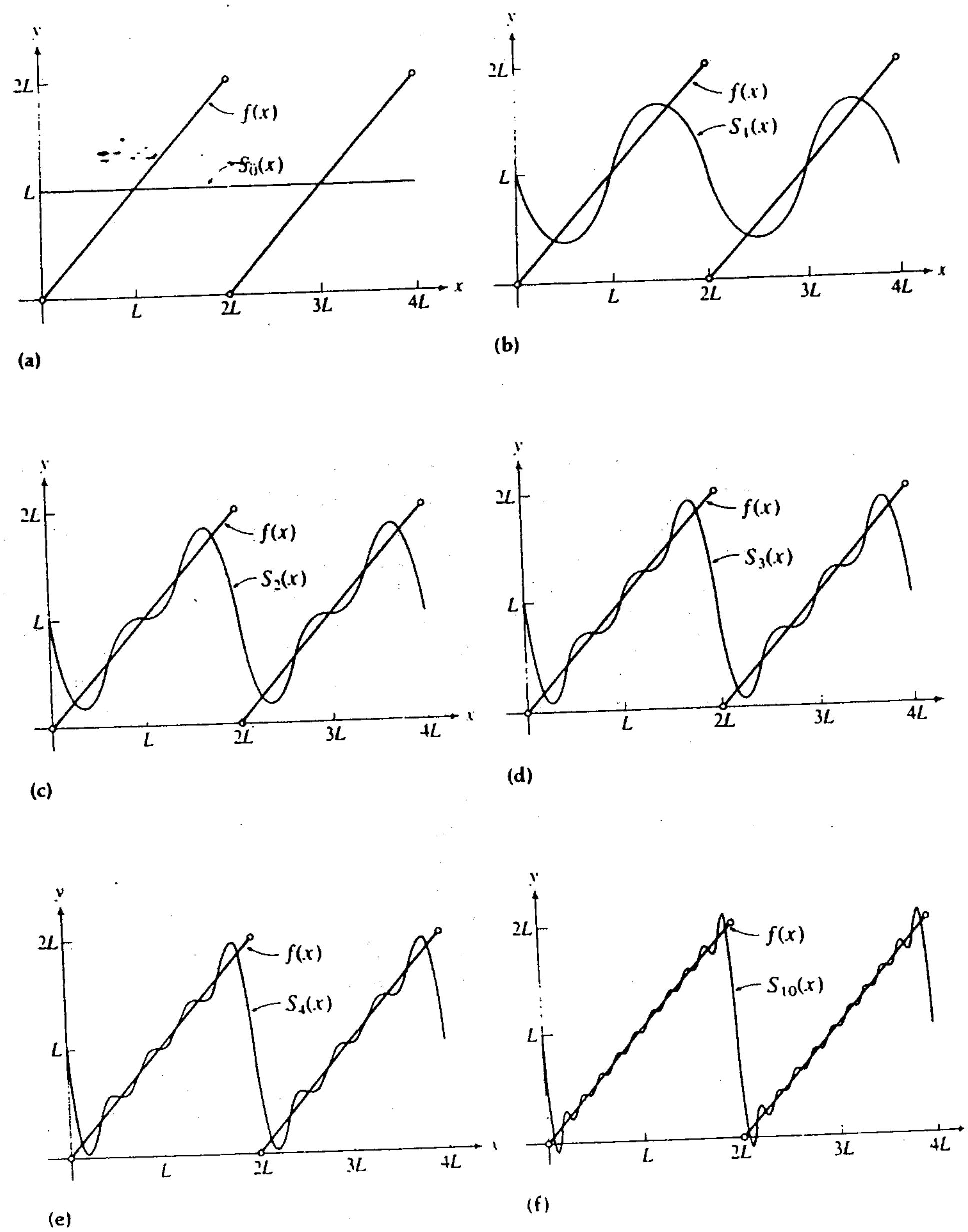

Figure 2.9

Proof: Because $f'(x)$ is piecewise smooth, its Fourier series converges to $[f'(x+)+f'(x-)]/2$ for each x,

$$\frac{f'(x+)+f'(x-)}{2} = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right),$$

where
$$A_n = \frac{1}{L}\int_0^{2L} f'(x)\cos\frac{n\pi x}{L}\,dx = \frac{1}{L}\left\{f(x)\cos\frac{n\pi x}{L}\right\}_0^{2L} + \frac{n\pi}{L^2}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx$$
$$= \frac{n\pi}{L}b_n, \qquad n > 0;$$
$$B_n = \frac{1}{L}\int_0^{2L} f'(x)\sin\frac{n\pi x}{L}\,dx = \frac{1}{L}\left\{f(x)\sin\frac{n\pi x}{L}\right\}_0^{2L} - \frac{n\pi}{L^2}\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx$$
$$= \frac{-n\pi}{L}a_n;$$
$$A_0 = \frac{1}{L}\int_0^{2L} f'(x)\,dx = \frac{1}{L}\{f(x)\}_0^{2L} = 0.$$

Consequently,

$$\frac{f'(x+)+f'(x-)}{2} = \frac{\pi}{L}\sum_{n=1}^{\infty} n\left(-a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right). \qquad \blacksquare$$

Example 4: The function in Figure 2.10(a) is the derivative of the function in Figure 2.7 (for Example 3). Find its Fourier series.

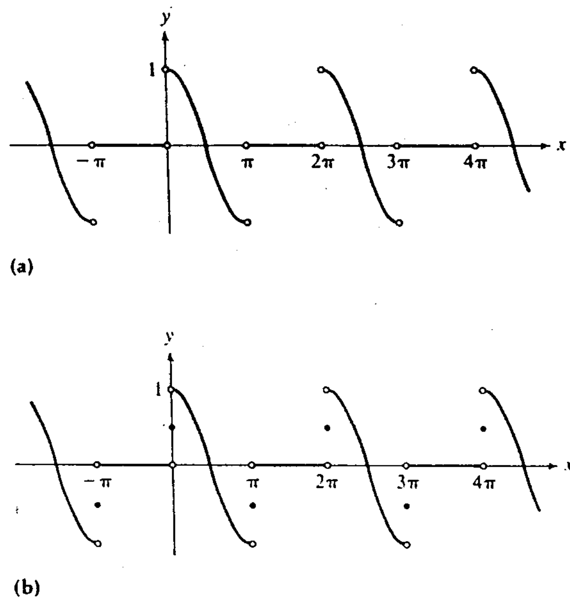

Figure 2.10

Solution: The function $f(x)$ in Figure 2.7 is continuous and has Fourier series

$$f(x) = \frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\cos 2nx}{4n^2 - 1}.$$

Since $f'(x)$ and $f''(x)$ are piecewise continuous, we may differentiate this series term by term and write

$$f'(x) = \frac{1}{2}\cos x - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{-2n\sin 2nx}{4n^2 - 1}$$
$$= \frac{1}{2}\cos x + \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2 - 1}\sin 2nx,$$

provided we understand that $f'(x)$ is the function in Figure 2.10(b), that is, provided we define $f'(n\pi) = (-1)^n/2$. ∎

Integration formulas (13b) for Fourier coefficients almost invariably involve integration by parts. These integrations can be combined by using what is called the complex form for a Fourier series. With the expressions $\cos\theta = (e^{i\theta} + e^{-i\theta})/2$ and $\sin\theta = (e^{i\theta} - e^{-i\theta})/(2i)$, we may express the Fourier series of a function $f(x)$ in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \frac{e^{n\pi xi/L} + e^{-n\pi xi/L}}{2} + b_n \frac{e^{n\pi xi/L} - e^{-n\pi xi/L}}{2i}\right)$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-n\pi xi/L} + \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{n\pi xi/L}$$
$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-n\pi xi/L} + \sum_{n=-1}^{-\infty}\left(\frac{a_{-n} - ib_{-n}}{2}\right)e^{-n\pi xi/L}$$

or

$$\boxed{f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-n\pi xi/L},} \tag{16a}$$

where $c_0 = a_0/2$, $c_n = (a_n + ib_n)/2$ when $n > 0$ and $c_n = (a_{-n} - ib_{-n})/2$ when $n < 0$. It is straightforward to verify using formulas (13b) that for all n,

$$\boxed{c_n = \frac{1}{2L}\int_0^{2L} f(x)e^{n\pi xi/L}\,dx.} \tag{16b}$$

This is called the *complex form* of Fourier series (13). Its compactness is evident, and only one integration is required to determine the complex Fourier coefficients c_n. In addition, Fourier coefficients a_n and b_n are easily extracted as real and imaginary parts of c_n (see Exercises 22–26).

Exercises 2.1

In Exercises 1–14, find the Fourier series for the function $f(x)$. Draw graphs of $f(x)$ and the function to which the series converges in Exercises 1–8, 13, and 14.

1. $f(x) = 3x + 2, \quad 0 < x < 4, \quad f(x + 4) = f(x)$
2. $f(x) = 2x^2 - 1, \quad 0 \le x < 2L, \quad f(x + 2L) = f(x)$
3. $f(x) = 2x^2 - 1, \quad -L \le x \le L, \quad f(x + 2L) = f(x)$
4. $f(x) = 3x, \quad 0 < x \le 2L, \quad f(x + 2L) = f(x)$
5. $f(x) = 3x, \quad -L < x \le L, \quad f(x + 2L) = f(x)$
6. $f(x) = \begin{cases} 2(L - x) & 0 \le x \le L \\ x - L & L < x < 2L \end{cases}, \quad f(x + 2L) = f(x)$
7. $f(x) = \begin{cases} 2 & 0 < x < 1 \\ 1 & 1 < x < 2, \\ 0 & 2 < x < 3 \end{cases} \quad f(x + 3) = f(x)$
8. $f(x) = \begin{cases} x & 0 \le x \le 2 \\ 2 & 2 \le x \le 4; \\ 6 - x & 4 \le x \le 6 \end{cases} \quad f(x + 6) = f(x)$
9. $f(x) = 1 + \sin x - \cos 2x$
10. $f(x) = 2\cos x - 3\sin 10x + 4\cos 2x$
11. $f(x) = \cos^2 2x$
12. $f(x) = 3\cos 2x \sin 5x$
13. $f(x) = e^x, \quad 0 < x < 4, \quad f(x + 4) = f(x)$
14. $f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ -2\sin x & \pi \le x \le 2\pi \end{cases}, \quad f(x + 2\pi) = f(x)$
15. Verify that the functions in Theorem 1 are indeed orthogonal.
16. A student was once heard to say that the Fourier series of a periodic function is not unique. For example, in Example 1 the Fourier series of the function in Figure 2.5 was found. The student stated that this function also has period $4L$ and therefore has a Fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{2L} + b_n \sin\frac{n\pi x}{2L}\right).$$

Is this series indeed different from that found in Example 1?

In Exercises 17–21, find the complex Fourier series for the given function.

17. The function in Example 2.
18. The function in Exercise 13.
19. The function in Exercise 7.
20. $f(x) = \begin{cases} 1 & 0 < x < L \\ -1 & L < x < 2L \end{cases}, \quad f(x + 2L) = f(x)$
21. $f(x) = \begin{cases} x & 0 < x < L \\ 2L - x & L < x < 2L \end{cases}, \quad f(x + 2L) = f(x)$

In Exercises 22–26, find the trigonometric Fourier series for the given function by calculating complex Fourier coefficients c_n and then taking real and imaginary parts.

22. The function in Example 2.
23. The function in Exercise 8.
24. The function in Exercise 7.
25. The function in Exercise 2.

26. The function in Exercise 6.

27. Is

$$f(x) = \sum_{n=-\infty}^{\infty} d_n e^{n\pi xi/L}$$

where
$$d_n = \frac{1}{2L}\int_0^{2L} f(x) e^{-n\pi xi/L}\, dx$$

an alternative to equation (16) for the complex form of the Fourier series of a function $f(x)$? How is d_n related to a_n and b_n in this case?

28. A function $f(x)$ is said to be *odd-harmonic* if $f(x + L) = -f(x)$.
    (a) Prove that such a function is $2L$-periodic.
    (b) Illustrate an odd-harmonic function graphically.
    (c) Show that the Fourier series for an odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} \left[a_{2n-1} \cos\frac{(2n-1)\pi x}{L} + b_{2n-1} \sin\frac{(2n-1)\pi x}{L} \right],$$

where
$$a_{2n-1} = \frac{2}{L}\int_0^L f(x) \cos\frac{(2n-1)\pi x}{L}\, dx$$

and
$$b_{2n-1} = \frac{2}{L}\int_0^L f(x) \sin\frac{(2n-1)\pi x}{L}\, dx.$$

2.2 Fourier Sine and Cosine Series

When $f(x)$ is a $2L$-periodic, piecewise smooth function, it has a Fourier series representation as in (13a) with coefficients defined by (13b). If, in addition, $f(x)$ is an even function, it is a simple exercise to show that its Fourier coefficients satisfy

$$\boxed{a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\, dx, \qquad b_n = 0} \tag{17b}$$

(see, for instance, Example 2). Thus, the Fourier series of an even function has only cosine terms,

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L},} \tag{17a}$$

and is called a *Fourier cosine series.*

When $f(x)$ is an odd function, its Fourier coefficients are

$$\boxed{a_n = 0, \qquad b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\, dx,} \tag{18b}$$

and therefore the Fourier series of an odd function has only sine terms,

$$\boxed{f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L},} \tag{18a}$$

and is called a *Fourier sine series.*

Example 5: Find the Fourier series for the function $f(x)$ in Figure 2.11.

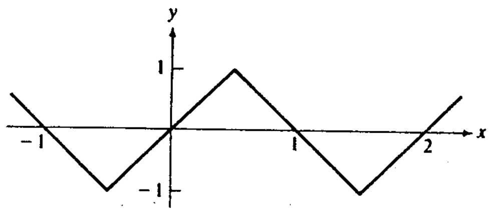

Figure 2.11

Solution: Because $f(x)$ is an odd function of period 2, its Fourier series must be a sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin n\pi x$$

with coefficients

$$\begin{aligned} b_n &= \frac{2}{1}\int_0^1 f(x)\sin n\pi x\,dx = 2\int_0^{1/2} 2x\sin n\pi x\,dx + 2\int_{1/2}^1 -2(x-1)\sin n\pi x\,dx \\ &= 4\left\{\frac{-x}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right\}_0^{1/2} \\ &\quad - 4\left\{\frac{-(x-1)}{n\pi}\cos n\pi x + \frac{1}{n^2\pi^2}\sin n\pi x\right\}_{1/2}^1 \\ &= \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

Because $f(x)$ is continuous for all x, the Fourier series of $f(x)$ converges to $f(x)$ for all x; that is,

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}\sin n\pi x = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2}\sin(2n-1)\pi x.$$ ■

Example 6: Find the Fourier series for the function $f(x)$ in Figure 2.12.

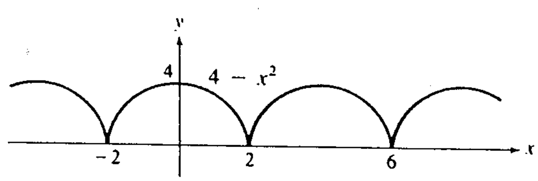

Figure 2.12

Solution: Because $f(x)$ is an even function of period 4, its Fourier series must be a cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{2}$$

with coefficients

$$a_0 = \frac{2}{2}\int_0^2 (4 - x^2)\,dx = \left\{4x - \frac{x^3}{3}\right\}_0^2 = \frac{16}{3};$$

$$\begin{aligned} a_n &= \frac{2}{2}\int_0^2 (4 - x^2)\cos\frac{n\pi x}{2}\,dx \\ &= \left\{\frac{2}{n\pi}(4 - x^2)\sin\frac{n\pi x}{2} - \frac{8x}{n^2\pi^2}\cos\frac{n\pi x}{2} + \frac{16}{n^3\pi^3}\sin\frac{n\pi x}{2}\right\}_0^2 \\ &= \frac{16(-1)^{n+1}}{n^2\pi^2}. \end{aligned}$$

Because $f(x)$ is continuous for all x, we may write

$$\begin{aligned} f(x) &= \frac{8}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^{n+1}}{n^2\pi^2}\cos\frac{n\pi x}{2} \\ &= \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{2}. \end{aligned}$$

Alternatively, we could have noted that this function is 4 minus the function in Example 2 when L is set equal to 2. Hence,

$$\begin{aligned} f(x) &= 4 - \left(\frac{2^2}{3} + \frac{4(2)^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\frac{n\pi x}{2}\right) \\ &= \frac{8}{3} + \frac{16}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}\cos\frac{n\pi x}{2}. \end{aligned}$$

■

Because we have treated Fourier sine and cosine series as special cases of the full Fourier series in Section 2.1, they have been approached from the following point of view: *Can an even (or odd) 2L-periodic function $f(x)$ be expressed in a Fourier series of form (17a) [or (18a)]?*

When sine and cosine series are used to solve (initial) boundary value problems, they arise in a different way. Sine series arise from a need to answer the following question: *Suppose a function $f(x)$ is defined for $0 < x < L$ and is piecewise smooth for $0 \le x \le L$. Is it possible to represent $f(x)$ in a series of the form*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \tag{19}$$

valid for $0 < x < L$?

Notice that $f(x)$ is not odd and it is not periodic; it is defined only between $x = 0$ and $x = L$. But by appropriately extending $f(x)$ outside the interval $0 < x < L$, we shall indeed be able to write it in form (19). First, we recognize that (19) resembles (18a), the Fourier sine series of an odd function. We therefore extend the domain of definition of $f(x)$ to include $-L < x < 0$ by demanding that the extension be odd; that is, we define $f(x) = -f(-x)$ for $-L < x < 0$. For example, if $f(x)$ is as shown in Figure 2.13(a), it is extended as shown in Figure 2.13(b). Next, we know that series (18a) represents a $2L$-periodic function. We therefore extend the domain of definition of $f(x)$ beyond $-L < x < L$ by making it $2L$-periodic [Figure 2.13(c)].

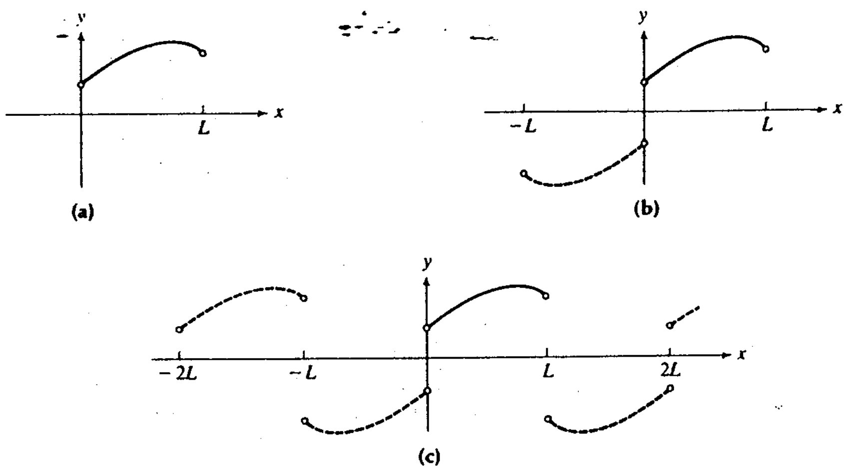

Figure 2.13

We have now extended $f(x)$, which was originally defined only for $0 < x < L$, to an odd, $2L$-periodic function. Because $f(x)$ was piecewise smooth on $0 \leq x \leq L$, the extended function is piecewise smooth for all x. As a result, the extended function can be represented in a Fourier sine series (18a), with coefficients defined by (18b), and this series converges to the average value of right- and left-hand limits at every point. Since the extension of $f(x)$ to an odd, periodic function does not affect its original values on $0 < x < L$, it follows that the Fourier sine series of this extension must represent $f(x)$ on $0 < x < L$. Thus, we should calculate the coefficients in (19) according to (18b). Finally, we should note that the series will converge to 0 at $x = 0$ and $x = L$.

In summary, when we are required to express a function $f(x)$, defined for $0 < x < L$, in form (19), we use the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$.

In a similar way, if we are required to express a function $f(x)$, piecewise smooth on $0 \leq x \leq L$, in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad 0 < x < L, \tag{20}$$

we use the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$. For the function $f(x)$ in Figure 2.13(a), this extension is as shown in Figure 2.14. The series will converge to $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to L^-} f(x)$ at $x = 0$ and $x = L$, respectively.

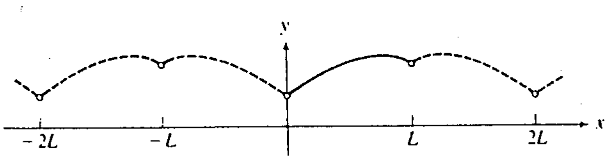

Figure 2.14

Example 7: Find coefficients b_n so that

$$1 + 2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{3}$$

for all x in the interval $0 < x < 3$.

Solution: Constants b_n must be the coefficients in the Fourier sine series of the extension of $1 + 2x$ to an odd function of period 6 (Figure 2.15). According to (18b),

$$b_n = \frac{2}{3}\int_0^3 f(x)\sin\frac{n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 (1 + 2x)\sin\frac{n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left\{\frac{-3}{n\pi}(1 + 2x)\cos\frac{n\pi x}{3} + \frac{18}{n^2\pi^2}\sin\frac{n\pi x}{3}\right\}_0^3 = \frac{2}{n\pi}[1 + 7(-1)^{n+1}].$$

Consequently,

$$1 + 2x = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1 + 7(-1)^{n+1}}{n}\sin\frac{n\pi x}{3}, \qquad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the series does not converge to $1 + 2x$; it converges to zero, the average value of right- and left-hand limits of the odd, periodic extension of $1 + 2x$.

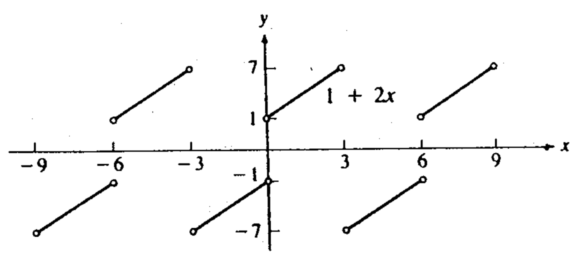

Figure 2.15

Example 8: Find coefficients a_n so that

$$1 + 2x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{3}$$

for all x in the interval $0 < x < 3$.

Solution: The constants a_n must be the coefficients in the Fourier cosine series of the extension of $1 + 2x$ to an even function $f(x)$ of period 6 (Figure 2.16). According to (17b),

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx = \frac{2}{3}\int_0^3 (1 + 2x)\,dx = \frac{2}{3}\left\{x + x^2\right\}_0^3 = 8;$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{n\pi x}{3}\,dx = \frac{2}{3}\int_0^3 (1 + 2x)\cos\frac{n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left\{\frac{3}{n\pi}(1 + 2x)\sin\frac{n\pi x}{3} + \frac{18}{n^2\pi^2}\cos\frac{n\pi x}{3}\right\}_0^3 = \frac{12}{n^2\pi^2}[(-1)^n - 1].$$

Consequently,

$$1 + 2x = 4 + \frac{12}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\frac{n\pi x}{3}.$$

Terms in the series vanish when n is even. To display only the odd terms, we replace n by $2n - 1$ and sum from $n = 1$ to infinity:

$$1 + 2x = 4 - \frac{24}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\cos\frac{(2n-1)\pi x}{3}, \qquad 0 < x < 3.$$

At $x = 0$ and $x = 3$, the series converges to 1 and 7, respectively (these being the average of right- and left-hand limits of the even, periodic extension), so that the series actually represents $1 + 2x$ for $0 \le x \le 3$. ■

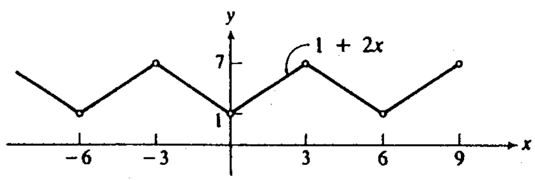

Figure 2.16

Exercises 2.2

In Exercises 1–5, find the Fourier series for the function $f(x)$. Draw graphs of $f(x)$ and the function to which the series converges in Exercises 2–5.

1. $f(x) = 2\sin 4x - 3\sin x$
2. $f(x) = |x|, \quad -\pi < x < \pi, \quad f(x + 2\pi) = f(x)$
3. $f(x) = \begin{cases} x & -4 \le x \le 4 \\ 8 - x & 4 \le x \le 12 \end{cases}, \quad f(x + 16) = f(x)$
4. $f(x) = 2x^2 - 1, \quad -L \le x \le L, \quad f(x + 2L) = f(x)$
5. $f(x) = \begin{cases} \cos x & -\pi/2 \le x \le \pi/2 \\ 0 & \pi/2 < x < 3\pi/2 \end{cases}, \quad f(x + 2\pi) = f(x)$

In Exercises 6–10, expand $f(x)$ in terms of the functions $\{\sin(n\pi x/L)\}$.

6. $f(x) = -x, \quad 0 < x < L$
7. $f(x) = \begin{cases} 1 & 0 < x < L/3 \\ 0 & L/3 < x < 2L/3 \\ -1 & 2L/3 < x < L \end{cases}$
8. $f(x) = \begin{cases} L/4 & 0 \le x \le L/4 \\ L/2 - x & L/4 < x \le L/2 \\ x - L/2 & L/2 < x < 3L/4 \\ L/4 & 3L/4 \le x \le L \end{cases}$
9. $f(x) = \sin(\pi x/L)\cos(\pi x/L)$
10. $f(x) = Lx - x^2, \quad 0 < x < L$

In Exercises 11–15, expand $f(x)$ in terms of the functions $\{1, \cos(n\pi x/L)\}$.

11. $f(x) = -x, \quad 0 < x < L$
12. $f(x) = \sin(\pi x/L)\cos(\pi x/L), \quad 0 < x < L$

13. $f(x) = Lx - x^2, \quad 0 < x < L$

14. $f(x) = 1, \quad 0 < x < L$

15. $f(x) = \begin{cases} 1 & 0 < x < L/2 \\ 0 & L/2 < x < L \end{cases}$

16. Find the Fourier series for the function $f(x) = |\sin x|$ by using the fact that the function has period π. What series is obtained if a period of 2π is used?

17. Under what additional condition is it possible to express a function $f(x)$ that is piecewise smooth on $0 \le x \le L$ in the form

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}?$$

18. Illustrate with graphs that when a function $f(x)$ defined on the interval $0 \le x \le L$ is continuous (from the right) at $x = 0$,
   (a) the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$ always converges to $f(0)$ at $x = 0$;
   (b) the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$ converges to $f(0)$ at $x = 0$ if and only if $f(0) = 0$.

   Are similar statements correct at $x = L$?

19. Suppose that $f(x)$ is continuous on the interval $0 \le x \le L$ with piecewise continuous derivatives $f'(x)$ and $f''(x)$.
   (a) Show that the Fourier sine series of the odd, $2L$-periodic extension $f_0(x)$ of $f(x)$ can be differentiated term by term to give a cosine series that converges to $[f_0'(x+) + f_0'(x-)]/2$ if $f(0) = f(L) = 0$. Does the differentiated series converge to $f'(0+)$ at $x = 0$ and $f'(L-)$ at $x = L$ when $f(0) = f(L) = 0$?
   (b) Show that the Fourier cosine series of the even, $2L$-periodic extension $f_e(x)$ of $f(x)$ can always be differentiated term by term to give a sine series that converges to $[f_e'(x+) + f_e'(x-)]/2$. Does the differentiated series converge to $f'(0+)$ at $x = 0$ and $f'(L-)$ at $x = L$?

20. (a) Find the Fourier series for the function

$$f(x) = \begin{cases} x & 0 < x < L \\ 2L - x & L < x < 2L \end{cases}, \qquad f(x + 2L) = f(x).$$

   Use this result to find Fourier series for the following functions:
   (b) $f_1(x) = L - |x|, \quad -L \le x \le L, \quad f(x + 2L) = f(x)$
   (c) $f_2(x) = 2L - |2L - x|, \quad 0 < x < 4L, \quad f(x + 4L) = f(x)$
   (d) $f_3(x) = x, \quad -L < x < L, \quad f(L + x) = f(L - x), \quad f(x + 4L) = f(x)$

21. (a) A function $f(x)$ is said to be odd and odd-harmonic if it satisfies the conditions

$$f(-x) = -f(x), \qquad f(L + x) = f(L - x).$$

   Show that such a function is $4L$-periodic.
   (b) Illustrate an odd, odd-harmonic function graphically. Is it symmetric about the line $x = L$?
   (c) Show that the Fourier series of an odd, odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{(2n-1)\pi x}{2L}$$

   where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{(2n-1)\pi x}{2L}\,dx.$$

22. (a) A function $f(x)$ is said to be even and odd-harmonic if it satisfies the conditions

$$f(-x) = f(x), \qquad f(L+x) = -f(L-x).$$

Show that such a function is $4L$-periodic.

(b) Illustrate an even, odd-harmonic function graphically. Is it antisymmetric about the line $x = L$?

(c) Show that the Fourier series of an even, odd-harmonic function takes the form

$$f(x) = \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)\pi x}{2L}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

2.3 Further Properties of Fourier Series

In Sections 2.1 and 2.2 we dealt with point-by-point convergence of Fourier series. In this section, and again in Chapter 5, uniform convergence of Fourier series is of considerable importance. It is appropriate, therefore, to give a brief review of uniform convergence, but we do so in a general setting rather than in the restrictive environment of Fourier series.

A series of functions $\sum_{n=1}^{\infty} u_n(x)$ converges to (or has sum) $S(x)$ if its sequence of partial sums $\{S_n(x)\}$ converges to $S(x)$. This is true if, given any $\varepsilon > 0$, there exists an integer N such that $|S_n(x) - S(x)| < \varepsilon$ whenever $n > N$. Usually N is a function of ε and x; in particular, the choice of N may vary from x to x. What this means is that convergence of $\{S_n(x)\}$ to $S(x)$ may be faster for some x's than for others. If it is possible to find an N, independent of x, such that $|S_n(x) - S(x)| < \varepsilon$ for all $n > N$ and all x in some interval I, then $\sum_{n=1}^{\infty} u_n(x)$ is said to converge *uniformly* to $S(x)$ in I. The word "uniform" is perhaps a misnomer. When N is independent of x, convergence is not necessarily uniformly fast for all x's; the rate of convergence may still vary from x to x. What we can say is that convergence does not become indefinitely slow for some x's in I. In practice, what often happens is that there is an x_0 in I at which convergence is slowest; for all other x's, convergence is more rapid than at this x_0. In this case, convergence is uniform. The most widely used test for uniform convergence of a series is the Weierstrass M-test.

Theorem 4 (Weierstrass M-Test)

If a convergent series of (positive) constants $\sum_{n=1}^{\infty} M_n$ can be found such that $|S_n(x)| \le M_n$ for each n and all x in I, then $\sum_{n=1}^{\infty} u_n(x)$ is uniformly convergent in I.

An excellent example to illustrate these ideas is the geometric series $\sum_{n=0}^{\infty} x^n$. It is well known that this series converges to $1/(1-x)$ on the interval $-1 < x < 1$. In Figure 2.17 we show the five partial sums $S_1(x) = 1$, $S_2(x) = 1 + x$, $S_3(x) = 1 + x + x^2$, $S_4(x) = 1 + x + x^2 + x^3$, and $S_5 = 1 + x + x^2 + x^3 + x^4$, as well as $S(x) = 1/(1-x)$. They indicate that convergence of the partial sums $S_n(x)$ to $S(x)$ is rapid for values of x close to zero, but as x approaches ± 1, convergence becomes much slower. We can demonstrate this algebraically by noting that

$$S(x) - S_n(x) = \frac{1}{1-x} - \frac{1-x^n}{1-x} = \frac{x^n}{1-x}.$$

This is the difference between the sum of the series and its nth partial sum. As x approaches 1, the difference becomes very large; near $x = -1$, it oscillates back and forth between numbers close to $\pm 1/2$.

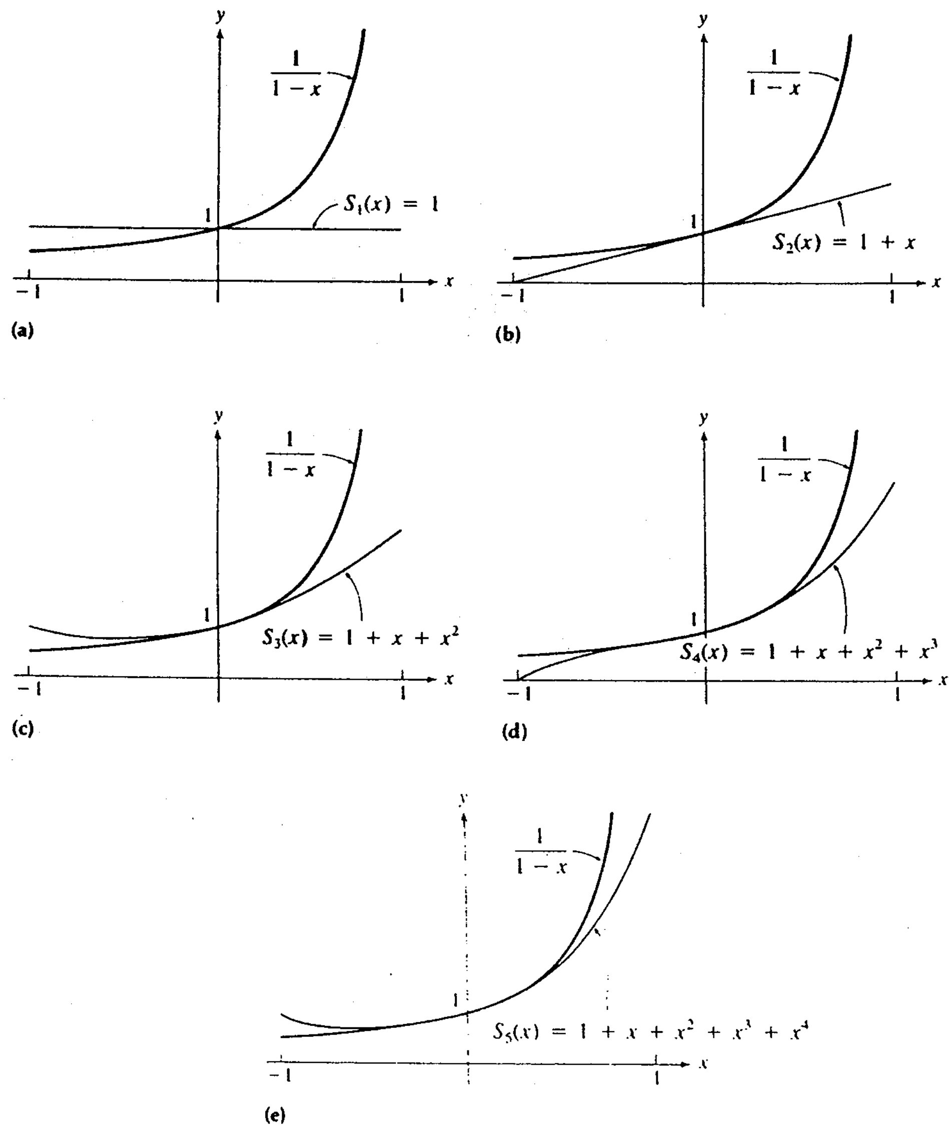

Figure 2.17

When x is confined to the interval $|x| \le a < 1$, we can state that $|x^n| \le a^n$, and since $\sum_{n=0}^{\infty} a^n$ converges, it follows that the geometric series $\sum_{n=0}^{\infty} x^n$ converges uniformly on $|x| \le a < 1$. Convergence is slowest at $x = a$; at all other points in $|x| \le a$, it converges more rapidly than it does at $x = a$. The series does not, however, converge uniformly on the interval $|x| < 1$; convergence becomes indefinitely slow as $x \to \pm 1$.

The Weierstrass M-test is easily generalized to series whose terms are functions of more than one variable. For example, $\sum_{n=1}^{\infty} u_n(x, y)$ is uniformly convergent for points (x, y) in a region R of the xy-plane if there exists a convergent series of constants $\sum_{n=1}^{\infty} M_n$ such that for each n and all (x, y) in R, $|u_n(x, y)| \le M_n$.

Series of the following form arise in almost all phases of our work:

$$\sum_{n=1}^{\infty} X_n(x) Y_n(y),$$

that is, series in which each term is a function $X_n(x)$ of x multiplied by a function $Y_n(y)$ of y. We find Abel's test useful in establishing uniform convergence of such series.

Theorem 5 (Abel's Test)

A series $\sum_{n=1}^{\infty} X_n(x) Y_n(y)$ converges uniformly in a region $\bar{R}$ of the xy-plane if:

(1) the series $\sum_{n=1}^{\infty} X_n(x)$ converges uniformly with respect to x for all x such that (x, y) is in $\bar{R}$;

(2) the functions $Y_n(y)$ are uniformly bounded† for all y such that (x, y) is in $\bar{R}$;

(3) for each y such that (x, y) is in $\bar{R}$, the sequence of constants $\{Y_n(y)\}$ is nonincreasing.

As further explanation of these conditions, suppose $\bar{R}$ is the "closed" region in Figure 2.18 consisting of the area R inside the curve plus the bounding curve $\beta(R)$. Condition (1) requires $\sum_{n=1}^{\infty} X_n(x)$ to be uniformly convergent for $a \le x \le b$. Conditions (2) and (3) must be satisfied for $c \le y \le d$. Of course, the roles of $X_n(x)$ and $Y_n(y)$ could be reversed.

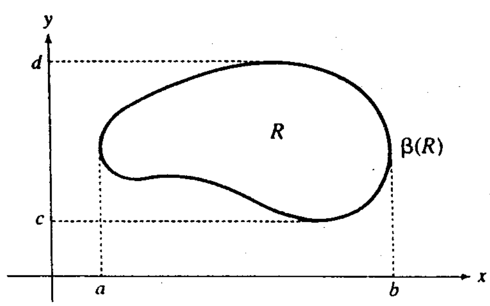

Figure 2.18

It is a well-known fact that the sum of finitely many continuous functions is a continuous function. On the other hand, the sum of infinitely many continuous

† A sequence of functions $\{Y_n(y)\}$ is said to be uniformly bounded on an interval I if there exists a constant M such that $|Y_n(y)| \le M$ for all x in I and all n.

functions may not be a continuous function. Fourier series are prime examples; each term in a Fourier series is continuous, but the sum of the terms may well be discontinuous (see Example 1). When convergence is uniform, the following result indicates that this cannot happen.

Theorem 6

A uniformly convergent series of continuous functions must converge to a continuous function.

This means that convergence of the Fourier series of a discontinuous function cannot be uniform over any interval that contains a point of discontinuity.

In many applications of series, it is necessary to integrate a series term by term. According to the following theorem, this is possible when the series converges uniformly.

Theorem 7

When a series $\sum_{n=1}^{\infty} u_n(x)$ of continuous functions converges uniformly to $S(x)$ on an interval $a \le x \le b$,

$$\int_a^b S(x)\,dx = \sum_{n=1}^{\infty} \int_a^b u_n(x)\,dx. \tag{21}$$

More important to our discussions of Fourier series and partial differential equations are sufficient conditions for term-by-term differentiability of a series. These are given in the next theorem.

Theorem 8

Suppose $\sum_{n=1}^{\infty} u_n(x) = S(x)$ for $a \le x \le b$. Then

$$S'(x) = \sum_{n=1}^{\infty} u_n'(x), \qquad a \le x \le b, \tag{22}$$

provided each $u_n'(x)$ is continuous for $a \le x \le b$ and the series $\sum_{n=1}^{\infty} u_n'(x)$ is uniformly convergent on $a \le x \le b$.

In the remainder of this section we discuss properties of Fourier series, which, although not directly related to partial differential equations, provide a better understanding of the manner in which Fourier series represent functions. We begin with the following result.

Theorem 9 (Bessel's Inequality)

If $f(x)$ is a piecewise continuous function on $0 \le x \le 2L$, its Fourier coefficients must satisfy the inequality

$$\frac{a_0^2}{2} + \sum_{k=1}^{n} (a_k^2 + b_k^2) \le \frac{1}{L} \int_0^{2L} [f(x)]^2\,dx. \tag{23}$$

Proof: When

$$S_n(x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right)$$

is the nth partial sum of the Fourier series of $f(x)$, orthogonality of the sine and cosine functions can be used to evaluate

$$\begin{aligned}\int_0^{2L} [f(x) - S_n(x)]^2\,dx &= \int_0^{2L} [f(x)]^2\,dx - 2\int_0^{2L} f(x)\left(\frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi x}{L}\right.\right.\\ &\quad \left.\left. + b_k \sin\frac{k\pi x}{L}\right)\right)dx + \int_0^{2L}\left(\frac{a_0^2}{4} + \sum_{k=1}^{n}\left[a_k^2\left(\cos\frac{k\pi x}{L}\right)^2\right.\right.\\ &\quad \left. + b_k^2\left(\sin\frac{k\pi x}{L}\right)^2\right] + a_0 \sum_{k=1}^{n}\left(a_k \cos\frac{k\pi x}{L} + b_k \sin\frac{k\pi x}{L}\right)\\ &\quad + 2\sum_{i,j=1}^{n} a_i b_j \cos\frac{i\pi x}{L}\sin\frac{j\pi x}{L}\\ &\quad \left. + 2\sum_{i>j=1}^{n}\left(a_i a_j \cos\frac{i\pi x}{L}\cos\frac{j\pi x}{L} + b_i b_j \sin\frac{i\pi x}{L}\sin\frac{j\pi x}{L}\right)\right)dx\\ &= \int_0^{2L} [f(x)]^2\,dx - 2\left(\frac{a_0}{2}(a_0 L) + \sum_{k=1}^{n}[a_k(a_k L) + b_k(b_k L)]\right)\\ &\quad + \frac{a_0^2}{4}(2L) + \sum_{k=1}^{n}[a_k^2(L) + b_k^2(L)].\end{aligned}$$

Consequently, for any n,

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) = \frac{1}{L}\left(\int_0^{2L} [f(x)]^2\,dx - \int_0^{2L} [f(x) - S_n(x)]^2\,dx\right).$$

Since the second integral on the right is nonnegative, it follows that for any n whatsoever,

$$\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2) \le \frac{1}{L}\int_0^{2L} [f(x)]^2\,dx. \qquad \blacksquare$$

By letting n become infinite, we can also state that

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{1}{L}\int_0^{2L} [f(x)]^2\,dx. \tag{24}$$

In Theorem 11 it is shown that inequality (24) may be replaced by an equality, the result being known as Parseval's theorem. Our proof of Parseval's theorem requires uniform convergence of the Fourier series of a function. Conditions that guarantee this are stated in the following theorem.

Theorem 10

If a $2L$-periodic function $f(x)$ is continuous and has a piecewise continuous first derivative, its Fourier series converges uniformly and absolutely to $f(x)$.

Proof: The conditions on $f(x)$ and $f'(x)$ ensure pointwise convergence of the Fourier series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

of $f(x)$ to $f(x)$ for each x. Since each term in this series may be expressed in the form

$$a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} = \sqrt{a_n^2 + b_n^2}\sin\left(\frac{n\pi x}{L} + \phi_n\right),$$

it follows that the series of absolute values

$$\left|\frac{a_0}{2}\right| + \sum_{n=1}^{\infty}\left|a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right| \le \frac{|a_0|}{2} + \sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}. \tag{25}$$

Uniform and absolute convergence of the Fourier series of $f(x)$ will be established once the series $\sum_{n=1}^{\infty}\sqrt{a_n^2 + b_n^2}$ is shown to be convergent. If

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

is the Fourier series for $f'(x)$, its Fourier coefficients are related to those of $f(x)$ by the equations

$$A_n = \frac{n\pi}{L}b_n, \qquad B_n = \frac{-n\pi}{L}a_n$$

(see the proof of Theorem 3). Thus,

$$\sum_{n=1}^{m}\sqrt{a_n^2 + b_n^2} = \sum_{n=1}^{m}\frac{L}{n\pi}\sqrt{A_n^2 + B_n^2} = \frac{L}{\pi}\sum_{n=1}^{m}\frac{\sqrt{A_n^2 + B_n^2}}{n}. \tag{26}$$

To proceed further, we require a result called Schwarz's inequality. It states that for arbitrary finite sequences $\{c_n\}$ and $\{d_n\}$, $n = 1, \ldots, m$ of nonnegative numbers,

$$\sum_{n=1}^{m} c_n d_n \le \left(\sum_{n=1}^{m} c_n^2\right)^{1/2}\left(\sum_{n=1}^{m} d_n^2\right)^{1/2}. \tag{27}$$

This result is verified in Exercise 1. When it is applied to the series $\sum_{n=1}^{m}\sqrt{A_n^2 + B_n^2}/n$ on the right side of equation (26), we obtain

$$\sum_{n=1}^{m}\sqrt{a_n^2 + b_n^2} \le \frac{L}{\pi}\left(\sum_{n=1}^{m}\frac{1}{n^2}\right)^{1/2}\left(\sum_{n=1}^{m}(A_n^2 + B_n^2)\right)^{1/2}.$$

Since $\sum_{n=1}^{m} 1/n^2 < \sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$ (see Example 2), it follows that

$$\sum_{n=1}^{m}\sqrt{a_n^2 + b_n^2} \le \frac{L}{\sqrt{6}}\left(\sum_{n=1}^{m}(A_n^2 + B_n^2)\right)^{1/2}.$$

But Bessel's inequality (24) applied to the Fourier series for $f'(x)$ gives

$$\sum_{n=1}^{m}(A_n^2 + B_n^2) < \sum_{n=1}^{\infty}(A_n^2 + B_n^2) \le \frac{1}{L}\int_0^{2L}[f'(x)]^2\,dx - \frac{A_0^2}{2}.$$

Consequently,

$$\sum_{n=1}^{m}\sqrt{a_n^2 + b_n^2} \le \frac{L}{\sqrt{6}}\left(\frac{1}{L}\int_0^{2L}[f'(x)]^2\,dx - \frac{A_0^2}{2}\right)^{1/2}.$$

Because this inequality is valid for any m whatsoever, it follows that the series $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges. Inequality (25) then indicates that the Fourier series of $f(x)$ converges uniformly and absolutely. ■

Continuity of $f(x)$ is indispensable to this theorem. A Fourier series cannot converge uniformly over an interval that contains a discontinuity because a uniformly convergent series of continuous functions always converges to a continuous function (Theorem 6). If $f(x)$ is defined only on the interval $0 \le x \le 2L$, continuity of its periodic extension requires that $f(2L) = f(0)$.

When $f(x)$ satisfies the conditions of Theorem 10, inequality (24) may be replaced by an equality. This result is contained in Theorem 11.

Theorem 11 (Parseval's Theorem)

If $f(x)$ is a $2L$-periodic function that is continuous and has a piecewise continuous first derivative, its Fourier coefficients satisfy

$$\boxed{\frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{L}\int_0^{2L} [f(x)]^2\,dx.} \tag{28}$$

Proof: With the conditions cited on $f(x)$, the Fourier series of $f(x)$,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right),$$

is uniformly convergent (Theorem 10). It may therefore be multiplied by $f(x)$ and integrated term by term between 0 and $2L$ to yield

$$\begin{aligned}\int_0^{2L} [f(x)]^2\,dx &= \frac{a_0}{2}\int_0^{2L} f(x)\,dx \\ &\quad + \sum_{n=1}^{\infty}\left(a_n\int_0^{2L} f(x)\cos\frac{n\pi x}{L}\,dx + b_n\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx\right) \\ &= \frac{a_0}{2}(a_0 L) + \sum_{n=1}^{\infty}(a_n(La_n) + b_n(Lb_n)).\end{aligned}$$

Thus,

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_0^{2L} [f(x)]^2\,dx.$$ ■

This theorem can also be proved (albeit by different methods) when $f(x)$ is only piecewise smooth and $2L$-periodic.

When $f(x)$ is continuous with a piecewise continuous first derivative, its Fourier series converges uniformly. This guarantees that the series can be integrated term by term between any two limits, and the resulting series of constants converges to the definite integral of $f(x)$ between the same two limits. The following theorem indicates that term-by-term integration of Fourier series is possible even when $f(x)$ is not continuous (and therefore the Fourier series is not uniformly convergent).

Theorem 12

When a $2L$-periodic function $f(x)$ is piecewise continuous, its Fourier series may be integrated term by term between any finite limits, and the resulting series converges to the definite integral of $f(x)$ between the same limits.

Proof: To prove the theorem, we must show that for any c and d,

$$\int_c^d f(x)\,dx = \frac{a_0}{2}\int_c^d dx + \sum_{n=1}^{\infty}\left(a_n\int_c^d \cos\frac{n\pi x}{L}\,dx + b_n\int_c^d \sin\frac{n\pi x}{L}\,dx\right)$$
$$= \frac{a_0}{2}(d-c) + \sum_{n=1}^{\infty}\left[\frac{La_n}{n\pi}\left(\sin\frac{n\pi d}{L} - \sin\frac{n\pi c}{L}\right) - \frac{Lb_n}{n\pi}\left(\cos\frac{n\pi d}{L} - \cos\frac{n\pi c}{L}\right)\right]$$

when a_0, a_n, and b_n are the coefficients in the Fourier series of $f(x)$.

Because $\int_c^d f(x)\,dx$ can always be expressed as the difference $\int_0^d f(x)\,dx - \int_0^c f(x)\,dx$, it suffices to show the result for integrals over the interval $(0, x)$, that is, to show that

$$\int_0^x f(t)\,dt = \frac{a_0 x}{2} + \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[a_n \sin\frac{n\pi x}{L} - b_n\left(\cos\frac{n\pi x}{L} - 1\right)\right].$$

To show this, we consider the function

$$F(x) = \int_0^x f(t)\,dt - \frac{a_0 x}{2}.$$

Since $f(x)$ is piecewise continuous, $F(x)$ is continuous and $F'(x)$ is piecewise continuous. Furthermore,

$$\begin{aligned}F(x+2L) &= \int_0^{x+2L} f(t)\,dt - \frac{a_0}{2}(x+2L)\\ &= \int_0^x f(t)\,dt + \int_x^{x+2L} f(t)\,dt - \frac{a_0 x}{2} - a_0 L\\ &= F(x) + \int_0^{2L} f(t)\,dt - a_0 L\\ &= F(x),\end{aligned}$$

since $a_0 L = \int_0^{2L} f(t)\,dt$. Thus $F(x)$ is $2L$-periodic. It follows that the Fourier series of $F(x)$ converges to $F(x)$ for all x; that is,

$$F(x) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\left(A_n\cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\right)$$

where, for $n > 0$,

$$\begin{aligned}A_n &= \frac{1}{L}\int_0^{2L} F(x)\cos\frac{n\pi x}{L}\,dx = \left\{\frac{1}{n\pi}F(x)\sin\frac{n\pi x}{L}\right\}_0^{2L} - \frac{1}{n\pi}\int_0^{2L} F'(x)\sin\frac{n\pi x}{L}\,dx\\ &= \frac{-1}{n\pi}\int_0^{2L}\left(f(x) - \frac{a_0}{2}\right)\sin\frac{n\pi x}{L}\,dx\\ &= \frac{-1}{n\pi}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}\,dx + \frac{a_0}{2n\pi}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_0^{2L} = \frac{-L}{n\pi}b_n.\end{aligned}$$

Similarly, $B_n = La_n/(n\pi)$. To obtain A_0, we evaluate the series for $F(x)$ at $x = 2L$:

$$\frac{A_0}{2} + \sum_{n=1}^{\infty} A_n = F(2L) = \int_0^{2L} f(t)\,dt - a_0 L = a_0 L - a_0 L = 0.$$

Thus,

$$A_0 = -2\sum_{n=1}^{\infty} A_n = \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n}.$$

The Fourier series for $F(x)$ is therefore

$$F(x) = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{b_n}{n} + \sum_{n=1}^{\infty}\left(\frac{-Lb_n}{n\pi}\cos\frac{n\pi x}{L} + \frac{La_n}{n\pi}\sin\frac{n\pi x}{L}\right);$$

that is

$$\int_0^x f(t)\,dt = \frac{a_0 x}{2} + \frac{L}{\pi}\sum_{n=1}^{\infty}\left[\frac{a_n}{n}\sin\frac{n\pi x}{L} + \frac{b_n}{n}\left(1 - \cos\frac{n\pi x}{L}\right)\right]. \qquad (29)$$

■

This theorem also provides the additional result that the antiderivative of a Fourier series is itself a Fourier series only when $a_0 = 0$.

Example 9: Illustrate that term-by-term integration of the Fourier series of the function $f(x)$ in Example 1 over the interval $0 \le x \le L$ gives the correct value for the integral of $f(x)$. You will need the fact that $\sum_{n=1}^{\infty} 1/(2n-1)^2 = \pi^2/8$.

Solution: The integral of $f(x)$ for $0 \le x \le L$ is

$$\int_0^L x\,dx = \frac{L^2}{2}.$$

On the other hand, term-by-term integration of the Fourier series of $f(x)$ (see Example 1) gives

$$\begin{aligned}
L\left(\int_0^L dx - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^L \sin\frac{n\pi x}{L}\,dx\right) &= L\left(L - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left\{\frac{-L}{n\pi}\cos\frac{n\pi x}{L}\right\}_0^L\right)\\
&= L\left(L + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}[(-1)^n - 1]\right)\\
&= L^2\left(1 + \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{-2}{(2n-1)^2}\right)\\
&= L^2\left(1 - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\right)\\
&= L^2\left[1 - \frac{4}{\pi^2}\left(\frac{\pi^2}{8}\right)\right] = \frac{L^2}{2}.
\end{aligned}$$

■

Exercises 2.3

1. In this exercise we verify Schwarz's inequality [(27)].

   (a) Show that (27) becomes an equality when terms in the sequences $\{c_n\}$ and $\{d_n\}$ are proportional, that is, when $d_n = \lambda c_n$ for all n ($\lambda > 0$).

(b) Now suppose that the sequences $\{c_n\}$ and $\{d_n\}$ are not proportional. Consider the finite series

$$\sum_{n=1}^{m} (c_n x + d_n)^2 = x^2 \sum_{n=1}^{m} c_n^2 + 2x \sum_{n=1}^{m} c_n d_n + \sum_{n=1}^{m} d_n^2.$$

Establish that the quadratic expression on the right has no zeros, and use this to verify (27).

2. (a) Prove that if a $2L$-periodic function is continuous with a piecewise continuous first derivative, its Fourier coefficients satisfy

$$\lim_{n\to\infty} na_n = 0 = \lim_{n\to\infty} nb_n.$$

(*Hint:* See Theorem 10.)

(b) Does the result in (a) hold if the function is only piecewise continuous?

3. Show that when $f(x)$ is a piecewise continuous function on $0 \le x \le L$,

(a)
$$\sum_{n=1}^{\infty} b_n^2 = \frac{2}{L}\int_0^L [f(x)]^2\,dx$$

when the b_n are calculated according to (18b) and

(b)
$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 = \frac{2}{L}\int_0^L [f(x)]^2\,dx$$

when the a_n are calculated according to (17b).

4. (a) A function $f(x)$ is continuous on the interval $0 \le x \le 2L$ and has a piecewise continuous first derivative. Does the Fourier series of the $2L$-periodic extension of $f(x)$ converge uniformly?

(b) A function $f(x)$ is continuous on the interval $0 \le x \le L$ and has a piecewise continuous first derivative. Does the Fourier sine series of the odd, $2L$-periodic extension of $f(x)$ converge uniformly?

(c) Is your conclusion in (b) the same for the Fourier cosine series of the even, $2L$-periodic extension of $f(x)$?

5. A sequence of functions $\{S_n(x)\}$ is said to *converge in the mean* to a function $f(x)$ on the interval $a \le x \le b$ if

$$\lim_{n\to\infty} \int_a^b [S_n(x) - f(x)]^2\,dx = 0.$$

A series of functions $\sum_{n=1}^{\infty} u_n(x)$ is said to converge in the mean to a function $f(x)$ on the interval $a \le x \le b$ if its sequence of partial sums converges in the mean to $f(x)$.

Use Theorems 9 and 11 to show that the Fourier series of a piecewise continuous function of period $2L$ converges in the mean to the function.

6. A piecewise continuous, $2L$-periodic function $f(x)$ is to be approximated by a sum of the form

$$S_n(x) = \frac{\alpha_0}{2} + \sum_{k=1}^{n}\left(\alpha_k \cos\frac{k\pi x}{L} + \beta_k \sin\frac{k\pi x}{L}\right).$$

One measure of the accuracy of this approximation is the quantity

$$E_n = \int_0^{2L} [f(x) - S_n(x)]^2\,dx,$$

called the *mean square error*. Suppose you are required to choose coefficients α_0, α_k, and β_k in such a way that E_n is as small as possible.

(a) Use the technique in Theorem 9 to show that E_n can be expressed in the form

$$E_n = \int_0^{2L} [f(x)]^2\,dx + L\left(\frac{\alpha_0^2}{2} + \sum_{k=1}^{n} (\alpha_k^2 + \beta_k^2)\right) - 2L\left(\frac{\alpha_0 a_0}{2} + \sum_{k=1}^{n} (\alpha_k a_k + \beta_k b_k)\right),$$

where a_0, a_k, and b_k are Fourier coefficients of $f(x)$.

(b) Regarding E_n as a function of $2n + 1$ variables α_0, α_j, and β_j, set its derivatives with respect to these variables equal to zero to find critical values of E_n. Show that the solution set is $\alpha_0 = a_0$, $\alpha_k = a_k$, and $\beta_k = b_k$. In other words, the partial sums of the Fourier series of a function approximate the function in the mean square sense better than any other trigonometric function of the same form.