CONTINUUM MECHANICS

Lecture Notes

by

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Mechanics is the study of the motion of matter and the forces that cause such motion. *Mechanics* is based on the concepts of time, space, force, energy, and matter.

A *material continuum* is a material for which the densities of mass, momentum, and energy exist in the mathematical sense. The mechanics of such a material continuum is *continuum mechanics.*

Y.C. Fung, "A first course in continuum mechanics"
Prentice-Hall Inc, 1977 ISBN 0-13-318311-4

MES 0571 -- References

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QA808.2 . E73 1980

Other References:

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2. L. E. Malvern, "Introduction to the Mechanics of a Continuous Medium", Prentice-Hall, 1969
3. C. Truesdell & R. A. Toupin, "The Classical Field Theories", Handbuch der Physik, Vol. III/1, 1960
4. C. Truesdell, "The Elements of Continuum Mechanics", Springer-Verlag, 1966
5. Y.C. Fung, "A first course in continuum mechanics", Prentice-Hall, 1977 FIU QA808.2 F85

I. Mathematical Preliminaries

A. Index Notation and Summation Convention

a_i one free index as a vector

α - scalar

A_{ij} 2 indices 2nd order

A_{ijk} 3 index, 3rd order

Matrix theory is concerned with operations with sets of numbers, i.e., arrays. Familiar examples of arrays are the column array and the square matrix:

A_{ij}
row column

$$\underset{\sim}{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \quad , \quad \underset{\sim}{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}$$

These arrays are denoted in index notation as a_i, A_{ij} $(i,j = 1,2,3)$. In matrix theory a linear set of equations is usually written

$$\underset{\sim}{A}\ \underset{\sim}{a} = \underset{\sim}{b} \qquad \text{(1A-1)}$$

where $\underset{\sim}{b}$ is another column array. This notation is called direct and leaves the multiplication $\underset{\sim}{A}\ \underset{\sim}{a}$ implicit. In index notation we make the multiplication explicit:

$$\sum_{j=1}^{3} A_{ij}\ a_j = b_i \qquad (i = 1,2,3) \qquad \text{(1A-2)}$$

The above summation on the index j conforms with the standard rules of matrix multiplication. In general, summation on the closest indices in any matrix product is the convention. In equation (1A-2) the repeated index j is called a dummy index, because any other index letter would suffice:

$$\sum_{j=1}^{3} A_{ij}\ a_j = \sum_{m=1}^{3} A_{im}\ a_m$$ (because the result will be always bi)

$$b_i = A_{ij} a_j \Rightarrow b_1 = A_{11}a_1 + A_{12}a_2 + A_{13}a_3$$

$i =$ free $\quad b_2 = A_{21}a_1 + A_{22}a_2 + A_{23}a_3$

$j =$ dummy $\quad b_3 = A_{31}a_1 + A_{32}a_2 + A_{33}a_3$

$$d_i = \underbrace{A_{ij} B_{jk}}_{C_{ik}} b_k \Rightarrow C_{11} = A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \quad (k=1)$$

$$C_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \quad (k=2)$$

$$C_{13} = A_{11}B_{13} + A_{12}B_{23} + A_{13}B_{33} \quad (k=3)$$

$$d_1 = C_{11}b_1 + C_{12}b_2 + C_{13}b_3 \qquad i=1 \quad k=1,2,3$$

$$\left[A_{ij} \right]\left[B_{jk} \right]\left[b_k \right] = \left[d_i \right]$$

Is d_i or d_k ?

$$\left[C_{ik} \right]\left[b_k \right] = \left[d_i \right]$$

The index i above is free, i.e., eqn. (1A-2) actually represents 3 eqns. corresponding to the three possible values of the free index. To simplify the notation we adopt summation convention: the summation sign is omitted and repeated indices are summed through their range of values 1,2,3. Then (1A-2) becomes

$$\Big[\ \underset{\text{free}}{A_{ij}}\ \underbrace{a_j}_{\text{dummy}} = b_i \Big] \tag{1A-3}$$

The following additional rules apply to summation convention:

Summation Convention

(i) An index (a letter name) must not appear more than twice in a product of matrices and/or vectors.

Valid: $A_{ij}\ B_{jk}\ b_k$ Invalid: $A_{jj}\ b_j$

(ii) The number and letter names of free indices in an eqn. must be the same on each side of the eqn.

Valid: $A_{ij}\ B_{jk} = C_{ik}$ Invalid: $A_{ij}\ b_j = c_k$

(iii) Any pair of repeated indices can be exchanged for another letter name.

Valid: $A_{ij}\ b_j = A_{ik}\ b_k$ Invalid: $A_{ij}\ b_j = A_{ik}\ b_j$

(iv) A free index in an eqn. may be exchanged for another letter name provided it is changed on both sides of the eqn.

Valid: Change $A_{ij}\ b_j = a_i$ to $A_{kj}\ b_j = a_k$

Invalid: Change $A_{ij}\ b_j = a_i$ to $A_{kj}\ b_j = a_i$

Products of the form $A_{ij}\ a_k$ need a special definition in direct notation, but cause no confusion in index notation and simply represent a higher ordered array, i.e.,

?

$$B_{ijk} = A_{ij}\ a_k$$

(there is no summation here)

$$\underset{\sim}{A}\,\underset{\sim}{a} = \underset{\sim}{b} \iff A_{ip}\, a_p = b_i$$

$$B\,A \iff B_{ip}A_{pj} \neq A_{ip}B_{pj}$$

?

$$B_{ijk} = A_{ij} a_k$$

What does this mean in matrix form.

Thus, $\underset{\sim}{B}$ is a 3rd order array having $3^3 = 27$ components. Clearly, index notation allows a simple representation of general products of arrays.

the raw become column and vise versa So $(A^T)_{ij} = A_{ji}$

In direct notation the transpose of a matrix $\underset{\sim}{A}$ is denoted by $\underset{\sim}{A}^T$ and is obtained by interchanging the rows and columns of $\underset{\sim}{A}$. In index notation we have

$$(\underset{\sim}{A}^T)_{ij} = A_{ji}$$

why we use Traspose

$A_{ip} B_{iq} = (A^T)_{pi} B_{iq} \Longleftrightarrow \underset{\sim}{A}^T \underset{\sim}{B}$

$A_{ip} B_{jp} = A_{ip} B^T_{pj} \Longleftrightarrow AB^T$

Some examples of the conversion between direct and index notation are the following:

$$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{C} \text{ or } A_{im} B_{mj} = C_{ij}$$

these two matricies are Not equal

$$\underset{\sim}{A}^T \underset{\sim}{B} = \underset{\sim}{C} \text{ or } (\underset{\sim}{A}^T)_{im} (\underset{\sim}{B})_{mj} = (\underset{\sim}{C})_{ij} \text{ or } \underbrace{A_{mi} B_{mj}}_{T} = C_{ij}$$

$$\underset{\sim}{A}\,\underset{\sim}{B}^T = \underset{\sim}{D} \text{ or } A_{im} B_{jm} = D_{ij}$$

$$\underset{\sim}{A}\,\underset{\sim}{B}\,\underset{\sim}{C} = \underset{\sim}{E} \text{ or } A_{im} B_{mn} C_{nj} = E_{ij}$$

Observe that in direct notation the convention for matrix products is that closest indices are summed.

Sym. Ant-S.

For an arbitrary matrix $\underset{\sim}{A}$ the symmetric and skew-symmetric parts of $\underset{\sim}{A}$ are defined as

Any Square Matrix $\underset{\sim}{A} = A^S + A^A \Longleftrightarrow A^S_{ij} + A^A_{ij}$

$$\boxed{\underset{\sim}{A}^S = \frac{1}{2}(\underset{\sim}{A} + \underset{\sim}{A}^T)} \qquad \boxed{\underset{\sim}{A}^A = \frac{1}{2}(\underset{\sim}{A} - \underset{\sim}{A}^T)} \tag{1A-4}$$

Symmetric Part of A   Anti Symmetric Part of A

$A^S = \frac{1}{2}(A + A^T)$

$A^A = \frac{1}{2}(A - A^T)$

It is clear that an arbitrary matrix can be uniquely decomposed into the sum of its symmetric and skew-symmetric parts since

$A^S_{ij} = \frac{1}{2}(A_{ij} + A^T_{ij}) = \frac{1}{2}(A_{ij} + A_{ij})$

$A^A_{ij} =$

$$\boxed{\underset{\sim}{A} = \underset{\sim}{A}^S + \underset{\sim}{A}^A} \tag{1A-5}$$

Symmetric Matrix $B_{ij} = B_{ji}$ $\bar{B} = \bar{B}^T$

then $\bar{B}^A = B_{[ij]} = 0$

(Skew) Anti-Symmetric $B_{ij} = -B_{ji}$ $\bar{B} = -\bar{B}^T$

then $B_{(ij)} = \bar{B}^S = 0$

We define the index notation equivalents of $\underset{\sim}{A}^S$, $\underset{\sim}{A}^A$ as $A_{(ij)}$, $A_{[ij]}$, respectively:

$$\boxed{A_{(ij)} = \frac{1}{2}\,(A_{ij} + A_{ji})} \qquad \boxed{A_{[ij]} = \frac{1}{2}\,(A_{ij} - A_{ji})} \tag{1A-6}$$

Symmetric Anti Symmetric

Then (1A-5) can be written as

$$\boxed{A_{ij} = A_{(ij)} + A_{[ij]}} \tag{1A-7}$$

Note that (1A-6) implies

$$A_{(ji)} \overset{\text{(defn.)}}{=} \frac{1}{2}\,(A_{ji} + A_{ij})$$

$$= \frac{1}{2}\,(A_{ij} + A_{ji}) \overset{\text{(defn.)}}{=} A_{(ij)}$$

$$A_{[ji]} \overset{\text{(defn.)}}{=} \frac{1}{2}\,(A_{ji} - A_{ij})$$

$$= -\frac{1}{2}\,(A_{ij} - A_{ji}) \overset{\text{(defn.)}}{=} -A_{[ij]}$$

Equivalently, since $(\underset{\sim}{A}^T)^T = \underset{\sim}{A}$ (1A-4) implies

$$(\underset{\sim}{A}^S)^T = \frac{1}{2}\,(\underset{\sim}{A}^T + \underset{\sim}{A}) = \frac{1}{2}\,(\underset{\sim}{A} + \underset{\sim}{A}^T) = \underset{\sim}{A}^S$$

$$(\underset{\sim}{A}^A)^T = \frac{1}{2}\,(\underset{\sim}{A}^T - \underset{\sim}{A}) = -\frac{1}{2}\,(\underset{\sim}{A} - \underset{\sim}{A}^T) = -\underset{\sim}{A}^A$$

If for a given $\underset{\sim}{B}$, $B_{ij} = B_{ji}$ (or $\underset{\sim}{B} = \underset{\sim}{B}^T$), then by defn. $B_{[ij]} = 0 \Rightarrow B_{ij} = B_{(ij)}$ and $\underset{\sim}{B}$ is a symmetric matrix. Similarly, if $B_{ij} = -B_{ji}$ (or $\underset{\sim}{B} = -\underset{\sim}{B}^T$) then $B_{(ij)} = 0$, $B_{ij} = B_{[ij]}$ and $\underset{\sim}{B}$ is skew symmetric.

The trace of $\underset{\sim}{A}$ is defined as

$$\mathrm{tr}\,\underset{\sim}{A} = A_{11} + A_{22} + A_{33} = A_{ii} \tag{1A-8}$$

Note that

$$\underline{\mathrm{tr}\,\underset{\sim}{A} = \mathrm{tr}\,\underset{\sim}{A}^T} \quad \text{because} \quad A_{\underset{\sim}{ii}} = A_{\underset{\sim}{jj}} \tag{1A-9}$$

both are dummy indices

Now if

$$\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B} \quad \text{or} \quad C_{ij} = A_{im}\,B_{mj}$$

then

$$\mathrm{tr}\,\underset{\sim}{C} = \mathrm{tr}\,\underset{\sim}{A}\,\underset{\sim}{B} = A_{im}\,B_{mi}$$

Now interchange $\underset{\sim}{A}$ and $\underset{\sim}{B}$:

$$\mathrm{tr}\,\underset{\sim}{B}\,\underset{\sim}{A} = B_{im}\,A_{mi} = A_{im}\,B_{mi}$$

Hence,

$$\boxed{\mathrm{tr}\,\underset{\sim}{A}\,\underset{\sim}{B} = \mathrm{tr}\,\underset{\sim}{B}\,\underset{\sim}{A}} \tag{1A-10}$$

Important.

Theorem 1 -- If A, B are symmetric and skew-symmetric respectively, i.e.

$$A_{ij} = A_{ji} \qquad B_{ij} = -B_{ji}$$

then

$$\boxed{A_{ij}\,B_{ij} = 0}$$

Note $A_{ij}\,B_{ij}$ can also be written $\underline{\mathrm{tr}\,\underset{\sim}{A}\,\underset{\sim}{B}^T = \mathrm{tr}\,\underset{\sim}{A}^T\,\underset{\sim}{B}}$

If A_{ij} and B_{ji} are the two matrices

Proof:

$$A_{ij}\,B_{ij} = A_{ji}\,(-\,B_{ji}) = -\,A_{ji}\,B_{ji}$$

Indices i,j are dummy on the right. Change i to j and j to i:

$$A_{ij}\,B_{ij} = -\,A_{ij}\,B_{ij}$$

which therefore must vanish. Q.E.D.

The Kronecker delta δ_{ij} and the alternator e_{ijk} are defined as

↳ or permutation symbol

Kronecker Delta
$$\delta_{ij} = \begin{Bmatrix} 1\ , & i=j \\ 0\ , & i\neq j \end{Bmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (1A\text{-}11)$$

Note the r.h.s. is just the identity matrix $\underset{\sim}{I}$.

Alternator
$$\varepsilon_{ijk} = e_{ijk} = \begin{cases} 1 \text{ if ijk are an even permutation of } 1,2,3 \\ -1 \text{ if ijk are an odd permutation of } 1,2,3 \\ 0 \text{ if 2 indices are equal} \end{cases}$$

Count the number of inversions in the entire set. (1A-12).

Even number ⇒ even permutation etc.

i.e.

$$\begin{array}{lcccc} & e_{123} = & e_{312} = & e_{231} = +1 & \\ \text{inversions} \Rightarrow & 0 & 2 & 2 & \qquad (1A\text{-}13) \\ & e_{132} = & e_{213} = & e_{321} = -1 & \\ \text{inversions} \Rightarrow & 1 & 1 & 3 & \end{array}$$

Note that δ_{ij} is symmetric: $\delta_{ij} = \delta_{ji}$. If $\underset{\sim}{a}$ is a vector, then $\underset{\sim}{I}\,\underset{\sim}{a} = \underset{\sim}{a}$. In index notation

The Kronecker Delta is <u>symmetric</u> in <u>j and i</u>

$$\delta_{ij}\, a_j = \delta_{i1}\, a_1 + \delta_{i2}\, a_2 + \delta_{i3}\, a_3$$

$$= \left\{ \begin{array}{lcl} a_1 & , & i=1 \\ a_2 & , & i=2 \\ a_3 & , & i=3 \end{array} \right\}$$

i.e. $\delta_{ij}\, a_j = a_i$ (1A-14)

Similarly, $\underset{\sim}{I}\, \underset{\sim}{A} = \underset{\sim}{A}$ becomes

ϵ_{ijk} is antisymmetric in
j and k
i and j

$$\delta_{im}\, A_{mj} = A_{ij}$$

Setting $\underset{\sim}{A} = \underset{\sim}{I}$ above implies

$$\delta_{im}\, \delta_{mj} = \delta_{ij}$$

From the defns. (1A-12), (1A-13), we see that

$$\underset{1\,2\,3}{e_{ijk}} = \underset{3\,1\,2}{e_{kij}} = \underset{2\,3\,1}{e_{jki}}$$ All even permutations!

(1A-15)

$$= -e_{jik} = -e_{ikj}$$ odd permutations

Note these relations are valid for any values of i,j,k. It can be shown that (by direct expansion)

Imp. theorem

$$\boxed{e_{ijk}\, e_{mnk} = \delta_{im}\, \delta_{jn} - \delta_{in}\, \delta_{jm}}$$ Important! (1A-16)

The operation of setting n=j above is called <u>contraction</u> and results in

Contraction

$$e_{ijk}\, e_{mjk} = \delta_{im}\, \delta_{jj} - \delta_{ij}\, \delta_{jm}$$

$$= 3\delta_{im} - \delta_{im}$$

which implies

$$e_{ijk}\,e_{mjk} = 2\delta_{im} \qquad (1A\text{-}17)$$

Similarly, m=i above implies

$$e_{ijk}\,e_{ijk} = 2\delta_{ii} = 6 \qquad (1A\text{-}18)$$

B. Determinants

The determinant of A_{ij} is defined as

$$\det \underset{\sim}{A} = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$\begin{aligned} &= A_{11}A_{22}A_{33} + A_{12}A_{23}A_{31} + A_{13}A_{21}A_{32} \\ &\quad - (A_{11}A_{23}A_{32} + A_{12}A_{\underset{21}{\cancel{23}}}A_{\underset{33}{\cancel{31}}} + A_{13}A_{\underset{22}{\cancel{21}}}A_{\underset{31}{\cancel{32}}}) \end{aligned} \qquad (1B\text{-}1)$$

Noting the signs and the ordering of indices and recalling the definition of the alternator (1A-12), (1B-1) can be expressed as

$$\det \underset{\sim}{A} = e_{ijk}\ A_{1i}\ A_{2j}\ A_{3k} \qquad (1B\text{-}2)$$

Interchanging the order of the products in (1B-1), we have the alternate form

$$\det \underset{\sim}{A} = e_{ijk}\ A_{i1}\ A_{j2}\ A_{k3} \qquad (1B\text{-}3)$$

More general expressions corresponding to (1B-2), (1B-3) are

Determinant of A_{ij}

$$\boxed{e_{mnp} \det \underset{\sim}{A} = e_{ijk}\ A_{mi}\ A_{nj}\ A_{pk}}$$

or

$$\boxed{e_{mnp} \det \underset{\sim}{A} = e_{ijk}\ A_{im}\ A_{jn}\ A_{kp}} \longrightarrow \text{Imp.} \qquad (1B\text{-}4)$$

These are verified by direct substitution: let m=2, n=1, p=3 in $(1B\text{-}4)_1$

Notes

$A_{1i} A_{1j}$ is symmetric in i and j because

NOTE: $|A| = |A^T|$

$$e_{213} \det \underset{\sim}{A} = e_{ijk} A_{2i} A_{1j} A_{3k}$$

$$- \det \underset{\sim}{A} = e_{ijk} A_{1j} A_{2i} A_{3k}$$

$$= - e_{jik} A_{1j} A_{2i} A_{3k}$$

Changing dummy names on the right,

$$\det \underset{\sim}{A} = e_{ijk} A_{1i} A_{2j} A_{3k}$$

which is identical with (1B-2). Now let m=1=n, p=2 in $(1B\text{-}4)_1$

$$\cancel{e_{112}}^{\,0} \det \underset{\sim}{A} = e_{ijk} A_{1i} A_{1j} A_{2k} \qquad (*)$$

Now e_{ijk} is skew-symmetric in indices ij for each k by (1A-15) and $A_{1i} A_{1j}$ is symmetric in ij, i.e., if

$$B_{ij} = A_{1i} A_{1j}$$

then $$B_{ji} = A_{1j} A_{1i} = A_{1i} A_{1j} = B_{ij}$$

Hence, Thm. 1 gives

$$\boxed{e_{ijk} A_{1i} A_{1j} = 0 \quad , \quad k = 1,2,3}$$

and (*) reduces to an identity.?

Eqns. (1B-4) can be solved for det $\underset{\sim}{A}$ upon multiplying by e_{mnp}:

$$e_{mnp} e_{mnp} \det \underset{\sim}{A} = 6 \det \underset{\sim}{A} = e_{mnp} e_{ijk} A_{mi} A_{nj} A_{pk}$$

Hence, $$\det \underset{\sim}{A} = \frac{1}{6} e_{mnp} e_{ijk} A_{mi} A_{nj} A_{pk} \qquad (1B\text{-}5)$$

Eqn. $(1B-4)_2$ yields the same result but with mnp exchanged with ijk.

It follows by inspection from (1B-1) that

$$\boxed{\det \underset{\sim}{A}^T = \det \underset{\sim}{A}} \qquad (1B-6)$$

Theorem 2 -- Given square matrices $\underset{\sim}{A}$, $\underset{\sim}{B}$, then

$$\det (\underset{\sim}{A}\,\underset{\sim}{B}) = \det \underset{\sim}{A} \det \underset{\sim}{B} \qquad (1B-7)$$

Proof: Let $\underset{\sim}{C} = \underset{\sim}{A}\,\underset{\sim}{B}$:

$$C_{ij} = A_{im}\,B_{mj}$$

Then (1B-2) implies

$$\begin{aligned}
\det (\underset{\sim}{A}\,\underset{\sim}{B}) = \det \underset{\sim}{C} &= e_{ijk}\,C_{1i}\,C_{2j}\,C_{3k} \\
&= e_{ijk}\,(A_{1m}B_{mi})(A_{2n}B_{nj})(A_{3p}B_{pk}) \\
&= (e_{ijk}B_{mi}B_{nj}B_{pk})\,A_{1m}\,A_{2n}\,A_{3p} \\
&\overset{(1B-4)_1}{=} e_{mnp} \det \underset{\sim}{B}\,A_{1m}\,A_{2n}\,A_{3p} \\
&\overset{(1B-2)}{=} \det \underset{\sim}{A} \det \underset{\sim}{B} \qquad \text{Q.E.D.}
\end{aligned}$$

It follows that

$$\begin{aligned}
\det (\underset{\sim}{B}\,\underset{\sim}{A}) &= \det \underset{\sim}{B} \det \underset{\sim}{A} = \det (\underset{\sim}{A}\,\underset{\sim}{B}) \\
\det (\underset{\sim}{A}\,\underset{\sim}{B}\,\underset{\sim}{C}) &= \det \underset{\sim}{A} \det \underset{\sim}{B} \det \underset{\sim}{C}
\end{aligned} \qquad (1B-8)$$

If det $\underset{\sim}{A}$ is expanded by the 1st row,

$$\det \underset{\sim}{A} = A_{11} \begin{vmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{vmatrix} - A_{12} \begin{vmatrix} A_{21} & A_{23} \\ A_{31} & A_{33} \end{vmatrix} + A_{13} \begin{vmatrix} A_{21} & A_{22} \\ A_{31} & A_{32} \end{vmatrix}$$

$$= A_{11}\, \alpha_{11} + A_{12}\, \alpha_{12} + A_{13}\, \alpha_{13} = A_{1i}\, \alpha_{1i} \qquad (1B-9)$$

The array α_{ij} is the cofactor of A_{ij}:

$\left\{ \begin{array}{l} \text{Co-factor of } A_{ij} \\ \text{or signed minor} \end{array} \right.$

Cofactors

$$\alpha_{ij} = \text{cofactor } (A_{ij}) = (-1)^{i+j} \Delta_{ij} \qquad (1B-10)$$

where Δ_{ij} is the 2x2 determinant obtained by deleting the ith row and jth column in det $\underset{\sim}{A}$. Expanding det $\underset{\sim}{A}$ about the other 2 rows gives

$$\det \underset{\sim}{A} = A_{2i}\, \alpha_{2i} = A_{3i}\, \alpha_{3i} \qquad (1B-11)$$

Can be proved

Similarly, expanding by columns gives

$$\det \underset{\sim}{A} = A_{i1}\, \alpha_{i1} = A_{i2}\, \alpha_{i2} = A_{i3}\, \alpha_{i3} \qquad (1B-12)$$

From (1B-2)

$$\det \underset{\sim}{A} = A_{1i}\, (e_{ijk}\, A_{2j}\, A_{3k}) \overset{(1B-9)}{=} A_{1i}\, \alpha_{1i}$$

which implies $\boxed{\alpha_{1i} = e_{ijk}\, A_{2j}\, A_{3k}}$ (1B-13)

Similarly (1B-2) and (1B-11) implies

even permutation

$$\left. \begin{array}{l} \alpha_{2i} = e_{ijk}\, A_{3j}\, A_{1k} \\ \alpha_{3i} = e_{ijk}\, A_{1j}\, A_{2k} \end{array} \right\} \qquad (1B-14)$$

Eqn. (1B-13) can be written as

$$\alpha_{1i} = \frac{1}{2} (e_{ijk} A_{2j} A_{3k} + e_{ikj} A_{2k} A_{3j})$$

(1A-15)

$$= \frac{1}{2} (e_{ijk} A_{2j} A_{3k} - e_{ijk} A_{2k} A_{3j})$$

$$\alpha_{1i} = \frac{1}{2} e_{ijk} (A_{2j} A_{3k} - A_{2k} A_{3j})$$

$$= \frac{1}{2} e_{ijk} e_{1np} A_{nj} A_{pk}$$

Similarly, (1B-14) implies

$$\alpha_{2i} = \frac{1}{2} e_{ijk} e_{2np} A_{nj} A_{pk} \qquad \alpha_{3i} = \frac{1}{2} e_{ijk} e_{3np} A_{nj} A_{pk}$$

Combining these three equations into a single expression, we have

$$\boxed{\alpha_{mi} = \frac{1}{2} e_{mnp} e_{ijk} A_{nj} A_{pk}} \qquad (1B\text{-}15)$$ Required

Eqns. (1B-12) lead to the same result.

If $\underset{\sim}{A}$, $\underset{\sim}{B}$ are square matrices and

$$\underset{\sim}{A}\,\underset{\sim}{B} = \underset{\sim}{B}\,\underset{\sim}{A} = \underset{\sim}{I}\ , \quad A_{im} B_{mj} = B_{im} A_{mj} = \delta_{ij} \qquad (1B\text{-}16)$$

then $\underset{\sim}{B}$ is the inverse to $\underset{\sim}{A}$ and denoted by

$$\underset{\sim}{B} = \underset{\sim}{A}^{-1}\ , \quad B_{ij} = A^{-1}_{ij}$$

If det $\underset{\sim}{A} \neq 0$, $\underset{\sim}{A}$ is called non-singular.

Theorem 3. -- If $\underset{\sim}{A}$ is a non-singular square matrix, then $\underset{\sim}{A}^{-1}$ is given by

det $A \neq 0$

Inverse Matrix

$$\boxed{A^{-1}_{ij} = \frac{\alpha_{ji}}{\det A}} \qquad (1B\text{-}17)$$

It can be directly by Cramer's Theorem $A_{ij} x_j = b_i$

1) IF det $A \neq 0$

2) α_{ij} is the cofactor of A_{ij} (1B-10)

where α_{ij} = cofactor (A_{ij})

Proof: Multiply (1B-15) by A_{mq}:

$$\alpha_{mi}\ A_{mq} = \frac{1}{2}\ e_{ijk}\ e_{mnp}\ A_{mq}\ A_{nj}\ A_{pk}$$

$$= \frac{1}{2}\ (e_{mnp}\ A_{mq}\ A_{nj}\ A_{pk})\ e_{ijk}$$

$$\overset{(1B-4)_2}{=} \frac{1}{2}\ (e_{qjk}\ \det \underset{\sim}{A})\ e_{ijk}$$

$$\alpha_{mi}\ A_{mq} \overset{(1A-17)}{=} \frac{1}{\cancel{2}}\ (\cancel{2}\delta_{qi})\ \det \underset{\sim}{A} \qquad (1B-18)$$

Hence,

$$\left(\frac{\alpha_{mi}}{\det \underset{\sim}{A}}\right) A_{mq} = \delta_{qi}$$

Comparison with $B_{im}\ A_{mq} = \delta_{iq}$ implies

$$B_{im} = A^{-1}_{im} = \frac{\alpha_{mi}}{\det \underset{\sim}{A}}$$

Using (1B-17), it can be shown that the other half of (1B-16), i.e., $B_{im}\ A_{mj} = \delta_{ij}$, is satisfied.

Theorem 4 — If $\underset{\sim}{A}$ is a square matrix with cofactor matrix $\underset{\sim}{\alpha}$, then

$$\frac{\partial}{\partial A_{ij}} (\det \underset{\sim}{A}) = \alpha_{ij} \qquad (1B-19)$$

The only time That ij apear in den.

Do not Devide

$(A^{-1})A = (A^{-1}) \ldots\ldots$

only mult. by the Inv.

Proof. Use (1B-9):

$$\frac{\partial}{\partial A_{ij}} (\det \underset{\sim}{A}) = \frac{\partial}{\partial A_{ij}} (A_{li}\ \alpha_{li})$$

$$= \frac{\partial}{\partial A_{1j}} (A_{11}\alpha_{11} + A_{12}\alpha_{12} + A_{13}\alpha_{13})$$

$$= \left\{ \begin{matrix} \alpha_{11} & , & j=1 \\ \alpha_{12} & , & j=2 \\ \alpha_{13} & , & j=3 \end{matrix} \right\} = \alpha_{1j}$$

From (1B-11)

$$\frac{\partial}{\partial A_{2j}} (\det \underset{\sim}{A}) = \alpha_{2j}$$

$$\frac{\partial}{\partial A_{3j}} (\det \underset{\sim}{A}) = \alpha_{3j}$$

Combining the 3 results:

$$\frac{\partial}{\partial A_{ij}} (\det \underset{\sim}{A}) = \alpha_{ij} \qquad \text{Q.E.D.}$$

Theorem 5 — A set of linear, homogeneous eqns

$A_{ij} u_j = b_i$ Kramer's theorem. $u_j = \frac{\alpha_{ji} b_i}{\det A}$ If = 0

$$A_{ij} u_j = 0 \qquad (*)$$

must be ≠

evens $u_1, u_2, u_3 \ldots = 0$ (trivial soln)

has a solution other than $u_j \equiv 0$ if and only if $\det \underset{\sim}{A} = 0$.

Proof: Let $\tilde{u}_j \neq 0$ be a solution of (*) and mult. (*) by α_{im}:

(1B-18)

$$0 = \alpha_{im} A_{ij} \tilde{u}_j = \delta_{mj} \det \underset{\sim}{A} \tilde{u}_j = \det \underset{\sim}{A} \tilde{u}_m$$

which is satisfied for $\tilde{u}_m \neq 0$ if and only if $\det \underset{\sim}{A} = 0$. Q.E.D.

C. Base Vectors, Orthogonal Transformations and Cartesian Tensors

Let x_i (i=1,2,3) be a right-handed rectangular cartesian coordinate system. We define a set of unit vectors $\underset{\sim}{e}_i$ along the coordinate axes x_i. Then $\underset{\sim}{e}_i$ forms an orthonormal triad, i.e.,

Orthonormal System
(Basis)

$$\begin{aligned} \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_1 = \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_2 = \underset{\sim}{e}_3 \cdot \underset{\sim}{e}_3 = 1 \\ \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_2 = \underset{\sim}{e}_1 \cdot \underset{\sim}{e}_3 = \underset{\sim}{e}_2 \cdot \underset{\sim}{e}_3 = 0 \end{aligned} \qquad (1C\text{-}1)$$

The $\underset{\sim}{e}_i$ are commonly called base vectors. Eqns. (1C-1) can be concisely written as

$$\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij} \qquad (1C\text{-}2)$$

Compact Notation

Since $\underset{\sim}{e}_i$ forms a right-handed system, we have

$$\begin{aligned} \underset{\sim}{e}_1 \times \underset{\sim}{e}_2 &= \underset{\sim}{e}_3 = - \underset{\sim}{e}_2 \times \underset{\sim}{e}_1 \ , \quad \underset{\sim}{e}_1 \times \underset{\sim}{e}_1 = 0 \\ \underset{\sim}{e}_2 \times \underset{\sim}{e}_3 &= \underset{\sim}{e}_1 = - \underset{\sim}{e}_3 \times \underset{\sim}{e}_2 \ , \quad \underset{\sim}{e}_2 \times \underset{\sim}{e}_2 = 0 \\ \underset{\sim}{e}_3 \times \underset{\sim}{e}_1 &= \underset{\sim}{e}_2 = - \underset{\sim}{e}_1 \times \underset{\sim}{e}_3 \ , \quad \underset{\sim}{e}_3 \times \underset{\sim}{e}_3 = 0 \end{aligned}$$

Using the alternator e_{ijk}, these equations can be written as

$$\underset{\sim}{e}_i \times \underset{\sim}{e}_j = e_{ijk} \underset{\sim}{e}_k \qquad (1C\text{-}3)$$

Compact Notation

Taking the dot product of (1C-3) with $\underset{\sim}{e}_k$ and using (1C-2),

$$\begin{aligned} \underset{\sim}{e}_i \times \underset{\sim}{e}_j \cdot \underset{\sim}{e}_k &= e_{ijm} \underset{\sim}{e}_m \cdot \underset{\sim}{e}_k \\ &\overset{(1C-2)}{=} e_{ijm} \delta_{mk} = e_{ijk} \end{aligned} \qquad (1C\text{-}4)$$

This equation expresses the fact that $\underset{\sim}{e}_i$ are a right-handed triad.

A set of vectors $\underset{\sim}{f}_i$ is called <u>linearly independent</u> if

$$\alpha_i \underset{\sim}{f}_i = 0 \text{ implies } \alpha_i = 0$$

Linear independent System for $\underset{\sim}{f}_i$

Using the defined set $\underset{\sim}{e}_i$, take the dot product of $\underset{\sim}{e}_j$ with the eqn. $\alpha_i \underset{\sim}{e}_i = 0$:

$$0 = \alpha_i \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j \overset{(1C-2)}{=} \alpha_i \delta_{ij} = \alpha_j$$

Hence, $\underset{\sim}{e}_i$ are linearly independent and forms a <u>basis</u> for the 3-dimensional space such that every vector $\underset{\sim}{a}$ can be uniquely expressed in terms of its components a_i with respect to $\underset{\sim}{e}_i$:

$$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3 = a_i \underset{\sim}{e}_i \tag{1C-5}$$

Note that the components a_j are obtained by dotting the above equation with $\underset{\sim}{e}_j$

$$\underset{\sim}{a} \cdot \underset{\sim}{e}_j = a_i \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j \overset{(1C-2)}{=} \delta_{ij} a_i = a_j$$

Given vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$, $\underset{\sim}{c}$ with components a_i, b_i, c_i we have the following products:

<u>Dot Product</u>

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = (a_i \underset{\sim}{e}_i) \cdot (b_j \underset{\sim}{e}_j)$$

$$\overset{(1C-2)}{=} a_i b_j \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = a_i b_j \delta_{ij} = a_i b_i$$

$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i$ Dot Product

Cross Product

$$\underset{\sim}{a} \times \underset{\sim}{b} = (a_i \underset{\sim}{e_i}) \times (b_j \underset{\sim}{e_j}) = a_i b_j \underset{\sim}{e_i} \times \underset{\sim}{e_j}$$

(1C-3)
$$= a_i b_j e_{ijk} \underset{\sim}{e_k} \qquad (Vector)$$

?

which implies the components of $\underset{\sim}{a} \times \underset{\sim}{b}$ are $a_i b_j e_{ijk}$ since

$$(\underset{\sim}{a} \times \underset{\sim}{b}) \cdot \underset{\sim}{e_m} = a_i b_j e_{ijk} \underset{\sim}{e_k} \cdot \underset{\sim}{e_m} = a_i b_j e_{ijk} \delta_{km}$$
$$= a_i b_j e_{ijm} \qquad (component)$$

Triple Product

$$\underset{\sim}{a} \times \underset{\sim}{b} \cdot \underset{\sim}{c} = (a_i \underset{\sim}{e_i}) \times (b_j \underset{\sim}{e_j}) \cdot (c_k \underset{\sim}{e_k})$$
$$= a_i b_j c_k \underset{\sim}{e_i} \times \underset{\sim}{e_j} \cdot \underset{\sim}{e_k}$$

(1C-4)
$$\underset{\sim}{a} \times \underset{\sim}{b} \cdot \underset{\sim}{c} = e_{ijk} a_i b_j c_k$$

→ We now consider a linear transformation of $\underset{\sim}{e_i}$ into $\underset{\sim}{\bar{e}_i}$:

$$\underset{\sim}{\bar{e}_i} = Q_{ij} \underset{\sim}{e_j} \qquad \text{Linear Transformation of Coordinate System from } \bar{e}_i \text{ to } e_j \qquad (1C\text{-}6)$$

where $\underset{\sim}{Q}$ is an <u>orthogonal</u> matrix, i.e.

$$\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^T$$

Then from (1B-16)

$$\underset{\sim}{Q}\,\underset{\sim}{Q}^T = \underset{\sim}{Q}^T \underset{\sim}{Q} = I \qquad (1C\text{-}7)$$

or
$$Q_{im} Q_{jm} = Q_{mi} Q_{mj} = \delta_{ij}$$

Since $\underset{\sim}{Q}$ is orthogonal, the new basis $\bar{\underset{\sim}{e}}_i$ is also orthonormal:

$$\bar{\underset{\sim}{e}}_i \cdot \bar{\underset{\sim}{e}}_j = (Q_{im}\,\underset{\sim}{e}_m) \cdot (Q_{jn}\,\underset{\sim}{e}_n)$$

$$\overset{(1C-2)}{=} Q_{im}\,Q_{jn}\,\delta_{mn} = Q_{im}\,Q_{jm} \overset{(1C-7)}{=} \delta_{ij} \checkmark$$

Now solve (1C-6) for $\underset{\sim}{Q}$, i.e., dot with $\underset{\sim}{e}_j$:

$$\bar{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_j = (Q_{im}\,\underset{\sim}{e}_m) \cdot \underset{\sim}{e}_j = Q_{im}\,\delta_{mj} = Q_{ij} \qquad (1C-8)$$

From the definition $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \cos\theta$

$$Q_{ij} = \bar{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_j = \cos(\bar{\underset{\sim}{e}}_i, \underset{\sim}{e}_j) \qquad (1C-9)$$

Direction Cosine Matrix

which implies $\underset{\sim}{Q}$ is the direction cosine matrix relating $\bar{\underset{\sim}{e}}_i$ to $\underset{\sim}{e}_j$. Note that from (1C-8)

$$Q_{ij} = \bar{\underset{\sim}{e}}_i \cdot \underset{\sim}{e}_j \neq \bar{\underset{\sim}{e}}_j \cdot \underset{\sim}{e}_i = Q_{ji}$$

Using (1C-7), we can solve (1C-6) for $\underset{\sim}{e}_i$, i.e., multiply by Q_{im}:

$$Q_{im}\,\bar{\underset{\sim}{e}}_i = Q_{im}\,Q_{ij}\,\underset{\sim}{e}_j \overset{(1C-7)}{=} \delta_{mj}\,\underset{\sim}{e}_j = \underset{\sim}{e}_m$$

which implies

$$\underset{\sim}{e}_i = Q_{ji}\,\bar{\underset{\sim}{e}}_j \qquad (1C-10)$$

Hence, we can transform in either direction:

$$\bar{\underset{\sim}{e}}_i = Q_{ij}\,\underset{\sim}{e}_j \;, \quad \underset{\sim}{e}_i = Q_{ji}\,\bar{\underset{\sim}{e}}_j \qquad (1C-11)$$

Note the order of the indices above.

Since $\underset{\sim}{\bar{e}}_i$ are orthonormal, we can associate a rectangular cartesian coordinate system $\bar{x}_i$ with $\underset{\sim}{\bar{e}}_i$. Then the coordinates of a point in space are different for the two systems but are related by $\underset{\sim}{Q}$. Consider the position vector $\underset{\sim}{r}$ of a point in space (see Fig. I-1). Then

$$\underset{\sim}{r} = \bar{x}_i \, \underset{\sim}{\bar{e}}_i = x_n \, \underset{\sim}{e}_n \qquad (1C-12)$$

$$\overset{(1C-11)}{=} x_n \, Q_{in} \, \underset{\sim}{\bar{e}}_i$$

that is

$$(\bar{x}_i - Q_{in} \, x_n) \, \underset{\sim}{\bar{e}}_i = 0$$

which implies

$$\bar{x}_i = Q_{ij} \, x_j \qquad (1C-13)$$

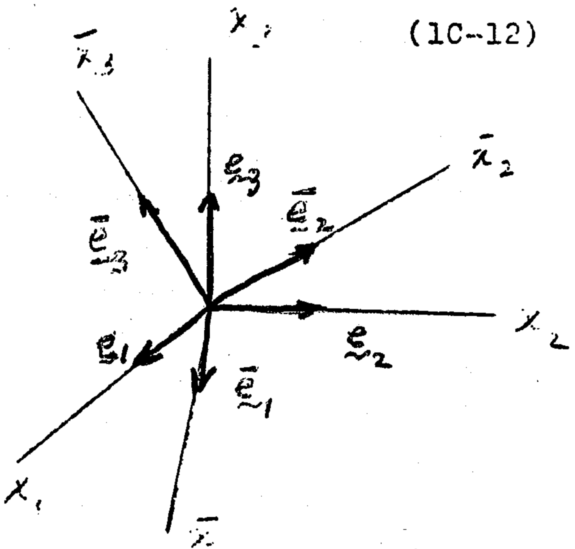


FIG. I-1

since $\underset{\sim}{\bar{e}}_i$ are linearly independent. Eqn. (1C-12) can be inverted to yield

$$x_i = Q_{ji} \, \bar{x}_j \qquad (1C-14)$$

An orthogonal transformation leaves the lengths of vectors invariant. Consider $\underset{\sim}{r} = \bar{x}_i \, \underset{\sim}{\bar{e}}_i$. Then

$$|\underset{\sim}{r}|^2 = \underset{\sim}{r} \cdot \underset{\sim}{r} = (\bar{x}_i \, \underset{\sim}{\bar{e}}_i) \cdot (\bar{x}_j \, \underset{\sim}{\bar{e}}_j)$$

$$= \bar{x}_i \, \bar{x}_j \, \delta_{ij} = \bar{x}_i \, \bar{x}_i$$

$$\overset{(1C-13)}{=} (Q_{im} \, x_m)(Q_{in} \, x_n)$$

$$= Q_{im} \, Q_{in} \, x_m \, x_n = \delta_{mn} \, x_m \, x_n = x_m \, x_m$$

Rotation: $\det \underset{\sim}{Q} = +1$

Reflection: $\det \underset{\sim}{Q} = -1$ (also Reflection combined with rotation)

Hence,

$$\bar{x}_i \bar{x}_i = x_i x_i$$

From (1C-7)

$$\det (Q_{im} Q_{jm}) = \det (\underset{\sim}{Q} \underset{\sim}{Q}^T) = (\det \underset{\sim}{Q})(\det \underset{\sim}{Q}^T)$$

$$= (\det \underset{\sim}{Q})^2 = \det \delta_{ij} = 1$$

Hence,

$$\det \underset{\sim}{Q} = \pm 1$$

When $\det \underset{\sim}{Q} = + 1$, then $\underset{\sim}{Q}$ is called a proper orthogonal matrix, and the transformation represents a rotation of coordinates. For example, if

$$Q_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then from (1C-6)

$$\underset{\sim}{\bar{e}}_1 = \underset{\sim}{e}_2 \;, \quad \underset{\sim}{\bar{e}}_2 = \underset{\sim}{e}_3 \;, \quad \underset{\sim}{\bar{e}}_3 = \underset{\sim}{e}_1$$

Note that the basis $\underset{\sim}{\bar{e}}_i$ is right handed. (See Fig. I-2). If $\det \underset{\sim}{Q} = - 1$, then $\underset{\sim}{Q}$ is called an improper orthogonal matrix and the transformation represents a reflection of coordinates or a rotation combined with a reflection. For example, the transformation

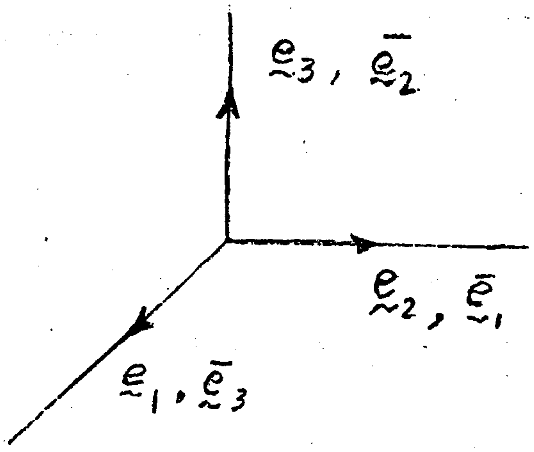


Fig. I-2

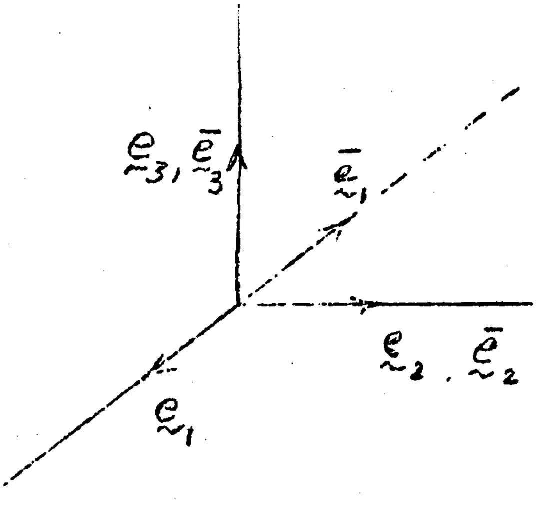


Fig. I-3

$$Q_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

implies from (1C-6)

$$\bar{\underset{\sim}{e}}_1 = - \underset{\sim}{e}_1 \quad , \quad \bar{\underset{\sim}{e}}_2 = \underset{\sim}{e}_2 \quad , \quad \bar{\underset{\sim}{e}}_3 = \underset{\sim}{e}_3$$

This is a reflection about the $\underset{\sim}{e}_2 - \underset{\sim}{e}_3$ plane, giving a left-handed basis $\bar{\underset{\sim}{e}}_i$. (See Fig. I-3).

Consider the triple product

$$\bar{\underset{\sim}{e}}_m \times \bar{\underset{\sim}{e}}_n \cdot \bar{\underset{\sim}{e}}_p \overset{(1C-6)}{=} (Q_{mi}\, \underset{\sim}{e}_i) \times (Q_{nj}\, \underset{\sim}{e}_j) \cdot (Q_{pk}\, \underset{\sim}{e}_k)$$

$$= Q_{mi}\, Q_{nj}\, Q_{pk}\, \underset{\sim}{e}_i \times \underset{\sim}{e}_j \cdot \underset{\sim}{e}_k$$

$$= Q_{mi}\, Q_{nj}\, Q_{pk}\, e_{ijk}$$

$$\overset{(1B-4)}{=} e_{mnp} \det \underset{\sim}{Q}$$

$$= \begin{cases} + e_{mnp} & \text{when } \underset{\sim}{Q} \text{ proper implies } \bar{\underset{\sim}{e}}_i \text{ right-handed} \\ - e_{mnp} & \text{when } \underset{\sim}{Q} \text{ improper implies } \bar{\underset{\sim}{e}}_i \text{ left-handed} \end{cases}$$

since for example

$$\bar{\underset{\sim}{e}}_1 \times \bar{\underset{\sim}{e}}_2 \cdot \bar{\underset{\sim}{e}}_3 = \begin{cases} + e_{123} = +1 \text{ when } \underset{\sim}{Q} \text{ proper} \\ - e_{123} = -1 \text{ when } \underset{\sim}{Q} \text{ improper} \end{cases}$$

We will consider only proper transformations (rotations) and call these <u>admissible</u>.

As we know, a vector is a quantity whose components depend on the basis used and change in a particular way when the basis is transformed. For example, the components of $\underset{\sim}{r}$, the position vector of a point in space, change according to the rule (1C-13). Note that the vector itself does not change: roughly speaking it is invariant. These ideas lead to the following general definitions of a vector:

Definition 1 -- A set of numbers u_i are components of a vector (or 1st order tensor) $\underset{\sim}{u}$ if under rotations of $x_i \rightarrow \bar{x}_i$, $u_i \rightarrow \bar{u}_i$ such that

$$\bar{u}_i = Q_{ij}\, u_j \qquad (1C\text{-}15)$$

Given any vector $\underset{\sim}{v}$, its components must satisfy (1C-15) since

$$\underset{\sim}{v} = \bar{v}_i\, \bar{\underset{\sim}{e}}_i = v_i\, \underset{\sim}{e}_i \overset{(1C\text{-}10)}{=} v_i\, Q_{mi}\, \bar{\underset{\sim}{e}}_m$$

Definitions of Vectors

i.e. $(\bar{v}_m - v_i\, Q_{mi})\, \bar{\underset{\sim}{e}}_m = 0$

which implies

$$\bar{v}_m = Q_{mi} \cdot v_i$$

For an inverted form equivalent to (1C-15), multiply by Q_{im}:

$$Q_{im}\, \bar{u}_i = Q_{im}\, Q_{ij}\, u_j = \delta_{mj}\, u_j = u_m$$

Definition 2 -- A number α is a scalar if it is invariant under rotations $x_i \rightarrow \bar{x}_i$:

$$\bar{\alpha} = \alpha$$

RCCS - Right Handed Cartesian Coordinate System.

Note that a set of 3 scalars (α, β, γ) are not components of a vector since $(\bar{\alpha}, \bar{\beta}, \bar{\gamma}) = (\alpha, \beta, \gamma)$ for all rotations. except if each equals zero it will form zero vector

Definition 3 -- A set of nine numbers A_{ij} are components of a <u>2nd order</u> <u>tensor</u> $\underset{\sim}{A}$ if under rotations of $x_i \rightarrow \bar{x}_i$, $A_{ij} \rightarrow \bar{A}_{ij}$ such that

$$\bar{A}_{ij} = \underbrace{Q_{im} Q_{jn}}_{2Q's} A_{mn} \quad \Big\} \text{ Tensor trans. law} \qquad (1C\text{-}16)$$

$$\bar{A}_{ijk} = Q_{ip} Q_{jq} Q_{kr} A_{pqr} \text{ and so on}$$

The inverted form is

$$A_{mn} = Q_{im} Q_{jn} \bar{A}_{ij}$$

The general transformation rule of an Nth order tensor is

$$\underbrace{\bar{B}_{ij\ldots}\qquad}_{N \text{ indices}} = (\overbrace{Q_{im} Q_{jn} \qquad\qquad}^{N \text{ products}}) \underbrace{B_{mn\ldots}}_{N \text{ indices}}$$

Since we are considering rotations of RCCS, these tensors are called <u>cartesian</u>. Proper transformations Only! (Rotations)

<u>Note</u>: From the above definitions, it follows that:

(a) If the components of a tensor vanish in one coordinate system, they vanish in all admissible systems, i.e., $v_i = 0$ implies

$$\bar{v}_i = Q_{ij} v_j = 0$$

to add 2 matrix
add corresp. ~~matrix~~ element. ✓

(b) The sum or difference of tensors of the <u>same</u> order is a tensor of that order, i.e., if A_{ij}, B_{ij} are 2nd order tensors, then

$$\bar{A}_{ij} \pm \bar{B}_{ij} = Q_{im}\, Q_{jn}\, A_{mn} \pm Q_{im}\, Q_{jn}\, B_{mn}$$
$$= Q_{im}\, Q_{jn}\, (A_{mn} \pm B_{mn})$$

which implies

$\underset{\sim}{A} \pm \underset{\sim}{B}$ 2nd order tensor

(c) If $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are tensors and the eqn.

$$A_{ij} = B_{ij} \qquad (*)$$

holds in one coordinate system x_i, then it holds in any admissible system $\bar{x}_i$, i.e.

$$\bar{A}_{ij} = \bar{B}_{ij}$$

Now properties (a), (b) imply (c). The quantities $A_{ij} - B_{ij}$ are components of a tensor by (b) and vanish in x_i system by (*). Then (a) implies $A_{ij} - B_{ij}$ must vanish in any admissible $\bar{x}_i$, i.e., $\bar{A}_{ij} = \bar{B}_{ij}$. Hence, like vector equations, tensor equations are independent of the particular RCCS used.

The following theorems illustrate how one determines tensor character of given arrays.

Theorem 6 -- If $\underset{\sim}{u}$ and $\underset{\sim}{v}$ are vectors, then the outer product $u_i\, v_j$ are components of a 2nd order tensor.

Proof: Since $\underset{\sim}{u}$, $\underset{\sim}{v}$ are vectors

Outer Product

$$\bar{u}_i = Q_{im}\, u_m \quad , \quad \bar{v}_j = Q_{jn}\, v_n$$

under rotations of $x_i \rightarrow \bar{x}_i$. Then

outer product $u_i v_j$
dot product $\underset{\sim}{u}\cdot\underset{\sim}{v}$ ~ $u_i v_i$
cross product $\underset{\sim}{u} \times \underset{\sim}{v}$

$$\bar{u}_i \; \bar{v}_j = (Q_{im}\, u_m)(Q_{jn}\, v_n)$$

$$= Q_{im}\, Q_{jn}\, u_m\, v_n \qquad (*)$$

which implies $u_i \, v_j$ is a 2nd order tensor by (1C-16). see (1A-16) Note that <u>contracting</u> (*) yields

$$\bar{u}_i \; \bar{v}_i = Q_{im}\, Q_{in}\, u_m\, v_n = \delta_{mn}\, u_m\, v_n$$

$$= u_m\, v_m = u_i\, v_i$$

Hence, the inner product $\underset{\sim}{u} \cdot \underset{\sim}{v}$ transforms as a scalar.

<u>Theorem 7</u> --- If $\underset{\sim}{u}$, $\underset{\sim}{v}$ are vectors and

$$u_i = A_{ij}\, v_j \qquad (1C\text{-}17)$$

tensor if satisfies the tensor trans. laws

is a tensor equation, i.e., <u>it holds in any coordinate system</u>, then $\underset{\sim}{A}$ is a 2nd order tensor.

<u>Proof</u>: Since $\underset{\sim}{u}$, $\underset{\sim}{v}$ are vectors, then under rotations $x_i \to \bar{x}_i$

$$\bar{u}_i = Q_{im}\, u_m \quad , \quad \bar{v}_j = Q_{jn}\, v_n$$

Hence,

$$\bar{u}_i = Q_{im}\, u_m \overset{(1C\text{-}17)}{=} Q_{im}\, A_{mn}\, v_n \qquad (1C\text{-}18)$$

But $\bar{v}_j = Q_{jn}\, v_n$ implies $v_n = Q_{jn}\, \bar{v}_j$ and (1C-18) becomes

$$\bar{u}_i = Q_{im}\, A_{mn}\, Q_{jn}\, \bar{v}_j \qquad (1C\text{-}19)$$

Now (1C-17) is valid in the barred system:

$$\bar{u}_i = \bar{A}_{ij}\, \bar{v}_j \overset{(1C\text{-}19)}{=} Q_{im}\, Q_{jn}\, A_{mn}\, \bar{v}_j$$

i.e. $$(\bar{A}_{ij} - Q_{im} Q_{jn} A_{mn}) \bar{v}_j = 0$$

Then for non-vanishing $\bar{v}_j$, we must have

$$\bar{A}_{ij} = Q_{im} Q_{jn} A_{mn} \qquad (\dagger)$$

Hence, A_{ij} are components of a 2nd order tensor by (1C-16). By contracting (†)

$$\bar{A}_{ii} = Q_{im} Q_{in} A_{mn} = A_{mm} = A_{ii}$$

i.e., $A_{ii} = \text{tr}\,\underset{\sim}{A}$ transforms as a scalar.

trace is the same in all coord. systems because being scalar

Note that the operation of contracting a tensor of order $N \geq 2$ always results in a tensor of order $N - 2$. For a 3rd order tensor B_{ijk} there are 3 ways of contracting: B_{iik}, B_{iji}, B_{ijj} with each a vector, i.e., since

$$\bar{B}_{ijk} = Q_{im} Q_{jn} Q_{kp} B_{mnp}$$

then

$$\bar{B}_{iik} = (Q_{im} Q_{in}) Q_{kp} B_{mnp} = Q_{kp} B_{mmp}$$

if one free index left so you have a vector

which implies B_{iik} satisfies the vector transformation law (1C-15).

1) A *tensor field* assigns a tensor $\tilde{T}(\tilde{x}, t)$ to every pair $(\tilde{x}, t)$ where the *position vector* $\tilde{x}$ varies over a particular region of space, and t varies over a particular interval of time.

2) A tensor field is said to be *continuous* (or differentiable) if the components of $\tilde{T}(\tilde{x}, t)$ are continuous functions of $\bar{x}$ and t.

from here to the end

tensor field if satisfies the tensor law

D. Tensor Fields

If the components of a cartesian tensor $\underset{\sim}{A}$ of order N are functions of x_i in some region R of space and $\underset{\sim}{A}$ satisfies the transformation law at each point in R, then $\underset{\sim}{A}(x_i)$ is called a tensor field. If the components of $\underset{\sim}{A}$ are differentiable, then $\frac{\partial \underset{\sim}{A}}{\partial x_i}$ is a cartesian tensor of order N+1. We prove this for a scalar field:

Theorem 8 -- If $\lambda(\underset{\sim}{x})$ is a scalar field, then $\frac{\partial\lambda}{\partial x_i}$ are the components of a vector.

Proof: Under rotations of $\underset{\sim}{x} \to \underset{\sim}{\bar{x}}$

$$x_j = Q_{mj}\,\bar{x}_m \tag{1D-1}$$

which implies $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{\bar{x}})$ and $\lambda(\underset{\sim}{x})$ becomes a new function of $\underset{\sim}{\bar{x}}$: $\lambda(\underset{\sim}{x}) = \lambda(\underset{\sim}{x}(\underset{\sim}{\bar{x}})) = \bar{\lambda}(\underset{\sim}{\bar{x}})$. But at each point in space with coordinates x_i or $\bar{x}_i$, λ is a scalar which implies

$$\bar{\lambda}(\underset{\sim}{\bar{x}}) = \lambda(\underset{\sim}{x}) \tag{1D-2}$$

Since $x = x(\bar{x})$,

$$\frac{\partial\bar{\lambda}}{\partial\bar{x}_i} = \frac{\partial\lambda}{\partial\bar{x}_i} = \frac{\partial\lambda}{\partial x_j}\,\frac{\partial x_j}{\partial\bar{x}_i} \tag{*}$$

But (1D-1) implies

$$\frac{\partial x_j}{\partial\bar{x}_i} = \frac{\partial}{\partial\bar{x}_i}\left(Q_{1j}\,\bar{x}_1 + Q_{2j}\,\bar{x}_2 + Q_{3j}\,\bar{x}_3\right)$$

$$= \begin{Bmatrix} Q_{1j}\ , & i=1 \\ Q_{2j}\ , & i=2 \\ Q_{3j}\ , & i=3 \end{Bmatrix} = Q_{ij}$$

?

Why switch to ij from mj

Hence (*) implies

$$\boxed{\frac{\partial \bar{\lambda}}{\partial \bar{x}_i} = Q_{ij} \frac{\partial \lambda}{\partial x_j}}$$

which is the <u>transformation law for vectors.</u> Q.E.D. The extension of this theorem to higher ordered tensors is proved in the same way. We note the special cases

v_i (vector) implies $\frac{\partial v_i}{\partial x_j}$ (2nd order tensor)

A_{ij} (2nd order tensor) implies $\frac{\partial A_{ij}}{\partial x_k}$, (3rd order tensor)

Since $\frac{\partial \lambda}{\partial x_i}$ are components of a vector, we define the <u>gradient</u> or <u>del</u> operator

$$\text{grad}(\ ) = \underset{\sim}{\nabla}(\ ) = \underset{\sim}{e}_i \frac{\partial}{\partial x_i} \qquad (1D\text{-}3)$$

Then grad λ is a vector having the representation

$$\text{grad } \lambda = \underset{\sim}{e}_i \frac{\partial \lambda}{\partial x_i}$$

<u>Notation</u>: Usually when working <u>with a single RCCS x_i</u>, partial derivatives are written as

$$\frac{\partial \lambda}{\partial x_i} = \lambda_{,i}$$

<u>Notation</u>

Using the $\underset{\sim}{\nabla}$ operator, we define <u>divergence</u> and <u>curl</u> of a vector $\underset{\sim}{v}$ as

<u>Notation</u>

$$\text{div } \underset{\sim}{v} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} \ , \quad \text{curl } \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} \qquad (1D\text{-}4)$$

From Schaum's

$$\frac{\partial \phi}{\partial x_i} = \phi_{,i}$$

$$\frac{\partial V_i}{\partial x_i} = V_{i,i}$$

$$\frac{\partial V_i}{\partial x_j} = V_{i,j}$$

$$\nabla \phi = \frac{\partial \phi}{\partial x_i} \bar{e}_i = \partial_i \phi = \phi_{,i}$$

$$\nabla \cdot \bar{V} = \partial_i V_i = V_{i,i}$$

$$\nabla \times \bar{V} = \epsilon_{ijk} \partial_j V_k = \epsilon_{ijk} V_{k,j}$$

$$\nabla^2 \phi = \nabla \cdot \nabla \phi = \partial_{ii} \phi = \phi_{,ii}$$

For a component form of div $\underset{\sim}{v}$, express $\underset{\sim}{v}$ in component form and use (1D-3)

$$\text{div } \underset{\sim}{v} = (\underset{\sim}{e}_i \frac{\partial}{\partial x_i}) \circ (v_j \underset{\sim}{e}_j)$$

$$= \underset{\sim}{e}_i \circ \underset{\sim}{e}_j \frac{\partial v_j}{\partial x_i} \qquad (\text{since } \underset{\sim}{e}_i \text{ constant})$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \delta_{ij} v_{j,i} = v_{i,i} \qquad \text{divergence} \qquad (1D\text{-}5)$$

Similarly for curl $\underset{\sim}{v}$:

$$\text{curl } \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} = (\underset{\sim}{e}_i \frac{\partial}{\partial x_i}) \times (v_j \underset{\sim}{e}_j)$$

$$= \frac{\partial v_j}{\partial x_i} \underset{\sim}{e}_i \times \underset{\sim}{e}_j$$

$$\underset{\sim}{\nabla} \times \underset{\sim}{v} \overset{(1C\text{-}3)}{=} e_{ijk} v_{j,i} \underset{\sim}{e}_k \qquad \text{Curl} \qquad (1D\text{-}6)$$

Hence, the components of curl $\underset{\sim}{v}$ are

$$e_{ijk} v_{j,i} = e_{kij} v_{j,i}$$

Expanding, these become

$$\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3} \quad , \quad \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1} \quad , \quad \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$$

Suppose we want a component form of div (grad λ). Using (1D-3), $(1D\text{-}4)_1$

$$\begin{aligned}\operatorname{div}\,(\operatorname{grad}\lambda) &= \underset{\sim}{\nabla}\cdot(\underset{\sim}{\nabla}\lambda)\\ &= (\underset{\sim}{e}_i\,\frac{\partial}{\partial x_i})\cdot(\underset{\sim}{e}_j\,\lambda_{,j})\\ &= \delta_{ij}\,\lambda_{,ji} = \lambda_{,ii}\\ &= \frac{\partial^2\lambda}{\partial x_1{}^2}+\frac{\partial^2\lambda}{\partial x_2{}^2}+\frac{\partial^2\lambda}{\partial x_3{}^2}\end{aligned}$$

Where does the j come in?

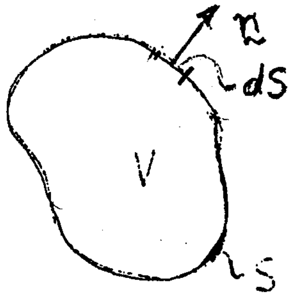
But by definition

$$\frac{\partial^2}{\partial x_1{}^2}+\frac{\partial^2}{\partial x_2{}^2}+\frac{\partial^2}{\partial x_3{}^2} = \nabla^2(\ )$$

is the Laplacian operator. Hence,

$$\boxed{\operatorname{div}\,(\operatorname{grad}\lambda) = \nabla^2\lambda}$$

Divergence Theorem -- Let $\underset{\sim}{u}$ be a continuously differentiable vector field defined throughout a region V with piecewise smooth bounding surface S and $\underset{\sim}{n}$ be the unit outer normal to S. Then



$$\int_V \operatorname{div}\,\underset{\sim}{u}\;dV = \int_S \underset{\sim}{u}\cdot\underset{\sim}{n}\;dS \tag{1D-7}$$

In component form

$$\int_V u_{i,i}\;dV = \int_S u_i\,n_i\;dS \tag{1D-8}$$

From Schaum's

The divergence theorem relates a volume integral to a surface integral. For a vector field $\tilde{v} = \bar{v}(\tilde{x})$

$$\int_V \operatorname{div} \bar{v} \, dV = \int_S \bar{n} \cdot \bar{v} \, dS$$ $\bar{n}$ outward normal (unit) vector

or

$$\int_V v_{i,i} \, dV = \int_S v_i n_i \, dS$$

For higher order tensors it becomes

$$\int_V T_{ijk\ldots p} \, dV = \int_S T_{ijk\ldots} n_p \, dS$$

More simply, the total divergence within the domain equals the net flux emerging from the domain.

Algebraically, how are tensors of higher order handled?

This theorem can be extended to the general form

$$\int_V (\ )_{,i}\ dV = \int_S (\ )\ n_i\ dS$$

where ( ) denotes any continuously differentiable tensor field. For example,

<u>Scalar Field</u>

$$\int_V \lambda_{,i}\ dV = \int_S \lambda\ n_i\ dS \qquad \text{(1D-9)}$$

<u>2nd Order Tensor</u>

$$\int_V A_{mn,i}\ dV = \int_S A_{mn}\ n_i\ dS \qquad \text{(1D-10)}$$

For a proof of (1D-7) see O. D. Kellogg, "Foundations of Potential Theory", Dover, 1929, page 84.

E. Isotropic Tensors

If the components of a tensor of order N are invariant under all rotations of $x_i \to \bar{x}_i$, then the tensor is called isotropic. All scalars are isotropic tensors of order 0 since $\lambda = \bar{\lambda}$. A 2nd order tensor $\underset{\sim}{A}$ is isotropic if

$$\bar{A}_{ij} = A_{ij}$$

Theorem 9 -- The Kronecker delta δ_{ij} and alternator e_{ijk} are isotropic tensors.

Proof: We must first show δ_{ij} and e_{ijk} are tensors. Consider the array $\bar{\underset{\sim}{e}}_i \cdot \bar{\underset{\sim}{e}}_j$:

$$\bar{\underset{\sim}{e}}_i \cdot \bar{\underset{\sim}{e}}_j \overset{(1C-6)}{=} (Q_{im}\, \underset{\sim}{e}_m) \cdot Q_{jn}\, \underset{\sim}{e}_n$$

$$= Q_{im}\, Q_{jn}\, \underset{\sim}{e}_m \cdot \underset{\sim}{e}_n$$

or

$$\bar{\delta}_{ij} = Q_{im}\, Q_{jn}\, \delta_{mn}$$

which implies the array $\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$ is a 2nd order tensor. Now using the orthogonality properties of $\underset{\sim}{Q}$:

$$\bar{\delta}_{ij} = Q_{im}\, Q_{jn}\, \delta_{mn} = Q_{im}\, Q_{jm} = \delta_{ij}$$

Hence, δ_{ij} is an isotropic 2nd order tensor. Now consider

$$\bar{\underset{\sim}{e}}_m \times \bar{\underset{\sim}{e}}_n \cdot \bar{\underset{\sim}{e}}_p = (Q_{mi}\, \underset{\sim}{e}_i) \times (Q_{nj}\, \underset{\sim}{e}_j) \cdot (Q_{pk}\, \underset{\sim}{e}_k)$$

$$= Q_{mi}\, Q_{nj}\, Q_{pk}\, \underset{\sim}{e}_i \times \underset{\sim}{e}_j \cdot \underset{\sim}{e}_k$$

or by (1C-4)

$$\bar{e}_{mnp} = Q_{mi}\, Q_{nj}\, Q_{pk}\, e_{ijk}$$

which implies e_{ijk} is a 3rd order tensor. Now

$$\bar{e}_{mnp} \overset{(1B-4)_1}{=} \underline{\det \underset{\sim}{Q}}\, e_{mnp} = e_{mnp}$$ under only proper transformation

Hence, e_{ijk} is an isotropic 3rd order tensor.

Theorem 10 -

(a) There are no isotropic tensors of 1st order.

(b) Isotropic tensors of orders 2, 3, 4 must have the forms

$$A_{ij} = a\,\delta_{ij}$$

$$B_{ijk} = b\,e_{ijk} \qquad (1E\text{-}1)$$

$$C_{ijk\ell} = \alpha\,\delta_{ij}\,\delta_{k\ell} + \beta\,\delta_{ik}\,\delta_{j\ell} + \gamma\,\delta_{i\ell}\,\delta_{jk}$$

three scalars and it built up from 2nd order only

important

Proof:

(a) Suppose $\underset{\sim}{u} \neq 0$ is an isotropic vector. Then

$$u_i = Q_{ij}\,u_j \qquad (1E\text{-}2)$$

must hold for arbitrary proper orthogonal $\underset{\sim}{Q}$. But (1E-2) implies

$$0 = u_i - Q_{ij}\,u_j = (\delta_{ij} - Q_{ij})\,u_j = 0$$

But $\underset{\sim}{u}$ is non-vanishing which implies $Q_{ij} = \delta_{ij}$. But this is a contradiction since $\underset{\sim}{Q}$ is arbitrary.

(b) If $\underset{\sim}{A}$ is an isotropic 2nd order tensor it must satisfy

$$A_{ij} = Q_{im}\,Q_{jn}\,A_{mn} \qquad (1E\text{-}3)$$

for arbitrary proper orthogonal $\underset{\sim}{Q}$. Hence, (1E-3) must hold if we choose $\underset{\sim}{Q}$ as

$$Q_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

This $\underset{\sim}{Q}$ is certainly admissible and represents a 180° rotation about the x_1-axis. Now let i=1, j=2 in (1E-3):

$$\begin{aligned} A_{12} &= Q_{1m}\, Q_{2n}\, A_{mn} \\ &= Q_{11}\, Q_{22}\, A_{12} = - A_{12} \end{aligned}$$

Hence, A_{12} must vanish. Similarly letting i,j have the values (2,1), (1,3) and (3,1) in turn implies $A_{21} = 0 = A_{13} = A_{31}$. Now choose

$$Q_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

From (1E-3) letting i=3, j=2:

$$A_{32} = Q_{3m}\, Q_{2n}\, A_{mn} = Q_{33}\, Q_{22}\, A_{32} = - A_{32}$$

Hence, $A_{32} = 0$ and letting i=2, j=3 implies $A_{23} = 0$. Thus, the two above choices for $\underset{\sim}{Q}$ imply all off-diagonal components of $\underset{\sim}{A}$ must vanish. Finally, choose

$$Q_{ij} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Letting i=j=1 in (1E-3):

$$A_{11} = Q_{1m}\, Q_{1n}\, A_{mn} = Q_{12}\, Q_{12}\, A_{22} = A_{22}$$ Bad proof

Similarly, i=j=2 and i=j=3 imply $A_{22} = A_{33}$ and $A_{33} = A_{11}$. Hence, letting $A_{11} = A_{22} = A_{33} = a$, then we have shown (1E-3) implies $A_{ij} = a\, \delta_{ij}$, where a is a scalar. Clearly, this form is isotropic, i.e.

$$\overline{A}_{ij} = \overline{a}\, \overline{\delta}_{ij} = a\, \delta_{ij} = A_{ij}$$

From Schaum's,

For every symmetric Tensor, T_{ij}, there is associated a vector V_i, with direction n_j, such that

$$V_i = T_{ij} n_j$$

If the direction is chosen such that V_i is parallel to n_i then the inner product may be expressed as a scalar multiple of n_i, hence

$$T_{ij} n_j = \lambda n_i \qquad (1)$$

where $n_i \equiv$ principal direction or principal axis of T_{ij}

Using $n_i \equiv \delta_{ij} n_j$, (1) becomes

$$(T_{ij} - \delta_{ij} \lambda) = 0$$

which represents a system of (3) eqns with 4 unknowns.

~~Expansion~~ For a non trivial solution $|T_{ij} - \delta_{ij}\lambda| = 0$

Expansion of the determinant gives a cubic polynomial in λ,

$$\lambda^3 - I_T \lambda^2 + II_T \lambda - III_T = 0$$

which is the characteristic eqn of T_{ij}. The scalar coefficients are the 1st, 2nd, and 3rd invariants, I_T, II_T, III_T, respectively. The three roots are $\lambda_{(1)}$, $\lambda_{(2)}$, and $\lambda_{(3)}$ are the principal values of T_{ij}

F. Eigenvalues of Real Matrices

Consider an arbitrary real 3 × 3 array A_{ij}. The characteristic determinant of $\underset{\sim}{A}$ is defined as det $(A_{ij} - a\delta_{ij})$. The characteristic equation of $\underset{\sim}{A}$ is

$$\det (A_{ij} - a\delta_{ij}) = 0 \qquad (1F\text{-}1)$$

Expansion of (1F-1) gives a cubic equation in the parameter a and can be written in the form

$$a^3 - I_A\, a^2 + II_A\, a - III_A = 0 \qquad (1F\text{-}2)$$

where I_A, II_A, III_A are the principal invariants of $\underset{\sim}{A}$ defined as

$$\begin{aligned} I_A &= A_{ii} = \mathrm{tr}\, \underset{\sim}{A} \\ II_A &= \frac{1}{2} (A_{ii}\, A_{jj} - A_{ij}\, A_{ji}) \\ &= \frac{1}{2} [(\mathrm{tr}\, \underset{\sim}{A})^2 - \mathrm{tr}(\underset{\sim}{A})^2] \qquad (1F\text{-}3) \\ III_A &= \det \underset{\sim}{A} \end{aligned}$$

The solutions a_α ($\alpha = 1,2,3$) to the characteristic equation are called the principal values or eigenvalues of $\underset{\sim}{A}$. Since (1F-2) is a cubic there exists at least one real eigenvalue of any real 3 × 3 array. Associated with each a_α is a principal direction or eigenvector $\underset{\sim}{n}^{(\alpha)}$ determined by the linear homogeneous equations

$$(A_{ij} - a_\alpha\, \delta_{ij})\, n_j^{(\alpha)} = 0 \quad , \quad (\alpha = 1,2,3) \qquad (1F\text{-}4)$$

Since the $\underset{\sim}{n}^{(\alpha)}$ define directions, it is sufficient to normalize the solutions of (1F-4), i.e., without losing generality, we take the $\underset{\sim}{n}^{\alpha}$ vectors to have unit length:

$$n_i^{(\alpha)} n_i^{(\alpha)} = 1 \quad , \quad (\alpha = 1,2,3) \qquad (1F-5)$$

Then (1F-4), (1F-5) determine a set of three unit vectors associated with the three principal values.

Given an arbitrary matrix $\underset{\sim}{A}$, and given a non-singular matrix $\underset{\sim}{B}$, the matrix $\underset{\sim}{C} = \underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1}$ is called <u>similar</u> to $\underset{\sim}{A}$ for the following reason: $\underset{\sim}{A}$ and $\underset{\sim}{C}$ have the same principal values. ✓ important

Consider

$$\underset{\sim}{C} - \lambda\underset{\sim}{I} = \underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1} - \lambda\underset{\sim}{I} = \underset{\sim}{B}\,\underset{\sim}{A}\,\underset{\sim}{B}^{-1} - \lambda\underset{\sim}{B}\ I\ \underset{\sim}{B}^{-1}$$
$$= \underset{\sim}{B}\,(\underset{\sim}{A} - \lambda\underset{\sim}{I})\,\underset{\sim}{B}^{-1}$$

Then

$$\det(\underset{\sim}{C} - \lambda\underset{\sim}{I}) = \det\underset{\sim}{B}\,\det(\underset{\sim}{A} - \lambda\underset{\sim}{I})\,\det\underset{\sim}{B}^{-1} \qquad (*)$$

But $\underset{\sim}{B}\,\underset{\sim}{B}^{-1} = \underset{\sim}{I}$ implies

$$\det\underset{\sim}{B}^{-1} = \frac{1}{\det\underset{\sim}{B}} \qquad (1F-6)$$

Hence, (*) becomes

$$\det(\underset{\sim}{C} - \lambda\underset{\sim}{I}) = \det(\underset{\sim}{A} - \lambda\underset{\sim}{I})$$

✓

This implies $\underset{\sim}{C}$, $\underset{\sim}{A}$ have the same characteristic equations and hence the same principal values. As a special case, we can take $\underset{\sim}{B}$ as any orthogonal matrix $\underset{\sim}{Q}$ (certainly non-singular); then $\underset{\sim}{A}$ and $\underset{\sim}{Q}\,\underset{\sim}{A}\,\underset{\sim}{Q}^T$ are similar matrices. Imp.

Imp

Theorem 11 -- Given any real, symmetric matrix $\underset{\sim}{A}$, then

(a) The principal values a_α are all real.

(b) The principal directions $\underset{\sim}{n}^{(\alpha)}$ are orthogonal, provided the a_α are distinct.

(c) The vectors $\underset{\sim}{n}^{(\alpha)}$ form the columns of an orthogonal matrix $\underset{\sim}{Q}$ such that $\underset{\sim}{D} = \underset{\sim}{Q}^T \underset{\sim}{A}\,\underset{\sim}{Q}$ is diagonal with the a_α as the diagonal elements.

Proof:

(a) Suppose a + ib is a root of (1F-2) and $\underset{\sim}{n}$ (possibly a vector with complex components) is the corresponding direction. Then (1F-4) implies

$$\underset{\sim}{A}\,\underset{\sim}{n} = (a + ib)\,\underset{\sim}{n}$$

Take the dot product with $\underset{\sim}{n}$:

$$\underset{\sim}{n} \cdot \underset{\sim}{A}\,\underset{\sim}{n} = (a + ib)\,\underset{\sim}{n} \cdot \underset{\sim}{n} \qquad (*)$$

Since $\underset{\sim}{n}$ may be complex, then in components the left hand side is

$$\lambda = \underset{\sim}{n} \cdot \underset{\sim}{A}\,\underset{\sim}{n} = A_{ij}\,\bar{n}_i\,n_j$$

where $(\bar{\ })$ denotes complex conjugate of any quantity. Since $\underset{\sim}{A}$ is symmetric

$$A_{ij}\,\bar{n}_i\,n_j = A_{ji}\,\bar{n}_i\,n_j = A_{ji}\,n_j\,\bar{n}_i$$
$$= A_{ij}\,n_i\,\bar{n}_j$$

That is, $\lambda = \bar{\lambda}$. Therefore λ and the left hand side of (*) are real numbers. Now on the right hand side of (*) $\underset{\sim}{n} \cdot \underset{\sim}{n} = n_i\,\bar{n}_i = |\underset{\sim}{n}|^2$, which is real. Hence,

$$\mathrm{Im}\,(\underset{\sim}{n} \cdot \underset{\sim}{A}\,\underset{\sim}{n}) = 0 = b\,|\underset{\sim}{n}|^2$$

which implies $b=0$. Q.E.D.

(b) Let a_1 and a_2 be distinct roots of (1F-2). Then (1F-4) implies

$$A_{ij}\,n_j^{(1)} = a_1\,n_i^{(1)} \quad , \quad A_{ij}\,n_j^{(2)} = a_2\,n_i^{(2)}$$

Multiply the 1st equation by $n_i^{(2)}$, the 2nd by $n_i^{(1)}$ and subtract:

$$A_{ij}(n_j^{(1)}\,n_i^{(2)} - n_i^{(2)}\,n_j^{(1)}) = (a_1 - a_2)\,n_i^{(1)}\,n_i^{(2)}$$

Now the left hand side vanishes by Theorem 1 since $\underset{\sim}{A}$ is symmetric: $2A_{ij}\,n_{[j}^{(1)}\,n_{i]}^{(2)} = 0$. Hence, $n_i^{(1)}\,n_i^{(2)} = 0$, i.e., $\underset{\sim}{n}^{(1)}$ and $\underset{\sim}{n}^{(2)}$ are orthogonal. Q.E.D.

(c) Define

$$Q_{ij} = \begin{pmatrix} n_1^{(1)} & n_1^{(2)} & n_1^{(3)} \\ n_2^{(1)} & n_2^{(2)} & n_2^{(3)} \\ n_3^{(1)} & n_3^{(2)} & n_3^{(3)} \end{pmatrix}$$

(Note that $\underset{\sim}{Q}$ is orthogonal since the $\underset{\sim}{n}^{(\alpha)}$ are orthonormal.) Then the columns of $\underset{\sim}{A}\ \underset{\sim}{Q}$ are $A_{ij}\ n_j^{(1)}$, $A_{ij}\ n_j^{(2)}$, $A_{ij}\ n_j^{(3)}$. But by (1F-4), i.e., $A_{ij}\ n_j^{(\alpha)} = a_\alpha\ n_i^{(\alpha)}$, the columns of $\underset{\sim}{A}\ \underset{\sim}{Q}$ must equal $a_\alpha\ n_i^{(\alpha)}$:

$$\underset{\sim}{A}\ \underset{\sim}{Q} = \begin{pmatrix} a_1\ n_1^{(1)} & a_2\ n_1^{(2)} & a_3\ n_1^{(3)} \\ a_1\ n_2^{(1)} & a_2\ n_2^{(2)} & a_3\ n_2^{(3)} \\ a_1\ n_3^{(1)} & a_2\ n_3^{(2)} & a_3\ n_3^{(3)} \end{pmatrix}$$

$$= \underset{\sim}{Q} \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}$$

$$= \underset{\sim}{Q}\ \underset{\sim}{D}$$

Hence,

$$\underset{\sim}{Q}^T\ \underset{\sim}{A}\ \underset{\sim}{Q} = \underset{\sim}{D}$$

is a diagonal matrix with a_α as the diagonal entries. Q.E.D.

We note the following: (i) If the a_α are not distinct, say $a_1 = a_2 \neq a_3$, then $\underset{\sim}{n}^{(3)}$ is determined by (1F-4), (1F-5) (except for sign), while the same equations can be satisfied by choosing $\underset{\sim}{n}^{(1)}$ and $\underset{\sim}{n}^{(2)}$ to be any orthonormal vectors lying in the plane orthogonal to $\underset{\sim}{n}^{(3)}$. If all the principal values a_α are equal, then any orthonormal triad $\underset{\sim}{n}^{(\alpha)}$ can be chosen such that (1F-4), (1F-5) are satisfied. (ii) The signs of the $\underset{\sim}{n}^{(\alpha)}$ vectors are usually chosen to make the triad a

right-handed system, i.e., $\underset{\sim}{n}_1 \times \underset{\sim}{n}_2 \quad \underset{\sim}{n}_3 = 1$. Then $\underset{\sim}{Q}$ defined above will be proper orthogonal and hence represents a rotation. The vectors $\underset{\sim}{n}^{(\alpha)}$ are then called the principal axes of $\underset{\sim}{A}$. (iii) The procedure of determining $\underset{\sim}{D} = \underset{\sim}{Q}^T \underset{\sim}{A} \underset{\sim}{Q}$ is called diagonalizing the matrix $\underset{\sim}{A}$, since in principal axes $\underset{\sim}{A}$ becomes diagonal.

Theorem 12 -- The extremal values of the quadratic form $\lambda = A_{ij}\, n_i\, n_j$ are the principal values a_α of $\underset{\sim}{A}$ and occur along the principal axes $\underset{\sim}{n}^{(\alpha)}$.

Proof: Suppose we seek the extremal values (i.e., maxima, minima, minimax) of the quadratic form ($\underset{\sim}{A}$ is real and symmetric)

$$\lambda = A_{ij}\, n_i\, n_j \tag{1F-7}$$

subject to the condition that $\underset{\sim}{n}$ is a unit vector

$$n_i\, n_i = 1 \tag{1F-8}$$

Since λ is a function of $\underset{\sim}{n}$, then the extremal values of λ (if any exist) will occur for certain directions. We employ the method of Lagrange Multipliers (Reference: R. Courant, "Differential and Integral Calculus, Interscience), i.e., necessary conditions that λ take on extremal values subject to $\underset{\sim}{n} \circ \underset{\sim}{n} = 1$ are that

$$\frac{\partial F}{\partial n_i} = 0 \tag{1F-9}$$

where F is defined as

$$F(\underset{\sim}{n}) = \lambda - a(\underset{\sim}{n} \cdot \underset{\sim}{n} - 1)$$
$$= A_{ij}\, n_i\, n_j - a(n_i\, n_i - 1) \qquad (1F\text{-}10)$$

and a is an unknown Lagrange Multiplier. From (10)

$$\frac{\partial F}{\partial n_i} = A_{ij}\, n_j - a\, n_i = (A_{ij} - a\delta_{ij})\, n_j$$

Hence, (9) implies

$$(A_{ij} - a\delta_{ij})\, n_j = 0 \qquad (1F\text{-}11)$$

are necessary conditions for extremal values of λ. But the solution to (11) and (8) yield the principal values and directions of $\underset{\sim}{A}$. Hence, λ assumes extremal values in the direction of the principal axes. It remains to show that the extremal values of λ are the principal values of $\underset{\sim}{A}$. Consider a_1, $\underset{\sim}{n}^{(1)}$ which satisfy

$$A_{ij}\, n_j^{(1)} = a_1\, n_i^{(1)}$$

Then λ becomes

$$\lambda\Big|_{\underset{\sim}{n}^{(1)}} = A_{ij}\, n_i^{(1)}\, n_j^{(1)} = a_1\, n_i^{(1)}\, n_i^{(1)} = a_1$$

Similarly, for a_2, $\underset{\sim}{n}^{(2)}$ and a_3, $\underset{\sim}{n}^{(3)}$. Q.E.D.

$A \Rightarrow B$ suffic

$A \Leftrightarrow$ necc. & suff.

$A \Leftarrow B$ nec.

II Deformation of Continuous Bodies

A. Deformation Tensors

Let a continuous body B_0 at time t=0 have volume V_0 with boundary S_0. If forces act on B_0, it will be deformed into a new configuration B(t) with volume V(t) and boundary S(t). We call B_0 a <u>reference</u> configuration and assume that the position of every material point in B_0 is known. We define two fixed RCCS X_K, x_k with right-handed orthonormal bases $\underset{\sim}{I}_K$, $\underset{\sim}{i}_k$, respectively. Capital indices will denote components with respect to $\underset{\sim}{I}_K$ and lower case indices with respect to $\underset{\sim}{i}_k$. Then a typical point in the body has position vectors (see Fig. II 1).

$$\underset{\sim}{R} = X_K \underset{\sim}{I}_K \quad , \quad \underset{\sim}{r} = x_k \underset{\sim}{i}_k \tag{2A-1}$$

The X_K are <u>material coordinates</u> of P and x_k the <u>spatial</u> coordinates. The deformation of B_0 into B(t) is described by the mapping

$$x_i = x_i(X_K,t) \tag{2A-2}$$

We assume that this mapping and its inverse

$$X_K = X_K(x_i,t) \tag{2A-3}$$

are one-to-one (implying one point in B_0 is mapped into one point in B(t) and visa-versa) and continuously differentiable in their arguments. The inverse (2A-3) will exist throughout B(t) provided the Jacobian of the mapping (2A-2) is non-vanishing at every point of B_0:

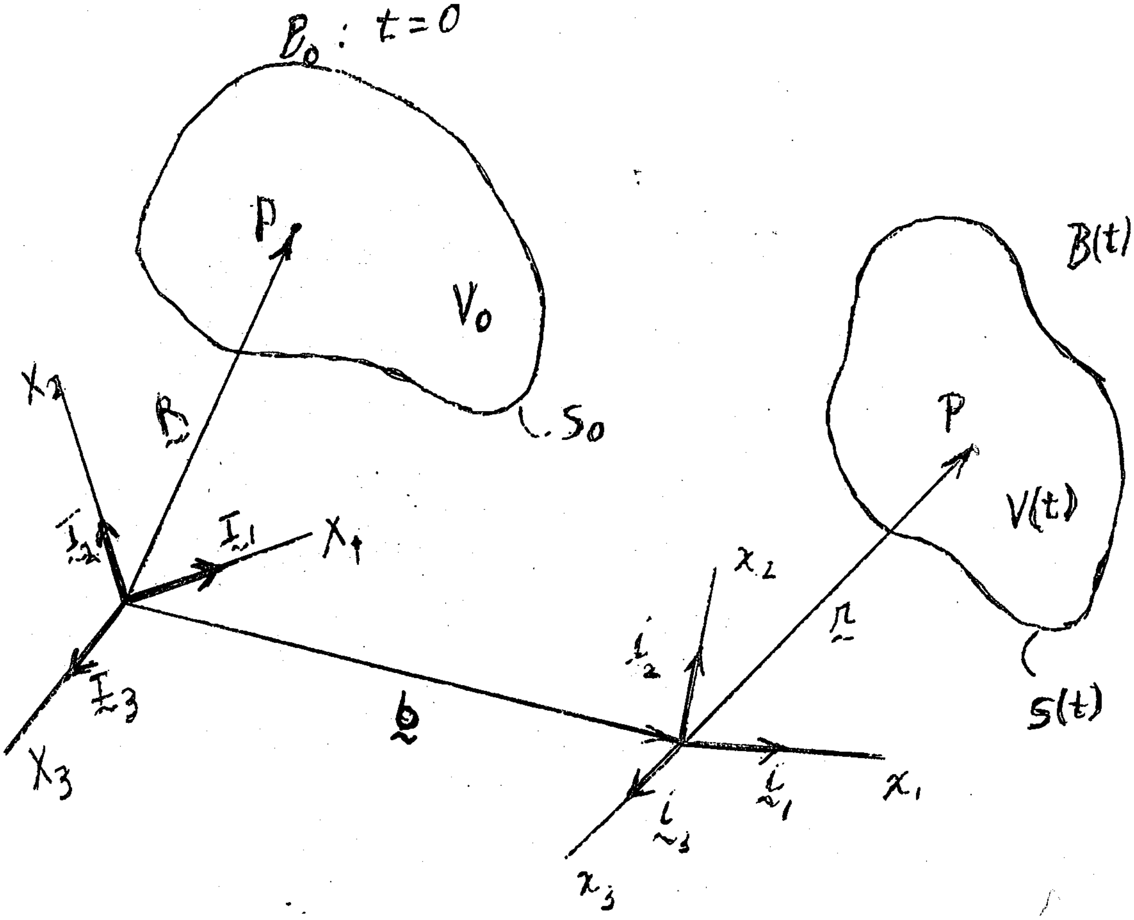

$B_0 : t=0$
P
V_0
S_0
X_2
$\underset{\sim}{R}$
$\underset{\sim}{I}_2$
$\underset{\sim}{I}_1$
X_1
$\underset{\sim}{I}_3$
X_3
$\underset{\sim}{b}$
$B(t)$
P
$V(t)$
$\underset{\sim}{r}$
x_2
$\underset{\sim}{i}_2$
$S(t)$
x_1
$\underset{\sim}{i}_1$
$\underset{\sim}{i}_3$
x_3

Fig. II-1

$$J(\underset{\sim}{X},t) = \det\left(\frac{\partial x_i}{\partial X_K}\right) \neq 0 \qquad (2A\text{-}4)$$

Since $B_0 = B(0)$, then

$$\underset{\sim}{R} = \underset{\sim}{b} + \underset{\sim}{r}\Big|_{t=0}$$

where $\underset{\sim}{b}$ is the constant position vector of the x_i system with respect to X_K system. Using (2A-1) and (2A-2)

$$X_K\, \underset{\sim}{I}_K = \underset{\sim}{b} + x_k(\underset{\sim}{X},0)\, \underset{\sim}{i}_k$$

Now dot $\underset{\sim}{I}_M$ with both sides:

$$X_K\, \underset{\sim}{I}_K \cdot \underset{\sim}{I}_M = \underset{\sim}{b} \cdot \underset{\sim}{I}_M + x_k(\underset{\sim}{X},0)\, \underset{\sim}{i}_k \cdot \underset{\sim}{I}_M$$

or

$$X_K = b_K + x_k(\underset{\sim}{X},0)\, \underset{\sim}{i}_k \cdot \underset{\sim}{I}_m \qquad (2A\text{-}5)$$

We define the direction cosine matrix

$$\underset{\sim}{i}_k \cdot \underset{\sim}{I}_M = \alpha_{kM} \qquad (2A\text{-}6)$$

Direction Cosine Matrix

which is orthogonal, i.e.

$$\alpha_{kM}\, \alpha_{kN} = \delta_{MN}\ ,\quad \alpha_{kM}\, \alpha_{iM} = \delta_{ki} \qquad (2A\text{-}7)$$

Then (2A-5) implies

$$X_M = b_M + x_k(\underset{\sim}{X},0)\, \alpha_{kM}$$

$$\frac{\partial X_m}{\partial X_p} = \frac{\partial}{\partial X_p} \hat{e}_p \left(X_m \hat{e}_m \right) = \frac{\partial X_m}{\partial X_p} \underbrace{\hat{e}_p \cdot \hat{e}_m}_{\delta_{mp}}$$

$\swarrow$

$$\frac{\partial X_m}{\partial X_p} = \begin{cases} 1 & m = p \\ 0 & m \neq p \end{cases} \qquad \delta_{mp} = \begin{cases} 1 & m = p \\ 0 & m \neq p \end{cases}$$

Now differentiate with respect to X_P:

$$\frac{\partial X_M}{\partial X_P} = \cancelto{0}{\frac{\partial b_K}{\partial X_P}} + \frac{\partial x_k}{\partial X_P}(\underset{\sim}{X},0)\ \alpha_{kM}$$

recalling that $\underset{\sim}{b}_K$ is a constant position vector.

i.e.

$$\delta_{MP} = \frac{\partial x_k}{\partial X_P}(\underset{\sim}{X},0)\ \alpha_{kM}$$

how is δ_{MP} obtained.

Multiply by α_{iM} and use $(2A-7)_2$:

$$\delta_{MP}\ \alpha_{iM} = \frac{\partial x_k}{\partial X_P}(\underset{\sim}{X},0)\ \underbrace{\alpha_{kM}\ \alpha_{iM}}_{\delta_{ki}}$$

i.e.

$$\frac{\partial x_i}{\partial X_P}(\underset{\sim}{X},0) = \alpha_{iP}$$

Taking the determinant:

$$\boxed{J(\underset{\sim}{X},0) = \det \frac{\partial x_i}{\partial X_P}(\underset{\sim}{X},0) = \det(\alpha_{iP}) = 1} \qquad (2A-7A)$$

since $\underset{\sim}{\alpha}$ is proper orthogonal. Recall that $\underset{\sim}{I}_K$, $\underset{\sim}{i}_k$ are right handed. Now $J(\underset{\sim}{X},t)$ is a continuous function of t which never vanishes and equals 1 at t=0. Hence,

$$\underline{\underline{J(\underset{\sim}{X},t) > 0 \quad \text{for all } \underset{\sim}{X},t}} \qquad (2A-8)$$

Consider an infinitesimal line element $d\underset{\sim}{R}$ at any point in B_0 which is mapped into $d\underset{\sim}{r}$ in $B(t)$. (See Fig. II-2).

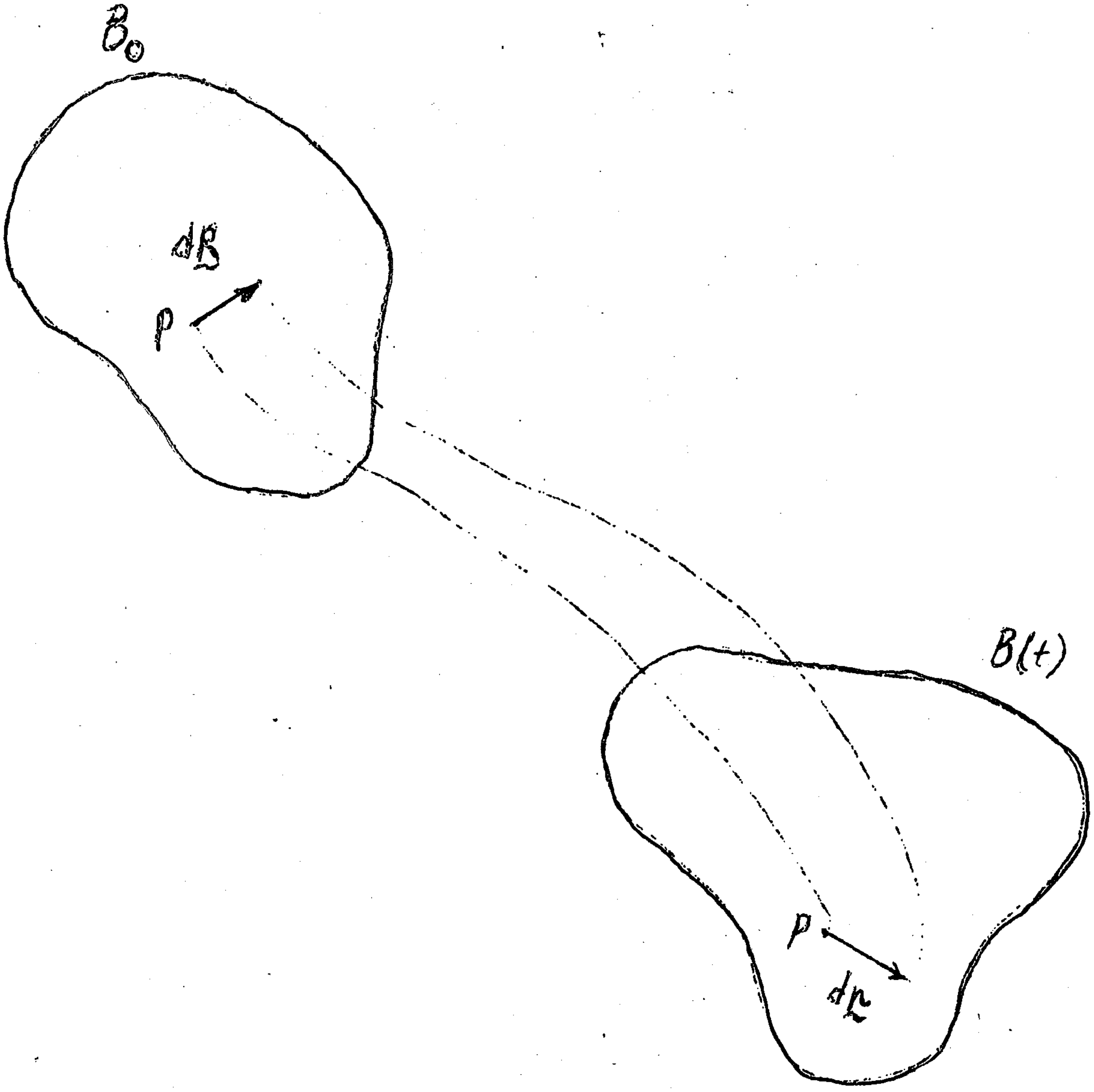


Fig. II-2

In general $d\underset{\sim}{R}$ is stretched and rotated by the deformation. From (2A-1)

$$d\underset{\sim}{R} = dX_K \underset{\sim}{I}_K \quad , \quad d\underset{\sim}{r} = dx_k \underset{\sim}{i}_k \tag{2A-9}$$

By the mapping (2A-2)

$$\boxed{dx_i = \frac{\partial x_i}{\partial X_K} dX_K = x_{i,K}\, dX_K} \tag{2A-10}$$

The derivatives $x_{i,K}$ are called <u>deformation gradients</u> and map dX_K into dx_i. By (2A-3)

Deformation Gradients.

$$\boxed{dX_K = \frac{\partial X_K}{\partial x_i} dx_i = X_{K,i}\, dx_i} \tag{2A-11}$$

and $X_{K,i}$ are the <u>inverse</u> deformation gradients mapping dx_i back into dX_K. Now the arrays $x_{i,K}$, $X_{K,i}$ are inverses to one another, i.e.

$$x_{i,K} X_{K,j} = \frac{\partial x_i}{\partial X_K}\frac{\partial X_K}{\partial x_j} = \frac{\partial x_i}{\partial x_j} = \delta_{ij} \qquad (2A\text{-}12)_1$$

Similarly,

$$X_{K,i}\, x_{i,M} = \delta_{KM} \qquad (2A\text{-}12)_2$$

— Application of Cramer's Theorem

$X_{K,i} = \frac{A_{iK}}{J}$ and has solution if $J \neq 0$

Hence, recalling (1B-17) and considering $x_{i,K}$ as given quantities, then $X_{K,i}$ is determined by

$$X_{K,i} = \frac{\text{cofactor } x_{i,K}}{\det x_{i,K}} = \frac{\text{cofactor } x_{i,K}}{J} \tag{2A-13}$$

Let the lengths of $d\underset{\sim}{R}$, $d\underset{\sim}{r}$ be denoted by dS, ds:

$$dS = |d\underset{\sim}{R}| \quad , \quad ds = |d\underset{\sim}{r}|$$

Now dS in B_0 is determined by

$$dS^2 = d\underset{\sim}{R} \cdot d\underset{\sim}{R} \overset{(2A-9)}{=} (dX_K \underset{\sim}{I}_K) \circ (dX_M \underset{\sim}{I}_M)$$

$$= \delta_{KM}\, dX_K\, dX_M \qquad (2A\text{-}14)$$

Similarly for $d\underset{\sim}{r}$ in B(t):

$$ds^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} \overset{(2A-9)}{=} \delta_{ij}\, dx_i\, dx_j \qquad (2A\text{-}15)$$

Now (2A-10), (2A-11) give relationships between dx_i, dX_K. Use (2A-10) in (2A-14):

$$ds^2 = \delta_{ij}(x_{i,K}\, dX_K)(x_{j,M}\, dX_M)$$

$$= x_{i,K}\, x_{i,M}\, dX_K\, dX_M$$

<u>Green's</u> <u>Deformation</u> <u>Tensor</u> C_{KM} is defined as

$$C_{KM}(\underset{\sim}{X}, t) = x_{i,K}\, x_{i,M} = \frac{\partial x_i}{\partial X_k} \frac{\partial x_i}{\partial X_m} \qquad (2A\text{-}16)$$

?

Note C_{KM} is nonlinear in the deformation gradients $x_{i,K}$.

Then ds^2 becomes

$$ds^2 = C_{KM}\, dX_K\, dX_M \qquad (2A\text{-}17)$$

Hence C_{KM} is a measure of the deformation of $d\underset{\sim}{R}$ into $d\underset{\sim}{r}$, i.e., knowing $\underset{\sim}{C}$ and $d\underset{\sim}{R}$, $ds = |d\underset{\sim}{r}|$ can be determined. We also note

that (2A-17) is a quadratic form in dX_K and that $ds^2 > 0$ implies C_{KM} is a positive definite array. This means, among other things, that the three eigenvalues of C_{KM} are always positive. Now C_{KM} transforms as a 2nd order tensor under rotations of the material coordinates X_K. Consider a proper orthogonal $\underset{\sim}{Q}$ such that

$$\bar{X}_K = Q_{KP}\, X_P \quad \text{or} \quad X_P = Q_{KP}\, \bar{X}_K \qquad (*)$$

Then

$$\frac{\partial x_i}{\partial \bar{X}_K} = \frac{\partial x_i}{\partial X_P}\,\frac{\partial X_P}{\partial \bar{X}_K} \overset{(*)}{=} Q_{KP}\,\frac{\partial x_i}{\partial X_P}$$

which implies $x_{i,K}$ transforms as a vector for each $i = 1,2,3$ under rotations of $\underset{\sim}{X} \to \underset{\sim}{\bar{X}}$. Similarly, we can show $x_{i,K}$ transforms as a vector for each $K = 1,2,3$ under rotations of $\underset{\sim}{x} \to \underset{\sim}{\bar{x}}$. Now compute the components of $\underset{\sim}{C}$ in the $\bar{X}_K$ system:

$$\bar{C}_{KM} = \frac{\partial x_i}{\partial \bar{X}_K}\,\frac{\partial x_i}{\partial \bar{X}_M} = \left(\frac{\partial x_i}{\partial X_P}\,\frac{\partial X_P}{\partial \bar{X}_K}\right)\left(\frac{\partial x_i}{\partial X_N}\,\frac{\partial X_N}{\partial \bar{X}_M}\right)$$

$$\overset{(*)}{=} \left(Q_{KP}\,\frac{\partial x_i}{\partial X_P}\right)\left(Q_{MN}\,\frac{\partial x_i}{\partial X_N}\right)$$

Hence, by (2A-16)

$$\bar{C}_{KM} = Q_{KP}\, Q_{MN}\, C_{PN} \qquad (2A\text{-}18)$$

which is the transformation law for 2nd order tensors. Now use (2A-11) in (2A-14)

$$dS^2 = \delta_{KM}(X_{K,i}\ dx_i)(X_{M,j}\ dx_j)$$

$$= X_{K,i}\ X_{K,j}\ dx_i\ dx_j$$

We now define Cauchy's Deformation Tensor:

$$c_{ij}(\underset{\sim}{x},t) = X_{K,i}\ X_{K,j} = \frac{\partial X_K}{\partial x_i}\frac{\partial X_K}{\partial x_j} \qquad (2A\text{-}19)$$

Note that c_{ij} is nonlinear in the inverse deformation gradients $X_{K,i}$. Then

$$dS^2 = c_{ij}\ dx_i\ dx_j \qquad (2A\text{-}20)$$

Since $dS^2 > 0$, $\underset{\sim}{c}$ is also positive definite with 3 positive eigenvalues. Eqn. (2A-20) implies given $d\underset{\sim}{r}$ in $B(t)$, then $dS = |d\underset{\sim}{R}|$ can be determined, i.e., c_{ij} is a measure of the deformation of line elements at any point of the body. We can show that c_{ij} transforms as a 2nd order tensor under rotations of $\underset{\sim}{x} \rightarrow \underset{\sim}{\bar{x}}$.

Note that $\underset{\sim}{C}$, $\underset{\sim}{c}$ are symmetric tensors:

$$\boxed{\begin{aligned} C_{MK} &\overset{\text{defn.}}{=} x_{i,M}\ x_{i,K} = x_{i,K}\ x_{i,M} \overset{\text{defn.}}{=} C_{KM} \\ c_{ji} &= X_{K,j}\ X_{K,i} = X_{K,i}\ X_{K,j} = c_{ij} \end{aligned}}$$

Also, in the special case that

$$C_{KM} = \delta_{KM}\ , \quad c_{ij} = \delta_{ij} \qquad \text{Rigid Body Motion} \qquad (2A\text{-}21)$$

at every point of the body, then dS = ds, i.e.

$$C_{KM} = \delta_{KM} \overset{(2A\text{-}17)}{\text{implies}} ds^2 = \delta_{KM}\, dX_K\, dX_M \overset{(2A\text{-}14)}{=} dS^2$$

$$c_{ij} = \delta_{ij} \overset{(2A\text{-}20)}{\text{implies}} dS^2 = \delta_{ij}\, dx_i\, dx_j \overset{(2A\text{-}15)}{=} ds^2$$

Then the mapping of B_0 into $B(t)$ is called a <u>rigid body motion</u>. Note that $d\underset{\sim}{R}$ can suffer rotation and translation but no change in length. If (2A-21) holds only at a single material point, then the motion is <u>locally rigid</u>.

$d x_i = \partial$

B. Strain at a Point (Nonlinear Theory)

By "strain" we mean changes in length and relative orientation of line elements under the deformation. A measure of length change is $ds^2 - dS^2$:

$$ds^2 - dS^2 \overset{(2A-14)}{\underset{(2A\cdot 17)}{=}} (C_{KM} - \delta_{KM})\, dX_K\, dX_M$$

$$ds^2 - dS^2 = 2E_{KM}\, dX_K\, dX_M \tag{2B-1}$$

where E_{KM} is Lagrange's strain tensor:

$$2E_{KM}(X,t) = C_{KM} - \delta_{KM} \tag{2B-2}$$

Note that $\underset{\sim}{E}$ is a symmetric 2nd order tensor in X_K coordinates since $\underset{\sim}{C}$, $\underset{\sim}{\delta}$ are. Another expression for $ds^2 - dS^2$ is

$$ds^2 - dS^2 \overset{(2A-15)}{\underset{(2A-20)}{=}} (\delta_{ij} - c_{ij})\, dx_i\, dx_j$$

$$ds^2 - dS^2 = 2e_{ij}\, dx_i\, dx_j \tag{2B-3}$$

where e_{ij} is Euler's strain tensor:

$$2e_{ij}(x,t) = \delta_{ij} - c_{ij} \tag{2B-4}$$

Again note that $\underset{\sim}{e}$ is a (symmetric) 2nd order tensor under rotations of the x_i coordinates. Consider (2B-3):

$$ds^2 - dS^2 = 2e_{ij}\, dx_i\, dx_j \overset{(2A-10)}{=} 2e_{ij}\, x_{i,K}\, x_{j,M}\, dX_K\, dX_M$$

$$\overset{(2B-1)}{=} 2E_{KM}\, dX_K\, dX_M$$

i.e.

$$2(E_{KM} - e_{ij}\, x_{i,K}\, x_{j,M})\, dX_K\, dX_M = 0$$

which implies for arbitrary $dX_K \neq 0$

$$E_{KM} = e_{ij}\, x_{i,K}\, x_{j,M} \qquad (2B\text{-}5)$$

Similarly, using (2A-11) in (2B-1) and comparing with (2B-3).

$$e_{ij} = E_{KM}\, X_{K,i}\, X_{M,j} \qquad (2B\text{-}6)$$

This can be established directly from (2B-5) using (2A-12), i.e., multiply (2B-5) by $X_{K,m}\, X_{M,n}$

$$X_{K,m}\, X_{M,n}\, E_{KM} = e_{ij}\underbrace{(x_{i,K}\, X_{K,m})}_{\delta_{im}}\underbrace{(x_{j,M}\, X_{M,n})}_{\delta_{jn}} \quad \text{by (2A-12)}$$

$$= e_{mn} \qquad \text{Q.E.D.}$$

Note that under a <u>rigid body motion</u>,

$$\boxed{\begin{array}{l} C_{KM} = \delta_{KM} \overset{(2B\text{-}2)}{\text{implies}} E_{KM} = 0 \\[2ex] c_{ij} = \delta_{ij} \overset{(2B\text{-}4)}{\text{implies}} e_{ij} = 0 \end{array}}$$

Rigit Body Motion

$E_{KM} \equiv$ Lagranges Strain Tensor

$e_{ij} \equiv$ Euler's Strain Tensor

To interpret the diagonal components of C_{KM}, E_{KM}, we define a <u>unit</u> vector $\underset{\sim}{N}$ along $d\underset{\sim}{R}$:

$$N_K = \frac{dX_K}{|d\underset{\sim}{R}|} = \frac{dX_K}{dS} \qquad (2B\text{-}7)$$

$$\boxed{\Lambda_{(N)} = \left|\frac{dr}{dR}\right| = \frac{ds}{dS}}$$

Now the stretch of $\underset{\sim}{dR}$ with direction $\underset{\sim}{N}$ is defined as

Stretch

$$\Lambda_{(N)} = \frac{|\underset{\sim}{dr}|}{|\underset{\sim}{dR}|} = \frac{ds}{dS} \tag{2B-8}$$

$$E(N) = \Lambda(N) - 1$$

and the extension is

$$\boxed{\delta(N) = E_{(N)} = \Lambda_{(N)} - 1 = \frac{ds}{dS} - 1 = \frac{ds - dS}{dS}} \tag{2B-9}$$

Extension

Divide (2A-17) by dS^2:

(2B-7)

$$\frac{ds^2}{dS^2} = C_{KM} \frac{dX_K}{dS} \frac{dX_M}{dS} = C_{KM} N_K N_M$$

i.e., using (2B-8)

$$\boxed{\Lambda^2_{(N)} = C_{KM} N_K N_M} \tag{2B-10}$$

which implies C_{KM} is a measure of the stretch of $\underset{\sim}{dR}$ with direction $\underset{\sim}{N}$ in B_0. Since $N_K \neq 0$ and $\underset{\sim}{C}$ positive definite, then $\Lambda^2_{(N)}$ is certainly positive. Note (2B-10) is a sum, in general involving all the C_{KM} components.

Consider an element originally along the X_1 or $\underset{\sim}{I}_1$ direction. Then $N_K = (1,0,0)$ and (2B-10), (2B-9) imply

$$\Lambda_{(1)} = \sqrt{C_{11}} \quad , \quad E_{(1)} = \Lambda_{(1)} - 1 = \sqrt{C_{11}} - 1 \tag{2B-11}$$

Now (2B-2) implies $C_{KM} = 2E_{KM} + \delta_{KM}$ and hence

$$C_{11} = 2E_{11} + 1$$

Thus, in terms of E_{11}

$$\Lambda_{(1)} = (2E_{11} + 1)^{1/2} \quad , \quad E_{(1)} = (2E_{11} + 1)^{1/2} - 1 \qquad (2B-12)$$

Eqns. (2B-11), (2B-12) imply C_{11}, E_{11} are <u>measures</u> of the stretch and extension of an element originally along the $\underset{\sim}{I}_1$ direction. By taking elements along $\underset{\sim}{I}_2$, $\underset{\sim}{I}_3$ the other diagonal components of $\underset{\sim}{C}$, $\underset{\sim}{E}$ have similar interpretations.

For the off-diagonal components of $\underset{\sim}{C}$, $\underset{\sim}{E}$, consider two line elements $d\underset{\sim}{R}^{(1)}$, $d\underset{\sim}{R}^{(2)}$ at a point in B_0 which are deformed into $d\underset{\sim}{r}^{(1)}$, $d\underset{\sim}{r}^{(2)}$. By (2A-10) the components of the elements are related by

$$dx_i^{(1)} = x_{i,K}\, dX_K^{(1)} \quad , \quad dx_i^{(2)} = x_{i,K}\, dX_K^{(2)} \qquad (*)$$

We choose $d\underset{\sim}{R}^{(1)}$, $d\underset{\sim}{R}^{(2)}$ along the $\underset{\sim}{I}_1$, $\underset{\sim}{I}_2$ directions:

$$dX_K^{(1)} = (dS_1, 0, 0) \quad , \quad dX_K^{(2)} = (0, dS_2, 0)$$

Then (*) becomes

$$dx_i^{(1)} = x_{i,1}\, dS_1 \quad , \quad dx_i^{(2)} = x_{i,2}\, dS_2 \qquad (2B-13)$$

Now the angle between $d\underset{\sim}{r}^{(1)}$, $d\underset{\sim}{r}^{(2)}$ in B(t) is found from the inner product $d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)}$:

$$\cos\theta_{12} = \frac{d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)}}{|d\underset{\sim}{r}^{(1)}|\,|d\underset{\sim}{r}^{(2)}|}$$

From (2B-13)

$$d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(1)} = dx_i^{(1)}\, dx_i^{(1)} \overset{(2B-13)}{=} (x_{i,1}\, dS_1)(x_{i,1}\, dS_1)$$

$$\overset{(2A-16)}{=} C_{11}\, dS_1{}^2 \overset{(2B-11)}{=} \Lambda^2_{(1)}\, dS_1{}^2$$

$$d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)} = x_{i,1}\, x_{i,2}\, dS_1\, dS_2 = C_{12}\, dS_1\, dS_2$$

$$d\underset{\sim}{r}^{(2)} \cdot d\underset{\sim}{r}^{(2)} = x_{i,2}\, x_{i,2}\, dS_2{}^2 = C_{22}\, dS_2{}^2 = \Lambda^2_{(2)}\, dS_2{}^2$$

Hence,

$$\cos\theta_{12} = \frac{C_{12}\, dS_1\, dS_2}{\Lambda_{(1)}\, \Lambda_{(2)}\, dS_1\, dS_2} = \frac{C_{12}}{\Lambda_{(1)}\, \Lambda_{(2)}} = \frac{2E_{12}}{\Lambda_{(1)}\, \Lambda_{(2)}} \qquad (2B-14)$$

We define the <u>shear</u> Γ_{12} as the change in angle between the two elements:

$$\Gamma_{12} = \frac{\pi}{2} - \theta_{12}$$

which implies $\sin\Gamma_{12} = \cos\theta_{12}$, and (2B-14) becomes

$$\sin\Gamma_{12} = \frac{C_{12}}{\Lambda_{(1)}\, \Lambda_{(2)}} = \frac{2E_{12}}{\Lambda_{(1)}\, \Lambda_{(2)}} \qquad (2B-15)$$

i.e., C_{12}, E_{12} are measures of the shear between 2 elements originally along $\underset{\sim}{I}_1$, $\underset{\sim}{I}_2$. Note that Γ_{12} depends on the stretches. Using (2B-12)

$$\sin\Gamma_{12} = \frac{2E_{12}}{(2E_{11} + 1)^{1/2}\,(2E_{22} + 1)^{1/2}} \qquad (2B\text{-}16)$$

Similar expressions can be derived in terms of the other off-diagonal components of $\underset{\sim}{C}$, $\underset{\sim}{E}$.

From Schaums (3.13)

The Lagrangian and Eulerian linear strain tensors are symmetric 2nd order Cartesian tensors. The principal direction of a strain tensor is one for which the orientation of an element at a given point is not altered by a pure strain deformation. The principal strain value is simply the unit relative displacement (normal strain) that occurs in the principal direction.

From Schaum's (3.13)

The 1st invariant of the Lagrangian strain tensor can be written as

$$I_L = L_{ii} = L_{(1)} + L_{(2)} + L_{(3)}$$

The change in volume per unit origonal volume of a differential element whose sides are parallel the the principal strain directions is called the cubical ~~directi~~ dilation, given by

$$D_0 = \frac{\Delta V_0}{V_0} = \frac{dX_1(1+L_{(1)})\,dX_2(1+L_{(2)})\,dX_3(1+L_{(3)}) - dX_1\,dX_2\,dX_3}{dX_1\,dX_2\,dX_3}$$

For small strain theory, the 1st order approx. of D_0 is:

$$\boxed{D_0 = L_{(1)} + L_{(2)} + L_{(3)} = I_L}$$

After ended [illegible]

C. Principal Strains at a Point

We have shown that the deformation tensors C_{KM}, c_{ij} and the strain tensors E_{KM}, e_{ij} are real and symmetric. Hence, by Theorem 11 these tensors have three real principal values and a corresponding triad of principal axes. Note that these principal values and axes vary from point to point in B_0, or $B(t)$ since in general the above tensors are functions of the points in the body.

Suppose we focus attention on the material strain tensor $\underset{\sim}{E}$. The principal strains E_α and corresponding directions $\underset{\sim}{N}^{(\alpha)}$ are determined by (see Section F of Chapter I):

$$(E_{KM} - E_\alpha \delta_{KM}) N_M^{(\alpha)} = 0$$

$$N_K^{(\alpha)} N_K^{\alpha} = 1 \qquad (2C\text{-}1)$$

$$E^3 - I_E E^2 + II_E E - III_E = 0 \qquad (2C\text{-}2)$$

where the principal invariants of $\underset{\sim}{E}$ are

$$I_E = E_{KK}$$

$$II_E = \frac{1}{2} (E_{KK} E_{MM} - E_{KM} E_{KM}) \qquad (2C\text{-}3)$$

$$III_E = \det \underset{\sim}{E}$$

If we define the normal strain in the direction $\underset{\sim}{N}$ as

$$E_N = E_{KM} N_K N_M \qquad \text{Normal Strain} \qquad (2C\text{-}4)$$

Deformation tensor C_{KM} c_{ij}
Strain Tensor E_{KM} e_{ij} } Real and symmetric

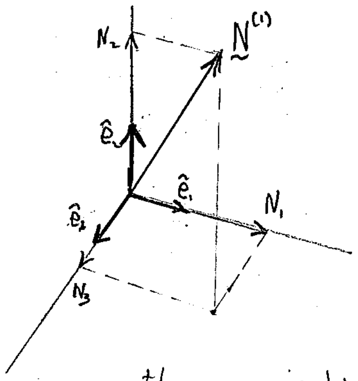
$$(E_{KM} - E_{\alpha}\delta_{KM})N_M^{(\alpha)} = 0$$

$E_\alpha = E_{(1)}, E_{(2)}, E_{(3)}$ are the principal strains

$$(E_{11} - E^{\alpha}\delta_{11})N_1^{\alpha} + (E_{12} - E^{\alpha}\delta_{12})N_2^{\alpha} + (E_{13} - E^{\alpha}\delta_{12})N_3^{\alpha} = 0$$
$$(E_{21} - E^{\alpha}\delta_{21})N_1^{\alpha} + (E_{22} - E^{\alpha}\delta_{22})N_2^{\alpha} + (E_{23} - E^{\alpha}\delta_{23})N_3^{\alpha} = 0$$
$$(E_{31} - E^{\alpha}\delta_{31})N_1^{\alpha} + (E_{32} - E^{\alpha}\delta_{32})N_2^{\alpha} + (E_{33} - E^{\alpha}\delta_{33})N_3^{\alpha} = 0$$

Normal strain in the direction $\underset{\sim}{N}$ is:

$$E_N = E_{KM}N_K N_M$$
$$= E_{11}N_1N_1 + E_{12}N_1N_2 + E_{13}N_1N_3 + E_{21}N_2N_1$$
$$+ E_{22}N_2N_2 + E_{23}N_2N_3 + E_{31}N_3N_1$$
$$+ E_{32}N_3N_2 + E_{33}N_3N_3$$



$\underset{\sim}{N}$ has components N_1, N_2, N_3, or N_i along the three coordinate axes ($\hat{e}_1, \hat{e}_2, \hat{e}_3$ or e_i) (X_1, X_2, X_3, or X_i)

thus in indicial notation $N_K N_M$ means $N_K N_M \underbrace{\hat{e}_K \cdot \hat{e}_M}_{\delta_{KM}}$

hence only the terms where $K=M$ are non-zero, so

$$E_N = E_{11}N_1N_1 + E_{22}N_2N_2 + E_{33}N_3N_3$$

but it is assumed that $N_K N_K = 1$ (2C-1) so

$$E_N = E_{11} + E_{22} + E_{33}$$

Therefore, in the principal axes, the shearing strains (E_{12}, E_{31}, etc) <u>vanish</u>.

Consider a free body with stresses acting on the faces ⊥ to the x, y, and z axes. Stresses on these faces produce infinitesimal force vectors, dR_x, dR_y, dR_z hence

$$\begin{aligned} d\bar{R}_x &= -\sigma_x\, dA_x\, \hat{i} + \tau_{xy}\, dA_x\, \hat{j} - \tau_{zx}\, dA_x\, \hat{k} \\ d\bar{R}_y &= -\tau_{xy}\, dA_y\, \hat{i} \mp \sigma_{yy}\, dA_y\, \hat{j} - \tau_{yz}\, dA_y\, \hat{k} \\ d\bar{R}_z &= -\tau_{zx}\, dA_z\, \hat{i} - \tau_{yz}\, dA_z\, \hat{j} - \sigma_z .\, dA_z\, \hat{k} \end{aligned} \qquad (1)$$

For equilibrium of the elemental volume, a resultant force is required giving

$$d\bar{R} + d\bar{R}_x + d\bar{R}_y + d\bar{R}_z = 0 \qquad (2)$$

Combining (1) and (2) gives

$$d\bar{R} = dR_1\, \hat{i} + dR_2\, \hat{j} + dR_3\, \hat{k} \qquad (3)$$

where

$$\begin{aligned} dR_1 &= \sigma_x\, dA_x + \tau_{xy}\, dA_y + \tau_{zx}\, dA_z \\ dR_2 &= \tau_{xy}\, dA_x + \sigma_y\, dA_y + \tau_{yz}\, dA_z \\ dR_3 &= \tau_{zx}\, dA_x + \tau_{yz}\, dA_y + \sigma_z\, dA_z \end{aligned}$$

the unit normal vector along the face where $d\bar{R}$ acts is

$$\hat{N} = l\,\hat{i} + m\,\hat{j} + n\,\hat{k}$$

where l, m, n are the direction cosines.

Thus $dA_x = l\,dA \quad dA_y = m\,dA \quad dA_z = n\,dA$

then

$$T_x = \frac{dR_1}{dA} = l\sigma_x + m\tau_{xy} + n\tau_{zx}$$

$$T_y = \frac{dR_2}{dA} = l\tau_{xy} + m\sigma_y + n\tau_{yz} \qquad (4)$$

$$T_z = \frac{dR_3}{dA} = l\tau_{zx} + m\tau_{yz} + n\sigma_z$$

T_x T_y and T_z are surface tractions; stresses on the face dA in the x, y, and z directions.

If face dA is the principal plane then it carries only the principal stress in direction $\hat{N}$ thus

$$T_x = l\sigma \quad T_y = m\sigma \quad T_z = n\sigma \qquad (5)$$

Combining (4) and (5) gives

$$\begin{aligned} l(\sigma_x - \sigma) + m\tau_{xy} + n\tau_{zx} &= 0 \\ l\tau_{xy} + m(\sigma_y - \sigma) + n\tau_{yz} &= 0 \\ l\tau_{zx} + m\tau_{yz} + n(\sigma_z - \sigma) &= 0 \end{aligned} \qquad (6)$$

Taking the determinant of this system gives

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0 \qquad (7) \qquad \text{Charact. Eqn}$$

where

$$\text{Invariants}\left\{\begin{aligned} I_1 &= \sigma_x + \sigma_y + \sigma_z \\ I_2 &= \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 \\ I_3 &= \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 \end{aligned}\right.$$

Now on the principal planes, shear stress is zero so

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_3 = \sigma_1\sigma_2\sigma_3$$

then by Theorem 12, the extremal values of E_N are the principal strains E_α and occur in the direction of the principal axes defined by $\underset{\sim}{N}^{(\alpha)}$. Recall that $\underset{\sim}{E}$ is a 2nd tensor under rotations of the material axes X_K into $\bar{X}_K$, i.e.

$$\bar{E}_{KM} = Q_{KP}\, Q_{MN}\, E_{PN}$$

Note this equation has the equivalent direct notation form $\underset{\sim}{\bar{E}} = \underset{\sim}{Q}\,\underset{\sim}{E}\,\underset{\sim}{Q}^T$. Hence, if $\underset{\sim}{Q}$ is chosen to be the proper orthogonal array whose columns are $\underset{\sim}{N}^{(1)}$, $\underset{\sim}{N}^{(2)}$, $\underset{\sim}{N}^{(3)}$, then $\underset{\sim}{Q}$ rotates the X_K axes into the principal axes at each point. But by Theorem 11 $\underset{\sim}{\bar{E}}$ is diagonal with E_α as the diagonal entries. Hence, in principal axes the shearing strains (E_{12}, E_{13}, E_{23}) vanish and the normal strains (E_{11}, E_{22}, E_{33}) assume extremal values.

The principal values and directions of C_{KM} are determined by equations similar to (2C-1), (2C-2) with corresponding invariants as in (2C-3). Recalling (2B-10): $\Lambda^2_{(N)} = C_{KM}\, N_K\, N_M$ and Theorem 12, then the extremal values of the stretch squared are the principal values C_α and occur along the principal axes of $\underset{\sim}{C}$. Also, $\underset{\sim}{C}$ is diagonalized when transformed to principal axes. In view of (2B-2): $\underset{\sim}{C} = 2\underset{\sim}{E} + \underset{\sim}{\delta}$, then if $\underset{\sim}{E}$ is diagonalized, $\underset{\sim}{C}$ must also be diagonal. Hence, $\underset{\sim}{C}$ and $\underset{\sim}{E}$ have the same principal axes. The principal values of $\underset{\sim}{C}$, $\underset{\sim}{E}$ are related by (using (2B-2))

$$C_\alpha = 2E_\alpha + 1 \qquad \text{(2C-5)}$$

By assuming $\underset{\sim}{C}$, $\underset{\sim}{E}$ are expressed in principal axes, then the invariants of $\underset{\sim}{C}$, $\underset{\sim}{E}$ can be shown to satisfy:

$$\begin{aligned} I_C &= 3 + 2I_E \\ II_C &= 3 + 4I_E + 4II_E \\ III_C &= 1 + 2I_E + 4II_E + 8III_E \end{aligned} \tag{2C-6}$$

let $\underset{\sim}{A} = A_{ij} = \dfrac{\partial x_i}{\partial X_j}$ hence

$$[A] = \begin{bmatrix} \dfrac{\partial x_1}{\partial X_1} & \dfrac{\partial x_2}{\partial X_1} & \dfrac{\partial x_3}{\partial X_1} \\ \dfrac{\partial x_1}{\partial X_2} & \dfrac{\partial x_2}{\partial X_2} & \dfrac{\partial x_3}{\partial X_2} \\ \dfrac{\partial x_1}{\partial X_3} & \dfrac{\partial x_2}{\partial X_3} & \dfrac{\partial x_3}{\partial X_3} \end{bmatrix}$$

thus $\det \underset{\sim}{A} = \det (x_{i,j})$

using $e_{mnp} \det \underset{\sim}{A} = e_{ijk} A_{im} A_{jn} A_{kp}$ it is implied that

$$e_{mnp} \det (x_{i,j}) = e_{ijk} \, x_{i,m} \, x_{j,n} \, x_{k,p}$$

switching indices as follows:

$m \to K$
$n \to M$
$p \to N$
$j \to L$

gives

$$e_{ijk} \, x_{i,K} \, x_{j,M} \, x_{k,N} = e_{KMN} \det (x_{i,L})$$

from (2A-7A)

$$J = \det (x_{i,p}) \cdot \qquad \text{where at } t=0 \quad J(\underset{\sim}{X}, 0) = 1$$

so

$$\underbrace{d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)} \times d\underset{\sim}{r}^{(3)}}_{dV} = J \underbrace{d\underset{\sim}{R}^{(1)} \cdot d\underset{\sim}{R}^{(2)} \times d\underset{\sim}{R}^{(3)}}_{dV_0}$$

$$\boxed{dV = J \, dV_0}$$

D. Deformation of a Volume Element

Consider three line elements $d\underset{\sim}{R}^{(\alpha)}$ in B_0 which are deformed into $d\underset{\sim}{r}^{(\alpha)}$ in $B(t)$. By (2A-10) the components of these elements are related by

$$dx_i^{(\alpha)} = x_{i,K}\, dX_K^{(\alpha)} \qquad (\alpha = 1,2,3) \qquad (*)$$

From calculus the volume of the parallelepiped (6-sided prism with parallelogram faces) whose edges are $d\underset{\sim}{R}^{(\alpha)}$ is given by the magnitude of the scalar triple product $d\underset{\sim}{R}^{(1)} \cdot d\underset{\sim}{R}^{(2)} \times d\underset{\sim}{R}^{(3)}$. Computing the volume in $\mathcal{B}(t)$.

$$d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)} \times d\underset{\sim}{r}^{(3)} = e_{ijk}\, dx_i^{(1)} dx_j^{(2)} dx_k^{(3)}$$

using (2A-9)
$d\underset{\sim}{r} = dx_k\, \underset{\sim}{i}_k$
$d\underset{\sim}{R} = dX_k\, \underset{\sim}{I}_k$

$$\overset{(*)}{=} e_{ijK}\, x_{i,K}\, x_{j,M}\, x_{k,N}\, dX_K^{(1)} dX_M^{(2)} dX_N^{(3)}$$

$$\overset{(1B\text{-}4)\ ?}{=} e_{KMN} \det(x_{i,L})\, dX_K^{(1)} dX_M^{(2)} dX_N^{(3)}$$

(1B-4) $e_{mnp} \det \underset{\sim}{A} = e_{ijk} A_{im} A_{jn} A_{kp}$

i.e.

$$d\underset{\sim}{r}^{(1)} \cdot d\underset{\sim}{r}^{(2)} \times d\underset{\sim}{r}^{(3)} = J\, d\underset{\sim}{R}^{(1)} \cdot d\underset{\sim}{R}^{(2)} \times d\underset{\sim}{R}^{(3)}$$

Taking magnitudes and recalling that J is positive, we find

$$\boxed{dV = J\, dV_0} \qquad (2D\text{-}1)$$

Note that dV_0 is independent of time, but dV and J depend on $\underset{\sim}{X}$,t. For an alternate form of (2D-1) we use (2A-16):

$C_{KM} = x_{i,K}\, x_{i,M}$

$$\det(C_{KM}) = \det(x_{i,K}\, x_{i,M}) = (\det x_{i,K})^2 = J^2$$

Also, by definition

$$III_C = \det(C_{KM})$$

Hence,

$$J^2 = III_C \quad ; \quad J = \sqrt{III_C} \qquad (2D-2)$$

(we can show $III_C > 0$) and (2D-1) becomes

$$dV = \sqrt{III_C}\, dV_0 \qquad (2D-3)$$

Definition of Isochoric Deformation

If for all $\underset{\sim}{\lambda} \in B_0$ and all $t > 0$,

$$J = \sqrt{III_C} = 1 \qquad (2D-4)$$

then the deformation is called isochoric or volume preserving. Note that (2D-1) or (2D-3) then imply $dV = dV_0$ and $V = V_0$.

E. Homogeneous Deformations

Choose the x_i, X_K coordinate systems coincident with common origin. Then a static homogeneous deformation is given by

Common origin

$$\int dx_i = \int \underbrace{x_{i,K}}_{\text{if const. } D_{iK}} dX_K \quad \therefore x_i = D_{iK} x_K + \overset{=0}{C}$$

$$x_i = D_{iK} X_K \qquad (2E\text{-}1)$$

where D_{iK} is a constant, non-singular matrix. Note that $x_{i,K} = D_{iK}$ are independent of the material point, as are $C_{KM} = D_{iK} D_{iM}$ and $\mathcal{E}_{KM}$. It can be shown that (2E-1) implies that (finite) lines deform into lines, planes deform into planes, ellipses deform into ellipses. We now consider some special cases.

Case (a) Uniform Dilatation

$$D_{iK} = \lambda\delta_{iK} \quad , \quad \lambda = \text{const.}$$

$$x_1 = \lambda X_1 \quad , \quad x_2 = \lambda X_2 \quad , \quad x_3 = \lambda X_3$$

This mapping deforms a sphere of radius R in B_0 into a sphere of radius λR in B:

$$X_K X_K = X_1^2 + X_2^2 + X_3^2 = R^2$$

$$\left(\frac{x_1}{\lambda}\right)^2 + \left(\frac{x_2}{\lambda}\right)^2 + \left(\frac{x_3}{\lambda}\right)^2 = R^2$$

which implies

$$x_i x_i = \lambda^2 R^2$$

Case (b) -- Uniaxial Strain

$$D_{iK} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_1 = \lambda X_1 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3$$

Under this deformation a bar with axis in X_1 direction is stretched or compressed with no deformations in transverse planes. (See Fig. II-3). Note

$$C_{KM} = D_{iK} D_{iM} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$E_{KM} = \frac{1}{2} (C_{KM} - \delta_{KM})$$

$$= \frac{1}{2} \begin{pmatrix} \lambda^2-1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Note these arrays imply the X_K axes are principal axes at all points of B_0. We observe that the plane $X_1 = L$ is deformed into the plane $x_1 = \lambda L$.

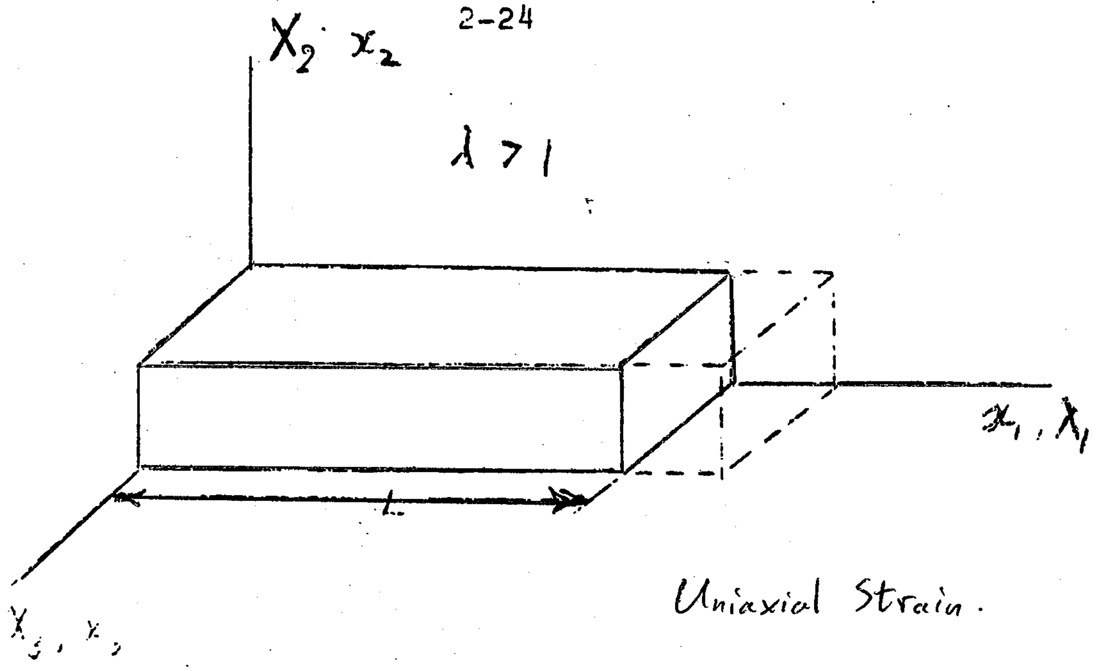


Fig. II-3

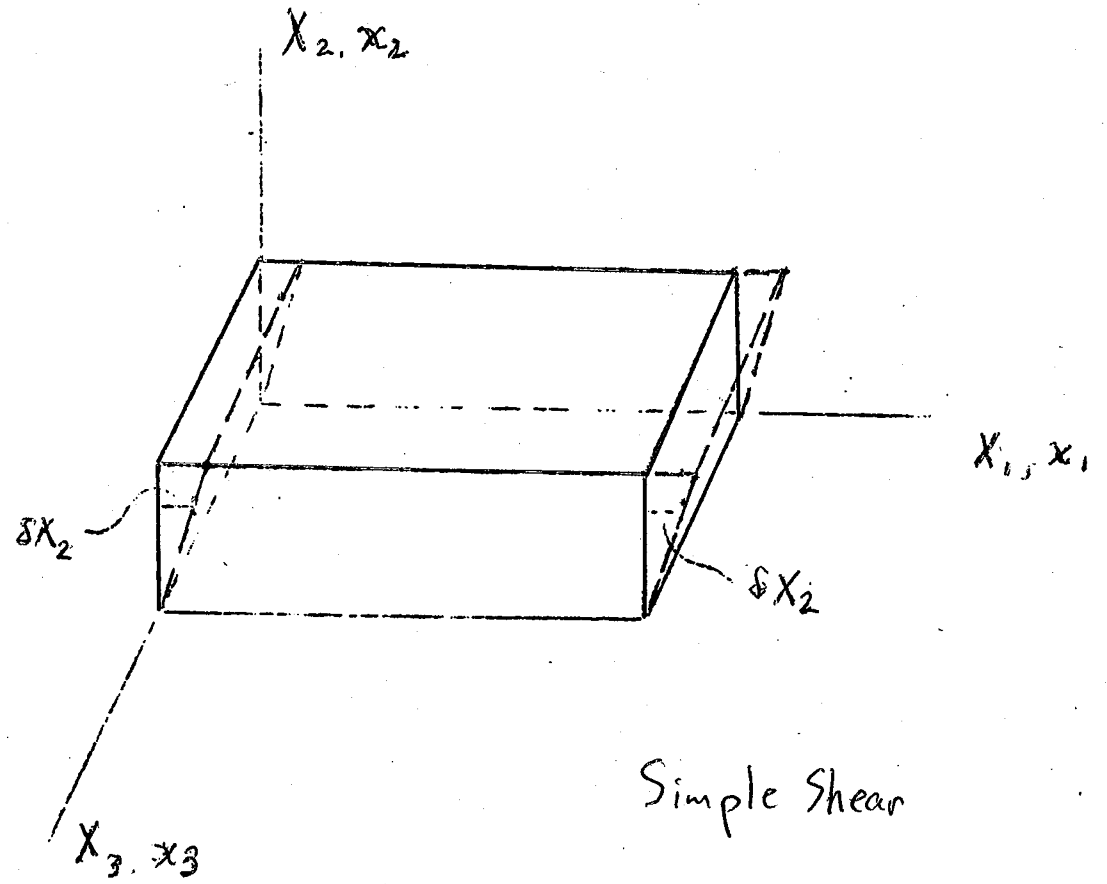


Fig. II-4

Case (c) -- Simple Extension

$$D_{iK} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & K\lambda & 0 \\ 0 & 0 & K\lambda \end{pmatrix}$$

$$x_1 = \lambda X_1 \quad , \quad x_2 = K\lambda X_2 \quad , \quad x_3 = K\lambda X_3$$

This case is similar to the previous case but with transverse planes of the bar suffering deformations. Again the X_K axes are principal axes.

$$C_{KM} = D_{iK}\, D_{iM} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & K^2\lambda^2 & 0 \\ 0 & 0 & K^2\lambda^2 \end{pmatrix}$$

$$E_{KM} = \frac{1}{2} \begin{pmatrix} \lambda^2-1 & 0 & 0 \\ 0 & K^2\lambda^2-1 & 0 \\ 0 & 0 & K^2\lambda^2-1 \end{pmatrix}$$

Case (d) -- Simple Shear

$$D_{iK} = \begin{pmatrix} 1 & S & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_1 = X_1 + SX_2 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3$$

Consider this deformation applied to a rectangular block. (See Fig. II-4). Note that

$$J = \det x_{i,K} = \det D_{iK} = 1$$

which implies simple shear is an *isochoric deformation*.

$$C_{KM} = x_{i,K}\, x_{i,M} \qquad \underset{\sim}{r} = x_k\, \underset{\sim}{i}_k \qquad \underset{\sim}{R} = X_K\, \underset{\sim}{I}_K$$

$$= \frac{\partial x_i}{\partial X_K}\frac{\partial x_i}{\partial X_M} = \frac{\partial \underset{\sim}{r}}{\partial X_K}\frac{\partial \underset{\sim}{r}}{\partial X_M}\underbrace{\underset{\sim}{i}_i \cdot \underset{\sim}{i}_i}_{1}$$

$\underset{\sim}{R} + \underset{\sim}{u} = \underset{\sim}{b} + \underset{\sim}{r}$, taking $\frac{\partial}{\partial X_K}$ of both sites yields $(\underset{\sim}{R} + \underset{\sim}{u})_{,K} = (\underset{0}{\cancel{b}} + \underset{\sim}{r})_{,K} = \underset{\sim}{r}_{,K}$

replace $\underset{\sim}{R}$ with $\underset{\sim}{X}$ and $\bar{r}$ with $\underset{\sim}{x}$

$$\Big[\underbrace{X_{P,K}}_{\frac{\partial X_P}{\partial X_K} = \delta_{PK}} + U_{P,K}\Big]\underbrace{\underset{\sim}{I}_{P,K}}_{\underset{\sim}{I}_P} = x_{k,K}\, \underset{\sim}{i}_k$$

$\left[\delta_{PK} + U_{P,K}\right] \underset{\sim}{I}_P = x_{k,K}\, \underset{\sim}{i}_k$ dotting both sides with $\underset{\sim}{i}_m$? why $\underset{\sim}{i}_m$

$$\left[\delta_{PK} + U_{P,K}\right] \underbrace{\underset{\sim}{I}_P \cdot \underset{\sim}{i}_m}_{\alpha_{mP}} = x_{k,K} \underbrace{\underset{\sim}{i}_k \cdot \underset{\sim}{i}_m}_{\delta_{Km}} = \begin{cases} 1 & K = m \\ 0 & K \neq m \end{cases}$$

? switch $x_{k,K} \to x_{m,K}$

$$x_{m,K} = \left[\delta_{PK} + U_{P,K}\right] \alpha_{mP}$$

Similarly for the $x_{i,M}$ term, using N instead of P yields $(K \to M)$

$x_{m,M} = \left[\delta_{NM} + U_{N,M}\right] \alpha_{mN}$ hence $C_{KM} = x_{m,K}\, x_{m,M}$ becomes

$$C_{KM} = \left[\delta_{PK} + U_{P,K}\right]\left[\delta_{NM} + U_{N,M}\right] \underbrace{\alpha_{mN}\, \alpha_{mP}}_{\delta_{PN} \text{ by (2A-7)}}$$

Non trivial solutions only exist when $P = N$ thus replacing P with N gives

$$C_{KM} = \left[\delta_{NK} + U_{N,K}\right]\left[\delta_{NM} + U_{N,M}\right] \quad \therefore$$

F. Strain-Displacement Equations

We introduce a displacement vector $\underset{\sim}{u}$ to define the motion of a material point. From Fig. II-5:

$$\underset{\sim}{R} + \underset{\sim}{u} = \underset{\sim}{b} + \underset{\sim}{r} \tag{2F-1}$$

Now the deformation tensors C_{KM}, c_{ij} and the strain tensors E_{KM}, e_{ij} can be expressed in terms of $\underset{\sim}{u}$ by (2F-1). Since $\underset{\sim}{b}$ is a constant vector,

$$(\underset{\sim}{R} + \underset{\sim}{u})_{,K} = (\underset{\sim}{b} + \underset{\sim}{r})_{,K} = \underset{\sim}{r}_{,K} \tag{2F-2}$$

Now $\underset{\sim}{u}$ can be expressed in components with respect to either $\underset{\sim}{i}_m$, $\underset{\sim}{I}_M$:

$$\underset{\sim}{u} = u_k \, \underset{\sim}{i}_k = U_K \, \underset{\sim}{I}_K \tag{2F-3}$$

$$u_k = \underset{\sim}{u} \cdot \underset{\sim}{i}_k \quad , \quad U_K = \boxed{\underset{\sim}{u}} \cdot \underset{\sim}{I}_K \quad ?$$

and we take

$$u_k = u_k(\underset{\sim}{x},t) \quad , \quad U_K = U_K(\underset{\sim}{X},t) \tag{2F-4}$$

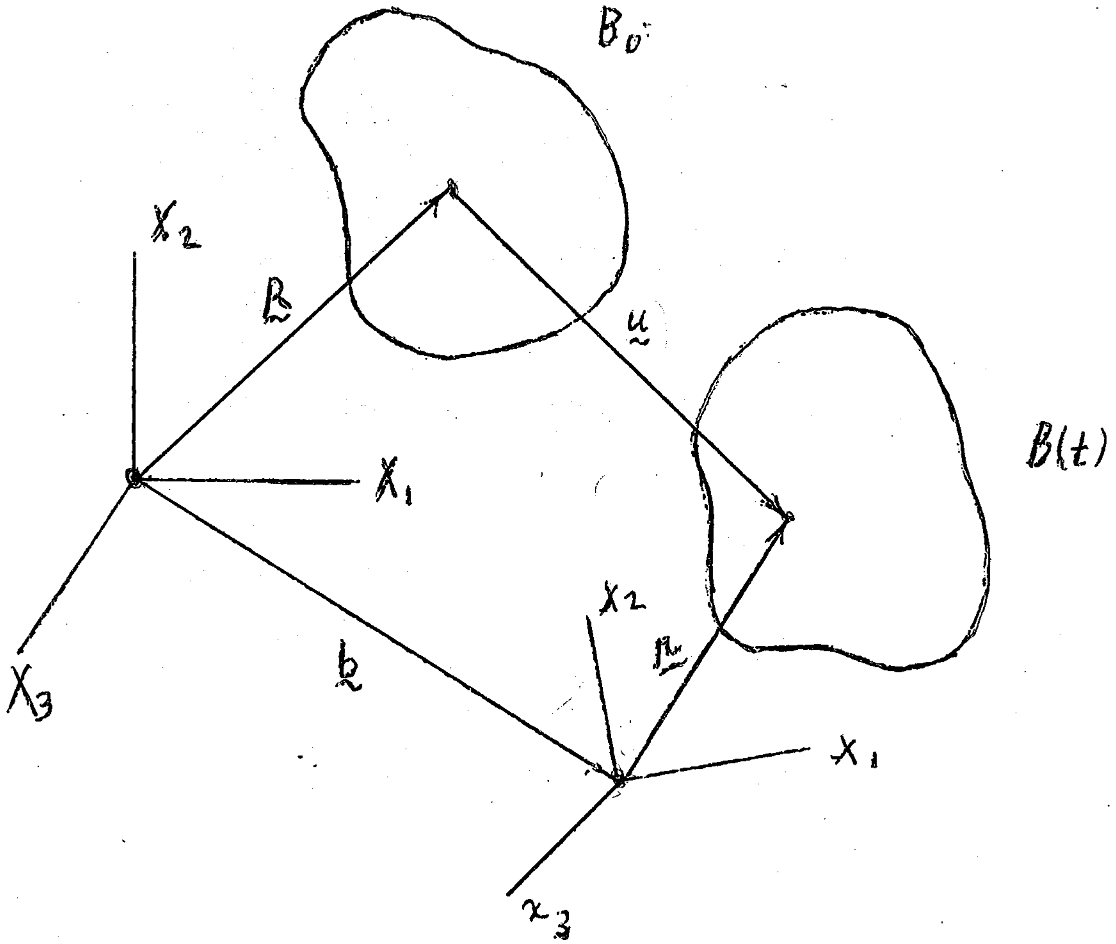
Then in component form (2F-2) becomes

$$(x_k \, \underset{\sim}{i}_k)_{,K} = [(X_P + U_P) \, \underset{\sim}{I}_P]_{,K}$$

$$x_{k,K} \, \underset{\sim}{i}_k = (X_{P,K} + U_{P,K}) \, \underset{\sim}{I}_P = (\delta_{PK} + U_{P,K}) \, \underset{\sim}{I}_P$$

Dot this equation with $\underset{\sim}{i}_m$:

$$x_{m,K} = (\delta_{PK} + U_{P,K}) \, \underset{\sim}{i}_m \cdot \underset{\sim}{I}_P \overset{(2A-6)}{=} (\delta_{PK} + U_{P,K}) \, \alpha_{\boxed{mp}} \quad P \text{ cap?} \tag{2F-5}$$



$\underset{\sim}{u}$ = displacement vector of point

Fig. II-5

which implies

$$C_{KM} = x_{m,K}\, x_{m,M} = \overbrace{\alpha_{mP}(\delta_{PK} + U_{P,K})\, \alpha_{mN}}^{\delta_{PN} \text{ by (2A-7)}}(\delta_{NM} + U_{N,M})$$

$$= (\delta_{NK} + U_{N,K})(\delta_{NM} + U_{N,M})$$

i.e.

$$C_{KM} = \delta_{KM} + U_{K,M} + U_{M,K} + U_{N,K}\, U_{N,M} \qquad \text{(2F-6)}$$

and

$$E_{KM} = \frac{1}{2}(C_{KM} - \delta_{KM})$$

$$= \frac{1}{2}(U_{K,M} + U_{M,K}) + \frac{1}{2} U_{N,K}\, U_{N,M}$$

$$\leftarrow E_{KM} = U_{(K,M)} + \frac{1}{2} U_{N,K}\, U_{N,M} \qquad \text{Material form (Lagrangian)} \quad \text{(2F-7)}$$

This is the material form of the strain-displacement equations. For the spatial form, take $\frac{\partial}{\partial x_i}$ of (2F-1) with $\underset{\sim}{u} = u_k\, \underset{\sim}{i}_k$ and form c_{ij}, e_{ij}. This leads to

$$\leftarrow e_{ij} = u_{(i,j)} - \frac{1}{2} u_{m,i}\, u_{m,j} \qquad \text{Spatial form (Eulerian)} \quad \text{(2F-8)}$$

---

proof: $(\underset{\sim}{R} + \underset{\sim}{U})_{,i} = (\underset{\sim}{b} + \underset{\sim}{r})_{,i} \Rightarrow X_{K,i}\, \underset{\sim}{I}_K + U_{k,i}\, \underset{\sim}{i}_k = x_{m,i}\, \underset{\sim}{i}_m$

dot in I_M $\Rightarrow$ $X_{M,i} = \alpha_{iM} - \alpha_{kM} U_{k,i}$

$$C_{ij} = (\alpha_{iM} - \alpha_{kM} U_{k,i})(\alpha_{jM} - \alpha_{pM} U_{R,j}) = \overset{\delta_{ij}}{\underbrace{\alpha_{iM}\alpha_{jM}}} - \overset{\delta_{ip}}{\underline{\alpha_{iM}\alpha_{pM}}}\, U_{p,j} - \overbrace{\alpha_{kM} U_{k,i}\, \alpha_{jM}}^{\delta_{kj}}$$

$$+ \underbrace{\alpha_{kM}\alpha_{pM}}_{\delta_{kp}} U_{k,i} U_{p,j}$$

$$\therefore\ C_{ij} = \delta_{ij} - U_{i,j} - U_{j,i} + U_{R,i} U_{k,j}$$

and $e_{ij} = \frac{1}{2}(\delta_{ij} - C_{ij}) = \frac{1}{2}[u_{i,j} + u_{j,i} - U_{k,i} U_{k,j}]$

$$= u_{(i,j)} - \frac{1}{2} U_{m,i} U_{m,j}$$

q.e.d.

Mehta

G. Small Deformations

In order to specify the conditions under which a deformation is infinitesimal, we define the norm of an array $\underset{\sim}{A}$ as

$$||\underset{\sim}{A}|| = [\mathrm{tr}(\underset{\sim}{A}\ \underset{\sim}{A}^T)]^{1/2} \qquad (2G-1)$$

Defining $\underset{\sim}{H}$ to be the 2nd order tensor with components $U_{K,M}$, we have

$$||\underset{\sim}{H}|| = [U_{K,P}\ U_{K,P}]^{1/2} \qquad (2G-2)$$

Define $||\underset{\sim}{H}|| = \varepsilon$ and let $\underset{\sim}{F}(\underset{\sim}{H})$ be any 2nd order tensor function of $\underset{\sim}{H}$ whose norm is less than $C\ \varepsilon^n$ (where C is a positive constant).

$$||\underset{\sim}{F}(\underset{\sim}{H})|| < C\ \varepsilon^n \qquad (2G-3)$$

Then $\underset{\sim}{F}$ is said to be <u>of order ε^n</u> and is denoted by

$$\underset{\sim}{F}(\underset{\sim}{H}) = \underset{\sim}{O}(\varepsilon^n) \qquad (2G-4)$$

Note (2G-3) and (2G-4) are equivalent, i.e., one implies the other. Also, $\underset{\sim}{H} = \underset{\sim}{O}(\varepsilon)$ since

$$||\underset{\sim}{H}|| = \varepsilon < C\ \varepsilon \quad , \quad C > 1 \qquad (\dagger)$$

<u>Definition</u> - If $||\underset{\sim}{H}|| = \varepsilon << 1$, the deformation is said to be small or infinitesimal. Note that each element of the array $\underset{\sim}{H}$ or $U_{K,M}$ is small when $\varepsilon << 1$. We now define a tensor $\tilde{\underset{\sim}{E}}$

$$\tilde{\underset{\sim}{E}} = \underset{\sim}{H}^S \quad \text{or} \quad \tilde{E}_{KM} = \frac{1}{2}(U_{K,M} + U_{M,K}) = U_{(K,M)} \qquad (2G-5)$$

It follows that for small deformations

$$||\underset{\sim}{\tilde{E}}|| = ||\underset{\sim}{H}^S|| < C\,\varepsilon \quad , \quad \underset{\sim}{\tilde{E}} = \underset{\sim}{O}(\varepsilon)$$

Now use (2G-5) in (2F-7):

$$\underset{\sim}{E} = \underset{\sim}{\tilde{E}} + \frac{1}{2}\,\underset{\sim}{H}^T\,\underset{\sim}{H} \quad \text{or} \quad E_{KM} = \tilde{E}_{KM} + \frac{1}{2}\,U_{N,K}\,U_{N,M}$$

Using (†), we can write this as

$$\underset{\sim}{E} = \underset{\sim}{\tilde{E}} + \underset{\sim}{O}(\varepsilon^2)$$

Then neglecting terms of order ε^2 compared to those of order ε, we have

$$\underset{\sim}{E} \cong \underset{\sim}{\tilde{E}} \quad \text{or} \quad E_{KM} \cong \tilde{E}_{KM} \qquad \text{Small Deformation Theory} \qquad (2G-6)$$

Based on (2G-6) $\tilde{E}_{KM}$ is called the <u>linearized material strain tensor</u>.

Recall the expressions for the stretches, extensions and shears. From (2B-12) for small deformations

$$\Lambda_{(1)} = (1 + 2E_{11})^{1/2} \cong (1 + 2\tilde{E}_{11})^{1/2} \qquad (*)$$

Since $\tilde{E}_{11} << 1$, then expanding (*) in a binomial series

$$\Lambda_{(1)} \cong 1 + \frac{1}{2}\,(2\tilde{E}_{11}) + \frac{1}{2!}\cdot\frac{1}{2}\,(\frac{1}{2} - 1)(2\tilde{E}_{11})^2 + \ldots$$

$$\cong 1 + \tilde{E}_{11} \qquad (2G-7)$$

and

$$E_{(1)} = \Lambda_{(1)} - 1 \cong \tilde{E}_{11} \qquad (2G\text{-}8)$$

Hence, for small deformations $\tilde{E}_{11}$, $\tilde{E}_{22}$, $\tilde{E}_{33}$ are approximately equal to the extensions of elements originally having directions along $\underset{\sim}{I}_1$, $\underset{\sim}{I}_2$, $\underset{\sim}{I}_3$.

We now consider the shears for small deformations. From (2B-16)

$$\sin \Gamma_{12} = 2E_{12}(1 + 2E_{11})^{-1/2}(1 + 2E_{22})^{-1/2}$$

$$\cong 2\tilde{E}_{12}(1 - \frac{1}{2} \cdot 2\tilde{E}_{11} + \ldots)(1 - \frac{1}{2} \cdot 2\tilde{E}_{22} + \ldots)$$

$$\cong 2\tilde{E}_{12}(1 - \tilde{E}_{11})(1 - \tilde{E}_{22}) \cong 2\tilde{E}_{12}$$

This implies $\sin \Gamma_{12}$ is small and can be approximated by Γ_{12}. hence

$$\Gamma_{12} \cong 2\tilde{E}_{12} \qquad (2G\text{-}9)$$

Thus, for small deformations $\tilde{E}_{12}$, $\tilde{E}_{13}$, $\tilde{E}_{23}$ approximately equal half the shears, i.e., half the change in angle between pairs of elements originally along the $(\underset{\sim}{I}_1, \underset{\sim}{I}_2)$, $(\underset{\sim}{I}_1, \underset{\sim}{I}_3)$, $(\underset{\sim}{I}_2, \underset{\sim}{I}_3)$ directions.

Consider the deformation of dV_0 into dV. From (2C-6) and (2D-3)

$$\frac{dV}{dV_0} = \sqrt{III_C} = (1 + 2I_E + 4II_E + 8III_E)^{1/2}$$

$$\cong (1 + 2I_{\tilde{E}})^{1/2} \cong 1 + \frac{1}{2} \cdot 2I_{\tilde{E}} \cong 1 + I_{\tilde{E}} \qquad (2G\text{-}10)$$

Recall that $II_{\tilde{E}} = O(\varepsilon^2)$ and $III_{\tilde{E}} = O(\varepsilon^3)$. Now

$$\frac{dV - dV_0}{dV_0} = \frac{dV}{dV_0} - 1 \overset{(2G\text{-}10)}{\cong} I_{\tilde{E}} = \tilde{E}_{KK} \qquad (2G\text{-}11)$$

Hence, $I_{\tilde{E}}$ equals the approximate change in volume per unit undeformed volume. Recalling (2D-2): $J = \sqrt{III_C}$, then (2G-10) implies

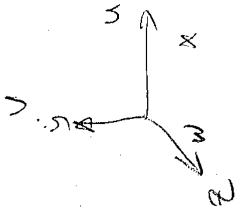
$$J \cong 1 + I_{\tilde{E}} \qquad (2G\text{-}12)$$

To determine how the material and spatial strain tensors are related for small deformations, we begin by dotting (2F-3) with $\underset{\sim}{I}_M$:

$$U_K \underset{\sim}{I}_K \cdot \underset{\sim}{I}_M = u_k \underset{\sim}{i}_k \cdot \underset{\sim}{I}_M \overset{(2A-6)}{=} \alpha_{kM} u_k$$

i.e.

$$U_M(\underset{\sim}{X},t) = \alpha_{kM} u_k(\underset{\sim}{x},t)$$



Taking $\frac{\partial}{\partial X_K}$

$$U_{M,K} = \alpha_{kM} u_{k,M} = \alpha_{kM} u_{k,i} x_{i,K}$$

$$\overset{(2F-3)}{=} \alpha_{kM} \alpha_{iP} u_{k,i}(\delta_{PK} + U_{P,K})$$

$$\cong \alpha_{kM} \alpha_{iP} \delta_{PK} u_{k,i} = \alpha_{kM} \alpha_{iK} u_{k,i}$$

$u_{(i,j)} =$

$u_{(1,1)} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) = \varepsilon_{xx}$

$u_{(1,2)} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = \varepsilon_{xy}$

$1 \equiv x$
$2 \equiv y$
$3 \equiv z$

Since $\underset{\sim}{\alpha}$ is orthogonal, inversion gives

$$u_{k,i} \cong \alpha_{kM} \alpha_{iK} U_{M,K} \qquad (*)$$

Because $U_{M,K}$ is of order ε, each term on the right hand side is of order ε. Hence,

$$u_{k,i} = \underset{\sim}{O}(\varepsilon)$$

Then from (2F-8): $e_{ij} = u_{(i,j)} - \frac{1}{2} u_{m,i} u_{m,j}$, the 2nd term is $\underset{\sim}{O}(\varepsilon^2)$ and

$$e_{ij} \cong u_{(i,j)} \equiv \tilde{e}_{ij} \qquad (2G-13)$$

Small Deformation Theory

where $\tilde{e}_{ij}$ is the <u>linearized spatial strain tensor</u>. Taking the symmetric part of (*)

$$u_{(k,i)} \cong \alpha_{(kM}\ \alpha_{i)K}\ U_{M,K}$$

$$= \frac{1}{2}\ (\alpha_{kM}\ \alpha_{iK} + \alpha_{iM}\ \alpha_{kK})\ U_{M,K}$$

$$= \frac{1}{2}\ (\alpha_{kM}\ \alpha_{iK}\ U_{M,K} + \alpha_{iK}\ \alpha_{kM}\ U_{K,M})$$

$$= \frac{1}{2}\ (U_{M,K} + U_{K,M})\ \alpha_{kM}\ \alpha_{iK}$$

which implies

$$u_{(k,i)} \cong \alpha_{kM}\ \alpha_{iK}\ U_{(M,K)}$$

or by (2G-5) and (2G-13)

$$\tilde{e}_{ki} \cong \alpha_{kM}\ \alpha_{iK}\ \tilde{E}_{MK} \tag{2G-14}$$

Since we can always choose $\alpha_{kM} = \delta_{kM}$ by taking the coordinate axes coincident, then

$$\tilde{e}_{ki} \cong \delta_{kM}\ \delta_{iK}\ \tilde{E}_{MK} \tag{2G-15}$$

i.e.

$$\tilde{e}_{11} \cong \tilde{E}_{11}\ ,\quad \tilde{e}_{12} \cong \tilde{E}_{12}\ ,\quad \text{etc.}$$

Hence, for small deformations there is no distinction between the material and spatial strain tensors; their physical interpretations being the same when $\alpha_{kM} = \delta_{kM}$. Upon contracting (2G-15):

$$\tilde{e}_{kk} \cong \alpha_{kM}\ \alpha_{kK}\ \tilde{E}_{MK} = \delta_{MK}\ \tilde{E}_{MK}$$

$$\cong \tilde{E}_{KK}$$

Small Deformation Theory

which implies

$$I_{\overset{\approx}{E}} \cong I_{\tilde{e}}$$

so that (2G-11), (2G-12) become

$$\frac{dV - dV_0}{dV_0} \cong I_{\tilde{e}}$$

$$J \cong 1 + I_{\tilde{e}} \qquad (2G-16)$$

III. Kinematics of Motion

A. Basic Concepts: Dual Descriptions, Material Derivatives, etc.

Consider any tensor field $\underset{\sim}{F}$ associated with the deformation of B_0 into $B(t)$. Since the deformation can be specified by either $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$ or $\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x},t)$, $\underset{\sim}{F}$ can be expressed in the material description

$$\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{X},t) \tag{3A-1}$$

or the spatial description

$$\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{x},t) \tag{3A-2}$$

It is understood that the functional forms of $\underset{\sim}{F}$ in (3A-1), (3A-2) are in general different. If we choose X_K = const., then (3A-1) gives the value of $\underset{\sim}{F}$ at time t at the particle P in B(t) having initial coordinates X_K in B_0. This means we are following a given particle P with $\underset{\sim}{F}$ changing as P moves through space. Choosing x_i = const., then (3A-2) gives the value of $\underset{\sim}{F}$ at the particle in B(t) having spatial position x_i at time t. In this case we are viewing a fixed spatial point with $\underset{\sim}{F}$ changing as different particles move past the point as t changes.

Lagrangian

Eulerian

The time rate of change of $\underset{\sim}{F}$ along a given particle P: X_K = const. is the material derivative of $\underset{\sim}{F}$ and is denoted by $\frac{D\underset{\sim}{F}}{Dt} \equiv \dot{\underset{\sim}{F}}$. From (3A-1)

$$\frac{D\underset{\sim}{F}}{Dt} \equiv \dot{\underset{\sim}{F}} = \frac{\partial \underset{\sim}{F}}{\partial t}(\underset{\sim}{X},t)\Big|_{\underset{\sim}{X}} \tag{3A-3}$$

$$\frac{D\underset{\sim}{F}}{Dt} = \underbrace{\left.\frac{\partial \underset{\sim}{F}}{\partial t}\right|_{\underset{\sim}{x}}}_{\text{Local rate of change}} + \underbrace{v_i \frac{\partial \underset{\sim}{F}}{\partial x_i}}_{\text{Convective rate of change}}$$

$$\underbrace{\frac{Dv_i}{Dt}(\underset{\sim}{x}, t)}_{\text{Acceleration}} = \underbrace{\left.\frac{\partial v_i}{\partial t}\right|_{\underset{\sim}{x}}}_{\text{Local Acceleration}} + \underbrace{v_j \frac{\partial v_i}{\partial x_j}}_{\text{Velocity gradients}}$$

By choosing $\underset{\sim}{F} = \underset{\sim}{x}(\underset{\sim}{X},t)$, we obtain the definition of velocity of the particle $\underset{\sim}{X}$:

Velocity

$$v_i(\underset{\sim}{X},t) = \frac{Dx_i}{Dt} = \dot{x}_i = \frac{\partial x_i}{\partial t}(\underset{\sim}{X},t)\Big|_{\underset{\sim}{X}} \tag{3A-4}$$

To compute $\frac{D\underset{\sim}{F}}{Dt}$ from the spatial description (3A-2), write $\underset{\sim}{F} = \underset{\sim}{F}(x_i(\underset{\sim}{X},t),\ t)$. Then

$$\frac{D\underset{\sim}{F}}{Dt} = \frac{\partial \underset{\sim}{F}}{\partial t}\Big|_{\underset{\sim}{x}} + \frac{\partial \underset{\sim}{F}}{\partial x_i}\frac{\partial x_i}{\partial t}\Big|_{\underset{\sim}{X}}$$

$$\boxed{\frac{D\underset{\sim}{F}}{Dt} \overset{(3A-4)}{=} \frac{\partial \underset{\sim}{F}}{\partial t}\Big|_{\underset{\sim}{x}} + v_i\frac{\partial \underset{\sim}{F}}{\partial x_i}} \tag{3A-5}$$

In this equation $\frac{\partial \underset{\sim}{F}}{\partial t}\Big|_{\underset{\sim}{x}}$ is the local rate of change of $\underset{\sim}{F}$ at $\underset{\sim}{x}$ and $v_i\frac{\partial \underset{\sim}{F}}{\partial x_i}$ is the convected rate of change of $\underset{\sim}{F}$ due to the particle moving past the point $\underset{\sim}{x}$. For the special case that $\underset{\sim}{F} = \underset{\sim}{F}(\underset{\sim}{x})$, $\frac{\partial \underset{\sim}{F}}{\partial t} = 0$ and the field $\underset{\sim}{F}$ is called steady.

Applying $\frac{D}{Dt}$ to (3A-4), we obtain the material form of the acceleration:

$$\dot{v}_i(\underset{\sim}{X},t) = \frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t}\Big|_{\underset{\sim}{X}} \tag{3A-6}$$

But by the inverse mapping $\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x},t)$, v_i can be expressed in spatial description $v_i(\underset{\sim}{x},t)$. Then $\dot{v}_i$ follows from (3A-5):

$$\boxed{\dot{v}_i(\underset{\sim}{x},t) = \frac{Dv_i}{Dt}(\underset{\sim}{x},t) = \frac{\partial v_i}{\partial t}\Big|_{\underset{\sim}{x}} + v_j\frac{\partial v_i}{\partial x_j}} \tag{3A-7}$$

Acceleration

where $\frac{\partial v_i}{\partial x_j} = v_{i,j}$ are velocity gradients.

Now consider the time rate of change of an element $d\underset{\sim}{r}$ in B(t) as it deforms: $d\underset{\sim}{r} = d\underset{\sim}{x}\,\underset{\sim}{i}$ or $d\underset{\sim}{r} = dx_i$

$$\frac{D}{Dt}(dx_i) = \frac{D}{Dt}(x_{i,K}\, dX_K) = \frac{D}{Dt}(x_{i,K})\, dX_K \qquad (*)$$

since dX_K is independent of t. Now

$$\frac{D}{Dt}(x_{i,K}) = \frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_K}\right)\bigg|_{\underset{\sim}{X}} = \frac{\partial}{\partial X_K}\left(\frac{\partial x_i}{\partial t}\bigg|_{\underset{\sim}{X}}\right)$$

$$= v_{i,K} = v_{i,j}\, x_{j,K} \qquad (3A-8)$$

by expressing v_i in spatial form. Then (*) implies

$$\boxed{\frac{D}{Dt}(dx_i) = v_{i,j}\, x_{j,K}\, dX_K \overset{(2A-10)}{=} v_{i,j}\, dx_j} \qquad (3A-9)$$

Time Rate of Change of Element $d\underset{\sim}{r}$

Consider the time rate of change of a deforming volume element

$$\frac{D}{Dt}(dV) \overset{(2D-1)}{=} \frac{D}{Dt}(J\, dV_0) = \dot{J}\, dV_0 \qquad (3A-10)$$

But since $J = \det(x_{i,K})$, then

$$\dot{J} = \frac{\partial J}{\partial x_{i,K}}\, \overline{\dot{x_{i,K}}} = \frac{\partial J}{\partial x_{i,K}}\, v_{i,j}\, x_{j,K} \qquad (*)$$

From Theorem 4 [Eqn. (1B-19)]:

$$\frac{\partial J}{\partial x_{i,K}} = \lambda_{iK}$$

where λ_{iK} is the cofactor of $x_{i,K}$. By (1B-16) and (1B-17)

$$\frac{\partial J}{\partial x_{i,K}} x_{j,K} = \lambda_{iK} x_{j,K} = J \delta_{ij}$$

Then (*) implies

$$\boxed{\dot{J} = J v_{i,j} \delta_{ij} = J v_{i,i} = J \operatorname{div} \underset{\sim}{v}} \qquad (3A\text{-}11)$$

and (3A-10) becomes

$$\boxed{\frac{D}{Dt}(dV) = J \operatorname{div} \underset{\sim}{v}\, dV_0 \overset{(2D\text{-}1)}{=} \operatorname{div} \underset{\sim}{v}\, dV} \qquad (3A\text{-}12)$$

Time Rate of Change of a Deforming Volume Element

We note that for an isochoric deformation $J = 1$, $\dot{J} = 0$ and (3A-11) implies

↳ simple shear

$$\nabla \cdot \underset{\sim}{v} = 0 \qquad \boxed{\operatorname{div} \underset{\sim}{v} = 0} \qquad \operatorname{div} \underset{\sim}{v} = v_{i,i} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} \qquad (3A\text{-}13)$$

Transport Theorem.- Let $\underset{\sim}{F}(\underset{\sim}{x}, t)$ be a tensor field, continuously differentiable in $\underset{\sim}{x}, t$ and $V(t)$ be a deforming material volume. Consider the material derivative of $\int_{V(t)} \underset{\sim}{F}\, dV$. Using (2D-1) we can map the integration back to the reference configuration $\mathcal{B}_0$:

$$\int_{V(t)} \underset{\sim}{F}\, dV \overset{(2D\text{-}1)}{=} \int_{V_0} \underset{\sim}{F}\, J\, dV_0$$

Then

$$\frac{D}{Dt}\int_{V(t)} \underset{\sim}{F}\, dV = \int_{V_0} \dot{\overline{\underset{\sim}{F} J}}\, dV_0 \qquad ?$$

$$= \int_{V_0} (\dot{\underset{\sim}{F}} J + F \dot{J})\, dV_0$$

$$\overset{(3A\text{-}11)}{=} \int_{V_0} (\dot{\underset{\sim}{F}} + F \operatorname{div} \underset{\sim}{v})\, J\, dV_0$$

i.e.

$$\boxed{\frac{D}{Dt}\int_{V(t)} \underset{\sim}{F}\, dV = \int_{V(t)} (\dot{\underset{\sim}{F}} + \underset{\sim}{F}\ \mathrm{div}\ \underset{\sim}{v})\, dV} \qquad (3A\text{-}14)$$

This result is Reynolds' Transport Theorem. Some component forms are

$$\frac{D}{Dt}\int_{V(t)} \varphi(\underset{\sim}{x},t)\, dV = \int_{V(t)} (\dot{\varphi} + \varphi\, v_{i,i})\, dV$$

$$\frac{D}{Dt}\int_{V(t)} f_i(\underset{\sim}{x},t)\, dV = \int_{V(t)} (\dot{f}_i + f_i\, v_{j,j})\, dV$$

Now expand $\dot{\underset{\sim}{F}}$ in (3A-14) using (3A-5)

$$\begin{aligned}\frac{D}{Dt}\int_{V(t)} \underset{\sim}{F}\, dV &= \int_{V(t)} \left(\frac{\partial \underset{\sim}{F}}{\partial t} + \underset{\sim}{F}_{,j}\, v_j + \underset{\sim}{F}\, v_{j,j}\right) dV \\ &= \int_{V(t)} \left[\frac{\partial \underset{\sim}{F}}{\partial t} + (\underset{\sim}{F}\, v_j)_{,j}\right] dV \\ &= \int_{V(t)} \frac{\partial \underset{\sim}{F}}{\partial t}\, dV + \int_{S(t)} \underset{\sim}{F}\, v_j\, n_j\, dA \qquad (3A\text{-}15)\end{aligned}$$

by the divergence theorem. This result is an alternate form of Reynolds' Theorem.

B. Stretching and Spin Tensors

We define the <u>stretching tensor</u> d_{ij} and the <u>spin tensor</u> w_{ij} as

$$d_{ij} = v_{(i,j)} \quad , \quad w_{ij} = v_{[i,j]} \qquad \text{(3B-1)}$$

Then the velocity gradients can be written as

$$v_{i,j} = d_{ij} + w_{ij} \qquad \text{(3B-2)}$$

and (3A-9) becomes

$$\frac{D}{Dt}(dx_i) = \dot{\overline{dx_i}} = (d_{ij} + w_{ij})\, dx_j \qquad \text{(3B-3)}$$

To interpret the elements of d_{ij}, consider an element $d\underset{\sim}{r}$ at some point in B(t) and take the material derivative of ds^2 recalling $ds^2 = d\underset{\sim}{r} \cdot d\underset{\sim}{r} = dx_i\, dx_i$:

$$\frac{D}{Dt}(ds^2) = \frac{D}{Dt}(dx_i\, dx_i)$$

$$\cancel{2}\, ds\, \dot{\overline{ds}} = \cancel{2}\, \dot{\overline{dx_i}}\, dx_i$$

$$ds\, \dot{\overline{ds}} \overset{(3B-3)}{=} (d_{ij} + w_{ij})\, dx_i\, dx_j = d_{ij}\, dx_i\, dx_j$$

since $w_{ij}\, dx_i\, dx_j$ vanishes by Theorem 1. Now divide by ds^2:

$$\frac{\dot{\overline{ds}}}{ds} = d_{ij} \frac{dx_i}{ds} \frac{dx_j}{ds} \qquad (*)$$

If we define

$$n_i = \frac{dx_i}{ds} \quad , \quad d_{(n)} = \frac{\dot{\overline{ds}}}{ds} \quad ,$$

then n_i is a unit vector along $d\underset{\sim}{r}$ and $d_{(n)}$ is called the stretching of $d\underset{\sim}{r}$. Then (*) becomes

$$d_{(n)} = d_{ij}\, n_i\, n_j \tag{3B-4}$$

For an element $d\underset{\sim}{r}$ along $\underset{\sim}{i}_1$ at time t, $n_i = (1,0,0)$ and (3B-4) gives

$$\boxed{d_{(1)} = \frac{\dot{\overline{ds}}}{ds} = d_{11}}$$

i.e., d_{11} gives the rate of change of length per unit length of an element instantaneously along $\underset{\sim}{i}_1$. Similar interpretations apply to d_{22}, d_{33}. For an interpretation of the off-diagonal elements of d_{ij} we compute the angle between elements $dx_i^{(1)}$, $dx_i^{(2)}$ in B(t) from their dot product:

$$dx_i^{(1)}\, dx_i^{(2)} = ds_1\, ds_2 \cos\theta_{12}$$

Taking the material derivative of this expression, using (3B-3), (3B-4), we can show that

$$\begin{aligned} 2\, d_{ij}\, n_i^{(1)}\, n_j^{(2)} = (d_{(n_1)} + d_{(n_2)}) \cos\theta_{12} \\ - \dot{\theta}_{12} \sin\theta_{12} \end{aligned} \tag{3B-5}$$

Now we choose the elements $d\underset{\sim}{r}^{(1)}$, $d\underset{\sim}{r}^{(2)}$ instantaneously along $\underset{\sim}{i}_1$, $\underset{\sim}{i}_2$, respectively. Then

$$n_i^{(1)} = (1,0,0) \quad , \quad n_i^{(2)} = (0,1,0) \quad , \quad \theta_{12} = \pi/2$$

and (3B-5) implies

$$\boxed{2\ d_{12} = -\ \dot{\theta}_{12}}$$

For $d_{12} > 0$ then the angle between the elements is instantaneously decreasing. Thus d_{12} is half the rate of decrease of the angle between elements instantaneously along $\underset{\sim}{i}_1$, $\underset{\sim}{i}_2$. Similar interpretations apply to d_{23}, d_{13}.

Since d_{ij} is real and symmetric, then by Theorem 11 we can determine a principal axes system $\bar{x}_i$ in which d_{ij} is diagonalized. Hence, in this system elements along $\bar{x}_i$ suffer only stretchings with no rates of change of the angles between them.

For a detailed interpretation of the spin tensor w_{ij} see Eringen, pp. 79-81. Roughly speaking w_{ij} is a measure of the rate of rotation of elements in the neighborhood of each point. A special case arises when $d_{ij} = 0$ at some point P. Then (3B-4) implies $d_{(n)} = \frac{\dot{\overline{ds}}}{ds} = 0$ or $\dot{\overline{ds}} = 0$, i.e., the motion is locally rigid. Thus all elements at P are instantaneously being rigidly rotated. Then (3B-3) implies

Spin Tensor

$$\dot{\overline{dx_i}} = w_{ij}\ dx_j \quad , \quad \dot{\overline{ds}} = 0$$

i.e., w_{ij} gives the rate of rotation of $d\underset{\sim}{r}$ for a locally rigid motion.

Since $w_{ij} = -\ w_{ji}$ has 3 independent components, we can define a vector $\underset{\sim}{w}_i$ such that

$$w_i = e_{imn}\, w_{nm} \quad , \quad 2\, w_{mn} = e_{nmi}\, w_i \qquad (3B\text{-}6)$$

i.e.

$$w_1 = 2\, w_{\underset{23}{\cancel{32}}} \quad , \quad w_2 = 2\, w_{13} \quad , \quad w_3 = 2\, w_{21}$$

Then $\underset{\sim}{w}$ is called the vorticity vector. From $(3B\text{-}1)_2$

$$\boxed{w_i = e_{imn}\, v_{[n,m]} = e_{imn}\, v_{n,m}}$$

Vorticity Vector

or

$$\underset{\sim}{w} = \text{curl } \underset{\sim}{v} = \nabla \times \underset{\sim}{V} \qquad (3B\text{-}7)$$

When $w_i = 0 = w_{ij}$ throughout the body, the motion is called irrotational.

IV. Balance Laws

A. Conservation of Mass

Let $\rho_0(\underset{\sim}{X})$ and $\rho(\underset{\sim}{x},t)$ be the mass densities of B_0, $B(t)$. Each volume element dV_0 is mapped into dV under the deformation. Hence, the elements of mass associated with dV_0, dV are

$$dM = \rho_0(\underset{\sim}{X})\ dV_0 \qquad dm = \rho(\underset{\sim}{x},t)\ dV$$

Consider the total mass of an arbitrary material subvolume of the body $\overline{V}_0$ at $t=0$ deformed into $\overline{V}(t)$ at $t > 0$:

$$M = \int_{\overline{V}_0} \rho_0(\underset{\sim}{X})\ dV_0 \quad , \quad m = \int_{\overline{V}(t)} \rho(\underset{\sim}{x},t)\ dV \qquad (*)$$

Postulate I -- Conservation of mass -- The total mass of any deforming subvolume of the body is constant:

$$M = m$$

i.e., from (*)

$$\int_{\overline{V}(t)} \rho(\underset{\sim}{x},t)\ dV = \int_{\overline{V}_0} \rho_0(\underset{\sim}{X})\ dV_0 \qquad (4A\text{-}1)$$

By (2D-1): $dV = J\ dV_0$, we can change variables using the mapping $\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t)$ and integrate the left hand side over $\overline{V}_0$:

$$\int_{\overline{V}_0} \rho(\underset{\sim}{X},t)\ J(\underset{\sim}{X},t)\ dV_0 = \int_{\overline{V}_0} \rho_0(\underset{\sim}{X})\ dV_0$$

or

$$\int_{\bar{V}_0} [\rho(\underset{\sim}{X},t)\; J(\underset{\sim}{X},t) - \rho_0(\underset{\sim}{X})]\; dV_0 = 0$$

If the integrand is a continuous function of $\underset{\sim}{X}$, then because $\bar{V}_0$ is an arbitrary subvolume, the integrand must vanish:

$$\boxed{\rho(\underset{\sim}{X},t)\; J(\underset{\sim}{X},t) = \rho_0(\underset{\sim}{X})} \qquad (4A\text{-}2)$$

This is the material form of the conservation of mass.

Consider (4A-1) and take the material derivative, noting that the right hand side is independent of t:

$$\frac{D}{Dt}\int_{\bar{V}(t)} \rho(\underset{\sim}{x},t)\; dV = 0$$

Applying the transport theorem (3A-14), we obtain

$$\int_{\bar{V}(t)} (\dot{\rho} + \rho\, v_{i,i})\; dV = 0 \qquad (\dagger)$$

Assuming continuity of the integrand, then for arbitrary $\bar{V}(t)$, this implies

Continuity Equation

$$\boxed{\dot{\rho} + \rho\, v_{i,i} = 0 \quad \text{or} \quad \dot{\rho} + \rho\, \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0} \qquad (4A\text{-}3)$$

Note that all variables are expressed in the spatial description, since the integration in (†) is over a spatial subvolume $\bar{V}(t)$. Hence, (4A-3) is called the spatial form of the conservation of mass or the continuity equation. Expanding the $\dot{\rho}$ term, we have the alternate forms:

$$\frac{\partial \rho}{\partial t} + v_i \, \rho_{,i} + \rho \, v_{i,i} = 0$$

$$\boxed{\frac{\partial \rho}{\partial t} + (\rho \, v_i)_{,i} = 0}$$ Alternate form of continuity equ (4A-4)

For <u>steady density</u>: $\rho = \rho(\underset{\sim}{x})$, $\frac{\partial \rho}{\partial t} = 0$ and (4A-4) implies

$$(\rho \, v_i)_{,i} = \underset{\sim}{\nabla} \cdot (\rho \, \underset{\sim}{v}) = \operatorname{div} (\rho \, \underset{\sim}{v}) = 0 \qquad (4A\text{-}5)$$

<u>If ρ = constant everywhere, then the deformation is called</u> <u>incompressible</u> and (4A-4) implies

note: isochoric def. $\operatorname{div} \underset{\sim}{v} = 0$
incompressible $\operatorname{div} \underset{\sim}{v} = 0$

$$v_{i,i} = \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \operatorname{div} \underset{\sim}{v} = 0 \qquad (4A\text{-}6)$$ Incompressible ℱ

By virtue of the conservation of mass (4A-3), we can obtain an alternate form of the material derivative of a volume integral over a deforming volume. Hence, replace $\underset{\sim}{F}$ by $\rho \, \underset{\sim}{F}$ in Transport Theorem (3A-14):

$$\frac{D}{Dt} \int_{V(t)} \rho \, \underset{\sim}{F} \, dV = \int_{V(t)} [\dot{\overline{(\rho \, \underset{\sim}{F})}} + \rho \, \underset{\sim}{F} \operatorname{div} \underset{\sim}{v}] \, dV$$

$$= \int_{V(t)} [\cancelto{0 \text{ by } (4.3)}{(\dot{\rho} + \rho \operatorname{div} \underset{\sim}{v})} \, F + \rho \, \dot{\underset{\sim}{F}}] \, dV$$

i.e. Alt. form for material derivative ~~over a~~ of a volume integral over a deforming volume.

$$\boxed{\frac{D}{Dt} \int_{V(t)} \rho \, \underset{\sim}{F} \, dV = \int_{V(t)} \rho \, \dot{\underset{\sim}{F}} \, dV} \qquad (4A\text{-}7)$$

Another proof:

$$\frac{D}{Dt} \int_{V(t)} \rho \underset{\sim}{F} \, dV = \frac{D}{Dt} \int_{V(t)} \rho \underset{\sim}{F} J \, dV_0 = \int_{V(t)} \dot{\overline{\rho \underset{\sim}{F} J}} \, dV_0$$

$$= \int_{V(t)} \dot{\overline{\rho J}} \underset{\sim}{F} \, dV_0 + \int_{V(t)} \rho J \dot{\underset{\sim}{F}} \, dV_0$$

$$= \cancel{\int \Big(\dot{\rho} \frac{\overset{1}{J}}{J} + \rho \frac{\overset{v_{i,i}}{\dot{J}}}{J}\Big) J F \, dV_0} + \int_{V(t)} \rho \dot{\underset{\sim}{F}} \, dV = \int_{V(t)} \rho \dot{\underset{\sim}{F}} \, dV$$

B. Linear Momentum Balance and the Stress Tensor

Consider an arbitrary subvolume $\bar{V}(t)$ with surface $\bar{S}$ of B(t). (See Fig. IV-1.) We assume that a distribution of stress vectors $\underset{\sim}{t}^{(n)}$ acts on $\bar{S}$ such that the force on dS is $\underset{\sim}{t}^{(n)}$ dS. Also, let there act a body force density $\underset{\sim}{f}$ at each point of $\bar{V}$ such that the force on dV is $\rho \underset{\sim}{f}$ dV. The linear momentum of dm is $\rho \underset{\sim}{v}$ dV. Summing forces and applying Newton's 2nd Law: $\Sigma \underset{\sim}{F} = \overline{m\dot{\underset{\sim}{v}}}$, we have

Postulate II -- Linear Momentum Balance

$$\frac{D}{Dt}\int_{\bar{V}} \rho \, \underset{\sim}{v} \, dV = \int_{\bar{V}} \rho \, \underset{\sim}{f} \, dV + \int_{\bar{S}} \underset{\sim}{t}^{(n)} \, dS \qquad (4B\text{-}1)$$

Applying (4A-7) on the left hand side, we have

$$\int_{\bar{V}} \rho \, \dot{\underset{\sim}{v}} \, dV = \int_{\bar{V}} \rho \, \underset{\sim}{f} \, dV + \int_{\bar{S}} \underset{\sim}{t}^{(n)} \, dS \qquad (4B\text{-}2)$$

We now apply (4B-2) to a small tetrahedron at any point x_i of $\bar{V}$. (See Fig. IV-2.)

Applying the mean value theorem for integrals to (4B-2) and assuming continuity of all functions, we have

$$\bar{\underset{\sim}{t}}^{(n)} A + \bar{\underset{\sim}{t}}^{(-1)} A_1 + \bar{\underset{\sim}{t}}^{(-2)} A_2 + \bar{\underset{\sim}{t}}^{(-3)} A_3 + \overline{\rho(\underset{\sim}{f} - \dot{\underset{\sim}{v}})} \, \frac{1}{3} h A = 0$$

where $\bar{\underset{\sim}{t}}^{(n)}$, $\bar{\underset{\sim}{t}}^{(-1)}$, etc., are mean values. Now take the limit as h → 0, noting that $\bar{\underset{\sim}{t}}^{(n)} \to \underset{\sim}{t}^{(n)}$, $\bar{\underset{\sim}{t}}^{(-1)} \to \underset{\sim}{t}^{(-1)}$, etc.

$$\underset{\sim}{t}^{(n)} A + \underset{\sim}{t}^{(-j)} A_j = 0$$

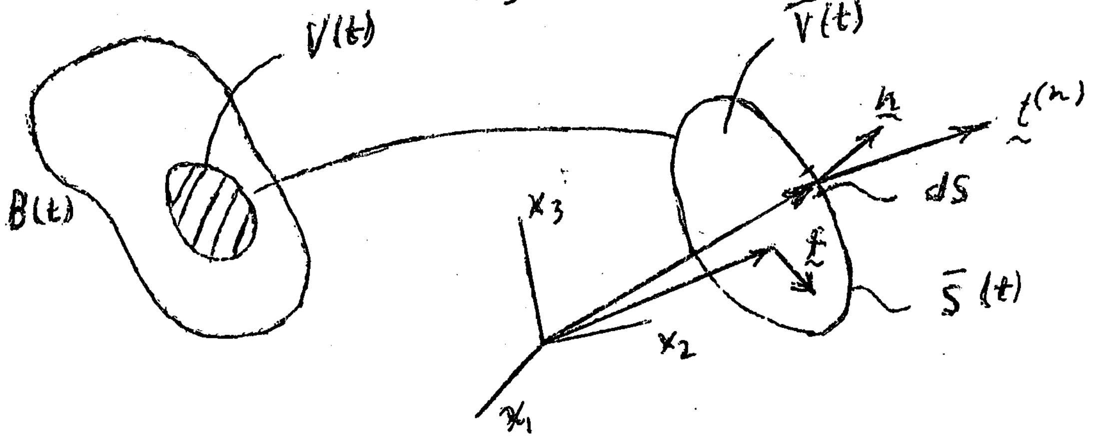


Fig. IV-1

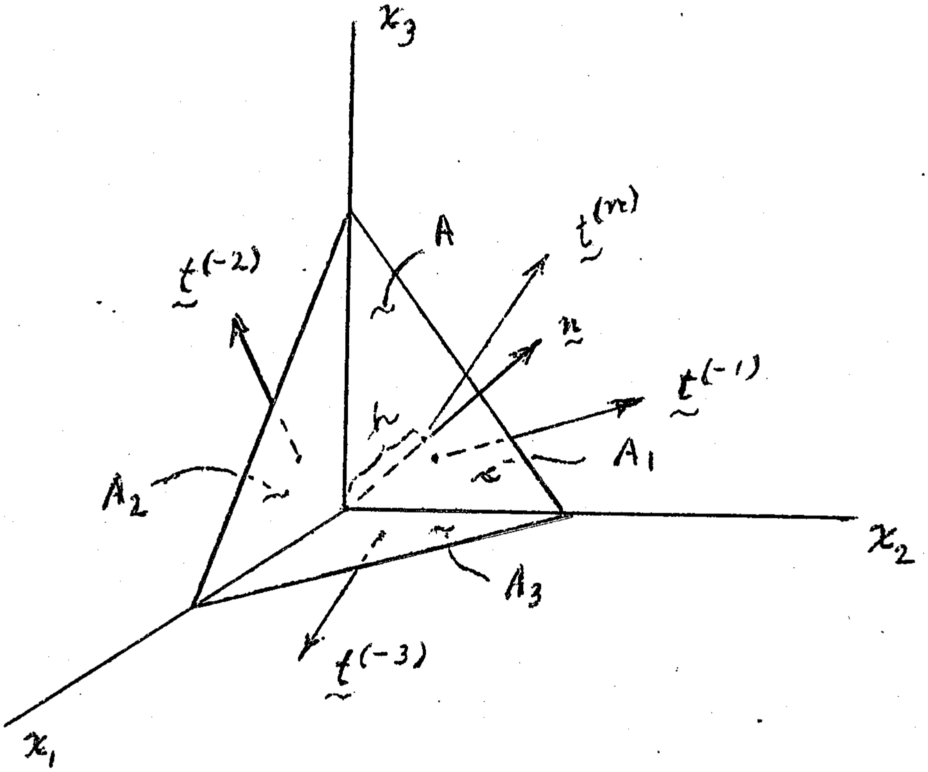


Fig. IV-2

But from solid geometry it can be shown that $A_j = A\, n_j$, hence

$$\underset{\sim}{t}^{(n)} = -\,\underset{\sim}{t}^{(-j)}\, n_j \qquad (4B\text{-}3)$$

If we let $\underset{\sim}{n}$ be replaced by $-\underset{\sim}{n}$ in (4B-3), then

$$\underset{\sim}{t}^{(-n)} = \underset{\sim}{t}^{(-j)}\, n_j \overset{(4B\text{-}3)}{=} -\,\underset{\sim}{t}^{(n)} \qquad (4B\text{-}4)$$

This is an expression of Newton's 3rd Law: at any point the stress vectors acting on opposite sides of a surface element are equal in magnitude and opposite in direction. Applying (4B-4) to $\underset{\sim}{t}^{(-1)}$, $\underset{\sim}{t}^{(-2)}$, $\underset{\sim}{t}^{(-3)}$, we have

$$\underset{\sim}{t}^{(-j)} = -\,\underset{\sim}{t}^{(j)}$$

Then (4B-3) becomes

$$\underset{\sim}{t}^{(n)} = \underset{\sim}{t}^{(j)}\, n_j \qquad (4B\text{-}5)$$

This result is Cauchy's Fundamental Theorem: All stress vectors $\underset{\sim}{t}^{(n)}$ at a point are determined from the stress vectors acting on 3 mutually orthogonal planes at the point.

We now define components of the stress vectors $\underset{\sim}{t}^{(j)}$ as follows

$$\underset{\sim}{t}_{jk} = \underset{\sim}{t}^{(j)} \circ \underset{\sim}{i}_k \quad , \quad \underset{\sim}{t}^{(j)} = t_{jk}\, \underset{\sim}{i}_k$$

Hence, t_{jk} is the kth component of the stress vector which acts on coordinate plane x_j = constant. Then (4B-5) becomes

$\underset{\sim}{t}^{(n)} = t_{jk}\, n_j\, \underset{\sim}{i}_k$. Dotting with $\underset{\sim}{i}_m$ gives Cauchy's Formula

$$t_i^{(n)} = t_{ji}\, n_j \qquad (4B\text{-}6)$$

Since $\underset{\sim}{t}^{(n)}$ and $\underset{\sim}{n}$ are vectors, then (4B-6) implies t_{ij} is a 2nd order tensor called the stress tensor. This follows from Theorem 7 of Chapter I.

Sign Convention: If $\underset{\sim}{n}$ for the plane x_i = constant is in positive (negative) coordinate direction, then a positive stress component acts in the positive (negative) coordinate direction. (See Fig. IV-3.)

Using Cauchy's Formula (4B-6) in the linear momentum balance (4B-2), we have

$$\int_{\overline{V}} \rho\, \dot{v}_i\, dV = \int_{\overline{V}} \rho\, f_i\, dV + \int_{\overline{S}} t_{ji}\, n_j\, dS$$

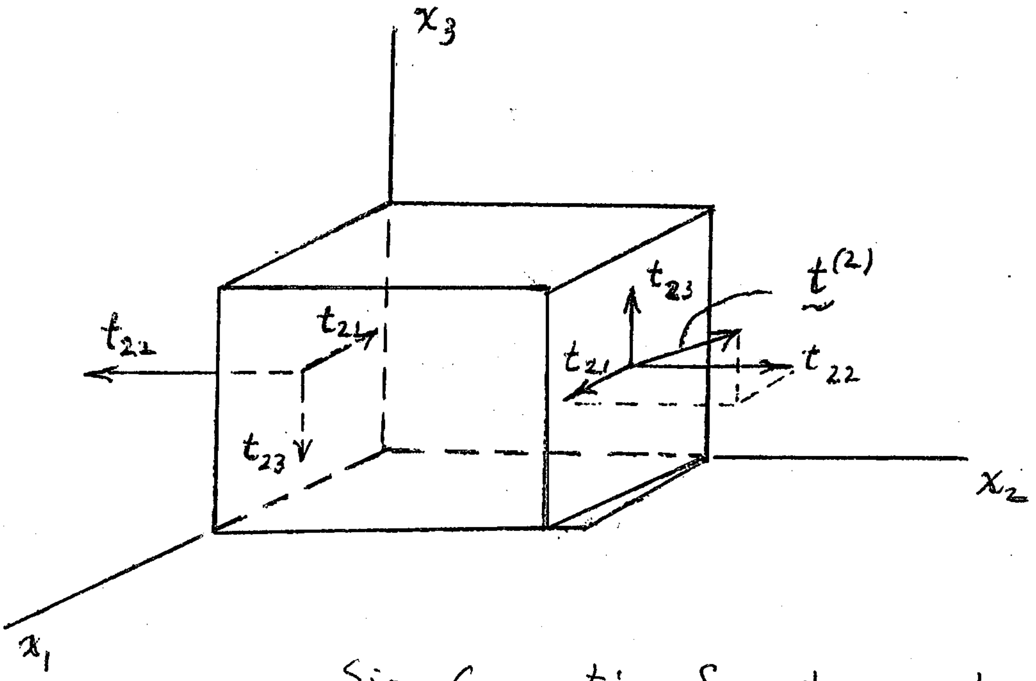
$$= \int_{\overline{V}} \rho\, f_i\, dV + \int_{\overline{V}} t_{ji,j}\, dV$$

i.e.

$$\int_{\overline{V}} (t_{ji,j} + \rho\, f_i - \rho\, \dot{v}_i)\, dV = 0$$

which implies the Local Form of linear momentum balance:

$$\boxed{t_{ji,j} + \rho\, f_i = \rho\, \dot{v}_i} \qquad (4B\text{-}7)$$



Sign Convention for stress vector components.

Fig. IV-3

C. Moment of Momentum Balance

The elemental moments acting on dV, dS are $\underset{\sim}{r} \times (\rho\ \underset{\sim}{f}\ dV)$ and $\underset{\sim}{r} \times (\underset{\sim}{t}^{(n)}\ dA)$, respectively. The angular momentum of dm is $\underset{\sim}{r} \times (\rho\ \underset{\sim}{v}\ dV)$. Hence, we have

Postulate III -- Moment of Momentum Balance

$$\frac{D}{Dt} \int_{\bar{V}} \underset{\sim}{r} \times \rho\ \underset{\sim}{v}\ dV = \int_{\bar{V}} \underset{\sim}{r} \times \rho\ \underset{\sim}{f}\ dV + \int_{\bar{S}} \underset{\sim}{r} \times \underset{\sim}{t}^{(n)}\ dA \qquad (4C\text{-}1)$$

Applying (4A-1) on the left hand side,

$$\frac{D}{Dt} \int_{\bar{V}} (\underset{\sim}{r} \times \rho\ \underset{\sim}{v})\ dV = \int_{\bar{V}} \rho(\overset{\cdot}{\overline{\underset{\sim}{r} \times \underset{\sim}{v}}})\ dV = \int_{\bar{V}} \rho\ \underset{\sim}{r} \times \dot{\underset{\sim}{v}}\ dV$$

Use this result and (4B-6) in (4C-1) in component form

$$\int_{\bar{V}} \rho\ e_{ijk} x_j \dot{v}_k\ dV = \int_{\bar{V}} \rho\ e_{ijk} x_j f_k\ dV + \int_{\bar{S}} e_{ijk} x_j t_{pk} n_p\ dS$$

$$= e_{ijk} \int_{\bar{V}} [\rho\ x_j f_k + (x_j t_{pk})_{,p}]\ dV$$

or

$$e_{ijk} \int_{\bar{V}} [x_j(t_{pk,p} + \rho\ f_k - \rho\ \dot{v}_k) + t_{jk}]\ dV = 0 \quad (*)$$

But the term in ( ) vanishes by the linear momentum balance (4B-7). Hence (*) yields

$$\int_{\bar{V}} e_{ijk}\ t_{jk}\ dV = 0$$

By the usual argument this implies

$$e_{ijk} \, t_{jk} = 0$$

or

$$t_{ij} = t_{ji} \qquad \text{Stress Tensor is Symmetric} \qquad (4C\text{-}2)$$

Hence, provided the linear momentum balance is satisfied, Postulate III implies the stress tensor must be symmetric. Since t_{ij} is real and symmetric, then by Theorem 11 of Chapter I, $\underset{\sim}{t}$ has three real principal values (the principal stresses) and a corresponding set of principal axes determined from

$$\underline{\det (t_{ij} - t \, \delta_{ij}) = 0} \qquad (4C\text{-}3)$$

and

$$(t_{ij} - t_\alpha \, \delta_{ij}) \, n_j^{(\alpha)} = 0$$

$$n_i^{(\alpha)} \, n_i^{(\alpha)} = 1 \qquad (4C\text{-}4)$$

From Schaum's (6.10), the Fourier Law of Conduction gives

$$C_i = -k\, T_{,i} \quad \text{or} \quad C_i = -k \frac{\partial T}{\partial x_i} \qquad k \equiv \text{thermal conductivity}$$

In eqn (40-3) $C_i = q_i$, also $r \Rightarrow$ per unit mass per unit time

Heat Energy Entering $\bar{V}$ (40-3)

$$P_H = \int_{\bar{V}} \rho r \, dV - \int_{\bar{S}} q_i n_i \, dS$$

Is $r = r(\underset{\sim}{x}, t)$? From fourier's law $q_i = -k \, T_{,i} = -k \dfrac{\partial T}{\partial x_i}$

hence

$$\int_{\bar{S}} q_i n_i \, dS = k \int_{\bar{S}} \left[n_1 \frac{dT}{dx_1} + n_2 \frac{dT}{dx_2} + n_3 \frac{dT}{dx_3} \right] dS$$

where $T = T(\underset{\sim}{x}, t)$

If $\bar{n}$ is a function of the location of dS along $\bar{S}$ then is $\bar{n} = \bar{n}(\underset{\sim}{x}, t) = n_i \hat{e}_i$?

And is: $\bar{n} = n_1(\underset{\sim}{x}, t)\, \hat{e}_1 + n_2(\underset{\sim}{x}, t)\, \hat{e}_2 + n_3(\underset{\sim}{x}, t)\, \hat{e}_3$?

Thus

$$\int_{\bar{S}} q_i n_i \, dS = k \int_{\bar{S}} \left[n_1(\underset{\sim}{x}, t) \frac{\partial T(\underset{\sim}{x}, t)}{\partial x_1} + n_2(\underset{\sim}{x}, t) \frac{\partial T(\underset{\sim}{x}, t)}{\partial x_2} + n_3(\underset{\sim}{x}, t) \frac{\partial T(\underset{\sim}{x}, t)}{\partial x_3} \right] dS$$

D. Energy Balance (1st Law of Thermodynamics)

We define the total kinetic and internal energies of any subvolume $\bar{V}(t)$ as

$$\boxed{K = \frac{1}{2}\int_{\bar{V}} \rho\ v_i\ v_i\ dV \quad , \quad E = \int_{\bar{V}} \rho\ e\ dV} \qquad (4D\text{-}1)$$

where e is the internal energy density per unit mass, which accounts for the "energy of deformation" or energy stored in the material. The rate at which work is done on $\bar{V}$ by the external forces is

$$\boxed{P_E = \int_{\bar{V}} \rho\ f_i\ v_i\ dV + \int_{\bar{S}} t_i^{(n)}\ v_i\ dS} \quad \text{Work} \qquad (4D\text{-}2)$$

We have assumed here that there are no distributed couples acting in $\bar{V}$ or on $\bar{S}$. For theories which include magnetic effects, for example, these couples would have to be included. For a theory which includes thermal effects the rate at which heat energy is entering $\bar{V}$ is defined as

$$\boxed{P_H = \int_{\bar{V}} \rho\ r\ dV - \int_{\bar{S}} q_i\ n_i\ dS} \quad \text{Heat Entering } \bar{V} \qquad (4D\text{-}3)$$

where r is the heat source density function in $\bar{V}$ and q_i is the heat flux vector acting across $\bar{S}$ such that $q_i\ n_i$ is the rate at which heat energy / unit area is <u>leaving</u> $\bar{V}$.

Postulate IV -- Energy Balance

$$\frac{D}{Dt}(K + E) = P_E + P_H \qquad (4D\text{-}4)$$

From (4D-1) - (4D-3) we can express (4D-4) as

$$\frac{D}{Dt}\int_{\overline{V}} \rho(e + \frac{1}{2} v_i v_i)\, dV = \int_{\overline{V}} \rho(f_i v_i + r)\, dV + \int_{\overline{S}} (t_i^{(n)} v_i - q_i n_i)\, dS \tag{4D-5}$$

If (4A-7) is applied on the left hand side and Cauchy's Formula (4B-6) is employed on the right, we find

$$\int_{\overline{V}} \rho(\dot{e} + v_i \dot{v}_i)\, dV = \int_{\overline{V}} \rho(f_i v_i + r)\, dV + \int_{\overline{S}} (t_{ji} v_i - q_j)\, n_j\, dS$$

Applying the divergence theorem and collecting terms under one integral, there results

$$\int_{\overline{V}} [\rho\, \dot{e} + v_i(\rho\, \dot{v}_i - \rho\, f_i - t_{ji,j}) - t_{ji}\, v_{i,j} + q_{i,i} - \rho\, r]\, dV = 0$$

But if linear momentum balance is satisfied, then the ( ) vanishes by (4B-7) and by the usual argument, we obtain the <u>local balance of energy</u>:

$$\boxed{\rho\, \dot{e} = t_{ji}\, v_{i,j} - q_{i,i} + \rho\, r} \tag{4D-6}$$

Since the stress tensor is symmetric, Theorem 1 gives

$$t_{ji}v_{i,j} = t_{ij}v_{i,j} = t_{ij}v_{(i,j)} \overset{(3B-1)}{=} t_{ij}d_{ij} = \Phi \qquad (4D-7)$$

where Φ is the <u>stress power</u>. Hence (4D-6) becomes

$$\rho \dot{e} = t_{ij} d_{ij} - q_{i,i} + \rho r \qquad (4D-8)$$

If the heat flux vector and source term vanish everywhere ($q_i = 0 = r$), then we have the <u>adiabatic case</u> and (4D-8) reduces to

$$\boxed{\rho \dot{e} = t_{ij} d_{ij}} \qquad (4D-9)$$

Adiabatic Case

Local Energy Balance

V. Constitutive Equations

To complete the governing equations of a continuum, we must develop equations which describe the response of materials to deformation, i.e., constitutive equations. These equations relate the stress tensor to deformation measures, e.g., strains, strain rates, etc. We consider here only classical theories of elastic① solids and Stokesian② fluids.

A. Elasticity (Isothermal)

For an elastic solid it is assumed that for the adiabatic case ($q_i = 0 = r$) temperature is constant (isothermal) and that the stress tensor is a function of the strain tensor. There are two approaches, i.e., the methods of Green and Cauchy:

Green's Method -- The existence of a strain energy function W per unit volume is assumed such that (Hyperelasticity)

E_{Kn} ≡ Lagrange's Strain Tensor

$$W = W(\underset{\sim}{X}, E_{KM}) = \rho_o e \tag{5A-1}$$

Note that in general W can be a nonlinear function of its arguments; the functional form will depend on the particular material. When W depends explicitly on $\underset{\sim}{X}$, the material is called inhomogeneous; otherwise homogeneous. Now the energy balance (4D-9) and (5A-1) imply a relationship between t_{ij} and W. Computing $\dot{e}$, we have

$$\dot{e} = \frac{1}{\rho_o} \dot{W} = \frac{1}{\rho_o} \frac{\partial W}{\partial E_{KM}} \dot{E}_{KM}. \tag{5A-2}$$

$$2\dot{E}_{KM} = \frac{D}{Dt}\left(x_{i,K}\, x_{i,M}\right)$$

Derivation from 5-2

①

$$= \dot{\overline{x_{i,K}}}\; x_{i,M} + x_{i,K}\, \dot{\overline{x_{i,M}}}$$

recall eq (3A-8) $$\frac{D}{Dt}\phi = \frac{\partial \phi}{\partial t}\Big|_{\underset{\sim}{X}} + v_i \frac{\partial \phi}{\partial x_i}$$

so

$$= \left[\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_K}\right)\Big|_{\underset{\sim}{X}} + v_i \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial X_K}\right)\right] x_{i,M}$$

$$+ \left[\frac{\partial}{\partial t}\left(\frac{\partial x_i}{\partial X_M}\right)\Big|_{\underset{\sim}{X}} + v_i \frac{\partial}{\partial x_i}\left(\frac{\partial x_i}{\partial X_M}\right)\right] x_{i,K}$$

Since $\partial \underset{\sim}{X}$ is independent of t it can be moved out of $\frac{\partial}{\partial t}(\;)$ so

$$= \left[\frac{\partial}{\partial X_K}\frac{\partial x_i}{\partial t}\Big|_{\underset{\sim}{X}} + \frac{\partial v_i}{\partial X_K}\right] x_{i,M} + \left[\frac{\partial}{\partial X_M}\frac{\partial x_i}{\partial t}\Big|_{\underset{\sim}{X}} + \frac{\partial v_i}{\partial X_M}\right] x_{i,K}$$

by definition $v_i = \frac{\partial x_i}{\partial t}\Big|_{\underset{\sim}{X}}$ so

$$= \left[\frac{\partial v_i}{\partial X_K} + \frac{\partial v_i}{\partial X_K}\right] x_{i,M} + \left[\frac{\partial v_i}{\partial X_M} + \frac{\partial v_i}{\partial X_M}\right] x_{i,K}$$

note that $\boxed{\frac{\partial v_i}{\partial X_K} = \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_K} \quad \text{and} \quad \frac{\partial v_i}{\partial X_M} = \frac{\partial v_i}{\partial x_j}\frac{\partial x_j}{\partial X_M}}$ this is ok ?

The second routine is designed to read the output file back into SIMPLER. This can be used for initializing or "seeding" the arrays before computation.

```
c ============================Read in Seed File for U,V,P,T,etc.
      entry seedfile
c ==============================================================

      open (unit=3,file='cvout.dat',status='old')

        read(3,*)
        read(3,91) L1,M1,mode
        read(3,*)
        read(3,*)
        read(3,93) (x(i),i=1,L1)
        read(3,*)
        read(3,*)
        read(3,93) (y(j),j=1,M1)
        read(3,*)

        do 997 nf=1,10
          if(.not.lprint(nf)) go to 997
            read(3,*)
            do i=1,L1
                  read(3,93) (f(i,j,nf),j=1,m1)
            end do
            read(3,*)
 997    continue
      return
      end
```

IMPORTANT NOTES:

1) Both routines write and read to a common file (CVOUT.DAT) which should be **renamed to a more unique file name**. Otherwise, subsequent solutions will be overwritten in the same file.

2) The **LPRINT flags in the USER subroutine of SIMPLER must be set to TRUE** for the data to be written to the file. However, the node location data and the header is always written.

3) Make sure the **TITLE() data is set in the USER subroutine** so that each set of contour data will have a unique name.

so

~~$\partial\dot{E}_{km} = 2\frac{\partial v_i}{\partial x_j}\frac{\partial x}{\partial x_k}$~~

E

$$\cancel{2}\dot{E}_{km} = \cancel{2} v_{i,j}\, x_{j,k}\, x_{i,m} + \cancel{2} v_{i,j}\, x_{j,m}\, x_{i,k}$$

$$E_{km} = V_{i,j}\left[x_{j,k}\, x_{i,m} + x_{j,m}\, x_{i,k}\right]$$

~~$x_{j,m}\, x_{i,k} + x_{i,m}\, x_{j,k}$~~

expanding ~~yields~~ the [ ] term on RHS ~~only~~ yields

$$\cancel{E_{km}} = x_{1,k}\, x_{1,m} + x_{2,k}\, x_{1,m} + x_{3,k}\, x_{1,m}$$

$$+ x_{1,k}\, x_{2,m} + x_{2,k}\, x_{2,m} + x_{3,k}\, x_{2,m}$$

$$+ x_{1,k}\, x_{3,m} + x_{2,k}\, x_{3,m} + x_{3,k}\, x_{3,m}$$

$$+ x_{1,m}\, x_{1,k} + x_{1,m}\, x_{2,k} + x_{1,m}\, x_{3,k}$$

$$+ x_{2,m}\, x_{1,k} + x_{2,m}\, x_{2,k} + x_{2,m}\, x_{3,k}$$

$$+ x_{3,m}\, x_{1,k} + x_{3,m}\, x_{2,k} + x_{3,m}\, x_{3,k}$$

thus

$$\boxed{x_{j,k}\, x_{i,m} = x_{j,m}\, x_{i,k}}$$

ok ?

so

$$\dot{E}_{km} = 2V_{i,j}\left[x_{i,k}\, x_{j,m}\right]$$

```
return
```

$$2V_{i,j} = V_{i,j} + V_{i,j}$$

$$= 2V_{(i,j)} + 2V_{[i,j]}$$

$$= \cancel{V_{1,1}} + \cancel{V_{1,1}} + \cancel{V_{1,2}} + \cancel{V_{2,1}} + \cancel{V_{1,3}} + \cancel{V_{3,1}}$$

$$+ \cancel{V_{2,1}} + \cancel{V_{1,2}} + \cancel{V_{2,2}} + \cancel{V_{2,2}} + \cancel{V_{2,3}} + \cancel{V_{3,2}}$$

$$+ \cancel{V_{3,1}} + \cancel{V_{1,3}} + \cancel{V_{3,2}} + \cancel{V_{2,3}} + \cancel{V_{3,3}} + \cancel{V_{3,3}}$$

$$= 2\left[V_{1,1} + V_{2,1} + V_{3,1} + V_{1,2} + V_{1,3} + V_{2,2} + V_{3,3} + V_{3,2} + V_{2,3} \right]$$

hence $2V_{i,j} = V_{i,j} + V_{j,i}$

where $V_{j,i} = \left[V^{*}_{i,j}\right]^T$

by definition $d_{i,j} = V_{(i,j)} = \frac{1}{2}\left[V_{i,j} + V_{j,i}\right]$

so

$$\boxed{\cancel{2} V_{i,j} = \tfrac{1}{2}\left[V_{i,j} + V_{j,i}\right] = V_{(i,j)} = d_{i,j}}$$ ok ?

hence

$$\dot{E}_{km} = \textcircled{2}\, d_{i,j}\left[x_{i,k}\, x_{j,m}\right]$$

? where does it come from?

2.0 CREATING DATA FILES FROM SIMPLER

To avoid errors in the format of the data file, FORTRAN code is included (CVP_F.FOR) in the CVPLOT package which should be added to existing versions of SIMPLER. The code is two small sub-programs which can be placed anywhere in SIMPLER, although it is recommended that the routines be placed in the SUPPLY subroutine.

The first routine creates the data file which is compatible with CVPLOT.

```
c =================Solution Output in CVPLOT.EXE file format
      entry cvpfile
c ============================================================

   91   format(10x,i2,14x,i2,14x,i2)
   92   format(1h )
   93   format(4x,1p5e15.5)
   94   format(4x,'START HEADER')
   95   format(4x,'START X')
   96   format(4x,'START THETA')
   97   format(4x,'START Y')
   98   format(4x,'START RADIUS')
   99   format(4x,'START UVEL')
  100   format(4x,'START TVEL')
  101   format(4x,'START VVEL')
  102   format(4x,'START RVEL')
  103   format(4x,'START CONTOUR',1x,a30,1x)

c ---------------------------------------------------------------

      open (unit=1,file='cvout.dat',status='new')

        write(1,94)
        write(1,91) L1,M1,mode
        write(1,92)

        if(mode.eq.1.or.mode.eq.2) write(1,95)
        if(mode.eq.3) write(1,96)

        write(1,93) (x(i),i=1,L1)
        write(1,92)

        if(mode.eq.1.or.mode.eq.2) write(1,97)
        if(mode.eq.3) write(1,98)

        write(1,93) (y(j),j=1,M1)
        write(1,92)

        do 998 nf=1,10
          if(.not.lprint(nf)) go to 998

            if(nf.eq.1.and.mode.ne.3) write(1,99)
            if(nf.eq.1.and.mode.eq.3) write(1,100)
            if(nf.eq.2.and.mode.ne.3) write(1,101)
            if(nf.eq.2.and.mode.eq.3) write(1,102)

            if(nf.ge.3) then
              write(1,103) title(nf)
            end if

            do i=1,L1
                  write(1,93) (f(i,j,nf),j=1,m1)
            end do

            write(1,92)
  998   continue
```

$t_{i,j} = \frac{\rho}{\rho_0} \frac{\partial W}{\partial E_{kn}} x_{i,k} x_{j,n}$

Stress tensor

Change in Strain Energy w.r.t. the Strain Tensor

measure of deformation?

rates of deformation in the body?

Now (2B-2): $2E_{KM} = C_{KM} - \delta_{KM}$ implies

$C_{KM} \equiv$ Green's Deformation Tensor

From (2A-16) $C_{KM} = x_{i,K}\, x_{i,M}$

$$2\dot{E}_{KM} = \dot{C}_{KM} = \frac{D}{Dt}(x_{i,K}\, x_{i,M})$$

$$= \overline{\dot{x}_{i,K}}\, x_{i,M} + x_{i,K}\, \overline{\dot{x}_{i,M}}$$

$$\overset{(3A-8)}{=} v_{i,j}\, x_{j,K}\, x_{i,M} + x_{i,K}\, v_{i,j}\, x_{j,M}$$

$$= (v_{i,j} + v_{j,i})\, x_{i,K}\, x_{j,M}$$

Hence,

$$\dot{E}_{KM} = d_{ij}\, x_{i,K}\, x_{j,M}$$

Then (5A-2) becomes

$$\dot{e} = \frac{1}{\rho_o} \frac{\partial W}{\partial E_{KM}}\, d_{ij}\, x_{i,K}\, x_{j,M}$$

Now substitute this into (4D-9):

$$\rho\, \dot{e} = \frac{\rho}{\rho_o} \frac{\partial W}{\partial E_{KM}}\, x_{i,K}\, x_{j,M}\, d_{ij} = t_{ij}\, d_{ij}$$

i.e.

$$\left(t_{ij} - \frac{\rho}{\rho_o} \frac{\partial W}{\partial E_{KM}}\, x_{i,K}\, x_{j,M}\right) d_{ij} = 0$$

For energy balance this must hold for arbitrary deformations, i.e., for arbitrary d_{ij}. Hence, we obtain

<u>Nonlinear</u>

$$\boxed{t_{ij} = \frac{\rho}{\rho_o} \frac{\partial W}{\partial E_{KM}}\, x_{i,K}\, x_{j,M}} \quad \text{Stress Tensor} \quad (5A-3)$$

These <u>nonlinear</u> constitutive equations are attributed to <u>Boussinesq</u>.

In order to reduce (5A-3) for a small deformation theory, we assume a series expansion for W in the arguments E_{KM}:

$$W = W_o + A_{KM} E_{(KM)} + \frac{1}{2} B_{KMLN} E_{KM} E_{LN} + \dots \qquad (5A-4)$$

where we assume the homogeneous case, i.e., the above coefficients are constants. Since $\underset{\sim}{E}$ is symmetric, we can take

$$A_{KM} = A_{MK} \quad , \quad B_{KMLN} = B_{MKLN} = B_{KMNL} = B_{LNKM} \qquad (5A-5)$$

Then

$$\frac{\partial W}{\partial E_{KM}} = A_{KM} + B_{KMLN} E_{LN} + \dots \qquad (5A-6)$$

For small deformations $E_{KM} \cong \tilde{E}_{KM}$, $|U_{K,M}| << 1$ and (4A-2) along with (2G-12) implies

$$\frac{\rho}{\rho_o} = \frac{1}{J} \cong (1 + I_{\underset{\sim}{E}})^{-1} \cong 1 - I_{\underset{\sim}{E}} \cong 1 - I_{\underset{\sim}{e}} \qquad (5A-7)$$

and (2F-5) implies

$$x_{i,K} = \delta_{iP} (\delta_{PK} + U_{P,K}) \qquad (5A-8)$$

provided the X_K, x_i coordinate systems are taken coincident. Now we substitute (5A-6) - (5A-8) in (5A-3):

$$t_{ij} = (1-I_{\underset{\sim}{E}})[A_{KM} + B_{KMLN}\tilde{E}_{LN} + \dots][\delta_{PK} + U_{P,K}][\delta_{QM} + U_{Q,M}] \, \delta_{iP}\delta_{jQ}$$

Retaining only 1st order terms in $\underset{\sim}{\tilde{E}}$ and $U_{K,M}$, we find

$$t_{ij} = [(1-I_{\tilde{E}})\, A_{PQ} + A_{KQ}U_{P,K} + A_{PM}U_{Q,M} + A_{PQLN}\tilde{E}_{LN}]\, \delta_{iP}\delta_{jQ} \qquad (5A\text{-}9)$$

At initial time t=0, $U_K \equiv 0 \equiv \tilde{E}_{KM}$ which implies

$$t_{ij}\Big|_{t=0} = A_{PQ}\, \delta_{iP}\, \delta_{jQ}$$

Hence, A_{KM} in (5A-4) represents an initial state of stress. We assume $A_{KM} = 0$ implying B_o is a stress-free natural state; then (5A-9) gives

$$t_{ij} = B_{PQLN}\, \tilde{E}_{LN}\, \delta_{iP}\, \delta_{jQ} \qquad (5A\text{-}10)$$

Finally, recall that (2G-14) implies for $\alpha_{km} = \delta_{kM}$:

$$\tilde{E}_{LN} = \tilde{e}_{mn}\, \delta_{mL}\, \delta_{nN}$$

and (5A-10) becomes

$$t_{ij} = (B_{PQLN}\, \delta_{iP}\, \delta_{jQ}\, \delta_{mL}\, \delta_{nN})\, \tilde{e}_{mn}$$

or

$$\underline{t_{ij} = b_{ijmn}\, \tilde{e}_{mn}} \qquad \text{Valid for anisotropic materials.} \qquad (5A\text{-}11)$$

where

$$b_{ijmn} = B_{PQLN}\, \delta_{iP}\, \delta_{jQ}\, \delta_{mL}\, \delta_{nN} \qquad (*)$$

Since $\underset{\sim}{t}$, $\underset{\sim}{\tilde{e}}$ are 2nd order tensors, we can show that $\underset{\sim}{b}$ is a 4th order tensor under rotations of $x_i \to \bar{x}_i$. By (5A-5) and (*) $\underset{\sim}{b}$ satisfies

$$\boxed{b_{ijmn} = b_{jimn} = b_{ijnm} = b_{mnij}} \qquad (5A-12)$$

These conditions imply that there are 21 independent components of $\underset{\sim}{b}$ for the general linear theory of elasticity.

Cauchy's Method -- Here we assume directly that

$$t_{ij} = t_{ij}(e_{ij})$$

Stress is a function of strain only

Then for a linear theory about a stress free state:

Generalized Hooke's Law

$$t_{ij} = c_{ijmn}\,\tilde{e}_{mn} \qquad (5A-13)$$

[Valid for anisotropic materials]

where $\underset{\sim}{c}$ is a 4th order tensor under rotations of $x_i \rightarrow \bar{x}_i$. Since $\underset{\sim}{t}$, $\underset{\sim}{\tilde{e}}$ are symmetric, $\underset{\sim}{c}$ must satisfy

$$\boxed{c_{ijmn} = c_{jimn} = c_{ijnm}} \qquad (5A-14)$$

These conditions imply there are 36 independent components of $\underset{\sim}{c}$. The forms (5A-11) or (5A-13) are valid for anisotropic materials, i.e., materials whose elastic properties (expressed by $\underset{\sim}{b}$, $\underset{\sim}{c}$) depend on direction. For isotropic materials the properties are independent of direction. This condition is expressed by requiring (5A-11) or (5A-13) to have the same form under arbitrary rotations of $x_i \rightarrow \bar{x}_i$, i.e., from (5A-13)

$$\bar{t}_{ij} = \bar{c}_{ijmn}\,\bar{\tilde{e}}_{mn} = c_{ijmn}\,\bar{\tilde{e}}_{mn}$$

which implies

$$\bar{c}_{ijmn} = c_{ijmn}$$

$$\underset{\sim}{t} = \underset{\sim}{c}\,\underset{\sim}{e} \qquad \text{or} \qquad t_{ij} = C_{ijmn}\,\tilde{e}_{mn}$$

$$t_{ij} = C_{ijKM}$$

$$\text{stress} = (\text{constant})(\text{strain})$$

E_{ij} ~~Er~~ Green's Strain tensor

e_{ij} Euler's (Cauchy's) ~~Cauchy~~'s Strain tensor

$\tilde{E}_{KM} \equiv$ Linearized Material Strain Tensor

$\tilde{e}_{mn} \equiv$ linearized Spatial Strain Tensor

In hydrodynamics E_{ij} is Lagrangian, and e_{ij} is Eulerian, in description. Both are symmetric! $E_{ij} = E_{ji}$ $e_{ij} = e_{ji}$

Recall

$$E_{ij} = \frac{1}{2}\left(\delta_{\alpha\beta}\frac{\partial x_\alpha}{\partial a_i}\frac{\partial x_\beta}{\partial a_j} - \delta_{ij}\right)$$

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \delta_{\alpha\beta}\frac{\partial a_\alpha}{\partial x_i}\frac{\partial a_\beta}{\partial x_j}\right)$$

Or in text notation,

$$\left.\begin{aligned} E_{KM} &= \frac{1}{2}\left(\delta_{ij}\frac{\partial x_i}{\partial X_K}\frac{\partial x_j}{\partial X_M} - \delta_{KM}\right) \\ \tilde{e}_{ij} \cong e_{ij} &= \frac{1}{2}\left(\delta_{ij} - \delta_{KM}\underbrace{\frac{\partial X_K}{\partial x_i}\frac{\partial X_M}{\partial x_j}}\right)\end{aligned}\right\} \text{Strain}$$

$$\frac{\partial X_1}{\partial x_1} \quad \frac{\partial X_2}{\partial x_1} \quad \frac{\partial X_3}{\partial x_1} \quad \text{etc}$$

~~note that~~ expanding e_{ij}

$$e_{11} = \frac{1}{2}\left\{1 - \frac{\partial X_1}{\partial x_1}\frac{\partial X_1}{\partial X_1}\right\} \cancel{= \frac{1}{2} - \frac{\partial X_1}{\partial x_1}} - \frac{\partial X_2}{\partial x_1}\frac{\partial X_2}{\partial x_1} - \underbrace{\frac{\partial X_3}{\partial x_1}\frac{\partial X_3}{\partial x_1}}\Big]$$

$$e_{12} = \frac{1}{2}\left[-\frac{\partial X_1}{\partial x_1}\frac{\partial X_1}{\partial x_2} - \frac{\partial X_2}{\partial x_1}\frac{\partial X_1}{\partial x_2} - \frac{\partial X_3}{\partial x_1}\frac{\partial X_3}{\partial x_2}\right]$$

$$e_{13} = \frac{1}{2}\left[-\frac{\partial X_1}{\partial x_1}\frac{\partial X_1}{\partial x_3} - \frac{\partial X_2}{\partial x_1}\frac{\partial X_2}{\partial x_3} - \frac{\partial X_3}{\partial x_1}\frac{\partial X_3}{\partial x_2}\right]$$

from $\underset{\sim}{c}_{ij}$ Cauchy's Deformation tensor

etc

$$t_{11} = \lambda(e_{11} + e_{22} + e_{33}) + 2\mu e_{11}$$

$$t_{12} = 2\mu e_{12}$$

$$\sigma_K = C_{KM} \epsilon_M$$

$$\sigma_1 = (\lambda + 2\mu) e_1 + \lambda e_2 + \lambda e_3 + 0(e_4) \ldots$$

$$\sigma_1 = \sigma_{11} = \lambda(e_1 + e_2 + e_3) + 2\mu e_{11}$$

$$\sigma_2 = \sigma_{22} = \lambda(e_1 + e_2 + e_3) + 2\mu e_{22}$$

$$\sigma_3 = \sigma_{33} = \lambda(e_1 + e_2 + e_3) + 2\mu (e_{33})$$

$$\sigma_{xx} = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu \epsilon_x$$

$$\sigma_{yy} = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu \epsilon_y$$

$$\sigma_{zz} = \lambda(\epsilon_x + \epsilon_y + \epsilon_z) + 2\mu \epsilon_z$$

~~defining~~ using $\lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$

and $\mu = \dfrac{E}{2(1+\nu)}$ $\Bigg\}$ PG 217 Fung.

By definition $\underset{\sim}{C}$ and $\underset{\sim}{c}$ are symmetric.

$$C_{KM} = x_{i,M}\, x_{i,K} = x_{i,K}\, x_{i,M} = C_{MK}$$

$$c_{ij} = X_{K,j}\, X_{K,i} = X_{K,i}\, X_{K,j} = c_{ji}$$

$c_{ij} \equiv$ is a measure of the deformation of line elements at any point of a body.

Using the mapping transformations gives

$$x_i = x_i(X_K, t) \quad \text{so} \quad dx_i = \frac{\partial x_i}{\partial X_K}\, dX_K = x_{i,K}\, dX_K$$

likewise

$$dX_K = \frac{\partial X_K}{\partial x_i}\, dx_i = X_{K,i}\, dx_i$$

the derivatives $x_{i,K}$ are the deformation gradients and map dX_K into dx_i. $X_{K,i}$ maps dx_i into dX_K.

Since the gradients are inverses to one another then

$$\delta_{ij} = x_{i,K} X_{K,j} \quad \text{and} \quad \delta_{KM} = X_{K,i}\, x_{i,M}$$

the length of line elements in the body can be represented by ds and dS. where

$$\underbrace{ds^2 = C_{KM}\, dX_K\, dX_M}_{\text{Deformation}} \quad \text{and} \quad \underbrace{dS^2 = c_{ij}\, dx_i\, dx_j}_{\text{Deformation}}$$

gives

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right]$$

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

which are common in undergraduate texts.

Hence $\underset{\sim}{c}$ must be an isotropic 4th order tensor. Then by Theorem 10, Chapter I, $\underset{\sim}{c}$ must have the form (using 1E-1) (pg 1-35)

$$c_{ijmn} = \lambda\ \delta_{ij}\ \delta_{mn} + \mu\ \delta_{im}\ \delta_{jn} + \gamma\ \delta_{in}\ \delta_{jm}$$

But the symmetric conditions (5A-14) imply that $\gamma=\mu$. Hence, substituting (5A 13)

$$t_{ij} = [\lambda\ \delta_{ij}\ \delta_{mn} + \mu(\delta_{im}\ \delta_{jn} + \delta_{in}\ \delta_{jm})]\tilde{e}_{mn}$$

$$= \lambda\ \delta_{ij}\ I_{\tilde{e}} + \mu(\tilde{e}_{ij} + \tilde{e}_{ji})$$

$$\boxed{t_{ij} = \lambda\ I_{\tilde{e}}\ \delta_{ij} + 2\mu\ \tilde{e}_{ij}} \qquad \text{Hooke's Law} \qquad (5A\text{-}15)\ \underline{\text{Linear}}$$

Where λ, μ are called <u>Lamé's constants</u>. We can show (5A-11) yields exactly the same form (5A-15) for the isotropic case.

To complete the governing equations for the linear theory, we assume that u_i and its derivatives are small in absolute value of order ε, i.e., of the same order as the displacement gradients $u_{i,j}$. Then from (2F-1) $\underset{\sim}{v} = \dot{\underset{\sim}{r}} = \dot{\underset{\sim}{u}}$ and

$$v_i = \dot{u}_i = \frac{\partial u_i}{\partial t} + v_j \frac{\partial u_i}{\partial x_j} \approx \frac{\partial u_i}{\partial t}$$

$$\dot{v}_i = \frac{\partial^2 u_i}{\partial t^2} + v_j \frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial t}\right) \cong \frac{\partial^2 u_i}{\partial t^2} \qquad (*)$$

Recalling (5A-7) and assuming the body force vector $\underset{\sim}{f}$ is of order ε, then $\rho\ \underset{\sim}{f} \cong \rho_o\ \underset{\sim}{f}$ and the linear momentum balance (4B-7) becomes using also (*).

$$t_{ij,j} + \rho_o\ f_i = \rho_o \frac{\partial^2 u_i}{\partial t^2} \qquad (5A\text{-}16)$$

Equations (5A-15), (5A-16) together with the strain displacement equations (2G-13):

$$\tilde{e}_{ij} = u_{(i,j)} \tag{5A-17}$$

are the complete set of governing equations for small deformations (isothermal) of a homogeneous, isotropic elastic solid. Note that there are 15 equations and 15 unknowns: t_{ij}, u_i, $\tilde{e}_{ij}$. Also, ρ is not considered an unknown since it is given by (5A-7) after u_i is determined.

Combining equations (5A-15) - (5A-17), we obtain <u>Navier's displacement equations of motion</u>:

$$\mu\, u_{i,jj} + (\lambda+\mu)\, u_{j,ji} + \rho_o\, f_i = \rho_o \frac{\partial^2 u_i}{\partial t^2} \tag{5A-18}$$

or in direct notation

$$\mu\, \nabla^2 \underset{\sim}{u} + (\lambda+\mu)\, \underset{\sim}{\nabla} (\underset{\sim}{\nabla} \cdot \underset{\sim}{u}) + \rho_o\, \underset{\sim}{f} = \rho_o \frac{\partial^2 \underset{\sim}{u}}{\partial t^2}$$

<u>These equations must be solved subject to initial conditions on u_i, $\frac{\partial u_i}{\partial t}$ and given boundary conditions on the surface S which are of three types:</u>

(a) <u>Displacement:</u> $u_i = \overline{u}_i$ on S

(b) <u>Stress:</u> $t_{ij}\, n_j = \overline{t}_i$ on S

(c) <u>Mixed:</u>

$$u_i = \bar{u}_j \qquad \text{on } S_u$$

$$t_{ij}\, n_j = \bar{t}_i \qquad \text{on } S_t$$

where in (c) S_u and S_t are disjoint subsets of S such that $S_u + S_t = S$ and $\bar{u}_i$, $\bar{t}_i$ are prescribed functions.

B. Stokesian Fluids

Classical fluid theories are based on the assumptions of Stokes:

(a) The stress tensor $\underset{\sim}{t}$ is a continuous function of the stretching tensor $\underset{\sim}{d}$.

(b) When $\underset{\sim}{d}$ vanishes, the stress must reduce to a hydrostatic pressure: $\underset{\sim}{t} = - p \underset{\sim}{I}$.

(c) material isotropy.

The general form satisfying (a) and (b) is

$$t_{ij} = - p \delta_{ij} + \tau_{ij}(\underset{\sim}{d}) \quad , \quad \tau_{ij}(0) = 0 \qquad (5B-1)$$

The function $\underset{\sim}{\tau}$ above can of course be nonlinear in $\underset{\sim}{d}$. When $\underset{\sim}{\tau}$ is a <u>linear</u> function of $\underset{\sim}{d}$, the fluid is called <u>Newtonian</u> and

$$t_{ij} = - p \delta_{ij} + b_{ijmn} d_{mn} \qquad (5B-2)$$

Since $\underset{\sim}{t}$ and $\underset{\sim}{d}$ are symmetric tensors, then $\underset{\sim}{b}$ must be symmetric in the 1st pair of indices and by Theorem 1 of Chapter I $b_{ij[mn]} d_{mn}$ always vanishes. Hence, we take

$$b_{ijmn} = b_{jimn} = b_{ijmn} \qquad (5B-3)$$

Assumption (c) implies that (5B-2) must have the same form for arbitrary rotations of $x_i \rightarrow \overline{x}_i$, i.e., $b_{ijmn} = \overline{b}_{ijmn}$ so that $\underset{\sim}{b}$ must be an isotropic 4th order tensor. By Theorem 10 of Chapter I and (5B-3), then

$$b_{ijmn} = \lambda \delta_{ij} \delta_{mn} + \mu (\delta_{im} \delta_{jn} + \delta_{in} \delta_{jm}) \qquad (5B-4)$$

where the parameters λ,μ for Newtonian fluids are called viscosity coefficients and are determined by experiment. Substitution of (5B-4) into (5B-2) yields the constitutive equations for Newtonian fluids:

$$\boxed{t_{ij} = (-p + \lambda I_d)\,\delta_{ij} + 2\mu\, d_{ij}} \qquad (5B-5)$$

In general the viscosity coefficients λ,μ are temperature dependent.

From thermodynamical considerations we can show that p, ρ and the absolute temperature θ are related by an equation of state

$$f(p,\rho,\theta) = 0 \qquad (5B-6)$$

and the internal energy function depends on θ, ρ via a caloric equation of state:

$$e = e(\theta,\rho) \qquad (5B-7)$$

The particular form for (5B-6) and (5B-7) depends on the material and must be determined experimentally. An example of (5B-6) is the perfect gas law: $p = \rho R \theta$ where R is the gas constant. One more constitutive equation is needed for heat conducting fluids, i.e., an equation relating heat flux to temperature and the deformation measures. The simplest form of this relationship is Fourier's Law of Heat Conduction:

$$q_i = -k\,\theta_{,i} \qquad (5B-8)$$

where the constant k is the thermal conductivity. The governing equations for heat conducting, compressible, Newtonian fluids are now complete and consist of the continuity equation (4A-3), linear momentum balance (4B-7), energy balance (4D-8), the definition of d_{ij} (3B-1):

$$\left\{\begin{array}{lr}
\dot{\rho} + \rho\, v_{i,i} = 0 \text{ — Continuity} & \text{(5B-9)} \\[2ex]
t_{ij,j} + \rho\, f_i = \rho\, \dot{v}_i \text{ — linear m. balance} & \text{(5B-10)} \\[2ex]
d_{ij} = v_{(i,j)} \text{ — definit} & \text{(5B-11)} \\[2ex]
\rho\, \dot{e} = t_{ij}\, d_{ij} - q_{i,i} + \rho\, r \text{ — energy bal.} & \text{(5B-12)}
\end{array}\right.$$

and the constitutive equations (5B-5), (5B-6) and (5B-8). We find there are 22 equations for the unknowns ρ, t_{ij}, v_i, e, q_i, θ, d_{ij}, p.

Isothermal Flows

The non-heat conducting case is specified by

$$q_i = 0 = r \quad , \quad \theta = \text{const.}$$

Then λ, μ are constants in (5B-5), eqn. (5B-8) is satisfied identically and the energy balance along with the caloric equation of state determine e by integration provided ρ, $\underset{\sim}{t}$, $\underset{\sim}{d}$ are determined first. The governing equations then reduce to (5B-6), which becomes a pressure-density relation since temperature is constant, i.e.

$$p = g(\rho) \tag{5B-13}$$

the continuity equation (5B-9) and equations (5B-5), (5B-10) and (5B-11) which when combined yield the Navier-Stokes Equations:

navier stokes eqns

$$\boxed{\mu \nabla^2 v_i + (\lambda+\mu) v_{j,ji} - p_{,i} + \rho f_i = \rho \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right)} \quad (5B\text{-}14)$$

-(7-18)

Note that nonlinearities occur in (5B-9), (5B-13) and the inertia terms of (5B-14). The appropriate boundary conditions are that fluid particles must adhere to solid boundaries S past which a fluid flow occurs, i.e.

$$v_i = 0 \quad \text{on } S \qquad (5B\text{-}15)$$

if S is fixed and

$$v_i = V_i \quad \text{on } S \qquad (5B\text{-}16)$$

if S moves with velocity $\underset{\sim}{V}$. We now consider some special isothermal flow equations.

Incompressible Flows

For many flow problems, e.g., liquids at sufficiently low flow velocities, a good approximation is incompressibility: $\rho = \rho_o$ = constant. Then the continuity equation (5B-9) reduces to

$$\underline{I_d = v_{i,i} = 0} \qquad (5B\text{-}17)$$

1st inv.

and $v_{i,ij} = 0$ so that (5B-14) becomes

$$\boxed{\mu \nabla^2 v_i - p_{,i} + \rho_o f_i = \rho_o \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right)} \qquad (5B\text{-}18)$$

In addition, since ρ is a constant, (5B-13) no longer applies, but p is still an unknown which is determined by applying the boundary conditions to the solution of (5B-18). Note also that (5B-17) implies (5B-5) reduces to

$$\boxed{t_{ij} = - p\, \delta_{ij} + 2\, \mu\, d_{ij}} \quad \text{Const. eqn} \qquad (5B\text{-}19)$$

For these flows only one viscosity coefficient μ appears in the governing equations.

Ideal Incompressible Flows

In some problems viscosity effects are dominant only in the neighborhood of a solid boundary, called the boundary layer. The flow outside this region can be considered non-viscous, i.e., $\mu \doteq 0$. Then the governing equations reduce to (from (5B-17) and (5B-18))

$$v_{i,i} = 0 \quad , \quad - p_{,i} + \rho_o f_i = \rho \left(\frac{\partial v_i}{\partial t} + v_{i,j} v_j\right) \qquad (5B\text{-}20)$$

while from (5B-19) the stress field reduces to a hydrostatic pressure:

$$t_{ij} = - p\, \delta_{ij} \qquad (5B\text{-}21)$$

If boundary layer effects are neglected as a further approximation, then the boundary condition on $\underset{\sim}{v}$ is that the component of velocity normal to a solid boundary S must vanish:

$$v_n = \underset{\sim}{v} \cdot \underset{\sim}{n} = 0 \qquad (5B\text{-}22)$$

where $\underset{\sim}{n}$ is the unit normal vector to S.

Now we can show that the acceleration vector can be expressed as

$$\frac{D\underset{\sim}{v}}{Dt} = \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{w} \times \underset{\sim}{v} + \frac{1}{2} \underset{\sim}{\nabla} v^2 \qquad (5B\text{-}23)$$

where $v^2 = \underset{\sim}{v} \cdot \underset{\sim}{v}$ and $\underset{\sim}{w}$ is the vorticity vector: (recall (3B-7))

$$\underset{\sim}{w} = \text{curl } \underset{\sim}{v} = \underset{\sim}{\nabla} \times \underset{\sim}{v} \qquad (5B\text{-}24)$$

Then (5B-20) become

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 0 \quad , \quad -\frac{1}{\rho_o} \underset{\sim}{\nabla} p + \underset{\sim}{f} = \frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{w} \times \underset{\sim}{v} + \frac{1}{2} \underset{\sim}{\nabla} v^2 \quad (5B\text{-}25)$$

We consider now the special case of <u>steady</u>, <u>irrotational</u> flow of an ideal incompressible fluid. For this case p, $\underset{\sim}{v}$ are functions of x_i alone, i.e., $\frac{\partial}{\partial t} = 0$, and the vorticity vanishes:

$$\underset{\sim}{\nabla} \times \underset{\sim}{v} = 0 \qquad (5B\text{-}26)$$

This condition is necessary and sufficient for the existence of a <u>velocity potential</u> function $\varphi(x_i)$ such that

$$\underset{\sim}{v} = \underset{\sim}{\nabla} \varphi \qquad (5B\text{-}27)$$

The velocity field, however, must still satisfy continuity $(5B-25)_1$; hence

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v} \overset{(5B-27)}{=} \underset{\sim}{\nabla} \cdot \underset{\sim}{\nabla} \varphi = \nabla^2 \varphi = 0 \qquad (5B-28)$$

Therefore, φ must satisfy Laplace's equation (5B-28). The boundary condition (5B-22) now becomes

$$\underset{\sim}{v} \cdot \underset{\sim}{n} = \underset{\sim}{\nabla} \varphi \cdot \underset{\sim}{n} = \frac{\partial \varphi}{\partial n} = 0 \qquad (5B-29)$$

Thus, the velocity field is completely determined by (5B-27) after solving (5B-28) and (5B-29). The linear momentum balance $(5B-25)_2$ is then the governing equation for the pressure. For steady, irrotational flow this becomes

$$-\frac{1}{\rho_o} \underset{\sim}{\nabla} p + \underset{\sim}{f} - \frac{1}{2} \underset{\sim}{\nabla} v^2 = 0 \qquad (5B-30)$$

For cases in which the body force either vanishes or is conservative, (5B-30) can be integrated explicitly. Let

$$\underset{\sim}{f} = -\underset{\sim}{\nabla} F \qquad (5B-31)$$

where $F = F(\underset{\sim}{x})$ is a body force potential function. Then (5B-30) becomes

$$-\underset{\sim}{\nabla} \left(\frac{p}{\rho_o} + \frac{1}{2} v^2 + F\right) = 0$$

Integrating,

$$\frac{p}{\rho_o} + \frac{1}{2} v^2 + F = \text{const.} \qquad (5B-32)$$

This result is <u>Bernoulli's Equation</u> for the steady, irrotational flow of an ideal, incompressible fluid and determines $p(\underset{\sim}{x})$ after $\underset{\sim}{v}$ is known. The constant in (5B-32) is evaluated at any point in the flow for which p and $\underset{\sim}{v}$ are known.

VI. Thermodynamics of Continuous Media

In Chapter V we treated constitutive equations only to an extent sufficient to formulate the classical constitutive equations for linear, isothermal elasticity and for Newtonian fluids. In this chapter and the succeeding one we present a more general framework which allows us to formulate consistent nonlinear constitutive equations for fluids and solids undergoing non-isothermal deformations.

A. Homogeneous Processes

References: "The Elements of Continuum Mechanics", C. Truesdell, Springer-Verlag, 1966 and "Rational Thermodynamics", C. Truesdell, McGraw-Hill, 1969.

In order to motivate the ideas of continuum thermodynamics, we begin with the special case of homogeneous processes, in which bodies suffer no local deformation or variation in temperature. Thus, all quantities introduced will depend on time alone and not on location within the body. The resulting theory is closely related to "classical" thermodynamics.

We begin by assuming that a <u>temperature</u> $\theta(t) > 0$ can be associated with every body undergoing a homogeneous process. Such a temperature function is called <u>absolute</u>, since its greatest lower bound is zero.

In rigid body mechanics the concept of the configuration of a rigid body, i.e., position of mass center and angular orientation specified, is fundamental. In thermodynamics a body is described by n real parameters $\nu_\alpha(t)$, $\alpha = 1,2,...,n$.

These parameters are selected on the basis of the physical problem one wishes to treat. For example, we could specify one parameter $\nu = 1/\rho$, the <u>specific volume</u> of the body, for a theory of dilute gases. For the purpose of developing the theory, no specific interpretation is necessary.

The <u>thermodynamic state</u> of the body at a given time is specified by the set of $n+1$ parameters $\theta > 0$, ν_α. For now, a process can be thought of as a sequence of changes in state, specified by continuously differentiable functions $\theta(t) > 0$, $\nu_\alpha(t)$. We assume the body is unconstrained in the sense that these functions are arbitrary.

We now postulate the first of two basic principles governing the thermodynamics of homogeneous processes. Let K be the total kinetic energy of the body, E the total internal energy, P the total rate at which work is done by external forces and Q the total rate at which work is done due to thermal effects. We will refer to P simply as the <u>power</u> and Q as the <u>heating</u>. Then we have the <u>Balance of Energy</u> (1st Law of Thermodynamics)

$$\dot{E} + \dot{K} = P + Q \tag{6A-1}$$

Note that in Section 4D we defined K, E, P, Q as integrals in terms of certain densities, i.e., internal energy density e, body force density, etc. For homogeneous processes it suffices to deal with K, E, P and Q directly. Noting that P, $\dot{K}$ are due to mechanical effects alone, we define

$$W = P - \dot{K} \tag{6A-2}$$

as the net working, i.e., the power not used up in producing motion. Then (6A-1) gives

$$\dot{E} = W + Q \tag{6A-3}$$

so that the change in internal energy is the sum of the net working and the heating.

The balance of energy is a statement of the equivalence of heat and work. But experience suggests that while energy and work may always be converted into heat, there is a limit to the amount of heat which may be converted into mechanical work. For example, consider the work done in compressing a spring made of viscoelastic material. We know that a portion of the work done goes into increasing the strain energy of the spring with the remainder going into heating the spring according to the balance of energy (6A-3). But some of the heating is dissipated and cannot be reconverted into mechanical work. This irreversibility inherent in processes involving real materials leads to the postulate that there exists an upper bound B for the heating Q according to the 2nd Law of Thermodynamics:

$$Q \leq B \tag{6A-4}$$

In terms of the bound B, it is convenient to introduce a quantity H, called the entropy in classical terms, such that

$$H = \int \frac{B}{\theta} dt \quad , \quad \theta\dot{H} = B \tag{6A-5}$$

Note that the units of H are energy per unit time per unit absolute temperature. Then the 2nd law (6A-4) can be expressed as

$$\theta \dot{H} \geq Q \qquad (6A\text{-}6)$$

which is also called the entropy production inequality. Equivalently, by (6A-3)

$$\dot{E} - W \leq \theta \dot{H} \qquad (6A\text{-}7)$$

We now define a thermodynamical process explicitly as a set of functions $\theta(t)$, $\nu_\alpha(t)$, $W(t)$, $E(t)$, $Q(t)$, $H(t)$ which satisfy the two laws of thermodynamics (6A-3) and (6A-6) (or its alternate form 6A-7). A thermodynamical process, which we will refer to simply as a process, is reversible if equality holds in (6A-6) or (6A-7); otherwise it is called irreversible. We also introduce the following terminology for processes:

isothermal: $\dot{\theta} = 0$
adiabatic : $Q = 0$
isentropic: $\dot{H} = 0$

These definitions along with the two laws imply the following results:

(a) If $Q = 0$, then (6A-6), (6A-3) imply

$$\dot{H} \geq 0 \quad , \quad \dot{E} = W \qquad (6A\text{-}8)$$

Hence, in an adiabatic process the work done equals the change in internal energy. For a reversible adiabatic process, the entropy is constant, otherwise, it increases.

(b) If $\dot{H} = 0$, then (6A-6) and (6A-3) imply

$$Q \leq 0 \quad , \quad \dot{E} \leq W \tag{6A-9}$$

This implies that a reversible isentropic process is adiabatic. In an irreversible isentropic process the internal energy change is less than the work done and some heat is lost.

(c) Integrating (6A-6) by parts, we find

$$\int_0^t Q \, dt \leq \theta H \Big|_0^t - \int_0^t \dot{\theta} H \, dt$$

If $\dot{\theta} = 0$, then

$$H(t) - H(0) \geq \frac{1}{\theta} \int_0^t Q \, dt \tag{6A-10}$$

Hence, in a reversible isothermal process, the increase in entropy is greater than the heat gained per unit absolute temperature. In an irreversible isothermal process the increase in entropy is greater than a reversible process at the same temperature and same heat is gained.

A little reflection indicates that in any change in state (θ, ν_α) the nature of the material of the body will determine the change in internal energy E, the power not used up in producing motion W, the heating Q, the heating bound B (and hence the change in entropy). This implies a functional relationship between E, W, Q and H and the state functions θ, ν_α, expressed mathematically as

$$E = E(\theta,\nu_\alpha,\dot\theta,\dot\nu_\alpha,\ldots) \quad ; \quad W = W(\theta,\nu_\alpha,\dot\theta,\dot\nu_\alpha,\ldots)$$
$$Q = Q(\theta,\nu_\alpha,\dot\theta,\dot\nu_\alpha,\ldots) \quad ; \quad H = H(\theta,\nu_\alpha,\dot\theta,\dot\nu_\alpha,\ldots) \qquad (6A\text{-}11)$$

These are constitutive equations; the particular form being dependent on the material. We now define an admissible thermodynamic process as process for which the constitutive equations (6A-11) are satisfied.

Implicit in the previous statement of the 2nd Law is that it holds for all processes which the material can undergo, consistent with its constitution. We now make this explicit: the reduced dissipation inequality (6A-7) or (6A-13) must hold for all admissible processes. In particular, this means that at an arbitrary value of time, the state functions θ, ν_α and their rates of change $\dot\theta$, $\dot\nu_\alpha$ may take on any real values whatever, so long as $\theta > 0$.

Noting that in the form (6A-7) Q has been eliminated, we can omit the constitutive equation for Q in (6A-11), and regard the heating as determined by the energy balance (6A-3): $Q = \dot E - W$. An admissible process is then a process in which constitutive equations of the form (6A-11) for E, W and H are satisfied.

For later convenience, we introduce a combined measure of internal energy and entropy, namely the free energy:

$$\Psi = E - \theta H \qquad (6A\text{-}12)$$

which we can regard as replacing E in (6A-11). From (6A-12)

$$\dot\Psi = \dot E - \dot\theta H - \theta\dot H$$

Then by (6A-7)

$$\dot{\Psi} + \theta H - W \leq 0 \qquad (6A-13)$$

This inequality, as well as (6A-7), is sometimes called the reduced dissipation inequality, since the heating Q has been eliminated via the energy balance.

We consider the following example, which is related to the "equations of state" in classical thermodynamics. Let the constitutive equations for Ψ, H and W have the special form

$$\Psi = \Psi(\theta,\nu_\alpha) \quad , \quad H = H(\theta,\nu_\alpha) \qquad (6A-14)$$
$$W = -\sum_{\alpha=1}^{n} \omega_\alpha(\theta,\nu_\beta)\dot{\nu}_\alpha$$

Note that the net working W is a homogeneous linear function of the rates $\dot{\nu}_\alpha$; in classical terms, the coefficients ω_α are called thermodynamic pressures. Now the constitutive equations (6A-14) must satisfy the reduced dissipation inequality (6A-13) for all admissible processes. Substituting (6A-14) into (6A-13), we find

$$(H + \frac{\partial\Psi}{\partial\theta})\dot{\theta} + \sum_{\alpha=1}^{n} (\omega_\alpha + \frac{\partial\Psi}{\partial\nu_\alpha})\dot{\nu}_\alpha \leq 0 \qquad (6A-15)$$

The coefficients of $\dot{\theta}$, $\dot{\nu}_\alpha$ above are functions of θ, ν_α alone. Now (6A-15) must hold for all admissible processes implying that θ, ν_α, $\dot{\theta}$, $\dot{\nu}_\alpha$ may be given arbitrary values at any given time. For this to be true, each coefficient of $\dot{\theta}$, $\dot{\nu}_\alpha$ must vanish:

$$H = -\frac{\partial\Psi}{\partial\theta} \quad , \quad \omega_\alpha = -\frac{\partial\Psi}{\partial\nu_\alpha} \tag{6A-16}$$

These conditions are <u>necessary</u> for (6A-15) to hold for all admissible processes. It is easy to see that conditions (6A-16) are also <u>sufficient</u>. Note that equality must hold in (6A-15). This means that materials described by the constitutive equations (6A-14), subject to (6A-16), can undergo <u>only reversible processes</u>. We view conditions (6A-16) as <u>thermodynamic restrictions on the form of the assumed constitutive equations</u> (6A-14). Note that for the material being considered the entropy H and working W are determined entirely from the free energy $\Psi(\theta,\nu_\alpha)$ as a potential function.

We now consider some additional terminology. Let the <u>internal dissipation</u> be defined as the excess of the heating bound over the heating:

$$\Delta = B - Q \tag{6A-17}$$

By (6A-5) in terms of the entropy we have

$$\Delta = \theta\dot{H} - Q \tag{6A-18}$$

Since $\dot{H}$ is the rate of change of entropy, we call the ratio $\frac{\Delta}{\theta}$ the <u>net entropy production</u> Γ:

$$\Gamma = \frac{\Delta}{\theta} = \dot{H} - \frac{Q}{\theta} \tag{6A-19}$$

Here, $\frac{Q}{\theta}$ is regarded as an influx of entropy due to heating. Hence, Γ is the rate of change of H less the influx of

entropy $\frac{Q}{\theta}$. Note that by the 2nd law in the form (6A-4): $Q \leq B$ and (6A-19), we have

$$\Delta \geq 0 \quad , \quad \Gamma \geq 0 \tag{6A-20}$$

Recall that for reversible processes, equality holds in the 2nd law and hence in (6A-20): $\Delta = 0 = \Gamma$. That is, in reversible processes the internal dissipation and net entropy production vanish. From (6A-3) we eliminate Q from (6A-18):

$$\Delta = \theta\dot{H} - \dot{E} + W \overset{(6A-20)}{\geq} 0 \tag{6A-21}$$

Note this is equivalent to (6A-7), which was called the reduced dissipation inequality. Alternatively, in terms of the free energy function (6A-12), (6A-21) becomes

$$\Delta = W - (\dot{\Psi} + \dot{\theta}H) \geq 0 \tag{6A-22}$$

which is seen to be equivalent to (6A-13).

B. Non-Homogeneous Processes

Reference: B. D. Coleman and V. J. Mizel, "Existence of Caloric Equations of State in Thermodynamics", Journal of Chemical Physics, Vol. 40, 1116-1125, 1964.

If a process is non-homogeneous, then we deal with field variables which vary from point to point in the continuum. In addition to the variables such as stress, strain, internal energy, etc., introduced previously, we define an absolute temperature field $\theta(\underset{\sim}{X},t) > 0$ and a specific entropy field $\eta(\underset{\sim}{X},t)$ such that the total entropy of the body is

$$H = \int_V \rho\eta \, dV \tag{6B-1}$$

In terms of the field variables the postulates of mass balance, linear momentum balance, angular momentum balance and energy balance are expressed in their local or pointwise forms, which we summarize here:

$$\dot{\rho} + \rho v_{i,i} = 0 \quad , \quad \rho J = \rho_o \tag{6B-2}$$

$$t_{ij,j} + \rho f_i = \rho \dot{v}_i \quad , \quad t_{ij} = t_{ji} \tag{6B-3}$$

$$\rho \dot{e} = t_{ij} \, v_{i,j} - q_{i,i} + \rho r \tag{6B-4}$$

We now make explicit the motion of a thermodynamical process for the non-homogeneous case.

Definition -- A thermodynamical process is a set of functions of $\underset{\sim}{X},t$: x_i, θ, $\underset{\sim}{t}$, $\underset{\sim}{q}$, η, e, $\underset{\sim}{f}$, r which satisfy the balance equations (6B-3) and (6B-4).

We will again call a thermodynamical process simply a process. Note that a process is given when x_i, θ, $\underset{\sim}{t}$, $\underset{\sim}{q}$, η and e alone are specified since $\underset{\sim}{f}$, r can then be determined by the linear momentum and energy balances, respectively. In addition, we have considered ρ to be known from the conservation of mass: $\rho J = \rho_o$ when $x_i(\underset{\sim}{X},t)$ is given.

Recalling Chapter 5, constitutive equations are required for $\underset{\sim}{t}$, $\underset{\sim}{q}$ and e. We add the entropy η to the list, based on the discussion in the previous section. These variables depend functionally on the thermodynamic state of the material, i.e., the fields $\theta(\underset{\sim}{X},t)$, $x_i(\underset{\sim}{X},t)$ and possibly their space and time derivatives. Hence, we postulate that

$$\begin{aligned} e &= e(S) \quad , \quad \eta = \eta(S) \\ \underset{\sim}{t} &= \underset{\sim}{t}(S) \quad , \quad \underset{\sim}{q} = \underset{\sim}{q}(S) \end{aligned} \tag{6B-5}$$

where S represents a set of kinematic and thermodynamic variables. We will be concerned with two specific examples, namely, heat conducting elastic solids with argument set S_1 and heat conducting Stokesian fluids with set S_2 where

$$\begin{aligned} S_1 &= \{x_{i,K}, \theta, \theta_{,K}\} \\ S_2 &= \{\tfrac{1}{\rho}, v_{i,j}, \theta, \theta_{,i}\} \end{aligned} \tag{6B-6}$$

For inhomogeneous materials $\underset{\sim}{X}$ is included in the argument sets.

Definition -- A process is admissible provided the constitutive equations (6B-5) are satisfied.

We state some properties of admissible processes. First, for every choice of deformation function $x_i(\underset{\sim}{X},t)$ and temperature $\theta(\underset{\sim}{X},t)$ there exists a unique admissible process. This follows since with x_i, θ given, then e, η, $\underset{\sim}{t}$, $\underset{\sim}{q}$ are determined from (6B-5) and $\underset{\sim}{f}$, r from $(6B\text{-}3)_1$ and (6B-4). Second, at any point $P \in B(t)$: $\underset{\sim}{X} = \overline{\underset{\sim}{X}}$, there exists at least one admissible process such that θ, $\theta_{,i}$, $x_{i,K}$ have arbitrary values $\alpha(t)$, $a_i(t)$, $A_{iK}(t)$. Consider

$$\begin{aligned} x_i(\underset{\sim}{X},t) &= \delta_{iK}\,\overline{X}_K + A_{iK}(t)(X_K - \overline{X}_K) \\ \theta(\underset{\sim}{X},t) &= \alpha(t) + a_i(t)\,A_{iK}(t)(X_K - \overline{X}_K) \end{aligned} \qquad (6B\ 7)$$

These functions along with the constitutive equations (6B-5) and balance laws (6B-3), (6B-4) certainly generate an admissible process for arbitrary $\alpha(t)$, $a_i(t)$, $A_{iK}(t)$. From (6B-7)

$$\begin{aligned} x_{i,K} &= A_{iK}(t) \quad , \quad \text{for all } \underset{\sim}{X} \\ \theta(\overline{X},t) &= \alpha(t) \end{aligned} \qquad (6B\text{-}8)$$

Using $(6B\text{-}7)_1$ in $(6B\text{-}7)_2$, we find θ in spatial form:

$$\theta(\underset{\sim}{x},t) = \alpha(t) + a_i(t)(x_i - \delta_{iK}\,\overline{X}_K) \qquad (6B\text{-}9)$$

which implies

$$\theta_{,i} = a_i(t) \quad \text{for all } \underset{\sim}{X}$$

Hence, θ, $\theta_{,i}$, $x_{i,K}$ have arbitrary values $\alpha(t)$, $a_i(t)$, $A_{iK}(t)$ at $\underset{\sim}{X} = \underset{\sim}{\bar{X}}$. Further, at any given time $t = \bar{t}$, the values $\dot{\alpha}(\bar{t})$, $\ddot{\alpha}(\bar{t})$, ... (up to a finite number of derivatives) are arbitrary and hence independent. To see this, suppose we seek a function $\alpha(t)$ with

$$\alpha(\bar{t}) = c_0 \quad , \quad \dot{\alpha}(\bar{t}) = c_1 \quad , \quad \ddot{\alpha}(\bar{t}) = c_2 \qquad (*)$$

where c_0, c_1, c_2 are arbitrary numbers. Then (*) is satisfied by the function

$$\alpha(t) = c_0 + c_1(t-\bar{t}) + \frac{1}{2} c_2(t-\bar{t})^2$$

By the same argument a_i, $\dot{a}_i$, $\ddot{a}_i$, ... and A_{iK}, $\dot{A}_{iK}$, $\ddot{A}_{iK}$, ... can be arbitrarily assigned at a given time. Note that at $\underset{\sim}{X} = \underset{\sim}{\bar{X}}$

$$\dot{x}_{i,K} = \dot{A}_{iK} \quad , \quad \dot{\theta} = \dot{\alpha} \quad , \quad \dot{\theta}_{,i} = \dot{a}_i$$

Hence, θ, $\theta_{,i}$, $x_{i,K}$, $\dot{\theta}$, $\dot{\theta}_{,i}$, $\dot{x}_{i,K}$, ... are arbitrary quantities at any given time at a point $\underset{\sim}{\bar{X}}$ with (6B-7) still defining a unique admissible process.

Entropy Production Inequality

Recall that the heating of the body has the form (from Section 4D with P_H replaced by Q):

$$Q = \int_{\bar{V}} \rho r \, dV - \int_{\bar{S}} \underset{\sim}{q} \cdot \underset{\sim}{n} \, dS \qquad (6B-10)$$

The heating arises from distributed sources r within $\overline{V}$ and the heat flux vector q_i across the boundary $\overline{S}$. Hence, at any point P ε V(t) the heating is ρr dV, and in analogy with the interpretation of $\frac{Q}{\theta}$ in (6A-19), we take $\frac{\rho r}{\theta}$ dV as the influx of entropy at P due to heat sources (radiation). Then the total influx of entropy in V(t) is $\int \frac{\rho r}{\theta}\, dV$. Similarly, the total influx of entropy due to heat flux through the boundary S is $-\int \frac{\underset{\sim}{q}\circ\underset{\sim}{n}}{\theta}\, dS$. We define the <u>net entropy production</u> of the body as Γ:

$$\Gamma = \frac{d}{dt}\int_{\overline{V}} \rho\eta\, dV - \int_{\overline{V}} \frac{\rho r}{\theta}\, dV + \int_{\overline{S}} \frac{\underset{\sim}{q}\circ\underset{\sim}{n}}{\theta}\, dS \qquad (6B\text{-}11)$$

In analogy with $(6A\text{-}20)_2$, we postulate the

<u>Global Entropy Production Inequality</u>

The net entropy production must be non-negative for every subvolume $\overline{V}$ of the body and for <u>all admissible processes</u>:

$$\Gamma \geq 0 \qquad (6B\text{-}12)$$

This inequality is also known as the Clausius-Duhem inequality or the 2nd law of thermodynamics (for non-homogeneous processes).

If equality holds in (6B-12), we call the process <u>reversible</u>, otherwise <u>irreversible</u>. Using the transport theorem (4A-7) and the divergence theorem (1D-8) in (6B-11), we obtain the local entropy production inequality from (6B-12), by the usual argument:

$$\rho\dot{\eta} - \frac{\rho r}{\theta} + \left(\frac{q_i}{\theta}\right)_{,i} \geq 0 \qquad (6B\text{-}13)$$

or expanding the last term

$$\rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta} q_{i,i} - \frac{1}{\theta^2} q_i \theta_{,i} \geq 0 \qquad (6B\text{-}14)$$

Corresponding to Γ, we introduce the <u>specific</u> net entropy production γ such that

$$\Gamma = \int_V \rho\gamma \, dV \qquad (6B\text{-}15)$$

Then we find from (6B-11)

$$\rho\gamma = \rho\dot{\eta} - \frac{\rho r}{\theta} + \frac{1}{\theta} q_{i,i} - \frac{1}{\theta^2} q_i \theta_{,i} \geq 0 \qquad (6B\text{-}16)$$

Recalling (6A-19): $\Delta = \theta\Gamma$, we define the local <u>internal dissipation</u> δ as

$$\delta = \rho\theta\gamma \qquad (6B\text{-}17)$$

Since $\theta > 0$, we can multiply (6B-16) by θ with the result

$$\delta = \rho\theta\dot{\eta} - \rho r + q_{i,i} - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \qquad (6B\text{-}18)$$

Note that the internal dissipation is non-negative. From the local balance of energy (6B-4), $q_{i,i} - \rho r = t_{ij} v_{i,j} - \rho\dot{e}$. Hence, an alternate form of (6B-14) is, after multiplying by θ and using (6B-18):

$$\delta = \rho\theta\dot{\eta} + t_{ij} v_{i,j} - \rho\dot{e} - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \qquad (6B\text{-}19)$$

This is a reduced dissipation inequality similar to (6A-21) in the sense that the stress power $t_{ij}\, v_{i,j}$ and the internal energy rate $\dot{e}$ explicitly enter the inequality.

We define the free energy function ψ as

$$\psi = e - \theta\eta \tag{6B-20}$$

Then since $\dot{\psi} = \dot{e} - \dot{\theta}\eta - \theta\dot{\eta}$, the energy balance (6B-4) becomes

$$\rho\dot{\psi} + \rho\theta\dot{\eta} + \rho\eta\dot{\theta} = t_{ij}\, v_{i,j} - q_{i,i} + \rho r \tag{6B-21}$$

In addition, the reduced form (6B-19) becomes

$$\delta = -\rho\eta\dot{\theta} + t_{ij}\, v_{i,j} - \rho\dot{\psi} - \frac{1}{\theta} q_i\, \theta_{,i} \geq 0 \tag{6B-22}$$

Finally, we adopt the following terminology for non-homogeneous processes. Recall that equality in (6B-12) implied a reversible process, and hence equality in any of the alternate forms (6B-13,14,16,18,19,22) also implies a reversible process. Also, based on the definitions made for homogeneous processes, we have

$$\begin{aligned} &\text{isothermal:} && \dot{\theta} = 0 && \text{for all } \underset{\sim}{X} \in B, \text{ all } t \\ &\text{adiabatic:} && q_i = 0 = r && \text{for all } \underset{\sim}{X} \in B, \text{ all } t \\ &\text{isentropic:} && \dot{\eta} = 0 && \text{for all } \underset{\sim}{X} \in B, \text{ all } t \end{aligned} \tag{6B-23}$$

C. Thermodynamical Restrictions on Constitutive Equations

1. Heat Conducting Elastic Solids

In Chapter V we generated constitutive equations for isothermal elasticity by assuming $\underset{\sim}{t}$ was a function of the nonlinear strain tensor $\underset{\sim}{E}$ and proceeded using either Green's and Cauchy's method. A more general initial assumption for nonlinear, isothermal elasticity is to assume $\underset{\sim}{t}$ is a function of the displacement gradients $x_{i,K}$. For the heat conducting case constitutive equations are also needed for $\underset{\sim}{q}$, η and e or ψ, which must depend in some way on the temperature field. Hence, we assume constitutive equations in the form

$$\psi = \psi(S) \quad , \quad \eta = \eta(S)$$

$$\underset{\sim}{t} = \underset{\sim}{t}(S) \quad , \quad \underset{\sim}{q} = \underset{\sim}{q}(S) \tag{6C-1}$$

where

$$S = \{x_{i,K},\ \theta,\ \theta_{,K}\} \tag{6C-2}$$

It turns out that $\theta_{,K}$ must be included in S to account for heat conduction. From (6C-1) and (6C-2), we find

$$\dot{\psi} = \frac{\partial\psi}{\partial x_{i,K}}\,\dot{\overline{x_{i,K}}} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,K}}\,\dot{\overline{\theta_{,K}}}$$

Recalling the identity (3A-8): $\dot{\overline{x_{i,K}}} = v_{i,j}\ x_{j,K}$, we have

$$\dot{\psi} = \frac{\partial\psi}{\partial x_{i,K}}\,x_{j,K}\,v_{i,j} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,K}}\,\dot{\theta}_{,K}$$

Substituting this result into the entropy production inequality (6B-22) (reduced form) and collecting terms, we find

$$\delta = -\rho(\eta + \frac{\partial\psi}{\partial\theta})\dot{\theta} + (t_{ij} - \rho\frac{\partial\psi}{\partial x_{i,K}} x_{j,K})v_{i,j}$$
$$- \rho\frac{\partial\psi}{\partial\theta_{,K}}\dot{\theta}_{,K} - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \tag{6C-3}$$

Now this inequality must hold for all $\underset{\sim}{X} \varepsilon B$, all t and all admissible processes. Based on the discussion of admissible processes in Section 6B, we write (6C-3) in the form

$$-\rho[\eta(x_{i,K}, \theta, \theta_{,K}) + \frac{\partial\psi}{\partial\theta}(x_{i,K}, \theta, \theta_{,K})]\dot{\theta}$$
$$+ f(x_{i,K}, v_{i,j}, \theta, \dot{\theta}_{,K}, \theta_{,i}) \geq 0 \tag{*}$$

Consider an admissible process in which θ, $x_{i,K}$, $v_{i,j} = \dot{x}_{i,K} X_{K,j}$, $\theta_{,i}$ and $\dot{\theta}_{,K} = \dot{\overline{\theta_{,i} x_{i,K}}}$ are *fixed* numbers, but $\dot{\theta}$ is *arbitrary* at arbitrary $\bar{\underset{\sim}{X}}$, $\bar{t}$. That such an admissible process exists, follows from Section 6B; in particular, in terms of (6B-7), we have assigned fixed numbers to $\alpha(\bar{t})$, $A_{iK}(\bar{t})$, $\dot{A}_{iK}(\bar{t}) A^{-1}_{Kj}(\bar{t})$, $a_i(\bar{t})$ and $\dot{a}_i(\bar{t}) A_{iK}(\bar{t}) + a_i(\bar{t}) \dot{A}_{iK}(\bar{t})$ while $\dot{a}(\bar{t})$ is an arbitrary number. Now we write (*) in the form

$$a\dot{\theta} + b \geq 0 \tag{†}$$

where a,b are fixed numbers, representing the coefficient of $\dot{\theta}$ and the function f in (*). The inequality (†) must hold for all $\dot{\theta}$, positive or negative. Clearly, the inequality will be violated for some $\dot{\theta}$, unless $a \equiv 0$. This implies

$$\eta = -\frac{\partial\psi}{\partial\theta} \tag{6C-4}$$

This a <u>necessary</u> condition for (6C-3) to hold. Note that (6C-4) holds for all $\underset{\sim}{X} \varepsilon B$ and all t, since $\underset{\sim}{\bar{X}}$, $\bar{t}$ were arbitrary. Now use (6C-4) in (6C-3):

$$\delta = (t_{ij} - \rho \frac{\partial\psi}{\partial x_{i,K}} x_{j,K}) v_{i,j} - \rho \frac{\partial\psi}{\partial\theta_{,K}} \dot{\theta}_{,K} - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \qquad (6C\text{-}5)$$

Now write this in the form

$$- \rho \frac{\partial\psi}{\partial\theta_{,K}} (x_{i,M}, \theta, \theta_{,M}) \dot{\theta}_{,K} + g(x_{i,M}, v_{i,j}, \theta, \theta_{,i}) \geq 0 \qquad (**)$$

We consider an admissible process in which θ, $x_{i,K}$, $v_{i,j}$, $\theta_{,i}$ are <u>fixed</u> numbers, but $\dot{\theta}_{,K}$ is a triplet of arbitrary numbers. Then from (**) we have

$$C_K \dot{\theta}_{,K} + d \geq 0 \qquad (\dagger\dagger)$$

where C_K and d are fixed numbers. Let $\dot{\theta}_{,1}$ be arbitrary and non-zero, while $\dot{\theta}_{,2} = 0 = \dot{\theta}_{,3}$. Then (††) implies $c_1 = 0$. Similarly, $c_2 = c_3 = 0$. Hence, <u>necessary</u> conditions for (††) are

$$\frac{\partial\psi}{\partial\theta_{,K}} \equiv 0 \qquad (6C\text{-}6)$$

for all $\underset{\sim}{X} \varepsilon B$ and all t. These conditions imply from (6C-1) and (6C-4) that

$$\psi = \psi(S_o) \quad , \quad \eta = \eta(S_o) \tag{6C-7}$$

where

$$S_o = \{x_{i,K}, \theta\} \tag{6C-8}$$

Hence, the free energy and entropy cannot depend on the temperature gradient $\theta_{,K}$. Equation $(6C\text{-}7)_1$ is called the <u>caloric equation of state</u>.

Now using (6C-6) in (6C-5), we find

$$\delta = (t_{ij} - \rho \frac{\partial\psi}{\partial x_{i,K}} x_{j,K})v_{i,j} - \frac{1}{\theta} q_i \ \theta_{,i} \geq 0 \tag{6C-9}$$

We write this inequality in the form

$$\begin{aligned} &[t_{ij}(x_{k,M}, \theta, \theta_{,K}) - \rho \frac{\partial\psi}{\partial x_{i,K}} (x_{k,M}, \theta)x_{j,K}]v_{i,j} \\ &\quad + h(x_{i,K}, \theta, \theta_{,i}) \geq 0 \end{aligned} \tag{*}$$

Consider an admissible process in which θ, $x_{i,K}$, $\theta_{,i}$ are fixed numbers, while $v_{i,j}$ is a 3×3 array of arbitrary numbers. Then (*) has the form

$$B_{ij} \ v_{i,j} + C \geq 0 \tag{†}$$

where B_{ij} and C are fixed. It follows from (†) that $B_{ij} \equiv 0$. Hence

$$t_{ij} = \rho \frac{\partial\psi}{\partial x_{i,K}} x_{j,K} \tag{6C-10}$$

for all $\underset{\sim}{X} \varepsilon$ B and all t. Note from (6C-7) that

$$\underset{\sim}{t} = \underset{\sim}{t}(S_o) \tag{6C-11}$$

i.e., the stress tensor cannot depend on temperature gradient $\theta_{,K}$. In addition, ψ is a potential function for stress, similar to the result that the strain energy function is a potential for stress in the isothermal case (see (5A-3)). Recall that $\underset{\sim}{t}$ must be symmetric as required by angular momentum balance. Hence, ψ must satisfy the restriction

$$\frac{\partial\psi}{\partial x_{[i,K}} x_{j],K} = 0 \tag{6C-12}$$

Now using (6C-10) in (6C-9), we find

$$\delta = -\frac{1}{\theta} q_i \; \theta_{,i} \geq 0 \tag{6C-13}$$

Since q_i depends on $\theta_{,K} = \theta_{,i} \, x_{i,K}$ and hence on $\theta_{,i}$, (6C-13) does <u>not</u> imply $q_i \equiv 0$. On the other hand, if $\theta_{,K}$ had not been included in the argument set S: (6C-2), then heat conduction would not be possible. This would define a different class of materials. Note that (6C-13) implies that the class of materials considered in general undergo irreversible processes. However, the dissipation vanishes, implying a reversible process, if either $\underset{\sim}{q} = 0$ or $\theta_{,i} = 0$.

For easy reference, we summarize the results:

$$\psi = \psi(S_o) \quad , \quad \eta = -\frac{\partial\psi}{\partial\theta} = \eta(S_o)$$

$$t_{ij} = \rho \frac{\partial\psi}{\partial x_{i,K}} x_{j,K} = t_{ij}(S_o) \tag{6C-14}$$

$$q_i \; \theta_{,i} \leq 0$$

These conditions were shown to be necessary for (6C-3) to hold for all admissible processes. Substitution back into (6C-3), clearly implies they are also sufficient. Hence, we have obtained the thermodynamical restrictions on the assumptions (6C-1) and (6C-2).

A further important result can be obtained from (6C-13). By $(6C\text{-}14)_4$ and the identity $\theta_{,K} = \theta_{,i}\, x_{i,K}$, we have

$$q_i(x_{j,K}, \theta, \theta_{,K})\, X_{K,i}\, \theta_{,K} \leq 0 \qquad (*)$$

Consider $x_{i,K}$ and θ fixed. Then (*) must hold for arbitrary $\theta_{,K}$, and therefore certainly holds for $\theta_{,K}$ replaced by $\alpha\, \theta_{,K}$, where α is arbitrary. Define the function

$$f(\alpha) = \alpha\, q_i(x_{j,K}, \theta, \alpha\, \theta_{,K}) X_{K,i}\, \theta_{,K}$$

By (*) the maximum value of $f(\alpha)$ is zero. Assuming q_i continuously differentiable in $\theta_{,K}$, then q_i is a continuous function of $\theta_{,K}$ and $f(0) = 0$. Hence, $f(\alpha)$ achieves its maximum value for $\alpha = 0$, and $f'(0)$ must vanish. Compute $f'(\alpha)$:

$$f'(\alpha) = q_i(x_{j,K}, \theta, \alpha\, \theta_{,K}) X_{K,i}\, \theta_{,K} + \alpha \frac{\partial q_i}{\partial g_P}\, \theta_{,P}\, X_{K,i}\, \theta_{,K}$$

where $g_P = \alpha\, \theta_{,P}$. Then

$$f'(0) = q_i(x_{i,K}, \theta, 0) X_{K,i}\, \theta_{,K} \equiv 0$$

which must hold for arbitrary $\theta_{,K}$. Since $X_{K,i}$ is fixed, then it is necessary that

$$q_i(x_{i,K}, \theta, 0) = 0 \qquad (6C-15)$$

i.e., the heat flux must vanish with temperature gradient. Note that this result, as with (6C-14), is valid only for the class of materials considered.

Recalling the energy equation (6B-21), the results (6C-14) imply

$$\rho\theta\dot{\eta} = - q_{i,i} + \rho r \qquad (6C-16)$$

This reduced form of the energy equation is called the <u>heat conduction equation</u> and is the governing equation for the temperature field when particular forms of the constitutive equations for ψ, q_i are given.

We consider a special case of the above results in which the material is <u>incompressible</u>. An example of this type of material is rubber. The incompressibility condition is

$$\rho = \rho_o \quad , \quad J = \det(x_{i,K}) = 1 \qquad (6C-17)$$

This implies the deformation gradients $x_{i,K}$ are not independent quantities. Hence, in computing $\underset{\sim}{t}$ from ψ via $(6C-14)_3$, we must ensure that the constraint J = 1 is satisfied. This is most easily accomplished by the method of Lagrange multipliers, i.e., replace ψ in $(6C-14)_3$ by

$$\tilde{\psi} = \psi - \frac{p}{\rho_o}(J-1) \qquad (*)$$

where p is an unknown multiplier, independent of $x_{i,K}$, but generally depending on the material point and time. From (*)

$$\frac{\partial\tilde{\psi}}{\partial x_{i,K}} = \frac{\partial\psi}{\partial x_{i,K}} - \frac{p}{\rho_o}\frac{\partial J}{\partial x_{i,K}}$$

Recalling the identity $\frac{\partial J}{\partial x_{i,K}} x_{j,K} = J\,\delta_{ij}$, and that J = 1,

$$t_{ij} = \rho\,\frac{\partial\tilde{\psi}}{\partial x_{i,K}} x_{j,K} = -\,p\,\delta_{ij} + \rho_o\,\frac{\partial\psi}{\partial x_{i,K}} x_{j,K} \qquad (6C\text{-}18)$$

Hence, (6C-18) replaces $(6C\text{-}14)_3$ for an incompressible, heat conducting elastic material. The unknown $p(\underset{\sim}{X},t)$ is called <u>mechanical pressure</u>, and is determined, along with the other unknowns, from solving the complete set of field equations.

Consider the special case when the material is restricted to <u>isothermal</u> and <u>adiabatic</u> processes, i.e., $\theta = \theta_o$ = const. and $r = 0 = q_i$: Then the constitutive equation $(6C\text{-}14)_2$ for η no longer is valid, while the energy balance (6C-16) implies η = const. Also, $(6C\text{-}14)_1$ implies $\psi = \psi(x_{i,K}, \theta_o)$, and $(6C\text{-}14)_3$ implies $\underset{\sim}{t}$ depends only on $x_{i,K}$. This gives rise to a purely mechanical theory for which we can define a strain energy function W such that

$$W = W(x_{i,K}) = \rho_o\,\psi(x_{i,K}, \theta_o)$$

Then $(6C\text{-}14)_3$ is replaced by

$$t_{ij} = \frac{\rho}{\rho_o}\frac{\partial W}{\partial x_{i,K}} x_{j,K} \qquad (6C\text{-}19)$$

(Compare this form with (5A-3)).

2. Heat Conducting Stokesian Fluids

In Section 5B a class of Stokesian fluids were treated by assuming $\underset{\sim}{t}$ was a function of the stretching tensor $\underset{\sim}{d}$ such that when $\underset{\sim}{d}$ vanished, $\underset{\sim}{t}$ reduced to a pressure. Here, we adopt a more general starting point with $\underset{\sim}{t}$ assumed to be a function of the specific volume $\frac{1}{\rho}$, the velocity gradients $v_{i,j}$, and θ, $\theta_{,i}$ to account for heat conduction. In addition, the entropy, free energy and heat flux vector are assumed to be functions of the same set of arguments. That is, we postulate constitutive equations:

$$\begin{aligned} &\psi = \psi(S) \quad , \quad \eta = \eta(S) \\ &\underset{\sim}{t} = \underset{\sim}{t}(S) \quad , \quad \underset{\sim}{q} = \underset{\sim}{q}(S) \qquad\qquad (6C\text{-}20) \\ &S = \{\rho^{-1},\ v_{i,j},\ \theta,\ \theta_{,i}\} \end{aligned}$$

To obtain the thermodynamic restrictions on (6C-20), we employ the entropy production inequality (6B-22). From (6C-20)

$$\dot{\psi} = \frac{\partial\psi}{\partial\rho^{-1}}\left(-\frac{\dot{\rho}}{\rho^2}\right) + \frac{\partial\psi}{\partial v_{i,j}}\,\dot{\overline{v_{i,j}}} + \frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \frac{\partial\psi}{\partial\theta_{,i}}\,\dot{\overline{\theta_{,i}}}$$

By the continuity equation $\frac{\dot{\rho}}{\rho} = -\,v_{i,i} = -\,\delta_{ij}\,v_{i,j}$. Hence,

$$\rho\dot{\psi} = \frac{\partial\psi}{\partial\rho^{-1}}\,\delta_{ij}\,v_{i,j} + \rho\,\frac{\partial\psi}{\partial v_{i,j}}\,\dot{\overline{v_{i,j}}} + \rho\,\frac{\partial\psi}{\partial\theta}\,\dot{\theta} + \rho\,\frac{\partial\psi}{\partial\theta_{,i}}\,\dot{\overline{\theta_{,i}}}$$

Using this result in (6B-22) and collecting terms, we find

$$\begin{aligned} \delta = &-\rho\left(\eta + \frac{\partial\psi}{\partial\theta}\right)\dot{\theta} + \left(t_{ij} - \delta_{ij}\,\frac{\partial\psi}{\partial\rho^{-1}}\right)v_{i,j} \qquad (6C\text{-}21) \\ &-\rho\,\frac{\partial\psi}{\partial\theta_{,i}}\,\dot{\overline{\theta_{,i}}} - \rho\,\frac{\partial\psi}{\partial v_{i,j}}\,\dot{\overline{v_{i,j}}} - \frac{1}{\theta}\,q_i\,\theta_{,i} \geq 0 \end{aligned}$$

We note that by (6C-20) the coefficients of $\dot{\theta}$, $\overline{\dot{\theta}_{,i}}$, $\overline{\dot{v}_{i,j}}$ above are dependent on ρ^{-1}, $v_{i,j}$, θ, $\theta_{,i}$. Hence, by an argument similar to that used for heat conducting elastic solids, it is necessary that

$$\frac{\partial\psi}{\partial\theta_{,i}} = 0 = \frac{\partial\psi}{\partial v_{i,j}} \quad , \quad \eta = -\frac{\partial\psi}{\partial\theta}$$

so that

$$\psi = \psi(S_o) \quad , \quad \eta = -\frac{\partial\psi}{\partial\theta} = \eta(S_o)$$
$$S_o = \{\rho^{-1}, \theta\} \qquad (6C-22)$$

Then (6C-21) reduces to

$$\delta = (t_{ij} - \delta_{ij}\frac{\partial\psi}{\partial\rho^{-1}})v_{i,j} - \frac{1}{\theta} q_i \theta_{,i} \geq 0 \qquad (6C-23)$$

The coefficients of $v_{i,j}$ and $\theta_{,i}$ are dependent on these quantities so that (6C-23) cannot be reduced further. Since the term $\delta_{ij}\frac{\partial\psi}{\partial\rho^{-1}}$ represents a hydrostatic state of stress, we define the <u>thermodynamic pressure</u>

$$\pi = -\frac{\partial\psi}{\partial\rho^{-1}} = \pi(\rho^{-1}, \theta) \qquad (6C-24)$$

In addition, the 1st term in (6C-23) represents mechanical dissipation, and hence it is natural to define the coefficient of $v_{i,j}$ as the <u>dissipative stress</u>:

$$_{D}t_{ij} = t_{ij} + \pi\delta_{ij} \qquad (6C-25)$$

where (6C-24) was used. Now (6C-23) has the form

$$\delta = {}_D t_{ij} \, v_{i,j} - \frac{1}{\theta} q_i \, \theta_{,i} \geq 0 \tag{6C-26}$$

It is clear that in heat conducting Stokesian fluids dissipation arises from both mechanical and thermal effects, namely the dissipative part of the stress tensor and heat conduction. Note that there is no <u>mechanical</u> dissipation in heat conducting elastic solids. However, it would be reasonable to expect mechanical dissipation in viscoelastic materials. Note that if either ${}_D\underset{\sim}{t}$ or $\underset{\sim}{q}$ is non-vanishing, (6C-26) implies irreversible processes. For reversible processes exclusively, then

${}_D\underset{\sim}{t} = 0 = \underset{\sim}{q}$ and (6C-25) implies

$$t_{ij} = - \pi(\theta, \rho^{-1})\delta_{ij} \tag{6C-27}$$

This special class of materials is called <u>ideal</u> <u>compressible</u> fluids.

Summarizing the results, we have

$$\begin{aligned} &\psi = \psi(S_o) \quad ; \quad \eta = - \frac{\partial\psi}{\partial\theta} = \eta(S_o) \\ &t_{ij} = - \pi\delta_{ij} + {}_D t_{ij} \quad , \quad \pi = - \frac{\partial\psi}{\partial\rho^{-1}} = \pi(S_o) \\ &S_o = \{\rho^{-1}, \theta\} \\ &\delta = {}_D t_{ij} \, v_{i,j} - \frac{1}{\theta} q_i \, \theta_{,i} \geq 0 \end{aligned} \tag{6C-28}$$

These conditions were shown to be necessary for the entropy production inequality to hold for every admissible process. Inspection shows that they are also sufficient conditions. Comparing the above results with those for thermoelastic solids, we note that ψ is a potential function for η and π, but not for $\underset{\sim}{t}$. Also, ψ and hence η, π cannot depend on velocity gradients or temperature gradient. Equations $(6C\text{-}28)_{1,4}$ are equations of state, comparable to (5B-6), (5B-7) introduced without thermodynamic justification. Note that generally stress depends on temperature gradient, a basic difference between these fluids and thermoelastic solids.

Further information can be extracted from $(6C\text{-}28)_6$, which we write as

$$\begin{aligned} &{}_Dt_{ij}(\rho^{-1}, v_{m,n}, \theta, \theta_{,k})\, v_{i,j} \\ &\quad - \frac{1}{\theta} q_i(\rho^{-1}, v_{m,n}, \theta, \theta_{,k})\, \theta_{,i} \geq 0 \end{aligned} \tag{*}$$

Consider ρ and θ fixed. Then (*) must hold for arbitrary $v_{i,j}$ and $\theta_{,i}$, in particular for $v_{i,j}$ replaced by $\alpha v_{i,j}$ and $\beta\theta_{,i}$, where α, β are arbitrary. Define the function

$$\begin{aligned} f(\alpha,\beta) = {} & \alpha\, {}_Dt_{ij}(\rho^{-1}, \alpha v_{m,n}, \theta, \beta\theta_{,k})\, v_{i,j} \\ & - \beta \frac{1}{\theta} q_i(\rho^{-1}, \alpha v_{m,n}, \theta, \beta\theta_{,k})\, \theta_{,i} \geq 0 \end{aligned} \tag{$\dagger$}$$

By (*) the minimum value of $f(\alpha,\beta)$ is zero. With sufficient continuity properties of the functions ${}_D\underset{\sim}{t}$ and $\underset{\sim}{q}$, then $f(0,0) = 0$. Hence, f achieves its minimum value at $\alpha = 0 = \beta$. This implies $\frac{\partial f}{\partial \alpha}$ and $\frac{\partial f}{\partial \beta}$ must vanish at $\alpha = 0 = \beta$. Then from (†) we have

$$\frac{\partial f}{\partial \alpha}(0,0) = {}_Dt_{ij}(\rho^{-1}, 0, \theta, 0)\, v_{i,j} = 0$$

$$\frac{\partial f}{\partial \beta}(0,0) = -\frac{1}{\theta}\, q_i(\rho^{-1}, 0, \theta, 0)\, \theta_{,i} = 0$$

These conditions must hold for arbitrary $v_{i,j}$ and $\theta_{,i}$. Hence, for (*) to hold it is necessary that the functions ${}_D\underset{\sim}{t}$ and $\underset{\sim}{q}$ satisfy

$$\begin{aligned} {}_D\underset{\sim}{t}(\rho^{-1}, 0, \theta, 0) &= 0 \\ \underset{\sim}{q}(\rho^{-1}, 0, \theta, 0) &= 0 \end{aligned} \qquad (6C\text{-}29)$$

i.e., the dissipative stress and heat flux vector must vanish with velocity gradients and temperature gradient. The state defined by $v_{i,j} = 0 = q_i$ is called the <u>thermal and mechanical equilibrium state</u>.

We now return to the energy equation (6B-21). From (6C-28)

$$\rho\dot{\psi} = -\pi\delta_{ij}\, v_{i,j} - \rho\eta\dot{\theta}$$

Hence, from (6B-21)

$$\rho\theta\dot{\eta} = (t_{ij} + \pi\delta_{ij})\, v_{i,j} - q_{i,i} + \rho r$$

or

$$\rho\theta\dot{\eta} = {}_Dt_{ij}\, v_{i,j} - q_{i,i} + \rho r \qquad (6C\text{-}30)$$

This is the heat conduction equation for heat conducting Stokesian fluids and is the governing equation for temperature θ when particular forms of the constitutive equations for ψ, $\underset{\sim}{t}$ and $\underset{\sim}{q}$ are specified.

For the case of an incompressible material, $\rho = \rho_o$ = constant, $\dot{\rho} = 0$. Then $\frac{\partial\psi}{\partial\rho^{-1}}$ does not appear in (6C-21) and (6C-24) no longer holds. The constraint $\rho = \rho_o$ is satisfied by the method of Lagrange multipliers, i.e., replace ψ by

$$\tilde{\psi} = \psi - p(\frac{1}{\rho} - \frac{1}{\rho_o})$$

where p is the unknown multiplier, independent of ρ, but generally a function of $\underset{\sim}{x}$,t. Then, we can compute π using $\tilde{\psi}$ and (6C-24):

$$\pi = - \frac{\partial\tilde{\psi}}{\partial\rho^{-1}} = - (-p) = p$$

Hence, π is replaced by p, the mechanical pressure, which has no constitutive equation. Then from (6C-28) $\underset{\sim}{t}$ has the form

$$t_{ij} = - p\delta_{ij} + {}_{D}t_{ij} \qquad (6C-31)$$

VII. Theory of Constitutive Equations

A. General Principles

In Chapter VI we considered thermomechanical materials for which ψ, η, $\underset{\sim}{t}$, $\underset{\sim}{q}$ are <u>functions</u> of x_i, θ and their gradients. These constitutive equations depend only on the argument sets at the present time, and are said to define materials <u>without memory</u>. For materials with memory the constitutive equations also depend on past values of the argument set. We will not consider memory effects in this chapter.

We have seen that the entropy production inequality places restrictions on the assumed form of the constitutive equations. Other restrictions arise from additional principles, motivated largely on physical grounds.

1. <u>Equipresence</u> All constitutive equations should have the same argument set, unless a fundamental postulate of continuum mechanics is violated.

This principle is intended to serve as a starting point for the development of constitutive equations, so that fundamental 'cause-effect relationships are not inadvertently omitted. For example, if the deformation gradients $x_{i,K}$ had been omitted in the argument set for heat flux q_i thermomechanical coupling effects would not be present in the heat conduction equation for thermoelastic materials. Note that equipresence was satisfied by (6C-1) and (6C 20). It was then shown that for thermoelastic materials ψ could not depend on $\theta_{,i}$, otherwise the entropy production inequality would be violated.

2. Material Symmetry

Consider linear transformations of the material coordinate system of the form

$$\overline{X}_K = H_{KM} X_M \tag{7A-1}$$

where $\underset{\sim}{H}$ is orthogonal:

$$\underset{\sim}{H}\,\underset{\sim}{H}^T = \underset{\sim}{H}^T\,\underset{\sim}{H} = \underset{\sim}{I} \tag{7A-2}$$

Recall that (7A 1) represents of rotation if det $\underset{\sim}{H}$ = +1 and a rotation possibly combined with a reflection if det $\underset{\sim}{H}$ = -1. By (7A-1) B_o, the initial configuration of the body, is referred to a new coordinate system $\overline{X}_K$. Inverting (7A-1), we find

$$X_P = H_{KP} \overline{X}_K \tag{7A-3}$$

Then the deformation function becomes an implicit function of $\underset{\sim}{\overline{X}}$: $x_i(\underset{\sim}{X}(\underset{\sim}{\overline{X}}),t)$. Consider now a homogeneous, hyperelastic material with strain energy function $W = W(x_{i,K})$. Under (7A-1), we have

$$\frac{\partial x_i}{\partial \overline{X}_K} = H_{KP} \frac{\partial x_i}{\partial X_P} \; , \quad \frac{\partial x_i}{\partial X_K} = H_{PK} \frac{\partial x_i}{\partial \overline{X}_P} \tag{7A-4}$$

and in general W becomes a different function of $\dfrac{\partial x_i}{\partial \overline{X}_P}$.

$$W = W(x_{i,K}) = W\left(H_{PK} \frac{\partial x_i}{\partial \overline{X}_P}\right) = \overline{W}\left(\frac{\partial x_i}{\partial \overline{X}_P}\right) \tag{*}$$

If the functional form of W does not change under the transformation (7A-1) for some $\underset{\sim}{H}$, i.e.,

$$\bar{W}(\frac{\partial x_i}{\partial \bar{X}_P}) = W(\frac{\partial x_i}{\partial \bar{X}_P})$$

then W is called form invariant with respect to $\underset{\sim}{H}$. Then (*) becomes

$$W(\frac{\partial x_i}{\partial X_P}) = W(\frac{\partial x_i}{\partial \bar{X}_P})$$

or from (7A-4)

$$W(\frac{\partial x_i}{\partial X_K}) = W(H_{KP} \frac{\partial x_i}{\partial X_P}) \qquad (7A\text{-}5)$$

The symmetry group $\{\underset{\sim}{H}\}$ of the material is defined as the group of all orthogonal $\underset{\sim}{H}$ for which W is form invariant, i.e., (7A-5) satisfied. If $\{\underset{\sim}{H}\}$ equals all orthogonal $\underset{\sim}{H}$ such that det $\underset{\sim}{H} = \pm 1$, then $\{\underset{\sim}{H}\}$ is called the full orthogonal group, and the material is called isotropic. Otherwise, the material is called anisotropic. We now state the

Material Symmetry Postulate -- The constitutive equations $\psi(S)$, $\eta(S)$, $\underset{\sim}{t}(S)$ and $\underset{\sim}{q}(S)$ must be form invariant under the symmetry group of the material i.e.,

$$\begin{aligned} \psi(S) &= \psi(\bar{S}) \quad , \quad \eta(S) = \eta(\bar{S}) \\ \underset{\sim}{t}(S) &= \underset{\sim}{t}(\bar{S}) \quad , \quad \underset{\sim}{q}(S) = \underset{\sim}{q}(\bar{S}) \end{aligned} \qquad (7A\text{-}6)$$

where $\bar{S}$ is determined by subjecting the argument set S to the transformation (7A-1) for all $\underset{\sim}{H} \epsilon \{\underset{\sim}{H}\}$.

This postulate imbodies the fact that the functional forms of the constitutive equations, and hence, the <u>material response</u>, are generally dependent upon the orientation of the reference configuration B_o relative to the X_K axes, but are restricted in form by the orientations determined by the symmetry group. Note that the symmetry group must be determined experimentally.

3. Material Frame Indifference (Objectivity)

We define a <u>change</u> <u>in</u> <u>frame</u> by

$$\begin{aligned} x_i^*(\underset{\sim}{X},t^*) &= Q_{ij}(t)\ x_j(\underset{\sim}{X},t) + b_i(t) \\ t^* &= t - a \end{aligned} \qquad (7A\text{-}7)$$

where a, $\underset{\sim}{b}(t)$, $\underset{\sim}{Q}(t)$ are respectively an arbitrary number, vector and orthogonal tensor:

$$\underset{\sim}{Q}(t)\ \underset{\sim}{Q}^T(t) = \underset{\sim}{Q}^T(t)\ \underset{\sim}{Q}(t) = \underset{\sim}{I} \qquad (7A\text{-}8)$$

A change in frame <u>does</u> <u>not</u> represent a deformation of the body B(t), but rather defines an arbitrary <u>time</u> <u>dependent</u> change in the spatial reference frame (see Fig. VII-1). An observer fixed in the starred frame sees the actual deformation $x_i(\underset{\sim}{X},t)$ as $x_i^*(\underset{\sim}{X},t^*)$ due to the motion of his frame of reference.

Tensor quantities ϕ, u_i, V_{ij} are called <u>frame</u> <u>indifferent</u> (or objective) if under the change in frame (7A-7), they transform according to

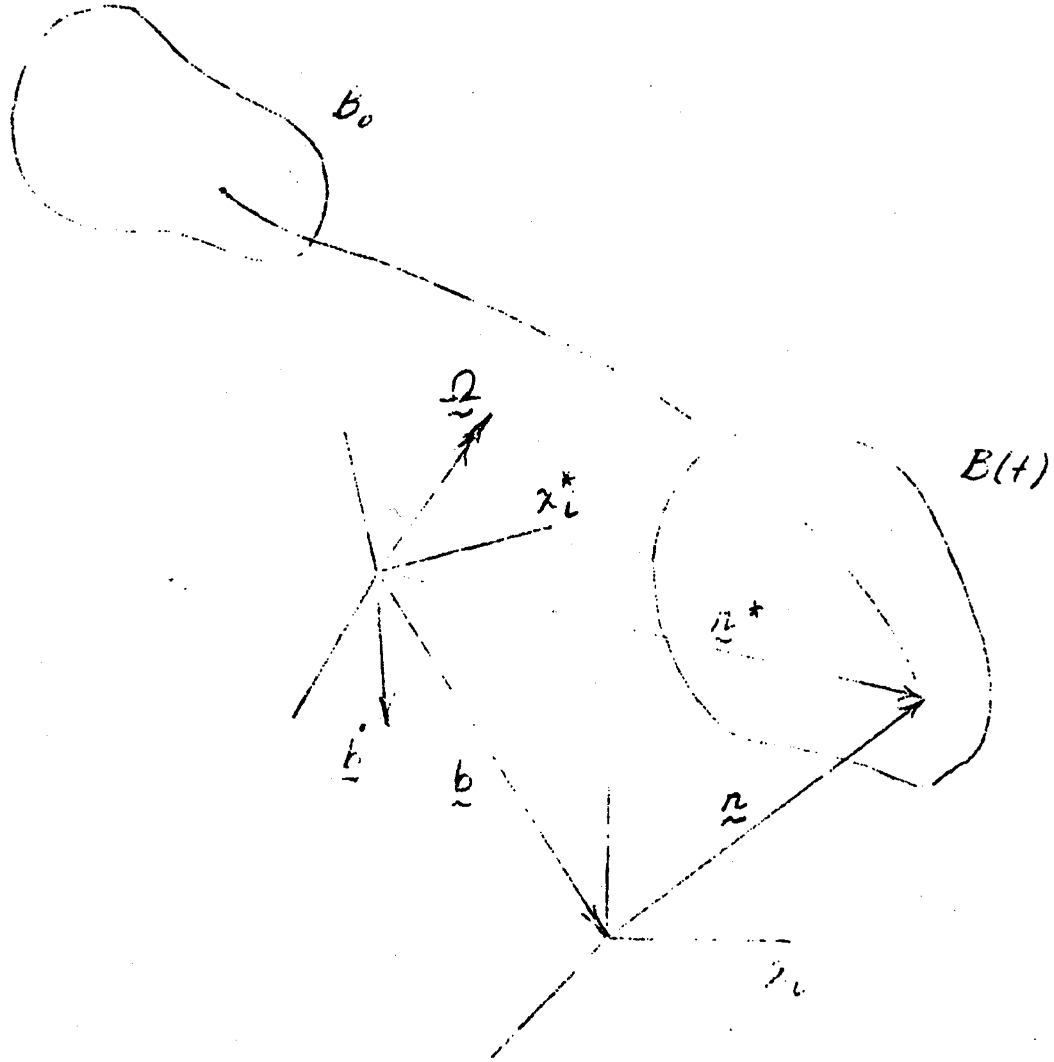


Fig. VII-1

$$\phi^* = \phi \quad , \quad u_i^* = Q_{ij}(t)\, u_j$$

$$V_{ij}^* = Q_{im}(t)\, Q_{jn}(t)\, V_{mn} \tag{7A-9}$$

This implies that frame indifferent tensor quantities satisfy the appropriate tensor transformation law under the change in (spatial) frame. Some of the kinematical quantities introduced previously are frame-indifferent, while others are not. From (7A-7) we have

$$v_i^* = \dot{x}_i^* = \dot{Q}_{ij}\, x_j + Q_{ij}\, v_j + \dot{b}_i \tag{7A-10}$$

Hence, the velocity $\underset{\sim}{v}$ is not frame-indifferent because of the presence of the first and last terms in (7A-10). Inverting $(7A\text{-}7)_1$, we have

$$x_j = Q_{nj}(x_n^* - b_n) \tag{7A-11}$$

Using this in (7A-10), we obtain $\underset{\sim}{v}^*$ as a function of $\underset{\sim}{x}^*$,t:

$$v_i^* = \dot{Q}_{ij}\, Q_{nj}(x_n^* - b_n) + Q_{ij}\, v_j(\underset{\sim}{x}(\underset{\sim}{x}^*,t),t) + \dot{b}_i \tag{7A-12}$$

From (7A-8)

$$\underset{\sim}{Q}\, \dot{\underset{\sim}{Q}}^T + \dot{\underset{\sim}{Q}}\, \underset{\sim}{Q}^T = 0 \tag{*}$$

Defining the quantity

$$\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\, \underset{\sim}{Q}^T \quad , \quad \Omega_{ij} = \dot{Q}_{im}\, Q_{jm} \tag{7A-13}$$

then (*) becomes

$$\underset{\sim}{\Omega} + \underset{\sim}{\Omega}^T = 0 \quad \text{or} \quad \underset{\sim}{\Omega} = -\, \underset{\sim}{\Omega}^T$$

The tensor $\underset{\sim}{\Omega}$ determines the angular velocity of the starred frame. To see this, under (7A-7) we have

$$\underset{\sim}{i}_n^* = Q_{nm}(t)\,\underset{\sim}{i}_m \quad , \quad \underset{\sim}{i}_m = Q_{pm}\,\underset{\sim}{i}_p^*$$

where $\underset{\sim}{i}_n$, $\underset{\sim}{i}_n^*$ are the orthonormal bases associated with the x_i, x_i^* coordinate systems. Note that $\underset{\sim}{i}_n^*$ is time dependent, while $\underset{\sim}{i}_n$ is fixed. Now we have

$$\begin{aligned}\frac{d}{dt}\,\underset{\sim}{i}_n^* &= \dot{Q}_{nm}\,\underset{\sim}{i}_m = \dot{Q}_{nm}\,Q_{pm}\,\underset{\sim}{i}_p^* \\ &= \Omega_{np}\,\underset{\sim}{i}_p^*\end{aligned} \qquad (*)$$

We introduce a vector quantity Ω_i such that

$$\Omega_{np} = e_{npm}\,\Omega_m \quad ; \quad 2\Omega_m = e_{npm}\,\Omega_{np}$$

Then we have from (*)

$$\frac{d}{dt}\,\underset{\sim}{i}_n^* = e_{npm}\,\Omega_m\,\underset{\sim}{i}_p^* \qquad (\dagger)$$

But since $\underset{\sim}{i}_n^*$ is orthonormal, recalling (1C-3) we have

$$e_{npm}\,\underset{\sim}{i}_p^* = e_{mnp}\,\underset{\sim}{i}_p^* = \underset{\sim}{i}_m^* \times \underset{\sim}{i}_n^*$$

Hence, (†) becomes

$$\frac{d}{dt}\,\underset{\sim}{i}_n^* = \Omega_m\,\underset{\sim}{i}_m^* \times \underset{\sim}{i}_n^* = \underset{\sim}{\Omega} \times \underset{\sim}{i}_n^* \qquad (7A\text{-}14)$$

From the theory of rotating coordinates, then Ω_i is the angular velocity of the starred frame. Now using (7A-13) in (7A-12), we find

$$v_i^*(\underset{\sim}{x}^*,t) = Q_{ij}\ v_j(\underset{\sim}{x}(\underset{\sim}{x}^*,t),t) + \Omega_{ij}(x_j^* - b_j) + \dot{b}_i \qquad (7A\text{-}15)$$

Note that in (7A-15) $\underset{\sim}{Q}\ \underset{\sim}{v}$ are the components of $\underset{\sim}{v}$ in the * frame, $\underset{\sim}{\Omega}(\underset{\sim}{x}^* - \underset{\sim}{b})$ represent the effect of the rotation of the * frame, and $\dot{\underset{\sim}{b}}$ represents the effect of the translation of the * frame. Now consider the velocity gradients relative to the * frame:

$$\frac{\partial v_i^*}{\partial x_j^*} = Q_{im}\frac{\partial v_m}{\partial x_j^*} + \Omega_{ij} \qquad (*)$$

From (7A-11)

$$\frac{\partial x_j}{\partial x_n^*} = Q_{nj} \qquad (7A\text{-}16)$$

and

$$\frac{\partial v_m}{\partial x_j^*} = \frac{\partial v_m}{\partial x_n}\frac{\partial x_n}{\partial x_j^*} = Q_{jn}\frac{\partial v_m}{\partial x_n}$$

Hence, (*) becomes

$$\frac{\partial v_i^*}{\partial x_j^*} = Q_{im}\ Q_{jn}\frac{\partial v_m}{\partial x_n} + \Omega_{ij} \qquad (7A\text{-}17)$$

This implies that while the velocity gradients $v_{i,j}$ are the components of a 2nd order tensor in the spatial system, $v_{i,j}$ is not frame-indifferent. By definition of the stretching tensor d_{ij} and spin tensor w_{ij}, we can show from (7A-17)

$$\begin{aligned} d_{ij}^* &= Q_{im}\ Q_{jn}\ d_{mn} \\ w_{ij}^* &= Q_{im}\ Q_{jn}\ w_{m,n} + \Omega_{ij} \end{aligned} \qquad (7A\text{-}18)$$

which imply that d_{ij} is frame-indifferent, while w_{ij} is not. Based on (7A 7), we can also show

$$\frac{\partial x_i^*}{\partial X_K} = Q_{ij}\,\frac{\partial x_j}{\partial X_K}\ , \quad C_{KM}^* = C_{KM}\ , \quad E_{KM}^* = E_{KM} \qquad (7A\ 19)$$

Hence, $X_{i,K}$, C_{KM} and E_{KM} are frame indifferent.

Now consider the temperature field θ. From the material description $\theta(\underset{\sim}{X},t)$, the value of θ in the * frame follows from (7A 7):

$$\theta(\underset{\sim}{X},t) = \theta(\underset{\sim}{X},t^*+a) = \theta^*(\underset{\sim}{X},t^*) \qquad (7A\ 20)$$

Hence, the material gradient of θ is

$$\theta_{,K}(\underset{\sim}{X},t) = \theta^*_{,K}(\underset{\sim}{X},t^*) \qquad (7A\ 21)$$

These results imply that θ and $\theta_{,K}$ transform as scalars under the change in frame, and hence are frame indifferent. Consider θ in the spatial description. From (7A-7)

$$\theta(\underset{\sim}{x},t) = \theta[\underset{\sim}{Q}^T(\underset{\sim}{x}^*\ \underset{\sim}{b})\ t^*+a] = \theta^*(\underset{\sim}{x}^*,t^*) \qquad (7A\text{-}22)$$

Taking the spatial gradient, we find

$$\frac{\partial\theta}{\partial x_i} = \frac{\partial\theta^*}{\partial x_j^*}\,\frac{\partial x_j^*}{\partial x_i} = Q_{ji}\,\frac{\partial\theta^*}{\partial x_j^*} \qquad (*)$$

by (7A 7). Upon inverting (*), we have

$$\frac{\partial\theta^*}{\partial x_i^*} = Q_{ij}\,\frac{\partial\theta}{\partial x_j} \qquad (7A\ 23)$$

Hence, (7A-22) implies θ transforms as a scalar field, while (7A-23) implies $\theta_{,i}$ transforms as a vector under the change in frame, i.e., θ, $\theta_{,i}$ are frame indifferent.

We now state the postulate of

Material Frame Indifference -- The constitutive equations $\psi(S)$, $\eta(S)$, $\underset{\sim}{t}(S)$, $\underset{\sim}{q}(S)$ must be frame indifferent, form-invariant functions of the argument set S for every change in frame of the form (7A-7), i.e.,

$$\begin{aligned} &\psi^* = \psi(S^*) = \psi(S) \quad , \quad \eta^* = \eta(S^*) = \eta(S) \\ &\underset{\sim}{t}^* = \underset{\sim}{t}(S^*) = \underset{\sim}{Q}\,\underset{\sim}{t}(S)\,\underset{\sim}{Q}^T \quad , \quad \underset{\sim}{q}^* = \underset{\sim}{q}(S^*) = \underset{\sim}{Q}\,\underset{\sim}{q}(S) \end{aligned} \qquad (7A\text{-}24)$$

where S^* is obtained by subjecting S to the change in frame (7A-7).

This postulate is an expression of the idea that the values of ψ, η, $\underset{\sim}{t}$ and $\underset{\sim}{q}$ existing in a body undergoing a given process must not be affected by the observer's frame of reference. In the next two sections we consider the ramifications of the postulates of this section on the constitutive equations for thermoelastic solids and heat conducting Stokesian fluids.

B. Thermoelastic Solids

Recalling that the principle of equipresence was satisfied by the initial constitutive assumption for thermoelastic solids (6C-1), we now turn to the restrictions imposed on (6C-14) by the principles of material frame-indifference (MFI) and material symmetry (MS).

From (6C-14), (7A-24) and $(7A-19)_1$, (7A-20), (7A-21), MFI requires that

$$\begin{aligned} \psi^* &= \psi(S_o^*) = \psi(S_o) \quad , \quad \eta^* = \eta(S_o^*) = \eta(S_o) \\ t_{ij}^* &= t_{ij}(S_o^*) = Q_{im}\, Q_{jn}\, t_{mn}(S_o) \\ S_o^* &= \{x^*_{i,K}, \theta^*\} = \{Q_{ij}\, x_{j,K}, \theta\} \\ S_o &= \{x_{i,K}, \theta\} \end{aligned} \tag{7B-1}$$

and

$$\begin{aligned} q_i^* &= q_i(S^*) = Q_{ij}\, q_j(S) \\ S^* &= \{S_o^*, \theta^*_{,K}\} = \{Q_{ij}\, x_{j,K}, \theta, \theta_{,K}\} \end{aligned} \tag{7B-2}$$

for every change in frame (7A-7), i.e., <u>for all orthogonal</u> $\underset{\sim}{Q}(t)$.

We first show that if $(7B-1)_1$ is satisfied, then $(7B-1)_{2,3}$ are automatically satisfied. By the constitutive equation for η $(6C-14)_2$ we have

$$\eta^* = -\frac{\partial\psi^*}{\partial\theta^*}$$

That is,

$$\eta(S_o^*) = -\frac{\partial\psi(S_o^*)}{\partial\theta^*} = -\frac{\partial\psi(S_o)}{\partial\theta} = \eta(S_o)$$

Hence, $(7B\ 1)_2$ is satisfied if $(7B\text{-}1)_1$ is. Now from $(6C\text{-}14)_3$

$$t^*_{ij} = t_{ij}(S^*_o) = \rho \frac{\partial \psi^*}{\partial x^*_{i,K}} x^*_{j,K} \qquad (*)$$

From (7B 1)

$$\frac{\partial \psi^*}{\partial x^*_{i,K}} = \frac{\partial \psi(S^*_o)}{\partial x^*_{i,K}} = \frac{\partial \psi(S_o)}{\partial x^*_{i,K}} = \frac{\partial \psi}{\partial x_{n,P}} \frac{\partial x_{n,P}}{\partial x^*_{i,K}}$$

But inverting $(7A\text{-}19)_1$, we have

$$x_{i,K} = Q_{ji} x^*_{j,K} \qquad (7B\text{-}3)$$

and

$$\frac{\partial x_{n,P}}{\partial x^*_{i,K}} = Q_{jn} \frac{\partial x^*_{j,P}}{\partial x^*_{i,K}} = Q_{jn}\, \delta_{ij}\, \delta_{PK}$$

$$\frac{\partial \psi^*}{\partial x^*_{i,K}} = Q_{in} \frac{\partial \psi}{\partial x_{n,K}}$$

$$\frac{\partial \psi^*}{\partial x^*_{i,K}} x^*_{j,K} = Q_{in}\, Q_{jm} \frac{\partial \psi}{\partial x_{n,K}} x_{m,K}$$

by $(7A\text{-}19)_1$. Hence (*) becomes

$$t_{ij}(S^*_o) = Q_{in}\, Q_{jm}\, \rho \frac{\partial \psi}{\partial x_{n,K}} x_{m,K} = Q_{in}\, Q_{jm}\, t_{mn}(S_o)$$

which implies $(7B\text{-}1)_3$ is satisfied provided $(7B\text{-}1)_1$ is.

Now write the requirement on ψ in expanded form

$$\psi(Q_{ij}\, x_{j,K}, \theta) = \psi(x_{j,K}, \theta) \qquad (7B\ 4)$$

We seek the solution to this equation, i.e., what form must ψ have in order to satisfy (7B-4) for <u>all</u> orthogonal $\underset{\sim}{Q}$? This problem is one in the theory of algebraic invariants. Some pertinent results are given in Appendix A. By definition (7B-4) means that ψ must be an isotropic scalar function of the three vectors $x_{i,1}$, $x_{i,2}$, $x_{i,3}$. By Theorem 1 of Appendix A ψ must be expressible as a function of the inner products $x_{i,K}\, x_{i,M} = C_{KM}$. Hence, the solution to (7B-4) is

$$\psi = \psi(C_{KM},\theta) \tag{7B-5}$$

We note that this form certainly satisfies (7B-1) with $\tilde{S}_o = \{C_{KM}, \theta\}$, i.e., $\psi(\tilde{S}_o^*) = \psi(\tilde{S}_o)$, since by $(7A\text{-}19)_2$ $C^*_{KM} = C_{KM}$. Now from $(6C\text{-}14)_2$

$$\eta = -\frac{\partial\psi}{\partial\theta} = \eta(C_{KM},\theta) \tag{7B-6}$$

To determine the new form of the constitutive equation for $\underset{\sim}{t}$ under (7B-5), we have

$$\frac{\partial\psi}{\partial x_{i,K}} = \frac{\partial\psi}{\partial C_{MN}} \frac{\partial C_{MN}}{\partial x_{i,K}}$$

We can show that

$$\frac{\partial C_{MN}}{\partial x_{i,K}} = \delta_{MK}\, x_{i,N} + \delta_{NK}\, x_{i,M}$$

so that

$$\frac{\partial\psi}{\partial x_{i,K}} = \left(\frac{\partial\psi}{\partial C_{KN}} + \frac{\partial\psi}{\partial C_{NK}}\right) x_{i,N}$$

$$= 2\, \frac{\partial\psi}{\partial C_{NK}}\, x_{i,N} \tag{*}$$

provided ψ is defined such that

$$\frac{\partial\psi}{\partial C_{KN}} = \frac{\partial\psi}{\partial C_{NK}} \qquad (\dagger)$$

Then using (*) in $(6C\text{-}14)_3$, we find

$$t_{ij} = 2\rho \frac{\partial\psi}{\partial C_{KM}} x_{i,K} x_{j,M} \qquad (7B\text{-}7)$$

We can show that angular momentum balance and hence (6C-12) are ensured automatically when ψ is defined such that (†) is satisfied. Hence, the constitutive equations (7B-5,6,7) are forms necessary and sufficient to satisfy the MFI requirements (7B 1).

Alternate forms of (7B-5,6,7) result if S_o is replaced by E_{KM}, θ. By $(7A\text{-}19)_3$ $E^*_{KM} = E_{KM}$ which implies MFI can be satisfied with a different function of E_{KM}, θ:

$$\psi = \psi(E_{KM}, \theta) \qquad (7B\text{-}8)$$

Then (7B-6) is replaced by

$$\eta = -\frac{\partial\psi}{\partial\theta} = \eta(E_{KM}, \theta) \qquad (7B\text{-}9)$$

and since $\frac{\partial\psi}{\partial E_{KM}} = 2\frac{\partial\psi}{\partial C_{KM}}$, (7B-7) is replaced by

$$t_{ij} = \rho \frac{\partial\psi}{\partial E_{KM}} x_{i,K} x_{j,M} \qquad (7B\text{-}10)$$

We note this is quite similar to (5A-3) for isothermal elastic solids, and in fact implies that for the heat conducting case $W(E_{KM})$ is replaced by $\rho_o \psi(E_{KM}, \theta)$. Based on (6C-18) for the incompressible case (7B-7) and (7B-10) are replaced by

$$t_{ij} = -p\delta_{ij} + 2\rho_o \frac{\partial\psi}{\partial C_{KM}} x_{i,K}\, x_{j,M}$$

$$t_{ij} = -p\delta_{ij} + \rho_o \frac{\partial\psi}{\partial E_{KM}} x_{i,K}\, x_{j,M} \qquad (7B\text{-}11a)$$

We now consider the restrictions on q_i: (7B-2). In order to satisfy this define functions $Q_K(S)$ such that

$$q_i(S) = x_{i,K}\, Q_K(S) \qquad (7B\text{-}11)$$

Now (7B 2) requires that $q_i(S^*) = Q_{ij}\, q_j(S)$ But from (7B-11)

$$q_i(S^*) = x^*_{i,K}\, Q_K(S^*)$$

$$= Q_{ij}\, x_{j,K}\, Q_K(S^*) \qquad (*)$$

by $(7A\ 19)_1$ Also by (7B 11)

$$Q_{ij}\, q_j(S) = Q_{ij}\, x_{j,K}\, Q_K(S) \qquad (**)$$

By (*) and (**) (7B 2) becomes

$$Q_{ij}\, x_{j,K}\, Q_K(S^*) = Q_{ij}\, x_{j,K}\, Q_K(S)$$

or

$$Q_{ij}\, x_{j,K}[Q_K(S^*) - Q_K(S)] = 0 \qquad (†)$$

Regarding these equations as linear homogeneous, then since $\det(Q_{ij}\, x_{j,K}) = \det \underset{\sim}{Q} \cdot \det(x_{i,K}) = \pm J \neq 0$, the only solution to (*) is the trivial solution·

$$Q_K(S^*) = Q_K(S)$$

Written out in terms of S, we have

$$Q_K(Q_{ij}\, x_{j,K},\ \theta,\ \theta_{,K}) = Q_K(x_{i,K},\ \theta,\ \theta_{,K}) \qquad (7B\text{-}12)$$

It is easy to show that satisfaction of (7B-12) implies (7B-2) is satisfied. Now (7B-12) must be satisfied for all orthogonal Q_{ij}, i.e., each Q_K must be an isotropic scalar function of the three vectors $x_{i,1}$, $x_{i,2}$, $x_{i,3}$. By Theorem 1 of Appendix A each Q_K must be expressible as a function of the inner products of these vectors, i.e., $x_{i,K}\, x_{j,M} = C_{KM}$. Hence

$$Q_K = Q_K(C_{MN},\ \theta,\ \theta_{,N})$$

Using this in (7B-11), we find

$$q_i = x_{i,K}\, Q_K(C_{MN},\ \theta,\ \theta_{,N}) \qquad (7B\text{-}13)$$

Recalling the restriction (6C-15) imposed on q_i by the entropy product inequality, we find that Q_K must satisfy

$$Q_K(C_{MN},\ \theta,\ 0) = 0 \qquad (7B\text{-}14)$$

Also, in terms of Q_K $(6C\text{-}14)_4$ becomes

$$q_i\, \theta_{,i} = x_{i,K}\, Q_K\, \theta_{,i} = Q_K\, \theta_{,K} \leq 0 \qquad (7B\text{-}15)$$

The restrictions under MFI are now satisfied. The constitutive equations are (7B-5), (7B-6), (7B-7) and (7B-13) with the restriction (7B-14). Alternately, we have (7B-8), (7B-9), (7B-10) and

$$q_i = x_{i,K} \bar{Q}_K(E_{MN}, \theta, \theta_{,N}) \tag{7B-16}$$

[illegible]

where $\bar{Q}_K(E_{MN}, \theta, \theta_{,N}) = Q_K(2E_{MN} + \delta_{MN}, \theta, \theta_{,N})$. These forms are valid for homogeneous, anisotropic materials.

We consider now the restrictions imposed on the constitutive equations of material symmetry. We suppose that the symmetry group of the material is non-empty. Recalling the transformation of material coordinates (7A-1), we have

$$\bar{C}_{KM} = H_{KP} H_{MQ} C_{PQ}, \quad \bar{G}_K = H_{KP} \theta_{,P}$$

or (7B-17)

$$\underset{\sim}{\bar{C}} = \underset{\sim}{H}\, \underset{\sim}{C}\, \underset{\sim}{H}^T, \quad \underset{\sim}{\bar{G}} = \underset{\sim}{H}\, \underset{\sim}{G}$$

where $\underset{\sim}{G} = \theta_{,K} \underset{\sim}{I}_K$. Then applying (7A-6) to (7B-5), (7B-6), (7B-7) we find

$$\psi(\underset{\sim}{H}\, \underset{\sim}{C}\, \underset{\sim}{H}^T, \theta) = \psi(\underset{\sim}{C}, \theta)$$

$$\eta(\underset{\sim}{H}\, \underset{\sim}{C}\, \underset{\sim}{H}^T, \theta) = \eta(\underset{\sim}{C}, \theta) \tag{7B-18}$$

$$\hat{g}(\underset{\sim}{H}\, \underset{\sim}{C}\, \underset{\sim}{H}^T, \theta) = \underset{\sim}{g}(\underset{\sim}{C}, \theta)$$

must be satisfied for all $\underset{\sim}{H} \in \{\underset{\sim}{H}\}$.

Since ψ is a potential function for $\eta, \underset{\sim}{t}$ via (7B-6) and (7B-7), it can be shown that satisfaction of the first of (7B-18) implies satisfaction of the remaining two conditions. We treat only the special case of an <u>isotropic</u> material. Then $\{\underset{\sim}{H}\}$ must be the full orthogonal group, i.e., all orthogonal $\underset{\sim}{H}$. Condition $(7B\text{-}18)_1$ then implies that ψ must be an isotropic scalar function of the symmetric 2nd order tensor $\underset{\sim}{C}$ and the scalar θ. By Theorem 2 of Appendix A, ψ must be expressible as a function of θ and the principal invariants of $\underset{\sim}{C}$:

$$\psi = \psi(\theta, I_C, II_C, III_C) \qquad (7B\text{-}19)$$

Then η and $\underset{\sim}{t}$ must also reduce to functions of the same arguments. From (7B-6)

$$\eta = -\frac{\partial\psi}{\partial\theta} = \eta(\theta, I_C, II_C, III_C) \qquad (7B\text{-}19a)$$

It can be shown that (7B-19) and (7B-7) imply

$$t_{ij} = 2\rho(\alpha_0\, \delta_{KM} + \alpha_1\, C_{KM} + \alpha_2\, C_{KP}\, C_{PM})\, x_{i,K}\, x_{j,M} \qquad (7B\text{-}20)$$

where the coefficients $\alpha_0, \alpha_1, \alpha_2$ are functions of θ, I_C, II_C, III_C and are given in terms of ψ by certain differential relationships.

We note in passing that ψ can also be expressed as a function of the moment invariants.

$$\bar{I}_C = \text{tr}\, \underset{\sim}{C} \quad , \quad \overline{II}_C = \text{tr}\, \underset{\sim}{C}^2 \quad , \quad \overline{III}_C = \text{tr}\, \underset{\sim}{C}^3 \qquad (7B\text{-}21)$$

which are related to the principal invariants by

$$\bar{I}_C = I_C \, , \quad \overline{II}_C = I_C^2 - 2II_C$$
$$\overline{III}_C = I_C^3 - 3I_C \, II_C + 3III_C \tag{7B-22}$$

The 1st two identities follow by definition, while the 3rd can be shown using the <u>Cayley-Hamilton Theorem</u>: Any matrix $\underset{\sim}{A}$ satisfies its own characteristic equation, i.e.

$$\underset{\sim}{A}^3 - I_A \, \underset{\sim}{A}^2 + II_A \, \underset{\sim}{A} - III_A \, \underset{\sim}{I} = 0 \tag{7B-23}$$

Finally, we consider the constitutive equation for heat flux for an isotropic material. We have from (7B-14) that

$$q_i(\bar{S}) = \frac{\partial x_i}{\partial \bar{X}_K} Q_K(\bar{S})$$
$$= H_{KM} \, x_{i,M} \, Q_K(\bar{S})$$

Hence, $(7A\ 6)_4$: $q_i(\bar{S}) = q_i(S)$ requires that

$$H_{KM} \, x_{i,M} \, Q_K(\bar{S}) = x_{i,K} \, Q_K(S)$$

or

$$x_{i,M}[H_{KM} \, Q_K(\bar{S}) - Q_M(S)] = 0$$

Since det $x_{i,M} = J \neq 0$, then

$$Q_K(\bar{S}) = H_{KM} \, Q_M(S)$$

Recall that $S = \{C_{KM}, \theta, \theta_{,K}\}$ and use (7B-17):

$$\underset{\sim}{Q}(\underset{\sim}{H} \, \underset{\sim}{C} \, \underset{\sim}{H}^T, \theta, \underset{\sim}{H} \, \underset{\sim}{G}) = \underset{\sim}{H} \, \underset{\sim}{Q}(\underset{\sim}{C}, \theta, \underset{\sim}{G}) \tag{7B-24}$$

which must hold for all orthogonal $\underset{\sim}{H}$ for an isotropic material. This implies $\underset{\sim}{Q}$ must be an isotropic vector-valued function of a symmetric 2nd order tensor $\underset{\sim}{C}$, a vector $\underset{\sim}{G}$ and a scalar θ. By Theorem 4 of Appendix A, $\underset{\sim}{Q}$ must be expressible in the form

$$\underset{\sim}{Q} = (\phi_0 \underset{\sim}{I} + \phi_1 \underset{\sim}{C} + \phi_2 \underset{\sim}{C}^2) \underset{\sim}{G}$$

or

$$Q_K = (\phi_0 \delta_{KM} + \phi_1 C_{KM} + \phi_2 C_{KP} C_{PM}) \theta_{,M} \qquad (7B-25)$$

where ϕ_0, ϕ_1, ϕ_2 are (nonlinear) functions of θ and the invariants

$$\begin{gathered} I_C, II_C, III_C \\ C_{KM} \theta_{,K} \theta_{,M}, C_{KP} C_{PM} \theta_{,K} \theta_{,M} \end{gathered} \qquad (7B-26)$$

Note that as a result of the assumption of material isotropy, Q_K automatically vanishes with $\theta_{,K}$, provided α_0, α_1, α_2 are bounded functions of $\theta_{,K}$. Also, using (7B-25) in (7B-15) we find

$$(\phi_0 \delta_{KM} + \phi_1 C_{KM} + \phi_2 C_{KP} C_{PM}) \theta_{,K} \theta_{,M} \leq 0 \qquad (7B-27)$$

which is a restriction imposed by the entropy production inequality on the functional form of ϕ_0, ϕ_1, ϕ_2. Finally, we record the form of q_i resulting from (7B-25):

$$q_i = x_{i\ K}(\phi_0 \delta_{KM} + \phi_1 C_{KM} + \phi_2 C_{KP} C_{PM}) \theta_{,M} \qquad (7B-28)$$

We mention that in view of the identities (2C-6) relating the principal invariants of $\underset{\sim}{C}$ and $\underset{\sim}{E}$, the constitutive equations (7A-8), (7B 9), (7B-10), expressed in terms of $\underset{\sim}{E}$, can easily be reduced to forms valid for isotropic materials. The incompressible case (7B-11a) is treated similarly.

C. Heat Conducting Stokesian Fluids

For these materials the constitutive equations were (6C-28). Applying the principle of MFI (7A-24) we have

$$\psi(S_o^*) = \psi(S_o) \quad , \quad \eta(S_o^*) = \eta(S_o) \quad , \quad \pi(S_o^*) = \pi(S_o) \qquad (7C-1)$$

$$S_o^* = \{\tfrac{1}{\rho^*}, \theta^*\} = \{\tfrac{1}{\rho}, \theta\} = S_o$$

$$ {}_Dt^*_{ij} = {}_Dt_{ij}(S^*) = Q_{im}\, Q_{jn}\, {}_Dt_{mn}(S)$$

$$q^*_i = q_i(S^*) = Q_{ij}\, q_i(S) \qquad (7C-2)$$

$$S^* = \{S^*_o, v^*_{i,j}, \theta^*_{,i}\} = \{\tfrac{1}{\rho}, \theta, Q_{im}\, Q_{jn}\, v_{m,n} + \Omega_{ij}, Q_{ij}\, \theta_{,j}\}$$

where (7A-17, 23) have been used in S^*. Since $S_o = S_o^*$, then any function $\psi = \psi(S_o)$ satisfies the MFI requirement $\psi^* = \psi$. Recalling that ψ is a potential function for η and π, then it follows that $\eta^* = \eta$, $\pi^* = \pi$. Consider now the requirement on the dissipative stress ${}_D\underset{\sim}{t}$ and heat flux $\underset{\sim}{q}$. Conditions (7C-2) must hold for all orthogonal $\underset{\sim}{Q}$. In particular, they must hold for the particular choice

$$Q_{ij} = \delta_{ij} \quad , \quad \Omega_{ij} = - w_{ij} \qquad \text{at} \quad t = \bar{t}$$

where $\bar{t}$ is an arbitrary fixed time. Then we find that

$$S^* = \{S_o, v_{i,j} - w_{ij}, \theta_{,i}\} = \{S_o, d_{ij}, \theta_{,i}\}$$

Hence, ${}_{D}\tilde{t}$ and $\tilde{q}$ can depend on $v_{i,j}$ only through the stretching tensor d_{ij}, i.e., we must replace S by $S_1 = \{S_o, d_{ij}, \theta_{,i}\}$. Then by $(7A\text{-}18)_1$, we have

$$S_1^* = \{S_o, d_{ij}^*, \theta^*_{,i}\} = \{S_o, Q_{im} Q_{jn} d_{mn}, Q_{ij} \theta_{,j}\}$$

and conditions (7C-2) are replaced by, using direct notation

$$\begin{aligned} {}_{D}\tilde{t}(\rho^{-1}, \theta, \tilde{Q}\, \tilde{d}\, \tilde{Q}^T, \tilde{Q}\, \tilde{g}) &= \tilde{Q}\; {}_{D}\tilde{t}(\rho^{-1}, \theta, \tilde{d}, \tilde{g})\, \tilde{Q}^T \\ \tilde{q}(\rho^{-1}, \theta, \tilde{Q}\, \tilde{d}\, \tilde{Q}^T, \tilde{Q}\, \tilde{g}) &= \tilde{Q}\, \tilde{q}(\rho^{-1}, \theta, \tilde{d}, \tilde{g}) \end{aligned} \qquad (7C\text{-}3)$$

Here, we have let $\tilde{g} = \text{grad}\ \theta = \theta_{,k}\, \tilde{i}_k$. Necessary conditions for (7C-3) follow by letting $\tilde{Q} = -\tilde{I}$:

$$\begin{aligned} {}_{D}\tilde{t}(\rho^{-1}, \tilde{\theta}, \tilde{d}, -\tilde{g}) &= {}_{D}\tilde{t}(\rho^{-1}, \theta, \tilde{d}, \tilde{g}) \\ \tilde{q}(\rho^{-1}, \tilde{\theta}, \tilde{d}, -\tilde{g}) &= -\tilde{q}(\rho^{-1}, \theta, \tilde{d}, \tilde{g}) \end{aligned} \qquad (7C\text{-}4)$$

The first of these equations implies dissipative stress must be an <u>even</u> function of spatial temperature gradient. The second implies heat flux must be an <u>odd</u> function of spatial temperature gradient. In particular, when $\tilde{g} = 0$

$$\tilde{q}(\rho^{-1}, \theta, \tilde{d}, 0) = 0 \qquad (7C\text{-}5)$$

i.e., regardless of the motion of the fluid, there is no heat flux when $\theta_{,i}$ vanishes. Note that (7C-5) is a stronger requirement on $\tilde{q}$ than $(6C\text{-}29)_2$, derived earlier from the entropy production inequality. Also, there is no result on ${}_{D}\tilde{t}$ similar to (7C-5). The above necessary conditions on ${}_{D}\tilde{t}$, $\tilde{q}$ must be

satisfied by any general solution of (7C-3). Equation $(7C-3)_1$ implies that $\underset{\sim}{D}^t$ is a symmetric 2nd order isotropic tensor function of a vector $\underset{\sim}{q}$ and a symmetric order tensor $\underset{\sim}{d}$. Similarly, $(7C-3)_2$ implies $\underset{\sim}{q}$ must be an isotropic vector function of the same arguments. The solutions follow from Theorems 4 and 5 of Appendix A:

$$q_i = (\kappa_0\ \delta_{ij} + \kappa_1\ d_{ij} + \kappa_2\ d_{im}\ d_{mj})\ \theta_{,j} \qquad (7C-6)$$

$$\begin{aligned} D^t{}_{ij} = \beta_0\ \delta_{ij} + \beta_1\ d_{ij} + \beta_2\ d_{im}\ d_{mj} + \beta_3\ \theta_{,i}\ \theta_{,j} \\ + \beta_4(\theta_{,i}\ d_{jm}\ \theta_{,m} + d_{im}\ \theta_{,m}\ \theta_{,j}) \qquad (7C-7) \\ + \beta_5(\theta_{,i}\ d_{jm}\ d_{mn}\ \theta_{,n} + d_{im}\ d_{mn}\ \theta_{,n}\ \theta_{,j}) \end{aligned}$$

where the κ's and β's are (nonlinear) functions of ρ^{-1}, θ and the invariants

$$\begin{aligned} I_d,\ II_d,\ III_d \quad , \quad \theta_{,i}\ \theta_{,i} \\ d_{ij}\ \theta_{,i}\ \theta_{,j} \quad , \quad d_{im}\ d_{mj}\ \theta_{,i}\ \theta_{,j} \end{aligned} \qquad (7C-8)$$

Note that (7C-6) satisfies (7C-5). The requirement (6C-29) on $\underset{\sim}{D}^t$ implies the coefficient β_0 must satisfy

$$\beta_0 \Big|_{\underset{\sim}{d}=0=\underset{\sim}{q}} = 0 \qquad (7C-9)$$

In addition, the functions (7C-6) and (7C-7) must satisfy the dissipative inequality (6C-26), which we repeat here for reference

$$\delta = D^t{}_{ij}\ d_{ij} - \frac{1}{\theta} q_i\ \theta_{,i} \geq 0 \qquad (7C-10)$$

This places restrictions on the form of the coefficient functions κ and β in (7C-6) and (7C-7).

The total stress t_{ij} follows by adding the term $- \pi \, \delta_{ij}$ to (7C-7). For an incompressible fluid $\rho = \rho_o$, $I_d = 0$ and π is undefined. Then $- \pi + \beta_o$ is also undefined and is replaced by the mechanical pressure p. In addition, I_d drops out of the set of invariants (7C-8).

Consider now the material symmetry requirements. Under a transformation of material coordinates of the form (7A-1), we have

$$\overline{S}_o = S_o \quad , \quad \overline{S}_1 = S_1$$

i.e., all the arguments of S_o, S_1 transform as scalars. Hence, the requirements (7A-6) are satisfied <u>for all orthogonal</u> $\underset{\sim}{H}$. This implies heat conducting Stokesian fluids, as defined by the given constitutive equations, are <u>isotropic</u> materials.

Since the nonlinear constitutive equations (7C-6,7) are difficult to work with in applications, various special cases are of interest. We discuss two of these.

(a) $\underset{\sim}{D}^t$ is independent of $\theta_{,i}$ and $\underset{\sim}{q}$ is independent of $\underset{\sim}{d}$
Then from (7C-6,7,8), we have

$$q_i = \kappa_o \, \theta_{,i} \quad , \quad \kappa_o = \kappa_o(\rho^{-1}, \theta, \theta_{,i} \, \theta_{,i}) \tag{7C-11}$$

$$\begin{aligned} {}_D t_{ij} &= \alpha_0 \, \delta_{ij} + \alpha_1 \, d_{ij} + \alpha_2 \, d_{im} \, d_{mj} \\ \alpha_K &= \alpha_K(\rho^{-1}, \theta, I_d, II_d, III_d) \quad , \quad K = 0,1,2 \end{aligned} \tag{7C-12}$$

with (7C-9) replaced by

$$\alpha_o(\rho^{-1}, \theta, 0, 0, 0) = 0 \tag{7C-13}$$

Note that (7C-10) reduces to the separate inequalities

$$_Dt_{ij}\ d_{ij} \geq 0 \quad , \quad q_i\ \theta_{,i} \leq 0 \tag{7C-14}$$

since $\theta_{,i}$ and d_{ij} are arbitrary at a given place and time. $(7C\text{-}14)_2$ places a restriction on the coefficient function κ_o:

$$\kappa_o(\rho^{-1}, 0, \theta_{,i}\ \theta_{,i}) \leq 0 \tag{7C-15}$$

From (7C-12) we see that when $\underset{\sim}{d}$ is diagonalized, then $\underset{\sim}{t} = - \pi \underset{\sim}{I} + {}_D\underset{\sim}{t}$ is also diagonalized. Hence, for this class of materials, <u>the principal axes of stress and stretching coincide</u>. Now using (7C-12) in $(7C\text{-}14)_1$, we find

$$_Dt_{ij}\ d_{ij} = \alpha_0 \text{ tr } \underset{\sim}{d} + \alpha_1 \text{ tr } (\underset{\sim}{d}^2) + \alpha_2 \text{ tr } (\underset{\sim}{d}^3) \geq 0 \tag{7C-16}$$

Again, this is viewed as a restriction on the form of the coefficient functions α_K. Using (7C-18,19), we record the heat conduction equation from (6C-30):

$$\begin{aligned} \rho\theta\dot{\eta} = {} & \alpha_0 \text{ tr } \underset{\sim}{d} + \alpha_1 \text{ tr } (\underset{\sim}{d}^2) + \alpha_2\ (\underset{\sim}{d}^3) \\ & - (\kappa_0\ \theta_{,i})_{,i} + \rho r \end{aligned} \tag{7C-17}$$

(b) Case (a) with ${}_D\underset{\sim}{t}$ linear in $\underset{\sim}{d}$ and $\underset{\sim}{q}$ linear in $\theta_{,i}$

From (7C-11), (7C-12), for linear constitutive equations

$$\kappa_0 = - \kappa(\rho^{-1},\theta) \quad , \quad \alpha_0 = \lambda(\rho^{-1},\theta)\ I_d \quad , \quad \alpha_1 = 2\mu(\rho^{-1},\theta) \tag{7C-18}$$

with all other coefficients vanishing. Then

$$q_i = -\kappa(\rho^{-1},\theta)\ \theta_{,i} \tag{7C-19}$$

$$D^t{}_{ij} = \lambda(\rho^{-1},\theta)\ I_d\ \delta_{ij} + 2\mu(\rho^{-1},\theta)\ d_{ij} \tag{7C-20}$$

where κ is the thermal conductivity and λ and μ are viscosity coefficients. From (7C-15) κ must satisfy

$$\kappa(\rho^{-1},\theta) \geq 0 \tag{7C-21}$$

Using (7C-17) in (7C-16), we find

$$\lambda\ I_d{}^2 + 2\mu\ \mathrm{tr}\ (\underset{\sim}{d}^2) \geq 0 \tag{7C-22}$$

By transforming this result into the principal axes of $\underset{\sim}{d}$ and using some elementary properties of quadratic forms, we can show that (7C-22) is satisfied, if and only if

$$3\lambda + 2\mu \geq 0 \quad , \quad \mu \geq 0 \tag{7C-23}$$

Finally, the heat conduction equation (7C-17) reduces to

$$\rho\theta\dot{\eta} = \lambda\ I_d{}^2 + 2\mu\ \mathrm{tr}\ (\underset{\sim}{d}^2) + (\kappa\theta_{,i})_{,i} + \rho r \tag{7C-24}$$

where $\eta = -\frac{\partial\psi}{\partial\theta} = \eta(\rho^{-1},\theta)$. Note that this equation is non-linear. An alternate form of (7C-24) is

$$\begin{aligned}\rho\ c_v\ \dot{\theta} = -\ \tilde{c}_\theta\ I_d + \lambda\ I_d{}^2 + 2\mu\ \mathrm{tr}\ (\underset{\sim}{d}^2) \\ + (\kappa\theta_{,i})_{,i} + \rho r\end{aligned} \tag{7C-25}$$

where

$$c_v = \theta \left.\frac{\partial\eta}{\partial\theta}\right|_{\rho^{-1}} = c_v(\rho^{-1},\theta)$$

$$\tilde{c}_\theta = \theta \left.\frac{\partial\eta}{\partial\rho^{-1}}\right|_{\theta} = \tilde{c}_\theta(\theta,\rho^{-1})$$

are the <u>specific heat</u> at constant (specific) volume and the <u>compressibility</u> at constant temperature, respectively.

Appendix A -- Some Results on Isotropic Functions

Definition -- Scalar, vector and 2nd order tensor valued functions ϕ, $\underset{\sim}{f}$, $\underset{\sim}{F}$ of n vectors $\underset{\sim}{u}^{(\alpha)}$ and m 2nd order tensors $\underset{\sim}{A}^{(\alpha)}$ which satisfy, respectively

$$\phi[\underset{\sim}{Q}\,\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{Q}\,\underset{\sim}{A}^{(\alpha)}\,\underset{\sim}{Q}^T] = \phi[\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{A}^{(\alpha)}]$$

$$\underset{\sim}{f}[\underset{\sim}{Q}\,\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{Q}\,\underset{\sim}{A}^{(\alpha)}\,\underset{\sim}{Q}^T] = \underset{\sim}{Q}\,\underset{\sim}{f}[\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{A}^{(\alpha)}] \qquad (A1)$$

$$\underset{\sim}{F}[\underset{\sim}{Q}\,\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{Q}\,\underset{\sim}{A}^{(\alpha)}\,\underset{\sim}{Q}^T] = \underset{\sim}{Q}\,\underset{\sim}{F}[\underset{\sim}{u}^{(\alpha)},\ \underset{\sim}{A}^{(\alpha)}]\,\underset{\sim}{Q}^T$$

for all orthogonal $\underset{\sim}{Q}$, are called isotropic functions.

Theorem 1 -- (Ref: A. L. Cauchy, Mém. Acad. Sci. Paris, Vol. 22, p. 615-654, 1850.)

A scalar valued function ϕ of three vectors $\underset{\sim}{u}^{(\alpha)}$, $\alpha = 1,2,3$ is isotropic if and only if it is expressible as a function of the six inner products $\underset{\sim}{u}^{(\alpha)} \cdot \underset{\sim}{u}^{(\beta)}$, $\alpha,\beta = 1,2,3$

$$\phi = \phi(\underset{\sim}{u}^{(\alpha)} \cdot \underset{\sim}{u}^{(\beta)}) \qquad (A2)$$

Theorem 2 -- (Ref C. Truesdell and W. Noll, Handbuch der Physik, Vol. III/3, p. 28, 1965.)

A scalar valued function ψ of a symmetric 2nd order tensor $\underset{\sim}{A}$ is isotropic if and only if it is expressible as a function of the three principal invariants of $\underset{\sim}{A}$:

$$\psi = \psi(I_A,\ II_A,\ III_A) \qquad (A3)$$

Theorem 3 -- (Preceeding reference, p. 32.)

A symmetric 2nd order tensor function $\underset{\sim}{f}$ of a single symmetric 2nd order tensor $\underset{\sim}{A}$ is isotropic if and only if it is expressible as

$$\underset{\sim}{f} = \alpha_0 \underset{\sim}{I} + \alpha_1 \underset{\sim}{A} + \alpha_2 \underset{\sim}{A}^2 \qquad (A4)$$

where α_0, α_1, α_2 are functions of the principal invariants of $\underset{\sim}{A}$, e.g.

$$\alpha_0 = \alpha_0(I_A, II_A, III_A) \qquad (A5)$$

Theorem 4 -- (Preceeding reference, p. 35.)

A vector-valued function $\underset{\sim}{g}$ of a symmetric 2nd order tensor $\underset{\sim}{A}$ and a vector $\underset{\sim}{u}$ is isotropic if and only if it is expressible as

$$\underset{\sim}{g} = (\phi_0 \underset{\sim}{I} + \phi_1 \underset{\sim}{A} + \phi_2 \underset{\sim}{A}^2) \underset{\sim}{u} \qquad (A6)$$

where ϕ_0, ϕ_1, ϕ_2 are functions of the invariants

$$\begin{aligned} &I_1 = I_A \quad , \quad I_2 = II_A \quad , \quad I_3 = III_A \\ &I_4 = \underset{\sim}{u} \cdot \underset{\sim}{u} \quad , \quad I_5 = \underset{\sim}{u} \cdot \underset{\sim}{A} \underset{\sim}{u} \qquad (A7) \\ &I_6 = \underset{\sim}{u} \cdot \underset{\sim}{A}^2 \underset{\sim}{u} \end{aligned}$$

Theorem 5 -- (Ref: R. S. Rivlin and J. L. Ericksen, J. Rational Mech. & Anal., Vol. 4, p. 323-425, 1955.)

A symmetric 2nd order tensor valued function $\underset{\sim}{H}$ of a symmetric 2nd order tensor $\underset{\sim}{A}$ and a vector $\underset{\sim}{u}$ is isotropic if and only if it is expressible as

VIII. Some Exact Solutions for Fluids

A. Newtonian Fluids (Isothermal Flows, Incompressible Materials)

We summarize the governing equations for incompressible Newtonian fluids.

$$v_{i,i} = 0 \qquad (8A-1)$$

$$t_{ij,j} + \rho f_i = \rho \dot{v}_i \qquad (8A-2)$$

$$t_{ij} = - p\delta_{ij} + 2\mu\ d_{ij} \qquad (8A-3)$$

$$\mu = \mu(\theta) \geq 0 \qquad (8A-4)$$

$$\rho\ c_v(\theta)\ \dot{\theta} = 2\mu(\theta)\ tr(\underset{\sim}{d}^2) + \rho r \qquad (8A-5)$$

$$c_v = c_v(\theta)$$

The constitutive equations (8A-3) follow from (6C-31),(7C-20) and (8A-4) from (7C-23). We have assumed no heat conduction and incompressibility in obtaining (8A-5) from (7C-25). If (8A-3) is substituted into (8A-2), we obtain the <u>Navier-Stokes equations</u> (Navier 1827, Stokes 1845):

$$\mu v_{i,jj} - p_{,i} + \rho f_i = \rho(\frac{\partial v_i}{\partial t} + v_{i,j}\ v_j) \qquad (8A-6)$$

Note that temperature changes are still possible even though heat conduction is non-existent. There is coupling between equations (8A-5) and (8A-6) through the temperature dependence of μ in (8A-6) and the term $2\mu\ tr(\underset{\sim}{d}^2)$ in (8A-5). If the heat source term r vanishes, then for incompressible materials

we can assume changes in θ are small such that μ is approximately constant. Then equations (8A-6) are independent of θ and suffice to determine the velocity field, subject to appropriate boundary conditions, without using (8A-5). Hence, the governing equations for (8A-1) and (8A-6).

The boundary conditions for Newtonian fluids are

$$t_{ij} n_j = \tilde{t}_i \quad \text{on} \quad S_t$$

$$v_i = \tilde{v}_i \quad \text{on} \quad S_v = S - S_t$$

where $\tilde{t}_i$, $\tilde{v}_i$ are prescribed functions. In particular, if S_v is a fixed, solid surface, we have the no-slip condition

$$v_i = 0 \quad \text{on} \quad S_v \tag{8A-7}$$

We record the component forms of the basic equations in rectangular cartesian and cylindrical coordinates.

Rectangular Cartesian Coordinates (x,y,z)

Let $\underset{\sim}{v} = (u,v,w)$.

Continuity Equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{8A-8}$$

Navier-Stokes Equations

$$\mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) - \frac{\partial p}{\partial x} + \rho f_x = \rho(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z})$$

$$\mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}) - \frac{\partial p}{\partial y} + \rho f_y = \rho(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}) \quad (8A-9)$$

$$\mu(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}) - \frac{\partial p}{\partial z} + \rho f_z = \rho(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z})$$

Constitutive Equations

$$t_{xx} = -p + 2\mu \frac{\partial u}{\partial x} \quad , \quad t_{yy} = -p + 2\mu \frac{\partial v}{\partial y}$$

$$t_{zz} = -p + 2\mu \frac{\partial w}{\partial z} \quad , \quad t_{xy} = \mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \quad (8A-10)$$

$$t_{xz} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \quad , \quad t_{yz} = \mu(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y})$$

Linear Momentum

$$\frac{\partial t_{xx}}{\partial x} + \frac{\partial t_{xy}}{\partial y} + \frac{\partial t_{xz}}{\partial z} + \rho f_x = \rho(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z})$$

$$\frac{\partial t_{xy}}{\partial x} + \frac{\partial t_{yy}}{\partial y} + \frac{\partial t_{yz}}{\partial z} + \rho f_y = \rho(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z}) \quad (8A-11)$$

$$\frac{\partial t_{xz}}{\partial x} + \frac{\partial t_{yz}}{\partial y} + \frac{\partial t_{zz}}{\partial z} + \rho f_z = \rho(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z})$$

Cylindrical Coordinates (r,θ,z)

Let $\underset{\sim}{v} = (u,v,w)$. The following equations are given in terms of the physical components of the tensor quantities.

Continuity Equation

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z} = 0 \quad (8A-12)$$

Navier-Stokes Equations

$$\begin{aligned}
&\mu\{\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(ru)] + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} - \frac{2}{r^2}\frac{\partial v}{\partial\theta} + \frac{\partial^2 u}{\partial z^2}\} \\
&\qquad - \frac{\partial p}{\partial r} + \rho f_r = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}) \\
&\mu\{\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(rv)] + \frac{1}{r^2}\frac{\partial^2 v}{\partial\theta^2} + \frac{2}{r^2}\frac{\partial u}{\partial\theta} + \frac{\partial^2 v}{\partial z^2}\} \\
&\qquad - \frac{1}{r}\frac{\partial p}{\partial\theta} + \rho f_\theta = \rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + \frac{uv}{r} + w\frac{\partial v}{\partial z}) \\
&\mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 w}{\partial\theta^2} + \frac{\partial^2 w}{\partial z^2}] - \frac{\partial p}{\partial z} \\
&\qquad + \rho f_z = \rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial\theta} + w\frac{\partial w}{\partial z})
\end{aligned} \tag{8A-13}$$

Constitutive Equations

$$\begin{aligned}
t_{rr} &= -p + 2\mu\frac{\partial u}{\partial r}, \quad t_{\theta\theta} = -p + 2\mu(\frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r}) \\
t_{zz} &= -p + 2\mu\frac{\partial w}{\partial z}, \quad t_{r\theta} = \mu[r\frac{\partial}{\partial r}(\frac{v}{r}) + \frac{1}{r}\frac{\partial u}{\partial\theta}] \\
t_{\theta z} &= \mu(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial\theta}), \quad t_{rz} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r})
\end{aligned} \tag{8A-14}$$

Linear Momentum

$$\begin{aligned}
&\frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta} + \frac{\partial t_{rz}}{\partial z} + \frac{1}{r}(t_{rr} - t_{\theta\theta}) + \rho f_r = \\
&\qquad \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial\theta} - \frac{v^2}{r} + w\frac{\partial u}{\partial z}) \\
&\frac{\partial t_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial\theta} + \frac{\partial t_{\theta z}}{\partial z} + \frac{2}{r}t_{r\theta} + \rho f_\theta = \\
&\qquad \rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial\theta} + \frac{uv}{r} + w\frac{\partial v}{\partial z})
\end{aligned} \tag{8A-15}$$

$$\frac{\partial t_{rz}}{\partial r} + \frac{1}{r}\frac{\partial t_{\theta z}}{\partial \theta} + \frac{\partial t_{zz}}{\partial z} + \frac{1}{r} t_{rz} + \rho f_z =$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + w\frac{\partial w}{\partial z}\right)$$

Stretching Tensor

$$d_{rr} = \frac{\partial u}{\partial r} , \quad d_{\theta\theta} = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} , \quad d_{zz} = \frac{\partial w}{\partial z}$$

$$d_{r\theta} = \frac{1}{2}\left[r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial \theta}\right] = d_{\theta r}$$

$$d_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right) = d_{zr} \tag{8A-16}$$

$$d_{\theta z} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{1}{r}\frac{\partial w}{\partial \theta}\right) = d_{z\theta}$$

1. Couette Flow in a Channel

We consider the steady flow of an incompressible Newtonian fluid in a channel between two infinite, parallel plates, one fixed and the other moving at velocity U. (See Fig. VIII-1). We assume zero body forces and a velocity field of the form

$$\underset{\sim}{v} = (u(y), 0, 0) \tag{8A-17}$$

Then the continuity equation (8A-8) is satisfied, and the Navier-Stokes equations (8A-9) give

$$\mu\frac{d^2u}{dy^2} - \frac{\partial p}{\partial x} = 0 \tag{8A-18}$$

$$\frac{\partial p}{\partial y} = 0 = \frac{\partial p}{\partial z}$$

Hence, p = p(x) and (8A-18) implies

$$\mu\, u''(y) = \frac{dp}{dx} = \text{constant} \quad , \quad (\ )' = \frac{d}{dy}$$

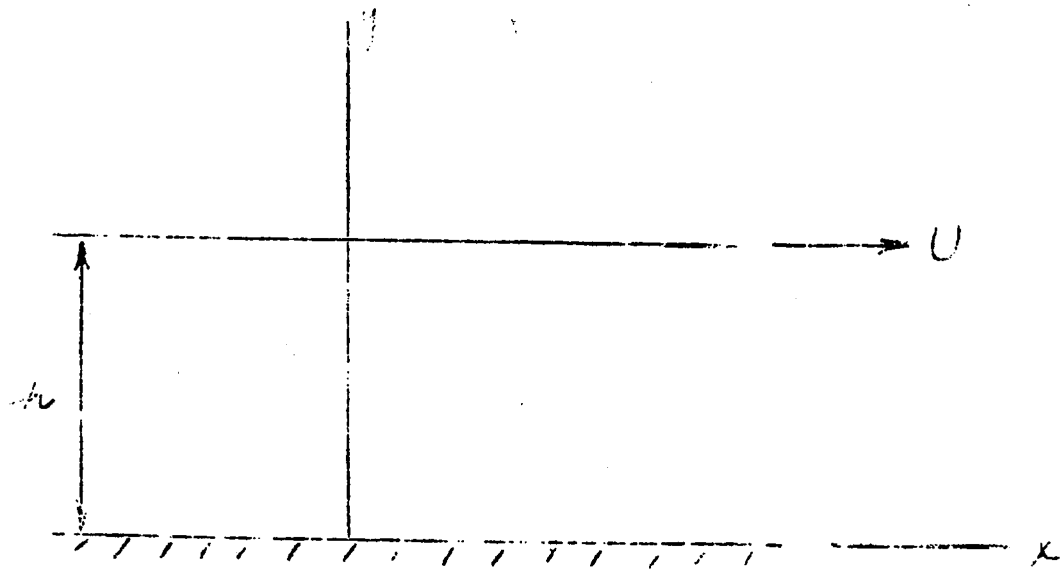


Fig. VIII-1

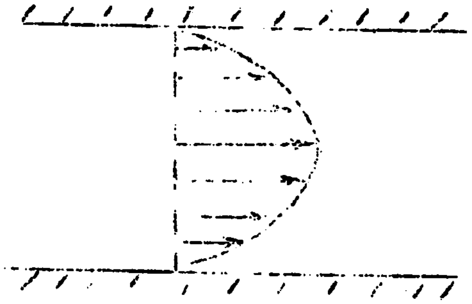


Fig. VIII-2

Integrating, we find

$$\mu\ u(y) = \frac{1}{2}\frac{dp}{dx}\ y^2 + C_1\ y + C_2 \qquad (*)$$

The boundary conditions are

$$u(0) = 0 \quad , \quad u(h) = U \qquad (8A-19)$$

Applying these conditions to (*), we find

$$C_2 = 0 \quad , \quad C_1 = \frac{\mu U}{h} - \frac{1}{2}\frac{dp}{dx}\ h$$

Hence, (*) becomes

$$u(y) = \frac{U}{h}\ y - \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\ (1 - \frac{y}{h}) \qquad (8A-20)$$

There are two special cases.

(a) Upper plate fixed, U = 0.

Then the general solution reduces to

$$u(y) = -\ \frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}\ (1 - \frac{y}{h}) \qquad (8A-21)$$

and the flow is induced by the pressure gradient, regarded as given. The velocity profile (8A-21) is parabolic. (See Fig. VIII-2). This simple solution suggests that μ could be determined experimentally by measuring velocity under a known pressure gradient.

(b) Simple Couette Flow

Let the pressure gradient vanish. Then the flow is induced by the motion of the upper plate. From (8A-20) we have the linear velocity profile

$$u(y) = \frac{U}{h} y \qquad (8A-22)$$

The quantity $\frac{U}{h}$ is called the <u>shear</u> <u>rate</u>. Note that u is independent of μ, so that this solution exists for all incompressible Newtonian fluids. If the stresses are calculated using (8A-10) we find

$$\begin{aligned} t_{xx} &= t_{yy} = t_{zz} = - p = \text{const.} \\ t_{xz} &= 0 = t_{yz} \quad , \quad t_{xy} = \mu \frac{U}{h} \end{aligned} \qquad (8A-23)$$

so that a measure of viscosity here is the in-plane force applied to the upper plane, which in fact produces the plate velocity U. The components of the stress vector acting on the fluid at the surface $y = h$ are given by $t_i = t_{ij} n_j$ where $\underset{\sim}{n} = (0,1,0)$. Hence,

$$t_i = t_{iy} \quad \text{at} \quad y = h$$

Then (8A-23) implies

$$t_x = t_{xy} = \mu \frac{U}{h} \; , \; t_y = t_{yy} = - p \; , \; t_z = t_{zy} = 0 \qquad (8A-24)$$

The pressure p corresponds to a normal force applied to the upper plate. If this force vanishes, then p = 0, and the normal stresses t_{xx}, t_{yy}, t_{zz} all vanish throughout the flow.

For the general case of (8A-20), we define a dimensionless pressure gradient

$$P = \frac{h^2}{2\mu U}\left(-\frac{dp}{dx}\right)$$

Then (8A-20) becomes

$$u(y) = \frac{U}{h}y + P\,U\,\frac{y}{h}\left(1 - \frac{y}{h}\right) \qquad (8A\text{-}25)$$

When $P > 0$, i.e., $\frac{dp}{dx} < 0$ implying pressure decreasing in the +x direction, then u is positive over the entire width of the channel. For $P < -1$, i.e., $\frac{dp}{dx} > 0$, pressure increasing with +x, then u is negative near the lower fixed plate. This "back-flow" is caused by the "adverse" pressure gradient which overcomes the dragging effect of the upper plate. These results are shown in the figure from "Boundary Layer Theory", H. Schlichting, McGraw-Hill, 1960, pg. 68.

2. Poiseuille Flow in a Pipe

We consider the steady flow of an incompressible fluid in a pipe under zero body forces. Let z be defined along the axis of the pipe with (r,θ) defined in a plane cross section. Then we assume an axisymmetric velocity field of the form

$$\underset{\sim}{v} = (0,\ 0,\ v(r))$$

has some importance in the hydrodynamic theory of lubrication. The flow in the narrow clearance between journal and bearing is, by and large, identical with Couette flow with a pressure gradient (*cf.* Sec. VIc).

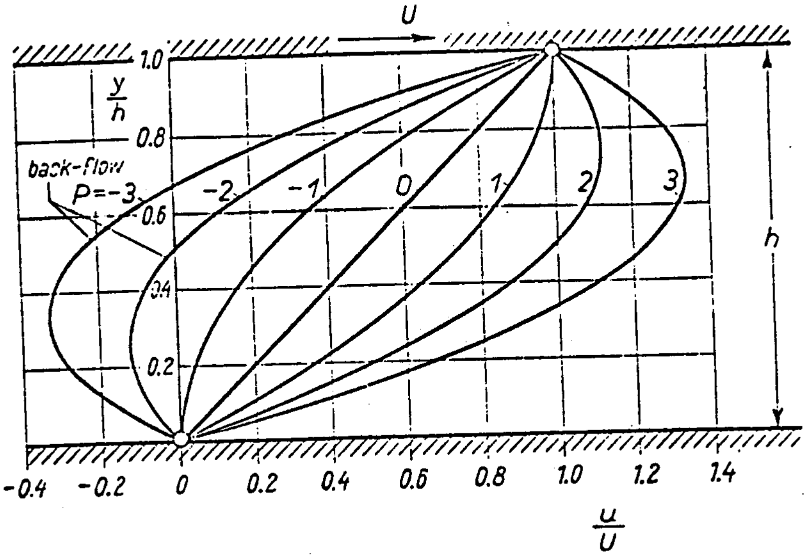


Fig. 5.2. Couette flow between two parallel flat walls
$P > 0$, pressure decrease in direction of wall motion; $P < 0$, pressure increase; $P = 0$, zero pressure gradient

2. The Hagen-Poiseuille theory of flow through a pipe. The flow through a straight tube of circular cross-section is the case with rotational symmetry which corresponds to the preceding case of two-dimensional flow through a channel. Let the x-axis be selected along the axis of the pipe, Fig. 1.2, and let y denote the radial co-ordinate measured from the axis outwards. The velocity components in the tangential and radial directions are zero; the velocity component parallel to the axis, denoted by u, depends on y alone, and the pressure is constant in every cross-section. Of the three Navier-Stokes equations in cylindrical co-ordinates, eqns. (3.33), only the one for the axial direction remains, and it simplifies to

$$\mu\left(\frac{d^2u}{dy^2}+\frac{1}{y}\frac{du}{dy}\right)=\frac{dp}{dx}\,, \qquad (5.6)$$

the boundary condition being $u = 0$ for $y = R$. The solution of eqn. (5.6) gives the velocity distribution

$$u\,(y)=-\frac{1}{4\,\mu}\frac{dp}{dx}\,(R^2-y^2)\,, \qquad (5.7)$$

where $-dp/dx = (p_1-p_2)/l$ = const is the pressure gradient, to be regarded as given. Solution (5.7), which was obtained here as an exact solution of the Navier-

Then the continuity equation (8A-12) is satisfied identically, while the Navier-Stokes equations (8A-13) yield

$$\frac{\partial p}{\partial r} = 0 = \frac{\partial p}{\partial \theta}$$

which implies p = p(r) and

$$\mu(w'' + \frac{1}{r} w') = \frac{dp}{dz} = \text{constant} \quad , \quad (\;)' = \frac{d}{dr} \qquad (*)$$

To integrate (*), write

$$\frac{1}{r} (r w')' = \frac{1}{\mu} \frac{dp}{dz}$$

Then we have

$$rw' = \frac{1}{\mu} \frac{dp}{dz} (\frac{1}{2} r^2 + C_1)$$

$$w' = \frac{1}{\mu} \frac{dp}{dz} (\frac{1}{2} r + \frac{C_1}{r})$$

$$w(r) = \frac{1}{\mu} \frac{dp}{dz} (\frac{1}{4} r^2 + C_1 \ln r + C_2) \qquad (**)$$

The boundary conditions are

$$w(a) = 0 \quad , \quad w(0) \text{ bounded}$$

where a is the radius of the pipe. These conditions imply

$$C_1 = 0 \quad , \quad C_2 = - \frac{1}{4\mu} \frac{dp}{dz} a^2$$

Hence, (**) becomes

$$w(r) = - \frac{1}{4\mu} \frac{dp}{dz} (a^2 - r^2) \qquad (8A-26)$$

Note that the flow is induced by the pressure gradient $\frac{dp}{dz}$. The velocity distribution is a paraboloid of revolution, with maximum velocity at $r = 0$:

$$w_{max} = - \frac{1}{4\mu} \frac{dp}{dz} a^2$$

The solution (8A-26) again suggests a simple experiment for the determination of the viscosity μ. It turns out that this can be done for small pipe diameters and small velocities. For large diameters the velocity profile is observed to be nearly uniform. In addition, the flow must remain laminar, i.e. without fluctuations of the flow field. It is found that the flow is laminar as long as the Reynold's number R is less than a critical value:

$$R = \frac{\bar{w}d}{\nu} < R_c = 2300$$

where $\bar{w} = \frac{1}{2} w_{max}$ is the mean velocity, d is the pipe diameter and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

3. Couette Flow Between Rotating Cylinders

We consider the steady flow of an incompressible fluid between two concentric cylinders rotating at different, constant angular velocities. Choosing cylindrical coordinates r, θ, z as shown in Fig. VIII-3, we assume an axisymmetric solution such that

$$\underset{\sim}{v} = (0, v(r), 0) \quad , \quad p = p(r)$$

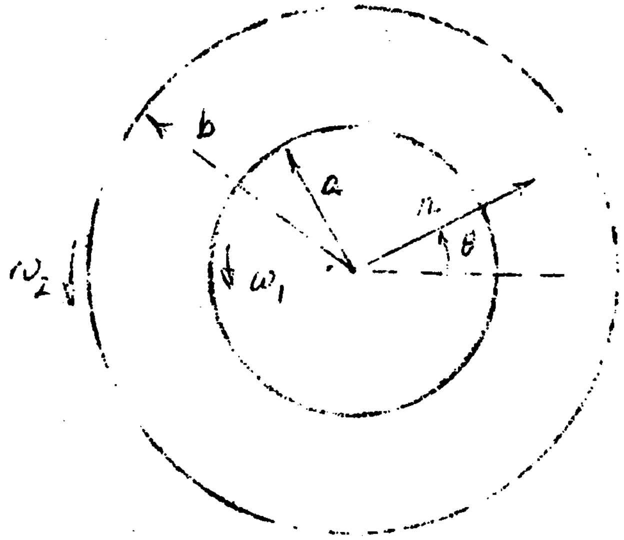


Fig. VIII-3

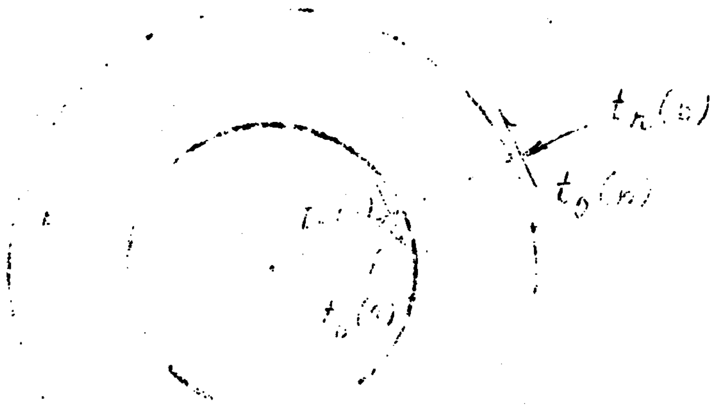


Fig. VIII-4

Then the continuity equation (8A-12) is satisfied identically, while the Navier-Stokes equations (8A-13) give

$$p' = \rho \frac{v^2}{r} \quad , \quad (\ )' = \frac{d}{dr} \tag{8A-27}$$

$$v'' + \frac{1}{r} v' - \frac{v}{r^2} = 0 \tag{8A-28}$$

Integrating (8A-28), we find

$$v(r) = \frac{1}{2} C_1 r + \frac{C_2}{r} \tag{*}$$

The boundary conditions are

$$v(a) = a\,\omega_1 \quad , \quad v(b) = b\,\omega_2 \tag{8A-29}$$

From (*), we find

$$\frac{1}{2} C_1 = \frac{b^2\omega_2 - a^2\omega_1}{b^2 - a^2} \quad , \quad C_2 = -\frac{a^2b^2(\omega_2 - \omega_1)}{b^2 - a^2} \tag{8A-30}$$

and v(r) becomes

$$v(r) = \frac{b^2\omega_2 - a^2\omega_1}{b^2 - a^2}\, r - \frac{a^2b^2(\omega_2 - \omega_1)}{b^2 - a^2}\, \frac{1}{r} \tag{8A-31}$$

Note that as in plane Couette flow v(r) is independent of the viscosity μ. The pressure is determined from (8A-27) and (8A-31):

$$p' = \frac{\rho}{r} \left(\frac{1}{2} C_1 r + \frac{C_2}{r}\right)^2$$

$$p(r) = \frac{1}{8} \rho C_1{}^2 r^2 + \rho C_1 C_2 \ln r - \frac{\rho C_2{}^2}{2r^2} + C_3 \qquad (8A-32)$$

where C_1, C_2 are given by (8A-30). The stresses follow from (8A-14):

$$t_{rr} = t_{\theta\theta} = t_{zz} = -p \quad , \quad t_{\theta z} = t_{rz} = 0 \qquad (8A-33)$$

and

$$t_{r\theta} = ur \left(\frac{v}{r}\right)' = -\frac{2\mu C_2}{r^2} \qquad (8A-34)$$

The stress vectors acting on the fluid at $r = a,b$ are given by $t_i = t_{ij} n_j$. At $r = b$, $\underset{\sim}{n} = (1,0,0)$ and $t_i = t_{ir}(b)$, implying

$$t_r(b) = t_{rr}(b) = -p(b) \quad , \quad t_z = t_{rz}(b) = 0$$

$$t_\theta(b) = t_{r\theta}(b) = -\frac{2\mu C_2}{b^2} \qquad (8A-35)$$

The distribution of stress vectors is shown in Fig. 4. Now consider the torque acting on the fluid by the outer cylinder. The moment of the force t_θ dS is $t_\theta b$ dS, where dS is the element of lateral surface area on the cylinder $r = b$. Hence, the torque is

$$M_2 = t_\theta(b)\, b(2\pi b)h$$

where h is the length of the outer cylinder. Hence, from (8A-35) we find

$$M_2 = -\frac{2\mu C_2}{b^2}(2\pi b^2)h = 4\pi\mu h\,\frac{a^2b^2(\omega_2 - \omega_1)}{b^2 - a^2} \tag{8A-36}$$

Similarly, at $r = a$, $\underset{\sim}{n} = (-1,0,0)$ and

$$t_r(a) = -t_{rr}(a) = p(a) \quad , \quad t_z = 0$$
$$t_\theta(a) = -t_{r\theta}(a) = \frac{2\mu C_2}{a^2} \tag{8A-37}$$

$$M_1 = -4\pi\mu h\,\frac{a^2b^2(\omega_2 - \omega_1)}{b^2 - a^2} = -M_2 \tag{8A-38}$$

Note for $\omega_2 > \omega_1$, M_1 is negative and M_2 positive. Since M_1 is torque <u>exerted on the fluid</u> by the inner cylinder, an equal and opposite torque must be applied to the inner cylinder in order to maintain its constant angular velocity ω_1 and to resist the dragging effect of the fluid. A measure of viscosity μ in this solution is the torque which must be applied to the cylinders. We consider two special cases.

(a) Inner cylinder at rest

Then $\omega_1 = 0$ and

$$v(r) = \frac{b^2\omega_2}{b^2-a^2}\,r - \frac{a^2b^2\,\omega_2}{b^2-a^2}\,\frac{1}{r}$$
$$M_2 = 4\pi\mu h\,\frac{a^2b^2}{b^2-a^2}\,\omega_2 \tag{8A-39}$$

If in addition $a \to 0$, then $v(r) = r\omega_2$, $M_2 = 0$, i.e., the fluid rotates with the outer cylinder as a rigid body.

(b) **Rotating Cylinder in an Infinite Fluid**

First, consider the case when the outer cylinder is at rest. Then $\omega_2 = 0$ and (8A-30) implies

$$\frac{1}{2} C_1 = -\frac{a^2\omega_1}{b^2-a^2} \quad , \quad C_2 = \frac{a^2b^2\omega_1}{b^2-a^2} = \frac{a^2\omega_1}{1 - \frac{a^2}{b^2}} \qquad (*)$$

The solution for $v(r)$, $p(r)$, $t_{r\theta}(r)$ and M_1 follows from (8A-31,32,34,38). Now letting $b \to \infty$ in (*), we have

$$C_1 = 0 \quad , \quad C_2 = a^2\omega_1$$

and

$$v(r) = \frac{a^2\omega_1}{r}$$

$$p(r) = -\frac{\rho a^4\omega_1^2}{2r^2} + C_3$$

$$t_{r\theta} = -\frac{2\mu a^2\omega_1}{r^2} \qquad (8A\ 40)$$

$$M_1 = 4\pi\ \mu\ h\ a^2\ \omega_1$$

This case corresponds to the velocity distribution produced by a line vortex in a non-viscous fluid.

4. Suddenly Accelerated Plane Wall

We consider an incompressible fluid in a non-steady flow generated by an infinite flat plate which is suddenly accelerated at t=0. (See Fig. VIII-5). We assume zero body forces and a velocity field of the form

$$\underset{\sim}{v} = (u(y,t),\ 0,\ 0)$$

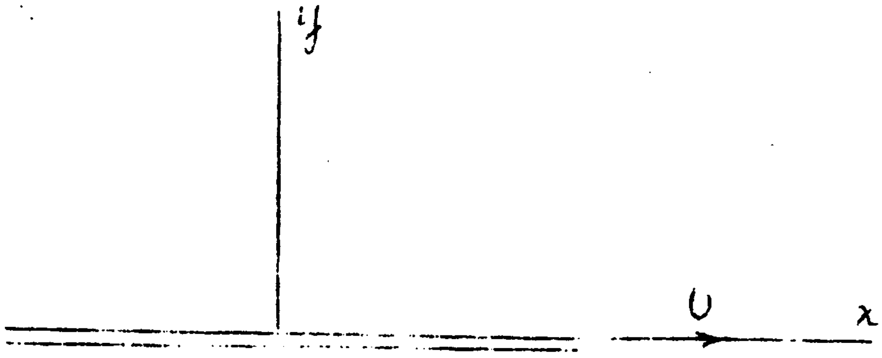


Fig. VIII-5

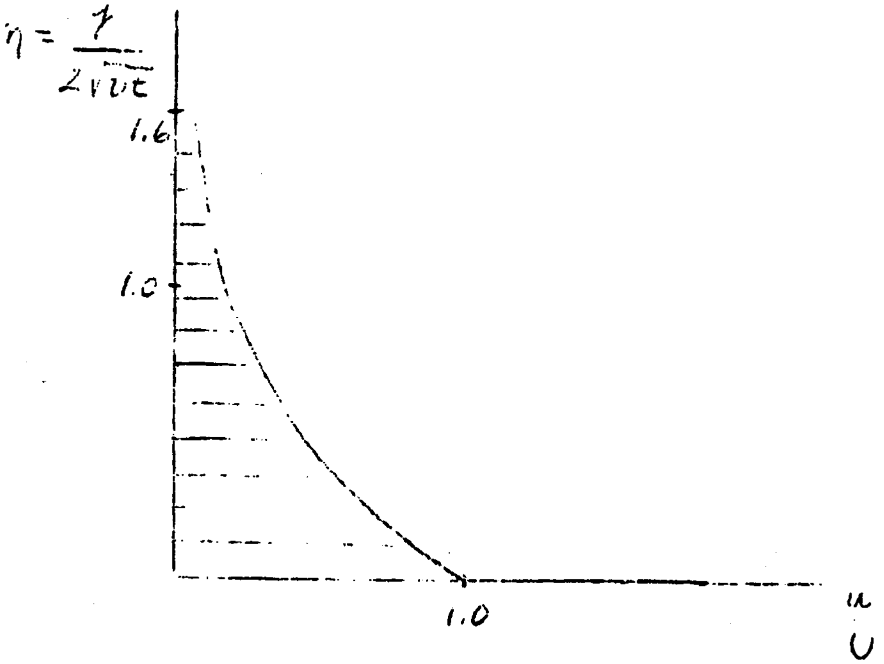


Fig. VIII-6

Then the continuity equation (8A-8) is identically satisfied, and the Navier-Stokes equations (8A-9) yield

$$\frac{\partial p}{\partial y} = 0 = \frac{\partial p}{\partial z}$$

which implies p = p(x) and

$$\mu \frac{\partial^2 u}{\partial y^2} - \frac{dp}{dx} = \rho \frac{\partial u}{\partial t} \tag{8A-41}$$

We assume that the pressure gradient $\frac{dp}{dx}$ vanishes, so that the flow is induced solely by the plate motion. Then (8A-21) yields

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \tag{8A-42}$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. The boundary and initial conditions are

$$u(y,0) = 0 \quad , \quad y > 0$$

$$u(0,t) = U \quad , \quad t > 0 \tag{8A-43}$$

$$u(y,t) \text{ bounded for } y \to \infty \quad , \quad t > 0$$

Equation (8A-42) is identical to the classical one-dimensional heat conduction equation, and conditions (8A-43) correspond to suddenly heated wall at y=0. We solve this problem by a similarity transformation:

$$u = \tilde{u}(\eta) \quad , \quad \eta = \frac{y}{2\sqrt{\nu t}}$$

Then

$$\frac{\partial u}{\partial t} = - \frac{\nu y}{4} (\nu t)^{-3/2} \tilde{u}' \quad , \quad \frac{\partial^2 u}{\partial y^2} = \frac{\tilde{u}''}{4\nu t}$$

and (8A-42) becomes

$$\tilde{u}'' + 2\eta\, \tilde{u}' = 0 \tag{8A-44}$$

with boundary conditions

$$\tilde{u}(0) = U \quad , \quad \tilde{u}(\infty) = 0 \tag{8A-45}$$

We non-dimensionalize by defining

$$v(\eta) = \frac{\tilde{u}(\eta)}{U} \tag{8A-46}$$

Then from (8A-44,45)

$$\begin{gathered} v'' + 2\eta\, v' = 0 \\ v(0) = 1 \quad , \quad v(\infty) = 0 \end{gathered} \tag{8A-47}$$

The solution to this boundary value problem is the complimentary error function erfc, which is a tabulated function. Hence,

$$u(y,t) = U \operatorname{erfc} \eta = U \operatorname{erfc} \frac{y}{2\sqrt{\nu t}} \tag{8A-48}$$

where by definition

$$\operatorname{erfc} \eta = 1 - \operatorname{erf} \eta = 1 - \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\xi^2}\, d\xi$$

and

$$\operatorname{erfc} 0 = 1 \quad , \quad \operatorname{erfc} \infty = 0$$

Note that the initial condition $(8A-43)_1$ is satisfied since for $y > 0$, $t \rightarrow 0$ implies $\eta \rightarrow \infty$. The velocity distribution given by (8A-48) is shown in Fig. VIII-6. Note that the velocity profiles at different times are "similar , i.e., reducible to the same curve by changing the η scale. Consider a fixed time t. At $\bar{t} = 2t$ then

$$\bar{\eta} = \frac{y}{2\sqrt{\nu \bar{t}}} = \frac{1}{\sqrt{2}} \frac{y}{2\sqrt{\nu t}} = \frac{1}{\sqrt{2}} \eta$$

and the curve at time t applies to time 2t if η is replaced by $\sqrt{2}\ \bar{\eta}$ on the graph.

We note that u approaches zero rapidly as η increases. This implies the viscosity of the fluid is predominant "near" the wall. Since erfc $2.0 \cong .01$, then $u \cong .01U$, and if for $\eta = 2.0$ the value of y is δ, then $\delta/2\sqrt{\nu t} = 2.0$ or $\delta = 4\sqrt{\nu t}$. The quantity δ defines the <u>boundary layer thickness</u> and is of order $\sqrt{\nu t}$.

5. Flow Near an Oscillating Flat Plate

Let the plate located at y=0 in the previous case undergo the harmonic motion $U \cos \omega t$. Assuming a solution of the same form:

$$\underset{\sim}{v} = (u(y,t),\ 0,\ 0)$$

we obtain (8A-42) with the boundary conditions

$$u(0,t) = U \cos \omega t \quad , \quad u(y,t) \text{ bounded for } y \rightarrow \infty \qquad (8A-49)$$

An initial condition is not imposed, since we seek a steady-state solution. The solution of the boundary value problem (8A-42) and (8A-49) is known from the theory of heat conduction and is

$$u(y,t) = U\ e^{-ky} \cos(\omega t - ky) \qquad (*)$$

Substitution of this form into (8A-42), we find

$$k = \sqrt{\frac{\omega}{2\nu}}$$

Noting that k has dimensions of inverse length, we let $ky = \sqrt{\frac{\omega}{2\nu}}\ y = \eta$ and obtain from (*)

$$u(\eta,t) = U\ e^{-\eta} \cos(\omega t - \eta) \qquad (8A\text{-}50)$$

For each fixed value of t this velocity profile has the form of a damped harmonic oscillation in the variable η. The amplitude of the oscillation is $U \exp(-\ y \sqrt{\frac{\omega}{2\nu}})$. The fluid layer at a distance y from the wall has a phase lag $\eta = y \sqrt{\frac{\omega}{2\nu}}$ relative to the wall. Two fluid layers a distance $\lambda = 2\pi/k = 2\pi \sqrt{\frac{2\nu}{\omega}}$ apart oscillate in phase. Thus λ can be regarded as a wave length, called the <u>depth of penetration</u> of the wave, and is of order $\sqrt{\frac{\nu}{\omega}}$. A non-dimensional plot of velocity profiles for various values of time are shown in figure from Boundary Layer Theory, H. Schlichting, McGraw-Hill, 1960, pg. 76.

The solution which satisfies the previous boundary conditions as well as the present initial conditions is known from the study of heat conduction [22]. Assuming that at $y = 0$ $u = U_0 \sin nt$ so that at $t = 0$ we have $u = 0$ at $y = 0$ we obtain

$$u(y, t) = U_0 e^{-\eta} \sin(nt - \eta) - \frac{2\nu U_0}{\pi} \int_0^\infty \frac{n}{n^2 + \nu^2 \xi^4} e^{-\nu \xi^2 t} \xi \cos \xi y \, d\xi \qquad (5.26b)$$

where ξ is the variable of integration. The solution is seen to consist of a steady-state term, similar to eqn. (5.26a), and a transient which dies out as $t \to \infty$.

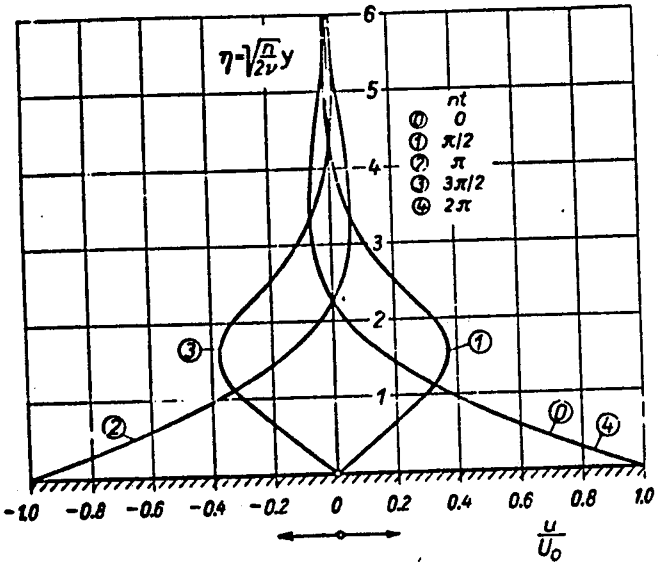


Fig. 5.8. Velocity distribution in the neighbourhood of an oscillating wall

A solution can also be given for the case of a plate oscillating parallel to another plate at a distance h from it. Supposing that the plate at $y = 0$ is at rest and that at $y = h$ performs a harmonic motion given by $U_0 \sin nt$ we obtain

$$\frac{u}{U_0} = A \sin(nt + \phi) - 2\pi\nu \sum_{k=1}^{\infty} \frac{k(-1)^k nh^2}{\nu^2 k^4 \pi^4 - n^2 h^2} \sin \frac{k\pi y}{h} e^{-\nu^2 k^2 \pi^2 t/h^2} \qquad (5.26c)$$

with

$$A = \left[\frac{\cosh 2(n/2\nu)^{1/2} y - \cos 2(n/2\nu)^{1/2} y}{\cosh 2(n/2\nu)^{1/2} h - \cos 2(n/2\nu)^{1/2} h}\right]^{1/2} \qquad (5.26d)$$

and

$$\phi = \arg\left[\frac{\sinh(n/2\nu)^{1/2}(1+i)}{\sinh(n/2\nu)^{1/2}(1-i)}\right]. \qquad (5.26e)$$

The solution consists again of a steady-state term and a transient which dies out with increasing time.

Bodies of various shapes, performing torsional oscillations under the influence of an elastic restoring couple exerted by a suspension wire, have been often used to measure the viscosity of fluids. The viscosity of the fluid is deduced from the period, $T = 2\pi/n$, and from the loga-

B. Stokesian Fluids (Isothermal Flows, Incompressible Materials)

We consider Stokesian or non-Newtonian fluids, i.e., materials having stress constitutive equations which are nonlinear in the stretching tensor $\underset{\sim}{d}$. For incompressible, non-heat conducting materials we have from (6C-31) and (7C-12)

$$t_{ij} = - p\delta_{ij}\ t_D t_{ij} = - p\delta_{ij} + \alpha_1 d_{ij} + \alpha_2 d_{im} d_{mj} \qquad (8B\text{-}1)$$

where α_1, α_2 are functions of II_d, III_d:

$$\begin{aligned} &\alpha_K = \alpha_K(I_1, I_2) \quad , \quad K = 1,2 \\ &I_1 = II_d \quad , \quad I_2 = III_d \end{aligned} \qquad (8B\text{-}2)$$

and subject to the inequality $(7C\text{-}14)_1$:

$$_D t_{ij}\ d_{ij} = \alpha_1\ tr(\underset{\sim}{d}^2) + \alpha_2\ tr(\underset{\sim}{d}^3) \geq 0 \qquad (8B\text{-}3)$$

Note that α_1, α_2 are assumed independent of θ, as in Section A. The governing equations are the continuity equation (8A-1), the linear momentum equations (8A-2) and the constitutive equations (8B-1). We reconsider some of the flows of Section A with the aim of assessing the effect of the nonlinear constitutive equations. Some references for the developments of this section are

J. Serrin, "Mathematical Principles of Classical Fluid Mechanics", Handbuch der Physik, Vol. 8/1, 1959, p. 241-243.

C. Truesdell, "The Mechanical Foundations of Elasticity and Fluid Dynamics", J. Rational Mech. Anal., Vol. 1, 1952, p. 123-130.

A. C. Eringen, "Nonlinear Theory of Continuous Media", McGraw-Hill, 1962, p. 224-232.

1. Plane Couette Flow

Consider a Stokesian fluid between two parallel plates as in Fig. VIII-1. We seek the conditions under which the simple Couette flow solution for Newtonian fluids (8A-22):

$$\underset{\sim}{v} = (ky,\ 0,\ 0) \quad , \quad k = \frac{U}{h} \tag{8B-4}$$

is also a solution for Stokesian fluids. We assume vanishing body forces and that p = constant, so that the flow is induced solely by the plate motion. Recall that the continuity equation (8A-1) is satisfied by (8B-4). Now from (8B-4), we have

$$d_{ij} = v_{(i,j)} = \begin{pmatrix} 0 & \frac{1}{2}k & 0 \\ \frac{1}{2}k & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{8B-5}$$

which implies

$$I_d = 0 \quad , \quad II_d = -\frac{1}{4}k^2 = I_2 \quad , \quad III_d = 0 = I_3 \tag{8B-6}$$

Note that these invariants are constant so that the functions α_1, α_2 are constants in this flow. Also, from (8B-5)

$$d_{im}\, d_{mj} = \begin{pmatrix} \frac{1}{4}k^2 & 0 & 0 \\ 0 & \frac{1}{4}k^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \underset{\sim}{d}^2 \qquad (8B\text{-}7)$$

and hence

$$\text{tr}(\underset{\sim}{d}^2) = \frac{1}{2}k^2 \qquad (8B\text{-}8)$$

$$\text{tr}(\underset{\sim}{d}^3) = I_d^3 - 3I_d\, II_d + 3III_d = 0$$

Then (8B-3) reduces to

$$\frac{1}{2}\alpha_1 k^2 \geq 0 \qquad \text{or} \qquad \alpha_1 \geq 0$$

i.e., α_1 must be a non-negative function of $I_2 = -\frac{1}{4}k^2$ and $I_2 = 0$:

$$\alpha_1(-\tfrac{1}{4}k^2,\ 0) \geq 0 \qquad (8B\text{-}9)$$

Since $III_d = 0$, no restriction is placed on the form of the coefficient α_2 in this flow by the entropy production inequality. Also, (8B-9) is a necessary condition for (8B-3), since it follows from a particular flow. Using (8A-5,7) in (8B-1), the stresses are given by

$$\underset{\sim}{t} = -p\underset{\sim}{I} + \frac{1}{2}\alpha_1 k \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4}\alpha_2 k^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (8B\text{-}1$$

where

$$\alpha_1 = \alpha_1(-\tfrac{1}{4}k^2,\ 0)\ ,\quad \alpha_2 = \alpha_2(-\tfrac{1}{4}k^2,\ 0) \qquad (8B\text{-}11)$$

Note that the stress components are constants. This along with the fact that $\dot{\underset{\sim}{v}} = 0 = \underset{\sim}{f}$ implies that the linear momentum equations (8A-2) or (8A-11) are identically satisfied. Hence, it remains to consider the boundary conditions. The stress vector acting on the fluid at the upper plate has components

$$y = h \quad , \quad t_x = t_{xy} = \frac{1}{2}\alpha_1 k \quad ,$$
$$t_y = t_{yy} = -p + \frac{1}{4}\alpha_2 k^2 \quad , \quad t_z = 0 \qquad (8B\text{-}12)$$

Now the particular functional form of α_1, α_2 depends on the fluid. But in any case, to produce simple Couette flow in a Stokesian fluid, the stress vector components which must be applied to the upper plate are an in-plane component whose magnitude generally depends non-linearly on the shear rate and <u>in addition</u> a normal component which exceeds the pressure by amount

$$\frac{1}{4} k^2 \alpha_2(-\frac{1}{4} k^2, 0)$$

This extra stress is tensile wherever α_2 is a positive function, and it is inferred for this case that in the absence of the extra stress the plates would tend to move together. This is called the <u>Poynting effect</u> and is a consequence of the nonlinearity of the Stokesian constitutive equations. (Recall (8A-29) for Newtonian fluids.) In the simplest case when α_1, α_2 are constants independent of k, then t_x is linear in shear rate, while the extra normal stress is quadratic. Since Stokesian fluids involve two

viscosity functions, α_1 and α_2, rather than two constants as for Newtonian fluids, a single experiment does not suffice to determine these parameters. Instead, each simple solution obtained, with corresponding experiment, will determine information on α_1 and α_2 for that particular flow.

2. Poiseuille Flow in a Pipe

We consider an incompressible Stokesian fluid in steady flow through a circular pipe. Body forces are assumed to vanish, and the axisymmetric velocity distribution

$$\underset{\sim}{v} = (0, 0, w(r)) \tag{8B-13}$$

is again assumed. Then recall that the continuity equation (8A-12) is satisfied. From (8A-16) the stretching tensor is given by

$$d_{ij} = \begin{pmatrix} 0 & 0 & \frac{1}{2} w' \\ 0 & 0 & 0 \\ \frac{1}{2} w' & 0 & 0 \end{pmatrix} \tag{8B-14}$$

and hence

$$I_d = 0 \ , \ II_d = - \frac{1}{4} w'^2 = I_1 \ , \ III_d = 0 = I_2 \tag{8B-15}$$

Also, from (8B-14)

$$d_{im} \, d_{mj} = \begin{pmatrix} \frac{1}{4} w'^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} w'^2 \end{pmatrix} = \underset{\sim}{d}^2 \tag{8B-16}$$

which implies

$$\begin{aligned} tr(\underset{\sim}{d}^2) &= \frac{1}{2} w'^2 \\ tr(\underset{\sim}{d}^3) &= I_d^3 - 3I_d II_d + 3III_d = 0 \end{aligned} \qquad (8B\text{-}17)$$

Hence, from (8B-3) we have

$$\frac{1}{2} \alpha_1 \; w'^2 \geq 0 \quad \text{or} \quad \alpha_1(-\frac{1}{4} w'^2, 0) \geq 0 \qquad (8B\text{-}18)$$

As in Couette flow, no restriction is placed on the function α_2 in this flow. Using (8B-14,16) in (8B-1), the stresses are

$$\underset{\sim}{t} = - p\underset{\sim}{I} + \frac{1}{2} \alpha_1 w' \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{4} \alpha_2 w'^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (8B\text{-}19)$$

where

$$\alpha_1 = \alpha_1(-\frac{1}{4} w'^2, 0) \quad , \quad \alpha_2 = \alpha_2(-\frac{1}{4} w'^2, 0) \qquad (8B\text{-}20)$$

Note that the stress components are not constants. Using (8B-19) and the fact that $\dot{\underset{\sim}{v}} = 0 = \underset{\sim}{f}$, the linear momentum equations (8A-15) yield $\frac{\partial p}{\partial \theta} = 0$ which implies $p = p(r,z)$ and

$$\begin{aligned} &\frac{\partial}{\partial r} (- p + \frac{1}{4} \alpha_2 \; w'^2) + \frac{1}{r} (\frac{1}{4} \alpha_2 \; w'^2) = 0 \\ &\frac{\partial}{\partial r} (\frac{1}{2} \alpha_1 \; w') - \frac{\partial p}{\partial z} + \frac{1}{r} (\frac{1}{2} \alpha_1 \; w') = 0 \end{aligned} \qquad (8B\text{-}21)$$

We rewrite these equations in the form

$$-\frac{\partial p}{\partial r} + \frac{1}{r}\left(\frac{1}{4}\, r\, \alpha_2\, w'^2\right)' = 0 \qquad (8B-22)$$

$$-\frac{\partial p}{\partial z} + \frac{1}{r}\left(\frac{1}{2}\, r\, \alpha_1\, w'\right)' = 0 \qquad (8B-23)$$

We assume the flow to be driven by constant axial pressure gradient $\frac{\partial p}{\partial z}$. Then (8B-23) implies

$$\frac{\partial p}{\partial z} = \frac{1}{r}\left(\frac{1}{2}\, r\, \alpha_1\, w'\right)' = C_1 = \text{constant} \qquad (*)$$

where $C_1 < 0$ for p decreasing with increasing z. Now (*) gives

$$p(r,z) = C_1\, z + f(r) \qquad (8B-24)$$

and

$$\alpha_1\, w' = C_1\, r + \frac{2C_2}{r} \qquad (8B-25)$$

where f(r) is an arbitrary function. Since w(r) and w'(r) are required to be bounded functions throughout the flow, C_2 must vanish. Equation (8B-25) is a nonlinear ordinary differential equation for w(r)

$$\alpha_1\left(-\frac{1}{4}\, w'^2,\ 0\right) w' = C_1\, r \qquad (8B-26)$$

subject to the boundary condition

$$w(a) = 0 \qquad (8B-27)$$

Note that w(r) is independent of the function $\alpha_2(-\frac{1}{4}\, w'^2,\ 0)$. Now the function f(r) is determined by (8B-22); hence, substitute (8B-24) in (8B-22):

$$- f' + \frac{1}{r} (\frac{1}{4} r \alpha_2 w'^2)' = 0$$

or

$$- f' + (\frac{1}{4} \alpha_2 w'^2)' + \frac{1}{r} (\frac{1}{4} \alpha_2 w'^2) = 0$$

But (8B-26) implies $w'^2 = (\frac{C_1 r}{\alpha_1})^2$. Hence,

$$f' = (\frac{1}{4} \alpha_2 w'^2)' + \frac{C_1{}^2 \alpha_2 r}{4\alpha_1{}^2}$$

Integrating, we find

$$f(r) = \frac{1}{4} \alpha_2 w'^2 + \frac{1}{4} C_1{}^2 \int \frac{\alpha_2}{\alpha_1{}^2} r \, dr + C_3$$

Hence, from (8B-24) we have

$$p(r,z) = C_1 z + \frac{1}{4} \alpha_2 w'^2 + \frac{1}{4} C_1{}^2 \int \frac{\alpha_2}{\alpha_1{}^2} r \, dr + C_3 \quad (8B\text{-}28)$$

Note that the pressure is not constant over a cross section, in contrast with the classical solution, due to the presence of the non-linear viscosity function α_2. Substitution of (8B-28) in (8B-19) gives the normal stresses

$$t_{rr} = t_{zz} = - C_1 z - \frac{1}{4} C_1{}^2 \int \frac{\alpha_2}{\alpha_1{}^2} r \, dr - C_3$$

$$(8B\text{-}29)$$

$$t_{\theta\theta} = - C_1 z - \frac{1}{4} \alpha_2 w'^2 - \frac{1}{4} C_1{}^2 \int \frac{\alpha_2}{\alpha_1{}^2} r \, dr - C_3$$

Recall that in the classical solution the normal stresses are all equal to -p. Here, the extra terms involving α_2 give rise to excess normal stresses which are pressures when α_2 is a positive function. This nonlinear effect is called the Poynting effect for Poiseuille flow. We consider two special cases.

(a) Constant Constitutive Functions α_1, α_2

Assume that the functions α_1 and α_2 in the general theory reduce to constants, i.e., let

$$\alpha_1 = 2\mu \quad , \quad \alpha_2 = \beta$$

where μ is assumed positive to satisfy (8B-18). Then the boundary value problem (8B-26), (8B-27) is linear and yields

$$w(r) = -\frac{C_1}{4\mu}(a^2-r^2) \tag{8B-30}$$

which is the classical solution (Newtonian fluids). From (8B-28,29,30) we find that

$$\begin{aligned} p(r,z) &= C_1 z + \frac{3\beta C_1{}^2 r^2}{32\mu^2} + C_3 \\ t_{rr} = t_{zz} &= -C_1 z - \frac{\beta C_1{}^2 r^2}{32\mu^2} - C_3 \\ t_{\theta\theta} &= -C_1 z - \frac{3\beta C_1{}^2 r^2}{32\mu^2} - C_3 \end{aligned} \tag{8B-31}$$

Hence, the normal stresses vary as r^2 over the cross section.

(b) Linear Constitutive Function α_1

Suppose that the function α_1 in the general theory is linear in II_d, III_d

$$\alpha_1(II_d, III_d) = D_1 II_d + D_2 III_d + D_3 \qquad (*)$$

For this flow $III_d = 0$, $II_d = -\frac{1}{4} w'^2$, and we write (*) as

$$\alpha_1(II_d, 0) = 2(\gamma_1 w'^2 + \gamma_2)$$

where γ_1, γ_2 are assumed to be positive constants to satisfy (8B-18). Then the differential equation (8B-26) becomes

$$\gamma_1 w'^3 + \gamma_2 w' - C_1 r = 0$$

which can be regarded as a cubic equation in $w'(r)$ with real coefficients. Since this cubic equation has either one or three real solutions for $w'(r)$, then for a given pressure gradient C_1, there will generally exist one or three velocity profiles.

We consider case (a) further. Suppose at some time $t = \tilde{t}$ the fluid exits the pipe at $z=0$ into atmosphere pressure p_0. Then the force exerted on the fluid by p_0 at the exit cross section is $-\pi a^2 p_0$. Since the flow is steady, this force is balanced by $\int t_z|_{z=0} dA$.

$$\int_A t_z|_{z=0} \, dA = \int_0^a t_{zz}|_{z=0} (2\pi r) dr = -\pi a^2 p_0 \qquad (!)$$

From (8B-31) we have

$$\int_0^a t_{zz}\Big|_{z=0} 2\pi r dr = -\left(\frac{\pi\beta C_1^{\,2}}{64\mu^2} a^4 + \pi a^2 C_3\right)$$

Hence, (†) allows calculation of C_3.

$$C_3 = p_0 - \frac{\beta C_1^{\,2} a^2}{64\mu^2} \qquad (8B-32)$$

On the fluid surface r=a there acts a radial stress vector component $t_r = t_{rr}(a,z)$. Hence, acting on the pipe we have a force of unit area $P = -t_{rr}(a,z)$. Hence, by (8B-31), (8B-32)

$$P = -t_{rr}(a,z) = C_1 z + \frac{\beta C_1^{\,2} a^2}{32\mu^2} + p_0$$

and

$$P - p_0 = C_1 z + \frac{\beta C_1^{\,2} a^2}{32\mu^2}$$

In particular, at the exit section z=0 at the instant the fluid reaches this section, we have

$$P - p_0\Big|_{z=0} = \frac{\beta C_1^{\,2} a^2}{32\mu^2}$$

Recall that we have no restriction on the sign of β. But if β > 0, there is a positive radial pressure difference at the exit section. It is inferred from this fact that as

the fluid exits into atmospheric pressure, it will tend to swell. This swelling phenomenon was observed experimentally by A. C. Merrington, Flow of Visco-elastic Materials in Capillaries, Nature, 152, 663, 1943. Note that this effect cannot be accounted for theoretically by Newtonian fluid theory since $\beta=0$.

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3. Couette Flow Between Rotating Cylinders

Consider an incompressible Stokesian fluid in steady flow between concentric rotating cylinders. We assume the body forces vanish and $\underset{\sim}{v} = (0, v(r), 0)$ as before. Recall that the continuity equation is then satisfied. From (8A-16) we have

$$\underset{\sim}{d} = \begin{pmatrix} 0 & \frac{1}{2} r(\frac{v}{r})' & 0 \\ \frac{1}{2} r(\frac{v}{r})' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (8B\text{-}33)$$

We let

$$f(r) = \frac{v(r)}{r} \qquad (8B\ 34)$$

Then

$$\underset{\sim}{d} = \begin{pmatrix} 0 & \frac{1}{2} rf' & 0 \\ \frac{1}{2} rf' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underset{\sim}{d}^2 = \begin{pmatrix} \frac{1}{4} r^2f'^2 & 0 & 0 \\ 0 & \frac{1}{4} r^2f'^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (8B\text{-}35)$$

and

$$I_d = 0 \quad , \quad II_d = -\frac{1}{4} r^2f'^2 \quad , \quad III_d = 0$$
$$tr(\underset{\sim}{d}^2) = \frac{1}{2} r^2f'^2 \quad , \quad tr(\underset{\sim}{d}^3) = 0 \qquad (8B\text{-}36)$$

From (8B-3)

$$\frac{1}{2} r^2\alpha_1(-\frac{1}{4} r^2f'^2, 0)f'^2 \geq 0 \qquad (8B\ 37)$$

and hence, α_1 must be a non-negative function of II_d, but again no restriction on α_2. From (8B-1) and (8B-35)

$$\underset{\sim}{t} = -p\underset{\sim}{I} + \frac{1}{2} r\alpha_1f' \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{4} r^2\alpha_2f'^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (8B\text{-}38)$$

where

$$\alpha_1 = \alpha_1(-\frac{1}{4} r^2f'^2, 0) \ , \ \alpha_2 = \alpha_2(-\frac{1}{4} r^2f'^2, 0) \qquad (8B\text{-}39)$$

Consider now the linear momentum equations (8A-15). Since the body forces vanish, we find $\frac{\partial p}{\partial z} = 0$ implying $p = p(r,\theta)$ and

$$\frac{\partial}{\partial r} (-p + \frac{1}{4} r^2\alpha_2f'^2) = -\rho \frac{v^2}{r} = -\rho rf^2$$

$$\frac{\partial}{\partial r} (\frac{1}{2} r\alpha_1f') - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{2}{r} (\frac{1}{2} r\alpha_1f') = 0$$

Assuming $p = p(r)$ as in the classical case, these equations reduce to

$$-p' + (\frac{1}{4} r^2\alpha_2f'^2)' = -\rho rf^2 \qquad (8B\text{-}41)$$

$$(\frac{1}{2} r\alpha_1f')' + \alpha_1f' = 0 \qquad (8B\text{-}42)$$

Noting that the latter equation can be written in the form

$$\frac{1}{r^2} [r^2(\frac{1}{2} r\alpha_1f')]' = 0$$

we have

$$\frac{1}{2} r\alpha_1 f' = \frac{1}{2} r\alpha_1\left(-\frac{1}{4} r^2 f'^2, 0\right) = \frac{A}{r^2} \qquad (8B\text{-}43)$$

where A is a constant. This equation is a nonlinear ordinary differential equation for the function $f(r) = \frac{v(r)}{r}$. The boundary conditions are

$$v(a) = af(a) = a\omega_1 \quad , \quad v(b) = bf(b) = b\omega_2$$

or

$$f(a) = \omega_1 \quad , \quad f(b) = \omega_2 \qquad (8B\text{-}44)$$

Note that as in Poiseuille pipe flow, the coefficient function α_2 has no effect on the velocity profile. By considering polynomial approximations to the function α_1, it can be shown that there are an <u>odd</u> number of velocity profiles possible for a given value of A in (8B-43). Now (8B-41) determines the pressure:

$$p(r) = \frac{1}{4} r^2\alpha_2 f'^2 + \int \rho r f^2 \, dr + B \qquad (8B\text{-}45)$$

where B is a constant. Using this result in (8B-38), the stresses become

$$t_{rr} = t_{\theta\theta} = -\int \rho r f^2 \, dr - B \qquad (8B\text{-}46)$$

$$t_{zz} = \frac{1}{4} r^2\alpha_2 f'^2 - \int \rho r f^2 \, dr - B \qquad (8B\text{-}47)$$

$$t_{r\theta} = \frac{1}{2} r\alpha_1 f' \overset{(8B\text{-}43)}{=} \frac{A}{r^2} \qquad (8B\text{-}48)$$

Comparing these values with the classical case (8A-33), (8A-34), we find that $t_{r\theta}$ still varies as $\frac{1}{r^2}$, but that the normal stresses are no longer all equal to -p. There is an excess axial normal stress $\bar{t}_{zz}$

$$\bar{t}_{zz} = -\frac{1}{4} r^2 \alpha_2 f'^2 \tag{8B 49}$$

which depends on α_2 (Poynting effect). The torque which must be applied to the outer cylinder follows as before

$$M_2 = 2\pi b^2 h \; t_{r\theta}(b) = 2\pi h \; A = \text{const.} \tag{8B 50}$$

Note that M_2 depends on both α_1 and the boundary conditions (8B-44) through the constant A.

We consider the special case when α_1 and α_2 are constant functions i.e.

$$\alpha_1 = 2\mu \quad , \quad \alpha_2 = \beta$$

where $\mu > 0$ to satisfy (8B-37). Then from (8B-43) we find the linear differential equation

$$\frac{1}{2} r(2\mu)f' = \frac{A}{r^2}$$

which implies $f' = \frac{A}{\mu r^3}$ and

$$f(r) = -\frac{A}{2\mu r^2} + C = \frac{v(r)}{r} \tag{8B 51}$$

where upon applying the boundary conditions (8B-44), we find

$$\frac{A}{2\mu} = -C_2 \quad , \quad C = \frac{1}{2} C_1 \tag{8B-52}$$

where C_1, C_2 are defined by (8A-30) for the classical case. Hence, (8B-51) and (8B-52) yield the same velocity profile as the classical solution (8A-31). To ascertain the qualitative effect of the excess normal axial stress, we suppose that the cylinders are of finite length with axis vertical. At the upper end there is a free surface of fluid exposed to atmospheric pressure p_0. We compute the balance of forces at the free surface as if it were a plane, and then infer from this balance the actual shape of the surface. The balance of forces is

$$\int_a^b t_{zz} \, 2\pi r \, dr = -p_0 \pi (b^2 - a^2) \tag{8B-53}$$

This allows determination of the constant B in (8B-47). We find

$$B = p_0 - K \tag{*}$$

where

$$K = \frac{1}{\pi(b^2 - a^2)} \int_a^b [-\bar{t}_{zz} + \int \rho r f^2(r) \, dr] \, 2\pi r \, dr \tag{8B-54}$$

and where $\bar{t}_{zz}$ is given by (8B-49). Now using (8B-54), (*) and (8B-47), we have

$$t_{zz} = \bar{t}_{zz} - \int \rho r f^2 \, dr - p_0 - K \tag{8B-55}$$

We now define the function

$$N(r) = t_{zz} + p_0 = \bar{t}_{zz} - \int \rho r f^2 \, dr - K \qquad (8B\text{-}56)$$

i.e., the amount by which the end boundary condition $t_{zz} = -p_0$ is not satisfied. Note $N(r)$ follows the same sign convention as t_{zz}, i.e., $N(r) < 0$ implies a pressure acting on the end plane. From the solution (8B-51), we have

$$\bar{t}_{zz} = - \frac{\beta}{r^4} \left(\frac{A}{2\mu}\right)^2$$

and $N(r)$ becomes

$$N(r) = - \frac{\beta}{r^4} \left(\frac{A}{2\mu}\right)^2 - \int \rho r f^2 \, dr - K \qquad (8B\text{-}57)$$

Then

$$N'(r) = \frac{4\beta}{r^5} \left(\frac{A}{2\mu}\right)^2 - \rho r f^2 \qquad (8B\text{-}58)$$

We note that when $\beta > 0$, then $\bar{t}_{zz} < 0$, $K > 0$, $N(r) < 0$. For the sake of argument let the inner cylinder be at rest, $\omega_1 = 0$. For the <u>classical case</u> $\beta=0$ and

$$N'(r) = - \rho r f^2 < 0 \qquad (8B\text{-}59)$$

This implies $N(r)$ is a decreasing function over the interval $a \le r \le b$. From (8A-30) $C_1 > 0$, $C_2 < 0$ and from (8B-52) $\frac{A}{2\mu} > 0$, $C > 0$. Hence, the function

$$r^2 f^2 = r\left(- \frac{A}{2\mu r^2} + C\right)^2$$

and N'(r) are minimum in absolute value at r=a and maximum in absolute value at r=b. Hence, N(r) has the form shown in Fig. VIII-7. Since N(r) is maximum at r=a and minimum at r=b, it is inferred that the slope of the free surface would be upwards from the inner to the outer cylinder, as shown in Fig. VIII-7.

Continuing with $\omega_1 = 0$, we suppose that β is positive and large enough such that N'(r) > 0. Then N(r) is increasing in absolute value from r=a to r=b, and the slope of the free surface reverses as shown in Fig. VIII-8. Hence, the fluid tends to climb the inner cylinder. This is called the Weissenberg effect and was experimentally observed in certain oils, and solutions of rubber, starch, cellulose acetate, etc. See K. Weissenberg, "A Continuum Theory of Rheological Behavior", Nature, 159, 1947, p. 310-311.

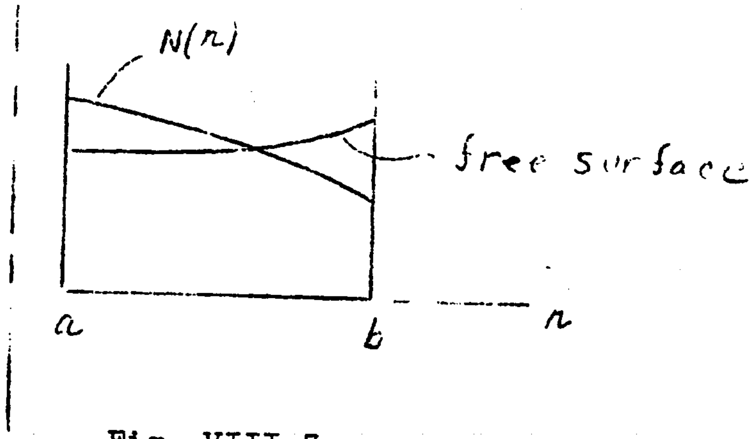


Fig. VIII-7

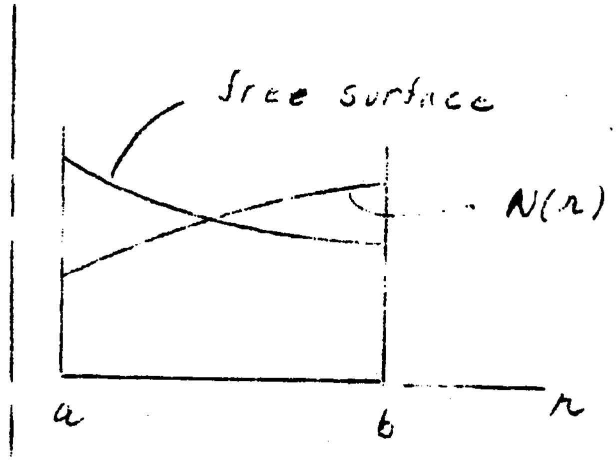


Fig. VIII-8

C. Heat-Conducting Newtonian Fluids

We consider incompressible fluids for which ${}_{D}\underset{\sim}{t}$ is linear in $\underset{\sim}{d}$ and $\underset{\sim}{q}$ linear in $\theta_{,i}$. Then equations (8A-1)-(8A-4) apply, but we modify (8A-5) to include heat conduction. Then from (7C-19), (7C-21)

$$q_i = -\kappa(\theta)\,\theta_{,i}\ , \quad \kappa(\theta) > 0 \tag{8C-1}$$

and from (7C-25)

$$\rho c_v(\theta)\dot{\theta} = 2\mu(\theta)\ \mathrm{tr}(\underset{\sim}{d}^2) + (\kappa\theta_{,i})_{,i} + \rho r \tag{8C-2}$$

We consider the special case when μ, κ, and $c_v = c$ are constants, independent of the temperature. Then (8C-1) and (8C-2) reduce to

$$q_i = -\kappa\theta_{,i}\ , \quad \kappa > 0 \tag{8C-3}$$

$$\rho c\dot{\theta} = 2\mu\ \mathrm{tr}(\underset{\sim}{d}^2) + \kappa\theta_{,ii} + \rho r \tag{8C-4}$$

The governing equations are then the Navier-Stokes equations (8A-6), which serve to determine the velocity field, and (8C-4) from which the temperature field can then be determined. In rectangular cartesian coordinates (8C-4) has the form

$$\begin{aligned}\rho c\left(\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} + w\frac{\partial\theta}{\partial z}\right)\\ = \kappa\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} + \frac{\partial^2\theta}{\partial z^2}\right) + \rho r\\ + 2\mu\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial z}\right)^2\right]\\ + \mu\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)^2\right]\end{aligned} \tag{8C-5}$$

In cylindrical coordinates (8C 4) has the form

$$\rho c(\frac{\partial\theta}{\partial t} + u\,\frac{\partial\theta}{\partial r} + \frac{v}{r}\,\frac{\partial\theta}{\partial\phi} + w\,\frac{\partial\theta}{\partial z}) =$$

$$\kappa(\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\,\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\,\frac{\partial^2\theta}{\partial\phi^2} + \frac{\partial^2\theta}{\partial z^2}) + \rho r$$

$$+ 2\mu[(\frac{\partial u}{\partial r})^2 + (\frac{1}{r}\,\frac{\partial v}{\partial\phi} + \frac{u}{r})^2 + (\frac{\partial w}{\partial z})^2] \qquad (8C\text{-}5a)$$

$$+ \mu\{[r\,\frac{\partial}{\partial r}\,(\frac{v}{r}) + \frac{1}{r}\,\frac{\partial u}{\partial\phi}]^2 + (\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r})^2$$

$$+ (\frac{\partial v}{\partial z} + \frac{1}{r}\,\frac{\partial w}{\partial\phi})^2\}$$

We consider two exact solutions

1. Simple Couette Flow

The solution of the Navier-Stokes equations is (8A-22)

$$u(y) = ky \quad , \quad k = \frac{U}{h}$$

$$v = 0 = w \qquad (8C\ 6)$$

We must now solve (8C 5) for the temperature field. We assume no heat sources, and since the flow is steady we assume the same for the temperature field:

$$\theta = \theta(y) \qquad (8C\ 7)$$

From (8C 5) we have

$$\kappa\theta'' = -\mu u'^2 = -\mu k^2$$

Hence,

$$\theta(y) = -\frac{\mu k^2}{2\kappa} y^2 + C_1 y + C_2 \qquad (8C-8)$$

We consider the two plates to be held at constant temperatures

$$\theta(0) = \theta_0 \quad , \quad \theta(h) = \theta_1 \quad , \quad \theta_1 > \theta_0 \qquad (8C-9)$$

Applying these boundary conditions to (8C-8) we find

$$C_2 = \theta_0 \quad , \quad C_1 = \frac{\theta_1-\theta_0}{h} + \frac{\mu k^2 h}{2\kappa} \qquad (8C-10)$$

and

$$\theta(y) = -\frac{\mu k^2}{2\kappa} y^2 + \left(\frac{\theta_1-\theta_0}{h} + \frac{\mu k^2 h}{2\kappa}\right) y + \theta_0 \qquad (8C-11)$$

which we put into the non-dimensional form

$$\frac{\theta-\theta_0}{\theta_1-\theta_0} = \frac{y}{h} + \frac{\mu U^2}{2\kappa(\theta_1-\theta_0)} \frac{y}{h} \left(1 - \frac{y}{h}\right) \qquad (8C-12)$$

For the special case when there is no flow, i.e., U=0, then θ(y) varies linearly:

$$\theta(y) = \frac{\theta_1-\theta_0}{h} y + \theta_0 \qquad (8C-13)$$

Also, if the upper and lower plates are held at the same temperature, i.e., $\theta_1 = \theta_0$, then θ(y) is parabolic:

$$\theta(y) = \frac{\mu U^2}{2\kappa} \frac{y}{h} \left(1 - \frac{y}{h}\right) + \theta_0 \qquad (8C-14)$$

with the maximum temperature generated by the flow at $y = \frac{h}{2}$.

Returning to the general solution and defining non-dimensional Prandtl-Eckert number

$$PE = \frac{\mu U^2}{\kappa(\theta_1-\theta_0)} \tag{8C-15}$$

we have from (8C-12)

$$\frac{\theta-\theta_0}{\theta_1-\theta_0} = \frac{y}{h} + \frac{1}{2} PE \frac{y}{h} \left(1 - \frac{y}{h}\right) \tag{8C-16}$$

A graph of this result is shown in Fig. VIII-9. We calculate the heat flux vector $q_i = -\kappa\theta_{,i}$ based on (8C-16):

$$q_x = 0 = q_z$$

$$q_y = -\kappa\theta' = -\frac{\kappa}{h}(\theta_1-\theta_0) - \frac{\kappa}{h} PE(\theta_1-\theta_0)\left(\frac{1}{2} - \frac{y}{h}\right)$$

A non-dimensional plot of q_y for various values of PE is shown in Fig. VIII-10. Evaluating the heat flux $\underset{\sim}{q}\cdot\underset{\sim}{n}$ acting across the fluid surface at $y=h$, we have $\underset{\sim}{n} = (0,1,0)$ and

$$\underset{\sim}{q}\cdot\underset{\sim}{n} = q_y(h) = -\frac{\kappa}{h}(\theta_1-\theta_0) + \frac{\kappa PE}{2h}(\theta_1-\theta_0)$$

$$= -\frac{\kappa}{2h}(\theta_1-\theta_0)(2-PE) \tag{8C-17}$$

When $U=0$, then $PE=0$ and $\underset{\sim}{q}\cdot\underset{\sim}{n} < 0$, implying cooling of the upper plate, i.e., heat flows from the upper to the lower plate. This remains true for $U \neq 0$ as long as U is sufficiently small, i.e., for

$$PE < PE^* = 2 \tag{8C-18}$$

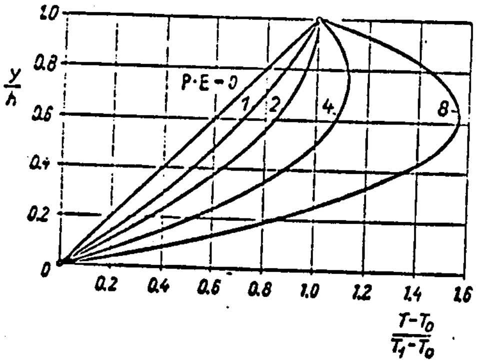


Fig. 14.5. Temperature distribution in Couette flow for various temperatures of both walls with heat generated by friction (T_0 = temperature of the lower wall, T_1 = temperature of the upper wall)

Fig. VIII-9

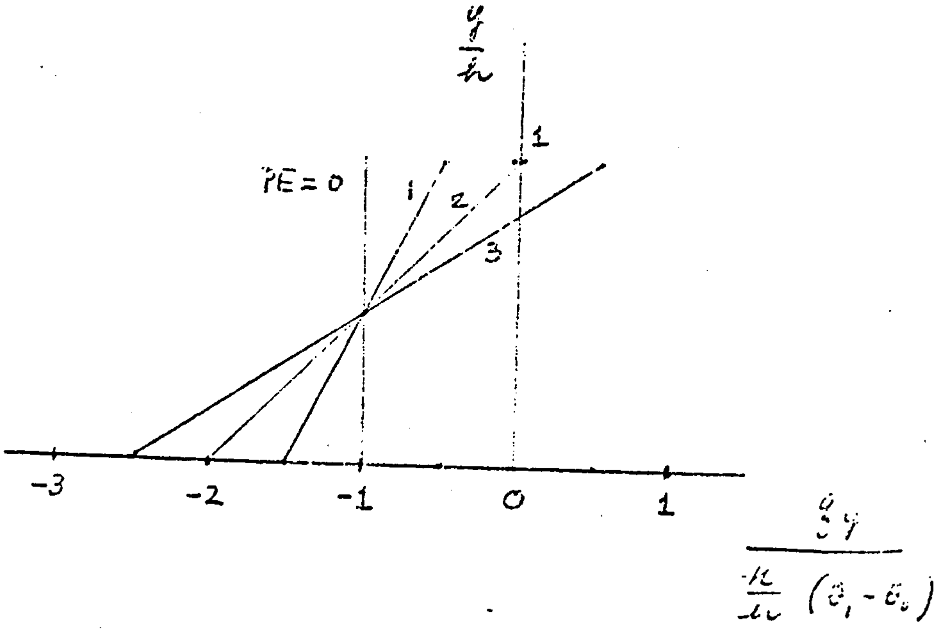


Fig. VIII-10

For PE = PE* there is no heat flux at the upper plate, while for PE > PE* heating of the upper plate occurs. For this case the temperature within the flow exceeds that of the upper plate due to the high shear rate $k = \frac{U}{h}$, so that heat energy must be removed at the upper plate in order that it be maintained at constant temperature.

Consider the case when no heat transfer is allowed at the lower wall, i.e., $q_y(0) = -\kappa\theta'(0) = 0$. Then the boundary conditions (8C-9) are replaced by

$$\theta'(0) = 0 \quad , \quad \theta(h) = \theta_1 \qquad (8C\ 19)$$

Then the general solution (8C-8) reduces to

$$\theta(y) - \theta_1 = \frac{\mu U^2}{2\kappa} (1 \quad \frac{y^2}{h^2}) \qquad (8C\ 20)$$

where the maximum temperature occurs at the lower wall:

$$\theta_{max} = \theta(0) = \theta_1 + \frac{\mu U^2}{2\kappa} \qquad (8C\text{-}21)$$

2. Plane Poiseuille Flow

The solution of the Navier-Stokes equations is given by (8A-21)

$$u(y) = -\frac{h^2}{2\mu}\frac{dp}{dx}\frac{y}{h}(1 - \frac{y}{h}) \qquad (8C\text{-}22)$$

$$v = 0 = w$$

If we shift the origin of coordinates to the midpoint of the channel, then (8C-22) becomes

$$u(y) = u_m\left(1 - \frac{y^2}{h^2}\right) \quad , \quad u_m = - \frac{h^2}{8\mu}\frac{dp}{dx} > 0 \tag{8C-23}$$

Assuming no heat sources and a steady temperature field of the form $\theta = \theta(y)$, then we find from (8C-5)

$$\kappa\theta'' = \mu u'^2 = - \frac{4\mu u_m^2}{h^4} y^2 \tag{8C-24}$$

which has the general solution

$$\theta(y) = \frac{\mu u_m^2}{3\kappa h^4} y^4 + C_1 y + C_2 \tag{8C-25}$$

Assuming equal temperatures at the boundaries

$$\theta(\pm \frac{h}{2}) = \theta_0 \tag{8C-26}$$

then we find

$$C_1 = 0 \quad , \quad C_2 = \theta_0 + \frac{\mu u_m^2}{3\kappa h^4}\left(\frac{h}{2}\right)^4 \tag{8C-27}$$

and

$$\theta(y) - \theta_0 = \frac{\mu u_m^2}{48\kappa}\left[1 - \left(\frac{y}{h/2}\right)^4\right] \tag{8C-28}$$

It is clear that the temperature variation is generated solely by the flow. The maximum temperature occurs where u is maximum, i.e., at $y = \frac{h}{2}$.

$$\theta_{max} = \theta_0 + \frac{\mu u_m^2}{48\kappa} \qquad (8C-29)$$

The velocity profile and temperature field are shown in Fig. VIII-11.

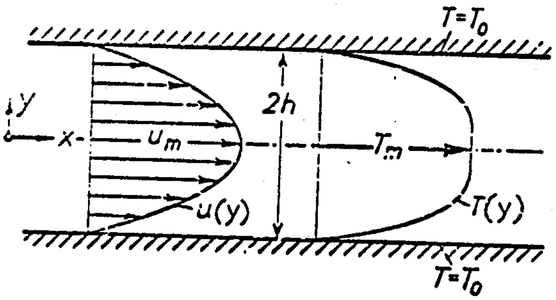


Fig. 14.6. Velocity and temperature distribution in a channel with flat walls with frictional heat taken into account

Fig. VIII-11

IX. Some Exact Solutions for Solids

A. Elasticity Governing Equations

We consider a homogeneous, isotropic elastic solid held in equilibrium by surface tractions. We assume the body is undeformed in the reference configuration and subjected to isothermal, adiabatic deformations without body forces.

1. Linear Theory

Equilibrium Equations:

$$t_{ij,j} = 0 \tag{9A 1}$$

Boundary Conditions:

$$u_i = u_i \qquad \text{on} \quad S_u \tag{9A 2}$$

$$t_i = t_{ij}\, n_j = \bar{t}_i \qquad \text{on} \quad S_t \tag{9A 3}$$

where u_i and $\bar{t}_i$ are prescribed functions of $\underset{\sim}{x}$.

Constitutive Equations

$$t_{ij} = \lambda\, u_{k,k}\, \delta_{ij} + 2\mu\, u_{(i,j)} \tag{9A 4}$$

Substitution of (4) into (1) yields Navier's equilibrium equations.

$$\mu\, u_{i,jj} + (\lambda+\mu)\, u_{j,ji} = 0 \tag{9A 5}$$

which must be integrated for $u_i(\underset{\sim}{x})$ subject to displacement boundary conditions (2) or traction conditions (from (3) and (4))

$$\lambda\, u_{k,k}\, n_i + 2\mu\, u_{(i,j)}\, n_j = \bar{t}_i \quad \text{on} \quad S_t \tag{9A 6}$$

2. Nonlinear Theory

Equations (1) (3) are unchanged, but we must add Conservation of Mass:

$$\rho J = \rho_0 \tag{9A 7}$$

Constitutive Equations (compressible materials)

$$t_{ij} = \gamma_{-1}\, b_{ij} + \gamma_0\, \delta_{ij} + \gamma_1\, c_{ij} \tag{9A 8}$$

where

$$b_{ij} = x_{i,K}\, x_{j,K} \qquad c_{ij} = X_{K,i}\, X_{K,j} \tag{9A 9}$$

and the response coefficients γ_{-1}, γ_0, γ_1 are functions of the invariants I_b, II_b, III_b

$$\gamma_\alpha = \gamma_\alpha(I_b, II_b, III_b) \qquad \alpha = -1, 0, 1 \tag{9A 10}$$

For an undeformed reference configuration $\underset{\sim}{b} = \underset{\sim}{c} = \underset{\sim}{I}$ and $\underset{\sim}{t}$ must vanish. Then from (8) and (10) we must require the response coefficients to satisfy

$$(\gamma_{-1} + \gamma_0 + \gamma_1)\Big|_{\underset{\sim}{b}=\underset{\sim}{I}} = 0 \tag{9A 11}$$

Since $J = \det x_{i,K}$ and $\det X_{K,i} = \frac{1}{J}$, we have from (9)

$$III_b = \det \underset{\sim}{b} = (\det x_{i,K})^2 = J^2$$

$$III_c = \det \underset{\sim}{c} = (\det X_{K,i})^2 = \frac{1}{J^2}$$

and hence

$$III_b = \frac{1}{III_c} \qquad (9A\text{-}12)$$

with the result that (7) can be expressed in the form

$$\rho \sqrt{III_b} = \rho_0 \qquad (9A\ 13)$$

For an incompressible material $J = 1$, $III_b = 1$ and (8), (10) are replaced by

$$t_{ij} = - p\, \delta_{ij} + \tilde{\gamma}_{-1}\, b_{ij} + \tilde{\gamma}_1\, c_{ij} \qquad (9A\text{-}14)$$

$$\tilde{\gamma}_\alpha = \tilde{\gamma}_\alpha(I_b, II_b) \qquad \alpha = -1, 1 \qquad (9A\text{-}15)$$

where $p(\underset{\sim}{x})$ is the pressure. In addition, (11) is replaced by

$$- p + (\gamma_{-1} + \gamma_1)\Big|_{\underset{\sim}{b}=\underset{\sim}{I}} = 0 \qquad (9A\ 16)$$

The linear case (4) can be recovered from (8) by using the approximate relationships.

$$b_{ij} \cong \delta_{ij} + 2u_{(i,j)} , \qquad c_{ij} \cong \delta_{ij} - 2u_{(i,j)} \qquad (9A\text{-}17)$$

$$\begin{aligned} \gamma_{-1} &\cong 2\mu + \lambda\, u_{i,i} \\ \gamma_0 &\cong - \mu(3 + u_{i,i}) \\ \gamma_1 &\cong \mu(1 + u_{i,i}) \end{aligned} \qquad (9A\ 18)$$

B. Simple Shear

We consider the deformation

$$x_1 = X_1 + KX_2 \quad , \quad x_2 = X_2 \quad , \quad x_3 = X_3 \quad , \quad K>0 \qquad (9B\ 1)$$

which corresponds to the two-dimensional shearing of a rectangular block (see Fig. IX-1).

1. Linear Theory

From (1) the displacements are

$$u_1 = x_1 - X_1 = KX_2 = Kx_2 \quad , \quad u_2 = u_3 = 0 \qquad (9B\text{-}2)$$

so that we have

$$u_{i,j} = \begin{pmatrix} 0 & K & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad , \quad u_{(i,j)} = \frac{1}{2}\begin{pmatrix} 0 & K & 0 \\ K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9B\ 3)$$

In terms of the norm (2G 2) we find $||u_{i,j}|| = K$, which implies that the linear theory is valid for $K << 1$. We note from (3) that

$$u_{i,i} = I_{\tilde{e}} = 0 \qquad (9B\ 4)$$

Hence, (2G-16) implies $J \cong 1$, and the deformation in the linear approximation is isochoric. Now from (9A-4), (3) and (4), we have

$$t_{ij} = 2\mu\ u_{(i,j)} = \begin{pmatrix} 0 & \mu K & 0 \\ \mu K & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9B\text{-}5)$$

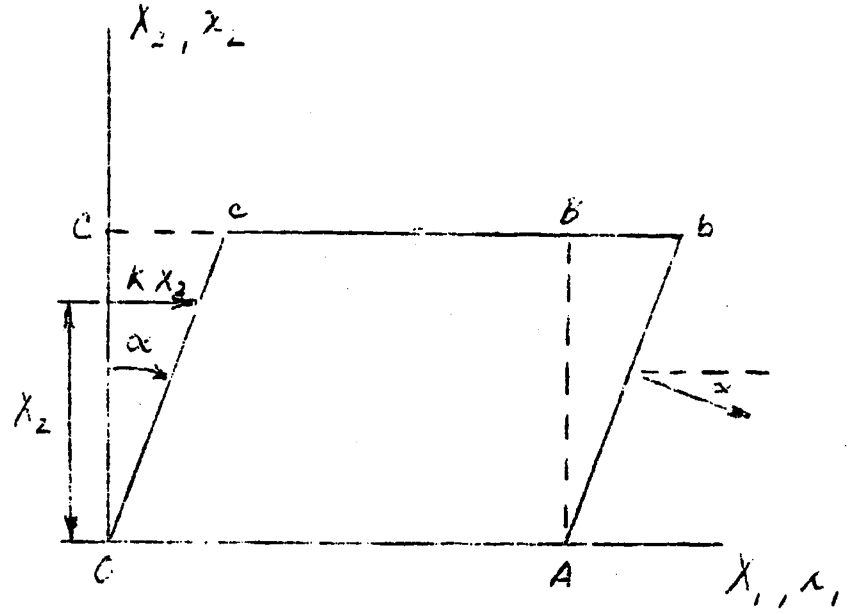


Fig. IX-1

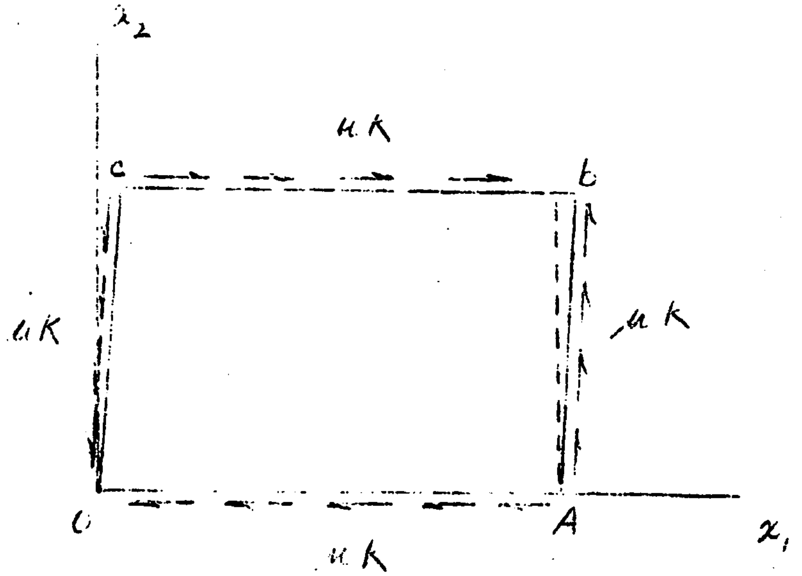


Fig. IX-2

Since the stress components are constant, the equilibrium equations (9A-1) are identically satisfied. We now investigate the boundary conditions in order to determine the tractions which must be applied to the surfaces of the block. The outer normals to the various planes are

OA: $n_1 = 0 = n_3 \;,\; n_2 = -1$

cb: $n_1 = 0 = n_3 \;,\; n_2 = 1$

Ab: $n_1 = \cos\alpha = (1+K^2)^{-1/2} \;,$

$$n_2 = -\sin\alpha = -K(1+K^2)^{-1/2} \;,\; n_3 = 0 \qquad (9B\text{-}6)$$

Oc: $n_1 = -(1+K^2)^{-1/2} \;,\; n_2 = K(1+K^2)^{-1/2} \;,\; n_3 = 0$

Hence, by (9A-3)

$$t_i = t_{ij}\, n_j = t_{i1}\, n_1 + t_{i2}\, n_2 + t_{i3}\, n_3 \qquad (9B\text{-}7)$$

and by (6) we have on

OA: $t_i = -t_{i2} = -(\mu K,0,0)$

cb: $t_i = t_{i2} = (\mu K,0,0)$

Ab: $n_1 \cong 1 \;,\; n_2 \cong 0$ since $K \ll 1$

$t_i = t_{i1} = (0,\mu K,0)$

Oc: $n_1 \cong -1 \;,\; n_2 \cong 0$ since $K \ll 1$

$t_i = -t_{i1} = -(0,\mu K,0)$

On the faces $X_3 = x_3 =$ const., $n_i = (0,0,\underline{+}1)$ and by (7)

$$t_i = \pm t_{i3} = 0$$

Hence, the block is held in equilibrium by the application of shearing stresses alone. Note that these tractions are directly proportional to K. (See Fig. IX-2.)

2. Nonlinear Theory (Ref.: A. C. Eringen, 'Nonlinear Theory of Continuous Media,' McGraw-Hill, 1962, pgs. 177-179 and C. Truesdell, Elements of Continuum Mechanics, Springer-Verlag, 1965, pgs 110-116)

For this case we impose the same deformation (1) without any magnitude restriction on K and then check to see if the governing equations can be satisfied for <u>all</u> elastic materials by the application of suitable applied tractions. From (1)

$$x_{i,K} = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (9B\ 8)$$

and J=1. Hence, the deformation is isochoric, and (9A 7) is satisfied by $\rho = \rho_0$. Now we invert (1)

$$X_1 = x_1 - Kx_2 \quad , \quad X_2 = x_2 \quad , \quad X_3 = x_3$$

and implies

$$X_{K,i} = \begin{pmatrix} 1 & -K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (9B\text{-}9)$$

By (9A-9) and (8), (9) we have

$$b_{ij} = x_{i,K}\, x_{j,K} = \begin{pmatrix} 1 & K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (9B\text{-}10)$$

$$c_{ij} = X_{K,i}\, X_{K,j} = \begin{pmatrix} 1 & 0 & 0 \\ -K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -K & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (9B\text{-}11)$$

and

$$\underset{\sim}{b}^2 = \begin{pmatrix} (1+K^2)^2+K^2 & (1+K^2)K+K & 0 \\ (1+K^2)K+K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence,

$$\mathrm{tr}(\underset{\sim}{b}^2) = (1+K^2)^2 + K^2 + 2 + K^2 = 3 + 4K^2 + K^4$$

and

$$I_b = \mathrm{tr}\, \underset{\sim}{b} = 3 + K^2$$

$$II_b = \frac{1}{2} [I_b^2 - \mathrm{tr}(\underset{\sim}{b}^2 0] = \frac{1}{2} [(3+K^2)^2 - 3 - 4K^2 - K^4] \qquad (9B\text{-}12)$$

$$= \frac{1}{2} (6 + 2K^2) = 3 + K^2 = I_b$$

$$III_b = 1 + K^2 - K^2 = 1$$

Now from (9A-10)

$$\gamma_\alpha = \gamma_\alpha(3+K^2,3+K^2,1) \equiv \hat{\gamma}_\alpha(K^2) \quad , \quad \alpha = -1,0,1 \qquad (9B\ 13)$$

Using (10), (11), (13) in (9A-8) we have

$$\underset{\sim}{t} = \hat{\gamma}_{-1}(K^2)\begin{pmatrix} 1+K^2 & K & 0 \\ K & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hat{\gamma}_0(K^2)\underset{\sim}{I} + \hat{\gamma}_1\begin{pmatrix} 1 & -K & 0 \\ -K & 1+K^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= (\hat{\gamma}_{-1}+\hat{\gamma}_0+\hat{\gamma}_1)\underset{\sim}{I} + K(\hat{\gamma}_{-1}-\hat{\gamma}_1)\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9B\text{-}14)$$

$$+ K^2\hat{\gamma}_{-1}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + K^2\hat{\gamma}_1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Since the stress components are again constants, the equilibrium equations (9A 1) are satisfied. Also, in the undeformed reference configuration $\underset{\sim}{b} = \underset{\sim}{c} = \underset{\sim}{I}$, $K = 0$ and (9A 11) along with (13) requires that

$$\hat{\gamma}_{-1}(0) + \hat{\gamma}_0(0) + \hat{\gamma}_1(0) = 0 \qquad (9B\ 15)$$

where

$$\hat{\gamma}_\alpha(0) = \gamma_\alpha(3,3,1)$$

From (14) we find

$$t_{11} = \tau(K^2) + K^2\hat{\gamma}_{-1}(K^2)$$

$$t_{22} = \tau(K^2) + K^2\hat{\gamma}_1(K^2)$$

$$t_{33} = \tau(K^2) \tag{9B-16}$$

$$t_{12} = K\hat{\mu}(K^2) \qquad t_{13} = 0 = t_{23}$$

where we have defined

$$\tau(K^2) = \hat{\gamma}_{-1} + \hat{\gamma}_0 + \hat{\gamma}_1$$

$$\hat{\mu}(K^2) = \hat{\gamma}_{-1} - \hat{\gamma}_1 \tag{9B-17}$$

The response function $\hat{\mu}(K^2)$ is called the *generalized shear modulus*. We note that the normal stresses are non-vanishing, even functions of K. Also, (16) implies

$$t_{11} - t_{22} = K\,t_{12} \tag{9B-18}$$

which is called a *universal relation* between stress components, since it holds independently of the material response functions $\hat{\gamma}_\alpha$; i.e., (18) holds for *all* nonlinear, compressible isotropic elastic materials in simple shear. We note in addition that the only non-vanishing shear stress component t_{12} is an odd function of K, leading to the expected result that t_{12} acts in the same direction as the shearing.

We now consider the boundary conditions. From (6) and (16) we have on

OA or cb: $t_i = \mp t_{i2}$

$$t_1 = \mp K\,\hat{\mu}(K^2)\ ,\ t_2 = \mp [\tau(K^2) + K^2\hat{\gamma}_{-1}(K^2)]\ ,\qquad (9B\text{-}19)$$

$$t_3 = 0$$

Ab. $t_i = t_{i1}(1+K^2)^{-1/2} - t_{i2}\,K(1+K^2)^{-1/2}$

$$t_1 = (1+K^2)^{-1/2}[\tau(K^2) + K^2\hat{\gamma}_{-1}(K^2) - K^2\hat{\mu}(K^2)]$$

$$t_2 = (1+K^2)^{-1/2}[K\hat{\mu}(K^2) - K\tau(K^2) - K^3\hat{\gamma}_1(K^2)] \qquad (9B\text{-}20)$$

$$t_3 = 0$$

Oc: reverse the signs in (20).

By $(17)_2$ $\hat{\gamma}_1 = \hat{\gamma}_{-1} - \hat{\mu}$ and (20) becomes

$$t_1 = (1+K^2)^{-1/2}(\tau + K^2\hat{\gamma}_{-1} - K^2\hat{\mu})$$

$$t_2 = (1+K^2)^{-1/2}[K(1+K^2)\hat{\mu} - K\tau - K^3\hat{\gamma}_{-1}] \qquad (9B\text{-}21)$$

$$t_3 = 0$$

We now resolve the applied tractions on face Ab into components N,T along the normal and tangential directions. Then

$$\begin{aligned} N &= \underset{\sim}{t} \quad \underset{\sim}{n} = t_1 n_1 + t_2 n_2 \\ &\overset{(6)_3}{=} (1+K^2)^{-1}[\tau + K^2\hat{\gamma}_{-1} - K^2\hat{\mu} - K^2(1+K^2)\hat{\mu} \\ &\qquad + K^2\tau + K^4\hat{\gamma}_{-1}] \end{aligned}$$

or

$$\begin{aligned} N = (1+K^2)^{-1}[&(1+K^2)\tau(K^2) - K^2(2+K^2)\hat{\mu}(K^2) \\ &+ K^2(1+K^2)\hat{\gamma}_{-1}(K^2)] \end{aligned} \qquad (9B-22)$$

Let $\underset{\sim}{e}$ be a unit vector along direction Ab:

$$e_1 = \sin\alpha = -n_2 = K(1+K^2)^{-1/2}$$

$$e_2 = \cos\alpha = n_1 = (1+K^2)^{-1/2}$$

Then

$$\begin{aligned} T &= \underset{\sim}{t} \cdot \underset{\sim}{e} \\ &= (1+K^2)^{-1}[K\tau + K^3\hat{\gamma}_{-1} - K^3\hat{\mu} + K(1+K^2)\hat{\mu} \\ &\qquad - K\tau - K^3\hat{\gamma}_{-1}] \end{aligned}$$

or

$$T = (1+K^2)^{-1}K\hat{\mu}(K^2) \qquad (9B-23)$$

Hence, from (21) - (23) we see that the nonlinear theory predicts that normal tractions must be applied to the faces in addition to the shearing tractions. (See Fig. IX 3.) Also, the magnitudes of N and T are dependent upon the material. On the faces $X_3 = x_3 =$ const., we have $\underset{\sim}{n} = (0,0,\pm 1)$ and

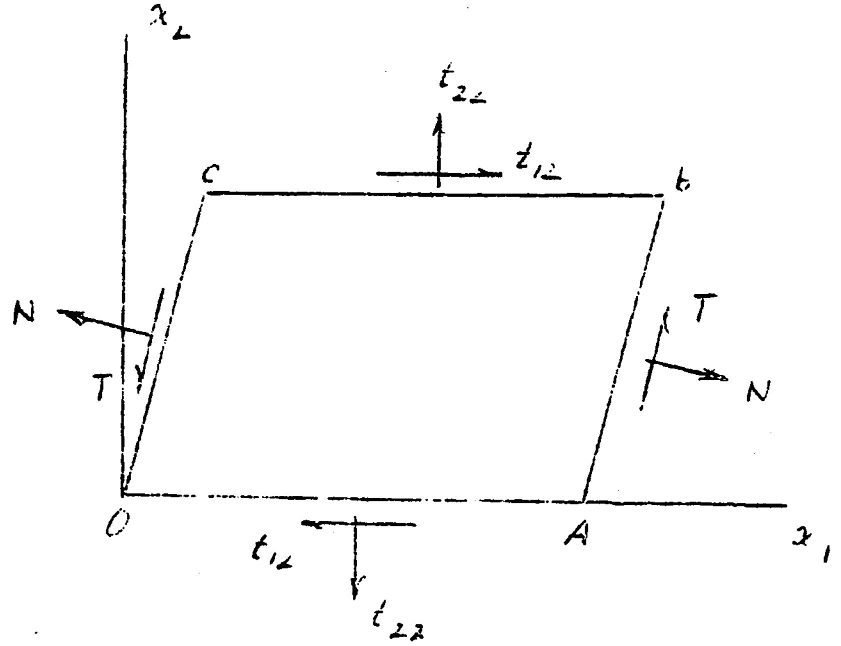


Fig. IX-3

$$t_i = \pm t_{i3} = (0, 0, \pm t_{33})$$

or

$$t_3 \overset{(16)}{=} \pm \tau(K^2) \qquad (9B \cdot 24)$$

so that a normal traction which is an even function of K must be applied to these faces. The presence of this traction is called the Kelvin effect. It was inferred by Kelvin that in the absence of t_3 the block will either expand or contract in the x_3 direction in an amount proportional to K^2. Note that in the linear theory approximation this effect is higher order. The appearance of the normal tractions $(19)_2$ and (22) is called the Poynting effect. In a series of experiments during the period 1905-1913 Poynting demonstrated the effect for the case of torsion of a circular cylinder. Without these normal tractions it is inferred that the faces Oc, Ab and OA, cb would either draw together or spread apart by an amount proportional to K^2.

Reducing to the case $K << 1$, we have by (9A·18), (4), (13)

$$\hat{\gamma}_{-1} \cong 2\mu \quad , \quad \hat{\gamma}_0 \cong -3\mu \quad , \quad \hat{\gamma}_1 \cong \mu \quad , \quad \mu = \text{const.}$$

Then (17) become

$$\hat{\mu} = \hat{\gamma}_{-1} - \hat{\gamma}_1 \cong \mu$$

$$\tau = \hat{\gamma}_{-1} + \hat{\gamma}_0 + \hat{\gamma}_1 \cong 0$$

and (22), (23) give

$$N \cong (1+K^2)^{-1}(-2K^2\mu + 2K^2\mu) \cong 0$$

$$T \cong (1+K^2)^{-1}K\mu \cong K\mu$$

Hence, in the limit as K → 0, N vanishes and T is linear in K. On face OA from (19)

$$t_1 \cong -K\mu \quad , \quad t_2 \cong -2K^2\mu = 0(K^2) \cong 0$$

and from (24) on $X_3 = x_3 = \text{const.}$

$$t_3 \cong 0$$

Thus, the results of the linear theory are recovered.

For incompressible materials from (9A-14,15), eqns. (13) and (14) are replaced by

$$\tilde{\gamma}_\alpha(3+K^2,\ 3+K^2) = \bar{\gamma}_\alpha(K^2) \quad , \quad \alpha = -1,1$$

$$\underset{\sim}{t} = (-p + \bar{\gamma}_{-1} + \bar{\gamma}_1)\underset{\sim}{I} + K\bar{\mu}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} K^2\bar{\gamma}_{-1} & 0 & 0 \\ 0 & K^2\bar{\gamma}_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9B\text{-}25)$$

where

$$\bar{\mu}(K^2) = \bar{\gamma}_{-1}(K^2) - \bar{\gamma}_1(K^2) \qquad (9B\text{-}26)$$

Generally, the arbitrary pressure p can vary with $\underset{\sim}{x}$, but for this problem the equilibrium equations (9A-1) require using (25)

$$-\frac{\partial p}{\partial x_i} = 0 \qquad \text{or} \qquad p = p_0 = \text{const.} \tag{9B-27}$$

where p_0 is arbitrary. From (25) and (27) the stress components are

$$\begin{aligned} t_{11} &= -p_0 + \bar{\gamma}_{-1} + \bar{\gamma}_1 + K^2\bar{\gamma}_{-1} \\ t_{22} &= -p_0 + \bar{\gamma}_{-1} + \bar{\gamma}_1 + K^2\bar{\gamma}_1 \\ t_{33} &= -p_0 + \bar{\gamma}_{-1} + \bar{\gamma}_1 \\ t_{12} &= K\bar{\mu} \quad , \quad t_{13} = t_{23} = 0 \end{aligned} \tag{9B-28}$$

Since p_0 is arbitrary, the applied tractions necessary to produce the deformation are not uniquely determined in contrast to the results for compressible materials. In particular, it is possible for any pair of parallel faces to be free of normal tractions. For example, if we choose

$$p_0 = \bar{\gamma}_{-1} + \bar{\gamma}_1 \tag{9B-29}$$

then from (28)

$$t_{11} = K^2\bar{\gamma}_{-1}(K^2) \quad , \quad t_{22} = K^2\bar{\gamma}_1(K^2) \quad , \quad t_{33} = 0 \tag{9B-30}$$

i.e., the faces $X_3 = x_3 =$ const. are traction free. Also, we can show from (30) that on

cb: $t_1 = t_{12} = (K\bar{\mu}, K^2\bar{\gamma}_1, 0)$

Oc: $t_1 = (1+K^2)^{-1/2}K^2(\bar{\gamma}_{-1} - \bar{\mu})$

$t_2 = (1+K^2)^{-1/2}[K(1+K^2)\bar{\mu} \quad K^3\bar{\gamma}_{-1}]$

C. Torsion of a Circular Cylinder

An example of a non-homogeneous large deformation is the uniform twist of a circular cylinder. If (R,Θ,Z) and (r,θ,z) are the cylindrical coordinates of a material point before and after deformation, then the uniform twist is specified by the mapping

$$r = R \quad , \quad \theta = \Theta + KZ \quad , \quad z = Z \qquad (9C\text{-}1)$$

where K is the twist. See Fig. IX-4. From (1) material points originally in the plane Z = const. remain in that plane after deformation. Cross sections Z = const. rotate relative to one another in an amount proportional to their axial distance from the end plane $Z = 0$. Material points originally on cylindrical surfaces R = const. remain on those surfaces after deformation.

1. Linear Theory

For this case the deformation is assumed small such that $KZ \leq K\ell << 1$, where ℓ is the length of the cylinder, i.e.

$$K << \frac{1}{\ell} \qquad (9C\text{-}2)$$

Because of this assumption, the linear theory can be treated using rectangular cartesian coordinates X_K and x_i. Let

$$\begin{aligned} x_1 &= r\cos\theta \quad , \quad x_2 = r\sin\theta \quad , \quad x_3 = z \\ X_1 &= R\cos\Theta \quad , \quad X_2 = R\sin\Theta \quad , \quad X_3 = Z \end{aligned} \qquad (9C\text{-}3)$$

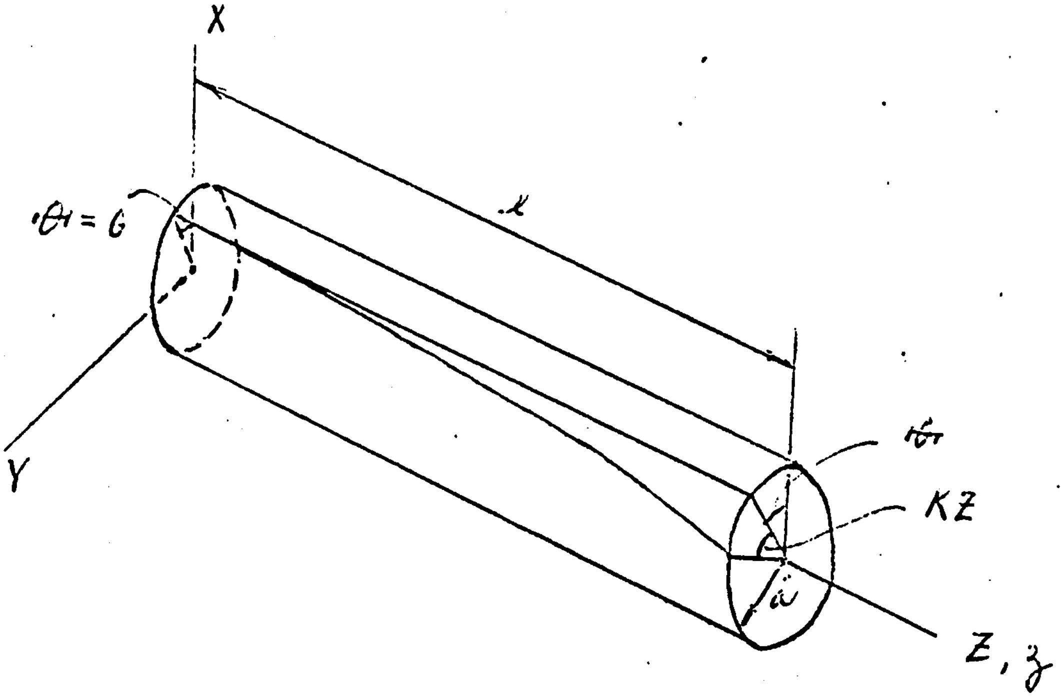


Fig. IX-4

Then from the deformation functions (1),

$$x_1 = r\cos\theta = R\cos(\Theta + KZ)$$
$$\cong R\cos\Theta - RKZ\sin\Theta$$
$$\cong X_1 - KX_2X_3 \qquad (9C\ 4)$$

Similarly,

$$x_2 \cong X_2 + KX_1X_3 \qquad (9C\text{-}5)$$

Then (4) and (5) imply

$$X_1 \cong x_1 + KX_2X_3 \overset{(1)}{=} x_1 + KX_2x_3$$
$$X_2 \cong x_2 - KX_1X_3 \overset{(1)}{=} x_2 - KX_1x_3$$

and

$$x_1 \cong X_1 - Kx_3(x_2 - KX_1x_3)$$
$$\cong X_1 - Kx_2x_3$$

$$x_2 \cong X_2 + Kx_3(x_1 + KX_2x_3)$$
$$\cong X_2 + Kx_1x_3$$

Hence, the displacements for the linear theory are

$$u_1 = -Kx_2x_3 \quad , \quad u_2 = Kx_1x_3 \quad , \quad u_3 = 0 \qquad (9C\text{-}6)$$

Then the displacement gradients and strain tensor are

$$u_{i,j} = \begin{pmatrix} 0 & -Kx_3 & -Kx_2 \\ Kx_3 & 0 & Kx_1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{e}_{ij} = u_{(i,j)} = \frac{1}{2}\begin{pmatrix} 0 & 0 & -Kx_2 \\ 0 & 0 & Kx_1 \\ -Kx_2 & Kx_1 & \end{pmatrix}$$

(9C 7)

Note that this deformation is non-homogeneous. From (7)

$$I_{\tilde{e}} = u_{i,i} = 0 \qquad (9C\text{-}8)$$

From (9A-4)

$$t_{ij} = 2\mu\, \tilde{e}_{ij} = \mu \begin{pmatrix} 0 & 0 & -Kx_2 \\ 0 & 0 & Kx_1 \\ -Kx_2 & Kx_1 & 0 \end{pmatrix} \qquad (9C\text{-}9)$$

The equilibrium equations (9A-1) are

$$t_{i1,1} + t_{i2,2} + t_{i3,3} = 0$$

which are identically satisfied by the stress field (9). Consider the boundary conditions. On the lateral surface r=a, we have (see Fig. IX-5)

$$n_1 = \cos\theta = \frac{x_1}{a}, \quad n_2 = \sin\theta = \frac{x_2}{a}, \quad n_3 = 0 \qquad (9C\text{-}10)$$

and from $t_i = t_{ij}n_j$ we find

$$t_i = t_{i1}n_1 + t_{i2}n_2 = t_{i1}\frac{x_1}{a} + t_{i2}\frac{x_2}{a} \qquad (9C\text{-}11)$$

Hence from (9)

$$t_1 = 0, \quad t_2 = 0, \quad t_3 = -\mu K x_2 \frac{x_1}{a} + \mu K x_1 \frac{x_2}{a} = 0$$

i.e., the lateral surface is free from surface tractions. Consider the end section $x_3 = \ell$ where $n_i = (0,0,1)$. Then

$$t_i = t_{i3} = (-K\mu x_2, K\mu x_1, 0) \qquad (9C\text{-}12)$$

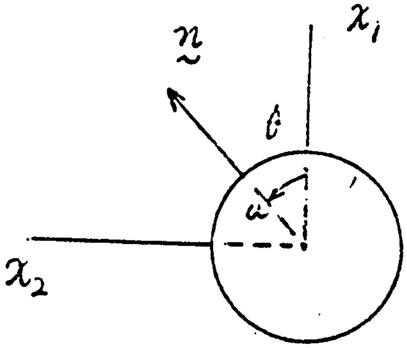


Fig. IX-5

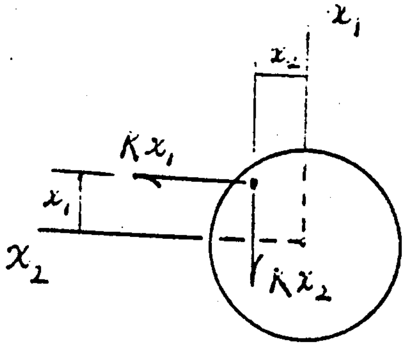


Fig. IX-6

which implies a distribution of shearing tractions on $x_3 = \ell$. (See Fig. IX 6) On $x_3 = 0$, $n_i = (0,0\ -1)$ and the signs in (12) reverse. The shearing tractions (12) are statically equivalent to a torque T about the x_3 axis. To see this, we compute the resultant force $\underset{\sim}{F}$ and moment $\underset{\sim}{M}$ due to (12). By definition we have

$$\underset{\sim}{F} = \int \underset{\sim}{t}\, dA \qquad \underset{\sim}{M} = \int \underset{\sim}{r} \times \underset{\sim}{t}\, dA$$

Then

$$F_1 = \int t_1\, dA = \int_0^{2\pi} \int_0^{a} (-K\mu r \sin\theta)\, r dr d\theta$$

$$= \int_0^{a} K\mu r^2 (\cos\theta)\Big|_0^{2\pi} dr = 0 \qquad (a)$$

$$F_2 = \int t_2\, dA = \int_0^{2\pi} \int_0^{a} K\mu r \cos\theta\, r dr d\theta$$

$$= \int_0^{a} K\mu r^2 (\sin\theta)\Big|_0^{2\pi} dr = 0 \qquad (b)$$

$$F_3 = \int t_3\, dA = 0$$

Hence, the resultant force on $x_3 = \ell$ (and also $x_3 = 0$) vanishes. Now for the moment

$$M_1 = \int e_{1jk} x_j t_k \, dA$$

$$= \int (\overset{0}{\cancel{x_2 t_3}} - x_3 t_2) \, dA$$

$$= - \ell \int t_2 \, dA \overset{(b)}{=} 0$$

$$M_2 = \int (x_3 t_1 \quad \overset{0}{\cancel{x_1 t_3}}) \, dA$$

$$= \ell \int t_1 \, dA \overset{(a)}{=} 0$$

$$T = M_3 = \int (x_1 t_2 - x_2 t_1) \, dA$$

$$= \int (K\mu x_1^2 + K\mu x_2^2) \, dA$$

$$= K\mu \int_0^{2\pi} \int_0^{a} r^2 \quad r dr d\theta = \frac{\pi}{2} K\mu a^4 \qquad (9C\ 13)$$

Thus, the bending moments M_1, M_2 vanish and a torque T <u>proportional to K</u> must be applied at the ends of the cylinder Eqn. (13) is the basis for experimental determination of the shear modulus μ through measurement of the twist K and the applied torque T. We note that the <u>torsional rigidity</u> of the cylinder is defined as the ratio

$$\frac{T}{K} = \frac{\pi}{2} \mu a^4 \qquad (9C\cdot 14)$$

2. Nonlinear Theory (Incompressible Materials)

We reconsider the deformation (1) when the twist K can take on finite values. Then the rectangular cartesian coordinates are not suitable, and we use the two curvilinear coordinate systems $X^K = (R,\theta,Z)$ and $x^i = (r,\theta,z)$. The metric tensor components for these systems are (see Appendix 3):

$$G_{KM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad G^{KM} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9C-15}$$

and

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9C-16}$$

In curvilinear coordinates the constitutive equations (9A-14) become

$$t_{ij} = - p\, g_{ij} + \tilde{\gamma}_{-1}\, b_{ij} + \tilde{\gamma}_1\, c_{ij} \tag{9C-17}$$

where $\underset{\sim}{b}, \underset{\sim}{c}$ are given by

$$b^{ij} = G^{KM} x^i{}_{,K} x^j{}_{,M} , \quad c_{ij} = G_{KM} X^K{}_{,i} X^M{}_{,j} \tag{9C-18}$$

and

$$\tilde{\gamma}_\alpha = \tilde{\gamma}_\alpha (I_b, II_b) , \quad \alpha = -1,1 \tag{9C-19}$$

From the deformation (1) we find

$$x^i{}_{,K} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix}, \quad X^K{}_{,i} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -K \\ 0 & 0 & 1 \end{pmatrix} \qquad (9C\text{-}20)$$

Note that J=1, so that the deformation is isochoric, and the conservation of mass (9A-7) is satisfied by $\rho = \rho_o$. From (15) and (20)

$$c_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -K & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -K \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & R^2 & -KR^2 \\ 0 & -KR^2 & K^2R^2+1 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & -Kr^2 \\ 0 & -Kr^2 & K^2r^2+1 \end{pmatrix} \qquad (9C\text{-}21)$$

Similarly, from (16) and (20)

$$b^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & K \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/R^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & K & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{R^2}+K^2 & K \\ 0 & K & 1 \end{pmatrix} \overset{(1)}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2}+K^2 & K \\ 0 & K & 1 \end{pmatrix} \qquad (9C\text{-}22)$$

It is interesting to note that $\underset{\sim}{b}, \underset{\sim}{c}$ depend on r while the deformation gradients are constant. From (16) and (22) the mixed components of $\underset{\sim}{b}$ are

$$b^i_j = b^{ik} g_{kj} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2}+K^2 & K \\ 0 & K & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+K^2r^2 & K \\ 0 & Kr^2 & 1 \end{pmatrix} \qquad (9C\text{-}23)$$

Now we compute $tr(\underset{\sim}{b}^2) = b^i_j \, b^j_i$. From (23)

$$b^i_k \, b^k_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+K^2r^2 & K \\ 0 & Kr^2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+K^2r^2 & K \\ 0 & Kr^2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & (1+K^2r^2)^2+K^2r^2 & (1+K^2r^2)K+K \\ 0 & Kr^2(1+K^2r^2)+K^2r^4 & K^2r^2+1 \end{pmatrix}$$

Hence,

$$tr(\underset{\sim}{b}^2) = 1+(1+K^2r^2)^2+2K^2r^2+1 = 3+4K^2r^2+K^4r^4 \qquad (9C\text{-}24)$$

and from (23)

$$I_b = tr\,\underset{\sim}{b} = b^i_i = 3 + K^2r^2$$

$$II_b = \frac{1}{2}\,[I_b^2 - tr(\underset{\sim}{b}^2)] = \frac{1}{2}\,[(3+K^2r^2)^2 - (3+4K^2r^2+K^4r^4)]$$

$$= \frac{1}{2}\,(6+2K^2r^2) = 3+K^2r^2 = I_b$$

We define

$$\hat{\gamma}_\alpha(K^2r^2) \equiv \tilde{\gamma}_\alpha(3+K^2r^2, 3+K^2r^2) \quad , \quad \alpha = -1,1 \qquad (9C\text{-}25)$$

We also need b_{ij} for (17):

$$b_{ij} = g_{ik}b^k_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1+K^2r^2 & K \\ 0 & Kr^2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2(1+K^2r^2) & Kr^2 \\ 0 & Kr^2 & 1 \end{pmatrix} \qquad (9C\text{-}26)$$

Hence, by (16),(21),(25),(26) and (17), the stress components become

$$t_{ij} = -p\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \hat{\gamma}_{-1}(K^2r^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2(1+K^2r^2) & Kr^2 \\ 0 & Kr^2 & 1 \end{pmatrix}$$

$$+ \hat{\gamma}_1(K^2r^2)\begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & -Kr^2 \\ 0 & -Kr^2 & K^2r^2+1 \end{pmatrix}$$

$$= (-p+\hat{\gamma}_{-1}+\hat{\gamma}_1)g_{ij} + (\hat{\gamma}_{-1}-\hat{\gamma}_1)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Kr^2 \\ 0 & Kr^2 & 0 \end{pmatrix}$$

$$+ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \hat{\gamma}_{-1}K^2r^4 & 0 \\ 0 & 0 & \hat{\gamma}_1K^2r^2 \end{pmatrix} \qquad (9C\text{-}27)$$

Note here that all terms above, except p, depend on r alone. Now we have the stress components

$$\begin{aligned} &\text{radial:} && t_{11} = \tau \\ &\text{tangential:} && t_{22} = r^2\tau + \hat{\gamma}_{-1}K^2r^4 \\ &\text{axial:} && t_{33} = \tau + \hat{\gamma}_1K^2r^2 \\ &\text{shear:} && t_{23} = \hat{\mu}Kr^2 \quad , \quad t_{12} = 0 = t_{13} \end{aligned} \tag{9C-28}$$

where we have defined the functions

$$\tau = -p + \hat{\gamma}_{-1} + \hat{\gamma}_1 \quad , \quad \hat{\mu}(K^2r^2) = \hat{\gamma}_{-1}(K^2r^2) - \hat{\gamma}_1(K^2r^2) \tag{9C-29}$$

Let T_{rr}, $T_{r\theta}$, etc. be physical components of stress such that

$$\begin{aligned} &T_{rr} = t_{11} \quad , \quad T_{\theta\theta} = \frac{1}{r^2}\, t_{22} \quad , \quad T_{zz} = t_{33} \\ &T_{r\theta} = \frac{1}{r}\, t_{12} \quad , \quad T_{rz} = t_{13} \quad , \quad T_{\theta z} = \frac{1}{r}\, t_{23} \end{aligned} \tag{9C-30}$$

From (28) we have

$$\begin{aligned} T_{rr} &= \tau \\ T_{\theta\theta} &= \tau + \hat{\gamma}_{-1}K^2r^2 \\ T_{zz} &= \tau + \hat{\gamma}_1K^2r^2 \\ T_{\theta z} &= \hat{\mu}Kr \quad , \quad T_{r\theta} = T_{rz} = 0 \end{aligned} \tag{9C 31}$$

In terms of these components the equilibrium equations in cylindrical coordinates are (these follow from (8A-15) with $t_{rr} = T_{rr}$, etc.)

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0$$

$$\frac{\partial T_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{2}{r} T_{r\theta} = 0 \qquad (9C\ 32)$$

$$\frac{\partial T_{rz}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta z}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{1}{r} T_{rz} = 0$$

From $(32)_{2,3}$ we find $\frac{\partial p}{\partial \theta} = 0 = \frac{\partial p}{\partial z}$ and hence

$$p = p(r) \qquad (9C\text{-}33)$$

Then from $(32)_1$

$$\tau' + \frac{1}{r}(-\hat{\gamma}_{-1}K^2r^2) = 0$$

Integrating, we find

$$\tau(K^2r^2) = -p + \hat{\gamma}_{-1} + \hat{\gamma}_1 = K^2 \int \hat{\gamma}_{-1}(K^2r^2)rdr + C \qquad (9C\ 34)$$

Note that this equation determines p, but not uniquely. The laterial surface can be rendered free of tractions by choosing

$$C = K^2 \int \hat{\gamma}_{-1}(K^2r^2)rdr\Big|_{r=a} \qquad (9C\text{-}35)$$

Then (34) implies

$$\tau(K^2r^2) = -K^2 \int_r^a \gamma_{-1}(K^2\xi^2)\xi d\xi \qquad (9C\text{-}36)$$

and

$$\tau(K^2a^2) = [-p + \hat{\gamma}_{-1}(K^2a^2) + \hat{\gamma}_1(K^2a^2)]_{r=a} = 0 \qquad (9C\text{-}37)$$

The traction boundary conditions in terms of tensor components are

$$t_i = t_{ij}\, n^j \tag{9C-38}$$

On the lateral surface $\underset{\sim}{n} = \underset{\sim}{g}_r = \underset{\sim}{g}^r$, $n^i = (1,0,0)$ and

$$t_i(a) = t_{i1}(a)$$

or from (28)

$$t_1(a) = t_{11}(a) = \tau(K^2a^2) \quad , \quad t_2(a) = 0 = t_3(a)$$

But by (37) $t_1(a) = 0$ and the lateral surface is traction free by the choice (35). On the end of the cylinder $z = \ell$, we have $\underset{\sim}{n} = \underset{\sim}{g}_z = \underset{\sim}{g}^z$, $n^i = (0,0,1)$ and (38) implies

$$t_i(r) = t_{i3}(r)$$

and from (28)

$$\begin{aligned} t_1 &= 0 \quad , \quad t_2 = t_{23}(r) = \hat{\mu}(K^2r^2)Kr \\ t_3(r) &= t_{33}(r) = \tau(K^2r^2) + \hat{\gamma}_1(K^2r^2)K^2r^2 \end{aligned} \tag{9C 39}$$

In addition to the shear traction t_{23}, an odd function of r, a <u>normal traction</u> $t_{33}(r^2)$ must be applied to the ends of the cylinder in order to maintain the deformation (1). This is the <u>Poynting effect</u> for uniform twist of a circular cylinder.

The tractions (39) are equivalent to a torque T about the z axis and an axial force N. Since the physical components corresponding to (39) are

$$T_{\theta z} = \hat{\mu} K r \quad ; \quad T_{zz} = \tau + \hat{\gamma}_1 K^2 r^2 \tag{9C-40}$$

on $z = \ell$. T and N are

$$T = 2\pi K \int_0^a \hat{\mu}(K^2 r^2) r^3 dr \tag{9C 41}$$

$$N = - 2\pi K^2 \int_0^a \left[\int_r^a \hat{\gamma}_{-1}(K^2 \xi^2) \xi d\xi + \hat{\gamma}_1 (K^2 r^2) r^2 \right] r dr \tag{9C-42}$$

We can show that for K infinitesimal, T is approximately proportional to K, while N is proportional to K^2. Hence, N is a 2nd order effect for K << 1. It is inferred from the presence of N for large twists that if N is not applied on $z = 0, \ell$, the cylinder will elongate or contract, depending on the character of the response functions $\hat{\gamma}_{-1}, \hat{\gamma}_1$.

This problem is an example of a finite non-homogeneous deformation which is an exact solution for all isotropic, homogeneous, incompressible elastic materials by the application of suitable surface tractions alone. Such solutions are called universal or controllable, since they exist independent of the particular response functions for the material.

At present the following universal solutions are known: (Reference: C. Truesdell, "The Elements of Continuum Mechanics", Springer-Verlag, 1966.)

Family 1 - Bending, stretching and shearing of a rectangular block

$$r = \sqrt{2AX} \quad , \quad \theta = BY \quad , \quad z = \frac{Z}{AB} - BCY \quad , \quad AB \neq 0$$

Family 2 - Straightening, stretching and shearing of a sector of a circular-cylindrical tube

$$x = \frac{1}{2} AB^2R^2 \quad , \quad y = \frac{\Theta}{AB} \quad , \quad z = \frac{Z}{B} + \frac{C\Theta}{AB} \quad , \quad AB \neq 0$$

Family 3 - Inflation or eversion, bending, torsion, extension and shear of a sector of a circular-cylindrical tube

$$r = (AR^2+B)^{1/2} \quad , \quad \theta = C\Theta + DZ \quad , \quad z = E\Theta + FZ$$

$$A(CF-DE) = 1$$

Family 4 - Inflation or eversion of a sector of a spherical shell

$$r = (\pm R^3 + A)^{1/3} \quad , \quad \theta = \pm \Theta \quad , \quad \phi = \Phi$$

Family 5

$$r = AR \quad , \quad \theta = B \log R + C\Theta \quad , \quad z = \frac{Z}{A^2C} \quad , \quad A^2C \neq 0$$

D. Thermoelasticity Governing Equations

We consider homogeneous isotropic, incompressible thermoelastic solids in mechanical equilibrium and subjected to steady state temperature fields. We assume that the body force and heat source functions vanish. The governing equations are.

1. Linear Theory

Equilibrium Equations.

$$t_{ij,j} = 0 \tag{9D-1}$$

Boundary Conditions:

$$t_{ij} n_j = t_i \tag{9D-2}$$

where t_i is specified.

Constitutive Equations:

$$t_{ij} = -(p+\beta T)\delta_{ij} + 2\mu u_{(i,j)} \tag{9D-3}$$

where
$$q_i = -\kappa T_{,i}$$
$$T = \theta - \theta_0 , \quad \beta = (3\lambda+2\mu)\alpha \tag{9D-4}$$

and α is the coefficient of thermal expansion.

Heat Conduction Equation:

$$q_{i,i} = 0 \tag{9D-5}$$

Boundary Condition:

$$q_i n_i = n \tag{9D-6}$$

where n is specified.

2. Nonlinear Theory

Equations (1), (2), (5) and (6) are unchanged. In addition we have

Constitutive Equations:

$$t_{ij} = -p\delta_{ij} + \tilde{\gamma}_{-1}b_{ij} + \tilde{\gamma}_{1}c_{ij} \tag{9D 7}$$

$$q_i = (\psi_{-1}b_{ij} + \psi_0\delta_{ij} + \psi_1 c_{ij})\theta_{,j} \tag{9D 8}$$

where

$$\tilde{\gamma}_\alpha = \tilde{\gamma}_\alpha(\theta, I_b, II_b) \quad ; \quad \alpha = -1, 1 \tag{9D 9}$$

$$\psi_\alpha = \psi_\alpha(I_b, II_b, I_1, I_2, I_3) \quad ; \quad \alpha = -1, 0, 1 \tag{9D 10}$$

$$I_1 = \theta_{,i}\theta_{,i} \ , \ I_2 = b_{ij}\theta_{,i}\theta_{,j} \ ; \ I_3 = c_{ij}\theta_{,i}\theta_{,j} \tag{9D-11}$$

Note that the heat flux vector is assumed independent of temperature θ. This is a special case of the general form of q_i. We make this assumption because it can be shown that if q_i depends explicitly on θ, then the only universal or controllable solutions for the nonlinear theory are those for which $\theta = \text{const}$. See H. J. Petroski and D. E. Carlson Archive Rational Mechanics and Analysis, Vol. 31, 1968.

E. Axial Heat Conductor

Ref. H. J. Petroski and D. E. Carlson, J. Appl. Mech., Vol. 37, 1151-1154, 1970.

We consider the deformation and temperature field

$$x_1 = \frac{1}{\sqrt{c}} X_1 \quad , \quad x_2 = \frac{1}{\sqrt{c}} X_2 \quad ; \quad x_3 = cX_3 \quad , \quad c > 1 \qquad (9E\text{-}1)$$

$$0 = T_0 + \frac{T_1}{\ell} x_3 \quad , \quad T_1 > T_0 \qquad (9E\text{-}2)$$

Equations (1) represent a homogeneous deformation of a circular cylinder of radius $\sqrt{c}$ a and length $\frac{\ell}{c}$ into a circular cylinder of radius a and length ℓ. The ends of the cylinder are maintained at constant temperatures T_0 and T_0+T_1, respectively, while the lateral surface is insulated ($q_1 = 0$).

1. Linear Theory

From (1) we have

$$u_1 = x_1-X_1 = (\frac{1}{\sqrt{c}} - 1)X_1 = (1 - \sqrt{c})x_1$$

$$u_2 = x_2-X_2 = (\frac{1}{\sqrt{c}} - 1)X_2 = (1 - \sqrt{c})x_2$$

$$u_3 = x_3-X_3 = (c-1)X_3 = (1 - \frac{1}{c})x_3$$

Hence,

$$u_{i,j} = \begin{pmatrix} 1 - \sqrt{c} & 0 & 0 \\ 0 & 1 - \sqrt{c} & 0 \\ 0 & 0 & 1 - \frac{1}{c} \end{pmatrix} = u_{(i,j)} \qquad (9E\text{-}3)$$

We now state the conditions of the small deformation theory: $|u_{i,j}| << 1$. Let

$$c = 1+\varepsilon \ , \quad 0 < \varepsilon << 1 \tag{9E-4}$$

Then

$$\sqrt{c} = (1+\varepsilon)^{1/2} \cong 1 + \frac{1}{2}\varepsilon$$

$$1 \cdot \sqrt{c} = -\frac{1}{2}\varepsilon$$

$$1 - c^{1} = 1 - (1+\varepsilon)^{1} \cong 1 - (1 \cdot \varepsilon) = \varepsilon$$

and (3) becomes

$$u_{(i,j)} \cong \frac{1}{2}\begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix} \tag{9E 5}$$

Letting θ_0 be the constant temperature of the reference configuration, then (2) implies

$$T = (\theta \cdot \theta_0) = T_0 + \frac{T_1}{\ell} x_3 \cdot \theta_0 \tag{9E 6}$$

Hence, for a small temperature rise we must require that $\frac{T_0+T_1}{\theta_0} << 1$. Using (5),(6) in (9D-3), the stress components are

$$t_{ij} = -[p + \beta(T_0 + \frac{T_1}{\ell} x_3 - \theta_0)]\delta_{ij} + \mu\begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix} \tag{9E 7}$$

Also, from (6) and (9D-4), the heat flux vector is

$$q_i = -\kappa(0, 0, \frac{T_1}{\ell}) = \text{const.} \qquad (9E\text{-}8)$$

Since q_i is a const. vector, the heat conduction eqn. (9D-5) is satisfied by the temperature field. From (7) and (9D-1) the equilibrium equations yield

$$[p + \beta(T_0 + \frac{T_1}{\ell} x_3 - \theta_0)]_{,i} = 0$$

which imply $p_{,1} = 0 = p_{,2}$ and

$$p(x_3) = -\beta(T_0 + \frac{T_1}{\ell} x_3 - \theta_0) + p_0 \qquad (9E\ 9)$$

where p_0 is an arbitrary constant. Then from (7) we have

$$t_{ij} = -p_0\delta_{ij} + \mu \begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 2\varepsilon \end{pmatrix} \qquad (9E\text{-}10)$$

We now consider the boundary conditions on the ends of the cylinder $x_3 = 0, \ell$. From (9D-2) and (9D 6) we have on $x_3 = \ell$:

$$t_i = t_{i3} \quad , \quad h = q_3 \qquad (9E\text{-}10a)$$

and by (8) and (10)

$$x_3 = \ell: \quad t_i = [0, 0, -(p_0 - 2\mu\varepsilon)] \qquad (9E\ 11)$$

$$h = -\kappa \frac{T_1}{\ell} \qquad (9E\text{-}12)$$

Strain is the change in length and relative orientation of line elements in the body. A measure of the change from $t=0$ to time t is $ds^2 - dS^2$ or

$$ds^2 - dS^2 = \underbrace{(C_{KM} - \delta_{KM})}_{\text{noted as } 2E_{KM}} dX_K \, dX_M$$

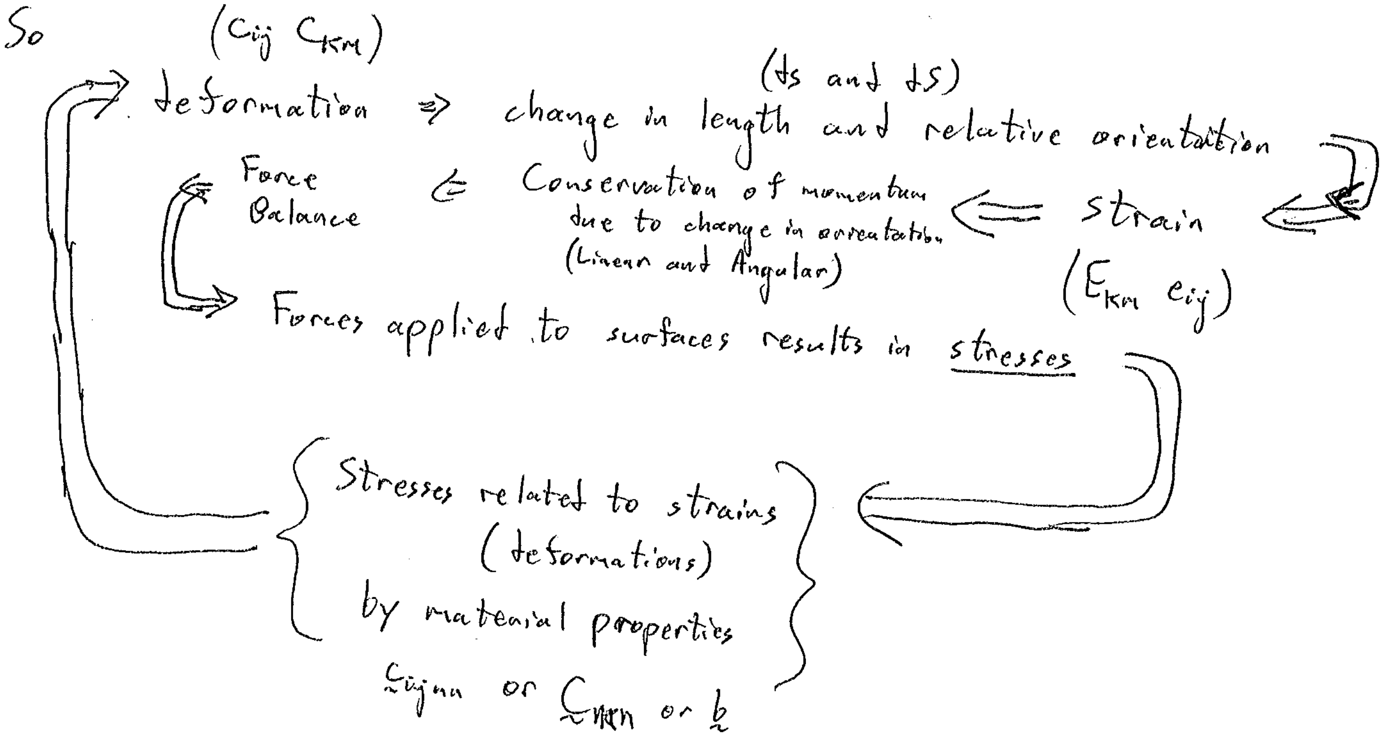
Lagrange's Strain tensor

Or

$$ds^2 - dS^2 = \underbrace{(\delta_{ij} - c_{ij})}_{\rightarrow 2e_{ij}} dx_i \, dx_j$$

Euler's Strain tensor

So



$(c_{ij} \; C_{KM})$

deformation ⇒ change in length and relative orientation

$(ds$ and $dS)$

strain

$(E_{KM} \; e_{ij})$

Conservation of momentum due to change in orientation (Linear and Angular)

Force Balance

Forces applied to surfaces results in stresses

Stresses related to strains (deformations) by material properties

$\underset{\sim}{c}_{ijmn}$ or $\underset{\sim}{C}_{MN}$ or $\underset{\sim}{b}$

$$\left[t_{ij} = \lambda I_{\tilde{e}} \delta_{ij} + 2\mu \tilde{e}_{ij} \right]$$

Hooke's Law

$$\delta_{MP} = \frac{\partial x_K}{\partial X_P}(\underset{\sim}{X},0)\,\alpha_{KM}$$

Prove that shearing strains vanish along the principal axes

$$\delta_{MP}\cancel{\alpha_{im}} = \frac{\partial x_K}{\partial X_P}(\underset{\sim}{X},0)\,\underbrace{\alpha_{Km}\cancel{\alpha_{im}}}_{\delta_{Ki}}$$

$\alpha_{mi}\ \alpha_{im}$

$\alpha_{KM}\,\alpha_{kp} =$

Thurs

11 am Sub

10 am El-Sayed

$$\cancel{\delta_{Ki}}\,\alpha_{Kp} = \frac{\partial x_K}{\partial X_P}\,\cancel{\delta_{Ki}}$$

$$dX_k = \frac{\partial X_K}{\partial x_i}\,dx_i = X_{k,i}\,dx_i$$

$$D_{ij} = \frac{1}{2}(D_{ij} + D_{ji}) + \frac{1}{2}(D_{ij} - D_{ji})$$

$$\frac{du_i}{dX_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right)$$

$$\frac{1}{v_j}\frac{du_i}{dX} = \frac{1}{2}(\qquad) + \frac{1}{2}(\qquad)$$

$dX_i = v_i\,dX$ so

$$du_i = \left[\frac{1}{2}(\ ) + \frac{1}{2}(\ )\right]\frac{dX_i}{\cancel{dX_j}}\cancel{dX}$$

$$du_i = \left[\frac{1}{2}(\ ) + \frac{1}{2}(\ )\right] dX_i$$

5-3 | Define small deformation theory, when is it valid? ✓
⇒ See 2-30 through 2-32

5-3 | If $\underset{\sim}{E}$ is symmetric then $A_{KM} E_{KM}$ infers that $A_{KM} = A_{MK}$ (~~or~~ i.e. $\underset{\sim}{A}$ is also symmetric) Is this correct? How is this inferred? A, B, ··· are coefficients matrices for E_{KM} and $E_{KM} E_{LN}$ respectively. Is that the reason? They would have to be symmetric.

5-4 | Discuss the technique of determining independent components from the equality ~~eq~~ relations e.g. (5A-12) and (5A-14)

5-5 | In 5A-13 e_{ij} represents the strain tensor (i.e. linear strain tensor) because of small deformation theory, there is no distinction between the Lagrangian and Eulerian descriptions. (Correct?)

Eqn 5A-15 | Is I_e the 1st invariant of $\tilde{e}$ or
$\mathrm{tr}\,\tilde{e} = e_{11} + e_{22} + e_{33}$
Is this eqn Hooke's law for isotropic materials only?

Discuss expanding eqn 5A-15 out and form C_{KM} (see schaums (pg 143))

Gen | 5A-15 can be expanded and rewritten to give the traditional form of hooke's law e.g. $\epsilon_{xx} = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$
Need to coalesce the development of hooke's law from the start to see the "big" picture.

Gen ~~Some confusion over the notation $\bar{t}_{ij}$ $\bar{c}_{ijmn}$ etc. Is this related to a "primed" conditions such as for a coordinate system and after a rotation, the primed coordinate system.~~

Gen ~~The stress tensor t_{ij} is never presented as T_{ij} is there no lagrangian description or are the the same.~~

③ ④

$$\sigma_{(ij)} = b_{(ij)(km)} \underline{e_{(km)}}$$

$$b_{ijkm} = b_{jikm} = b_{jimk}$$

$$A_{KM} \, E_{(\underline{MK})}$$

$$A_{KM} \, E_{KM}$$

$$\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{bmatrix} a_{11} & c & d \\ c & a_{22} & e \\ d & e & a_{33} \end{bmatrix}$$

$$\phi = (\underset{\sim}{A}_S + \underset{\sim}{A}_A) E_{(S)}$$

$$\phi = \underset{\sim}{A}_S \underset{\sim}{E}_{(S)} + \cancel{\underset{\sim}{A}_A E_{(S)}} \rightarrow 0$$

$$\underline{A_{KM}} E_{MK} = \underline{A_{MK}} E_{KM}$$

K, M

Note that (12) implies a non vanishing heat flux across $x_3 = \ell$ such that heat is entering the cylinder. This must occur to maintain the constant temperature T_0+T_1 on this surface. Since p_0 is an arbitrary constant, we can specify the end surfaces to be free of tractions. This implies

$$p_0 = 2\mu\varepsilon \qquad (9E\ 13)$$

Then (10) reduces to

$$t_{ij} = 3\mu \begin{pmatrix} -\varepsilon & 0 & 0 \\ 0 & -\varepsilon & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad (9E\ 14)$$

On the lateral surface of the cylinder $n_i = (\frac{x_1}{a}, \frac{x_2}{a}, 0)$ and

$$\begin{aligned} t_i &= t_{i1} \frac{x_1}{a} + t_{i2} \frac{x_2}{a} \\ &= (-3\mu\varepsilon \frac{x_1}{a}, -3\mu\varepsilon \frac{x_2}{a}, 0) \end{aligned}$$

These tractions are equivalent to a uniformly distributed radial traction:

$$t_n = t_i n_i = -3\mu\varepsilon \overset{(4)}{=} -3\mu(c-1) < 0 \qquad (9E\text{-}15)$$

which acts to compress the cylinder uniformly along its length. Also, from (8)

$$h = q_i n_i = q_1 \frac{x_1}{a} + q_2 \frac{x_2}{a} = 0 \qquad (9E\ 16)$$

which implies the lateral surface is insulated.

2. Nonlinear Theory

From (1) we have

$$x_{i,K} = \begin{pmatrix} \frac{1}{\sqrt{c}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{c}} & 0 \\ 0 & 0 & c \end{pmatrix}, \quad X_{K,i} = \begin{pmatrix} \sqrt{c} & 0 & 0 \\ 0 & \sqrt{c} & 0 \\ 0 & 0 & \frac{1}{c} \end{pmatrix} \qquad (9E\text{-}17)$$

It follows that the deformation is isochoric. From (17)

$$b_{ij} = x_{i,K}x_{j,K} = \begin{pmatrix} \frac{1}{c} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & c^2 \end{pmatrix} \qquad (9E\ 18)$$

$$c_{ij} = X_{K,i}X_{K,j} = \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix} \qquad (9E\text{-}19)$$

Then we have

$$I_b = \frac{2}{c} + c^2 = \frac{1}{c}(c^3+2)$$

$$II_b = \frac{1}{c^2}(2c^3+1) \qquad (9E\text{-}20)$$

It follows from (2) that

$$\theta_{i} = (0,\ 0,\ \frac{T_1}{\ell}) \qquad (9E\text{-}21)$$

Hence, from (9D 11) and (18),(19)

$$I_1 = \theta_{,i}\theta_{,i} = (\frac{T_1}{\ell})^2$$

$$I_2 = b_{ij}\theta_{,i}\theta_{,j} = b_{33}\theta_{,3}\theta_{,3} = (\frac{cT_1}{\ell})^2 \qquad (9E\ 22)$$

$$I_3 = c_{ij}\theta_{,i}\theta_{,j} = c_{33}\theta_{,3}\theta_{,3} = (\frac{T_1}{\ell c})^2$$

Hence, by (9D 9)

$$\begin{aligned} \tilde{\gamma}_\alpha &= \tilde{\gamma}_\alpha[T_0 + \frac{T_1}{\ell} x_3, \frac{1}{c}(c^3+2), \frac{1}{c^2}(2c^3+1)] \\ &\equiv \hat{\gamma}_\alpha(x_3) \end{aligned} \qquad (9E\ 23)$$

Using (18),(19) and (23) in (9D-7), we find

$$t_{ij} = -p\delta_{ij} + \hat{\gamma}_{-1} \begin{pmatrix} \frac{1}{c} & 0 & 0 \\ 0 & \frac{1}{c} & 0 \\ 0 & 0 & c^2 \end{pmatrix} + \hat{\gamma}_1 \begin{pmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & \frac{1}{c^2} \end{pmatrix} \qquad (9E\ 24)$$

Now from (20),(22) and (9D 10) we have

$$\psi_\alpha = \psi_\alpha[\frac{1}{c}(c^3+2), \frac{1}{c^2}(2c^3+1), (\frac{T_1}{\ell})^2, (\frac{cT_1}{\ell})^2, (\frac{T_1}{\ell c})^2] \qquad (9E\ 25)$$

which are constants for this problem. Using (18),(19) and (21),

$$b_{ij}\theta_{,j} = (0, 0, \frac{T_1c^2}{\ell}) \quad ; \quad c_{ij}\theta_{,j} = (0, 0, \frac{T_1}{\ell c^2})$$

so that by (9D-8)

$$q_1 = 0 = q_2 \quad , \quad q_3 = (\psi_{-1}c^2 + \psi_0 + \frac{1}{c^2}\psi_1)\frac{T}{\ell} \qquad (9E\text{-}26)$$

Hence, q_i is a constant vector and the heat conduction eqn. (9D-5) is identically satisfied. Eqns. (23),(24) together with the equilibrium eqns. (9D-1) imply

$$\frac{\partial p}{\partial x_1} = 0 = \frac{\partial p}{\partial x_2}$$

$$(-p + \hat{\gamma}_{-1}c^2 + \frac{1}{c^2}\hat{\gamma}_1)_{,3} = 0$$

Hence,

$$-p(x_3) + \hat{\gamma}_{-1}(x_3)c^2 + \frac{1}{c^2}\hat{\gamma}_1(x_3) = p_0 \qquad (9E\text{-}27)$$

where p_0 is an arbitrary constant. Then (24) reduces to the following stress components

$$t_{11}(x_3) = t_{22}(x_3) = p_0 + \frac{1}{c}(1-c^3)\hat{\gamma}_{-1} + \frac{1}{c^2}(c^3-1)\hat{\gamma}_1$$

$$t_{33} = p_0 \quad , \quad t_{12} = t_{23} = t_{13} = 0 \qquad (9E\text{-}28)$$

Considering the boundary conditions, we have on $x_3 = \ell$ from (10a), (26) and (28)

$$t_i = (0, 0, p_0)$$

$$h = (\psi_{-1}c^2 + \psi_0 + \frac{1}{c^2}\psi_1)\frac{T}{\ell} \qquad (9E\text{-}29)$$

Hence, in contrast to the linear theory, the ends are freed from tractions by choosing

$$p_0 = 0$$