Chapter 3

Basic Concepts in Statistics and Probability

3.6 Continuous Distributions

• Normal distribution
• Student t-distribution
• Exponential distribution
• Lognormal distribution
• Weibull distribution
• Extreme value distribution
• Gamma distribution
• Chi-square distribution
• Truncated normal distribution
• Bivariate and multivariate normal distribution

• F distribution
• Beta distribution
• Uniform distribution
The normal distribution (also called the Gaussian distribution) is by far the most commonly used distribution in statistics. This distribution provides a good model for many, although not all, continuous populations.

The normal distribution is continuous rather than discrete. The mean of a normal population may have any value, and the variance may have any positive value.

### Probability Density Function, Mean, and Variance of Normal Dist.

The probability density function of a normal population with mean $\mu$ and variance $\sigma^2$ is given by

$$f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad -\infty < x < \infty$$

If $X \sim N(\mu, \sigma^2)$, then the mean and variance of $X$ are given by

$$\mu_X = \mu$$

$$\sigma_X^2 = \sigma^2$$
68-95-99.7% Rule

This figure represents a plot of the normal probability density function with mean $\mu$ and standard deviation $\sigma$. Note that the curve is symmetric about $\mu$, so that $\mu$ is the median as well as the mean. It is also the case for the normal population.

- About 68% of the population is in the interval $\mu \pm \sigma$.
- About 95% of the population is in the interval $\mu \pm 2\sigma$.
- About 99.7% of the population is in the interval $\mu \pm 3\sigma$.

Standard Units

- The proportion of a normal population that is within a given number of standard deviations of the mean is the same for any normal population.
- For this reason, when dealing with normal populations, we often convert from the units in which the population items were originally measured to standard units.
- Standard units tell how many standard deviations an observation is from the population mean.
Standard Normal Distribution

In general, we convert to standard units by subtracting the mean and dividing by the standard deviation. Thus, if $x$ is an item sampled from a normal population with mean $\mu$ and variance $\sigma^2$, the standard unit equivalent of $x$ is the number $z$, where

$$z = \frac{(x - \mu)}{\sigma}.$$

The number $z$ is sometimes called the “z-score” of $x$. The z-score is an item sampled from a normal population with mean 0 and standard deviation of 1. This normal distribution is called the standard normal distribution.

Finding Areas Under the Normal Curve

- The proportion of a normal population that lies within a given interval is equal to the area under the normal probability density above that interval. This would suggest integrating the normal pdf, but this integral does not have a closed form solution.

- So, the areas under the curve are approximated numerically and are available in Table B. This table provides area under the curve for the standard normal density. We can convert any normal into a standard normal so that we can compute areas under the curve.

- The table gives the area in the right-hand tail of the curve between $\mu$ and $z$. Other areas can be calculated by subtraction or by using the fact that the normal distribution is symmetrical and that the total area under the curve is 1.
Normal Probabilities

Excel:
- NORM.DIST(x, mean, standard_dev, cumulative)
- NORM.INV(probability, mean, standard_dev)
- NORM.S.DIST(z)
- NORM.S.INV(probability)

Minitab:
- Calc → Probability Distributions → Normal

Linear Functions of Normal Random Variables

Let \( X \sim N(\mu, \sigma^2) \) and let \( a \neq 0 \) and \( b \) be constants. Then \( aX + b \sim N(a\mu + b, a^2 \sigma^2) \).

Let \( X_1, X_2, \ldots, X_n \) be independent and normally distributed with means \( \mu_1, \mu_2, \ldots, \mu_n \) and variances \( \sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2 \). Let \( c_1, c_2, \ldots, c_n \) be constants, and \( c_1 X_1 + c_2 X_2 + \ldots + c_n X_n \) be a linear combination. Then

\[
c_1 X_1 + c_2 X_2 + \ldots + c_n X_n \\
\sim N(c_1\mu_1 + c_2\mu_2 + \ldots + c_n\mu_n, c_1^2\sigma_1^2 + c_2^2\sigma_2^2 + \ldots + c_n^2\sigma_n^2)
\]
Distributions of Functions of Normal Random Variables

Let \( X_1, X_2, \ldots, X_n \) be independent and normally distributed with mean \( \mu \) and variance \( \sigma^2 \). Then

\[
\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right).
\]

Let \( X \) and \( Y \) be independent, with \( X \sim N(\mu_X, \sigma_X^2) \) and \( Y \sim N(\mu_Y, \sigma_Y^2) \). Then

\[
X + Y \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)
\]

\[
X - Y \sim N(\mu_X - \mu_Y, \sigma_X^2 + \sigma_Y^2)
\]

3.6.2 \( t \) Distribution

- If \( X \sim N(\mu, \sigma^2) \)

\[
Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}
\]  

(3.7)

- If \( \sigma \) is Not known, but \( n \geq 30 \)

\[
Z = \frac{\bar{X} - \mu}{s / \sqrt{n}}
\]  

(3.8)

- Let \( X_1, \ldots, X_n \) be a small \((n < 30)\) random sample from a normal population with mean \( \mu \). Then the quantity

\[
t = \frac{\bar{X} - \mu}{s / \sqrt{n}}
\]  

(3.9)

has a Student’s \( t \) distribution with \( n - 1 \) degrees of freedom (denoted by \( t_{n-1} \)).
More on Student’s t

- The probability density of the Student’s t distribution is different for different degrees of freedom.
- The t curves are more spread out than the normal.
- Table C, called a t table, provides probabilities associated with the Student’s t distribution.

\[ f(t) = \frac{1}{\sqrt{(n - 1)\pi}} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left[\frac{n - 1}{2}\right]} \left(1 + \frac{t^2}{n - 1}\right)^{-n/2} \]

\[ -\infty < t < \infty \]
$t = \frac{\hat{\theta} - \theta}{s_{\hat{\theta}}}$

Where $\theta$ is the parameter to be estimated, $\hat{\theta}$ is the estimator based on a sample, $s_{\hat{\theta}}$ is the sample estimator of $\sigma_{\hat{\theta}}$. 
3.6.3 Exponential Distribution

• The **exponential distribution** is a continuous distribution that is sometimes used to model the time that elapses before an event occurs (life testing and reliability).

• The probability density of the exponential distribution involves a parameter, which is the mean of the distribution, θ, whose value determines the density function’s location and shape.

Exponential R.V.: pdf, cdf, mean and variance

The pdf of an exponential r.v. is

\[ f(x) = \frac{1}{\theta} e^{-x/\theta} \quad x > 0 \]  (3.10)

The cdf of an exponential r.v. is

\[ F(x) = 1 - e^{-x/\theta} \quad x > 0 \]

The mean of an exponential r.v. is

\[ \mu_x = \theta \]

The variance of an exponential r.v. is

\[ \sigma_x^2 = \theta \]

Exponential Probability Density Function

\[ \lambda = \frac{1}{\theta} \]

Exponential Probabilities

Excel:

\[ \text{EXPONDIST}(x, \lambda, \text{cumulative}) \]

Minitab:

Calc \rightarrow Probability Distributions \rightarrow Exponential
Example

A radioactive mass emits particles according to a Poisson process at a mean rate of 15 particles per minute. At some point, a clock is started.

1. What is the probability that more than 5 seconds will elapse before the next emission?
2. What is the mean waiting time until the next particle is emitted?

Lack of Memory Property

The exponential distribution has a property known as the lack of memory property:

If $T \sim \text{Exp}(1/\theta)$, and $t$ and $s$ are positive numbers, then

$$P(T > t + s \mid T > s) = P(T > t).$$
Example

The lifetime of a transistor in a particular circuit has an exponential distribution with mean 1.25 years.

1. Find the probability that the circuit lasts longer than 2 years.

2. Assume the transistor is now three years old and is still functioning. Find the probability that it functions for more than two additional years.

3. Compare the probability computed in 1. and 2.

3.6.4 Lognormal Distribution

• For data that contain outliers, the normal distribution is generally not appropriate. The lognormal distribution, which is related to the normal distribution, is often a good choice for these data sets.

• If $X \sim N(\mu, \sigma^2)$, then the random variable $Y = e^X$ has the lognormal distribution with parameters $\mu$ and $\sigma^2$.

• If $Y$ has the lognormal distribution with parameters $\mu$ and $\sigma^2$, then the random variable $X = \ln Y$ has the $N(\mu, \sigma^2)$ distribution.
Lognormal pdf, mean, and variance

- The pdf of a lognormal random variable with parameters $\mu$ and $\sigma^2$ is
  \[ f(x) = \frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} \exp \left[ -\frac{1}{2\sigma^2} (\ln x - \mu)^2 \right] \quad x > 0 \]

- Mean:
  \[ E(X) = e^{\mu + \sigma^2/2} \]

- Variance:
  \[ V(Y) = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2} \]
Example

When a pesticide comes into contact with the skin, a certain percentage of it is absorbed. The percentage that is absorbed during a given time period is often modeled with a lognormal distribution. Assume that for a given pesticide, the amount that is absorbed (in percent) within two hours is lognormally distributed with a mean of 1.5 and standard deviation of 0.5. Find the probability that more than 5% of the pesticide is absorbed within two hours.
The Weibull distribution is a continuous random variable that is used in a variety of situations. A common application of the Weibull distribution is to model the lifetimes of components. The Weibull probability density function has two parameters, both positive constants, that determine the scale and shape. We denote these parameters $\alpha$ (scale) and $\beta$ (shape).

If $\beta = 1$, the Weibull distribution is the same as the exponential distribution.

The case where $\alpha=1$ is called the standard Weibull distribution.

The pdf of the Weibull distribution is

$$f(x) = \alpha \beta (\alpha x)^{\beta - 1} e^{-(\alpha x)^{\beta}} \quad x > 0$$

The mean of the Weibull is

$$\mu_x = \frac{1}{\alpha} \Gamma(1 + \frac{1}{\beta})$$

The variance of the Weibull is

$$\sigma_x^2 = \frac{1}{\alpha^2} \left\{ \Gamma \left( 1 + \frac{2}{\beta} \right) - \left[ \Gamma \left( 1 + \frac{1}{\beta} \right) \right]^2 \right\}$$
Weibull Probability Density Function

The cdf of the Weibull distribution (with $\alpha=1$) is

$$F(x) = 1 - e^{-x^\beta} \quad x > 0$$
Weibull Probabilities

Excel:
WEIBULL(x, alpha, beta, cumulative)

Minitab:
Calc→ Probability Distributions → Weibull

3.6.6 Extreme Value Distribution

Extreme value distributions are often used in reliability work.
Extreme Value Distribution

The pdf of the Extreme value distribution is
\[ f(x) = \frac{1}{b} \exp\left(\frac{x - a}{b}\right) \exp\left\{- \exp\left(\frac{x - a}{b}\right)\right\} \quad -\infty < x < \infty \]
with \(a\) the location parameter and \(b\) the scale parameter.

The cdf of the Extreme value distribution is
\[ F(x) = 1 - \exp\left\{- \exp\left(\frac{x - a}{b}\right)\right\} \]

- The Weibull distribution is the limiting distribution of the extreme value distribution.
- The natural log of a Weibull random time is an extreme value random observation.

http://www.itl.nist.gov/div898/handbook/apr/section1/apr163.htm
The gamma distribution is frequently a probability model for waiting times; for instance, in life testing, the waiting time until death is a random variable that is frequently modeled with a gamma distribution.

Gamma function is defined by

\[ f(x) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta} \quad x > 0 \]

Where \( \alpha > 0 \) (shape) and \( \beta > 0 \) (scale)

### Gamma Distribution

- Standard gamma distribution when \( \beta = 1 \)
- Mean: \( \alpha \beta \)  \( \text{Variance: } \alpha \beta^2 \)
- If \( k \) is an integer, then the distribution represents an Erlang distribution.
- If \( \alpha = 1 \), the gamma distribution is the same as the exponential.
- The gamma function has the following properties:
  1. If \( r \) is any integer, then \( \Gamma(r) = (r - 1)! \).
  2. For any \( r \), \( \Gamma(r + 1) = r \Gamma(r) \).
  3. \( \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \)
  4. \( \Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx \)
Gamma Probability Density Function

Gamma Probability Densities

\( \alpha = 1, \beta = 1 \)

\( \alpha = 3, \beta = 0.5 \)

\( \alpha = 5, \beta = 1 \)

Excel:

- `GAMMA.DIST(x, alpha, beta, cumulative)`
- `GAMMA.INV(probability, alpha, beta)`
- `GAMMALN(x)`

Minitab:

- Calc → Probability Distributions → Gamma
3.6.8 Chi-Square Distribution

• If \( \alpha = \beta/2 \), and \( \beta = 2 \) where \( r \) is a positive integer, the distribution is called a chi-square distribution with \( r \) degrees of freedom.
• The chi-square distribution results when \( \nu \) independent variables with standard normal distributions are squared and summed.
• The probability density function of the chi-square distribution is

\[
f(x) = \frac{e^{-x} x^{\frac{\nu}{2} - 1}}{2^{\frac{\nu}{2}} \Gamma\left(\frac{\nu}{2}\right)} \quad x \geq 0
\]
3.6.9 Truncated Normal Distribution

- Left truncated
- Right truncated
- Doubly truncated
**Left Truncated Normal Distribution**

- Left truncated normal distribution
  
  \[
  f(x) = \frac{k}{\sigma \sqrt{2\pi}} e^{-1/2[(x-\mu)/\sigma]^2} \quad x \geq a
  \]

  Where
  
  \[
  k = \left[1 - \Phi\left(\frac{a - \mu}{\sigma}\right)\right]^{-1}
  \]

  Where a is the truncation point, and \( \Phi \) is the cumulative area under the normal curve.

---

**Lift Truncated Normal Distribution**

- Let \( c = (a - \mu)/\sigma \)
- Expected value
  
  \[
  E(x) = \mu + k\sigma f[c]
  \]

  Where
  
  \[
  k = \left[1 - \Phi(c)\right]^{-1}
  \]

  \[
  f(c) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}c^2}
  \]

- Variance
  
  \[
  Var(x) = \sigma^2\{1 + cf(c)\left[1 - \Phi(c)\right]^{-1} - [f(c)]^2\left[1 - \Phi(c)\right]^{-2}\}
  \]

  (3.12)
Right Truncated Normal Distribution

- Right truncated normal distribution

\[ f(x) = \frac{k'}{\sigma\sqrt{2\pi}} e^{-1/2[(x-\mu)/\sigma]^2} \quad x \geq a \]

Where

\[ k' = \left[ \Phi(\frac{a' - \mu}{\sigma}) \right]^{-1} \]

Where \( a' \) is the truncation point, and \( \Phi \) is the cumulative area under the normal curve.

- Let \( c' = (a' - \mu)/\sigma \)
- Expected value

\[ E(x) = \mu - k'\sigma f[c'] \]

Where

\[ k = [\Phi(c')]^{-1} \]
\[ f(c') = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}c'^2} \]
- Variance

\[ Var(x) = \sigma^2 \{ 1 + c'f(c')[1 - \Phi(c')]^{-1} - [f(c')]^2[1 - \Phi(c')]^{-2} \} \]
3.6.10 Bivariate and Multivariate Normal Distribution

- Normal distribution
  \[ f(x) = \frac{1}{\sigma \sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty \]

- For 2 independent variables, \( x_1 \sim N(\mu_1, \sigma_1^2) \), \( x_2 \sim N(\mu_2, \sigma_2^2) \)
  \[ f(x_1, x_2) = \frac{1}{2\pi \sigma_1 \sigma_2} \exp[-(\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - (x_2-\mu_2)^2)] \]
  \[ (3.13) \]
Bivariate Normal Distribution

• Bivariate Normal distribution

\[ f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} \exp\left[ -\frac{(x_1 - \mu_1)^2}{2\sigma_1^2} - \frac{(x_2 - \mu_2)^2}{2\sigma_2^2} \right] \] (3.13)

• Let \( \mathbf{x} = [x_1 \ x_2] \), \( \mathbf{m} = [\mu_1 \ \mu_2] \), \( \mathbf{S} = \begin{bmatrix} \sigma_1^2 & 0 \\ 0 & \sigma_2^2 \end{bmatrix} \),

\[ f(\mathbf{x}) = \frac{1}{2\pi|\mathbf{S}|^{1/2}} \exp\left[ -\frac{1}{2} (\mathbf{x} - \mathbf{m})' \mathbf{S}^{-1} (\mathbf{x} - \mathbf{m}) \right] \]

Multivariate Normal Distribution

• Let \( \mathbf{x} = [x_1 \ \cdots \ x_p] \), \( \mathbf{m} = [\mu_1 \ \cdots \ \mu_p] \), \( \mathbf{S} = \begin{bmatrix} \sigma_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_p^2 \end{bmatrix} \)

\[ f(\mathbf{x}) = \frac{1}{(2\pi)^{p/2} |\mathbf{S}|^{1/2}} \exp\left[ -\frac{1}{2} (\mathbf{x} - \mathbf{m})' \mathbf{S}^{-1} (\mathbf{x} - \mathbf{m}) \right] \] (3.14)
3.6.11 F Distribution

- Let $W_1$ and $Y_1$ be independent $\chi^2$ random variables with $u$ and $\nu$ degrees of freedom, the ratio $F = (W/u)/(Y/\nu)$ has F distribution.
- Probability Density Function, with $u$, $\nu$ degrees of freedom,
  \[ f(x) = \frac{\Gamma\left(\frac{u + \nu}{2}\right)}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{u}{\nu}\right)^{\frac{u}{2}} x^{\frac{u}{2} - 1} \frac{\nu}{\nu + x}^{\frac{\nu}{2}} \quad 0 < x < \infty \]
- Mean
  \[ E(x) = \sqrt{\frac{\nu}{(\nu - 2)}} \quad \text{for } \nu > 2 \]
- Variance
  \[ V(x) = \frac{2\nu^2(\nu + \nu - 2)}{u(\nu - 2)^2(\nu - 4)} \quad \text{for } \nu > 4 \]
3.6.12 Beta Distribution

Beta function is defined by

\[ f(x) = \frac{(r+s-1)!}{(r-1)!(s-1)!} x^{r-1} (1-x)^{s-1} \quad 0 \leq x \leq 1 \]

Where \( r \) and \( s \) are shape parameters of Beta distribution

Mean of Beta distribution: \( r/(r+s) \)

Variance of Beta distribution: \( rs(r+s)^2(r+s+1)^{-1} \)

http://astarmathsandphysics.com/university_maths_notes/probability_and_statistics/probability_and_statistics_the_beta_distribution.html
3.6.12 Beta Distribution

Beta distribution and F distribution

\[ B(\alpha; r, s) = \frac{(2r/2s)F(\alpha;2r,2s)}{1+(2r/2s)F(\alpha;2r,2s)} \]

Where \( B(\alpha; r, s) \) denotes the \((1 - \alpha)\) percentile of a beta distribution with parameters \( r \) and \( s \).

3.6.13 Uniform Distributions

The **uniform distribution** has two parameters, \( a \) and \( b \), with \( a < b \). If \( X \) is a random variable with the continuous uniform distribution then it is uniformly distributed on the interval \((a, b)\). We write \( X \sim U(a,b) \).

The pdf is

\[
 f(x) = \begin{cases} 
 1, & a < x < b \\
 \frac{1}{b-a}, & 0, & \text{otherwise} 
\end{cases}
\]
Uniform Distribution: Mean and Variance

If $X \sim U(a, b)$.

Then the mean is

$$\mu_X = \frac{a + b}{2}$$

and the variance is

$$\sigma^2_X = \frac{(b - a)^2}{12}.$$