FLORIDA INTERNATIONAL UNIVERSITY
Mechanical Engineering Department

Summer 1987 — Analysis of Mechanical Systems — EGM 3311

COURSE CONTENT

1. Classification of 2nd order Partial Differential Equations (PDE); Characteristic equation and determination of characteristics;

2. Elliptic Equations in Fluids and Solids and their Solutions; Boundary Conditions of the Dirichlet, Neumann and Robin (or Churchill) type; Boundedness of Solution; Numerical Methods of Relaxation and Successive Replacement.

3. Parabolic Partial Differential Equations of the 2nd Order in Fluids and Solid Mechanics and their Solutions; Boundary and Initial Conditions.

4. Transform Methods--Laplace and Fourier.

5. Hyperbolic Partial Differential Equations of the 2nd Order in Fluids and Solid Mechanics and their Solutions; Boundary and Initial Conditions; Method of Characteristics.

6. Matrix methods and the Solution of Algebraic Equations Resulting From Mechanical Engineering problems

7. Statistical Methods as applied to Mechanical Engineering Problems and Resulting from Experimental Data; Correlation of Data and their interpretation.

Textbook: Advanced Calculus for Applications by F.B. Hildebrand, 2nd Edition

Grade will be determined on the basis of

3 Exams	20 % each
HW/Project	10 %
Final Exam	30 %

Grading Scheme:

90 and above	A	77 - 79	B-
87 - 89	A-	74 - 76	C+
83 - 86	B+	70 - 73	C
80 - 82	B	67 - 69	C-
		63 - 66	D+
		60 - 62	D
		Below 60	F

FLORIDA INTERNATIONAL UNIVERSITY
Mechanical Engineering Department

Fall 1987 Analysis of Mechanical Systems EGM 3311

COURSE CONTENT

1. Review of Ordinary Differential Equations: First Order and Second Order Linear Homogeneous and Non Homogeneous Equations.
   2.5 Lectures

2. Classification of 2nd order Partial Differential Equations (PDE); Characteristic equation and determination of characteristics;
   0.5 Lectures

3. Elliptic Equations in Fluids and Solids and their Solutions; Boundary Conditions of the Dirichlet, Neumann and Robin (or Churchill) type; Boundedness of Solution; Numerical Methods of Relaxation and Successive Replacement.
   5.0 Lectures

4. Parabolic Partial Differential Equations of the 2nd Order in Fluids and Solid Mechanics and their Solutions; Boundary and Initial Conditions.
   5.0 Lectures

5. Transform Methods--at least Laplace and possibly Fourier.
   2.0 Lectures

6. Hyperbolic Partial Differential Equations of the 2nd Order in Fluids and Solid Mechanics and their Solutions; Boundary and Initial Conditions; Method of Characteristics.
   5.0 Lectures

7. Numerical Methods for Parabolic and Hyperbolic Type Equations: Matrix methods and the Solution of Algebraic Equations Resulting From Mechanical Engineering problems.
   4.0 Lectures

8. **If Time is Available**
   Statistical Methods as applied to Mechanical Engineering Problems and Resulting from Experimental Data; Correlation of Data and their interpretation.

Textbook: Advanced Calculus for Applications by F.B. Hildebrand, 2nd Edition

Grade will be determined on the basis of

2 Exams	20 % each
HW/Project	10 %
Final Exam	30 %

Grading Scheme:			
90 and above A	77 - 79 B-	60 - 62 D
87 - 89 A-	74 - 76 C+	Below 60 F
83 - 86 B+	70 - 73 C	
80 - 82 B	67 - 69 C-	
	63 - 66 D+	

This is a _preliminary_ syllabus subject to change. All changes will be announced in class.

			CLASS #	
AUG	24	M	1	1ST order ODE
	26	W	2	1ST & 2nd order ODE
	31	M	3	2nd order ODE
SEPT	2	W	4	history - start of ELLIPTIC
	7	- LABOR DAY		NO CLASSES
	9	W	5	ELLIPTIC
	14	M	6	ELLIPTIC
	16	W	7	ELLIPTIC
	21	M	8	ELLIPTIC
	23	W	9	ELLIPTIC
	28	M	10	EXAM #1
	30	W	11	PARABOLIC
OCT	5	M	12	PARABOLIC
	7	W	13	PARABOLIC
	12	M	14	PARABOLIC
	14	W	15	PARABOLIC
	19	M	16	EXAM # 2
	21	W	17	HYPERBOLIC
	26	M	18	HYPERBOLIC
	28	W	19	HYPERBOLIC
NOV	2	M	20	HYPERBOLIC
	4	W	21	HYPERBOLIC
	9	M	22	HYP - TRANSFORMS
	11	- VETERAN'S DAY		
	16	M	23	EXAM # 3
	18	W	24	TRANSFORMS
	23	M	25	NUMERICAL
	25	W	26	NUMERICAL
	30	M	27	NUMERICAL
DEC	2	W	28	~~NUMERICAL~~ ~~EXAM~~ NUMERICAL
	4	F		FINAL EXAM
	7	Finals Week		

EXAMPLE

$y'' + y = \tan x$ let $y_P = u_1(x)\cos x + u_2(x)\sin x$

y_P': $u_1'\cos + u_2'\sin = 0$

$-u_1'\sin + u_2'\cos = \tan x$

$$\begin{pmatrix} \cos & s \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$

$$-\frac{\tan x \sin}{1} = u_1' \qquad \boxed{u_2' = \frac{\cos \tan}{1} = \sin x}$$

$$-\frac{\sin^2 x}{\cos x} = -\left[\frac{1-\cos^2 x}{\cos x}\right] = -\frac{1}{\cos x} + \cos x = u_1'$$

$u_2 = -\cos x$ $u_1 = -\ln|\sec x + \tan x| + \sin x$

$\therefore$ $y_P = -\ln| \quad |\cos x + \cos x \sin x - \sin x \cos x$

$y_P = -(\cos x)\ln|\sec x + \tan x|$

BASIC CONCEPTS

- VIBRATION OR OSCILLATION Any motion which repeats itself after an interval of time

  MOTION MAY BE REGULAR OR DETERMINISTIC

- IRREGULAR OR NONDETERMINISTIC
- WE STUDY THE MOTION OF BODIES OR SYSTEMS AS WELL AS THE FORCES THAT ACCOMPANY THIS MOTION OR IS CAUSED BY THIS OSCILLATORY MOTION

METHOD OF VARIATION OF PARAM. (LAGRANGE'S method)

used to solve most general form.

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

seek a solution of the form

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly indep solutions of the homog equation

$$y_p' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2' \qquad \text{Assume } u_1' y_1 + u_2' y_2 \equiv 0$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$\Rightarrow y_p'' + p y' + q y = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + p\,(u_1 y_1' + u_2 y_2') + q\,(u_1 y_1 + u_2 y$$

$$= u_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{=0} + u_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{=0} + u_1' y_1' + u_2' y_2' = g($$

Thus
$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(x) \end{aligned}\right\} \quad \text{first order system for } u_1', u_2'$$

solve for $u_1' = \dfrac{-\,y_2 g}{W(y_1, y_2)}$ $\qquad u_2' = \dfrac{y_1 g}{W(y_1, y_2)}$

since y_1, y_2 are lin indep $W(y_1, y_2) \neq 0$

$$u_1 = -\int^x \frac{y_2(t)\,g(t)\,dt}{W(y_1, y_2)(t)} \qquad u_2 = \int^x \frac{y_1(t)\,g(t)\,dt}{W(y_1, y_2)(t)}$$

and $y_p = u_1 y_1 + u_2 y_2$

SESSION #3

SUPPOSE we first look at SOLDE w/ const. coeff.

Suppose $g(x) = \sin nx,\ \cos nx$ & $r \neq \pm in$

assume $y_p = A \sin nx + B \cos nx$

$g(x) = ax^n$

assume $y_p = Ax^n + Bx^{n-1} + \dots + Ex + F$

$\longrightarrow$ $g(x) = ce^{-px}$ & $y_h = c_1 e^{+rx} + c_2 e^{sx}$

where $r \neq -p \quad s \neq -p$

assume $g(x) = Ae^{-px}$

EXAMPLE $\quad y'' - 3y' - 4y = 2\sin x \Rightarrow \overset{\text{CHAR EQ.}}{(r-4)(r+1) = 0}; \ y_h = c_1 e^{-x} + c_2 e^{4x}$

let $y_p = A\cos x + B \sin x$

put into DE

$$(-A - 3B - 4A)\cos x + (-B + 3A - 4B)\sin x = 2\sin x$$

$$\Rightarrow \quad \underset{0}{\parallel} \qquad\qquad \underset{2}{\parallel}$$

$$\Longrightarrow A = 3/17 \quad B = -5/17$$

$$y_p = \frac{1}{17}(3\cos x - 5\sin x)$$

Suppose $g(x)$ is a linear combination of y_h

$$y'' - 3y' - 4y = 4e^{-x} \qquad y_h = c_2 e^{+4x} + c_1 e^{-x}$$

if $y_p = Ae^{-x}$

$$[A + 3A - 4A]e^{-x} = 4e^{-x}$$ no solution using $y_p = Ae^{-x}$

take $y_p = Axe^{-x} + \quad = (Ax \quad)e^{-x}$

SINCE THE FN $g(x)$ contains a solution of the homog eq. must mult. the homog sol. by a fn whose ~~order~~ power is one higher than homog sol.

suppose y has solutions $(\sin x, \cos x)$

and $g($ $\quad x \quad \Rightarrow y_p = (Ax + \quad)\sin x + (Cx + \quad)\cos x$

$y_p = (Ax^2 + Bx)\sin x + (Cx^2 + Dx)\cos x$

Linear Independence of Solutions

For a Second Order Linear Diff Equation - Homogeneous

if $y_1 y_2' - y_1' y_2 \neq 0 = W_{12}(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

at ~~some~~ every pt in the interval of definition of the fn y

then $y = C_1 y_1 + C_2 y_2$

Non homogeneous - Second Order Linear Diff. Equation (SOLDE)

$$y'' + p(x) y' + q(x) y = g(x)$$

Solution

$$y = y_P + y_h$$

where y_h is solution to $y'' + p(x) y' + q(x) y = 0$

y_P is any solution of the nonhomog SOLDE

METHOD OF UNDETERMINED COEFFICIENTS

suppose $g(x) = \sum_{i=1}^{n} g_i(x)$

then $y_P = \sum_{i=1}^{n} y_{P_i}(x)$ when y_{P_i} is solution for $g_i(x)$

EXAMPLE

$$y'' + 4y = 1 + x + \sin x$$

homog solution to $y'' + 4y = 0$ is $y_h = C_1 \sin 2x + C_2 \cos 2x$

let $g_1(x) = 1$ $\quad y_{P_1} = C_1$ $\quad y''_{P_1} + 4y_{P_1} = 4C_1 = g_1(x) = 1$ $\quad C_1 = \frac{1}{4}$

let $g_2(x) = x$ $\quad y_{P_2} = C_2 x$ $\quad y''_{P_2} + 4y_{P_2} = 4C_2 x = g_2(x) = x$ $\quad C_2 = \frac{1}{4}$

let $g_3(x) = \sin x$ $\quad y_{P_3} = C_3 \sin x$ $\quad y''_{P_3} + 4y_{P_3} = 3C_3 \sin x = g_3(x) = \sin x$ $\quad C_3 = \frac{1}{3}$

$$y_P = \sum y_{P_i} = \tfrac{1}{4} + \tfrac{1}{4}x + \tfrac{1}{3}\sin x$$

$$y = y_P + y_h$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots + \frac{(i\theta)^n}{n!}$$

$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots + (-1)^n \frac{\theta^{2n}}{(2n!)} + i(-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$C_1 e^{i\frac{\sqrt{3}}{2}} + C_2 e^{-i\frac{\sqrt{3}}{2}} = C_1(\cos\theta + i\sin\theta) + C_2(\cos\theta - i\sin\theta)$$

$$= \cos\theta\,[C_1 + C_2] + i\sin\theta\,[C_1 - C_2]$$

$$\equiv C_1'\cos\theta + C_2'\sin\theta$$

C_1 & C_2 are complex no.
C_1' & C_2' are real no.

- note that e^{rx} for r being complex:
  satisfies $\frac{d}{dx}(e^{rx}) = re^{rx}$

THUS $$y(x) = e^{-x/2}\left\{C_1 e^{i\frac{\sqrt{3}x}{2}} + C_2 e^{-i\frac{\sqrt{3}x}{2}}\right\}$$

$$= e^{-x/2}\left\{C_1'\cos\frac{\sqrt{3}x}{2} + C_2'\sin\frac{\sqrt{3}x}{2}\right\}$$

THUS if the roots are complex conjugates $r_1 = \lambda + i\mu$ $r_2 = \lambda - i\mu$ then

$$y = e^{\lambda x}\left[C_1\cos\mu x + C_2\sin\mu x\right]$$

$$\mu = \frac{\sqrt{4ac - b^2}}{2a}$$
$$\lambda = -b/2a$$

IN OUR PROB $\quad p(t) = 4 \qquad y_1(s) = e^{-2s}$

$$y_2 = e^{-2x} \int^x \frac{1}{(e^{-2s})^2} \, e^{-\int^s 4\,dt} \, ds$$

$$= e^{-2x} \int^x \frac{1}{e^{-4s}} \, e^{-4t}\Big|^s \, ds$$

$$e^{-2x} \int^x \frac{\cancel{e^{-4s}}}{\cancel{e^{-4s}}} \, ds = x e^{-2x}$$

$$y_2 = x e^{-2x}$$

$\therefore$ FOR example 2 $\qquad y = C_1 e^{-2x} + C_2 x e^{-2x}$

HOMOG EQ W/
FOR CONSTANT COEFF IF ROOTS are same and one solution
is $y_1 = e^{rx} \qquad y_2 = x e^{rx}$

EXAMPLE #3

$$y'' + y' + y = 0 \implies r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + C_2 e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x}$$

$$= e^{-\frac{1}{2}x} \left\{ C_1 e^{i\frac{\sqrt{3}}{2}x} + C_2 e^{-i\frac{\sqrt{3}}{2}x} \right\}$$

EULER FORMULA
~~DE MOIVRE'S THEOREM~~

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

How do we find the second solution - METHOD OF REDUCTION OF ORDER

METHOD IS GOOD IN CASE OF THE GENERAL EQU.

$$y'' + p(x)y' + q(x)y = 0$$

ASSUME we know one solution $y_1(x)$

ASSUME the second solution is $y = v(x)y_1(x)$

$$y' = v'y_1 + vy_1' \qquad y'' = v''y_1 + 2v'y_1' + vy_1''$$

PUT INTO DE

$$v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + v'(2y_1' + py_1) + v''y_1 = 0$$

since y_1 satisfies ODE.

THIS IS A FIRST ORDER DE FOR v'

$$v'' + (2y_1'/y_1 + p)v' = 0$$

$$v' = C e^{\left[-\int^x (p(t) + 2y_1'/y_1)\,dt\right]}$$

$$= C e^{-\int^x p(t)dt} \underbrace{\underbrace{e^{-\int^x 2y_1'/y_1\,dt}}_{e^{-2\ln y_1}}}_{\frac{1}{y_1^2}}$$

$$v' = \frac{C}{y_1^2} e^{-\int^x p(t)dt} = C\,u(x).$$

$$v = C\int^x u(s)ds + \text{Const.}$$

$$y_2 = y_1 v = C\,y_1(x)\int^x u(s)ds + \cancel{\text{const.}\ y_1(x)}$$

DONT NEED SINCE y_2 must be indep of y_1

$$y_2 = y_1(x)\int^x \frac{1}{y_1^2(s)} e^{-\int^s p(t)dt} ds$$

SECOND order ODE

CONCENTRATED ON HOMOGENEOUS EQS w/ CONSTANT COEFF.

$ay'' + by' + cy = 0$ a sol. is $Ce^{rx} = y$

$\underbrace{(ar^2 + br + c)}_{=0} Ce^{rx} = 0$ $C \neq 0$ $e^{rx} \neq 0$

CHARACTERISTIC EQ.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac > 0$ real & unequal roots

$b^2 - 4ac = 0$ real & equal roots

if $b^2 - 4ac < 0$ imaginary roots that are complex conjugates

EXAMPLE 1 real roots - uneq

$y'' + 3y' + 2y = 0 \Rightarrow r^2 + 3r + 2 = 0$ $(r+1)(r+2) = 0$

$y = C_1 e^{-x} + C_2 e^{-2x}$ Solutions are linearly independent.

Since it's 2nd order have 2 unknown constants.

NEED 2 initial conditions to define y uniquely

EXAMPLE 2 real roots - equal

$y'' + 4y + 4 = 0 \Rightarrow r^2 + 4r + 4 = 0$ $(r+2)^2 = 0$

$y = C_1 e^{-2x} +$ 2nd solution

if we assume that at $x = 0$ $y = 1$ $y(x=0)=1$

$$y(0) = 1 = \frac{1}{3}(0) - \frac{1}{9} + C \cdot \cancel{e^0}^{\,1} = C - \frac{1}{9} \qquad C = \frac{10}{9}.$$

$$\therefore \; y(x) = \frac{1}{3}x - \frac{1}{9} + \frac{10}{9}e^{-3x}$$

EXAMPLE # 2

$$y' - 2xy = 1 \qquad \text{and} \quad IC \quad y(0) = 1$$

$$\Rightarrow \; y' + p(x)\,y = g(x) \qquad p(x) = -2x \qquad g(x) = 1$$

$$\mu(x) = C e^{\int^x p(t)dt} = C e^{\int^x -2t\,dt} = C e^{-t^2|^x}$$

$$\mu(x) = C e^{-x^2}$$

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)\, g(t)dt + \text{const.}\right]$$

$$= \frac{1}{Ce^{-x^2}}\left[\int^x \cancel{C}e^{-t^2}\, 1\, dt + \text{const}\right]$$

$$= e^{+x^2}\int^x e^{-t^2}dt + \text{const}\; e^{x^2}$$

$y(0) = 1$ $$1 = 1\int^0 e^{-t^2}dt + \text{const} \cdot 1$$

$$\text{const} = 1 - \int^0 e^{-t^2}dt$$

$$y(x) = e^{x^2}\int^x e^{-t^2}dt + e^{x^2} - e^{x^2}\int^0 e^{-t^2}dt$$

$$y(x) = e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2} \qquad + e^{x^2}\int_0 e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

$$\boxed{y(x) = \frac{\sqrt{\pi}}{2}\, e^{x^2}\,\mathrm{erf}(x) + e^{x^2}}$$

SUPPOSE

$$y' + p(x)\,y = g(x)$$

SUPPOSE $\exists$ a FN $\mu(x)$ $\ni$.

$$\mu y' + \mu p y = [\mu y]' = \mu g$$

$$\mu y' + \mu p y = \mu y' + \mu' y$$

$$\Rightarrow \quad \mu p = \mu' \quad \text{or} \quad p = \frac{d\mu/dx}{\mu} \quad \Rightarrow \quad p(x)dx = \frac{d\mu}{\mu}$$

$$\int^x p(t)\,dt = \int \frac{d\mu}{\mu} = \ln\mu - \ln C \quad \text{or} \quad \mu(x) = Ce^{\int^x p(t)\,dt}$$

$$\therefore \quad [\mu y]' = \mu(x)\,g(x)$$

$$\mu y = \int^x \mu(t)\,g(t)\,dt + \text{Const.}$$

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)\,g(t)\,dt + \text{const.}\right]$$

w/const $\quad y(x) = \frac{1}{\mu(x)}\left[\int_{x_0}^{x} \mu(t)g(t)dt + \mu(x_0)y_0\right]$

END LESSON # 1

SESSION #2

EXAMPLE

$$y' + 3y = x. \qquad \Rightarrow \quad y' + ay = g(x)$$

$$\mu(x) = Ce^{ax} \Rightarrow Ce^{3x}$$

$$y(x) = \frac{1}{\cancel{C}e^{3x}}\int^x \cancel{C}e^{3t}\,t\,dt + \text{Const}\,e^{-3x}$$

$$= e^{-3x}\int^x te^{3t}\,dt + \text{Const}\,e^{-3x}$$

$$\left[\begin{array}{ll} u = t & dv = e^{3t}dt \\ du = dt & v = \frac{1}{3}e^{3t} \end{array}\right.$$

$$\int u\,dv = uv - \int v\,du = \frac{1}{3}te^{3t}\Big|^x - \int^x \frac{1}{3}e^{3t}dt$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$$

$$y(x) = e^{-3x}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + \text{Const}\,e^{-3x}$$

$$= \frac{1}{3}x \qquad + e^{-3x}$$

- TO OBTAIN C NEED initial condition
- FOR EVERY ORDER OF DERIV NEED I.C.

$$y(x=x_0) = y_0$$

$$x = x_0 \quad y = Ce^{-ax_0} \Rightarrow C = y_0 e^{ax_0}$$

$$\therefore \quad y = y_0 e^{-a(x-x_0)}$$

AN ~~FIRST~~ O.D.E. w/ INITIAL CONDITION INITIAL VALUE PROBLEM

WHAT ABOUT IF THE RHS $\neq 0$ (NONHOMOGENEOUS PROBLEM

$$y' + ay = g(x)$$

MULTIPLY BOTH SIDES BY A FN $\mu(x)$ $\ni$ $\mu(x)[y'+ay] = [\mu y]'$

$$\mu' y + \mu y' = \mu(y' + ay) = \mu(x) g(x)$$

$$\mu' y = \mu a y \quad \text{or} \quad \frac{\mu'}{\mu} = a \quad \text{or} \quad (\ln \mu)' = a$$

$$\ln \mu = ax + \ln C \Rightarrow \mu = Ce^{ax}$$

$\mu(x)$ IS AN INTEGRATION FACTOR

$$\therefore \quad [\mu y]' = \mu g(x)$$

$$\mu y = \int^x \mu(t) g(t)\, dt + \text{Const}$$

$$y = \frac{1}{\mu(x)} \int^x \mu(t) g(t)\, dt + \frac{\text{Const}}{\mu(x)}$$

$$y(x) = e^{-ax} \int^x e^{at} g(t)\, dt + \text{Const}\, e^{-ax}$$

SINCE A CONSTANT REMAINS UNKNOWN NEED AN INITIAL CONDITION TO COMPLETELY DEFINE $y(x)$

IF $y(x_0) = y_0$ $\quad y(x) = e^{-ax} \int_{x_0}^{x} e^{at} g(t)\, dt + y_0 e^{a(x_0) - ax}$

ODE review

ORDER of DIF EQ. = HIGHEST DERIV. IN EQ.

$y = y(x)$ $\qquad y^{IV} + 2y'' = 0$ $\qquad$ 4th order

ORDINARY DIF EQ IF $y = y(x)$ only $\qquad$ 1 variable

PARTIAL DIF EQ $\quad$ if $y = y(x, z)$

$$\frac{\partial^2 y}{\partial x^2} + \alpha^2 \frac{\partial^2 y}{\partial z^2} = 0$$

IF COEF OF DERIVATIVES - DIF. EQ. W/ CONSTANT COEFF

$$u''(t) + c u'(t) + m u(t) = 0.$$

IF COEFF OF DERIV. ARE NOT CONSTANT

$$a(x) y'' + b(x) y' + c(x) y = 0$$

IF COEFF OF DERIV & FN ARE FNS OF X only & power of derivative is 1 => LINEAR

$$a(x) y'' + b(x) y' + c(x) y = 0$$

$$a(x) y'' + b(x)(y')^2 + c(x) y = 0 \qquad \text{NON LINEAR}$$

$$a(x) y'' + b(x) y' y = 0 \qquad \text{NONLINEAR}$$

IF RHS is ZERO IT IS A HOMOGENEOUS EQUATION

FIRST ORDER. DIFF. EQ. - CONST COEFF

$$y' + ay = 0$$

SOLUTION TO THIS is in the form $y = Ce^{px}$

$$\Rightarrow y' + ay = Ce^{px}[p + a] = 0 \qquad c \neq 0 \quad e^{px} \neq 0$$

$$\Rightarrow r \; -a \qquad \therefore$$

$$Ce^{-ax} = y \qquad \text{GENERAL SOLUTION}$$

LESSON # 4

- PHYSICAL PROCESSES NORMALLY VARY WITH TIME AND LOCATION
- TO UNDERSTAND THESE PROCESSES
    - IT WOULD BE NICE TO KNOW HOW THEY VARY IN TIME & SPACE
    - WHAT DRIVES THESE PROCESSES (HOW PROCESSES DEPEND ON SYSTEM PARAM)
    - WHERE THESE PROCESSES WILL BE AT SOME FUTURE TIME OR WHAT WILL HAPPEN AT SOME FUTURE LOCATION
- IT TURNS OUT THAT EQNS THAT DESCRIBE THESE PROCESSES ARE GENERALLY DIFFERENTIAL EQUATIONS
- WHEN THESE PROCESSES DEPEND ON THE VARIATION OF TWO OR MORE QUANTITIES, THEN THESE PROCESSES ARE GOVERNED BY PARTIAL DIFFERENTIAL EQUATIONS
- MANY PROCESSES IN NATURE ARE DESCRIBED BY 2^{nd} ORDER P.D.E.
- EXAMPLES VIBRATIONS OF A ROD (LONGITUDINAL VIBRATIONS)

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad c = \sqrt{E/\rho} \quad \text{BAR VELOCITY}$$

u - LONGITUDINAL DISPLACEMENT

HEAT TRANSFER

$$\frac{\partial T}{\partial t} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) \qquad \frac{k}{c\rho} = \alpha \quad \text{THERMAL DIFFUSIVITY}$$

T TEMPERATURE

POTENTIAL FLOW

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \nabla^2 \phi = 0 \qquad \text{LAPLACE'S EQN}$$

ϕ POTENTIAL $\quad \underset{\sim}{V} = \nabla \phi$

$v = \frac{\partial \phi}{\partial y} \quad u = \frac{\partial \phi}{\partial x}$

STEADY STATE INCOMPRESSIBLE IRROTATIONAL FLOW CONTINUITY EQN IS $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

IN THESE CASES u, T, ϕ FIELD VARIABLE DEPENDENT VARIABLE

x, y, t SPACE COORDINATES OR TIME INDEPENDENT VARIABLES

$$\frac{\partial}{\partial x} u(x,t) = \lim_{\Delta x \to 0} \frac{u(x+\Delta x,t) - u(x,t)}{\Delta x}$$

- THESE THREE EQNS CAN DESCRIBE MANY SYSTEMS IN MECH. ENG.
  - WAVE EQUATION ARISES FROM ACOUSTICS, VIBRATIONS, SHALLOW-WATER WAVE THEORY
  - HEAT EQN ARISES IN HEAT TRANSFER & ONE-DIMENSIONAL DIFFUSION PROBLEMS
  - LAPLACE'S EQN ARISES IN POTENTIAL FLOW THEORY, STEADY STATE HEAT TRANSFER, TORSION OF A BAR, STEADY STATE VIB. OF A MEMBRANE

- DO ALL THREE HAVE ANY COMMON IDEAS?
- HOW CAN THEY BE SOLVED? WHAT METHODS EXIST TO SOLVE THE EQNS
- HOW CAN I DERIVE THE MATHEMATICAL EQN?
- WHAT IS A WELL POSED PROBLEM - CAN I FIND A UNIQUE SOLUTION

- CHARACTERIZATION & CLASSIFICATION
  - MOST GENERAL 2^{nd} ORDER PDE OF A FN $u(x,y)$

$$\varphi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}) = 0 \qquad u_{xx} = \frac{\partial^2 u}{\partial x^2}$$

  - LINEAR WITH RESPECT TO HIGHEST DERIVATIVE

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + F(x, y, u, u_x, u_y) = 0$$

  - HERE a, b, c are fns of x, y only.
  - IF $F(x, y, u, u_x, u_y)$ quasilinear

    IF $F(x, y, u, u_x, u_y) = d\,u_x + e\,u_y + fu + g$ LINEAR

    IF $g = 0$ THEN IT IS HOMOGENEOUS

LINEAR 2^{nd} order PDE HOMOGENEOUS

$$a u_{xx} + b u_{yx} + c u_{yy} + d u_x + e u_y + f u = 0 \qquad (1)$$

- IF ALONG SOME CURVE $\varphi(x,y) =$ constant the equation $a\left(\frac{dy}{dx}\right)^2 - b\left(\frac{dy}{dx}\right) + c = 0$

THEN

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2 - 4ac}}{2a}$$ IS THE SLOPE OF THE CURVE φ IN THE x,y plane

- SO WHAT

IF $b^2 - 4ac > 0$ $\quad \frac{dy}{dx}$ has 2 values, real & (1) HYPERBOLIC TYPE

$b^2 - 4ac = 0$ $\quad \frac{dy}{dx}$ has 1 value, real & (1) PARABOLIC TYPE

$b^2 - 4ac < 0$ $\quad \frac{dy}{dx}$ has 2 complex values & (1) ELLIPTIC TYPE

- CHARACTERIZATION (CLASSIFICATION) OF EQN. IS DETERMINED BY $a u_{xx} + b u_{xy} + c u_{yy}$ portion of PDE

- $\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad a = 1 \quad b = 0 \quad c \to \frac{-1}{c^2} \quad d = e = f = 0$

$t \to y$

$$\frac{dt}{dx} = \frac{0 \pm \sqrt{0 - 4 \cdot 1 \cdot \frac{-1}{c^2}}}{2 \cdot 1} = \pm \frac{1}{c}$$ HYPERBOLIC

if $c =$ constant $\quad t = \pm \frac{1}{c}x +$ constant $\quad \therefore \quad t + \frac{1}{c}x = C_1 = \varphi_1(t,x)$

$t - \frac{1}{c}x = C_2 = \varphi_2(t,x)$

- $a^2 \frac{\partial^2 T}{\partial x^2} - \frac{\partial T}{\partial t} = 0 \qquad a \to a^2 \quad b = 0 \quad c = 0 \quad d = 0 \quad e = -1 \quad f = 0$

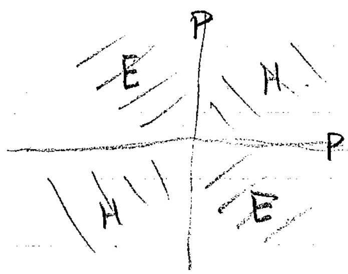
$t \to y$

$$\frac{dt}{dx} = \frac{0 \pm \sqrt{0 - 4a^2 \cdot 0}}{2a^2} = 0$$ PARABOLIC

$t =$ constant $= \varphi_1(t,x)$

$y u_{xx} - x u_{yy} + u_x + y u_y = 0$

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{0 - 4y(-x)}}{2y} = \pm \frac{2\sqrt{xy}}{2y} = \pm \sqrt{\frac{x}{y}}$$



$x=0$ para. excluding $x=0, y=0$

$x>0, y>0$ & $x<0$ $y<0$ hyper

$x<0$ $y>0$ & $x>0$ $y<0$ ellip

- $\dfrac{\partial^2\varphi}{\partial x^2} + \dfrac{\partial^2\varphi}{\partial y^2} = 0$ $\quad a=1 \quad b=0 \quad c=1 \quad d=e=f=0$

$$\frac{dy}{dx} = \frac{0 \pm \sqrt{0-4\cdot1\cdot1}}{2\cdot1} = \pm i \qquad \text{ELLIPTIC}$$

ELLIPTIC EQUATIONS $\longleftrightarrow$ LAPLACE TYPE

PARABOLIC " $\longleftrightarrow$ HEAT EQUATION

HYPERBOLIC " $\longleftrightarrow$ WAVE EQUATION

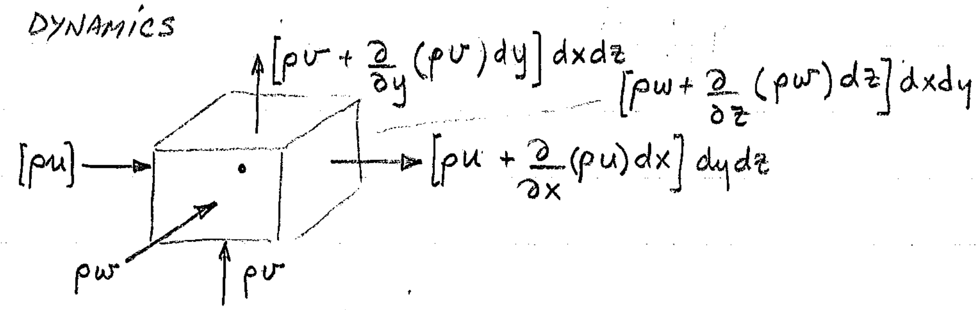
- WHY ARE THESE CURVES $\varphi(x,y)$ important? WHEN WE STUDY HYPERBOLIC EQUATIONS
  - WILL FIND $u = \varphi_1(x+ct) + \varphi_2(x-ct)$
  - $\dfrac{dt}{dx} = \dfrac{1}{c} \Rightarrow c = \dfrac{dx}{dt}$ determines speed at which information travels along characteristics
  - $x+ct$, $x-ct$ are characteristic lines along which information travels
  - NOTICE THAT FOR $c < \infty$ INFO TRAVELS AT FINITE SPEEDS
- FOR PARABOLIC EQNS
  - $\dfrac{dt}{dx} = 0 \Rightarrow \dfrac{dx}{dt} = \infty \Rightarrow c = \infty$ INFO TRAVELS AT ∞ SPEEDS

—— • —— • —— • —— LESSON # 5

- TALK ABOUT ELLIPTIC EQUATIONS FIRST & INVESTIGATE – FLUID DYNAMICS

LOOK AT AN ELEMENTAL VOLUME WHERE MASS MOVES THROUGH FIXED VOLUME

FLUID DYNAMICS



u, v, w are the velocity components of the fluid stream into this fixed volume

$\rho\,(\text{velocity}) \cdot \text{area it crosses} = \text{mass flow rate} = \rho \cdot \dfrac{\text{volume}}{\Delta t} = \dfrac{\text{mass}}{\Delta t}$

RATE OF CHANGE OF MASS IN VOLUME

Continuity says. $\dfrac{\text{mass in}}{\Delta t} = \dfrac{\text{mass out}}{\Delta t} + \dfrac{\text{mass stays in volume}}{\Delta t}$

RATE OF CHANGE OF MASS IN VOLUME = NET MASS INFLOW ; RATE OF DECREASE OF MASS IN VOL = MASS OUTFLOW

$$\frac{\partial}{\partial t}(\rho)\,dx\,dy\,dz = \frac{\text{mass in} - \text{mass out}}{\Delta t} = -\frac{\partial}{\partial x}(\rho u)\,dx\,dy\,dz - \frac{\partial}{\partial y}(\rho v)\,dx\,dy\,dz$$

$$-\frac{\partial}{\partial z}(\rho w)\,dx\,dy\,dz$$

$$\therefore \quad \frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

if $\rho \neq \rho(t) \implies \dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) + \dfrac{\partial}{\partial z}(\rho w) = 0$

IF ρ = constant $\dfrac{\partial}{\partial t}\rho = 0 \implies \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$

FOR 2-D PROBLEMS $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

VORTICITY IN 2-D is given by $\omega = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

IF FLOW IS IRROTATIONAL $\implies \vec{V} = u\vec{i} + v\vec{j} = \dfrac{\partial \phi}{\partial x}\bar{i} + \dfrac{\partial \phi}{\partial y}\bar{j} = \nabla\phi$ SCALAR VELOCITY POTENTIAL ϕ

DEFINE ϕ

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0$$

and continuity implies $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = \nabla^2\phi = 0$

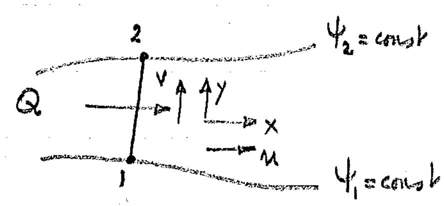
NOTE ALSO: WE CAN DEFINE A STREAM FUNCTION ψ TO SATISFY CONTINUITY

so that $\dfrac{\partial \psi}{\partial y} = u \qquad \dfrac{\partial \psi}{\partial x} = -v$

PUT INTO CONTINUITY. $\dfrac{\partial}{\partial x}\left(\dfrac{\partial \psi}{\partial y}\right) + \dfrac{\partial}{\partial y}\left(-\dfrac{\partial \psi}{\partial x}\right) = 0.$

PUT INTO VORTICITY $\omega = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \psi}{\partial x}\right) - \dfrac{\partial}{\partial y}\left(\dfrac{\partial \psi}{\partial y}\right) = -\nabla^2\psi$

IF IRROTATIONAL $\omega = 0 \quad \nabla^2\psi = 0$



VOLUME FLOW RATE for 2-D problem Q between 2 stream lines $= \psi_2 - \psi_1$

$$Q = -\int_1^2 (v\,dx - u\,dy) = \psi_2 - \psi_1$$

- Do - sep of var
- Show sep of var leads to $F(x+ct) + G(x-ct)$
- Go to deriv of hyperbolic & show characteristics
- define left going wave etc.
- Define domain of ~~inf~~ dep & Range of inf
- Talk about how result may be used for a string
- deriv $f(x) =$

  $g(x) =$

- give example $w_0(x) = \sin \frac{\pi x}{l}$ $\quad 0 \le x \le 1 \quad w_1(x) = 0$

Define for few reflections.

- f reflects as g at $x=0$ for $w=0$ but inverted
- $g(-\sigma)$ is defined for $-\sigma < 0$
- we know w_0 & w_1 $\quad 0 \le x \le l \quad \Rightarrow f, g$ defined for $x \ge 0$

  we want to extend def over the entire line
- center of wave lies along characteristic line

$w=0$ @ $x=l$ $\qquad 0 = f(l+ct) + g(l-ct)$

let $ct - l = \sigma$ $\qquad \sigma = -l \quad t=0$

$ct + l = \sigma + 2l$ $\qquad \sigma \to 0 \quad t >$

$-\sigma = l$

$-\sigma \to 0$

$l - ct = u$

$ct - l = -u$

$ct + 2l = -u + 2l.$

$0 = f(\sigma + 2l) + g(-\sigma)$

$f(\sigma) = -g(-\sigma) = f(\sigma + 2l)$

$t = 0$ to l/c

$u = l$ to 0

$-u + 2l = l$ to $2l$

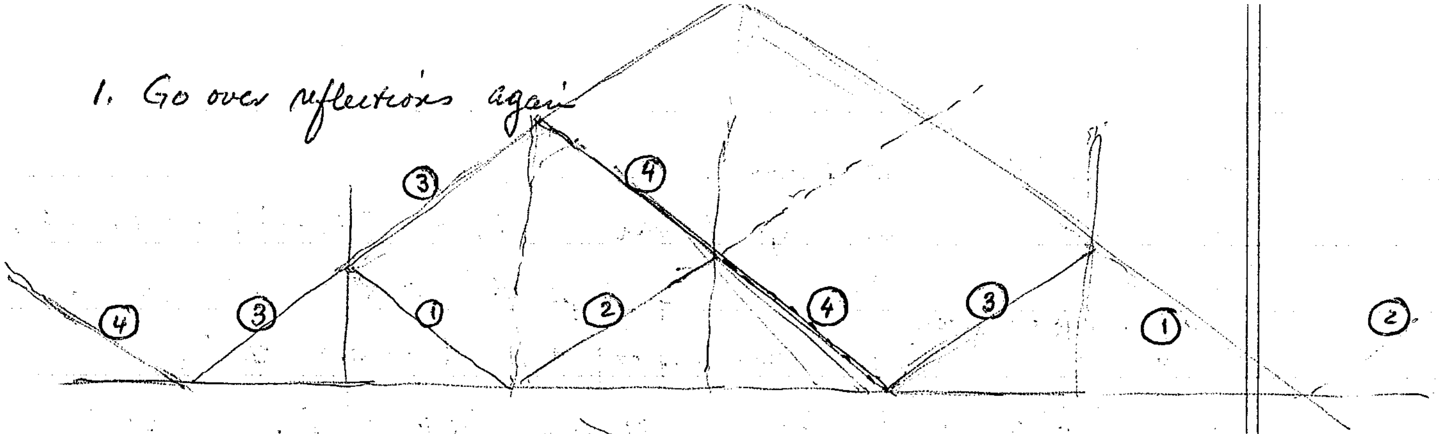
$f(-u+2l) + g(u) = 0$

$f(u) + g(-u+2l)$. $\qquad ct + l = u$

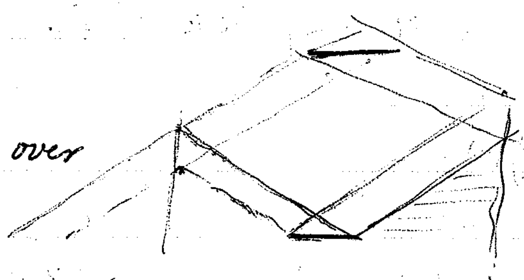
$ct - l = u - 2l.$

$-g(u) = f(-u+2l) = +f(-u)$ $\qquad g(-u) = -f(u) = g(-u+2l)$

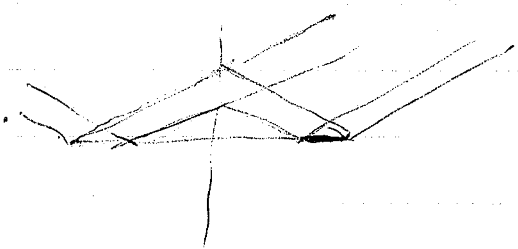
1. Go over reflections again



2. Go over



3.



1. Reflections for single bdry
2. Reflections for 2 boundaries
3. reflections of info.

4. Laplace Transforms

ALSO ψ lines are $\perp$ to ϕ lines

TO DETERMINE THE SOLUTION TO AN ELLIPTIC EQN COMPLETELY NEED BC.

GIVEN $\nabla^2\psi = 0$ MUST GIVEN ONE OF THE THREE

DIRICHLET 1. $\psi = f(x,y)$ on boundary — related to the flow rate

NEUMANN 2. $\dfrac{\partial\psi}{\partial n} = f(x,y)$ on boundary — GIVEN VELOCITY ON BDY.

CHURCHILL, ROBIN

$\rightarrow$ 3. $\dfrac{\partial\psi}{\partial n} = h(\psi - f)$ on boundary

$\dfrac{\partial\psi}{\partial x}$, $\dfrac{\partial\psi}{\partial y}$ FLOWRATE TO VELOCITY

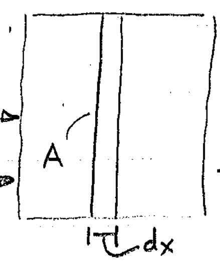
Heat Transfer problem

DID NOT DO

HEAT GENERATION WITHIN YOUR BODY = NET HEAT OUTFLOW

use fourier law $q_x = -kA\dfrac{\partial T}{\partial x}$ HEAT FLOW

1-D

$-kA\dfrac{\partial T}{\partial x} = q_{in} \longrightarrow$  $\longrightarrow q_{out} = -kA\dfrac{\partial T}{\partial x} + \dfrac{\partial}{\partial x}\left[-kA\dfrac{\partial T}{\partial x}\right]dx$

$q \longrightarrow \qquad \longrightarrow q + \dfrac{\partial q}{\partial x}dx$

1ST LAW THERMODYNAMICS $\dfrac{dQ}{dt} = \dfrac{dE}{dt} + \dfrac{dW}{dt}$ — work done by the control mass

into volume

TIME RATE OF CHANGE OF HEAT FLOW = CHANGE IN INTERNAL ENERGY + WORK DONE

$$\frac{dE}{dt} = \text{mass} \cdot \text{specific heat} \cdot \frac{\partial T}{\partial t} = \rho A\,dx \cdot c_v \frac{\partial T}{\partial t} \qquad c_v = \left(\frac{\partial u}{\partial T}\right)_v \text{ internal energy}$$

heat Generated in CV due to work. $\dfrac{dW}{dt} = -\dot{q}_{gen} A\,dx$ — since it is done on the control mass

$$\frac{dQ}{dt} = -kA\frac{\partial T}{\partial x} - \left[-kA\frac{\partial T}{\partial x} + \frac{\partial}{\partial x}\left(-kA\frac{\partial T}{\partial x}\right)dx\right] = \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)dx$$

$\therefore$ FROM 1ST LAW $\dfrac{\partial}{\partial x}\left(kA\dfrac{\partial T}{\partial x}\right)dx = -\dot{q}_{gen}A\,dx + \rho A\,dx \cdot c_v\dfrac{\partial T}{\partial t}$

IF k is constant & $A \neq A(x)$ $\quad k\dfrac{\partial^2 T}{\partial x^2} + \dot{q}_{gen} = \rho c_v \dfrac{\partial T}{\partial t}$

FOR A 2-D problem $\dfrac{dQ}{dt} = \left(\dfrac{dQ}{dt}\right)_{x\,dir} + \left(\dfrac{dQ}{dt}\right)_{y\,direction} = k\dfrac{\partial^2 T}{\partial x^2} + k\dfrac{\partial^2 T}{\partial y^2}$

$$\therefore \quad k\left[\nabla^2 T\right] + \dot{q}_{gen} = \rho C_v \frac{\partial T}{\partial t}$$

IF no source term in control volume $\dot{q}_{gen} = 0$ & steady state $\frac{\partial T}{\partial t} = 0$

$$\Rightarrow \quad \nabla^2 T = 0$$

—— o —— o —— o —— LESSON # 6

SIMILARLY AN INVESTIGATION OF STEADY STATE TEMPERATURE DISTRIBUTION LEADS TO

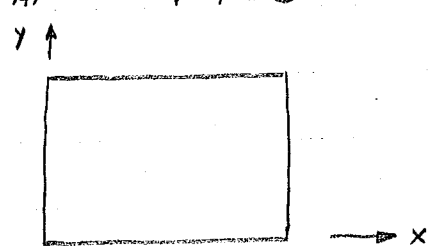
$$\nabla^2 T = 0$$

1. IS A TEMPERATURE B.C.
2. IS EQUIVALENT TO HEAT TRANSFER B.C.
3. IS EQUIVALENT TO A BALANCE CONVECTIVE = CONDUCTION

<u>BE CAREFUL</u> !! ILL POSED PROBLEMS, INCOMPLETELY POSED

- IF BOUNDARY CONDITIONS ARE MISSING INCOMPLETE
- IF TOO MUCH INFO OR NOT THE RIGHT TYPE GIVEN ILL POSED

LOOK AT $\nabla^2 T = 0$ in a plate $0 \le y \le H$ $0 \le x \le L$



Need to give B.C.

let $T = 0$ on $x = 0$ and $x = L$

$T = 0$ on $y = 0$ and

$T = f(x)$ on $y = H$

$T = f(x)$ on $y = H$ since y is fixed $T(x, y = H) = fn$ of x only

- ASSUME A SOLUTION $T = F(x)G(y)$

$$\frac{\partial^2 T}{\partial x^2} = F''G \qquad \frac{\partial^2 T}{\partial y^2} = FG'' \qquad \nabla^2 T = F''G + FG'' = 0$$

$$\frac{F''}{F} = -\frac{G''}{G} = -k^2$$

- CONVERT PDE into ODE
- THE ONLY WAY 2 FNS OF 2 INDEP VAR. ARE EQUAL TO EACH OTHER

$$F'' + k^2 F = 0$$

$$G'' - k^2 G = 0$$

LOOK AT B.C. $T(0,y) = 0 = F(0)G(y) \Rightarrow F(0) = 0$

$T(L,y) = 0 = F(L)\,G(y) \Rightarrow F(L) = 0$

$T(x,0) = 0 = F(x)\,G(0) \Rightarrow G(0) = 0$

$T(x,H) = f(x) = F(x)\,G(H).$

- REMEMBER FOR 2^nd ORDER ODE you need 2 CONDITIONS FOR SOLUTION TO BE COMPLETELY DEFINED

$F'' + k^2 F = 0 \qquad F(0) = F(L) = 0.$

$F(x) = A\cos kx + B\sin kx$

$F(0) \Rightarrow A = 0 \qquad F(L) = 0 \Rightarrow B = 0$ or $\sin kL = 0$

$B = 0$: TRIVIAL; $\sin kL = 0 \Rightarrow kL = n\pi \quad n = 0, 1, 2, \ldots \Rightarrow \frac{n\pi}{L} = k$

$\therefore \; F(x) = B \sin \frac{n\pi}{L} x$

$G'' - k^2 G = 0 \Rightarrow G(y) = C\sinh ky + D\cosh ky$ or $\tilde{C} e^{ky} + \tilde{D} e^{-ky}$

$G(0) = 0 \Rightarrow D = 0 \quad \therefore \; G(y) = C \sinh \frac{n\pi}{L} y$

$T_n(x,y) = \mathscr{C}_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \qquad \mathscr{C}_n = BC$

$$T(x,y) = \sum_{n=1}^{\infty} \mathscr{C}_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} = \Sigma T_n$$

for $n = 0 \Rightarrow k = 0 \Rightarrow F'' = 0 \qquad F_0 = A_0 x + B_0 \qquad F(0) = 0 \Rightarrow B_0 = 0 \quad F(L) = 0 \Rightarrow A_0 = 0$

$\therefore \; T_0(x,y) = F_0 \cdot G_0 = 0$

$$T(x,H) = \sum_{n=1}^{\infty} \mathscr{C}_n \sin\frac{n\pi x}{L} \sinh\frac{n\pi H}{L} = \sum_{n=1}^{\infty} \mathcal{B}_n \sin\frac{n\pi x}{L} = f(x).$$ fourier sine series

Given any fn $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{n\pi x}{L}\,dx \qquad n=0 \Rightarrow a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$$

IN OUR CASE $a_n = 0$ $\quad b_n = \mathcal{B}_n = \mathscr{C}_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L f(x) \sin\frac{n\pi x}{L}\,dx$

FOR EXAMPLE: IF $f(x)=1$ $\quad \mathscr{C}_n \sinh\frac{n\pi H}{L} = \frac{2}{L}\int_0^L 1 \sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\right)_0^L$

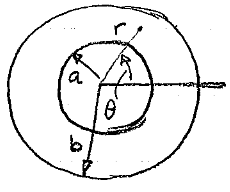
$$\mathscr{C}_n \sinh\frac{n\pi H}{L} = -\frac{2}{n\pi}\left(\cos n\pi - \cos 0\right) = -\frac{2}{n\pi}\left((-1)^n - 1\right) = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\therefore \quad \mathscr{C}_n = \frac{4}{n\pi}\frac{1}{\sinh\frac{n\pi H}{L}} \qquad n \text{ odd.}$$

$$\therefore \quad T(x,y) = \frac{4}{\pi}\sum_{n=1,3,5,7,\dots}^{\infty} \frac{\sin\frac{n\pi x}{L} \sinh\frac{n\pi y}{L}}{n \sinh\frac{n\pi H}{L}}$$

LESSON #7

LOOK AT TEMPERATURE DISTRIBUTION IN AN ANNULUS



$$\nabla^2 T = 0$$

$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

INSULATED INNER $\quad \frac{\partial T}{\partial r}(r_1,\theta) = 0$ and $T(r_2,\theta) = f(\theta)$ so that $f(\theta) = f(\theta + 2n\pi)$ periodicity

Choose $\quad T(r,\theta) = R(r)\,G(\theta)$

$$R''G + \frac{1}{r}R'G + \frac{1}{r^2}RG'' = 0$$ DIVIDE by RG

$$\frac{r^2R'' + rR'}{R} = -\frac{G''}{G} = k^2$$

$$G'' + k^2 G = 0 \quad \Rightarrow \quad G(\theta) = A\cos k\theta + B\sin k\theta$$
$$r^2R'' + rR' - k^2R = 0 \quad \Rightarrow \quad R(r) = Cr^k + Dr^{-k}$$
$\Big\}\ k \neq 0$

$$G'' = 0 \quad \Rightarrow \quad \bar{G}(\theta) = \bar{A} + \bar{B}\theta$$
$$r^2R'' + rR' = 0 \quad \Rightarrow \quad \bar{R}(r) = \bar{C} + \bar{D}\ln r$$
$\Big\}\ k = 0$

BC. $\frac{\partial T}{\partial r}(a,\theta) = 0 = R'(a)\,G(\theta) \quad \Rightarrow \quad R'(a) = 0 \qquad k=0 \Rightarrow \frac{\bar{D}}{a} = 0 \Rightarrow \boxed{\bar{D}=0}$

$T(a,0) = 0 \Rightarrow R(a)\,G(\theta) = 0$ or $R(a) = 0$

$R(a) = \bar{C} + \bar{D}\ln a = 0 \Rightarrow \boxed{\bar{C} = -\bar{D}\ln a}$ $\qquad k \neq 0 \Rightarrow Cka^{k-1} - Dka^{-(k+1)} = 0$

$R(a) = Ca^k + Da^{-k} = 0$ or $(Ca^{2k}+D)a^{-k} = 0 \Rightarrow \boxed{Ca^{2k} = -D} \quad \Rightarrow \boxed{(Ca^{2k} - D)(ka^{-(1+k)}) = 0}$

$Ca^{2k} = D$

at $r=b \quad T = f(\theta) = R(b)\,G(\theta) \quad \Rightarrow \quad G(\theta) = G(\theta + 2n\pi)$ periodic and $\Rightarrow k=n$ integer

$\Rightarrow \quad \bar{G}(\theta) = \bar{A}$

$$\therefore\ T(r,\theta) = \mathscr{C}_0 + \sum_{n=1}^{\infty}\left[\left(r^n + \frac{a^{2n}}{r^n}\right)\left(\tilde{A}_n\cos n\theta + \tilde{B}_n \sin n\theta\right)\right]$$

where $\mathscr{C}_0 = \bar{A}\bar{C} \qquad \tilde{A}_n = CA \quad \tilde{B}_n = CB$

$$T(b,\theta) = \mathscr{C}_0 + \sum_{n=1}^{\infty}\left[\left(b^n + \frac{a^{2n}}{b^n}\right)\left(\tilde{A}_n\cos n\theta + \tilde{B}_n\sin n\theta\right)\right] = f(\theta)$$

if $f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos n\theta + b_n \sin n\theta)$ $\qquad$ where $a_0 = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$

$$a_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta \qquad b_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta$$

$$a_0 = 2\mathscr{C}_0 \qquad a_n = \left(b^n + \frac{a^{2n}}{b^n}\right)\tilde{A}_n \qquad b_n = \left(b^n + \frac{a^{2n}}{b^n}\right)\tilde{B}_n$$

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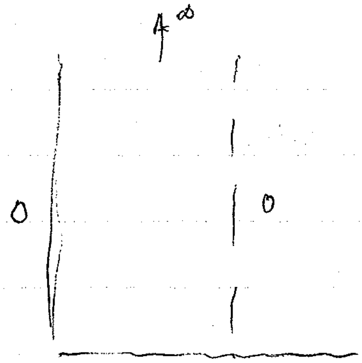
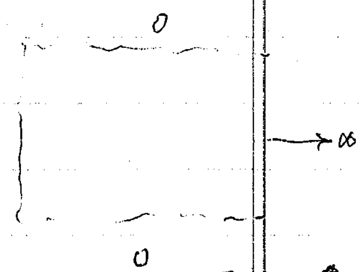


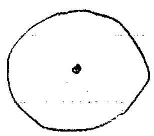
$F(x) = A\cos kx + B\sin kx$

$G(y) = \cancel{Ce^{ky}} + De^{-ky}$

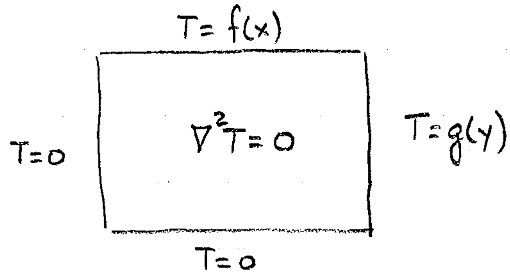


$F(x) = \cancel{A}e^{kx}$

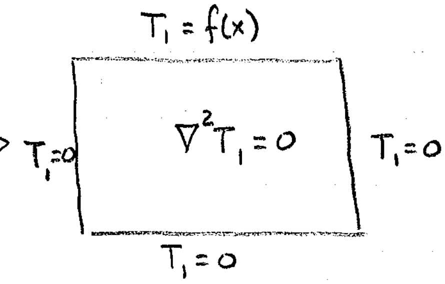
$G(y) = A\cosh y + B\sinh$

		$k=0$	$k \neq 0$
FOR 		$R(r) = \bar{C}\ln r + \bar{D}$	$C r^{k} + D r^{-k}$
		$G(\theta) = \bar{A}\theta + \bar{B}$	$A\cos k\theta + B\sin k\theta$
if	origin included	$R(r) = \bar{D}$	$C r^{k}$
		~~G(θ) remains~~	~~G(θ) B~~
if	infinity	$R(r) = \bar{D}$	$D r^{-k}$
		~~G(θ) remains~~	~~G(θ) remains~~
if	periodicity	~~R(r)~~	
		$G(\theta) = \bar{B}$	$G(\theta)$ requires $k =$ integer

HOW DO YOU SOLVE ?

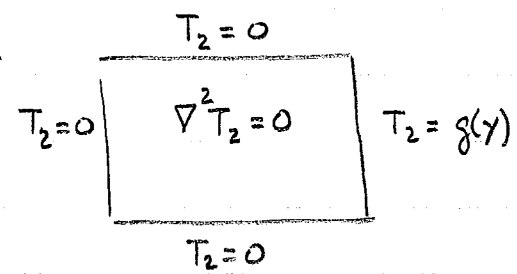


$\Rightarrow$



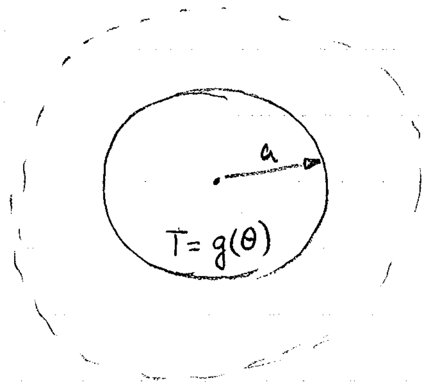
$T = T_1 + T_2$

+



HOW DO YOU SOLVE $\nabla^2 T = 0$ as $r \to \infty$

BOUNDEDNESS – SOLUTION MUST BE BOUNDED



FOR AN ANNULAR REGION

remember $R(r) = Cr^k + Dr^{-k}$

FOR $k \neq 0$ $\quad = Cr^k + \dfrac{D}{r^k}$

$G(\theta) = A\cos k\theta + B\sin k\theta$

FOR $k=0$ $\quad R(r) = \bar{C} + \bar{D}\ln r$

$G(\theta) = \bar{A} + \bar{B}\theta$

periodicity required $\bar{B} = 0$ and $k =$ integer $\quad G(\theta = \theta_0) = G(\theta = \theta_0 + 2\pi) = \bar{A} + \bar{B}\theta_0 = \bar{A} + \bar{B}(\theta_0 + 2\pi)$ $\Rightarrow \bar{B} = 0$

for bounded solutions $r^k \to \infty$ as $r \to \infty$ & $\ln r \to \infty$ as $r \to \infty$

FOR BOUNDEDNESS $\Rightarrow$ C & $\bar{D} \equiv 0$

$$\therefore \quad T = RG = \bar{A}\bar{C} + \sum_n r^{-n}\left[AD\cos n\theta + BD\sin n\theta\right]$$

$$= \mathscr{C}_0 + \sum_{n=1}^{\infty}\left[\mathscr{C}_n \cos n\theta + \mathscr{D}_n \sin n\theta\right] r^{-n}$$

at $r = a$ $\quad T = g(\theta) = \mathscr{C}_0 + \sum a^{-n}\left[\qquad\qquad\right] = \dfrac{a_0}{2} + \sum a_n \cos n\theta + b_n \sin$

where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{2\pi} g(\theta)\,d\theta$ $\quad b_n = \dfrac{2}{\pi}\displaystyle\int_0^{2\pi} g(\theta)\sin n\theta\,d\theta$ $\quad a_n = \dfrac{2}{\pi}\displaystyle\int_0^{2\pi} g(\theta)\cos n\theta\,d\theta$

$\dfrac{a_0}{2} = \mathscr{C}_0$ $\qquad a^{-n}\mathscr{D}_n$ $\qquad a^{-n}\mathscr{C}_n$

∇^2
Complicated Operator acting on T yields a constant times T

note that as $r \to \infty$ $\quad T \to \mathscr{C}_0$ $\quad$ Thus $T(r=\infty,\theta)$ is the mean of T prescribed on $r=a$.

— o — o — o —

LESSON #9

Suppose $\nabla^2 T = 1$ $\quad$ Poisson's equation

look at $\nabla^2 T$ in r,θ coordinates

$$\frac{\partial^2}{\partial r^2}T + \frac{1}{r}\frac{\partial}{\partial r}T + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 1$$

Choose $T = T_h + T_p$ $\quad$ so that $\nabla^2 T_h = 0$ and $\nabla^2 T_p = 1$

T_p : any solution that satisfies PDE

look at specific case : let $T_p = T_p(r)$ only - if we want periodic fn

$$\frac{\partial^2}{\partial r^2}T + \frac{1}{r}\frac{\partial}{\partial r}T = \frac{1}{r}\frac{d}{dr}\left(r\frac{d}{dr}T\right) = 1$$

$$\frac{d}{dr}\left(r\frac{d}{dr}T\right) = r \qquad r\frac{d}{dr}T = \frac{r^2}{2} + C_1 \qquad \frac{d}{dr}T = \frac{r}{2} + \frac{C_1}{r}$$

$$T = \underbrace{\frac{r^2}{4}}_{\text{solution To } \nabla^2 T = 1} + \underbrace{C_1 \ln r + C_2}_{\text{solution to } \nabla^2 T = 0}$$

in x,y coordinates $r^2 = x^2+y^2$ $\quad \therefore$ $\nabla^2 T = 1$ has $T_p = \frac{x^2}{4} + \frac{y^2}{4}$

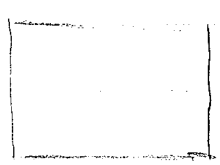
Suppose, $\frac{\partial^2 T}{\partial \theta^2} = r^2 \Rightarrow T_p = \frac{r^2\theta^2}{2} + \underbrace{A\theta + B}_{\text{solution to } \nabla^2 T = 0}$ $\qquad$ θ^2 is not periodic fun.

now find T_h as in the past & apply BC to $T_h + T_p$ $\quad$ <u>total solution</u>

Suppose $\nabla^2 T + \lambda^2 T = 0$ $\qquad$ Eigenvalue problem

boundary conditions for EV problems are $T=0$ all around bdry. Look at $T=0$

$\frac{\partial T}{\partial n} = 0$ or $\alpha T + \frac{\partial T}{\partial n} = 0$



let $T = F(x)\,G(y)$

$$F''G + FG'' + \lambda^2 FG = 0 \Rightarrow \frac{F''}{F} + \frac{G''}{G} + \lambda^2 = 0$$

$$\frac{F''}{F} = -\left(\frac{G''}{G} + \lambda^2\right) = -k^2$$

Note.

$F'' + k^2 F = 0$ $\quad$ and $\quad$ $F = A\sin kx + B\cos kx$

$T(0,y) = T(L,y) = T(x,0) = T(x,H) = 0$

$$\left.\begin{aligned} T(0,y) &= F(0)G(y) = 0 \Rightarrow F(0)=0 \\ T(L,y) &= F(L)G(y) = 0 \Rightarrow F(L)=0 \end{aligned}\right\} \quad \begin{aligned} &B=0 \\ &kL = n\pi \quad \therefore\ k = \frac{n\pi}{L} \qquad F_n = A_n \sin\frac{n\pi x}{L}. \end{aligned}$$

$T(x,0) = F(x)G(0) = 0 \Rightarrow G(0)=0$

$T(x,H) = F(x)G(H) = 0 \Rightarrow G(H)=0$

$-\left(\frac{G''}{G} + \lambda^2\right) = -k^2 \quad \therefore \quad \frac{G''}{G} = k^2-\lambda^2$ FOR 0 B.C. want sin, cos.

$= -(\lambda^2-k^2) = -\mu^2 \quad \therefore \quad k^2-\lambda^2 < 0.$

or $G'' + (\lambda^2-k^2)G = 0 \Rightarrow$

$G = C\sin\mu y + D\cos\mu y \qquad G(0)=0 \Rightarrow D=0$

$G(H)=0 \Rightarrow \mu H = m\pi$

$\therefore \quad \mu = \frac{m\pi}{H}$

$\therefore \quad G_m = C_m \sin\frac{m\pi}{H}y. \qquad \mu^2 = \lambda^2-k^2 \ \therefore\ \mu^2+k^2=\lambda^2$

$\lambda = \pi\sqrt{\frac{m^2}{H^2}+\frac{n^2}{L^2}}$

and $T_{mn} = C_m A_n \sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}$

when $k=0 \Rightarrow F''=0$ or $F = \bar{A}x+\bar{B}$; w/ BC $\Rightarrow \bar{A}=\bar{B}=0$

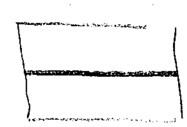
$G''+\lambda^2 G=0 \qquad G = \bar{C}\sin\lambda y + \bar{D}\cos\lambda y$; w/BC $\bar{D}=0$ $\lambda H = m\pi$

$\therefore \quad \mu=\lambda$ and $FG=0$ for $k=0$

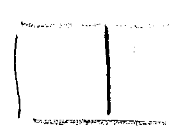
$\therefore \quad T = \sum\sum C_m A_n \sin\frac{m\pi y}{H}\sin\frac{n\pi x}{L}.$

Node lines $n=1, m=1$ $y=H$, $=0$ $x=L$, $=0$ $\therefore$ no node lines in region

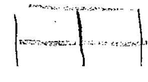
$n=1, m=2$ $y=0$, $y=H/2$, $=H$ $x=0$, L $\therefore$ 1 node line



$n=2, m=1$ $y=0$, $=H$ $x=L/2$, $=0$, L $\therefore$ 1 node line



$n=2, m=2$ 2 node lines



3 types of bc. $T = g(t)$ temp BC

$\frac{\partial T}{\partial x} = g(t)$ heat flow BC.

$\alpha \frac{\partial T}{\partial x} + T = g(t)$ convection = conduction

LESSON #10 - EXAM

LESSON #11

DERIVE THE 1-D HEAT EQN SEE LESSON #2

$$\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) = -\dot{q}_{gen}A + \rho A c_v \frac{\partial T}{\partial t}$$

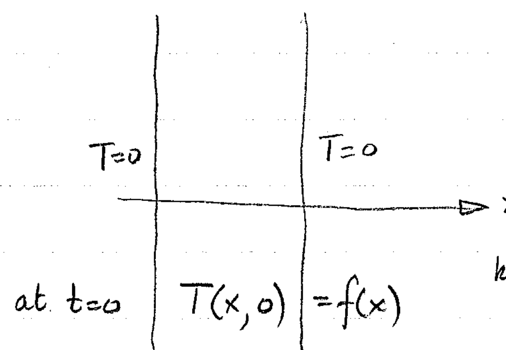
if $\dot{q}_{gen}=0$ and $kA \neq$ fn of x $\Rightarrow$ $\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\qquad \alpha^2 = \frac{k}{\rho c_v}$

FOR 2-D $\quad \frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(kA\frac{\partial T}{\partial y}\right) = -\dot{q}_{gen}A + \rho A c_v \frac{\partial T}{\partial t}$

if $\dot{q}_{gen}=0$ and $kA \neq$ fn of x & y $\Rightarrow$ $\alpha^2(\nabla^2 T) = \frac{\partial T}{\partial t}$

if $\frac{\partial}{\partial t}=0$ steady state $\Rightarrow$ $\nabla^2 T = 0$

LOOK AT 1-D HEAT EQN $\quad \alpha^2\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\qquad$ let $T = F(x)G(t)$



$$\alpha^2 F''G = FG' \Rightarrow \frac{F''}{F} = \frac{G'}{\alpha^2 G} = -k^2$$

$k \neq 0$ $\begin{cases} F'' + k^2 F = 0 & F = A\sin kx + B\cos kx \\ G' + \alpha^2 k^2 G = 0 \Rightarrow & G = Ce^{-\alpha^2 k^2 t} \end{cases}$

$k=0 \qquad F''=0 \Rightarrow F = \bar{A}x + \bar{B}\,;\; G'=0 \Rightarrow G = \bar{C}$

Need 1 IC for G and need 2 BC

$\left.\begin{array}{lll} T=0 \text{ @ } x=0 & \Rightarrow F(0)=0 & \Rightarrow B=0 \\ T=0 \text{ @ } x=L & \Rightarrow F(L)=0 & \Rightarrow kL = n\pi \quad k=\frac{n\pi}{L} \end{array}\right\} k\neq 0$

$\left.\begin{array}{ll} T=0 \text{ @ } x=0 & \Rightarrow \bar{B}=0 \\ T=0 \text{ @ } x=L & \Rightarrow \bar{A}=0 \end{array}\right\} k=0 \Rightarrow F(x)=0 \quad \therefore$ $k=0$ is not a solution

$$= \frac{2}{L}\int_0^L \left[f(x) - T_1 - (T_2-T_1)\frac{x}{L}\right]\sin\frac{n\pi x}{L}\,dx$$

$x = u \quad dv = \sin\frac{n\pi x}{L}dx$

$dx = du \quad v = \frac{-L}{n\pi}\cos\frac{n\pi x}{L}$

$\int u\,dv = uv - \int v\,du$

$= -\frac{xL}{n\pi}\cos\frac{n\pi x}{L} + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx$

$\cancel{\left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x}{L}}\Big|_0^L$

$$= \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx - \frac{2}{L}T_1\int_0^L \sin\frac{n\pi x}{L}dx - \frac{2}{L^2}(T_2-T_1)\int_0^L x\sin\frac{n\pi x}{L}dx$$

$$+ \frac{2}{L}T_1 \cos\frac{n\pi x}{L}\Big|_0^L \cdot \frac{L}{n\pi} - \frac{2}{L^2}(T_2-T_1)\left[-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_0^L$$

$$\frac{L}{n\pi}\frac{2}{L}T_1\left[\cos n\pi - 1\right]$$

$$+ \qquad - \frac{2}{\cancel{L^2}}(T_2-T_1)\left[-\frac{\cancel{L^2}}{n\pi}\cos n\pi\right]$$

$$+ \frac{2}{n\pi}(-T_1) + \frac{2}{n\pi}(T_2\cos n\pi)$$

$$\mathcal{B}_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx + \frac{2}{n\pi}(T_2\cos n\pi - T_1)$$

$$\therefore \quad T = \sum_{n=1}^{\infty} AC\, e^{-\alpha^2 k^2 t} \sin kx \qquad k = \frac{n\pi}{L}$$

@ $t=0$ $\quad T(x,0) = f(x) = \sum AC \sin kx$

But fourier series

$$f(x) = \frac{a_0}{2} + \sum a_k \cos kx + \sum b_k \sin kx \quad \Rightarrow a_0, a_k$$

$$b_k = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

as $t \to \infty$ $\quad T \to 0 \quad \therefore$ at $t \to \infty$ $\quad \frac{\partial T}{\partial x} \to 0$ and $q = 0$

LESSON # 12

what if the b.c. are not zero

$T = T_1$ $\quad \alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\quad T = T_2$ $\quad$ $T = f(x)$ $\quad = \quad$ $T = T_1$ $\quad \frac{\partial^2 T}{\partial x^2} = 0$ $\quad T = T_2$ $\quad$ 0 $\quad$ L $\quad$ $T = T_p$ $\quad + \quad$ $T = 0$ $\quad \alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\quad T = 0$

let $T = T_p + T_T$

$T = f(x) - T_p = p(x)$

$$T_p = Ax + B$$

$$B = T_1 \qquad A = \frac{T_2 - T_1}{L}$$

$$T_T = \sum \mathscr{C}_n e^{-\alpha^2 k^2 t} \sin kx \qquad k = \frac{n\pi x}{L}$$

let $T_p = T_1 + \frac{T_2 - T_1}{L} x$

$$\mathscr{C}_n = \frac{2}{L}\int_0^L p(x) \sin \frac{n\pi}{L} x$$

$$T = T_1 + \frac{T_2 - T_1}{L} x + \sum_{n=1} \mathscr{C}_n e^{-\alpha^2 \frac{n^2 \pi^2}{L^2} t} \sin \frac{n\pi x}{L} \qquad \text{as } t \to \infty \quad T \to T_p$$

$$\mathscr{C}_n = \frac{2}{L}\int_0^L \left\{ f(x) - \left(T_1 + \frac{T_2 - T_1}{L} x\right)\right\} \sin \frac{n\pi x}{L}\, dx = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx + \frac{2}{n\pi}(T_2 \cos n\pi - T_1$$

What if

$\frac{\partial T}{\partial x} = Q_1$ $\quad$ $T = f(x)$ $\quad$ $\frac{\partial T}{\partial x} = Q_2$

let $T = T_p + T_T$

now since $T = T_p + T_T$

$$\frac{\partial T}{\partial x} = \frac{\partial T_p}{\partial x} + \frac{\partial T_T}{\partial x} \quad ; \quad \text{at } x=0 \quad \frac{\partial T}{\partial x} = Q_1 = Q_1 + \frac{\partial T_T}{\partial x} \Rightarrow \frac{\partial T_T}{\partial x} = 0$$

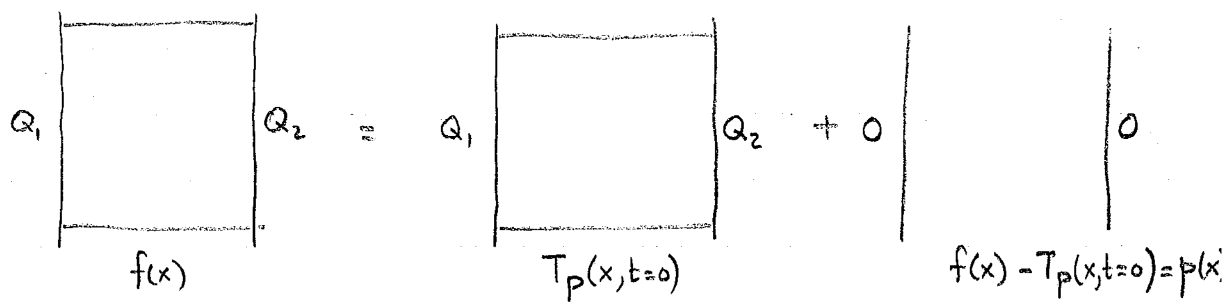
$$x = L \quad \frac{\partial T}{\partial x} = Q_2 = Q_2 + \frac{\partial T_T}{\partial x} \Rightarrow \frac{\partial T_T}{\partial x} = 0$$

$$T(x,t=0) = T_p(x,t=0) + T_T(x,t=0)$$

$$f(x) = \underbrace{\cancel{\frac{Q_2-Q_1}{2L}x^2 + Q_1 x}}_{T_p(x,t=0)} + T_T(x,t=0) \qquad \text{or } T_T(x,t=0) = f(x) - \Big[\overbrace{\cancel{\frac{Q_2-Q_1}{2L}x^2 + Q_1 x}}^{T_p(x,t=0)}\Big] = p(x).$$

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \Rightarrow \alpha^2\left(\frac{\partial^2 T_p}{\partial x^2} + \frac{\partial^2 T_T}{\partial x^2}\right) = \frac{\partial^2 T_p}{\partial t} + \frac{\partial^2 T_T}{\partial t}$$

$$\Rightarrow \alpha^2 \frac{\partial^2 T_T}{\partial x^2} = \frac{\partial^2 T_T}{\partial t}$$



CAN the steady state solution solve

let $T_p = Ax + B$ $\ni.$ $\frac{\partial^2 T_p}{\partial x^2} = 0$

$$\frac{\partial T_p}{\partial x} = A \qquad @\ x = 0 \quad \frac{\partial T_p}{\partial x} = Q_1 \qquad \Rightarrow Q_1 = A$$

$$T_p = Q_1 x + B \qquad \frac{\partial T_p}{\partial x} = Q_1 \quad ; \quad @\ x = L \ \frac{\partial T_p}{\partial x} = Q_2 \ \rightarrow\leftarrow$$

$\therefore$ T_p cannot be linear fn of x only (steady state soln).

$$\left.\frac{\partial T}{\partial x}=Q_1\right| \quad \alpha^2\frac{\partial^2 T_p}{\partial x^2}=\frac{\partial T_p}{\partial t} \quad \left|\frac{\partial T}{\partial x}=Q_2\right. \qquad + \qquad \left.\frac{\partial T}{\partial x}=0\right| \quad \alpha^2\frac{\partial^2 T_T}{\partial x^2}=\frac{\partial T_T}{\partial t} \quad \left|\frac{\partial T}{\partial x}=0\right.$$

$T_p(x,t=0)$

$$T_T(x,t=0)=f(x)-T_p(x,t=0)=p(x)$$

see next page for this derivation

let $T_p = t\,f(x)+g(x)$

$$\alpha^2\frac{\partial^2 T_p}{\partial x^2}=\frac{\partial T_p}{\partial t}: \quad \alpha^2\left[tf''+g''\right]=f \quad \Rightarrow \quad \underbrace{\alpha^2 t f''}=\underbrace{-\alpha^2 g''+f}_{\text{fn of } x}=0 \quad \Rightarrow \quad f''=0 \quad f=Ax+\bar{B}$$

$$g''=f/\alpha^2$$

$$\hookrightarrow g=\frac{1}{\alpha^2}\left[\frac{Ax^3}{6}+\frac{\bar{B}x^2}{2}+\bar{C}\right.$$

$$\therefore \quad T_p=t\,f(x)+g(x)=t\left[Ax+B\right]+\frac{1}{6\alpha^2}\left[Ax^3+3\bar{B}x^2+6\bar{C}x+6\bar{D}\right]$$

$$\Rightarrow \quad T_p = A\left[x^3+6\alpha^2xt\right]+3\bar{B}\left[x^2+2\alpha^2t\right]+6\bar{C}x+6\bar{D}$$

$$A\left[x^3+6\alpha^2xt\right]+B\left[x^2+2\alpha^2t\right]+Cx+D$$

$$\frac{\partial T_p}{\partial x}=A\left[3x^2+6\alpha^2t\right]+B\left[2x\right]+C$$

$$@\ x=0 \qquad Q_1=A\left[6\alpha^2t\right]+C \quad \text{must be true } \forall t \quad \Rightarrow \quad C=Q_1 \quad A=0$$

$$Q_2=A\left[3L^2\right]+B\left[2L\right]+Q_1 \quad \Rightarrow \quad B=\frac{Q_2-Q_1}{2L}$$

$$\therefore \quad T_p=\frac{Q_2-Q_1}{2L}\left[x^2+2\alpha^2t\right]+Q_1x+D$$

Now

$$T_p(x,t=0)=\frac{Q_2-Q_1}{2L}\left[x^2\right]+Q_1x \qquad \therefore \quad T_T(x,t=0)=f(x)-\left[\frac{Q_2-Q_1}{2L}x^2+Q_1x\right]$$

$$=p(x)$$

$$T=T_p+T_T=\frac{Q_2-Q_1}{2L}\left[x^2+2\alpha^2t\right]+Q_1x+\left\{\mathscr{C}_0+\sum\mathscr{C}_n e^{-\frac{\alpha^2n^2\pi^2t}{L^2}}\cos\frac{n\pi x}{L}\right.$$

$$\text{where} \quad \mathscr{C}_0=\frac{1}{L}\int_0^L p(x)\,dx \qquad \mathscr{C}_n=\frac{2}{L}\int_0^L p(x)\cos\frac{n\pi x}{L}\,dx$$

note that as $t\to\infty$ $T_p\to\infty$ as well $\Rightarrow$ no steady steady soln exists

LESSON # 13

$$\frac{\partial T}{\partial x} = Q_1 \ @\ x=0 \qquad \frac{\partial T}{\partial x} = Q_2 \ @\ x=L \quad \Rightarrow \quad \frac{\partial T}{\partial x} = Ax+B \qquad B = Q_1$$

$$A = \frac{Q_2 - Q_1}{L}$$

$$\therefore \frac{\partial T}{\partial x} = Q_1 + \frac{Q_2-Q_1}{L} x$$

note q is not a fn of time

Integrate $T_p = Q_1 x + \frac{Q_2-Q_1}{L}\frac{x^2}{2} + f(t)$

but T satisfies $\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$ $\quad \frac{\partial T}{\partial t} = f'(t)$.

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \alpha^2 \frac{Q_2-Q_1}{L} \qquad \therefore \qquad f'(t) = \alpha^2 \frac{Q_2-Q_1}{L}$$

$$f(t) = \alpha^2 \frac{Q_2-Q_1}{L} t + \text{const.} \qquad \therefore \qquad T = Q_1 x + \frac{Q_2-Q_1}{2L}\left(x^2 + 2\alpha^2 t\right) + \text{const}$$

$$T_p = Q_1 x + \frac{Q_2-Q_1}{2L}\left(x^2+2\alpha^2 t\right)$$

SELF-SIMILAR SOLUTIONS

1. SOMETIMES WE WANT TO FIND SOLUTION IN DIMENSIONLESS FORM

   WHY ?

2. RESULTS ARE INDEPENDENT OF SIZE OF SYSTEM

3. CHOOSE SOME LENGTH OR TIME SCALE THAT CHARACTERIZE PROBLEM IN TERMS OF INDEPENDENT VARIABLES.

4. $\frac{x}{L} = \chi$ LENGTH OF PROBLEM

   WHERE DO WE GET L, time

5. LENGTH OR TIME SCALE CAN COME FROM B.C. OR GEOMETRY

6. PROBLEMS WITH NATURAL CHARACTERISTIC SCALES ARE CALLED SCALE-SIMILAR.

   SCALE-SIMILAR SOLUTIONS FOR SYSTEMS OF DIFFERENT SIZES

   WILL HAVE SAME NONDIM. SOL. IF THEY HAVE SAME DIMENSIONLESS PARAM. BC & IC.

GIVE PDE HANDOUT

- CERTAIN PROBLEMS DO NOT HAVE NATURAL SCALE FOR INDEPENDENT VARIABLE

FOR EXAMPLE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_i \quad x>0$$
$$T(0,t) = T_s$$

and $x \to \infty \quad T \to T_i$

IF FOR INDEPENDENT VARIABLES OF THE PROBLEM $\nexists$ CLUE TO EXISTENCE

- NO CHARACTERISTIC LENGTH
- " " TIME

$\Big\} \Rightarrow$ SELF-SIMILAR SOLUTION.

- SOLUTION OF ALL PHYSICAL PROBLEMS MAY BE EXPRESSED IN DIMENSIONLESS FORM

~~FROM INDEPENDENT VARIABLES~~

- $\Rightarrow t, x$ must form a dimensionless group

FROM PDE : $x^2 = \alpha t$ or $\frac{x^2}{\alpha t}$, $\frac{\alpha t}{x^2}$, or $\frac{x}{\sqrt{\alpha t}}$ or $\frac{\sqrt{\alpha t}}{x}$

- IF SOL'N MADE DIMENSIONLESS BY COMBO OF INDEPENDENT VARIABLES INSTEAD OF GEOMETRY, BC OR IC, - PROBLEM IS SELF SIMILAR

- THERE IS A CHARACTERISTIC TEMP TO THE PROBLEM $T_s - T_i$

could guess

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \frac{x}{\sqrt{\alpha t}}\, h\left(\frac{x}{\sqrt{\alpha t}}\right)$$

- Could choose $\frac{T}{T_s - T_i} =$

- define $\eta = \frac{x}{\sqrt{\alpha t}}$ SIMILARITY VARIABLE

$\Rightarrow$ • ALL TEMPERATURE PROFILES FALL ONTO ONE GRAPH

CHOOSING a solution involving η

- REDUCES PDE TO AN ODE WHICH IS A FN OF η

- NOTE x, t (2 indep var) now becomes η (1 indep var)

  $\Rightarrow$ SELF SIMILARITY REDUCE # OF INDEPENDENT VAR. BY 1

METHOD OF APPROACH

- $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad w/ \quad T(0,t) = T_s$$

$$T(x,0) = T_i$$

$$T(x,t) \to T_i \quad \text{as } x \to \infty$$

IN GENERAL

Choose $\eta = \frac{Ax}{t^n}$ $\qquad$ A, n picked to reduce eqn to ODE

let $\frac{T - T_i}{T_s - T_i} = f(\eta)$

- NOTE: SINCE $T_s - T_i$ is a basic aspect of problem

  FORM OF η : SINCE $t = 0$ & $x = \infty$ give T_i

CHOOSE FORM OF η : NUMERATOR – INDEP VAR DIFFERENTIATED MOST OFTEN

DENOM – LEAST OFTEN

$$T = T_i + (T_s - T_i) f(\eta)$$

- $$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \frac{\partial \eta}{\partial x} = \frac{A}{t^n} \qquad \frac{\partial T}{\partial \eta} = (T_s - T_i)\frac{df}{d\eta}$$

$$= (T_s - T_i) f' \left(\frac{A}{t^n}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial \eta}\left[(T_s - T_i) f'(\eta)\left(\frac{A}{t^n}\right)\right] \cdot \frac{\partial \eta}{\partial x} = (T_s - T_i)\left(\frac{A^2}{t^{2n}}\right) f''(\eta)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = (T_s - T_i) f' \cdot \frac{-nAx}{t^{n+1}} = (T_s - T_i) f' \left[-\frac{n\eta}{t}\right]$$

$$(T_s - T_i)\frac{A^2}{t^{2n}} f''(\eta) = \frac{1}{\alpha}(T_s - T_i)\left(\frac{-\eta n}{t}\right) f'$$

$$(T_s - T_i)\frac{A}{t^{2n}}\left[f'' + \frac{n\eta}{\alpha A^2} t^{2n-1} f'\right] = 0$$

FOR ODE ; η, f, f', f'' $\qquad t^{2n-1} = 1 \Rightarrow n = 1/2$

$$\therefore \quad f'' + \frac{1}{2\alpha A^2}\eta f' = 0 \qquad \text{pick } A \Rightarrow f'' + \eta f' = 0$$

$\therefore$ choose $A = \frac{1}{\sqrt{2\alpha}}$

$$\therefore \quad \eta = \frac{x}{\sqrt{2\alpha t}}$$

LESSON #14

at $x = 0 \quad T = T_s \quad \Rightarrow \eta = 0 \quad \frac{T_s - T_i}{T_s - T_i} = 1 = f(\eta = 0)$

collapse of 2 conditions to 1
$t = 0 \quad T = T_i \quad \Rightarrow \eta = \infty \quad T = T_i \Rightarrow 0 = f(\eta = \infty)$
$x \to \infty \quad T \to T_i \quad \Rightarrow \eta \to \infty \quad T \to T_i \quad 0 \leftarrow f(\eta \to \infty)$

note $f'' + \eta f'$ (2^nd order ODE)

$$\frac{df'}{d\eta} + \eta f' = 0 \Rightarrow \frac{df'}{d\eta} = -\eta f' \quad \text{or} \quad \frac{df'}{f'} = -\eta\, d\eta$$

$$\ln f' = -\eta^2/2 + \ln C_1$$

$$f' = C_1 e^{-\eta^2/2}$$

$$df = C_1 e^{-\eta^2/2} d\eta \quad \text{or} \quad f = C_1 \int_0^{\eta} e^{-\sigma^2/2} d\sigma + C_2$$

when $\eta = 0 \quad f(\eta = 0) = 1 = C_2$

$f \to 0$ as $\eta \to \infty$ $\qquad$ NOW $\int_0^\infty e^{-\sigma^2/2} d\sigma = \sqrt{2}\int_0^\infty e^{-z^2} dz = \sqrt{\frac{\pi}{2}}$

$$0 = C_1 \cdot \sqrt{\frac{\pi}{2}} + 1 \qquad C_1 = -\sqrt{\frac{2}{\pi}} \qquad \therefore f = 1 - \sqrt{\frac{2}{\pi}}\int_0^\eta e^{-\sigma^2/2} d\sigma$$

$$= 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta/\sqrt{2}} e^{-z^2} dz$$

$$\frac{T - T_i}{T_s - T_i} = f(\eta) = \operatorname{erfc}(\eta/\sqrt{2})$$

$$\frac{T-T_i}{T_s-T_i} = \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

1. ALWAYS REMEMBER THAT IF THE PROBLEM IS INDEPENDENT OF LENGTH OR TIME SCALES — WE HAVE SELF SIMILAR SOLUTION

2. SOLUTION WILL REDUCE # OF INDEPENDENT VARIABLES BY 1 & $\underbrace{\text{PDE}}_{\text{OF 2 INDEP. VARIABLES}}$ BECOMES ODE

3. ASSUME GENERAL FORM OF TRANSFORMATION BASED ON B.C.'s & I.C.

4. IN SIMILARITY PARAMETER η, MOST DIFFERENTIATED VARIABLE SHOULD APPEAR IN NUMERATOR ie $A\frac{x}{t^n}$

5. TRANSFORM BC & IC USING SIMILARITY PARAMETER AND INSURE THEY ARE SATISFIED

   IF NOT ADD ADDITIONAL DEGREES OF FREEDOM

6. CONVERT DE INTO ONE THAT CONTAINS f & ITS DERIV, η and only one of the independent variables

   Determine parameters of η to reduce PDE order by one

7. Express BC & IC for reduced problem & solve.

let $\eta = \frac{Ax}{t^n}$ $\quad \frac{T}{T_i} = f(\eta)$ $\quad \frac{\partial T}{\partial x} = T_i f'(\eta) \frac{A}{t^n} = -q/k$.

$\frac{T}{T_i} = B t^m f(\eta) \implies m = n$.

$= B t^n f(\eta)$ $\quad \frac{\partial T}{\partial x} = T_i f'(0) A$ $\quad$ let $A = -\frac{q}{kT_i}$

$f'(0) = 1$

$$\frac{\partial T}{\partial t} = T_i t^{n-1} n f(\eta) + T_i t^n f' \cdot \frac{Ax(-n)}{t^{n+1}}$$

$$= T_i t^{n-1}(nf - \eta n f')$$

$$\frac{\partial T}{\partial x} = B T_i t^n f' \cdot \frac{A}{t^n}$$

$$\frac{\partial^2 T}{\partial x^2} = B \cancel{T_i} t^n f'' \frac{A^2}{t^{2n}} = \alpha \cancel{T_i} t^{n-1}(nf - \eta n f')$$

$$t^{-n} f'' A^2 - \alpha t^{n-1}(nf - \eta n f') = 0$$

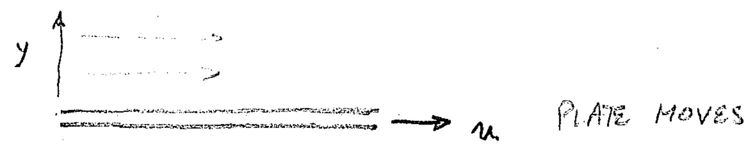
$-n = n-1 \quad n = \frac{1}{2}$. $\quad f'' - \frac{\alpha}{2A^2}(f - \eta f') = 0$.

$$f'' + (\eta f' - f)\frac{\alpha}{2A^2}$$

LESSON # 15

MOTION OF A VISCOUS FLUID OVER AN ∞ PLATE

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$



ν kinematic viscosity

let $u(y,t)$ $\qquad u(y=0,t) = at^b \qquad a, b$ fixed (1)

BC $\quad u(y,t) \to 0$ as $y \to \infty$ (2)

IC $\quad u(y,0) = 0$ (3)

let $\quad \eta = B\frac{y}{t^n} \qquad u = A\, f(\eta) \qquad A, B, n$ constant

- for behavior of $u \to 0$ for y large & for t small $\Rightarrow \eta \to \infty$ for y large, t small
  note that (2) & (3) are collapsed into 1 condition
- Check $u(y=0,t) = A\, f(0) = at^b \qquad y=0 \Rightarrow \eta = 0$
  impossible

Must add additional degree of freedom : let $u = At^m f(\eta)$
pick t^m since $u(y=0,t)$ involves t^b

$\therefore \quad u(y=0,t) = At^m f(0) = at^b$ must pick $m = b$
$A f(0) = a$ may pick $A = a \Rightarrow f(0) = 1$

Guidance try to get fns $f(0)$, $f(\infty)$ etc to be either $1, 0, \infty$ but not 2.735, 15.2 etc.

$\therefore \quad u = at^b f(\eta) \qquad \eta = By/t^n$

$\frac{\partial u}{\partial y} = at^b f'(\eta)\frac{d\eta}{dy} = at^b f' \cdot \frac{B}{t^n}$ $\qquad \Big\| \qquad \frac{\partial u}{\partial t} = abt^{b-1} f + at^b f' \cdot \frac{d\eta}{dt}$

$\partial^2 u/\partial y^2 = at^b f'' \, B^2/t^{2n}$ $\qquad \Big\| \qquad = abt^{b-1} f + at^b f'(-nBy/t^{n+1})$

Given $\quad y'' + p(x)y' + q(x)y = 0 \quad$ if y_1 is known & $y_2 = v y_1$

$$y_2 = y_1 \int^x \frac{1}{y_1^2(s)} e^{-\int^s p(t)dt} ds \qquad p(x) = x \qquad y_1(x) = x$$

$$= x \int^x \frac{1}{s^2} \underbrace{e^{-\int^s t dt}}_{e^{-s^2/2}} ds$$

$$= x \int^x \frac{1}{s^2} e^{-s^2/2} ds$$

$$\frac{\partial u}{\partial t} = abt^{b-1} f + at^{b-1} \cdot -n\eta f'$$

$$\therefore B^2 \nu a t^{b-2n} f'' = at^{b-1}[bf + f'(-n\eta)]$$

let $2n = 1 \quad \therefore n = 1/2 \qquad at^{b-1}$ cancel

$$B^2\nu f'' + \tfrac{1}{2}\eta f' - bf = 0$$

$$2B^2\nu f'' + \eta f' - 2bf = 0 \qquad \text{let } 2B^2\nu = 1 \Rightarrow B = \frac{1}{\sqrt{2\nu}}$$

$$\therefore \quad \eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f'' + \eta f' - 2bf = 0$$

from $u(y=0,t) = at^b = at^b f(0) \Rightarrow f(0) = 1$

irrespective of t: $u(y,t) \to 0$ as $y \to \infty \Rightarrow f(\eta \to \infty) \to 0$

in order to solve $f'' + \eta f' - 2bf = 0$ we need to know b

Suppose $b = 1/2 \qquad f'' + \eta f' - f = 0 \Rightarrow$ 2nd order ODE $\quad f = c_1 f_1 + c_2 f_2$

Use method of reduction in order - if you know a solution f_1 then $f_2 = f_1(\eta)\, g(\eta) \qquad f_2' = f_1' g + f_1 g' \quad f_2'' = f_1'' g + 2f_1' g' + f_1 g''$

PUT INTO ODE

$$f_1'' g + 2f_1' g_1' + f_1 g'' + \eta[f_1' g + f_1 g'] - f_1 g = 0$$

$$(f_1'' + \eta f_1' - f_1)\overset{\to 0}{g} + f_1 g'' + (\eta f_1 + 2f_1')g' = 0$$

$$\therefore \quad \underbrace{g''/g'}_{\frac{d}{d\eta}(\ln g')} = -\left(\frac{\eta f_1 + 2f_1'}{f_1}\right) = -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\frac{d}{d\eta}\left(\eta^2/2 + 2\ln f_1\right.$$

FIRST ORDER EQN IN g'

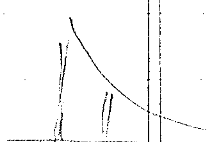
NOTICE $f_1 = \eta$ solves $\quad f_1' = 1 \quad f_1'' = 0$

$$\frac{dg'}{g'} = -\left(\eta + \frac{2}{\eta}\right)d\eta \Rightarrow \ln g' = -\eta^2/2 - 2\ln\eta$$

$$g' = \frac{1}{\eta^2}e^{-\eta^2/2} \qquad g(\eta) = \int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$

PICK ∞ as the lower limit since g' is bounded

$$f = C_1\eta + C_2\eta\int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$



: $f(\eta) \to 0$ as $\eta \to \infty$:

$$\infty > \sigma > \eta \qquad \frac{1}{\sigma^2} < \frac{1}{\eta^2} < \frac{1}{\eta}$$

$$\infty > \sigma^2 > \eta^2 > \eta$$

$$0 < \int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma < \int_\infty^\eta \frac{1}{\eta}e^{-\sigma^2/2}d\sigma \quad \text{and} \quad \frac{1}{\eta}\int_\infty^\eta e^{-\sigma^2/2}d\sigma \text{ is bounded}$$

$$f_2 = \eta\, g(\eta) < \int_\infty^\eta e^{-\sigma^2/2}d\sigma \qquad \text{as } \eta \to \infty \quad \int \to 0$$

$\therefore \quad f_2 \to 0$ as $\eta \to \infty$ but $f_1 = \eta \to \infty$ as $\eta \to \infty$ $\quad \therefore C_1 = 0$

and $$f = C_2\eta\int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$

→ INTEGRATE BY PARTS

let $\frac{1}{\sigma^2}d\sigma = dv \quad \omega = e^{-\sigma^2/2} \quad v = -\frac{1}{\sigma} \quad d\omega = e^{-\sigma^2/2}\cdot(-\sigma\, d\sigma)$

: $f(\eta = 0) = 1$

$$C_2\eta\left[-\frac{1}{\sigma}e^{-\sigma^2/2}\Big|_\infty^\eta - \int_\infty^\eta\left(-\frac{1}{\sigma}\right)(-\sigma)e^{-\sigma^2/2}d\sigma\right]$$

$$\left[-\frac{1}{\eta}e^{-\eta^2/2} - \int_\infty^\eta e^{-\sigma^2/2}d\sigma\right]$$

$$f = -C_2e^{-\eta^2/2} - C_2\eta\int_\infty^\eta e^{-\sigma^2/2}d\sigma$$

at $\eta = 0$ $$f = -C_2e^0 - C_2\cdot 0\cdot\int_\infty^0 e^{-\sigma^2/2}d\sigma = 1 \qquad C_2 = -1$$

$$\therefore \quad f(\eta) = e^{-\eta^2/2} + \eta\int_\infty^\eta e^{-\sigma^2/2}d\sigma$$

let $z = \sigma/\sqrt{2}$ $\qquad z^2 = \sigma^2/2 \qquad dz = \frac{d\sigma}{\sqrt{2}}$

$$\eta \cdot \sqrt{2} \int_{\infty}^{\eta/\sqrt{2}} e^{-z^2} dz$$

$$- \frac{\sqrt{\pi}}{2} \operatorname{erfc} \frac{\eta}{\sqrt{2}}$$

$$f(\eta) = e^{-\eta^2/2} - \eta \sqrt{\pi/2}\, \operatorname{erfc}\left(\eta/\sqrt{2}\right)$$

LESSON # [redacted]

LAPLACE TRANSFORMS – HANDOUT TABLES

CAN BE USED TO FIND SOLUTIONS OF PDE'S WHEN ONE OR MORE INDEPENDENT VARIABLES CAN RANGE FROM 0 TO ∞

ie. TIME $\quad t \geq 0$

$x \quad\quad x \geq 0$

used to solve: $\quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha} \frac{\partial T}{\partial t}$

we can define $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt \qquad s = \text{const}.$

note that $F(s) = \mathcal{L}(f(t))$

for every $f(t) \Longleftrightarrow F(s)$,

does $F(s)$ exist for any $f(t)$? NO.

$F(s)$ exists if • $f(t)$ is continuous or piecewise continuous in every finite interval $t_1 \leq t \leq T$ where $t_1 > 0$

• $t^n |f(t)|$ is bounded near $t = 0$ for some n $\quad n < 1$

• $e^{-s_0 t} |f(t)|$ is bounded for large t, for some value s_0

$|f| \leq \frac{M}{t^n} \Leftarrow$

$\left|\int f e^{-st} dt\right| < \int |f| e^{-st} dt$

$< \int \frac{M}{t^n} e^{-st} dt = M\left[\frac{t^{1-n} e^{-st}}{1-n} - \right.$

LAPLACE TRANSFORMS HAVE THE FOLLOWING PROPERTIES § 2.3.

$$\mathcal{L}\{a f_1(t) + b f_2(t)\} = a\mathcal{L}\{f_1(t)\} + b\mathcal{L}\{f_2(t)\} = a\cdot F_1(s) + b\cdot F_2(s)$$

$$\mathcal{L}\{\tfrac{d}{dt} f(t)\} = s\cdot F(s) - f(t=0+)$$

$$\mathcal{L}\{\tfrac{d^2}{dt^2} f(t)\} = s^2F(s) - s f(0+) - \tfrac{d}{dt}f(0+)$$

Convolution $\mathcal{L}\left\{\int_0^t f(t-a)\, g(a)\, da\right\} = F(s)G(s)$ where $G(s) = \int_0^\infty g(t)e^{-st}dt$

unit step fn. Heaviside step fn. $H(t) = \begin{cases} 1 & t>0 \\ 0 & t=0 \end{cases}$ $\mathcal{L}\{H(t)\} = \int_0^\infty H(t)e^{-st}dt = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = -\frac{1}{s}(-e^0) = \frac{1}{s}$

Suppose $\frac{d^2y}{dt^2} = k^2y$ where $y(0) = 0$ $\frac{dy}{dt} = 1$ at $t = 0$

$$\int \frac{d^2y}{dt^2} e^{-st}dt = \int k^2 y e^{-st}dt \Rightarrow s^2F(s) - s\underbrace{y(0+)}_{0} - \underbrace{y'(0+)}_{1} = k^2 F(s).$$

$$(s^2-k^2)F(s) = 1 \quad \therefore \quad F(s) = \frac{1}{s^2-k^2} = \frac{A}{(s-k)} + \frac{B}{(s+k)} \qquad A = \frac{1}{2k} \quad B = -\frac{1}{2k}$$

$$\therefore \quad F(s) = \frac{1}{2k}\left[\frac{1}{s-k} - \frac{1}{s+k}\right]$$

look at 29.3.8

$$y(t) = \frac{1}{2k}\left[e^{+kt} - e^{-kt}\right] = \frac{1}{k}\sinh kt \quad y(0) = \frac{1}{2k}[0]$$

$$y'(t) = +\frac{k}{2k}\left[e^{-kt} + e^{kt}\right] = \cosh kt \quad y'(0) = \frac{1}{2}[2] = 1.$$

Same result $\left\{ \begin{array}{l} y'' - k^2y = 0 \quad y = C_1e^{kt} + C_2e^{-kt} \quad @\ t=0 \quad 0 = C_1 + C_2 \\ y'(0) \quad 1 = k[C_1 - C_2] \Rightarrow C_2 = -\frac{1}{2k} \quad C_1 = \frac{1}{2k}. \end{array}\right.$

NOTE WE INCORPORATE : THE B.C. IN THE SOLUTION

JUST AS WE DID FOR O.D.E WE CAN DO FOR PDE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,t).$$

let $\mathcal{J}(x;s) = \int_0^\infty T(x,t)e^{-st}dt$ or $\mathcal{J}(x;s) = \mathcal{L}\{T(x,t)\}$.

$$\mathcal{L}\left\{\frac{\partial}{\partial x}T\right\} = \frac{\partial}{\partial x}\int_0^\infty T(x,t)e^{-st}dt = \frac{d}{dx}\mathcal{J}(x;s)$$

but $\mathcal{L}\left\{\frac{\partial}{\partial t}T\right\} = s\,\mathcal{J}(x;s) - T(x,t=0)$

$$\mathcal{L}\left\{\frac{\partial^2}{\partial x^2}T\right\} = \frac{d^2}{dx^2}\mathcal{T}(x;s)$$

$$\therefore \quad \mathcal{L}\left\{\frac{\partial^2}{\partial x^2}T = \frac{1}{\alpha}\frac{\partial T}{\partial t}\right\} \Rightarrow \frac{d^2}{dx^2}\mathcal{T}(x;s) = \frac{1}{\alpha}\left[s\mathcal{T}(x;s) - T(x,t=0)\right]$$

$$\frac{d^2}{dx^2}\mathcal{T} - \frac{s}{\alpha}\mathcal{T} = -\frac{1}{\alpha}T(x,t=0)$$

Gain by making PDE into ODE
Lose by making it into an inhomog ODE

look at the problem when @ $T(x,t=0) = T_i$ ①

$T(0,t) = T_s$ ②

$T(x,t) \to T_i$ as $x \to \infty$ ③

$\mathcal{T}'' - \frac{s}{\alpha}\mathcal{T} = -\frac{T_i}{\alpha}$ (①) let $\mathcal{T} = \mathcal{T}_H + \mathcal{T}_p$ let $\mathcal{T}_p = C$

$-\frac{s}{\alpha}C = -\frac{T_i}{\alpha} \Rightarrow C = \frac{T_i}{s} = \mathcal{T}_p$

$\mathcal{T}_H'' - \frac{s}{\alpha}\mathcal{T}_H = 0 \qquad \mathcal{T}_H = C_1 e^{-\sqrt{\frac{s}{\alpha}}x} + C_2 e^{\sqrt{\frac{s}{\alpha}}x}$

② $\int_0^\infty T(x=0,t)e^{-st}dt = \mathcal{T}(0;s) = \int_0^\infty T_s e^{-st}dt = T_s/s \Rightarrow \mathcal{T}(0;s) = T_s/s$

③ $\int_0^\infty T(x,t)e^{-st}dt = \mathcal{T}(x;s) = \int_0^\infty T_i e^{-st}dt = T_i/s \Rightarrow \mathcal{T}(x;s) \to \frac{T_i}{s}$ as $x \to \infty$

note $\mathcal{T} = \mathcal{T}_p + \mathcal{T}_H = \frac{T_i}{s} + C_1 e^{-\sqrt{\frac{s}{\alpha}}x} + C_2 e^{+\sqrt{s/\alpha}\,x}$ from ③ $\Rightarrow C_2 = 0$

from ② $\frac{T_s}{s} = \frac{T_i}{s} + C_1 e^0 \Rightarrow C_1 = \frac{T_s - T_i}{s}$

$\therefore \mathcal{T} = \frac{T_i}{s} + \frac{T_s - T_i}{s}e^{-\sqrt{\frac{s}{\alpha}}x}$ from 29.3.83 $k = \frac{x}{\sqrt{\alpha}}$

we want $T(x,t) = \mathcal{L}^{-1}\{\mathcal{T}(x;s)\} = T_i H(t) + T_s - T_i \operatorname{erfc}\left(\frac{x}{\sqrt{\alpha}}\cdot\frac{1}{2\sqrt{t}}\right)$

$\left(\frac{x}{\sqrt{2}\sqrt{2\alpha t}}\right)$

remember $\eta = \frac{x}{\sqrt{2\alpha t}} \quad \therefore \quad T(x,t) = T_i + T_s - T_i \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$

$$\frac{T - T_i}{T_s - T_i} = \operatorname{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$$

$$-T\sin\theta + (T+dT)\sin(\theta+d\theta) - \rho ds\, g = \rho ds \frac{\partial^2 w}{\partial t^2}$$

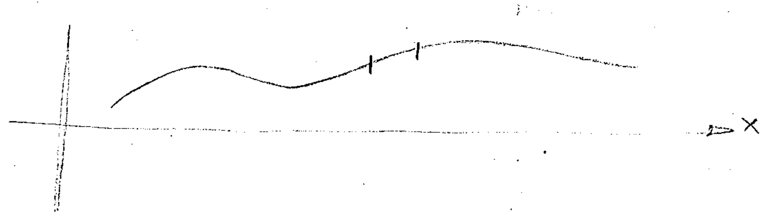
$$-T\sin\theta + (T+dT)(\sin\theta\cos d\theta + \cos\theta \sin d\theta) - \rho ds\, g = \rho ds \frac{\partial^2 w}{\partial t^2}$$ $\cos d\theta = 1 \quad \sin d\theta \cong d\theta$

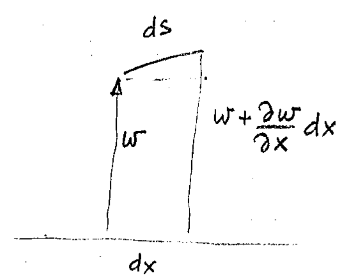
$$-T\sin\theta + [T\sin\theta + T\cos\theta\, d\theta + dT \sin\theta\cdot 1] - \rho ds g = \rho ds \frac{\partial^2 w}{\partial t^2}$$

$$\partial(T\sin\theta)$$

$$\frac{\partial}{\partial x}(\quad)dx - \rho dx\cdot g = \rho dx \frac{\partial^2 w}{\partial t^2}$$ $\sin\theta = \frac{\partial w}{\partial s} \cong \frac{\partial w}{\partial x}$

Derive wave eqn for a string - general





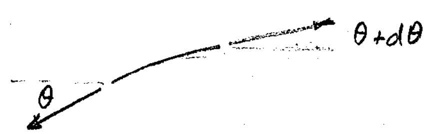
$dw^2 + dx^2 = ds^2$

$dx\sqrt{1+w'^2} = ds$

if $w' << 1$ $ds \approx dx$

let $T\cos\theta = F_x$

$F_x + \frac{\partial F_x}{\partial x} dx$



$\frac{\partial}{\partial x}[T\cos\theta]\, dx = 0.$

$T\cos\theta = \text{const}$ $\frac{\partial x}{\partial s} \cong 1$ if w' is small.

$\Sigma F_x = (T+dT)\cos(\theta + d\theta) - T\cos\theta = 0$

if $T\cos\theta\cos d\theta - T\sin\theta\sin d\theta$

$+ dT[\cos\theta\cos d\theta - \sin\theta\sin d\theta]$

$- T\cos\theta = 0$

if $d\theta$ is small $\cos d\theta \cong 1$ $\sin d\theta \cong d\theta$

$T\cos\theta - Td\theta\sin\theta + dT\cos\theta - T\cos\theta = 0$

$dT = Td\theta \cdot \tan\theta.$

if $\frac{\partial w}{\partial s} = \sin\theta = \frac{\partial w}{\partial x}$

$T\sin\theta = F_y$

$F_y + \frac{\partial F_y}{\partial \cancel{y}} d\cancel{x}$

$\Sigma F_y = -T\sin\theta + (T+dT)\sin(\theta + d\theta) - \rho d\cancel{x} g = \rho d\cancel{s}\, \ddot{w}$

$-T\sin\theta + T(\sin\theta \cdot 1 + d\theta\cos\theta) + dT(\sin\theta \cdot 1 + \cos\theta \cdot \cancel{d\theta})$

$T\cos\theta\, d\theta + \sin\theta \cdot dT$

$\frac{1}{\cos\theta}$ $[T\cos^2\theta\, d\theta + T\sin^2\theta\, d\theta] - \rho\frac{dx}{\cos\theta}g = \rho\frac{dx}{\cos\theta}\ddot{w}$

$[T\, d\theta] - \rho\, dx g = \rho dx\, \ddot{w}$

$T\frac{\partial\theta}{\partial x}dx$

$\frac{\partial}{\partial x}(T\sin\theta)\, dx$

$\frac{\partial}{\partial x}\left(T\frac{\partial w}{\partial x}\right) - \rho g = \rho\ddot{w}$

if $\theta = \frac{\partial w}{\partial x}$

$T\frac{\partial^2 w}{\partial x^2} - \rho g = \rho\frac{\partial^2 w}{\partial t^2}$

if $T = \text{const}$ $T\frac{\partial^2 w}{\partial x^2} - \rho g = \rho\ddot{w}$

$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \frac{\rho g}{T}$

$c = \sqrt{T/\rho}$

if $g << \frac{\partial^2 w}{\partial t^2}$

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad A = 1 \quad B = 0 \quad C = -\frac{1}{c^2}$$

$$\frac{dt}{dx} = \pm\frac{1}{c} \qquad \text{or} \qquad x + ct = \varphi_1 \qquad x - ct = \varphi_2$$

As it turns out w

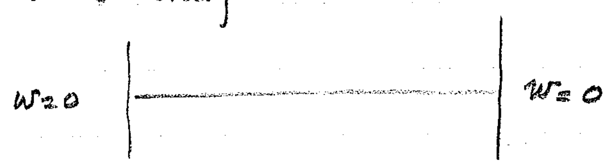
$$w = f(x+ct) + g(x-ct)$$

$$\frac{\partial w}{\partial x} = f'(\ )\cdot 1 + g'(\ )\cdot 1 \qquad \frac{\partial w}{\partial t} = [f'(\ ) - g'(\ )]c$$

$$\frac{\partial^2 w}{\partial x^2} = f''(\ )\cdot 1 + g''(\ )\cdot 1 \qquad \frac{\partial^2 w}{\partial t^2} = [f''(\ ) + g''(\ )]c^2$$

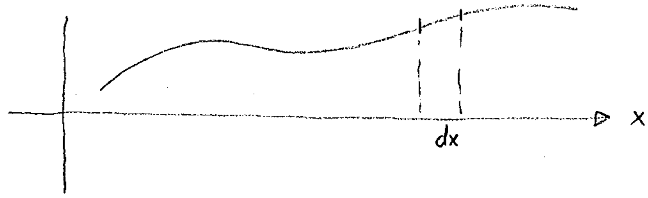
$\therefore$ we need two BC & 2 initial conditions

Lets look at string

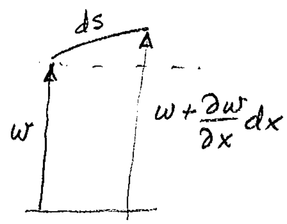


$$W'' - \frac{s^2}{c^2} W = -s w_0 - s^2 \dot{w}_0$$

LESSON # 16



KINEMATICS :



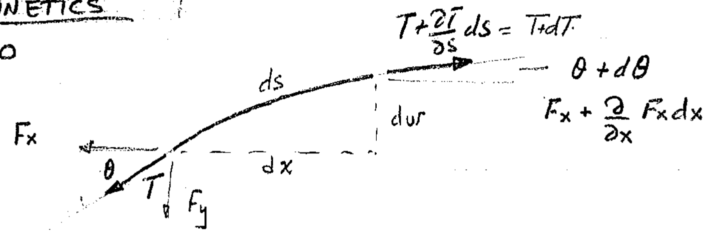
$$dw^2 + dx^2 = ds^2$$

$$dx\sqrt{1+w'^2} = ds$$

if $w' << 1$ $\quad ds \cong dx$

KINETICS

$\Sigma F_x = 0$



$$-F_x + \left(F_x + \frac{\partial}{\partial x} F_x dx\right) = 0 \text{ or}$$

$$dF_x = \frac{\partial}{\partial x}(F_x) dx = 0$$

$F_x = T\cos\theta \implies F_x = \text{constant}$

$$(T+dT)\cos(\theta+d\theta) - T\cos\theta = 0$$

$$(T+dT)(\cos\theta\cos d\theta - \sin\theta\sin d\theta) - T\cos\theta = 0$$

$$(T+dT)(\cos\theta - d\theta\sin\theta) - T\cos\theta = 0$$

$$-T d\theta\sin\theta + dT\cos\theta = 0 \text{ or } d(T\cos\theta) = 0$$

$$T\cos\theta = \text{const.}$$

if $W' << 1 \implies \frac{dx}{ds} \cong 1 = \cos\theta$

$$\frac{\partial w}{\partial s} \cong \frac{\partial w}{\partial x} = \sin\theta$$

$\Sigma F_y = m a_y$ $\qquad \rho\, ds = dm$

$$-F_y + F_y + \frac{\partial}{\partial x} F_y\, dx - \rho\, ds\, g = \rho\, ds \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial}{\partial x} F_y\, dx - \rho\, ds \cdot g = \rho\, ds \frac{\partial^2 w}{\partial t^2} \qquad F_y = T\sin\theta$$

$$\frac{\partial}{\partial x}\left(T \frac{\partial w}{\partial x}\right) dx - \rho\, dx \cdot g = \rho\, dx \frac{\partial^2 w}{\partial t^2} \qquad \text{or}$$

$$\frac{\partial}{\partial x}\left(T \frac{\partial w}{\partial x}\right) - \rho g = \rho \frac{\partial^2 w}{\partial t^2} \qquad \text{if } T = \text{const}$$

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \frac{g}{c^2} = \text{const.}$$

if $g/c^2 \sim 0$

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0 \qquad c = \sqrt{T/\rho}$$

BC. $W=0$ end is fixed

$\frac{\partial w}{\partial x}=0$ no force on the end.

IC. $w(x,t=0) = w_0(x)$

$\frac{\partial w}{\partial t}(x,t=0) = w_1(x)$

$T\frac{\partial w}{\partial x} = \sum$ forces in y direction

LOOK AT FREE VIBS OF A STRING

$\boxed{\text{Free Vibration of a string - Separation of variables}}$

$$w(x,t) = F(x)G(t)$$

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = F''G - \frac{1}{c^2}FG'' = 0 \implies c^2\frac{F''}{F} = +\frac{G''}{G} = -k^2$$

$$\left.\begin{aligned} &G''+k^2G=0 && G = A\cos kt + B\sin kt \\ &F''+\left(\frac{k}{c}\right)^2F=0 && F = C\cos\left(\frac{k}{c}x\right) + D\sin\left(\frac{k}{c}x\right) \end{aligned}\right\} k\neq 0$$

if $k=0$ $F''=0$ and $F(x)=\bar{C}x+\bar{D}$

$G''=0$ and $G(t)=\bar{A}t+\bar{B}$

if both ends are fixed for all time: $W(x=0,t) = F(0)G(t) = 0 \implies F(0)=0$

$W(x=\ell,t) = F(\ell)G(t) = 0 \implies F(\ell)=0$

look at $k=0$ solution $\left.\begin{aligned} &F(x=0) \implies \bar{D}=0 \\ &F(x=\ell) \implies \bar{C}=0 \end{aligned}\right\} F(x)=0$

$k\neq 0$ $F(x=0) \implies C=0$ and $F(x=\ell) = D\sin\left(\frac{k}{c}\ell\right)=0$ or $\frac{k\ell}{c}=n\pi$

$\therefore$ $k = \frac{n\pi c}{\ell}$ $\boxed{\text{eigenvalue}}$

$\therefore$ $F(x) = B\sin\frac{kx}{c} = B\sin\frac{n\pi x}{\ell}$ $\boxed{\text{eigenfunction}}$

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}\left[C_n\cos\frac{n\pi c}{\ell}t + D_n\sin\frac{n\pi c}{\ell}t\right] = \sum w_n(x,t)$$

$\boxed{\text{Must apply IC. on this fn to get } w(x,t)}$

$w_n(x,t) = \sin\frac{n\pi x}{\ell}\left[C_n\cos\frac{n\pi c}{\ell}t + D_n\sin\frac{n\pi c}{\ell}t\right]$ represents the n^{th} mode of vibration or n^{th} harmonic

$$\begin{cases} w(x,t=0) = w_0(x) = \sum C_n\sin\frac{n\pi x}{\ell} \\ \frac{\partial w}{\partial t}(x,t=0) = W_1(x) = \sum \frac{n\pi c}{\ell}D_n\sin n\pi x/\ell \end{cases}$$

$C_n = \frac{2}{\ell}\int_0^\ell w_0(x)\sin\frac{n\pi x}{\ell}dx$

$D_n = \frac{2}{\ell}\int_0^\ell \frac{\ell}{n\pi c}\left(W_1(x)\right)\sin\frac{n\pi x}{\ell}dx$

LESSON # 16 - EXAM

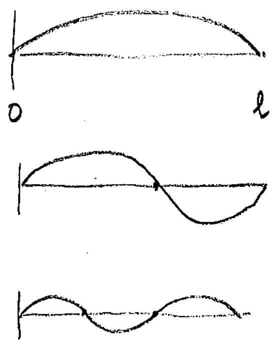
Given $w(x,t) = f(x+ct) + g(x-ct) = f(\eta) + g(\xi)$

$$\frac{\partial w}{\partial x} = f'(\eta)\frac{\partial \eta}{\partial x} + g'(\xi)\frac{\partial \xi}{\partial x} = f'(\eta) + g'(\xi) \qquad \frac{\partial w}{\partial t} = f'(\eta)\frac{\partial \eta}{\partial t} + g'(\xi)\frac{\partial \xi}{\partial t} = c\left[f'(\eta) - g'(\xi)\right]$$

$$\frac{\partial^2 w}{\partial x^2} = f''(\eta)\frac{\partial \eta}{\partial x} + g''(\xi)\frac{\partial \xi}{\partial x} = f''(\eta) + g''(\xi) \qquad \frac{\partial^2 w}{\partial t^2} = f''(\eta)\frac{\partial \eta}{\partial t} + g''(\xi)\frac{\partial \xi}{\partial t} = c^2\left[f''(\eta) + g''(\xi)\right]$$

$$w_{tt} = c^2 w_{xx}$$

$\sin \frac{n\pi x}{l}$



$n=1$

$\underline{k = \frac{\pi c}{l}}$ Fundamental frequency

NEED TO DO

node at $l/2$ $\qquad n=2$

nodes at $l/3,\ 2l/3$ $\qquad n=3$

To show general solution $\quad w_n(x,t) = C_n\left[\sin\frac{n\pi x}{l}\cos\left(\frac{n\pi c t}{l}\right)\right] + D_n\left[\sin\frac{n\pi x}{l}\sin\left(\frac{n\pi c t}{l}\right)\right]$

let $A = \frac{n\pi x}{l}$ $\quad B = \frac{n\pi c t}{l}$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$ $\qquad \cos(A+B) = \cos A\cos B - \sin A\sin B$

$\sin(A-B) = \sin A\cos B - \cos A\sin B$ $\qquad \cos(A-B) = \cos A\cos B + \sin A\sin B$

$\frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right] = \sin A\cos B$ $\qquad \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right] = \sin A\sin B$

$$w_n = \left\{\frac{C_n}{2}\sin(A+B) - \frac{D_n}{2}\cos(A+B)\right\} + \left\{\frac{C_n}{2}\sin(A-B) + \frac{D_n}{2}\cos(A-B)\right\}$$

$$= f\left(\frac{n\pi}{l}[x+ct]\right) + g\left(\frac{n\pi}{l}[x-ct]\right) = \boxed{\mathcal{F}(x+ct) + \mathcal{G}(x-ct)}$$

go back to $\quad W_{xx} - \frac{1}{c^2}W_{tt} = 0$ $\qquad a=1 \quad b=0 \quad c=-\frac{1}{c^2}$

$$\frac{dt}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{0 \pm \sqrt{0+4\frac{1}{c^2}}}{2} = \pm\frac{1}{c}$$

or $\quad dt \pm \frac{dx}{c} = 0 \quad$ or $\quad t \pm \frac{x}{c} = \text{const.}$ $\qquad ct + x = \text{constant}_1 = \varphi_1$
$\qquad ct - x = \text{constant}_2 = -\varphi_2$

characteristics

in general $\quad w(x,t) = f(\varphi_1) + g(\varphi_2)$

LESSON #18

NEED TO DO $\quad f(\varphi_1)$ represents a wave ~~a wave~~ moving in $-x$ dir, $g(\varphi_2)$ represents movement in $+x$ dir
with velocity c

WE CAN USE THIS RESULT TO FIND THE DISPLACEMENT OF AN ∞ LONG STRING

Using I.C. $\quad W(x,t=0) = W_0(x) \quad -\infty \le x \le \infty \qquad \frac{\partial W}{\partial t}(x,t=0) = W_1(x)$

$W(x,t=0) = f(x) + g(x) = W_0(x)$ $\qquad x$ is a dummy variable

$\frac{\partial W}{\partial t}(x,t=0) = c f'(x) - c g'(x) = W_1(x) \quad \Rightarrow \quad f'(x) - g'(x) = \frac{1}{c}W_1(x)$

$$f(x) - g(x) = \frac{1}{c}\int^x W_1(\xi)\,d\xi$$

$$f + g = W_0(x)$$
$$f - g = \frac{1}{c}\int_{x_0}^{x} W_1(\tilde{x})d\tilde{x}$$

$$f(x) = \frac{1}{2}\left[W_0(x) + \frac{1}{c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma\right]$$
$$g(x) = \frac{1}{2}\left[W_0(x) - \frac{1}{c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma\right]$$

but $$f(x+ct) = \frac{1}{2}\left[W_0(x+ct) + \frac{1}{c}\int_{x_0}^{x+ct} W_1(\sigma)\,d\sigma\right]$$

$$g(x-ct) = \frac{1}{2}\left[W_0(x-ct) - \frac{1}{c}\int_{x_0}^{x-ct} W_1(\sigma)\,d\sigma\right]$$

$$w(x,t) = f + g = \underbrace{\frac{1}{2}\left[W_0(x+ct) + W_0(x-ct)\right]}_{\text{displacement dependent term with } W_1 = 0} + \underbrace{\frac{1}{2c}\int_{x-ct}^{x+ct} W_1(\sigma)\,d\sigma}_{\text{velocity dependent term. with } W_0 = 0}$$

• Note that no boundary conditions are needed

Application: let $W_0(x) = \sin \pi x/L \quad 0 \le x \le L$ & $W_1(x) = 0$

$$w(x,t) = \frac{1}{2}\left[\sin\frac{\pi}{L}(x+ct) + \sin\frac{\pi}{L}(x-ct)\right]$$ — 2 wave forms. having ½ height of original, but same form.

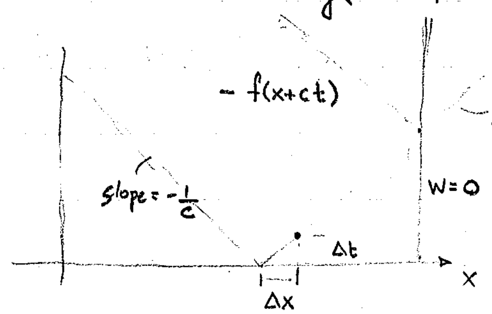
Can we use this result for a finite length string? yes, by reflections

TALK ABOUT
LEFT GOING WAVE
RIGHT GOING WAVE

$f(ct) + g(-ct) = 0 = w$
$g(-ct) = -f(ct)$

$f'(ct) + g'(ct) = 0 = \frac{\partial w}{\partial x}$
$f'(ct) = -g'(-ct)$
$f(ct) = g(-ct)$

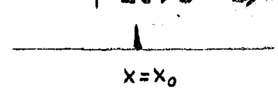
Look at a wave $g(x-ct)$



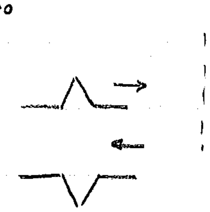
$x - ct = \text{const} \Rightarrow \Delta x - c\Delta t = 0$

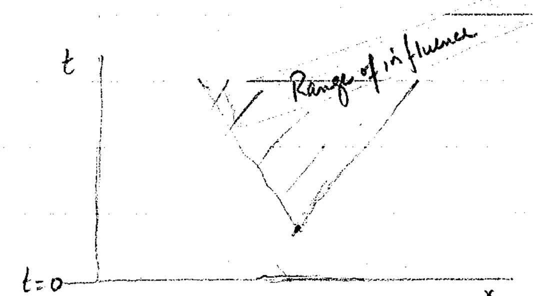
if $\Delta t > 0 \Rightarrow \Delta x > 0$ right moving

$x + ct = \text{const} \Rightarrow \Delta x + c\Delta t = 0$
if $\Delta t > 0 \Rightarrow \Delta x < 0$

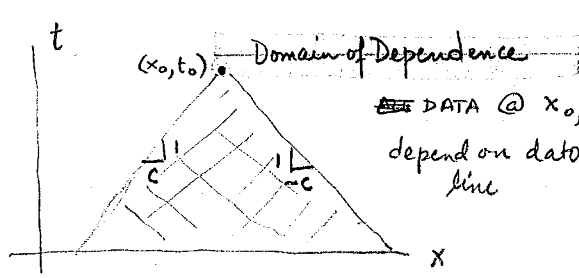


$g(x-ct)$ reflects as $-f(x+ct)$ $\Rightarrow$





DATA AT (x_0, t_0) influence.



DATA @ x_0, t_0 depend on data between line

LET $W(x,t) = f(x+ct) + g(x-ct)$ $\quad W(x,0)=F(x)$ $\quad \frac{\partial W}{\partial t}(x,0)=G(x)$ defined for $0 \le x \le l$ only!

Assume $W(0,t) = 0$ and $W(l,t)=0$; $0 \le x \le l$ $\quad t \ge 0$

$f(x) = \frac{1}{2}[w_0(x) + \frac{1}{c}\int_{x_0}^{x} w_1(\sigma)d\sigma]$

$g(x) = \frac{1}{2}[w_0(x) - \frac{1}{c}\int^{x} w_1(\sigma)d\sigma]$

since w_0 & w_1 are defined for $0 \le x \le L$
f & g are only defined $0 \le x \le L$

g

$f_{initial}$

$f_{reflect}$

g

$W(0,t)=0 = f(ct) + g(-ct)$ $\quad$ let $ct = u$.

$0 = f(u) + g(-u)$ $\quad$ f is reflected as a g wave

$g(-u) = -f(u)$ $\quad$ g is same form as u but $-$ive.

- ~~$g(u)$ is defined for $+u$.~~
- Only then ~~$+g(-u) = -f(u)$~~ is $g(-u)$ defined
- ~~$f(-u) = -g(u)$~~

REWRITE THESE TWO LESSONS

$W(l,t) = 0 = f(l+ct) + g(l-ct)$ $\quad$ let $ct-l = u$

$0 = f(u+2l) + g(-u)$ $\quad$ $ct+l = u+2l$.

$\therefore \quad f(u+2l) = -g(-u) = f(u)$

$-f(-u+2l) = g(u) = -f(-u)$

g is reflected as an f wave

reflected f wave = initial f wave but its argument is increased by $2l$

$f(u) = f(u+2l)$ is a periodicity condition

---

1) $f(u) = f(u+2l)$ & $f(u) = -g(-u)$ & $g(u) = -f(-u)$.

LESSON #19

Suppose $W(x,0) = F(x)$ $\quad \frac{\partial W}{\partial t}(x,0) = G(x)$ $\quad 0 \le x \le L$

$W(x,0) = F(x) = f(x) + g(x)$ $\qquad F(u) = f(u) + g(u)$

$\frac{\partial W}{\partial t}(x,0) = G(x) = c[f'(x) - g'(x)]$ $\qquad G(u) = c[f'(u) - g'(u)] = H'(u)$

integrate 1 time $\quad H(u) = c[f(u) - g(u)]$ $\qquad H(u) = \int_0^u G(\bar{u})d\bar{u}$

$f(u) = \frac{1}{2}[F(u) + \frac{1}{c}H(u)]$ $\qquad g(u) = \frac{1}{2}[F(u) - \frac{1}{c}H(u)]$

also $\quad -g(-u) = f(u) = -\frac{1}{2}[F(-u) - \frac{1}{c}H(-u)]$

$F(u) = -F(-u)$ or $F(u)$ is odd fn.

$H(u) = H(-u)$ or $H(u)$ is even but $\frac{dH}{du} = G(u)$ $\therefore$ G must be o.

$w_0(x) = \sin x \quad f(x) = \frac{1}{2}\sin x \quad \left.\begin{matrix} \\ \end{matrix}\right\} \; 0 \le x \le l$

$w_1(x) = 0 \quad g(x) = \frac{1}{2}\sin x$

if $w(0,t) = 0 \Rightarrow f(x) + g(-x) = 0 \Rightarrow f(x) = -g(-x)$

$\Rightarrow g(-x) = -f(x) = -\frac{1}{2}\sin x \quad \Big\} \; -l \le -x \le 0$

but $f(-x) = -g(x) = -\frac{1}{2}\sin x$

if $w(l,t) = 0 \Rightarrow f(l+ct) + g(l-ct) = 0$

let $l - ct = -u \qquad g(-u) + f(u+2l) = 0$

$\Rightarrow -f(u) + f(u+2l) = 0 \Rightarrow f(u+2l) = f(u)$

$\Rightarrow f(u+2l) = f(u) \Rightarrow \frac{1}{2}\sin x$

let $ct + l = \sigma \qquad l - ct = -\sigma + 2l$

$\therefore f(\sigma) + g(-\sigma+2l) = 0$ for $0 \le \sigma \le l$

$f(\sigma)\left[= -g(-\sigma)\right] + g(-\sigma+2l) = 0 \quad \sigma + 2l$ 0 to $2l$.

$\Rightarrow \underline{g(-\sigma) = g(-\sigma+2l)}$

$$f(x+ct) = \frac{1}{2}\left[F(x+ct) + \frac{1}{c}\int_0^{x+ct} G(\hat{u})\,d\hat{u}\right]$$

$$g(x-ct) = \frac{1}{2}\left[F(x-ct) - \frac{1}{c}\int_0^{x-ct} G(\hat{u})\,d\hat{u}\right] = \frac{1}{2}\left[F(x-ct) + \frac{1}{c}\int_{x-ct}^{0} G(\hat{u})\,d\hat{u}\right]$$

$$w(x,t) = f+g = \frac{1}{2}\left[F(x+ct) + F(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} G(\hat{u})\,d\hat{u}\right]$$

$f(u) = f(u+2l)$ $\qquad$ $g(-u) = -f(u)$ $\qquad$ B.C.

$\qquad\qquad\qquad\qquad\qquad f(-u) = -g(u)$

Use this method for small number of reflections (1 - 5)

Use SOV method for large number of reflections

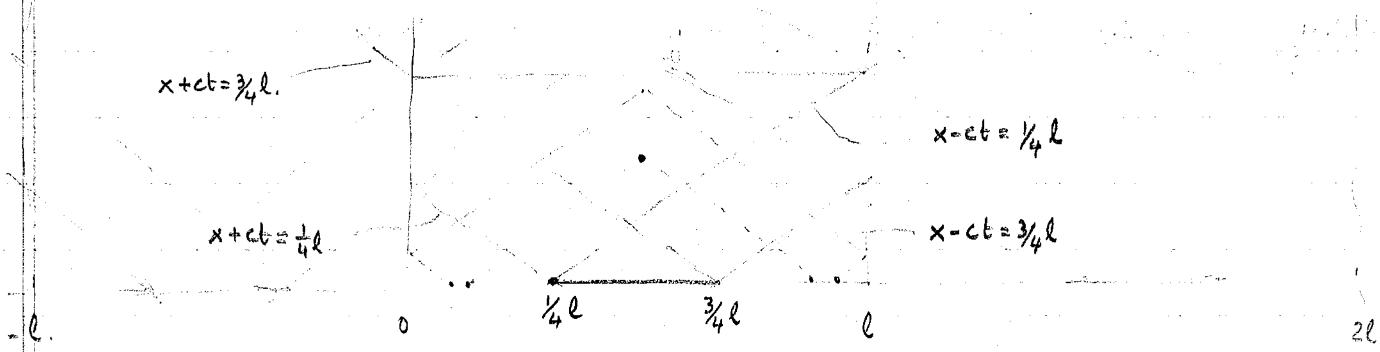
FOR EXAMPLE LET $F(x) = \sin x \quad \frac{1}{4}l \le x \le \frac{3}{4}l \qquad G(x)=0$

$f(x) = \frac{1}{2}F(x) = \frac{1}{2}\sin x \qquad f(x+ct) = \frac{1}{2}\sin(x+ct)$

$g(x) = \frac{1}{2}F(x) = \frac{1}{2}\sin x \qquad g(x-ct) = \frac{1}{2}\sin(x-ct)$

$f(-x) = -g(x) = -\frac{1}{2}\sin x \qquad f(x) = f(x+2l)$

$g(-x) = -f(x) = -\frac{1}{2}\sin x \qquad f(-x) = f(-x+2l) = -\frac{1}{2}\sin(x)$



BY LAPLACE TRANSFORMS $\qquad$ let $\mathcal{U}(x;s) = \int_0^t u(x,t)e^{-st}dt$

$u_{xx} - u_{tt} = 0 \qquad \mathcal{L}\{u_{xx}-u_{tt}\} = \mathcal{U}_{xx} - \{s^2\mathcal{U} - s\,u(x,0) - u_t(x,0)\} = 0$

$u(x,t=0) = xe^{-x} \qquad \mathcal{U}_{xx} - s^2\mathcal{U} + s\,x e^{-x} = 0$

$u_t(x,0) = 0 \qquad \mathcal{U}_{xx} - s^2\mathcal{U} = -s x e^{-x}$

$u(0,t) = 0 \qquad \mathcal{L}\{u(0,t)=0\} \Rightarrow \mathcal{U}(0;s) = 0$

Let $\mathcal{U} = \mathcal{U}_H + \mathcal{U}_P \qquad \mathcal{U}_H$ solves: $\mathcal{U}_{xx} - s^2\mathcal{U} = 0$

$\qquad \mathcal{U}_P$ solves: $\mathcal{U}_{xx} - s^2\mathcal{U} = s x e^{-x}$

$$\mathcal{U}_H = C_1 e^{-sx} + C_2 e^{+sx} \text{ and for the inhomogeneous take } \mathcal{U}_p = (Ax+B)e^{-x}$$

$$\mathcal{U}_{p_x} = A e^{-x} + (Ax+B)(-e^{-x}) = \left[A - Ax - B\right] e^{-x}$$

$$\mathcal{U}_{p_{xx}} = -Ae^{-x} + \left[A - Ax - B\right](-e^{-x}) = \left[-2A + Ax + B\right] e^{-x}$$

$$\frac{d^2\mathcal{U}_p}{dx^2} - s^2\mathcal{U}_p = \left\{\left[-2A + Ax + B\right] - s^2(Ax+B)\right\} e^{-x} = -sxe^{-x}$$

$$B - 2A - s^2B = 0 \qquad e^{-x} \qquad \Rightarrow \quad \frac{-2A}{s^2-1} = B$$

$$A - s^2A = -s \qquad xe^{-x}$$

$$A = \frac{s}{s^2-1} \qquad \nearrow \qquad \frac{-2s}{(s^2-1)^2} = B.$$

$$\therefore \quad \mathcal{U}_p = \left[\frac{sx}{s^2-1} - \frac{2s}{(s^2-1)^2}\right] e^{-x}$$

$$\therefore \quad \mathcal{U}_{TOT} = \mathcal{U}_H + \mathcal{U}_p = C_1 e^{-sx} + C_2 e^{sx} + \left[\frac{s}{s^2-1}\left(x - \frac{2}{s^2-1}\right)e^{-x}\right]$$

FOR BOUNDED SOLN $\Rightarrow C_2 = 0$.

ALSO $\quad \mathcal{U}(x=0;s) = 0 = C_1 e^0 + \left[\frac{s}{s^2-1}\left(0 - \frac{2}{s^2-1}\right)e^0\right] = C_1 - \frac{2s}{(s^2-1)^2} = 0$

$$\therefore \quad C_1 = \frac{2s}{(s^2-1)^2}$$

$$\mathcal{U}_{TOT} = \frac{2s}{(s^2-1)^2}\left[e^{-sx} - e^{-x}\right] + \frac{sx}{s^2-1}e^{-x}$$

PARTIAL FRACTION ANALYSIS

$$\frac{s}{s^2-1} = \frac{A}{s+1} + \frac{B}{s-1} \quad \Rightarrow \quad A(s-1) + B(s+1) = s \quad \text{when } s=1 \quad B(2)=1 \quad B = 1/2$$
$$s=-1 \quad A(-2) = -1 \quad A = 1/2.$$

$$\frac{s}{(s^2-1)^2} = \frac{A}{s+1} + \frac{B}{(s+1)^2} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \quad \Rightarrow \quad A(s+1)(s-1)^2 + B(s-1)^2 + C(s-1)(s+1)^2 + D(s+1)^2 =$$

when $s=1 \quad D(4) = 1 \quad D = 1/4$

$s=-1 \quad B(4) = -1 \quad B = -1/4$

also $\quad A = C = 0$

$$\therefore \quad \mathcal{U}_{TOT} = +\frac{1}{2}\left[\frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right]\left(e^{-sx} - e^{-x}\right) + \frac{1}{2}\left[\frac{1}{s+1} + \frac{1}{s-1}\right] xe^{-x}$$

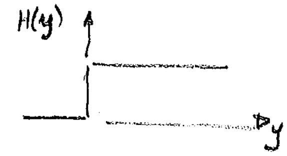
$$= -\frac{1}{2}\left[te^t - te^{-t}\right]e^{-x} + \frac{1}{2}\left[(t-x)e^{-(t-x)} + (t-x)e^{t-x}\right]H(t-x) + \frac{1}{2}\left[e^{-t} + e^t\right] xe^{-x}$$

$$= e^{-(x-t)}\left[\frac{x-t}{2}\right] + \frac{(t+x)}{2} e^{-(t+x)} + \frac{H(t-x)}{2}\left[(x-t)e^{x-t} - (x-t)e^{-(x-t)}\right]$$

Note $\quad e^{-xs}f(s) \iff F(t-x)H(t-x) \qquad \frac{1}{(s\mp1)^2} \Rightarrow te^{\pm t}$

$$\iff F(t-x)H(t-x) = (t-x)e^{\pm(t-x)}$$

$$\therefore \quad \frac{1}{2}\left[\frac{1}{(s-1)^2} - \frac{1}{(s+1)^2}\right] e^{-sx} = \frac{1}{2}\left[(t-x)e^{t-x} - (t-x)e^{-(t-x)}\right] H(t-x)$$



for $t=1$ $\quad t-x>0$ when $x<1 \Rightarrow H(t-x)=1$

when $x>1 \quad t-x<0 \quad H(t-x)=0$

$$u_{TOT} = e^{-(x-t)}\left(\frac{x-t}{2}\right)\left[1-H(t-x)\right] + \left(\frac{t+x}{2}\right)e^{-(x+t)} + \frac{1}{2}(x-t)e^{x-t}H(t-x)$$

if $x-t>0 \quad t-x<0 \quad H=0 \Rightarrow u_{TOT} = \underset{6}{e^{-(x-t)}\left(\frac{x-t}{2}\right)} + \underset{5}{\frac{t+x}{2}e^{-(x+t)}}$

$x-t<0 \quad t-x>0 \quad H=1 \Rightarrow u_{TOT} = \underset{8}{\frac{1}{2}(x-t)e^{x-t}} + \underset{5}{\frac{t+x}{2}e^{-(x+t)}}$

———o———o———o———o——— LESSON #19

EXAM

———o———o———o———o———

LESSONS # 20-24

GAUTAM DOES MATRIX SOLUTIONS VIA FINITE ELEMENT

———o———o———o———o———

EXAM - GAUTAM LESSON #25

———o———o———o———o——— LESSON # 26

FOURIER TRANSFORM OR REVIEW

FINAL EXAM. LESSON # 27

$a u_{xx} + \quad + c u_{yy} + d u_x + e u_y$

$a F''G + c FG'' + d F'G + e FG'$

$$a \frac{F''}{F} + d\frac{F'}{F} = -\left(\frac{cG''}{G} + e\frac{G'}{G}\right) = \text{const.}$$

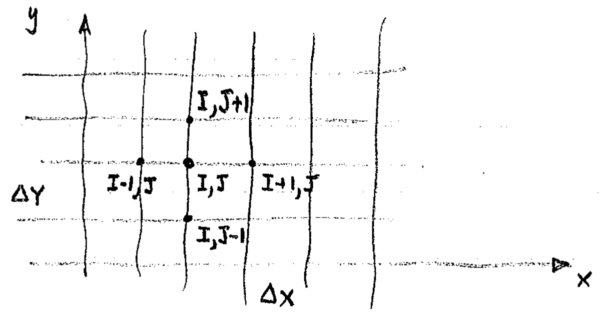
LESSON #5

NUMERICAL METHODS FOR STEADY STATE PROBLEMS

explicit methods { 1) RELAXATION
2) SUCCESSIVE APPROXIMATIONS } USED FOR BOUNDARY VALUE PROBLEMS

- *Numerical Methods used especially when region is not simple* (*simple region square, circular etc.*)
- FINITE DIFFERENCE NOTATION

OVER BDRY DRAW GRID. ΔY, ΔX GRID SIZE



LET $x_i = I\,\Delta X$

$y_j = J\,\Delta Y$

WANT TO WRITE $\dfrac{\partial^2 T}{\partial x^2} + \dfrac{\partial^2 T}{\partial y^2} = 0$ in terms of T

- USE TAYLORS APPROXIMATION ALONG CONSTANT X lines since $\left.\dfrac{\partial T}{\partial x}\right|_{y=\text{const.}}$

$$T_{I+1,J} = T_{I,J} + \left.\frac{\partial T}{\partial x}\right|_{I,J} \Delta x + \left.\frac{\partial^2 T}{\partial x^2}\right|_{I,J} \frac{\Delta x^2}{2} + A\,\Delta x^3 + B\,\Delta x^4$$

$$T_{I-1,J} = T_{I,J} - \left.\frac{\partial T}{\partial x}\right|_{I,J} \Delta x + \left.\frac{\partial^2 T}{\partial x^2}\right|_{I,J} \frac{\Delta x^2}{2} - A\,\Delta x^3 + B\,\Delta x^4$$

$$T_{I+1,J} - T_{I-1,J} = 2\Delta x \left(\frac{\partial T}{\partial x}\right)_{I,J} + 2A\,\Delta x^3$$

$$\left(\frac{\partial T}{\partial x}\right)_{I,J} = \frac{T_{I+1,J} - T_{I-1,J}}{2\Delta x} + \text{error}\,(\Delta x^2)$$ *Good approx for* $\dfrac{\partial T}{\partial x}$

ADD THE TWO

$$T_{I+1,J} + T_{I-1,J} = 2\,T_{I,J} + 2\,\frac{\Delta x^2}{2} \left.\frac{\partial^2 T}{\partial x^2}\right|_{I,J} + 2B\,\Delta x^4$$

$$\therefore \quad \left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} = \frac{T_{I+1,J} - 2T_{I,J} + T_{I-1,J}}{\Delta x^2} + \text{error}\,(\Delta x^2).$$ *Good approx for* $\dfrac{\partial^2 T}{\partial x^2}$

SIMILARLY

$$\left(\frac{\partial^2 T}{\partial y^2}\right)_{I,J} = \frac{T_{I,J+1} - 2T_{I,J} + T_{I,J-1}}{\Delta x^2} + \text{error}\,(\Delta y^2)$$ *for* $\dfrac{\partial^2 T}{\partial y^2}$

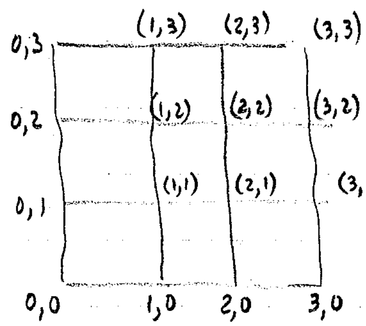
$$\therefore \quad \left(\frac{\partial^2 T}{\partial x^2}\right)_{I,J} + \left(\frac{\partial^2 T}{\partial y^2}\right)_{I,J} = \frac{T_{I+1,J} - 2T_{I,J} + T_{I-1,J}}{\Delta x^2} + \frac{T_{I,J+1} - 2T_{I,J} + T_{I,J-1}}{\Delta y^2} = 0$$

Note error is $O(\Delta x^2) + O(\Delta y^2)$

- IF Δx & Δy are small $\Rightarrow$ numerical representation of $\frac{\partial^2 T}{\partial x^2}, \frac{\partial^2 T}{\partial y^2} \rightarrow$ real solution

- IF $\Delta x, \Delta y \overset{=h}{}$ are the same. This template is good approx to $\nabla^2 T$ to $O(h^2)$

$$\nabla^2 T = T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1} - 4T_{I,J} = 0$$



For boundary value problems - BDRY VALUES ARE GIVEN

LET US ASSUME WE WANT TO FIND $T(x,y)$

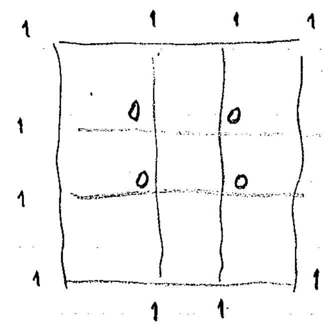
$\ni$. $\nabla^2 T = 0$ & $T = 1$ on bdry.

Relaxation Method

Define the residual $R_{I,J}$ to be $R_{I,J} = \frac{1}{4}\left(T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1}\right) - T_{I,J}$
and guess the value of T on the interior points

LET'S GUESS $T = 0$ @ 1,1 2,1 1,2 2,2. (interior points only!)

- FIND residual for interior points only.

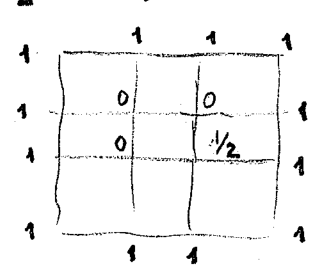


1st iter
$$R_{1,1} = \frac{1}{4}\left[0+1+0+1\right] - 0 = \tfrac{1}{2}$$
$$R_{1,2} = \frac{1}{4}\left[0+1+1+0\right] - 0 = \frac{1}{2}$$
$$R_{2,1} = \frac{1}{4}\left[1+0+0+1\right] - 0 = \frac{1}{2}$$
$$R_{2,2} = \frac{1}{4}\left[1+0+1+0\right] - 0 = \frac{1}{2}$$

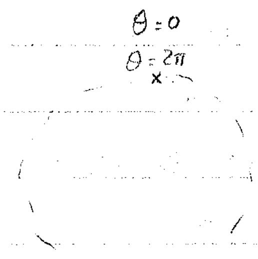
1st iter TAKE LARGEST Residual $R_{I,J}$ & set equal to zero by adjusting $T_{I,J}$ leaving all other pts unchanged

LET'S LOOK AT $R_{2,1} = \frac{1}{2}$ by letting $T_{2,1} = \frac{1}{2}$ $R_{2,1} = 0$

Now find

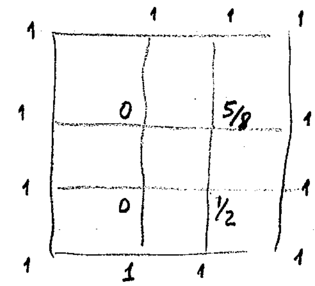


2nd iter
$$R_{1,1} = \frac{1}{4}\left(\frac{1}{2} + 1 + 0 + 1\right) - 0 = 5/8$$
$$R_{2,2} = \frac{1}{4}\left(1 + 0 + 1 + \frac{1}{2}\right) - 0 = \underline{5/8}$$
$$R_{1,2} = \frac{1}{4}\left(0 + 1 + 1 + 0\right) - 0 = 1/2$$

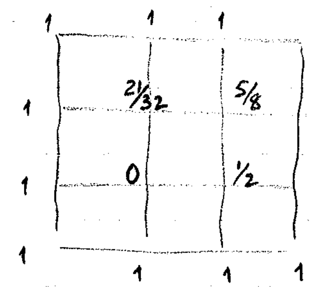

$\theta = 0$
$\theta = 2\pi$
x

2nd iter Let's look at largest residual and do same, make $R_{2,2} = 0 \Rightarrow T_{2,2} = 5/8$

3rd iter
$$\begin{cases} R_{1,1} = \frac{1}{4}\left[\frac{1}{2} + 1 + 0 + 1\right] - 0 = 5/8 \\ R_{1,2} = \frac{1}{4}\left[5/8 + 1 + 1 + 0\right] - 0 = \underline{\frac{21}{32}} \\ R_{2,1} = \frac{1}{4}\left[1 + 0 + \frac{5}{8} + 1\right] - \frac{1}{2} = \frac{5}{32} \end{cases}$$



3rd iter Let $R_{1,2} = 0 \Rightarrow$ Let $T_{1,2} = \frac{21}{32}$



4th iter
$$\begin{cases} R_{1,1} = \frac{1}{4}\left[\frac{1}{2} + 1 + \frac{21}{32} + 1\right] - 0 = \frac{101}{128} \\ R_{2,2} = \frac{1}{4}\left[1 + \frac{21}{32} + 1 + \frac{1}{2}\right] - \frac{5}{8} = \frac{21}{128} \\ R_{2,1} = \frac{1}{4}\left[1 + 0 + 5/8 + 1\right] - \frac{1}{2} = \frac{5}{32} \end{cases}$$

Let $R_{1,1} = 0 \Rightarrow T_{1,1} = \frac{101}{128}$

KEEP DOING THIS UNTIL $\left| T^{new}_{I,J} - T^{old}_{I,J} \right| < \varepsilon$

BASICALLY for largest residual $T^{new}_{I,J} = \frac{1}{4}\left[T_{I+1,J} + T_{I-1,J} + T_{I,J+1} + T_{I,J-1} \right]^{old.}$

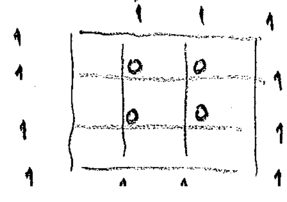
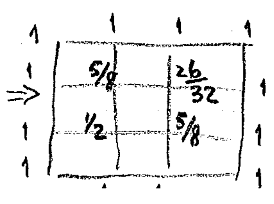
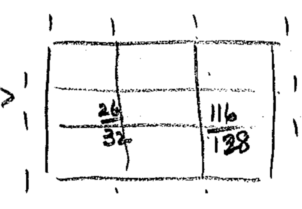
2 WAYS TO DO THIS: TO START ASSUME A SOLUTION $(T=0)$ FIND $R_{I,J}$ $\forall$ INTERIOR POINTS STARTING AT $(1,1)$. FORCE LARGEST RESIDUAL $\rightarrow 0$ & FIND REST OF $R_{I,J}$

THIS CHANGES ONE ~~TEMPERATURE~~ value AT A TIME

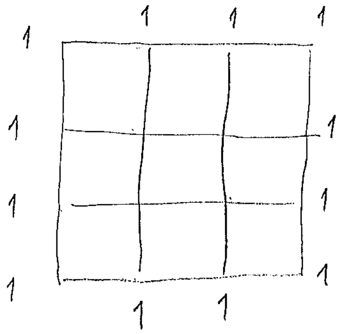
LESSON #6

THE OTHER – SUCCESSIVE REPLACEMENT

ASSUME A SOLUTION $(T=0)^{\text{everywhere in region}}$ FIND $R_{1,1}$. FORCE $R_{1,1} = 0$ (TO GET $T_{1,1}$)
and immediately use this value to find $R_{2,1}$. Force $R_{2,1} = 0$ (TO GET $T_{2,1}$)
$R_{1,2}$ Force $R_{1,2} = 0$ (TO GET $T_{1,2}$
and immediately use these values to find $R_{2,2}$


$\Rightarrow$

$\Rightarrow$


$T_{ij} = \frac{1}{4} \Diamond$



Start with 0 as initial

$R_{11} = \frac{1}{2} \longrightarrow$ let $T_{11} = \frac{1}{2}$

$R_{21} = \frac{1}{4}\left[\frac{1}{2}+1+1+0\right] - 0 = \frac{5}{8} \rightarrow$ let $T_{21} = \frac{5}{8}$

$R_{12} = \frac{1}{4}\left[1+1+0+\frac{1}{2}\right] - 0 = \frac{5}{8} \rightarrow$ let $T_{12} = \frac{5}{8}$

$R_{22} = \frac{1}{4}\left[\frac{5}{8}+1+1+\frac{5}{8}\right] - 0 = \frac{26}{32} \rightarrow$ let $T_{22} = \frac{26}{32}$

$R_{11} = \frac{1}{4}\left[1+\frac{5}{8}+\frac{5}{8}+1\right] - \frac{1}{2} = \frac{10}{32} \longrightarrow T_{11} = \frac{1}{2} + \frac{10}{32} = \frac{26}{32}$

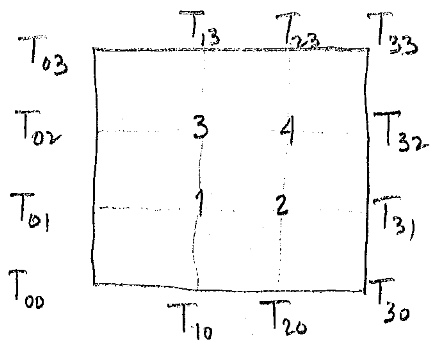
$R_{21} = \frac{1}{4}\left[\frac{26}{32}+1+1+\frac{26}{32}\right] - \frac{5}{8} = \frac{116}{128} - \frac{90}{128} = \frac{26}{128} \longrightarrow T_{21} = \frac{5}{8} + \frac{26}{128} = \frac{116}{128} \rightarrow \frac{29}{32}$

$R_{12} = \frac{1}{4}\left[\frac{116}{128}+1+1+\frac{26}{32}\right] = \frac{12}{128} + \frac{24}{128} = \frac{36}{128} = \frac{9}{128} \qquad \frac{121}{128}$

$R_{22} = \frac{1}{4}\left[\frac{121}{128}+1+1+\frac{29}{32}\right] = \frac{7}{128} + \frac{12}{128} = \frac{19}{128} \sim \frac{5}{128} \qquad \frac{123}{128}$

MATRIX METHOD

implicit



$4\left[T_{01} + T_{10} + T_2 + T_3\right] - T_1 = 0$

$4\left[T_1 + T_{20} + T_{31} + T_4\right] - T_2 = 0$

$4\left[T_{02} + T_{13} + T_4 + T_1\right] - T_3 = 0$

$4\left[T_3 + T_{23} + T_{32} + T_2\right] - T_4 = 0$

$$\begin{bmatrix} -1 & 4 & 4 & 0 \\ 4 & -1 & 0 & 4 \\ 4 & 0 & -1 & 4 \\ 0 & 4 & 4 & -1 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = -4 \begin{bmatrix} T_{01} + T_{10} \\ T_{20} + T_{31} \\ T_{02} + T_{13} \\ T_{23} + T_{32} \end{bmatrix}$$

$[\underset{\sim}{A}]\underset{\sim}{T} = \underset{\sim}{b} \qquad \underset{\sim}{T} = \underset{\sim}{A}^{-1}\underset{\sim}{b}$

HERE $\underset{\sim}{A}$ is known $\underset{\sim}{b}$ is known, If $\underset{\sim}{A}^{-1}$ exists then $\underset{\sim}{T}$ is known

lets look at parabolic equation $\dfrac{\partial^2 T}{\partial x^2} = \dfrac{1}{\alpha}\dfrac{\partial T}{\partial t}$

solution at time $t+\Delta t$ is dependent on t

$$T(x_i, t_j) = T_{IJ}$$

FORWARD DIFFERENCE $\dfrac{\partial T}{\partial t} = \dfrac{T_{I,J+1} - T_{I,J}}{\Delta t}$ order (Δt)

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{I+1,J} - 2T_{I,J} + T_{I-1,J}}{\Delta x^2} \quad \text{order } (\Delta x)^2$$

$$\Rightarrow \quad T_{I,J+1} = T_{I,J} + \frac{\alpha \Delta t}{\Delta x^2}\left[T_{I+1,J} - 2T_{I,J} + T_{I-1,J}\right]$$

for $\dfrac{\alpha \Delta t}{\Delta x^2} \leq 0.25$ solution is stable & nonoscillatory

$\dfrac{\alpha \Delta t}{\Delta x^2} \leq 0.5$ solution is stable

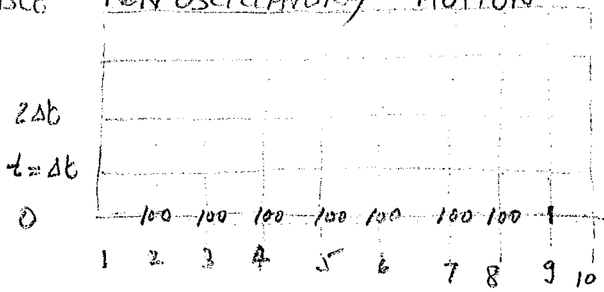
LET'S LOOK AT A ROD

$T = 0$ $\dfrac{\partial T}{\partial x} = 0$

$T(x, t=0) = 100\,°F$

$\rho = 168$ lb/ft^3
$c = .212$ BTU/lb°F
$k = .0370$ BTU/sec ft °F

FOR STABLE NON OSCILLATORY MOTION



FOR $\alpha = \dfrac{k}{c\rho} = .00104$ ft^2/s

THERMAL DIFFUSIVITY

$\Delta x = 1/12$ ft

TAKE $\Delta t = 1$ sec

$\dfrac{\alpha \Delta t}{\Delta x^2} = .15$

AT $X=0$ $t=0$ take T TO BE AVE OF $T(x,t=0)=100$ and $T(x=0,t)=0$
or <u>$50°F$</u>

@ $x=\ell$ $\frac{\partial T}{\partial x}=0$ $T_{8,j}=T_{10,j}$

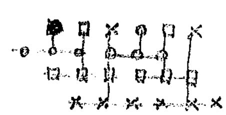
1. use $T_{I,J+1} = T_{I,J} + \frac{\alpha \Delta t}{\Delta x^2}\left[T_{I+1,J} - 2T_{I,J} + T_{I-1,J}\right]$

FOR PTS 2–8

2. use $T_{8,J} = T_{10,J}$ set up an extra column of imaginary pts at station 10.

3. $T_{I,J} = 100°$ FOR $I \geq 2$, $J=0$ FOR $I=1$ $T_{1,J=0} = \frac{0+100}{2}$

4. Evaluate across x at constant t then start again at next t



NOTE DATA AT J+1 ~~row~~ step depends on data at J'th ~~row~~ step – marching technique Need 2 arrays T1, T

T1 is an array of temp at J+1 step

T " " " " " " J step

let $u(I,J) = C y^I x^J$

$$C y^{I+1} x^J - C y^I x^J - \beta\left(C y^I x^{J+1} - 2C y^I x^J + C y^I x^{J-1}\right)]$$

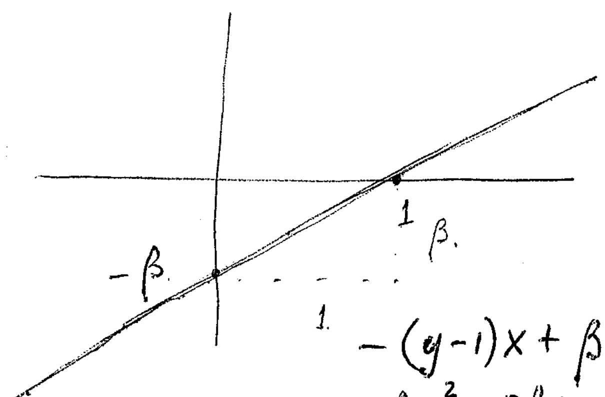
$$C y^I x^J \left[y - 1 - \beta\left(x - 2 + \frac{1}{x}\right)\right] = 0$$

$$xy - x - \beta(x^2 - 2x + 1) = 0$$
$$(x-1)y - \beta(x^2 - 2x + 1)$$
$$(x-1)y - \beta(x-1)^2$$

$(x-1)\left[y - \beta(x-1)\right] = 0$ either $x = 1$ or

$$\beta = \frac{y}{x-1}$$

for $y < 1$



$y - \beta x = -\beta$

$y = \beta(x-1)$

$$-(y-1)x + \beta(x^2 - 2x + 1) = 0$$
$$\beta x^2 - 2\beta x + \beta - yx + x = 0$$
$$\beta x^2 + x\left[1 - y - 2\beta\right] + \beta = 0$$

if $1 - y - 2\beta = 0$

$$\frac{1-y}{2} = \beta.$$

$x^2 + 1 = 0 \quad x = \pm i$

if $y = 0$ $\quad \beta = 1/2$

oscillation

$$(1-y-2\beta)^2 - 4\beta^2$$
$$(1-y)^2 - 4\beta(1-y) + \cancel{4\beta^2} - \cancel{4\beta^2} = 0$$

$$[1-y][1-y-4\beta] = 0$$

$\frac{1-y}{4} = \beta$ if $\beta = \frac{1}{4}$ when $y = 0$

$\beta = 0$ when $y = 1$

$0 \le \beta \le 1/4 \Rightarrow \quad x = \frac{(1-y-2\beta)}{2\beta} = \frac{1/2 - y}{1/2} = 1$

$$x = \frac{y-1+2\beta \pm \sqrt{[(1-y)-2\beta]^2 - 4\beta^2}}{2\beta}$$

$$(1-y)^2 - 4\beta(1-y) + \cancel{4\beta^2} - \cancel{4\beta^2}$$

x is real if $(1-y)[1-y-4\beta] > 0$

x is imag if $(1-y)[1-y-4\beta] < 0$

cut off is if $y=1$ or $\frac{1-y}{4} = \beta$

for conv.

$$0 \le y \le 1 \implies 0 \le \beta \le 1/4$$

$$x = \frac{y-1+2\beta}{2\beta} \qquad \begin{array}{ll} y=0 & \beta = 1/4 \\ y=1 & \beta=0 \end{array}$$

$$= \frac{-1+1/2}{1/2} = -1 \qquad \begin{array}{ll} y=0 & \beta = 1/4 \\ y=1 & \beta=0 \end{array}$$

$$x = \frac{-(1-y)\cancel{y-1} + \frac{1-y}{2}}{\frac{1-y}{2}} = \frac{\frac{1-y}{2}}{\frac{1-y}{2}} \; \frac{[1-2]}{1} = -1$$

let $x = e^{i\theta}$

$$A + iB = \sqrt{A^2+B^2}\, e^{i\theta}.$$

$$[(y-1)+2\beta]^2 + (1-y)^2 + 4\beta(1+y) \quad -1$$

$$\cancel{4\beta^2}$$

$$\cancel{2}(y-1)^2 + \cancel{8}\beta(y-1) + \cancel{4}\beta^2 - 4\beta^2 4\beta^2$$

$$(y-1)^2 + 4\beta(y-1) + 2\beta^2 - 4\beta^2 = 0$$

$$\tan\theta = \frac{B}{A} = \frac{\sqrt{}}{y-1+2\beta}$$

$$2(y-1)\Big/\left[y-1+4\beta\right]$$

$e^{i\pi/2} = 1$ $\quad \theta = \pi/2$

$\frac{1-y}{4} = \beta.$ for $|x| = 1$

$$\begin{array}{ll} y=1 & \beta=0 \\ y=0 & \beta=1/4 \end{array}$$

$$t = 2 - \frac{1}{t} + C\left[x - 2 + \frac{1}{x}\right]$$

$$\cdot xt \quad \left\{t - 2 + \frac{1}{t} - C\left[x - 2 + \frac{1}{x}\right]\right\} = 0$$

$$xt^2 - 2xt + x - C\left[x^2 t - 2xt + t\right]$$

$$xt^2 + x - C\left[x^2 t + t\right] = 0$$

$$\cancel{xt\left[t - Cx\right] + 1\left[x - ct\right]}$$

$$-Cx^2 + x(t^2+1) - ct = 0$$

$$x = \frac{-(t^2+1) \pm \sqrt{(t^2+1)^2 - 4t^2C^2}}{-2tC}$$

$$\text{if } (t^2+1)^2 = 4t^2C^2$$

$$\frac{t^2+1}{4t} = C$$

$t = 0 \quad C = \infty \quad x = \infty$

$t = 1 \quad C = \frac{1}{2} \quad x = \frac{-2}{-2 \cdot 1 \cdot \frac{1}{2}} =$

$$\cancel{tC} \quad \frac{(t^2+1)^2 + (t^2+1)^2 - 4t^2C^2}{4t^2C^2} = 1$$

$$\frac{\cancel{2}(t^2+1)^2}{4t^2C^2} = \cancel{2}\,1$$

$$\frac{t^2+1}{2t} = C$$

$-(t^2+1) \pm (t^2-1)$

$x = -2 / -2t = 1/t$

$x \quad -2t^2/-2t = t$

$$\cancel{\frac{\partial f}{\partial c} = 0 \Rightarrow 8t}$$

$$\cancel{(t^2+1)^2 - 4t^2C^2 = 0}$$

$$\cancel{\frac{\partial f}{\partial c} = -4t^2 \cdot 2c = 0 \Rightarrow c = 0 \text{ or } t = 0}$$

$$X = \frac{y-1+2\beta \pm \sqrt{(1-y)(1-y-4\beta)}}{2\beta} = A \pm Bi \Rightarrow Re^{i\theta} \quad \theta = \tan^{-1}\frac{B}{A}$$

for non osci stable $|R|<1$ & $\theta = 0 \Rightarrow B = 0 \Rightarrow \frac{1-y}{4} = \beta$

$$\frac{\frac{2(y-1)}{2} + \frac{1-y}{2}}{\frac{1-y}{2}} = -1 = \frac{-2+1}{1} = -1 = X$$

$0 \le \beta \le 1/4$

for oscill behavior $|x|<1$ $\quad \sqrt{A^2+B^2} < 1$

$$\left[(y-1)^2 + 4\beta(y-1) + 4\beta^2 + (1-y)^2 + (y-1)4\beta\right]/4\beta^2 = 1$$

$$2(y-1)^2 + 8\beta(y-1) + \cancel{4\beta^2} - \cancel{4\beta^2} = 0$$

$$2\cancel{\left[(y-1)^2 + 4\beta(y-1) + 2\beta^2\right]}$$

$$\cancel{\left[(y-1) + 2\beta\right]\left[(y-1) + \beta\right]}$$

$$\sqrt{2(y-1)\{y-1+4\beta\}} = 0 \qquad y=1 \text{ or } y-1+4\beta = 0$$

$$\beta = \frac{1-y}{4}$$

$$\cancel{\left[2(y-1)+2\beta\right]\left[(y-1)+4\beta\right]}$$

for stable osc.

let $\bar{y} = -2\beta$

$$2\{\bar{y}^2 + 4\beta\bar{y} + 2\beta^2]$$

$\bar{y} = 0 \quad 4\beta^2 \to 2\beta \qquad \bar{y} = -4\beta$

$$2\{16\beta^2 - 16\beta^2 + 2\beta^2$$

$4\beta^2 - 8\beta^2 + 2\beta^2$

$-4\beta^2$

Solution of PDE's W.C. Reynolds
Stanford Univ

Chapter 2

SELF-SIMILAR SOLUTIONS

2.1 Characteristic Scales; Scale-Similar Problems

It is often convenient to present the solution to a PDE problem in non-dimensional form. This makes the results independent of the size of the system for which the solution was obtained as well as independent of any choice of dimensional system. Non-dimensionalization is usually accomplished by choosing some length and time scales characterizing the problem, and then defining non-dimensional independent variables based on these scales. For example, the solution for fluid flow in a rotating sphere might be expressed non-dimensionally in terms of the dimensionless radius, $R = r/r_o$, where r_o is the radius of the sphere. Here r_o is the <u>characteristic length scale</u> of the problem. If the fluid is initially at rest, and at time zero it is put into rotation at angular velocity ω, then the period of rotation is $\tau = 2\pi/\omega$, and τ would be the <u>characteristic time scale</u>. Then a suitable dimensionless time would be $T = t/\tau$. Note that one of the characteristic scales for the independent variables (r_o) came from the <u>geometry</u> of the system, and the other (τ) from the <u>boundary conditions</u>.

The dependent variables also can be represented non-dimensionally. For example, in the rotating sphere problem the equatorial velocity is $u_o = \omega r_o$ and may be used as a characteristic velocity in the dimensionless velocity $\underline{U} = \underline{u}/u_o$.

The problem may also contain some parameters, such as the kinematic viscosity ν. The parameters also can be reduced to non-dimensional form, and in the case of viscosity it is customary to use a reciprocal dimensionless viscosity called the Reynolds number, $Re = u_o r_o/\nu$.

The solution for the velocity within the rotating sphere could then be expressed non-dimensionally as

$$\underline{U} = \underline{U}(R,T;Re)$$

This says that the dimensionless velocity (a vector) $\underline{U}$ will be a function of the dimensionless radial coordinate R, the dimensionless time coordinate T, and the parameter Re. It might also happen that the flow depends upon the polar angular coordinates ϕ and θ, which are additional non-dimensional independent variables.

Problems which have natural characteristic scales for the independent variables (here r_o and τ) are called scale-similar. Scale-similar solutions for systems of different size will have the same non-dimensional solution, provided that the two problems also have the same values of the dimensionless parameters and dimensionless boundary and initial conditions.

2.2 Self-Similarity

There are a few very interesting and important PDE problems for which no natural characteristic scales for the independent variables exist in the problem formulation. For example, consider the case of heat conduction in a semi-infinite slab initially at uniform temperature, subjected to a step increase in the surface temperature at time zero (Fig. 2.2.1). The appropriate PDE is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.2.1}$$

where α is a constant parameter called the thermal diffusivity of the medium. The initial condition is

$$T(x,0) = T_i \qquad x > 0 \tag{2.2.2}$$

The boundary condition at the surface is

$$T(0,t) = T_s \tag{2.2.3}$$

The temperature field must fall off to the initial temperature T_i as $x \to \infty$, giving a second boundary condition

$$T(x,t) \to T_i \qquad \text{as} \quad x \to \infty \tag{2.2.4}$$

There are no characteristic scales for either length or time in this problem. This fact is the clue that a self-similar solution must exist. Since the solution to all physical problems must be expressible in dimensionless form (nature is unaware of the length of a meter), there must be some way to non-dimensionalize the solution to this problem. The only possible way is for the variables to appear together in a non-dimensional group. Looking at the denominators in (2.2.1), it is readily apparent that x^2 and αt have the same dimensions, and therefore the quantity $x^2/(\alpha t)$ is dimensionless. Somehow the solution must be expressible in terms of this quantity, in order to have dimensionless form. Solutions made non-dimensional by combinations of the independent variables, rather than by characteristic scales imposed by the geometry, boundary, or initial conditions, are called self-similar solutions.

There is a characteristic temperature for this problem, namely the step increase in temperature $T_s - T_i$. Therefore, one might guess that the non-dimensional form of the solution is

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) \tag{2.2.5}$$

As we shall see, this guess is correct. In a moment we shall develop a systematic way of discovering the forms of self-similar solutions.

If (2.2.5) is indeed correct, then another fully equivalent form would be

$$\frac{T - T_i}{T_s - T_i} = g(x/\sqrt{\alpha t}) \tag{2.2.6}$$

and another would be

$$\frac{T - T_i}{T_s - T_i} = \frac{x}{\sqrt{\alpha t}}\, h(x/\sqrt{\alpha t}) \tag{2.2.7}$$

All of these solutions would really be the same, but the functions f, g, and h would be different.

In terms of the similarity variable, $\eta = x/\sqrt{\alpha t}$, the family of temperature profiles existing at different times will collapse to a single curve (Fig. 2.2.1b). This is the essence of self-similarity; the solution does not scale on the size of the system, instead it scales on itself.

At first glance, it may appear disadvantageous to seek a solution in terms of the non-linear combination of variables $\eta = x/\sqrt{\alpha t}$. However, note that a single function $g(\eta)$ would be involved, and therefore one would only have to deal with an ordinary differential equation (ODE). This is the practical advantage of a self-similar problem in two independent variables. The existence of self-similarity will always reduce the number of independent variables by one.

To summarize, self-similar solutions exist when a problem is not scale-similar, i.e. when characteristic scales for the independent variables do not exist in the problem formulation. In problems with two independent variables, self-similar solutions represent a collapse of the family of solutions as functions of the two variables to a single function of the similarity variable. The governing PDE is thereby reduced to an ODE, which may be solved by some appropriate analytical or numerical method. The proper form of the transformation depends upon the equation, the initial conditions, and the boundary conditions. The transformation can be discovered systematically, as we shall now illustrate by some examples.

2.3 Example with Constant Boundary Conditions

Consider the transient heat transfer problem discussed in section 2.2 The differential equation, boundary conditions, and initial conditions are (2.2.1)-(2.2.4). The solution must be expressible in terms of some similarity variable, which must be non-dimensional. Let's assume that the similarity variable is of the form

$$\eta = Ax/t^n \tag{2.3.1}$$

where A and n are constants to be chosen in a manner that reduces the PDE problem to an ODE problem. Now, suppose we assume that the dimensionless solution has the form

$$\frac{T - T_i}{T_s - T_i} = f(\eta) \tag{2.3.2}$$

This is suggested by the observation that the significant aspect is the difference between the temperature at any point $T(x,y)$ and the initial temperature T_i .* The form of η is suggested by the fact that the solution for $t=0$ and $x=\infty$ must give the same value of T , and hence must correspond to the same value of f , and hence to the same value of η . Now, we could have taken $\eta = Ax^m/t^n$ but this is no more general than the (2.3.1), since this η is just a power of the other η . Also, we could have taken $\eta = At/x^n$, which also is no more general. However, we will have to differentiate twice with respect to x , and only once with respect to t , and we will find our work easier if we keep the x-dependence of η as simple as possible. For this reason, we make η linear in x , and then divide by t to a power (to be chosen later).

The next step is to transform the PDE. Using the chain rule,

$$\frac{\partial T}{\partial x} = (T_s - T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial x} = (T_s - T_i)\, f' \cdot \frac{A}{t^n} \tag{2.3.3a}$$

$$\frac{\partial^2 T}{\partial x^2} = (T_s - T_i)\frac{A}{t^n}\frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = (T_s - T_i)\frac{A}{t^n}\, f'' \cdot \frac{A}{t^n} \tag{2.3.3b}$$

$$\frac{\partial T}{\partial t} = (T_s - T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = (T_s - T_i)\, f' \cdot \left(-\frac{Anx}{t^{n+1}}\right) \tag{2.3.3c}$$

Then, substituting in (2.2.1), we obtain

$$(T_s - T_i)\frac{A^2}{t^{2n}}\, f'' = -\frac{1}{\alpha}(T_s - T_i)\frac{Anx}{t^{n+1}}\, f'$$

---

*We could instead take

$$\frac{T}{T_s - T_i} = g(\eta\ ;\ T_s/T_i) \tag{2.3.2x}$$

The student should work through the problem with this starting assumption to verify that the same solution is obtained.

which simplifies to

$$f'' + \frac{1}{\alpha A^2} Anx\, t^{n-1} f' = 0 \tag{2.3.4}$$

Now, this is supposed to be an ODE for $f(\eta)$. Therefore, it can only contain f , f' , f'' , and η ; somehow we must make x and t disappear. To do this, we first replace x using (2.3.1), $x = t^n\eta/A$, and find

$$f'' + \frac{n}{\alpha A^2} t^{2n-1} \eta f' = 0 \tag{2.3.5}$$

Next, we can select the proper value of n as that which drops out t , namely $n = 1/2$. With this choice, (2.3.5) reduces to

$$f'' + \frac{1}{2\alpha A^2} \eta f' = 0 \tag{2.3.6}$$

This is an ODE, as desired. We still are free to choose A any way we like. To make (2.3.6) as simple as possible, let's pick

$$A = 1/\sqrt{2\alpha} \tag{2.3.7}$$

which reduces our ODE to

$$f'' + \eta f' = 0 \tag{2.3.8}$$

Note that η is a dimensionless variable. Now we have

$$\eta = x/\sqrt{2\alpha t} \tag{2.3.9}$$

We must also be able to express the boundary and initial conditions in terms of $f(\eta)$ in order to complete the self-similar transformation. Eqs. (2.2.2) and (2.2.4) both require

$$f(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{2.3.10}$$

And, (2.2.3) requires

$$f(0) = 1 \tag{2.3.11}$$

Eqs. (2.3.8), (2.3.10), and (2.3.11) define the ODE problem that we must solve.

Eqn. (2.3.8) can be written as

$$\frac{df'}{f'} = -\eta d\eta \tag{2.3.12}$$

Integrating,

$$\ell n\ f' = -\frac{\eta^2}{2} + C_0$$

or,

$$f' = C_1\ e^{-\eta^2/2} \tag{2.3.13}$$

Integrating again,

$$f = C_1 \int_{\infty}^{\eta} e^{-\sigma^2/2}\ d\sigma + C_2 \tag{2.3.14}$$

The lower limit is arbitrary, and ∞ is a good choice. We must be careful not to confuse the limit of integration (η) with the variable of integration, and therefore have introduced σ as the "dummy variable" of integration.

The boundary condition (2.3.10) requires $C_2 = 0$. The boundary condition (2.3.11) requires

$$1 = C_1 \int_{\infty}^{0} e^{-\sigma^2/2}\ d\sigma \tag{2.3.15}$$

Hence, we can write the solution as

$$f = \int_{\eta}^{\infty} e^{-\sigma^2/2}\ d\sigma \Bigg/ \int_{0}^{\infty} e^{-\sigma^2/2}\ d\sigma \tag{2.3.16}$$

We can express the solution in terms of known special functions by letting $z = \sigma/\sqrt{2}$. Then, $d\sigma = \sqrt{2}\ dz$, and

$$f = \int_{\eta/\sqrt{2}}^{\infty} e^{-z^2}\ dz \Bigg/ \int_{0}^{\infty} e^{-z^2}\ dz \tag{2.3.17}$$

The denominator has the value $\sqrt{\pi}/2$. The numerator is $\sqrt{\pi}/2 \ \text{erfc}\ (\eta/\sqrt{2})$, where erfc is the complementary error function.* Hence, the solution is

$$\frac{T - T_i}{T_s - T_i} = \text{erfc}\ \frac{x}{2\sqrt{\alpha t}} \tag{2.3.18}$$

2.4 Example with Variable Boundary Conditions

The motion of a viscous fluid, initially at rest, over an infinite plate that is set into motion at time zero is described by (Fig. 2.4.1)

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{2.4.1}$$

where u is the velocity tangential to the plate, and ν is the (constant) kinematic viscosity. Suppose the boundary condition at the plate $y=0$, is

$$u(0,t) = at^b \tag{2.4.2}$$

where a and b are fixed parameters. The other boundary condition is

$$u(y,t) \to 0 \qquad \text{as} \quad y \to \infty \tag{2.4.3}$$

The initial condition is

$$u(y,0) = 0 \tag{2.4.4}$$

There are no characteristic length or time scales in either the domain or boundary conditions of this problem, hence, we expect a self-similar solution. Suppose we assume

$$u = A\, f(\eta) \quad , \quad \eta = By/t^n \tag{2.4.5}$$

where A, B, and n are parameters that we will try to select to produce an ODE problem. The form of η is suggested by (2.4.3) and (2.4.4), which require

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*See HMF, Section 7.1.

that the solution have the same behavior for large y as for small t. However, when we try to fit the boundary condition (2.4.2) with this form, we get

$$A\, f(0) = a t^b \tag{2.4.6}$$

Since A and $f(0)$ will be constants, (2.4.6) can't be true except for the special case $b=0$ (which reduces this example to the previous one). Hence, (2.4.5) will not work.

We need to allow additional freedom. If we expect the curves of Fig. (2.3.1a) to collapse on a single non-dimensional curve, the value of the fluid velocity must somehow scale on the instantaneous wall velocity. This suggests that we try

$$u = A\, t^m f(\eta) \qquad \eta = By/t^n \tag{2.4.7}$$

Where now A, m, B, and n may be chosen to give us the desired self-similar solution.*

We can immediately determine m using (2.4.2),

$$u(0,t) = A\, t^m f(0) = a t^b \tag{2.4.8}$$

Hence, we must choose $m=b$. We may choose A any way we like. If we choose $A=a$, then we must impose the boundary condition

$$f(0) = 1 \tag{2.4.9}$$

Now, we have

$$u = a\, t^b f(\eta) \qquad \eta = By/t^n \tag{2.4.10}$$

which will fit the boundary conditions.

---

*We could have used $u = A\, y^k\, t^m\, g(\eta)$, or $u = A\, y^m\, h(\eta)$. These forms are equivalent to (2.4.7), with different functions f, g, and h. Eq. (2.4.7) is the simplest, since we must take two y derivatives and only one t derivative.

Next, we substitute (2.4.10) in the differential equation (2.4.1), and find ($f' = df/d\eta$, $f'' = d^2f/d\eta^2$)

$$\nu aB^2 t^{b-2n} f'' = abt^{b-1} f - at^{b-n-1} nByf' \tag{2.4.11}$$

As an ODE in $f(\eta)$, this may contain only f and its derivatives, η , and constants; y and t may not appear. So, we will replace y by

$$y = t^n \eta / B \tag{2.4.12}$$

Then, (2.4.11) reduces to

$$\nu aB^2 t^{b-2n} f'' = abt^{b-1} f - at^{b-1} n\eta f' \tag{2.4.13}$$

In order that t drop out, we must choose n such that

$$b-2n = b-1 \quad \text{or} \quad n = 1/2$$

With this choice, our ODE becomes

$$\nu B^2 f'' = bf - \frac{1}{2}\eta f' \tag{2.4.14}$$

Let's choose B such that $\nu B^2 = \frac{1}{2}$, or $B = 1/\sqrt{2\nu}$. Then we have

$$f'' + \eta f' - 2bf = 0 \tag{2.4.15}$$

and our similarity variable η is

$$\eta = y/\sqrt{2\nu t} \tag{2.4.16}$$

The boundary conditions on (2.4.15) are, from (2.4.9),

$$f(0) = 1 \tag{2.4.17a}$$

and, from (2.4.3),

$$f(\eta) \to 0 \qquad \text{as} \quad \eta \to \infty \tag{2.4.17b}$$

To complete the problem, we must solve (2.4.15) subject to (2.4.17). This will provide a good review of some ODE solution methods and will introduce us to some special functions.

In order to solve (2.4.15), one must be specific about the value of b. Let's first take $b = 1/2$, for which (2.4.15) becomes

$$f'' + \eta f' - f = 0 \tag{2.4.18}$$

The general solution will be of the form

$$f = C_1 f_1 + C_2 f_2 \tag{2.4.19}$$

where f_1 and f_2 are two linearly-independent solutions. For this case, $f_1 = \eta$ is one obvious solution; when the first solution to a second-order linear ODE is known, the second can always be constructed by setting

$$f_2(\eta) = f_1(\eta) \cdot g(\eta) \tag{2.4.20}$$

So, we assume

$$f_2(\eta) = \eta\, g(\eta)$$

Differentiating, and substituting in (2.4.18), we find

$$\eta g'' + 2g' + \eta(\eta g' + g) - \eta g = 0 \tag{2.4.21}$$

The zero-derivative terms cancel, which is why this methods works. So, we have

$$\eta g'' + (2+\eta^2) g' = 0 \tag{2.4.22}$$

which is really a <u>first</u>-order ODE for g'; separating the variables,

$$\frac{dg'}{g'} = -\left(\frac{2}{\eta} + \eta\right) d\eta \tag{2.4.23}$$

Integrating, and taking the exponential,*

$$g' = \exp\left(-2\,\ell n\,\eta - \frac{\eta^2}{2}\right) = \frac{1}{\eta^2}\, e^{-\eta^2/2} \tag{2.4.24}$$

Integrating again,

$$g(\eta) = \int_{\infty}^{\eta} \frac{1}{\sigma^2}\, e^{-\sigma^2/2}\, d\sigma \tag{2.4.25}$$

The lower limit choice is arbitrary, except that zero will cause problems; infinity is an "artistic" choice. So, we now have the general solution to (2.4.18) as

$$f = C_1\eta + C_2\eta \int_{\infty}^{\eta} \frac{1}{\sigma^2}\, e^{-\sigma^2/2}\, d\sigma \tag{2.4.26}$$

Note that again we were careful not to confuse the limit of integration (η) with the variable of integration (σ) .

We now apply the boundary condition (2.4.17b), which will require $C_1 = 0$ if we can show that the second solution f_2 is bounded as $\eta \to \infty$. We have

$$f_2(\eta) = \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2}\, e^{-\sigma^2/2}\, d\sigma < \eta \int_{\infty}^{\eta} \frac{1}{\eta}\, e^{-\sigma^2/2}\, d\sigma = \int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma$$

$$(\text{for } \eta > 1) \tag{2.4.27}$$

So, clearly $f_2(\eta) \to 0$ as $\eta \to \infty$. Therefore, C_1 is indeed zero.

---

*We choose the constant of integration to be 0. Any $g(\eta)$ will do since we can use any second solution.

The behavior of f_2 at $\eta = 0$ can be clarified through use of one of the most powerful tools of analysis-integration by parts.* With it, f_2 can be rewritten as

$$f_2 = \eta \left[-\frac{1}{\sigma} e^{-\sigma^2/2} \Big|_\infty^\eta - \int_\infty^\eta \left(-\frac{1}{\sigma}\right)(-\sigma)\, e^{-\sigma^2/2}\, d\sigma \right]$$

$$= -e^{-\eta^2/2} - \eta \int_\infty^\eta e^{-\sigma^2/2}\, d\sigma \qquad (2.4.28)$$

Now it is clear that $f_2(0) = -1$. Since (2.4.17a) requires that $f(0) = 1$, $C_2 = -1$. Therefore, the final solution is

$$f(\eta) = e^{-\eta^2/2} + \eta \int_\infty^\eta e^{-\sigma^2/2}\, d\sigma \qquad (2.4.29)$$

Using the charge of variables, $z = \sigma/\sqrt{2}$, this can be written as

$$f(\eta) = e^{-\eta^2/2} - \eta\sqrt{\frac{\pi}{2}}\, \mathrm{erfc}(\eta/\sqrt{2}) \qquad (2.4.30)$$

$$(\text{for } b = 1/2)$$

Next, let's consider the case $b = n/2$, where n is an integer. Eqn. (2.4.15) is then

$$f'' + \eta f' - nf = 0 \qquad (2.4.31)$$

If we let $z = \eta/\sqrt{2}$, then (2.4.31) becomes

$$\frac{d^2 f}{dz^2} + 2z\frac{df}{dz} - 2nf = 0 \qquad (2.4.32)$$

---

*Recall that $\int u\,dv = uv - \int v\,du$; this is called integration by parts; become adept at doing it, because it is tremendously useful and important.

The two linearly independent solutions of this equation are repeated integrals of the error function,*

$$f = C_1 i^n \mathrm{erfc}(z) + C_2 i^n \mathrm{erfc}(-z) \tag{2.4.33}$$

where the function $i^n\mathrm{erfc}(x)$ is**

$$i^n\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} e^{-t^2} dt \tag{2.4.34}$$

Hence, our solution is

$$f = C_1 i^n \mathrm{erfc}(\eta/\sqrt{2}) + C_2 i^n \mathrm{erfc}(-\eta/\sqrt{2}) \tag{2.4.35}$$

The boundary condition $f(\infty) = 0$ requires $C_2 = 0$, since $i^n\mathrm{erfc}(-\infty)$ is a constant. The boundary condition $f(0) = 1$ fixes C_1 as***

$$C_1 = \frac{1}{i^n\mathrm{erfc}(0)} = 2^n \, \Gamma(\tfrac{n}{2}+1) \tag{2.4.36}$$

where $\Gamma(x)$ is the <u>Gamma function</u>,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{2.4.37}$$

Hence, the solution is

$$f(\eta) = 2^n \, \Gamma(\tfrac{n}{2}+1) \, i^n\mathrm{erfc}(\eta/\sqrt{2}) \tag{2.4.38}$$

(for $b = n/2$)

---

*HMF Section 7.2.2.

**The student should verify (2.4.33) by substitution in (2.4.32). Integration by parts will be required.

***See HMF Section 7.2.7.

2.8 Summary

We have seen that self-similar solutions arise when there are no natural characteristic scales for the independent variables in the problem formulation. The self-similar transformation will always reduce the number of independent variables by one, so that in a problem with two independent variables the PDE will become an ODE. The steps used to systematically develop the self-similar solution are as follows:

(1) Assume a general form for the transformation, guided by the initial and boundary conditions. Use a form in which the variable that appears in the most complex way in the equations appears as simply as possible in the transformation.

(2) Express the boundary and initial conditions in terms of the similarity transformation, and verify that they can be satisfied by the assumed transformation. If they can not, add additional degrees of freedom.

(3) Remove one (or more) of the independent variables using the similarity variable. Then, determine the parameters of the transformation necessary to reduce the PDE order by one.

(4) Express the boundary and initial conditions for the reduced problem, and solve by appropriate methods.

In all of the examples worked here, the similarity variable involved forms like $y/\sqrt{x}$. The square-root behavior occurs frequently, but not exclusively. Some of the problems at the end of this chapter will require other powers in the similarity variable.

For Further Reading on Similarity Solutions

- Kline, S. J., Similitude and Approximation Theory, McGraw-Hill Book Co., New York, 1965.
- Hansen, A. G., Similarity Analysis of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- Sedov, L. I., Similarity and Dimensional Methods in Mechanics, Academic Press, New York, 1959.

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Exercises:

2.1 The temperature field $T(x,t)$ in a semi-infinite slab with a constant heat flux is described by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad ; \quad T(x,0) = T_i$$

$$T(x,t) \rightarrow T_i \text{ as } x \rightarrow \infty \quad ; \quad -k \frac{\partial T}{\partial x} = q \text{ at } x = 0$$

Solve for the temperature field for $x \geq 0$, $t \geq 0$.

2.2 The temperature field in the thermal boundary layer that grows within a hydrodynamic boundary layer at a step in wall temperature is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \text{ as } y \rightarrow \infty \quad ; \quad T(x,0) = T_w \ ;$$

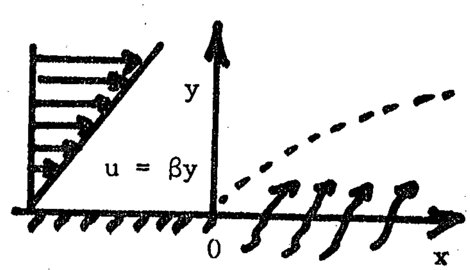
Solve for the temperature field for $x \geq 0$, $y \geq 0$.

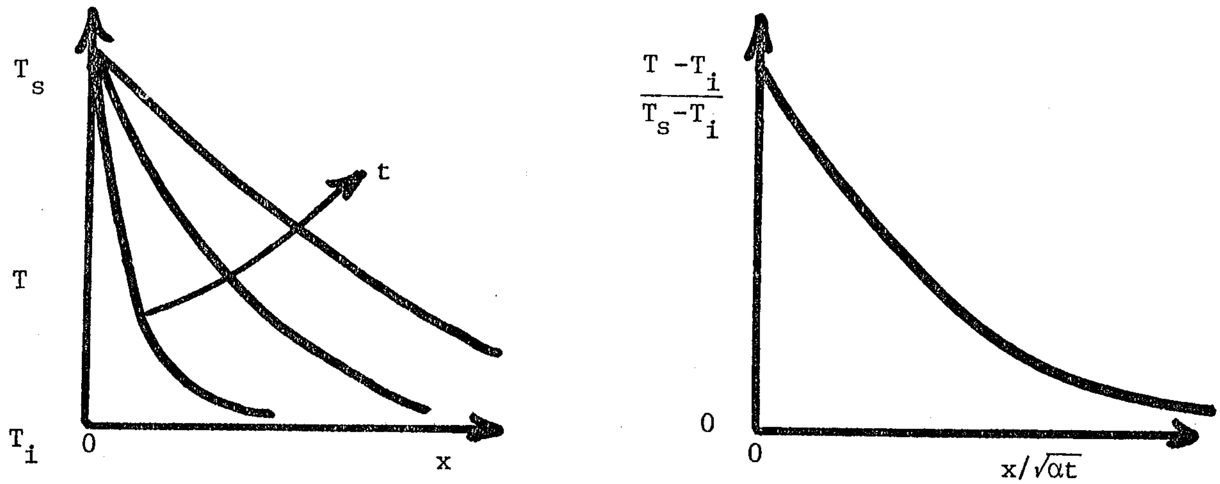
2.3 A device for measuring the velocity gradient in flows is shown in the figure. It consists of a heated plate at the wall, over which a thermal boundary layer grows. As long as the thermal boundary layer is confined to the region where the flow velocity u is linear $(u = \beta y)$, the problem is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \text{ as } y \rightarrow \infty \quad ; \quad -k \frac{\partial T}{\partial y} = q \text{ at } y = 0$$

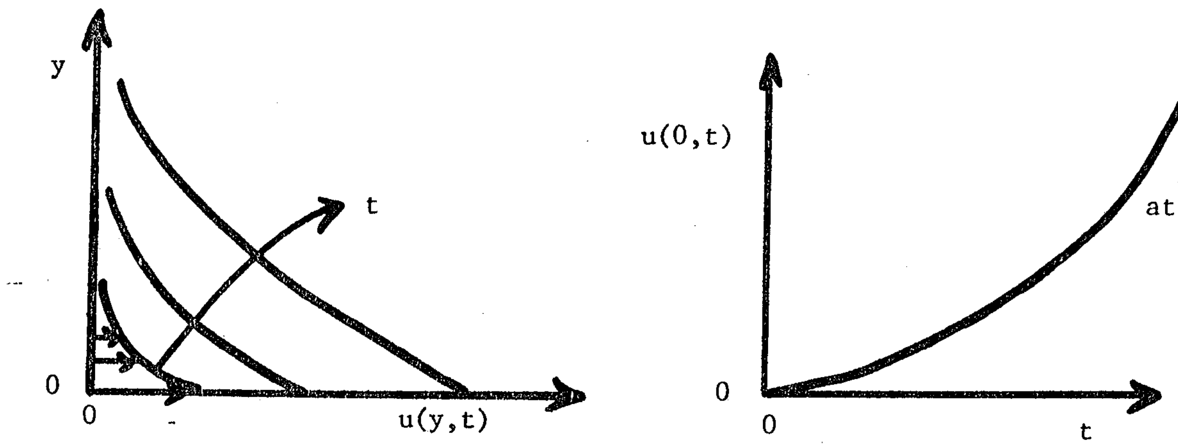
Derive an expression relating the <u>local</u> wall temperature, $T_w(x)$, to the flow parameters and x . Evaluate any constants in this expression. Hint: Γ .





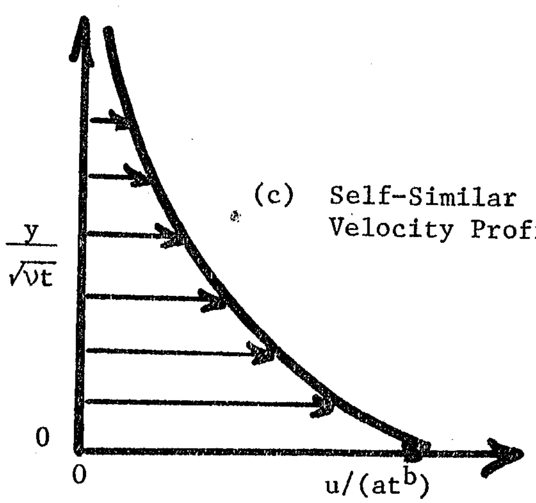
(a) (b)

Fig. 2.2.1 Temperature Field in a Semi-Infinite Slab



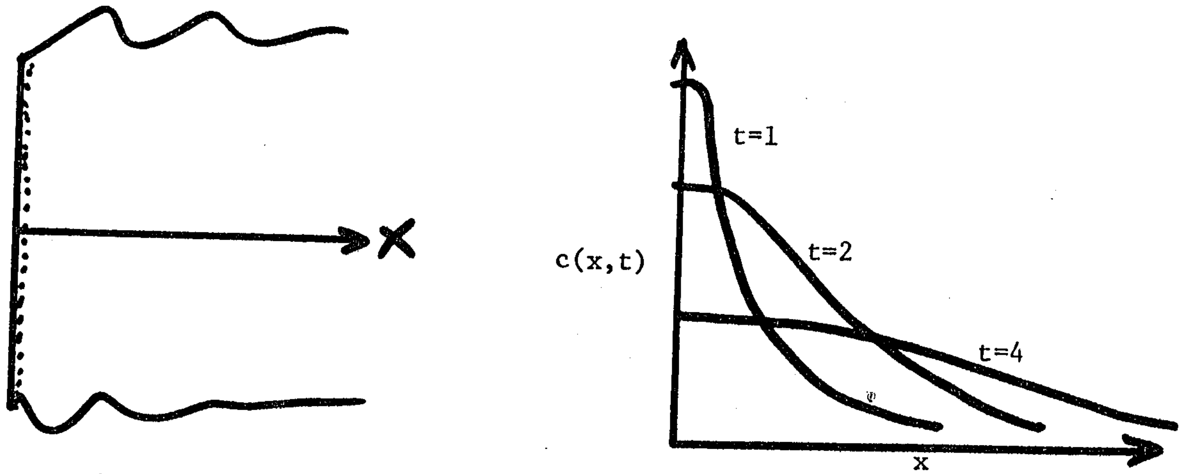
(a) Velocity Field

(b) Plate Velocity



(c) Self-Similar Velocity Profile

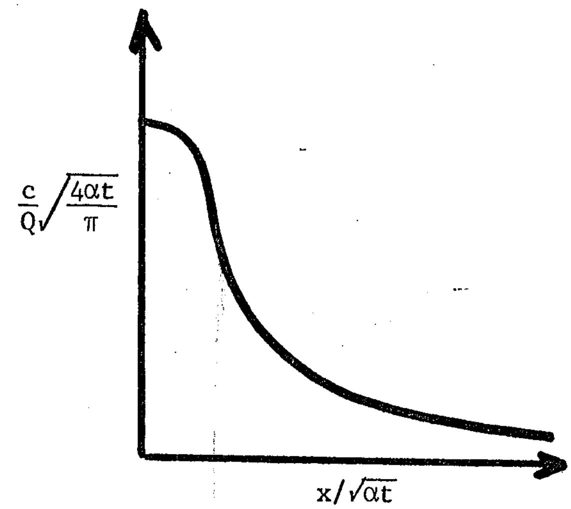
Fig. 2.4.1 Velocity Field in Viscous Flow over A Moving Plate



(a) The System

(b) Concentration Profiles

Fig. 2.5.1.



(c) Self-Similar Profile

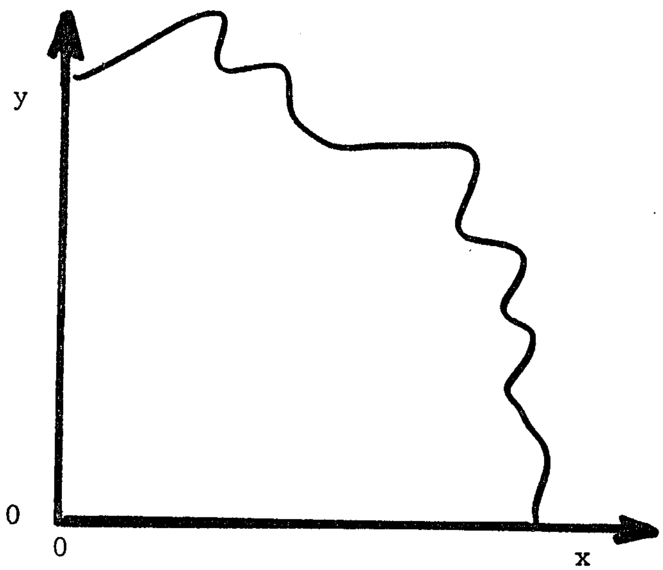


Fig. 2.7.1. Geometry for Analysis of Heating of a Corner

29. Laplace Transforms

9.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

9.1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a mplex variable. $F(t)$ is called the original funcon and $f(s)$ is called the image function. If the tegral in **29.1.1** converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

ists, then it converges for all s with $\mathscr{R}s>s_0$, and ie image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2
$$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \frac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function $F(t)$	*Image Function $f(s)$*
9.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
9.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
9.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
9.2.4	$F'(t)$	$sf(s)-F(+0)$
9.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
9.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
9.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
9.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
9.2.9	$-tF(t)$	$f'(s)$
9.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New ork, N.Y., 1958.

	Original Function $F(t)$	Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t} F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c} F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c} e^{(b/c)t}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t) \sum_{n=0}^{\infty} (-1)^n u(t-na)$	$\dfrac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\lvert F(t) \rvert$	$f(s) \coth \frac{as}{2}$
	Heaviside Expansion Theorem	
29.2.20	$\sum_{n=1}^{m} \frac{p(a_n)}{q'(a_n)} e^{a_n t}$	$\frac{p(s)}{q(s)}, \; q(s)=(s-a_1)(s-a_2) \ldots (s-a_m)$ $p(s)$ a polynomial of degree$<m$
29.2.21	$e^{at} \sum_{n=1}^{r} \frac{p^{(r-n)}(a)}{(r-n)!} \frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree$<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

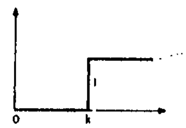
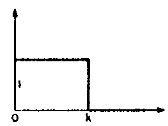
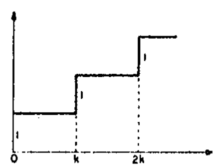
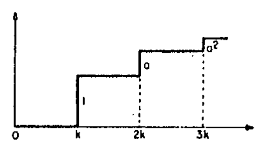
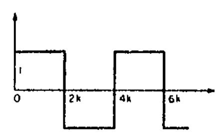
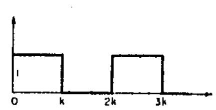
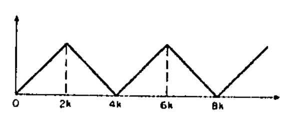
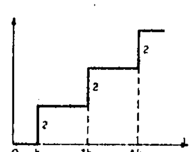
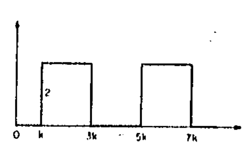
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

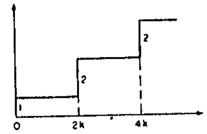
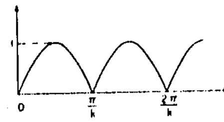
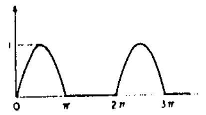
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
3	$\frac{1}{s^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}}{(n-1)!}$
9.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
9.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
9.3.6	$s^{-(n+\frac{1}{2})} \quad (n=1, 2, 3, \ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1 \cdot 3 \cdot 5 \ldots (2n-1)\sqrt{\pi}}$
9.3.7	$\frac{\Gamma(k)}{s^k} \quad (k>0)$	6	t^{k-1}
9.3.8	$\frac{1}{s+a}$		e^{-at}
9.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
9.3.10	$\frac{1}{(s+a)^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
9.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k>0)$	6	$t^{k-1}e^{-at}$
9.3.12	$\frac{1}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
9.3.13	$\frac{s}{(s+a)(s+b)} \quad (a \neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
9.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
9.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a}\sin at$
9.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
9.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a}\sinh at$
9.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
9.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
9.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
9.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at\cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	let $ib=a$ $\frac{s}{(s^2-b^2)^2} \Leftrightarrow \frac{t}{2ib}$: $\sin ibt$ $\frac{t}{2b}\sinh bt$
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)}$ $(a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{\frac{3}{2}}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf} a\sqrt{t}$	7

	$f(s)$		$F(t)$	
29.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$		$\frac{2}{a\sqrt{\pi}} e^{-a^2t}\int_0^{a\sqrt{t}} e^{\lambda^2}d\lambda$	7
29.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$		$e^{a^2t}[b-a \operatorname{erf} a\sqrt{t}]-be^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$		$e^{a^2t} \operatorname{erfc} a\sqrt{t}$	7
29.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$		$\frac{1}{\sqrt{b-a}} e^{-at} \operatorname{erf} (\sqrt{b-a}\sqrt{t})$	7
29.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$		$e^{a^2t}\left[\frac{b}{a} \operatorname{erf} (a\sqrt{t})-1\right]+e^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$		$\frac{n!}{(2n)!\sqrt{\pi t}} H_{2n}(\sqrt{t})$	22
29.3.47	$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$		$-\frac{n!}{(2n+1)!\sqrt{\pi}} H_{2n+1}(\sqrt{t})$	22
29.3.48	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$		$ae^{-at}[I_1(at)+I_0(at)]$	9
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$		$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}\, t\right)$	9
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}$	$(k>0)$ 6	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}\, t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$		$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}\, t\right)+I_1\left(\frac{a-b}{2}\, t\right)\right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$		$\frac{1}{t}\, e^{-at}I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$	$(k>0)$	$\frac{k}{t}\, e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}\, t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}$	$(\nu>-1)$	$\frac{1}{a^\nu}\, e^{-\frac{1}{2}at}I_\nu(\frac{1}{2}at)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$		$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}$	$(\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k}$	$(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$	$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t} J_k(at)$		9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$		9
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$		6, 10
29.3.61	$\frac{1}{s} e^{-ks}$	$u(t-k)$		
29.3.62	$\frac{1}{s^2} e^{-ks}$	$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu} e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)} u(t-k)$		6
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s} \tanh ks$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2} \tanh ks$	$tu(t)+2\sum_{n=1}^{\infty} (-1)^n (t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s \sinh ks}$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s \cosh ks}$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty}u(t-2nk)$ 	
.3.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\|\sin kt\|$ 	
.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
9.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
.3.80	$\frac{1}{s^{\mu}}e^{-\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}J_{\mu-1}(2\sqrt{kt})$	9
9.3.81	$\frac{1}{s^{\mu}}e^{\frac{k}{s}}\quad(\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}}I_{\mu-1}(2\sqrt{kt})$	9
9.3.82	$e^{-k\sqrt{s}}\quad(k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
9.3.83	$\frac{1}{s}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
9.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}}\quad(k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
9.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k\geq 0)$	$(4t)^{\frac{1}{2}n}\,\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}}\quad(n=0,1,2,\ldots;k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
9.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}}\quad(k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7

*See page II.

	$f(s)$	$F(t)$	
29.3.89	$\dfrac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})} \quad (k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\dfrac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\dfrac{k}{2\sqrt{t}}$	7
29.3.90	$\dfrac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})} \quad (k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\dfrac{k}{2\sqrt{t}}\right)$	7
29.3.91	$\dfrac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}} \quad (k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\frac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	9
29.3.92	$\dfrac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}} \quad (k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.93	$\dfrac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}} \quad (k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.94	$\dfrac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}} \quad (k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	9
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}} \quad (k>0)$	$\dfrac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks} \quad (k>0)$	$\dfrac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.97	$\dfrac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu} \quad (\nu>-1, k\geq 0)$	$\left(\dfrac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	9
29.3.98	$\dfrac{1}{s}\ln s$	$-\gamma-\ln t\,(\gamma=.57721\ 56649\ldots$ Euler's constant)	
29.3.99	$\dfrac{1}{s^k}\ln s \quad (k>0)$	$\dfrac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	6
29.3.100	$\dfrac{\ln s}{s-a} \quad (a>0)$	$e^{at}[\ln a+E_1(at)]$	5
29.3.101	$\dfrac{\ln s}{s^2+1}$	$\cos t\,\mathrm{Si}\,(t)-\sin t\,\mathrm{Ci}\,(t)$	5
29.3.102	$\dfrac{s\ln s}{s^2+1}$	$-\sin t\,\mathrm{Si}\,(t)-\cos t\,\mathrm{Ci}\,(t)$	5
29.3.103	$\dfrac{1}{s}\ln(1+ks) \quad (k>0)$	$E_1\left(\dfrac{t}{k}\right)$	5
29.3.104	$\ln\dfrac{s+a}{s+b}$	$\dfrac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\dfrac{1}{s}\ln(1+k^2s^2) \quad (k>0)$	$-2\,\mathrm{Ci}\left(\dfrac{t}{k}\right)$	5
29.3.106	$\dfrac{1}{s}\ln(s^2+a^2) \quad (a>0)$	$2\ln a-2\,\mathrm{Ci}\,(at)$	5

	$f(s)$			$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2)$	$(a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}(at)]$	5
29.3.108	$\ln\frac{s^2+a^2}{s^2}$			$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$			$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$			$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$			$\mathrm{Si}(kt)$	5
29.3.112	$e^{k^2s^2}\,\mathrm{erfc}\,ks$	$(k>0)$	7	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\,\mathrm{erfc}\,ks$	$(k>0)$	7	$\mathrm{erf}\,\frac{t}{2k}$	7
29.3.114	$e^{ks}\,\mathrm{erfc}\,\sqrt{ks}$	$(k>0)$	7	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\,\mathrm{erfc}\,\sqrt{ks}$	$(k\geq 0)$	7	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\,\mathrm{erfc}\,\sqrt{ks}$	$(k\geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\mathrm{erf}\,\frac{k}{\sqrt{s}}$		7	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\,\mathrm{erfc}\,\frac{k}{\sqrt{s}}$		7	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks)$	$(k>0)$	9	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s})$	$(k>0)$	9	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks)$	$(k>0)$	9	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s})$	$(k>0)$	9	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right)$	$(k>0)$	9	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks}I_0(ks)$	$(k>0)$	9	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks)$	$(k>0)$	9	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

	$f(s)$		$F(t)$
29.3.126	$e^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as) \quad (a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as) \quad (a>0; n=0,1,2,\ldots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$	5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms [4]

	$\phi(s)$	$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$	$\Phi(t)$
29.4.2	$e^{-ks} \quad (k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}} \quad (k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}} \quad (k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks} \quad (k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks} \quad (k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks \quad (k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)} \quad (k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)} \quad (k>0, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)} \quad (0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^b u[t+h-(2n+1)k] - e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s} \quad (0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x}\delta(t)dt=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.

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$a^2 u_{xx} - u_{tt} = 0$

Hence, the general solution of the wave equation is

$$u = F(x+at) + G(x-at) \qquad (7.6.4)$$

The general solution can be used to solve some problems, but it is a cumbersome approach for others (those handled better by separation of variables). For example, let's consider the problem where the initial conditions are specified, for $-\infty < x \leq +\infty$, as

$$u(x,0) = p(x) \qquad (7.6.5a)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = 0 \qquad (7.6.5b)$$

Applying (7.6.5b) to (7.6.4),

$$a[F'(x) - G'(x)] = 0 \qquad (7.6.6)$$

Therefore,

$$G(x) = F(x) + C_1 \qquad (7.6.7)$$

Now (7.6.5a) requires

$$F(x) + G(x) = p(x) \qquad (7.6.8)$$

Combining with (7.6.7),

$$F(x) = \frac{1}{2}\,p(x) - \frac{1}{2}\,C_1 \qquad (7.6.9a)$$

$$G(x) = \frac{1}{2}\,p(x) + \frac{1}{2}\,C_1 \qquad (7.6.9b)$$

So the solution satisfying (7.6.5) is

$$u(x,t) = \frac{1}{2}\,p(x+at) + \frac{1}{2}\,p(x-at) \qquad (7.6.10)$$

At point (x,t), the quantity $p(x+at)$ will have a value determined by the intercept of the "-" characteristic passing through point (x,t) with the line $t = 0$ (Fig. 7.6.1); similarly, the quantity $p(x-at)$ is constant along the "+" characteristic passing through (x,t). Therefore, for this problem the value of the solution at point 3 in Fig. 7.6.1 depends only upon the values of the initial data at points 1 and 2! The solution at point 3 is merely the average of the initial values at points 1 and 2.

For example, suppose that the initial distribution is a Gaussian pulse

$$u(x,0) = \exp(-x^2) \tag{7.6.11}$$

Then the solution at later times will be

$$u(x,t) = \frac{1}{2}\exp[-(x+at)^2] + \frac{1}{2}\exp[-(x-at)^2] \tag{7.6.12}$$

The solution says that the initial pulse splits into two parts, one which propagates to the left, the other to the right. The center of each pulse moves out along a characteristic line, so each pulse travels at the speed a.

7.7 Imaging in Wave Equation Solutions

Suppose we are interested in the reflection of a wave from a boundary. Eqn. (7.5.1) and the initial conditions (7.6.5) again govern the problem, but now we add the boundary condition

$$\left.\frac{\partial u}{\partial x}\right|_{x=o} = 0 \tag{7.7.1}$$

and restrict our interest to the domain $0 \le x \le \infty$. This problem can be solved by the general solution. We set

$$u = F(x+at) + G(x-at) \tag{7.7.2}$$

The initial conditions (7.6.5) require

$$F(x) + G(x) = p(x) \qquad x \geq 0 \tag{7.7.3a}$$

$$F'(x) - G'(x) = 0 \qquad x \geq 0 \tag{7.7.3b}$$

So (7.6.4) again give F and G, but <u>only</u> for positive arguments! Note that now the functions F and G are <u>not</u> defined for $x < 0$ by the initial conditions. Instead, we have, from (7.7.1),

$$F'(at) + G'(-at) = 0 \tag{7.7.4}$$

This must hold at all times. Therefore, for negative arguments the function G must be such that its derivative is the negative of the derivative of the function F for the same value of positive argument; i.e.,

$$G'(-\sigma) = -F'(\sigma) \quad \text{where } \sigma = at > 0 \tag{7.7.5}$$

This will be the case when G is the <u>mirror image</u> of F (Fig. 7.7.1). In mathematical terms,

$$\begin{aligned} G(-x) &= F(x) = \frac{p(x)}{2} && x>0 \qquad (7.7.6a) \\ G(x) &= F(-x) = \frac{p(-x)}{2} && x<0 \\ F(-x) &= G(x) = \frac{p(x)}{2} && x>0 \qquad (7.7.6b) \\ F(x) &= G(-x) = \frac{p(-x)}{2} && x<0. \end{aligned}$$

Therefore, since the c_1 terms cancel, we can take

$$\begin{aligned} (x+at) &\geq 0 & F(x+at) &= \frac{1}{2}\,p(x+at) \\ (x+at) &< 0 & F(x+at) &= \frac{1}{2}\,p[-(x+at)] \\ (x-at) &\geq 0 & G(x-at) &= \frac{1}{2}\,p(x-at) \\ (x-at) &< 0 & G(x-at) &= \frac{1}{2}\,p[-(x-at)] \end{aligned} \tag{7.7.7}$$

The solution (7.7.2) therefore can be thought of as a combination of four wave packets, as shown in Fig. 7.7.1. The first is half of the $p(x)$

wave, which moves to the right away from the reflecting boundary. The second is the other half of this wave, which moves to the left and passes through the boundary to negative x. The third is the mirror image of the $p(x)$ wave, which starts to the left of the reflecting wall (outside of the real problem) and travels to the right, entering the wall as its "mate" passes through going left. This image wave then appears in the domain of interest as a reflected wave. The fourth wave is the other half of the image $p(x)$ wave, which travels to the left and never enters the domain of interest.

Wave-equation solutions obtained by these imaging methods must be represented segmentally. If there are only one or two segments, this is not too difficult and is a convenient way to get the solution. However, if there are many reflections, such as would be the case for the solution of standing acoustic waves in a duct or the long-term vibration of a finite string, the approach becomes very cumbersome and the separation of variables technique usually is easier to execute and present.

7.8 Characteristics for the Laplace Equation

For the Laplace equation,

$$u_{xx} + u_{yy} = 0 \tag{7.8.1}$$

the characteristic slopes are $y' = \pm i$, so the characteristics are given by

$$x + iy = \text{const.} \quad \text{and} \quad x - iy = \text{const.} \tag{7.8.2}$$

On the surface this does not appear too useful, because the characteristics are not lines in the real x-y plane. However, we can learn something by transforming the equation to new variables ξ,η such that

$$\xi = x + iy \ , \qquad \eta = x - iy$$

$$u_x = u_\xi + u_\eta \ , \qquad u_y = i(u_\xi - u_\eta)$$

where $C^2 = kp/\rho$ is the isentropic sound speed. The independent variables are the velocity V, pressure p, and density ρ.

Develop the expressions for the slopes of the characteristics, and write pseudo-IDEs that apply on each characteristic. Organize an approximate numerical algorithm to solve this problem marching forward in time, using the method of characteristics.

7.4. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial conditions $u(x,0) = 0$, $u_t(x,0) = \exp(-x^2)$ in $-\infty \leq x \leq +\infty$. Derive an expression for the solution using the general solution of the wave equation.

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} e^{-\sigma^2} d\sigma = \frac{\sqrt{\pi}}{4}\left[\mathrm{erf}(x+t) - \mathrm{erf}(x-t)\right]$$

7.5. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial and boundary conditions

$$u(x,0) = xe^{-x} \qquad 0 \leq x \leq \infty$$

$$u_t(x,0) = 0$$

$$u(0,t) = 0$$

Develop (segmental) expressions for the solution to this problem in $0 \leq x \leq \infty$, and give an expression for the solution at $t = 1$. Interpret in terms of right- and left-running waves, using a sketch.

7.6. Consider the wave equation $u_{xx} - u_{tt} = 0$, with the initial and boundary conditions

$$u(x,0) = 0 \qquad 0 \leq x \leq 1$$

cos( )t terms disappear.

$$u_t(x,0) = \begin{cases} 1 & 0 \leq x \leq 1/2 \\ 0 & x \geq 1/2 \end{cases}$$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(1,t) &= 0 \end{aligned}\right\} \text{sin \& cos}$$

Develop this solution by the method of characteristics and by separation of variables, and compare.

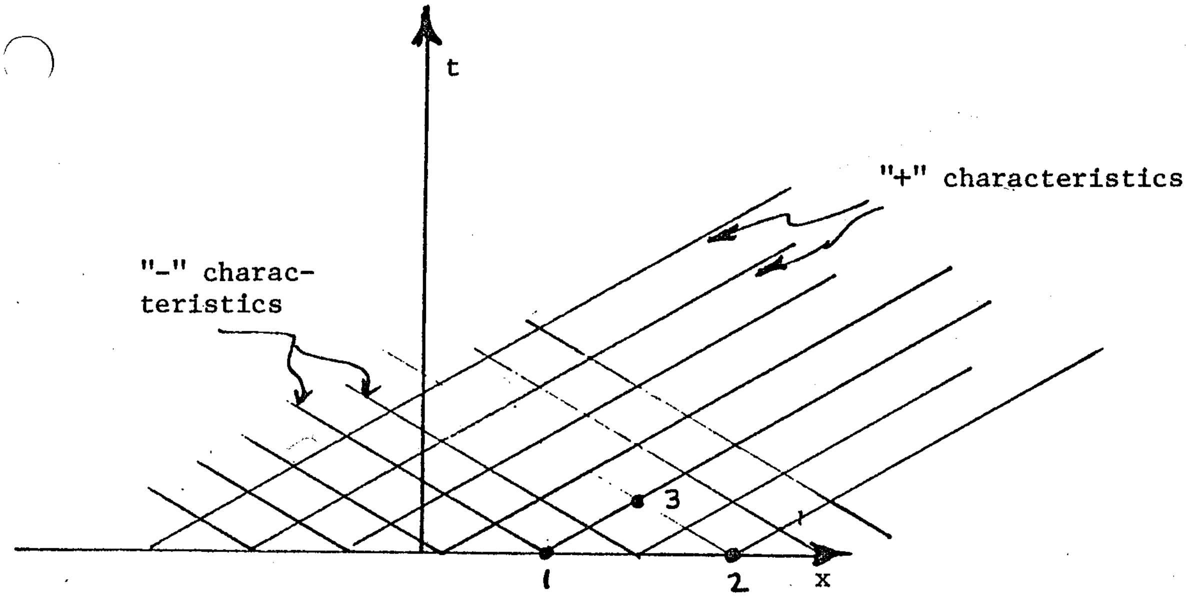


Fig. 7.5.1 The two sets of characteristics

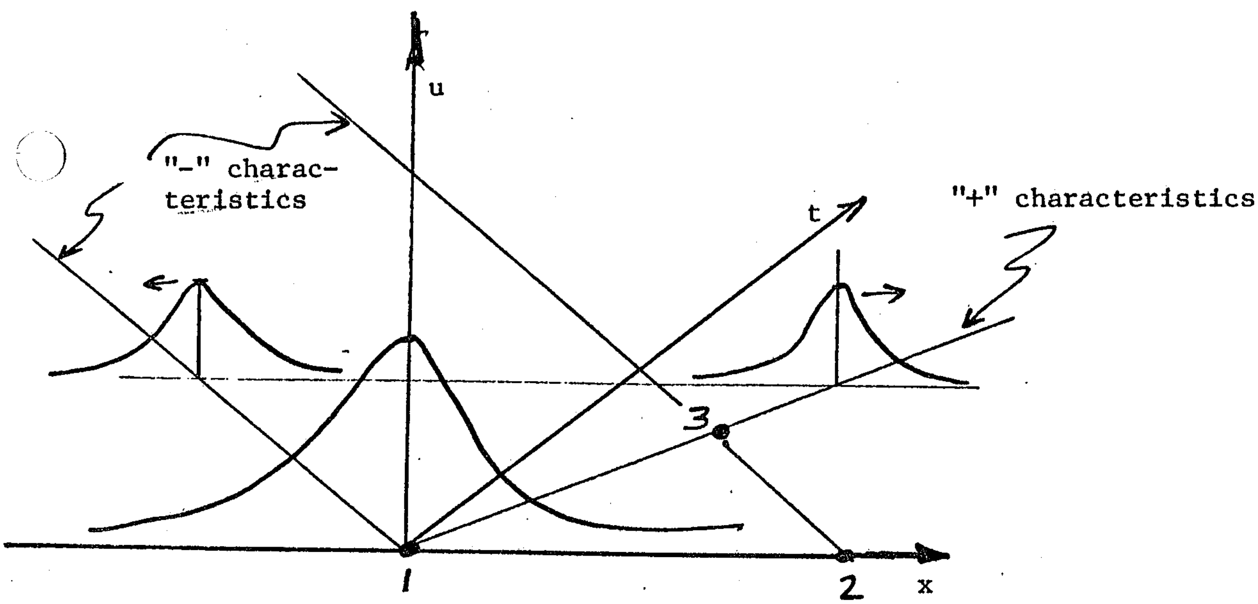


Fig. 7.6.1. Solution for an initial pulse

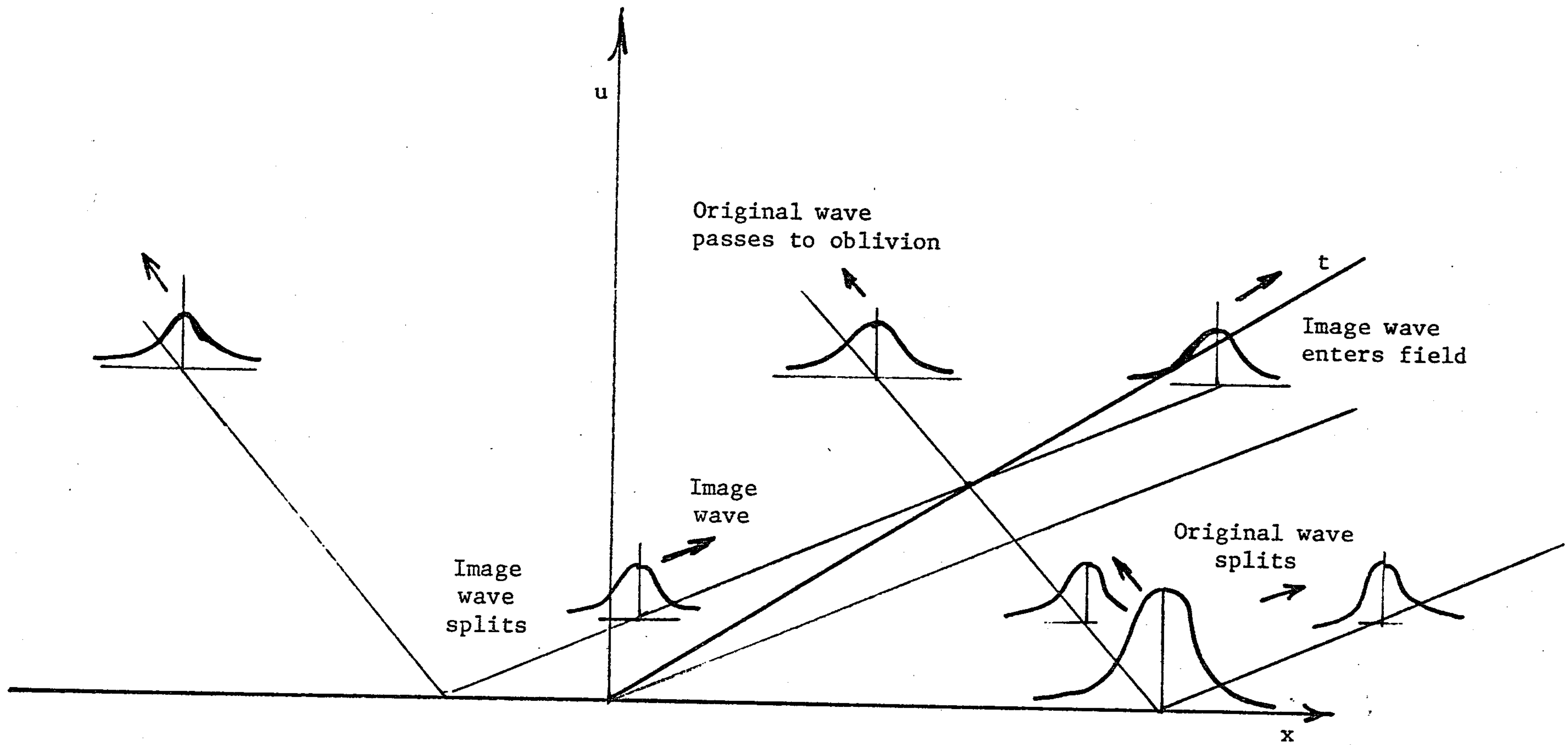


Fig. 7.7.1. Solution decomposition

HW #1 GIVEN END FIRST LECTURE

1. Solve $y' + \frac{1}{x} y = 3\cos 2x$

2. Solve $\frac{dy}{dx} = \frac{1}{e^y - x}$. Hint: Consider x as the dependent variable instead of y

3. Solve $xy' + y = x^2 - x + 1$ with $y(x=1) = 1$

4. Solve $y' + (\cot x) y = 2 \csc x$ with $y(\pi/2) = 1$

HW #2.

1. Show that e^x and e^{-x} form a fundamental set of solutions of the DE

$$y'' - y = 0$$

Hint: show that $W(y_1, y_2) \neq 0$ for any value of x

~~HW #2~~

2. Verify that $y_1(x) = x^{-1/2} \sin x$ is one solution of Bessel's equation of order 1/2

$$x^2 y'' + xy' + (x^2 - 1/4) y = 0$$

and determine a second solution via the method of reduction of order. Consider $0 < x < \infty$.

3. Determine the general solution to the following DE's.

a. $6y'' - y' - y = 0$

b. $4y'' + 4y' + y = 0$

c. $y'' - 2y' + 6y = 0$

d. $y'' + 2y' + 2y = 0$

FOR 4 [redacted] use the method of undetermined coefficients

4. Solve the following non-homogeneous differential equations

a. $u'' + \omega_0^2 u = \cos \omega t \qquad \omega \neq \omega_0$

b. $u'' + \omega_0^2 u = \cos \omega_0 t$

Is the particular solution of (4b) bounded as $t \to \infty$?

HW #3

1. Solve the differential equation by method of undetermined coefficients

$$u'' + \mu u' + \omega_0^2 u = \cos \omega t \qquad \mu^2 - 4\omega_0^2 < 0$$

2. Using the method of variation of parameters solve

a. $y'' - 5y' + 6y = 2e^x$

b. $y'' + y = \sec x$

3. Show that the particular solution to

$$y'' - 5y' + 6y = g(x)$$

is

$$y_p(x) = \int^x \left[e^{3(x-t)} - e^{2(x-t)} \right] g(t)\, dt$$

As a check let $g(x) = 2e^x$ and see if the result of [redacted] (3) match the results of (2a).

4. Consider the ODE

$$m\ddot{u} + c\dot{u} + ku = F_0 \cos \omega t \qquad \text{and let} \quad k/m = \omega_0^2$$

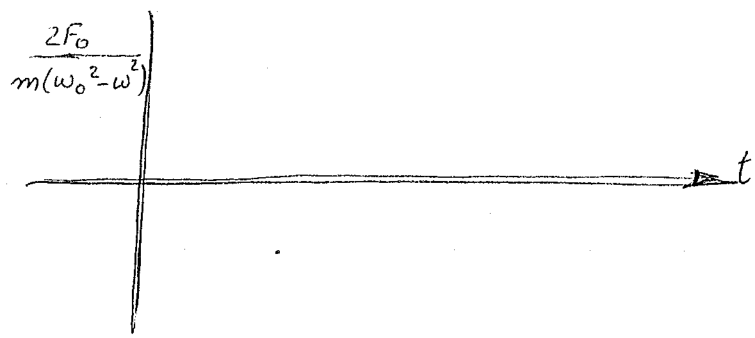
with initial conditions $u(0) = 0 \quad \dot{u}(0) = 0$

a. show that when $c = 0$, the constants of the homogeneous equation are $C_1 = \dfrac{-F_0}{m(\omega_0^2 - \omega^2)}$ and $C_2 = 0$ if $\omega_0 \neq \omega$

b. Show that ~~this~~ the solution can be written as

$$u = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{\omega_0 - \omega}{2} t \ \sin \frac{\omega_0 + \omega}{2} t .$$

c. Plot the results ~~where~~ for $0 \leq t \leq \frac{2\pi}{(\omega_0 - \omega)/2}$ when u is Normalized by $\frac{2F_0}{m(\omega_0^2 - \omega^2)}$ and let $\omega_0 = 1.1\,\omega$



HW #1

1. Determine the region in which

$$u_{xx} + y\,u_{yy} = 0$$

in hyperbolic, elliptic or parabolic.

Find $\varphi(x,y) = \text{constant}$ for the hyperbolic & parabolic case

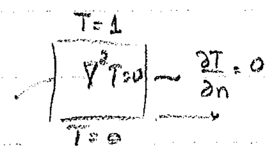
2. Determine the type of $\varphi(x,y) = \text{const}$

a. $x\,u_{xx} + 2\sqrt{xy}\,u_{xy} + y\,u_{yy} - u_x = 0$

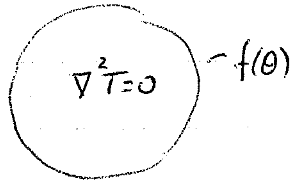
b. $u_{xx} = \frac{1}{a^2} u_y + \alpha u + \beta u_x$

c. $u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$

HW #2. Do



HW #3



HW #4 $T = T_1$ $\quad\Big|\; T_{xx} = \frac{1}{\alpha^2} T_t \;\Big|\quad$ $h\ell \frac{\partial T}{\partial x} + T = T_0$

HW #5 $T_{xx} = \frac{1}{\alpha} T_t$

$-k\,T_x = q$ @ $x = 0$

$T(x,0) = T_i$

$T(x,t) \Longrightarrow T_i$ as $x \to \infty$

Self symilar

HW #6 Do hw #5 using Laplace Transform

HW #1 Soln.

1. $u_{xx} + y\,u_{yy} = 0$ $\qquad a=1 \quad b=0 \quad c=y$

$b^2 - 4ac = 0 - 4\cdot1\cdot y = -4y$

if $y<0$ hyperbolic

$y=0$ parabolic

$y>0$ elliptic

$$\frac{dy}{dx} = \frac{b \pm \sqrt{b^2-4ac}}{2a} = \frac{\pm\sqrt{-4y}}{2} = \pm\sqrt{-y}.$$

Now hyperbolic let $\dfrac{dy}{\pm\sqrt{-y}} = dx \implies \mp 2\sqrt{-y} = x + c$

$\therefore \quad C_1 = 2\sqrt{-y} - x \qquad C_2 = 2\sqrt{-y} + x$

for parabolic

$\dfrac{dy}{dx} = 0 \implies dy = 0$ or $y = c$ but since $y=0$ for parabolic case $\implies C=0$

2a. $x u_{xx} + 2\sqrt{xy}\,u_{xy} + y\,u_{yy} - u_x = 0$ $\qquad a=x \quad b=2\sqrt{xy} \quad c=y$

$b^2-4ac = 4xy - 4xy = 0$ thus this is a parabolic PDE

$$\therefore \frac{dy}{dx} = \frac{b\pm 0}{2a} = \frac{2\sqrt{xy}}{2x} = \frac{\sqrt{y}}{\sqrt{x}}$$

or $\dfrac{dy}{\sqrt{y}} = \dfrac{dx}{\sqrt{x}}$ or $2\sqrt{y} = 2\sqrt{x} + c$ or $C = (\sqrt{y} - \sqrt{x})$

2b. $u_{xx} - \frac{1}{a^2}u_y + \beta u_x - \alpha u = 0$ $\qquad a=1 \quad b=0 \quad c=0$

$b^2 - 4ac = 0 - 4\cdot1\cdot0 = 0$ this is parabolic

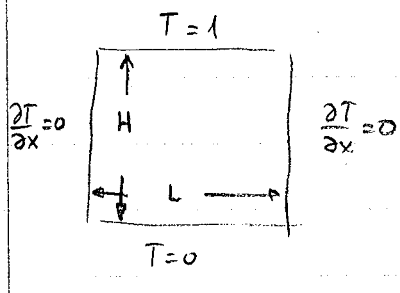
$\therefore \dfrac{dy}{dx} = \dfrac{b\pm 0}{2a} = 0 \quad \therefore dy = 0$ or $y =$ constant

2c $u_{xx} + 2u_{xy} + 4u_{yy} + 2u_x + 3u_y = 0$ $\qquad a=1 \quad b=2 \quad c=4$

$b^2 - 4ac = 4 - 4\cdot1\cdot4 = -12$ this is an elliptic PDE

$\therefore$ no real characteristic curves $\varphi(x,y)$ exist.

1	a, b, c 1 ea	3	
	$b^2 - 4ac$ 1	1	
	hyp, par, ellip 1 each	3	
	$\frac{dy}{dx}$ hyper 1 each	2	
	φ_1, φ_2 1 each	2	
	$\frac{dy}{dx}$ para 1	1	
	φ_1	1	
	$C = 0$	1	14
2a	a, b, c	3	
	$b^2 - 4ac$	1	
	$\frac{dy}{dx}$	1	
	φ_1	1	6
2b.	a, b, c	3	
	$b^2 - 4ac$	1	
	$\frac{dy}{dx}$ =	1	
	φ	1	6
2c	a, b, c	3	
	$b^2 - 4ac$	1	
	no real char	1	5
			31



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad \text{let } T = F(x)\,G(y) \qquad 1$$

$$\Rightarrow \quad F''G + FG'' = 0 \quad \text{or} \quad \frac{F''}{F} = -\frac{G''}{G} = -k^2$$

From $\frac{F''}{F} = -k^2 \Rightarrow F'' + k^2F = 0$ or $F(x) = A\cos kx + B\sin kx$ if $k \neq 0$ 1

$F'' = 0$ or $F(x) = \bar{A}x + \bar{B}$ if $k = 0$ 1

From $-\frac{G''}{G} = -k^2 \Rightarrow G'' - k^2G = 0$ or $G(y) = C\sinh ky + D\cosh ky$ if $k \neq 0$ 1

$G'' = 0$ or $G(y) = \bar{C}y + \bar{D}$ if $k = 0$ 1

B.C.

$\frac{\partial T}{\partial x} = 0$ on $x = 0$ $\quad \frac{\partial T}{\partial x} = F'(0)G(y) = 0 \Rightarrow F'(0) = 0$ 1

$\frac{\partial T}{\partial x} = 0$ on $x = L$ $\quad \frac{\partial T}{\partial x} = F'(L)G(y) = 0 \Rightarrow F'(L) = 0$ 1

$T = 0$ on $y = 0$ $\quad T = F(x)G(0) = 0 \Rightarrow G(0) = 0$ 1

for $k \neq 0$ $\quad F'(x) = k\left[-A\sin kx + B\cos kx\right]$ $\quad F'(0) = 0 \Rightarrow B = 0$ 1

$F'(L) = 0 \Rightarrow A = 0,\ k = 0$ or $kL = n\pi$ 3

$A = 0$ leads to $T = 0$ everywhere which contradicts B.C. $\therefore\ F(x) = A\cos\frac{n\pi}{L}x$

Now $k = 0$ is a possibility.

Look at $F(x) = \bar{A}x + \bar{B}$ $\quad F'(x) = \bar{A}$ $\quad F'(0) = 0 \Rightarrow \bar{A} = 0$ 1

with $\bar{A} = 0$ $\quad F'(L) = 0$ automatically $\therefore\ F(x) = \bar{B}$ for $k = 0$ 1

for $k \neq 0$ $\quad G(y) = C\sinh ky + D\cosh ky$ and $G(0) = 0 \Rightarrow D = 0 \quad \therefore\ G(y) = C\sinh ky$ 1

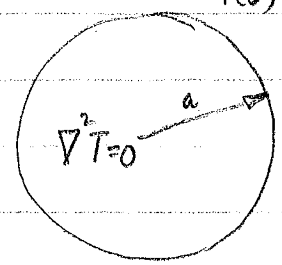
$k = 0$ $\quad G(y) = \bar{C}y + \bar{D}$ and $G(0) = 0 \Rightarrow \bar{D} = 0 \quad \therefore\ G(y) = \bar{C}y$ 1

$\therefore$

$$T = \bar{B}\bar{C}y + \sum_{n=1}^{\infty} A_nC_n \cos\frac{n\pi}{L}x\,\sinh\frac{n\pi}{L}y = \mathscr{C}y + \sum_{n=1}^{\infty} Q_n\cos\frac{n\pi x}{L}\,\sinh\frac{n\pi}{L}y \qquad 2$$

here $\mathscr{C} = \bar{B}\bar{C}$ and $Q_n = A_nC_n$

$f(\theta)$ periodic on the boundary



$$\nabla^2 T = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0$$

let $T = R(r)G(\theta)$ 1

$$\therefore \quad \nabla^2 T = R''G + \frac{1}{r}R'G + \frac{1}{r^2}RG'' = 0$$

Separate $\therefore \; r^2\frac{R''}{R} + \frac{r^2}{r}\frac{R'}{R} + \frac{G''}{G} = 0$ or $r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{G''}{G} = +k^2$

From this if $k \neq 0$ $\quad G'' + k^2 G = 0 \Rightarrow G = A\cos k\theta + B\sin k\theta$ 1

$r^2R'' + rR' - k^2R = 0 \Rightarrow R = Cr^k + Dr^{-k}$ 1

if $k = 0$

$G'' = 0 \Rightarrow \bar{A} + \bar{B}\theta$ 1

$r^2R'' + rR' = 0 \Rightarrow \bar{C} + \bar{D}\ln r$ 1

For bounded solutions at $r = 0 \Rightarrow D, \bar{D} = 0$ 2

For periodic solutions ie. $T(r,\theta) = T(r, \theta + 2\pi) \Rightarrow k = n$ and $\bar{B} = 0$ 2

$$\therefore \quad T(r,\theta) = \bar{A}\bar{C} + \sum_{n=1}^{\infty} r^n\left[CA\cos n\theta + CB\sin n\theta\right]$$ 3

let $\bar{A}\bar{C} = \mathcal{C}_0 \qquad CA = \mathcal{C}_n \qquad CB = \mathcal{D}_n$

$$T(r,\theta) = \mathcal{C}_0 + \sum_{n=1}^{\infty} r^n\left[\mathcal{C}_n\cos n\theta + \mathcal{D}_n \sin n\theta\right]$$

at $r = a$ $\quad T(a,\theta) = f(\theta) = \mathcal{C}_0 + \sum_{n=1} a^n\left[\mathcal{C}_n\cos n\theta + \mathcal{D}_n\sin n\theta\right]$

from fourier series, for a periodic function

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1} a_n \cos n\theta + \sum b_n \sin n\theta \qquad \text{where } a_0 = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\,d\theta$$

$a_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta$ and $b_n = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$. If we identify

3

$\mathcal{C}_0$ with $\frac{a_0}{2}$ $\quad \mathcal{C}_n a^n$ with a_n and $\mathcal{D}_n a^n$ with b_n and solve for $\mathcal{C}_0$, $\mathcal{C}_n$ and $\mathcal{D}_n$ we have the complete solution.

thus $T(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \left[a_n \cos n\theta + b_n \sin n\theta\right]$

where a_0, a_n & b_n are defined as on the other page.

Given

$T = T_1$ (at $x=0$), $\quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$ (for $0 < x < L$), $\quad h\ell \frac{\partial T}{\partial x} + (T - T_0) = 0$ (at $x = L$)

$T = f(x)$ @ $t=0$

find : T

WRITE THIS AS

$T = T_1$, $\quad \frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$, $\quad h\ell\frac{\partial T}{\partial x} + T = T_0$ (on 0 to L)

$T = f(x)$ @ $t=0$

Now let $\quad T = T_h + T_p \qquad T_h$: homogeneous solution

T_p : particular solution

For particular solution

(1) $T_p = T_1$ (at 0), $\quad h\ell\frac{\partial T_p}{\partial x} + T_p = T_0$ (at L)

Note BC here are not zero

For homogeneous

(2) $T_h = 0$ (at 0), $\quad h\ell\frac{\partial T_h}{\partial x} + T_h = 0$ (at L)

$p(x) = f(x) - T_p$ @ $t=0$

Note bc here are zero

(1) choose $\underline{T_p = \bar{A}x + \bar{B}}$, Note that $\frac{\partial^2 T_p}{\partial x^2} = 0$ & $\frac{\partial T_p}{\partial t} = 0 \quad \therefore$ it satisfies $\underline{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t} = 0}$

apply BC's of problem(1)

$T_p(x=0) = \bar{B} = T_1 \quad \Rightarrow \quad T_p = \bar{A}x + T_1$

$h\ell\frac{\partial T_p}{\partial x} + T_p = h\ell\bar{A} + (\bar{A}\ell + T_1) = T_0$ at $x = \ell \quad \Rightarrow \quad \underline{\bar{A} = \frac{T_0 - T_1}{\ell(1+h)}}$

$$\therefore \quad T_p = T_1 + \frac{T_0 - T_1}{1+h}\frac{x}{\ell} \quad \text{or} \quad \underline{T_1 - \frac{T_1 - T_0}{1+h}\frac{x}{\ell}}.$$

this is also the solution at $t=0$

the object here is to choose a solution that satisfies the BC. regardless of the PDE $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}$. If it satisfies it, great! If not, then we would need to do something else. In this case it does, so now we solve part (2)

(2) let $T_h = F(x)G(t)$ and put into $\frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T_h}{\partial t} \Rightarrow F''G = \frac{1}{\alpha^2}FG'$ or $\frac{F''}{F} = \frac{G'}{\alpha^2 G} = -1$

for $k=0$ case $\quad F'' = 0 \quad \Rightarrow \quad F' = \bar{A} \quad$ & $\quad \underline{F(x) = \bar{A}x + \bar{B}}$

$G' = 0 \quad \Rightarrow \quad \underline{G(t) = \bar{C}}$

for $k \neq 0 \quad F'' + k^2 F = 0 \quad$ and $\quad \underline{F(x) = A\cos kx + B\sin kx}$

Now look at BC's of problem ②

also $G' + \alpha^2 k^2 G = 0 \Rightarrow \underline{G(t) = Ce^{-\alpha^2 k^2 t}}$

Now $T_h(x=0,t) = 0$; $T_h(x,t) = F(x)G(t)$ & $T_h(0,t) = F(0)G(t) = 0$ for all t

$\Rightarrow \underline{F(0) = 0}$

Now $h\ell \dfrac{\partial T_h}{\partial x} + T_h = 0$ @ $x = \ell$; $T_h = FG$ $\dfrac{\partial T_h}{\partial x} = F'G$ since $\dfrac{\partial}{\partial x} G(t) = 0$

$\therefore h\ell F'(\ell)G + F(\ell)G = G(t)\left[h\ell F'(\ell) + F(\ell)\right] = 0$ for all $t \Rightarrow \underline{h\ell F'(\ell) + F(\ell) = 0}$

<u>For $k \neq 0$ case</u>

$F(0) = A\cos k\cdot 0 + B\sin k\cdot 0 = \underline{A = 0} \Rightarrow F(x) = B\sin kx$

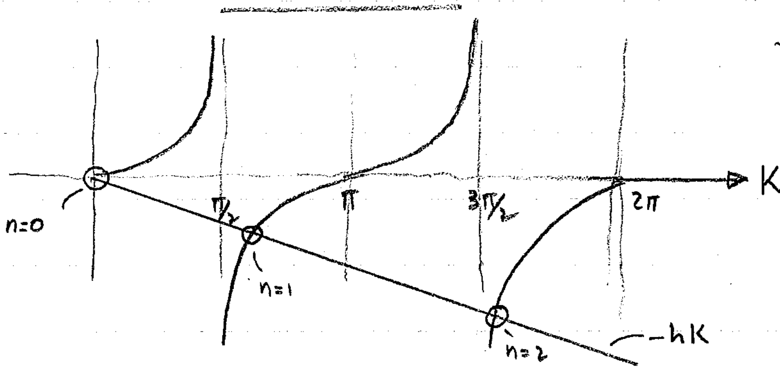
$h\ell F'(\ell) + F(\ell) = h\ell\left[Bk\cos k\ell\right] + B\sin k\ell = B\left[h(k\ell)\cos(k\ell) + \sin(k\ell)\right] = 0$ let $\underline{K = k\ell}$

12 + 2 fig

either $B = 0$ or $hK\cos K + \sin K = 0$. $B = 0 \Rightarrow F(x) = 0 \Rightarrow T(x,t) = 0$

which doesn't satisfy the initial condition. $\therefore hK\cos K + \sin K = 0 \Rightarrow \cos K\left[hK + \tan K\right] = 0$

$\Longrightarrow \underline{hK + \tan K = 0}$



the circled pts satisfy $\tan K = -hK$

but there are an ∞ no. of solutions

$\therefore \tan K_n = -hK_n \quad n = 1, 2, \ldots$

and $K_n = k_n \ell \Rightarrow k_n = \dfrac{K_n}{\ell}$

$\therefore F(x) = B\sin \dfrac{K_n}{\ell} x$ where K_n is a solution to $\tan K = -hK$

$$\therefore T_h(x,t) = \sum_{n=1}^{\infty} BC\, e^{-\alpha^2 \frac{K_n^2}{\ell^2} t^2} \sin \frac{K_n x}{\ell} \qquad K_n \neq 0$$

Now we check the $k = 0$ case to see if it contributes

<u>For $k = 0$ case</u>

$F(0) = 0 \Rightarrow \bar{A}\cdot 0 + \bar{B} = 0$ or $\underline{\bar{B} = 0}$ $\quad F'(x) = \bar{A}$

$0 = h\ell F'(\ell) + F(\ell) = h\ell\bar{A} + \bar{A}\ell = \bar{A}\,\ell(1+h) \Rightarrow \underline{\bar{A} = 0}$ $\quad \therefore \underline{F(x) = 0 \text{ which is not a solution}}$

$\therefore$ $n = 0$ pt on the graph $\Rightarrow k = 0$ & $K = 0$ is not a solution

Now $\quad T_h = \sum_{n=1}^{\infty} \mathscr{C}_n e^{-\frac{\alpha^2 K_n^2}{l}t} \sin \frac{K_n}{l} x \qquad$ where $\mathscr{C}_n = BC$

now look at the initial condition

$$T(x,t=0) = f(x) = T_p(x,t=0) + T_h(x,t=0)$$

but $T_p(x,t=0) = T_1 - \left(\frac{T_1 - T_0}{1+h}\right)\frac{x}{l} \qquad \therefore \qquad T_h(x,t=0) = f(x) - T_p$

Now $\underline{T_h(x,t=0)} = \sum_{n=1}^{\infty} \mathscr{C}_n e^0 \sin \frac{k_n x}{l} = \underline{\sum_{n=1}^{\infty} \mathscr{C}_n \sin \frac{k_n x}{l}} = f(x) - T_p = p(x)$

As with Fourier series, we can write $\underline{p(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)}$ if the $\varphi_n(x)$ fns are linearly independent

5 + 1 fig

They are linearly independent if $\underline{\int_0^l \varphi_n(x)\,\varphi_m(x)\,dx = 0 \text{ for } m \neq n}$

but when is this true? when $\underline{\varphi_n(x)}$ satisfies the condition $\left.\begin{array}{l}\varphi = 0 \text{ or} \\ \varphi' = 0 \text{ or} \\ \varphi + \text{const}\cdot\varphi' = 0\end{array}\right\}$ at either $x=0$ or $x=l$

in our case $\varphi_n(x) = \sin \frac{K_n x}{l}$ which satisfies $\varphi_n(0) = 0$ and $\varphi_n(l) + h\varphi_n'(l) = 0$

also if $p(x) = \sum_{n=1}^{\infty} a_n \varphi_n(x)$, then $\int_0^l p(x)\varphi_n(x)\,dx = a_n \int_0^l \varphi_n^2(x)\,dx$

and $a_n = \dfrac{\int_0^l p(x)\varphi_n(x)\,dx}{\int_0^l \varphi_n^2(x)\,dx}$ in our case $a_n = \mathscr{C}_n = \underline{\dfrac{\int_0^l p(x) \sin \frac{K_n}{l} x\,dx}{\int_0^l \sin^2 \frac{K_n x}{l}}}$

note that $\int_0^l \sin^2 \frac{K_n x}{l} dx \neq l/2$

Read the beginning section 5.7, then read section 5.6 in your books.

TOTAL: 27 gradables

Given $\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$

also $-k\frac{\partial T}{\partial x} = q$ when $x=0$

$T(x,0) = T_i$

$T(x,t) \to T_i$ as $x \to \infty$

} From these, let $\eta = \frac{Ax}{t^n}$

note for small t and large x $T \to T_i$

also let $T/T_i = Bf(\eta) \Rightarrow T = T_i B f(\eta)$ AS A GUESS. Now look @ $\frac{\partial T}{\partial x}$ @ $x=0$ $(\eta=0)$

$\therefore \frac{\partial T}{\partial x} = T_i B f'(\eta)\cdot\frac{\partial \eta}{\partial x} = T_i B f'(\eta)\cdot\frac{A}{t^n}$; at $x=0$ $\frac{\partial T}{\partial x} = -\frac{q}{K} = BT_i f'(\eta=0)\cdot\frac{A}{t^n}$

Since the left hand side is constant but the right hand side is note (due to t^n), we must add a degree of freedom. let $T/T_i = Bt^m f(\eta)$

Now $\frac{\partial T}{\partial x} = T_i B t^m f'(\eta)\frac{\partial \eta}{\partial x} = T_i B t^m f'(\eta)\cdot\frac{A}{t^n}$; @ $x=0$ $\frac{\partial T}{\partial x} = -\frac{q}{K} = T_i B A f'(0)\frac{t^m}{t^n}$

For both sides to be a constant $m=n$. BUT NOW CHECK TO SEE IF THIS SOLUTION SATISFIES THE OTHER CONDITIONS.

since $T(x,0) = T_i \Rightarrow T/T_i = 1 = B\cdot 0^m f(\eta\to\infty)$ note as $t\to 0$ $\eta = \frac{Ax}{t^n} \to \infty$

$\Rightarrow t^m\cdot f(\eta\to\infty)$ must be a constant (Remember t can be any value - it is a variable)

also since $T(x,t) \to T_i$ as $x\to\infty \Rightarrow T/T_i \to 1 = Bt^m f(\eta\to\infty)$ irrespective of t which would imply again that $t^m f(\eta\to\infty)$ must be a constant.

This causes a problem, for suppose $f(\eta) = A_k\eta^k + A_{k-1}\eta^{k-1} + A_{k-2}\eta^{k-2} + \ldots$, A power series in η, that would imply that $t^m f(\eta\to\infty) =$ constant cannot be satisfied. Therefore unlike what I said in class $T/T_i \neq Bt^m f(\eta)$.

NEW GUESS: suppose $\frac{T}{T_i} = C + Bt^m f(\eta)$ then $\frac{\partial T}{\partial x} = Bt^m f'(\eta)\cdot\frac{A}{t^n}$ as before

and

also as $T\to T_i$ for $x\to\infty$ (ie $\eta\to\infty$) $\Rightarrow \frac{T}{T_i} \to 1 = C + Bt^m f(\eta\to\infty)$

but since this must be true for any $t \Rightarrow f(\eta\to\infty) \equiv 0$ and $C=1$.

Also since at $t=0$ $T(x,0) = T_i$ $T/T_i = 1 = C + B\cdot 0\cdot f(\eta\to\infty) = C \therefore C=1$

Now lets look at the reason why this guess worked. Since the temperature initially was $T=T_i \Rightarrow T(x,t) = T_i +$ something, for $t>0$. This something is the function

we want. Therefore as a general rule look for the change in temperature $(T-T_i)$ and set that equal to $Bf(\eta)$, if all the conditions are given on T, or set $T-T_i = Bt^m f(\eta)$ if one of the conditions is given on $\frac{\partial T}{\partial x}$.

So for $T/T_i = C + Bt^m f(\eta) = 1 + Bt^m f(\eta)$

$$\frac{\partial T}{\partial x} = T_i B t^m f'(\eta)\frac{\partial \eta}{\partial x}, \text{ where } \frac{\partial \eta}{\partial x} = \frac{A}{t^n}, \text{ and at } x=0\ (\eta=0): \frac{\partial T}{\partial x} = \frac{-q}{K} = T_i B t^m f'(0)\cdot\frac{A}{t^n}$$

so for $\frac{\partial T}{\partial x}$ = constant take $m=n$ and $\left.\frac{\partial T}{\partial x}\right|_{x=0} = -\frac{q}{K} = T_i BA f(0)$

Now choose $T_i BA = -\frac{q}{K}$ and $\Rightarrow f'(0) = 1$. Remember that $f(\eta\to\infty) = 0$ from before. These are the two conditions on f needed.

Now $\frac{\partial^2 T}{\partial x^2} = T_i BA^2 f''\cdot\frac{t^n}{t^{2n}} = T_i BA^2 f'' t^{-n}$ since $m=n$

also

$$\frac{\partial T}{\partial t} = T_i B m t^{m-1} f + Bt^m f'\frac{\partial \eta}{\partial t} \quad \text{and } \frac{\partial \eta}{\partial t} = -\frac{nAx}{t^{n+1}} = -\frac{n\eta}{t}$$
$$= T_i B t^{n-1}[nf - n\eta f'] \quad \text{since } m=n$$

$\therefore$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \Rightarrow T_i BA^2 f'' t^{-n} = \frac{1}{\alpha} T_i B\, t^{n-1}[nf - n\eta f']$$

for this to be an ODE $\Rightarrow$ powers of t must be the same $\Rightarrow -n = n-1 \Rightarrow$

$$n = \tfrac{1}{2} = m$$

$\Rightarrow T_i B t^{-1/2}\left[A^2 f'' + \frac{1}{2\alpha}\eta f' - \frac{1}{2\alpha} f\right] = 0$ if we choose $\frac{1}{2\alpha A^2} = 1 \Rightarrow A = \frac{1}{\sqrt{2\alpha}}$

and

$f'' + \eta f' - f = 0$ which is the ODE with conditions $f'(0) = 1$ & $f(\infty) = 0$

Now since

$$T_i BA = -q/K \quad B = \frac{-q}{KT_i A} = -\frac{q\sqrt{2\alpha}}{KT_i}$$

Thus $T/T_i = 1 - \frac{q}{KT_i}\sqrt{2\alpha}\, t^m f(\eta) = 1 - \frac{q}{KT_i}\sqrt{2\alpha t}\, f(\eta)$ since $m = \tfrac{1}{2}$

We have already seen the above ODE and we've found the solution as

$$f(\eta) = C_1\eta + C_2\eta\int_\infty^\eta \frac{1}{s^2} e^{-s^2/2}\, ds$$

NOTE $\frac{d}{d\eta}\int_\infty^\eta \frac{1}{s^2} e^{-s^2/2} ds = \frac{1}{\eta^2} e^{-\eta^2/2}$

By application of $f'(0) = 1$ and $f(\eta=\infty) = 0$ we can find C_1 & C_2.

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PROBLEM: It is required to find the solution of the equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0$$

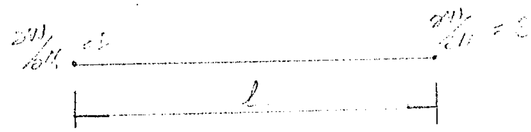
which satisfies the conditions:

$$\left.\begin{aligned} \frac{\partial w}{\partial x}(0,t) &= A \\ \frac{\partial w}{\partial x}(l,t) &= B \end{aligned}\right\} \text{ for all values of } t \quad 0 \le x \le l$$

$$\left.\begin{aligned} w(x,t=0) &= w_0 = 5x \\ \frac{\partial w}{\partial t}(x,t=0) &= w_1 = 0 \end{aligned}\right\} \quad 0 < x < l$$

Solution: since $w(x,t) = w_p(x,t) + w_h(x,t)$

<1> First look at w_p.



assume $\partial w_p/\partial x = Cx + D$

use B.C. to determine the C & D

$\partial w_p/\partial x\,(0,t) = C\cdot 0 + D = A \quad \Rightarrow \quad \underline{D = A} \quad \Rightarrow \quad \partial w/\partial x = Cx + A$

$\partial w_p/\partial x\,(l,t) = C\cdot l + A = B \quad \Rightarrow \quad C = \dfrac{B-A}{l}$

$\Rightarrow \quad \underline{\partial w_p/\partial x = \dfrac{B-A}{l}x + A}$ ✓ (It satisfies the B.C.)

$\Rightarrow \quad \underline{w_p = \dfrac{B-A}{2l}x^2 + Ax + f(t)}$ (w_p must satisfies the P.D.E)

$\partial w_p/\partial x = \dfrac{B-A}{l}x + A \qquad \dfrac{\partial^2 w_p}{\partial x^2} = \dfrac{B-A}{l}$

$\dfrac{\partial w_p}{\partial t} = f'(t) \qquad \dfrac{\partial^2 w_p}{\partial t^2} = f''(t)$

since $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2 w}{\partial t^2}$

$\Rightarrow \quad \dfrac{B-A}{l} = \dfrac{1}{c^2} f''(t) \quad \Rightarrow \quad f'(t) = c^2 \cdot \dfrac{B-A}{l} t + E$

$\Rightarrow \quad f(t) = \dfrac{c^2}{2}\dfrac{B-A}{l}t^2 + Et$

Therefore $\Rightarrow$

$$\underline{w_p = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\frac{B-A}{l}t^2 + Et} \checkmark \qquad \langle\langle 1 \rangle\rangle$$

zhao xin

$(w\ldots \quad \frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} \quad (w = B)$

$\frac{\partial w}{\partial x} = A \qquad \longleftarrow l \longrightarrow \qquad \frac{\partial w}{\partial x} = B$

$\left.\begin{array}{l} w(x, t=0) = w_0(x) \\ \frac{\partial w}{\partial t}(x, t=0) = w_1(x) \end{array}\right\}$ for while u

$\parallel$

$(w_p = A) \quad \frac{\partial^2 w_p}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_p}{\partial t^2} \quad (w_p = B$

$\frac{\partial w_p}{\partial x} = A \qquad \frac{\partial w_p}{\partial x} = B$

$w_p(x, t=0) = g_1(x)$

$\frac{\partial w_p}{\partial t}(x, t=0) = g_2(x)$

$+$

$(w_h = 0) \quad \frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2} \quad w_h = 0$

$\frac{\partial w_h}{\partial x} = 0 \qquad \frac{\partial w_h}{\partial x} = 0$

$w_h(x, t=0) = w_0(x) - g_1(x)$

$\frac{\partial w_h}{\partial t}(x, t=0) = w_1(x) - g_2(x)$

$\frac{\partial w}{\partial x} = 0$ — use I.C. when very end

Fn. <<1>> satisfies the B.C. and PDE

(2) look at w_h

we use the method of separation of variable

$$w_h(x,t) = F(x)G(t)$$

$$\frac{\partial w_h}{\partial x} = F'(x)G(t) \quad ; \quad \frac{\partial^2 w_h}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial w_h}{\partial t} = F(x)G'(t) \quad ; \quad \frac{\partial^2 w_h}{\partial t^2} = F(x)G''(t)$$

By $\frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2}$

$$F''(x)G(t) = \frac{1}{c^2}\cdot F(x)G''(t) \Rightarrow \frac{F''(x)G(t)}{F(x)G(t)} = \frac{F(x)G''(t)}{c^2\cdot F(x)G(t)} = -k^2$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''}{G} = -k^2$$

at $k \neq 0$ $\quad \begin{array}{l} F'' + k^2 F = 0 \\ G'' + c^2 k^2 F = 0 \end{array} \Rightarrow \begin{array}{l} F(x) = A\cos kx + B\sin kx \\ G(t) = C\cos kct + D\sin kct \end{array}$

at $k = 0$ $\quad \begin{array}{l} F'' = 0 \\ G'' = 0 \end{array} \Rightarrow \begin{array}{l} F(x) = \bar{A}x + \bar{B} \\ G(t) = \bar{C}t + \bar{D} \end{array}$

look at B.C. first

$$\frac{\partial w_h}{\partial x}(x=0, t) = 0 = F'(0)G(t) \Rightarrow F'(0) = 0$$

$$\frac{\partial w_h}{\partial x}(x=l, t) = 0 = F'(l)G(t) \Rightarrow F'(l) = 0$$

at $K=0$

By $F'(0)=0 \Rightarrow \bar{A}=0 \Rightarrow F(x)=\bar{B}$

$\boxed{\begin{array}{l} B_1=\bar{B}\bar{C} \\ B_2=\bar{B}\cdot\bar{D}\end{array}}$

Then: at $K=0$ $\quad W_h(x,t)=\bar{B}[\bar{C}t+\bar{D}] = \underline{B_1 t + B_2}$

at $K\neq 0$

By $F'(0)=0 = -AK\sin K\cdot 0 + BK\cos K\cdot 0 = 0 + BK \Rightarrow B=0$

By $F'(l)=0 = -AK\sin K\cdot l \Rightarrow Kl = n\pi \Rightarrow K=\frac{n\pi}{l}$

$\Rightarrow F(x) = A\cos\frac{n\pi}{l}x$

at $K\neq 0$ $\quad W_h(x,t) = \sum_{n=1}^{\infty} A\cos\frac{n\pi}{l}x\,[C\cos Kct + D\sin Kct]$

$= \underline{\sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n \sin Kct]}$

$\boxed{\begin{array}{l} b_n = AC \\ \mathcal{D}_n = AD\end{array}}$

so that general W_h is

$$W_h = B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n\sin Kct]$$

Therefore The whole fn. of w is:

$$W = W_p + W_h$$

$$= \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + Et + B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos Kct + \mathcal{D}_n\sin Kct]$$

$$= \underline{\underline{\frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + E_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[b_n\cos\frac{n\pi c}{l}t + \mathcal{D}_n\sin\frac{n\pi c}{l}t]}}$$

where. $\boxed{E_1 = E + B_1}$

use I.C. to determine the E_1, B_2, b_n and $\mathcal{D}_n$

<< 2 >>

By $W(x,t=0) = 5x = \frac{B-A}{2l}x^2 + Ax + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot b_n$

$\Rightarrow (5-A)x - \frac{B-A}{2l}x^2 = B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot b_n$

By Fourier series

$$B_2 = \frac{1}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]dx \checkmark$$

$$b_n = \frac{2}{l}\int_0^l\left[(5-A)x - \frac{B-A}{2l}x^2\right]\cdot\cos\frac{n\pi}{l}x\,dx \checkmark$$

By $\frac{\partial w}{\partial t}(x, t=0) = 0 = E_1 + \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x \cdot \mathcal{D}_n \cdot \frac{cn\pi}{l}$

$\Rightarrow \quad E_1 = 0 \;, \quad \mathcal{D}_n = 0$

Therefore :

$$w = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cos\frac{cn\pi}{l}t$$

* check the result :

B.C

$$\frac{\partial w}{\partial x}(0,t) = \frac{B-A}{l}\cdot 0 + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin\frac{n\pi}{l}\cdot 0 = \underline{\underline{A}}$$

$$\frac{\partial w}{\partial x}(l,t) = \frac{B-A}{l}\cdot l + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l}\sin n\pi = \underline{\underline{B}}$$

I.C

$$w(x,0) = \frac{B-A}{2l}x^2 + Ax + 0 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x = \underline{\underline{5x}}$$

(according to fn. of «2»)

$$\frac{\partial w}{\partial t}(x,0) = 0 + 0 + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l}\sin\frac{cn\pi}{l}\cdot 0\right) = \underline{\underline{0}}$$

PDE

$$\frac{\partial w}{\partial x} = \frac{B-A}{l}x + A + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \left(-\frac{n\pi}{l}\sin\frac{n\pi}{l}x\right)$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi}{l}x\right] \qquad «3»$$

$$\frac{\partial w}{\partial t} = c^2\frac{B-A}{l}t + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l}\cdot\sin\frac{cn\pi}{l}t\right)$$

$$\frac{\partial^2 w}{\partial t^2} = c^2\cdot\frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{cn\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right]$$

$$\Rightarrow \frac{\partial^2 w}{c^2\,\partial t^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right] \qquad «4»$$

By «3» = «4»

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$$

Therefore, Fn. w satisfies the B.C., I.C., and PDE.

so w is the solution of the problem.

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PROBLEM: Infinite string $\quad C = 1$

$$W_0(x) = 0$$
$$W_1(x) = e^{-x^2}$$

find $f(x)$, $g(x)$, $f(x+ct)$, $g(x-ct)$, $w(x,t)$

solution:

① $$\boxed{f(x) = \frac{W_0(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma}$$

$$f(x) = 0 + \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}\,d\sigma = \frac{1}{2}\left\{-\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \int_0^{x} e^{-\sigma^2}\,d\sigma\right\}$$

$$= \frac{1}{2}\,\frac{\sqrt{\pi}}{2}\left\{-\frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\sigma^2}\,d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x) - \operatorname{erf}(x_0)\right]$$

② $$\boxed{g(x) = \frac{W_0(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} W_1(\sigma)\,d\sigma}$$

$$g(x) = 0 - \left[\frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x) - \operatorname{erf}(x_0)\right]\right]$$

$$= \frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x_0) - \operatorname{erf}(x)\right]$$

③ $$\boxed{f(x+ct) = \frac{W_0(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} W_1(\sigma)\,d\sigma}$$

$$f(x+ct) = 0 + \frac{1}{2c}\int_{x_0}^{x+ct} e^{-\sigma^2}\,d\sigma$$

$$= \frac{1}{2c}\left\{-\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \int_0^{x+ct} e^{-\sigma^2}\,d\sigma\right.$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x_0)\right]$$

④ $$g(x-ct) = \frac{W_0(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} W_1(\sigma)\,d\sigma$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x_0) - \operatorname{erf}(x-ct)\right]$$

⑤ $$\boxed{W(x,t) = \frac{1}{2}\left[W_0(x+ct) - W_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} W_1(\sigma)\,d\sigma}$$

$$w(x,t) = 0 + \frac{1}{2c}\left\{-\int_{x_0}^{x-ct} W_1(\sigma)\,d\sigma + \int_{x_0}^{x+ct} W_1(\sigma)\,d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right]$$

/ −1

7.4. For the line $-\infty < x < +\infty$ where $u_{xx} - u_{tt} = 0$ the solution is given by

$$u(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t) + \int_{x-t}^{x+t} u_1(\sigma)\,d\sigma\right] \quad \text{for } c=1$$

where $u(x,t=0) = u_0(x)$

$\frac{\partial u}{\partial t}(x,t=0) = u_1(x)$; in our case $u_0(x) = 0$ and $u_1(x) = e^{-x^2}$

$$\therefore\ u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} e^{-\sigma^2}\,d\sigma = \frac{1}{2}\left[\int_0^{x+t} e^{-\sigma^2}\,d\sigma - \int_0^{x-t} e^{-\sigma^2}\,d\sigma\right]$$

with $\frac{2}{\sqrt{\pi}}\int_0^z e^{-\sigma^2}\,d\sigma = \mathrm{erf}(z) \Rightarrow \int_0^z e^{-\sigma^2}\,d\sigma = \frac{\sqrt{\pi}}{2}\mathrm{erf}(z)$

$$\therefore\ u(x,t) = \frac{1}{2}\left[\frac{\sqrt{\pi}}{2}\mathrm{erf}(x+t) - \frac{\sqrt{\pi}}{2}\mathrm{erf}(x-t)\right] = \underline{\frac{\sqrt{\pi}}{4}\left[\mathrm{erf}(x+t) - \mathrm{erf}(x-t)\right]}$$

7.5 Given: $u_{xx} - u_{tt} = 0$ with ① $u(x,t=0) = xe^{-x}$ $[= u_0(x)]$ ② $\frac{\partial u}{\partial t}(x,t=0) = 0$ $[= u_1(x)]$ $\Big\}$ $x \geq 0$ IC

③ with BC $u(x=0,t) = 0$. Find u for $t \geq 0, x \geq$

General solution is $u(x,t) = f(x+t) + g(x-t)$ since $c = 1$

now look at ③ : $u(0,t) = 0 = f(t) + g(-t) \Rightarrow g(-\sigma) = -f(\sigma)$ t or σ dum

this equation allows for the definition of g & f for negative arguments

Remember f & g are defined for + arguments only, thus we must extend the definition of f & g over the entire range of arguments (ie $-\infty < \sigma < \infty$) to use the ∞ line solution

$\therefore\ g(-\sigma) = -f(\sigma)$ when σ is a + argument $\Rightarrow$ g of $-$ argument $=$ $-f$ of $+$ argument. Now replace $-\sigma$ for $\sigma \Rightarrow g(\sigma) = -f(-\sigma)$; here, if σ is +, then f of $-$ argument $=$ $-g$ of + argument. Note also that $u(x,t=0) = f(x) + g(x) = u_0(x) \Rightarrow f(\sigma) = \frac{u_0}{2}(\sigma)$; $g(\sigma) = \frac{u_0}{2}(\sigma)$ ④

Now $u(x,t) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right]$ since $u_1 = 0$ &

$f(x+t) = \frac{1}{2}u_0(x+t)$ & $g(x-t) = \frac{1}{2}u_0(x-t)$ from ④

Thus for $\sigma = x+t > 0$ $\quad f(x+t) = \frac{1}{2}u_0(x+t) = \frac{1}{2}(x+t)e^{-(x+t)}$ ⑤

$\sigma = x-t > 0$ $\quad g(x-t) = \frac{1}{2}u_0(x-t) = \frac{1}{2}(x-t)e^{-(x-t)}$ ⑥

for - arguments $\quad f(\sigma) = -g(-\sigma) = -\frac{u_0}{2}(-\sigma)$ $\qquad$ here $-\sigma > 0$

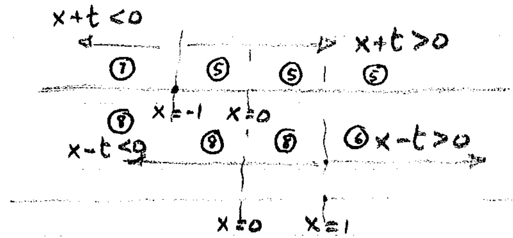
$$g(\sigma) = -f(-\sigma) = -\frac{u_0}{2}(-\sigma)$$

$\therefore$ for $\sigma = x+t < 0 \qquad f(x+t) = -\frac{u_0}{2}(-(x+t)) = -\frac{1}{2}\left[-(x+t)\,e^{(x+t)}\right]$ ⑦

$\sigma = x-t < 0 \qquad g(x-t) = -\frac{u_0}{2}(-(x-t)) = -\frac{1}{2}\left[-(x-t)\,e^{(x-t)}\right]$ ⑧

at $\quad t=1$: $\qquad x+t>0 \implies x>-1$

$\qquad\qquad\qquad x-t>0 \implies x>1$



$x+t>0$, $x-t>0$	$\therefore$ for $t=1$ & $x>1$	$u(x,t) =$ ⑤ + ⑥	
$x+t>0$, $x-t<0$	$t=1$ & $0 \le x < 1$	$u(x,t) =$ ⑤ + ⑧	remember we want $x \geq 0$ $t \geq 0$
	$t=1$ $-1 < x < 0$	$u(x,t) =$ ⑤ + ⑧	
	$x < -1$	$u(x,t) =$ ⑦ + ⑧	

for our problem since we want $u(x,t)$ for $x \geq 0$; the first two of these give us want we want . Note that at $\underline{\underline{x=0}}$ & $t=1$ we have the boundary condition

$x-t<0$, $x+t>0$

$$u(x,t) = \underbrace{\frac{1}{2}(x+t)\,e^{-(x+t)}}_{⑤} + \underbrace{\frac{1}{2}(x-t)\,e^{(x-t)}}_{⑧}\Bigg|_{\substack{x=0\\t=1}} = \frac{1}{2}e^{-1} - \frac{1}{2}e^{-1} = 0$$

taking $\mathcal{L}\left\{\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}\right\}$ gives

$$\alpha \frac{d^2}{dx^2} \mathcal{J}(x;s) = s\mathcal{J}(x;s) - T(x,0) \quad ; \text{ we define } \mathcal{J}(x;s) = \int_0^\infty T(x,t)\, e^{-st}\, dt$$

$T(x,0) = T_i$

or $\mathcal{J}'' - \frac{s}{\alpha}\mathcal{J} = \frac{T_i}{\alpha}$ $\qquad \mathcal{J}'' = \frac{d^2}{dx^2}\mathcal{J}$

Now transform the BC's:

@ $x=0$ $\quad -k\frac{\partial T}{\partial x} = q \quad \Rightarrow \quad -k\,\mathcal{J}'(0;s) = q/s \quad$ or $\quad \mathcal{J}'(0;s) = -q/ks \quad$ (1)

REMEMBER $\int_0^\infty \text{const}\, e^{-st} dt = \text{const}/s$; that's why →

ALSO $T(x,t) \to T_i$ as $x \to \infty \quad \Rightarrow \quad \mathcal{J}(x;s) \to T_i/s$ as $x \to \infty \quad$ (2)

Since ODE is not HOMOGENEOUS } let $\mathcal{J} = \mathcal{J}_H + \mathcal{J}_p$ where $\mathcal{J}_p'' - \frac{s}{\alpha}\mathcal{J}_p = \frac{T_i}{\alpha}$ choose $\mathcal{J}_p = C$

$0 - \frac{s}{\alpha} C = \frac{T_i}{\alpha} \quad \Rightarrow \quad \underline{C = T_i/s = \mathcal{J}_p}$

FOR THE HOMOGENEOUS PART

now $\mathcal{J}_H$ solves $\mathcal{J}_H'' - \frac{s}{\alpha}\mathcal{J}_H = 0 \quad \Rightarrow \quad \mathcal{J}_H = C_1 e^{-\sqrt{s/\alpha}\,x} + C_2 e^{\sqrt{s/\alpha}\,x}$

$$\therefore \quad \mathcal{J} = \mathcal{J}_p + \mathcal{J}_H = \frac{T_i}{s} + C_1 e^{-\sqrt{s/\alpha}\,x} + C_2 e^{+\sqrt{s/\alpha}\,x}$$

now from (2) $\Rightarrow$ $C_2 = 0$ since $\mathcal{J}(x;s) \to \frac{T_i}{s}$ as $x \to \infty$

To use (1) take $\frac{d}{dx}\mathcal{J}(x;s)$ and evaluate at $x=0$

$$\left.\frac{d}{dx}\mathcal{J}\right|_{x=0} = 0 + C_1 e^{-\sqrt{s/\alpha}\,x}\cdot\left(-\sqrt{\frac{s}{\alpha}}\right)\Big|_{x=0} = -C_1\sqrt{\frac{s}{\alpha}}\, e^0 = -C_1\sqrt{\frac{s}{\alpha}} = -\frac{q}{ks}$$

$$\therefore \quad C_1 = \frac{q}{k}\sqrt{\frac{\alpha}{s^3}}$$ 1

$$\therefore \quad \mathcal{J} = \frac{T_i}{s} + \frac{q}{k}\sqrt{\frac{\alpha}{s^3}}\, e^{-\sqrt{s/\alpha}\,x} = \frac{T_i}{s} + \frac{q\sqrt{\alpha}}{k}\cdot\frac{1}{\sqrt{s^3}}\, e^{-\frac{x}{\sqrt{\alpha}}\cdot\sqrt{s}}$$ 2

To find $T(x,t)$, find $\mathcal{L}^{-1}\{\mathcal{J}(x;s)\}$: $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ and

$\mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s^3}}\, e^{-p\sqrt{s}}\right\} = 2\sqrt{\frac{t}{\pi}}\, e^{-p^2/4t} - p\, \text{erfc}\left(\frac{p}{2\sqrt{t}}\right)$, where $p = \text{const} = \frac{x}{\sqrt{\alpha}}$ using 29.3.85

$$\therefore \quad T(x,t) = T_i \cdot 1 + \frac{q}{k}\sqrt{\alpha}\left[2\sqrt{\frac{t}{\pi}}\, e^{-\frac{x^2}{4\alpha t}} - \frac{x}{\sqrt{\alpha}}\, \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right]$$ 3

PROBLEM 108 Heat flow equation:

(Text book, P534) $\alpha^2 T_{xx} - T_t = 0$ for $t \geq 0$, $0 \leq x \leq l$

B.C. $T(x=0,t) = 0$ I.C. $T(x,t=0) = 0$

$T(x=l,t) = 1$

when $t>0$, by above condition. $T = A(x,t)$. where $A(x,t)$ is the fuction of which the serise (168) is an expansion.

solution:

ⓐ let the Laplace Transform $\bar{T}(x,s) = \int_0^\infty T(x,t)e^{-st}dt$

since $\mathcal{L}\{\frac{\partial}{\partial x}T\} = \frac{\partial}{\partial x}\int_0^\infty T(x,t)e^{-st}dt = \frac{d}{dx}\bar{T}(x,s)$

$\mathcal{L}\{\frac{\partial^2}{\partial x^2}T\} = \frac{d^2}{dx^2}\bar{T}(x,s)$

$\mathcal{L}\{\frac{\partial}{\partial t}T\} = s\bar{T}(x,s) - T(x,t=0+)$

$\because \mathcal{L}\{\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha^2}\frac{\partial T}{\partial t}\} = \frac{d^2}{dx^2}\bar{T}(x,s) = \frac{1}{\alpha^2}[s\bar{T}(x,s) - \bar{T}(x,t=0+)]$

$\Rightarrow \frac{d^2}{dx^2}\bar{T} - \frac{s}{\alpha^2}\bar{T} = -\frac{1}{\alpha^2}T(x,t=0+)$

By condition I.C. $T(x,t=0) = 0$ ①

B.C. $T(x=0,t) = 0$ ②

$T(x=l,t) = 1$ ③

By ① $T(x,t=0+) = 0 \Rightarrow$

$\underline{\bar{T}'' - \frac{s}{\alpha^2}\bar{T} = 0}$ $(\alpha^2\bar{T}_{xx} - s\bar{T} = 0)$

let $\bar{T} = \bar{T}_P + \bar{T}_H$

first do T_P By ①

not necessary since eqn is homogeneous only use T_p when non homogeneous

$\bar{T}_P'' - \frac{s}{\alpha^2}\bar{T}_P = 0$ let $\bar{T}_P = C \Rightarrow$

$\Rightarrow 0 - \frac{s}{\alpha^2}C = 0 \Rightarrow C = 0 \Rightarrow \bar{T}_P = 0$

second do T_H

$\bar{T}_H'' - \frac{s}{\alpha^2}\bar{T}_H = 0 \Rightarrow \bar{T}_H = C_1 e^{-\frac{\sqrt{s}}{\alpha}x} + C_2 e^{\frac{\sqrt{s}}{\alpha}x}$

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By ② $x=0$

$$\int_0^\infty T(x=0,t)\,e^{-st}dt = \bar T(0;s) = \int_0^\infty 0\cdot e^{-st}dt = 0 \quad\Rightarrow\quad \underline{\bar T(0,s)=0}$$

By ③ x.

$$\int_0^\infty T(x=l,t)\,e^{-st}dt = \bar T(x=l,s) = \int_0^\infty e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^\infty = \frac{1}{s} \quad\Rightarrow\quad \underline{\bar T(x=l,s)=\frac{1}{s}}$$

since $\bar T = \bar T_P + \bar T_H$

$$= 0 + c_1 e^{-\frac{\sqrt{s}}{\alpha}x} + c_2 e^{\frac{\sqrt{s}}{\alpha}x}$$

By $\bar T(0,s)=0 \quad\Rightarrow\quad \bar T(0,s) = c_1 + c_2 = 0 \quad\Rightarrow\quad c_1 = -c_2$ ④

By $\bar T(l,s) = 1/s$ let $\underline{s^{\frac{1}{2}}/\alpha = q} \quad\Rightarrow\quad c_1 e^{-ql} + c_2 e^{ql} = \frac{1}{s}$ ⑤

By ④ & ⑤ $\Rightarrow\quad c_1 = -\frac{1}{2\cdot\sinh ql}\cdot\frac{1}{s}\ ;\ c_2 = \frac{1}{2\sinh ql}\cdot\frac{1}{s}$

Therefore:

$$\bar T = c_1 e^{-\frac{\sqrt{s}}{\alpha}x} + c_2 e^{\frac{\sqrt{s}}{\alpha}x} = \frac{1}{s}\cdot\frac{1\times 2}{2\sinh ql}\left(\frac{e^{qx}-e^{-qx}}{2}\right)$$

$$= \underline{\frac{1}{s}\,\frac{\sinh qx}{\sinh ql}}$$

$$= \frac{1}{s}\,\frac{e^{qx}-e^{-qx}}{e^{ql}-e^{-ql}} = \frac{1}{s}\,\frac{e^{qx}}{e^{ql}}\,\frac{(1-e^{-2qx})}{(1-e^{-2ql})}$$

$$= \underline{\frac{1}{s}\,e^{-q(l-x)}\,\frac{1-e^{-2qx}}{1-e^{-2ql}}}$$

ⓑ The form of a series of ascending power is

$$\boxed{\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 \cdots} \qquad -1 < x < 1$$

here $x = -e^{-2ql}$ $\qquad -1 < -e^{-2ql} < 1 \longleftarrow$ ** $\underline{\text{here } q>0}$ (complex)

then $\dfrac{1}{1+(-e^{-2ql})} = 1 + e^{-2ql} + e^{-4ql} + e^{-6ql} + e^{-8ql} + \cdots\cdots$

Therefore: $\bar T(x,s) = \dfrac{1}{s}\cdot\dfrac{1}{1+(-e^{-2ql})}\cdot e^{-q(l-x)}(1-e^{-2qx})$

$$\Rightarrow \bar T(x,s) = \frac{1}{s}\left\{(1+e^{-2ql}+e^{-4ql}+e^{-6ql}+\cdots)(e^{-q(l-x)} - e^{-q(l+x)})\right\}$$

$$= \underline{\frac{1}{s}\cdot e^{-q(l-x)} - \frac{1}{s}e^{-q(l+x)} + \frac{1}{s}e^{-q(3l-x)} - e^{-q(3l+x)} + \cdots\cdots}$$

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(c) By (b)

$$\bar{T}(x,s) = \frac{1}{s} e^{-q(l-x)} - \frac{1}{s} e^{-q(l+x)} + \frac{1}{s} e^{-q(3l-x)} - \frac{1}{s} e^{-q(3l+x)} + \cdots$$

since $q = \frac{\sqrt{s}}{\alpha}$ $\Rightarrow$

$$\bar{T}(x,s) = \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(l-x)\right]} - \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(l+x)\right]} + \frac{1}{s} e^{-\sqrt{s}\left[\frac{1}{\alpha}(3l-x)\right]} - \cdots$$

By Laplace transform

$$\bar{T}(x,s) \Longrightarrow A(x,t)$$

By 29.3.83

$$f(s) = \frac{1}{s} e^{-k\sqrt{s}} = \operatorname{erfc} \frac{k}{2\sqrt{t}} = 1 - \operatorname{erf} \frac{k}{2\sqrt{t}}$$

In our problem

$k = \left[\frac{1}{\alpha}(l-x)\right]$ or $k = \left[\frac{1}{\alpha}(l+x)\right]$, or $\left[\frac{1}{\alpha}(3l-x)\right] \cdots\cdots$

Therefore By 29.3.83 we get

$$\bar{T}(x,s) \Longrightarrow A(x,t) = \left[1 - \operatorname{erf}\left(\frac{l-x}{2\alpha\sqrt{t}}\right)\right] + \left[1 - \operatorname{erf}\left(\frac{l+x}{2\alpha\sqrt{t}}\right)\right] + \left[1 - \operatorname{erf}\left(\frac{3l-x}{2\alpha\sqrt{t}}\right)\right] + \cdots\cdots$$

$$= T$$

excellent.

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \quad (1)$$

$u(0,t) = t/\beta \quad (2)$

$u(y,0) = 0 \quad (3)$

$u(y,t) \to 0$ as $y \to \infty$ (4)

Similar to the 2nd problem we did in class with $a = \frac{1}{\beta}$ and $b = 1$

choose $\eta = \frac{Ay}{t^n}$ note: as $y \to \infty$ & $t \to 0$, $\eta \to \infty$ and $u \to 0$ from (3) & (4)

Choose $u - u_\infty = Bt^m f(\eta)$ where u_∞ is velocity far from plate [ie $u_\infty = 0$ from (4)]

$u(y=0,t) = Bt^m f(\eta)$ as $y \to 0$ $\eta \to 0$

$= Bt^m f(0) = t/\beta$

since $f(0) = $ const. ; let $m = 1$ $B = \frac{1}{\beta}$ $\Rightarrow$ $f(0) = 1$

From (4): since $u \to 0$ as $y \to \infty$ $u = Bt^m f(\eta) \to Bt^m f(\eta \to \infty)$. Since $u \to 0$ as $y \to \infty$
$\Rightarrow f(\eta \to \infty) \Rightarrow 0$ irrespective of t

Note that if $f(\eta \to \infty) = 0$, then from (3) @ $t=0$ $u = B \cdot 0^m \cdot f(\eta \to \infty) = B \cdot 0 \cdot 0 = 0$ as requi

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$; $\frac{\partial \eta}{\partial y} = \frac{A}{t^n}$ $\frac{\partial u}{\partial \eta} = Bt^m f'(\eta)$ $\therefore \frac{\partial u}{\partial y} = Bt^m f'(\eta) \cdot \frac{A}{t^n}$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left[Bt^m f'(\eta)\cdot\frac{A}{t^n}\right]\cdot\frac{A}{t^n} = Bt^m f''(\eta)\cdot\frac{A^2}{t^{2n}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} ; \quad \frac{\partial \eta}{\partial t} = \frac{-nAy}{t^{n+1}} = -\frac{n}{t}\eta \quad \frac{\partial u}{\partial \eta} = Bt^m f'(\eta) \quad \therefore \frac{\partial u}{\partial t} = -\frac{n\eta}{t}Bt^m f'(\eta) + Bmt^{m-1}f$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \Rightarrow \cancel{B}\cancel{t^m} f''\frac{A^2}{\cancel{t^{2n}}} = \frac{1}{\alpha}\left[\frac{-n\eta}{\cancel{t}}\cancel{B}\cancel{t^m}f'(\eta)\right] + \frac{1}{\alpha}\cancel{B}m\cancel{t^{m-1}}f$$

for this eqn to be independent for $t \Rightarrow 2n = 1$ $n = 1/2$

or $0 = f''A^2 + \frac{\eta}{2\alpha}f'(\eta) - \frac{\cancel{B}mt^{0}}{\alpha}f \Rightarrow 0 = f'' + \frac{\eta}{2\alpha A^2}f' - \frac{\cancel{B}mf}{A^2\alpha} \Rightarrow$ $2\alpha A^2 = 1$, $A = \frac{1}{\sqrt{2\alpha}}$

and $f'' + \eta f' - \frac{\cancel{B}mf}{A^2\alpha} = 0$ thus

$$\eta = \frac{Ay}{t^n} = \frac{y}{\sqrt{2\alpha t}} \quad B = \frac{1}{\beta} \quad m = 1$$

$\therefore$ $u = \frac{1}{\beta}t\, f(\eta)$ where $f(\eta)$ satisfies $f'' + \eta f' - 2f = 0$ with.
$f(0) = 1$ and $f(\eta) \to 0$ as $\eta \to \infty$

$\frac{\partial T}{\partial x} = Q_1$ $\qquad\qquad\qquad\qquad$ $T = T_1$

Choose for $T_p = Ax + B$. $\quad \therefore \frac{\partial T_p}{\partial x} = A$. $\quad$ At $x = 0$ $\underline{\frac{\partial T_p}{\partial x} = Q_1 = A}$ $\quad$ also

$$T_p = Q_1 x + B \; . \quad @ \; x = L \quad T_p = T_1 = Q_1 L + B \quad \therefore \quad B = T_1 - Q_1 L$$

$$\therefore \quad T_p = Q_1 x + T_1 - Q_1 L = Q_1 (x - L) + T_1$$

since T_p is linear in x $\quad \frac{\partial T_p}{\partial x} = Q_1 \quad \frac{\partial^2 T_p}{\partial x^2} = 0$ $\quad$ also $\quad \frac{\partial T_p}{\partial t} = 0 \Rightarrow \frac{\partial^2 T_p}{\partial x^2} = 0 = \frac{1}{\alpha}\frac{\partial}{\partial t}$

Also since T_p is not a fn of t $\Rightarrow$ T_p is the solution irrespective of time; thus it must be the solution at $t = 0$ $\quad \therefore \quad \underline{T_p(x, t=0) = Q_1(x - L) + T_1}$

for T_h

$\frac{\partial T}{\partial x} = 0$ $\qquad\qquad\qquad$ $T = 0$

$T(x, t=0) = f(x) - T_p(x, t=0)$

Choose $\quad T_h = F(x)\, G(t)$

$$\therefore \quad \frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t} \Rightarrow F'' G = \frac{1}{\alpha} F G'$$

or $\quad \frac{F''}{F} = \frac{G'}{\alpha G} = -k^2$

thus $\left.\begin{array}{l} F'' + k^2 F = 0 \\ G' + k^2 \alpha G = 0 \end{array}\right\}$ for $k \neq 0$ $\quad$ and $\quad \begin{array}{l} F'' = 0 \\ G' = 0 \end{array}$ $\quad$ for $k = 0$

For $k = 0$ $\quad F = \bar{A}x + \bar{B}\, ; F' = \bar{A}$ $\quad$ for $k \neq 0$ $\quad F = A\cos kx + B\sin kx\, ; F' = k[-A\sin kx + B\cos$

$\qquad G = \bar{C}$ $\qquad\qquad\qquad\qquad\qquad G = Ce^{-\alpha k^2 t}$

<u>BC</u> $\quad \left.\frac{\partial T}{\partial x}\right|_{x=0} = F'(x)\, G(t)\Big|_{x=0} = 0$ for all time $\therefore$ $F'(0) = 0$

$\quad T\Big|_{x=L} = F(x)\, G(t)\Big|_{x=L} = 0$ for all time $\therefore$ $F(L) = 0$

for $k = 0$ $\quad F'(0) = \bar{A} = 0 \Rightarrow F = \bar{B}$, $\quad$ also $F(L) = 0 = \bar{B} \Rightarrow F(x) = 0$ for $k = 0$

trival solution

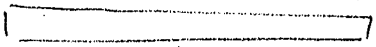
for $k \neq 0$ $\quad F'(0) = k[-A\overset{0}{\cancel{\sin 0}} + B\overset{1}{\cancel{\cos 0}}] = 0 \Rightarrow B = 0 \Rightarrow F(x) = A\cos kx$. Also $F(L) = 0 \Rightarrow A\cos kL =$

$\quad \Rightarrow kL = \frac{\pi}{2} n$ where n is odd. $\quad \therefore \quad k = \frac{n\pi}{2L}$

$$\therefore \quad T(x,t) = \sum AC\, e^{-\alpha k^2 t} \cos kx = \sum_{n = odd} \mathscr{C}_n\, e^{-\alpha \frac{n^2\pi^2}{4L^2} t} \cos \frac{n\pi}{2L} x$$

@ $t = 0$ $\quad T(x, 0) = f(x) - [Q_1(x - L) + T_1] = \sum \mathscr{C}_n \cos \frac{n\pi x}{2L} \Rightarrow \mathscr{C}_n = \frac{2}{L}\int_0^L h(x) \cos \frac{n\pi x}{2L}\, dx$

$\qquad = h(x)$

$\frac{\partial T}{\partial x} = Q_1$  $T = T_1$

Choose for $T_p = Ax + B$. $\therefore$ $\frac{\partial T_p}{\partial x} = A$. At $x = 0$ $\underline{\frac{\partial T_p}{\partial x} = Q_1 = A}$ also

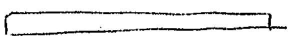
$T_p = Q_1 x + B$. @ $x = L$ $T_p = T_1 = Q_1 L + B$ $\therefore$ $B = T_1 - Q_1 L$

$\therefore$ $T_p = Q_1 \cancel{x} + T_1 - Q_1 L = Q_1 (x - L) + T_1$

since T_p is linear in x $\frac{\partial T_p}{\partial x} = Q_1$ $\frac{\partial^2 T_p}{\partial x^2} = 0$ also $\frac{\partial T_p}{\partial t} = 0 \Rightarrow \frac{\partial^2 T_p}{\partial x^2} = 0 = \frac{1}{\alpha}\frac{\partial}{\partial}$

Also since T_p is not a fn of t $\Rightarrow$ T_p is the solution irrespective of time; thus it must be the solution at $t = 0$ $\therefore$ $\underline{T_p(x, t=0) = Q_1(x - L) + T_1}$

for T_h

$\frac{\partial T}{\partial x} = 0$  $T = 0$

$T(x, t=0) = f(x) - T_p(x, t=0)$

Choose $T_h = F(x)\, G(t)$

$\therefore$ $\frac{\partial^2 T_h}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t} \Rightarrow F''G = \frac{1}{\alpha} F G'$

or $\frac{F''}{F} = \frac{G'}{\alpha G} = -k^2$

thus $\left.\begin{array}{l} F'' + k^2 F = 0 \\ G' + k^2 \alpha G = 0 \end{array}\right\}$ for $k \neq 0$ and $\begin{array}{l} F'' = 0 \\ G' = 0 \end{array}$ for $k = 0$

For $k = 0$ $F = \bar{A}x + \bar{B}$; $F' = \bar{A}$; $G = \bar{C}$ for $k \neq 0$ $F = A\cos kx + B\sin kx$; $F' = k[-A\sin kx + B\cos$; $G = C e^{-\alpha k^2 t}$

<u>BC</u> $\left.\frac{\partial T}{\partial x}\right|_{x=0} = \left. F'(x)\, G(t)\right|_{x=0} = 0$ for all time $\therefore$ $F'(0) = 0$

$\left. T\right|_{x=L} = \left. F(x)\, G(t)\right|_{x=L} = 0$ for all time $\therefore$ $F(L) = 0$

for $k = 0$ $F'(0) = \bar{A} = 0 \Rightarrow F = \bar{B}$, also $F(L) = 0 = \bar{B} \Rightarrow F(x) = 0$ for $k = 0$
trival solution

for $k \neq 0$ $F'(0) = k[-A\overset{0}{\cancel{\sin 0}} + B\overset{1}{\cancel{\cos 0}}] = 0 \Rightarrow B = 0 \Rightarrow F(x) = A\cos kx$. Also $F(L) = 0 \Rightarrow A\cos kL =$

$\Rightarrow kL = \frac{\pi}{2} n$ where n is odd. $\therefore$ $k = \frac{n\pi}{2L}$

$\therefore$ $T(x,t) = \sum AC\, e^{-\alpha k^2 t} \cos kx = \sum_{n=odd} \mathcal{C}_n\, e^{-\alpha \frac{n^2\pi^2}{4L^2} t} \cos \frac{n\pi}{2L} x$

@ $t = 0$ $T(x, 0) = f(x) - [Q_1(x - L) + T_1] = \sum \mathcal{C}_n \cos\frac{n\pi x}{2L} \Rightarrow \mathcal{C}_n = \frac{2}{L}\int_0^L h(x) \cos\frac{n\pi x}{2L}\, dx$
$= h(x)$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \quad (1)$$

$u(0,t) = t/\beta \quad (2)$

$u(y,0) = 0 \quad (3)$

$u(y,t) \to 0$ as $y \to \infty$ (4)

Similar to the 2nd problem we did in class with $a = \frac{1}{\beta}$ and $b = 1$

choose $\eta = \frac{Ay}{t^n}$ note: as $y \to \infty$ & $t \to 0$, $\eta \to \infty$ and $u \to 0$ from (3) & (4)

Choose $u - u_\infty = Bt^m f(\eta)$ where u_∞ is velocity far from plate [ie $u_\infty = 0$ from (4)]

$u(y=0,t) = Bt^m f(\eta)$ as $y \to 0$ $\eta \to 0$
$= Bt^m f(0) = t^1/\beta$

since $f(0) = $ const.; let $m = 1$ $B = \frac{1}{\beta} \Rightarrow \underline{\underline{f(0) = 1}}$

From (4): since $u \to 0$ as $y \to \infty$ $u = Bt^m f(\eta) \to Bt^m f(\eta \to \infty)$. Since $u \to 0$ as $y \to \infty$
$\Rightarrow f(\eta \to \infty) \equiv 0$ irrespective of t

Note that if $f(\eta \to \infty) = 0$, then from (3) @ $t = 0$ $u = B \cdot 0^m \cdot f(\eta \to \infty) = B \cdot 0 \cdot 0 = 0$ as req.

Now $\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$; $\frac{\partial \eta}{\partial y} = \frac{A}{t^n}$ $\frac{\partial u}{\partial \eta} = Bt^m f'(\eta)$ $\therefore \frac{\partial u}{\partial y} = Bt^m f'(\eta)\cdot\frac{A}{t^n}$

$$\frac{\partial^2 u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial \eta}\left(\frac{\partial u}{\partial y}\right)\frac{\partial \eta}{\partial y} = \frac{\partial}{\partial \eta}\left[Bt^m f'(\eta)\cdot\frac{A}{t^n}\right]\cdot\frac{A}{t^n} = Bt^m f''(\eta)\cdot\frac{A^2}{t^{2n}}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \eta}\cdot\frac{\partial \eta}{\partial t} ; \quad \frac{\partial \eta}{\partial t} = -n\frac{Ay}{t^{n+1}} = -\frac{n}{t}\eta \quad \frac{\partial u}{\partial \eta} = Bt^m f'(\eta) \quad \therefore \frac{\partial u}{\partial t} = -\frac{n\eta}{t}Bt^m f'(\eta) + Bm\ldots$$

$$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{1}{\alpha}\frac{\partial u}{\partial t} \Rightarrow \cancel{B}\cancel{t^m} f''\frac{A^2}{\cancel{t^{2n}}} = \frac{1}{\alpha}\left[\frac{-n\eta}{\cancel{t}}\cancel{B}\cancel{t^m}f'(\eta)\right] + \frac{1}{\alpha}\cancel{B}m\cancel{t^{m-1}}f$$

for this eqn to be independent for $t \Rightarrow 2n = 1$ $n = 1/2$

or $0 = f''A^2 + \frac{\eta}{2\alpha}f'(\eta) - \frac{\cancel{B}mt^{-1}}{\alpha}f \Rightarrow 0 = f'' + \frac{\eta}{2\alpha A^2}f' - \frac{\cancel{B}m}{A^2\alpha}f \Rightarrow$ $2\alpha A^2 = 1$ $A = \frac{1}{\sqrt{2\alpha}}$

and $f'' + \eta f' - \frac{\cancel{B}m}{A^2\alpha}f = 0$ thus $\eta = \frac{Ay}{t^n} = \frac{y}{\sqrt{2\alpha t}}$ $B = \frac{1}{\beta}$ $m = 1$

$\therefore$ $u = \frac{1}{\beta}t f(\eta)$ where $f(\eta)$ satisfies $f'' + \eta f' - 2f = 0$ with.
$f(0) = 1$ and $f(\eta) \to 0$ as $\eta \to$

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PROBLEM: Infinite string $c = 1$

$w_0(x) = 0$

$w_1(x) = e^{-x^2}$

find $f(x)$, $g(x)$, $f(x+ct)$, $g(x-ct)$, $w(x,t)$

Solution:

① $$\boxed{f(x) = \frac{w_0(x)}{2} + \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)\,d\sigma}$$

$$f(x) = 0 + \frac{1}{2}\int_{x_0}^{x} e^{-\sigma^2}\,d\sigma = \frac{1}{2}\left\{-\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \int_0^{x} e^{-\sigma^2}\,d\sigma\right\}$$

$$= \frac{1}{2}\frac{\sqrt{\pi}}{2}\left\{-\frac{2}{\sqrt{\pi}}\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \frac{2}{\sqrt{\pi}}\int_0^{x} e^{-\sigma^2}\,d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x) - \operatorname{erf}(x_0)\right]$$

② $$\boxed{g(x) = \frac{w_0(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} w_1(\sigma)\,d\sigma}$$

$$g(x) = 0 - \left[\frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x) - \operatorname{erf}(x_0)\right]\right]$$

$$= \frac{\sqrt{\pi}}{4}\left[\operatorname{erf}(x_0) - \operatorname{erf}(x)\right]$$

③ $$\boxed{f(x+ct) = \frac{w_0(x+ct)}{2} + \frac{1}{2c}\int_{x_0}^{x+ct} w_1(\sigma)\,d\sigma}$$

$$f(x+ct) = 0 + \frac{1}{2c}\int_{x_0}^{x+ct} e^{-\sigma^2}\,d\sigma$$

$$= \frac{1}{2c}\left\{-\int_0^{x_0} e^{-\sigma^2}\,d\sigma + \int_0^{x+ct} e^{-\sigma^2}\,d\sigma\right.$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x_0)\right]$$

④ $$g(x-ct) = \frac{w_0(x-ct)}{2} - \frac{1}{2c}\int_{x_0}^{x-ct} w_1(\sigma)\,d\sigma$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x_0) - \operatorname{erf}(x-ct)\right]$$

⑤ $$\boxed{w(x,t) = \frac{1}{2}\left[w_0(x+ct) - w_0(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} w_1(\sigma)\,d\sigma}$$

$$w(x,t) = 0 + \frac{1}{2c}\left\{-\int_{x_0}^{x-ct} w_1(\sigma)\,d\sigma + \int_{x_0}^{x+ct} w_1(\sigma)\,d\sigma\right\}$$

$$= \frac{\sqrt{\pi}}{4c}\left[\operatorname{erf}(x+ct) - \operatorname{erf}(x-ct)\right]$$

-1

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at $K=0$

By $F'(0)=0 \Rightarrow \bar{A}=0 \Rightarrow F(x)=\bar{B}$

Then: at $K=0$ $W_h(x,t) = \bar{B}[\bar{C}t+\bar{D}] = \underline{B_1 t + B_2}$

$\boxed{\begin{array}{l} B_1 = \bar{B}\bar{C} \\ B_2 = \bar{B}\cdot\bar{D} \end{array}}$

at $K \neq 0$

By $F'(0)=0 = -AK\sin K\cdot 0 + BK\cos K\cdot 0 = 0 + BK \Rightarrow B=0$

By $F'(l)=0 = -AK\sin K\cdot l \Rightarrow Kl = n\pi \Rightarrow K = \frac{n\pi}{l}$

$\Rightarrow F(x) = A\cos\frac{n\pi}{l}x$

at $K\neq 0$ $W_h(x,t) = \sum_{n=1}^{\infty} A\cos\frac{n\pi}{l}x\,[C\cos Kct + D\sin Kct]$

$= \underline{\sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x\,[\mathcal{B}_n\cos Kct + \mathcal{D}_n\sin Kct]}$

$\boxed{\begin{array}{l} \mathcal{B}_n = AC \\ \mathcal{D}_n = AD \end{array}}$

so that general W_h is

$$W_h = B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{B}_n\cos Kct + \mathcal{D}_n\sin Kct]$$

Therefore The whole fn. of W is:

$$W = W_p + W_h$$

$$= \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + Et + B_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{B}_n\cos Kct + \mathcal{D}_n\sin Kct]$$

$$= \underline{\underline{\frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\cdot\frac{B-A}{l}t^2 + E_1 t + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\,[\mathcal{B}_n\cos\frac{n\pi c}{l}t + \mathcal{D}_n\sin\frac{n\pi c}{l}t]}}$$

where. $\boxed{E_1 = E + B_1}$

use I.C. to determine the E_1, B_2, $\mathcal{B}_n$ and $\mathcal{D}_n$

By $W(x,t=0) = 5x = \frac{B-A}{2l}x^2 + Ax + B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot\mathcal{B}_n$ <<2>>

$\Rightarrow (5-A)x - \frac{B-A}{2l}x^2 = B_2 + \sum_{n=1}^{\infty}\cos\frac{n\pi}{l}x\cdot\mathcal{B}_n$

By Fourier series

$$B_2 = \frac{1}{l}\int_0^l [(5-A)x - \frac{B-A}{2l}x^2]\,dx$$ ✓

$$\mathcal{B}_n = \frac{2}{l}\int_0^l [(5-A)x - \frac{B-A}{2l}x^2]\cdot\cos\frac{n\pi}{l}x\,dx$$ ✓

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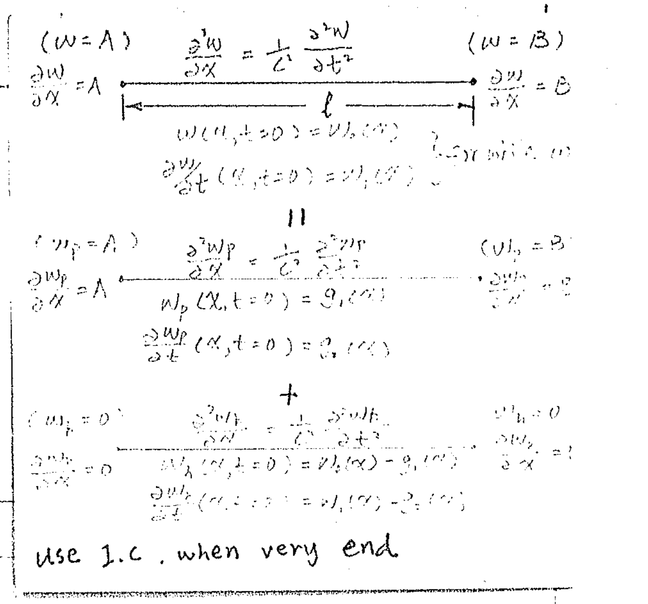
$Fn. \langle\langle 1 \rangle\rangle$ satisfies the B.C. and PDE



(2) look at w_h

we use the method of separation of variable

$$w_h(x, t) = F(x)G(t)$$

$$\frac{\partial w_h}{\partial x} = F'(x)G(t) \quad : \quad \frac{\partial^2 w_h}{\partial x^2} = F''(x)G(t)$$

$$\frac{\partial w_h}{\partial t} = F(x)G'(t) \quad : \quad \frac{\partial^2 w_h}{\partial t^2} = F(x)G''(t)$$

By $\frac{\partial^2 w_h}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w_h}{\partial t^2}$

$$F''(x)G(t) = \frac{1}{c^2}\cdot F(x)G''(t) \Rightarrow \frac{F''(x)G(t)}{F(x)G(t)} = \frac{F(x)G''(t)}{c^2\cdot F(x)G(t)} = -K^2$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{1}{c^2}\frac{G''}{G} = -K^2$$

at $K \neq 0$ $\quad F'' + K^2F = 0$, $\quad G'' + c^2K^2F = 0$ $\Rightarrow$ $F(x) = A\cos Kx + B\sin Kx$, $\quad G(t) = C\cos Kct + D\sin Kct$

at $K = 0$ $\quad F'' = 0$, $\quad G'' = 0$ $\Rightarrow$ $F(x) = \bar{A}x + \bar{B}$, $\quad G(t) = \bar{C}t + \bar{D}$

look at B.C. first

$$\frac{\partial w_h}{\partial x}(x=0, t) = 0 = F'(0)G(t) \Rightarrow F'(0) = 0$$

$$\frac{\partial w_h}{\partial x}(x=\ell, t) = 0 = F'(\ell)G(t) \Rightarrow F'(\ell) = 0$$

By $\frac{\partial w}{\partial t}(x, t=0) = 0 = E_1 + \sum_{n=1}^{\infty} \cos\frac{n\pi}{l}x \cdot \mathcal{D}_n \cdot \frac{cn\pi}{l}$

$\Rightarrow \quad E_1 = 0 \quad , \quad \mathcal{D}_n = 0$

Therefore :

$$w = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2} \cdot \frac{B-A}{l} t^2 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cos\frac{cn\pi}{l}t$$

* check the result :

B.C $\quad \frac{\partial w}{\partial x}(0,t) = \frac{B-A}{l} \cdot 0 + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l} \sin\frac{n\pi}{l} \cdot 0 = \underline{A}$

$\frac{\partial w}{\partial x}(l,t) = \frac{B-A}{l} \cdot l + A + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \frac{-n\pi}{l} \sin n\pi = \underline{B}$

I.C $\quad w(x,0) = \frac{B-A}{2l}x^2 + Ax + 0 + B_2 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x = \underline{5x}$

(according to fn. of «2»)

$\frac{\partial w}{\partial t}(x,0) = 0 + 0 + 0 + 0 + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l} \sin\frac{cn\pi}{l} \cdot 0\right) = \underline{0}$

PDE $\quad \frac{\partial w}{\partial x} = \frac{B-A}{l}x + A + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \left(-\frac{n\pi}{l} \sin\frac{n\pi}{l}x\right)$

$$\frac{\partial^2 w}{\partial x^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{cn\pi}{l}t \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{n\pi}{l}x\right] \qquad \text{«3»}$$

$\frac{\partial w}{\partial t} = c^2 \frac{B-A}{l} t + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \left(-\frac{cn\pi}{l} \cdot \sin\frac{cn\pi}{l}t\right)$

$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{cn\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right]$

$$\Rightarrow \frac{\partial^2 w}{c^2 \partial t^2} = \frac{B-A}{l} + \sum_{n=1}^{\infty} b_n \cos\frac{n\pi}{l}x \cdot \left[-\left(\frac{n\pi}{l}\right)^2 \cos\frac{cn\pi}{l}t\right] \qquad \text{«4»}$$

By «3» = «4»

$$\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2}$$

Therefore, Fn. w satisfies the B.C., I.C., and PDE.

so w is the solution of the problem.

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Problem: It is required to find the solution of the equation

$$\frac{\partial^2 w}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = 0$$

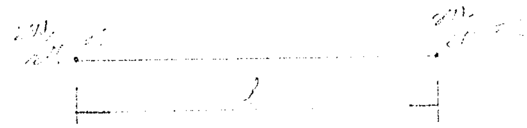
which satisfies the conditions:

$$\frac{\partial w}{\partial x}(0,t) = A \qquad \text{for all values of } t \qquad \Big\} \ 0 \le x \le l$$
$$\frac{\partial w}{\partial x}(l,t) = B$$

$$w(x,t=0) = w_0 = 5x \qquad \Big\} \ 0 < x < l$$
$$\frac{\partial w}{\partial t}(x,t=0) = w_1 = 0$$

solution: since $w(x,t) = w_p(x,t) + w_h(x,t)$

<1> First look at w_p.



assume $\partial w_p/\partial x = Cx + D$

use B.C to determine the C & D

$\partial w_p/\partial x\,(0,t) = C\cdot 0 + D = A \Rightarrow \underline{D = A} \Rightarrow \frac{\partial w_p}{\partial x} = Cx + A$

$\partial w_p/\partial x\,(l,t) = C\cdot l + A = B \Rightarrow C = \frac{B-A}{l}$

$\Rightarrow \underline{\frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A}$ ✓ (it satisfies the B.C)

$\Rightarrow \underline{w_p = \frac{B-A}{2l}x^2 + Ax + f(t)}$ (w_p must satisfies the P.D.E)

$$\frac{\partial w_p}{\partial x} = \frac{B-A}{l}x + A \qquad \frac{\partial^2 w_p}{\partial x^2} = \frac{B-A}{l}$$

$$\frac{\partial w_p}{\partial t} = f'(t) \qquad \frac{\partial^2 w_p}{\partial t^2} = f''(t)$$

since $\frac{\partial^2 w}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}$

$$\Rightarrow \frac{B-A}{l} = \frac{1}{c^2}f''(t) \Rightarrow f'(t) = c^2\cdot\frac{B-A}{l}t + E$$

$$\Rightarrow f(t) = \frac{c^2}{2}\frac{B-A}{l}t^2 + Et$$

Therefore $\Rightarrow$

$$\underline{w_p = \frac{B-A}{2l}x^2 + Ax + \frac{c^2}{2}\frac{B-A}{l}t^2 + Et} \ ✓ \qquad \langle\langle 1 \rangle\rangle$$

Cesar

for your information. Esam Hamad

EGM 3311 Analysis of Engineering Systems

FALL 2002

Final Exam

Name:________________________

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1. A mechanical part has an exponential time-to-failure distribution with mean time to failure of 10,000 hours. The part has already lasted for 15,000 hours. What is the probability that it will fail by 20,000 hours?

2. The outer diameters of 10 piston rings are found to be (in mm)

121.5 119.4 126.7 117.9 120.2 124.3 122.5 120.8 121.9 123.6

If the diameters follow normal distribution, find the 95% confidence interval for the outer diameters of the entire population of piston rings.

3. A new filtering device is being tested. Before its installation, a random sample yielded the following information about the percentage of impurity: $\overline{x_1} = 12.5$, $s_1^2 = 101.17$ and $n_1=8$. After installation, a random sample yielded $\overline{x_2} = 10.2$, $s_2^2 = 94.73$ and $n_2=9$.

a) Can you conclude that the two variances are equal?
b) Has the filtering device reduced the percentage of impurity significantly?

4. Find the solution of the following set of equations using the Gauss elimination method:

$$
\begin{aligned}
-5x_1 - x_2 + 2x_3 &= 1; \\
2x_1 + 6x_2 - 3x_3 &= 2; \\
2x_1 + x_2 + 7x_3 &= 32;
\end{aligned}
$$

5. A metal rod of length 1 m is initially at 100° C. The steady-state temperatures of the left and right ends of the rod are 150° C and 25° C, respectively. Using α^2=0.2, Δt=0.05 min and Δt=0.2 m, determine the temperature distribution in the rod at t=0.1 min. The temperature is governed by the following partial differential equation:

$$\alpha^2 \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$