Chapter 10:

Statistical Quality Control

Introduction

- As the marketplace for industrial goods has become more global, manufacturers have realized that quality and reliability of their products must be as high as possible for them to be competitive.
- It is now generally recognized that the most cost-effective way to maintain high quality is through constant monitoring of the production process.
- This monitoring is often done by sampling units of production and measuring some quality characteristic.
- In this chapter, we will focus on both the Shewart control charts and on cumulative sum (CUSUM) charts, since these are among the most powerful of the commonly used tools for statistical quality control.
Section 10.1: Basic Ideas

- The basic principle of control charts is that in any process there will always be variation in the output.
- Some of this variation is due to a wide variety of causes that are inherent to the process like natural variation in the process. These causes are called **common causes** or **chance causes**.
- When common causes are the only source of variation, the process is said to be in a state of statistical control, or in control.
- Processes must be continually monitored because they may become out of control at any time.

More Basics

- Sometimes special factors are present that produce additional variability.
- These could be machines that are malfunctioning, operator error, fluctuations in ambient conditions, and variations in the properties of raw materials.
- These factors are called **special causes** or **assignable causes**.
- Special causes generally produce a higher level of variability than do common causes, this variability is considered unacceptable.
- When a process is operating in the presence of one or more special factors, it is said to be out of statistical control.
Control Charts

- There are various control charts depending on what quality characteristic is being measured.
- There are charts for continuous variables, binary variables, or count variables.
- Control charts used for continuous variables are called variable control charts.
- Control charts used for binary or count variables are called attribute control charts.

Collecting Data

- Data to be used in the construction of a control chart are collected in a number of samples, taken over a period of time.
- These samples are called rational subgroups.
- The basic principle for selecting subgroups is that all the variability within the units of a rational subgroup should be due to common causes and none should be due to special causes.
- The two most popular methods are:
  1. Sample at regular time intervals, with all the items in each sample manufactured near the time the sampling is done.
  2. Sample at regular time intervals, with the items in each sample drawn from all the units produced since the last sample was taken.
More on Collecting Data

- For variables data, the number of units in each sample is small, often between three and eight.
- The number of samples should be at least 20.
- Many small samples taken frequently is usually better than a few large samples taken less frequently.
- For binary or count data, samples must in general be larger, say at least 20.

Control versus Capability

- There is a difference between process control and process capability.
- A process is in control if there are no special causes operating.
- The distinguishing feature of a process that is in control is that the values of the quality characteristic vary without any trend or pattern, since the common causes do not change over time.
- It is quite possible for a process to be in control, and yet to be producing output that does not meet a given specification.
More on Capability

- Example: Suppose that the process produces steel rods whose lengths vary randomly between 19.9 and 20.1 cm, with no apparent pattern to the fluctuation. This process is in a state of control. However, if the design specifications calls for a length between 21 and 21.2 cm, very little of the output would meet the specification.
- The ability of a process to produce output that meets a given specification is called the capability of a process.

Comments

- A process that is in control and capable at a given time may go out of control at a later time, as special causes re-occur. For this reason processes must be continually monitored.
- Control charts function in many ways like hypothesis test. The null hypothesis is that the process is in control. The control chart presents data that provide evidence about the truth of this hypothesis.
Section 10.2: Control Charts for Variables

- We have a process and assume that each of the samples we collect are a sample from a normal population with mean $\mu$ and standard deviation $\sigma$.
- The quantity $\mu$ is called the **process mean**, and $\sigma$ is called the **process standard deviation**.
- The idea behind control charts is that each value of $\bar{X}$ approximates the process mean during the time its sample was taken, while the values of $R$ and $s$ can be used to approximate the process standard deviation.

More on Control Charts

- If the process is in control, then the process mean and standard deviation are the same for each sample.
- If the process is out of control, the process mean $\mu$ or the process standard deviation $\sigma$, or both, will differ from sample to sample.
- Therefore the values of $\bar{X}$, $R$, and $s$ will vary less when the process is in control than when the process is out of control.
- If the process is in control, the values of $\bar{X}$, $R$, and $s$ will almost always be contained within computable limits, called **control limits**.
Control Charts

- If the process is out of control, the values of $\bar{X}$, $R$, and $s$ will be more likely to exceed these limits.
- A control chart plots the values of $\bar{X}$, $R$, or $s$, along with the control limits, so that it can be easily seen whether the variation is large enough to conclude that the process is out of control.
- Since the variation within each sample is supposed to be due only to common causes, this variation should not be too much different from one sample to another.
- Therefore the first thing to do is to check to make sure that the amount of variation within each sample, measured either by the sample range or sample standard deviation, does not vary too much from sample to sample.

$R$ Chart

- The $R$ chart can be used to assess variation in the sample range, or the $S$ chart can be used to assess variation in the sample standard deviation.
- The $R$ chart has a center line at $\bar{R}$ and upper and lower lines indicate the $3\sigma$ upper and lower control limits.
- The control limits are drawn so that when the process is in control, almost all the points will lie within the limits.
- A point plotting outside the control limits is evidence that the process is out of control.
- The control limits are based on the fact that for most populations, it is unusual to observe a value that differs from the mean by more than three standard deviations.
Examining a Chart

- Once the control limits have been calculated and the points plotted, the $R$ chart can be used to assess whether the process is in control with respect to variation.
- At sample number 6 there is a signal since there is a point that plotted outside of the control limits. This indicates that there is a special cause operating and that the process variation is not in control.
- The appropriate action is to determine the nature of the special cause, and then delete the out-of-control sample and re-compute the control limits.
Now, that the process variation has been brought into control, we can assess whether the process mean is in control by plotting the \( \bar{X} \)-chart.

- The sample means are plotted on the vertical axis.
- The sample 6 has not been used in this chart since it had to be deleted in order to bring the process variation under control. Like all control charts, the \( \bar{X} \)-chart has a center line and upper and lower control limits.
To compute the center line and control limits, we can assume that the process standard deviation is the same for all samples, since the R chart has been used to bring the process variation into control.

If the process mean \( \mu \) is in control, then it too is the same for all samples.

In this case, the samples means are drawn from a normal population with mean \( \mu_X = \mu \) and standard deviation \( \sigma_X = \sigma / \sqrt{n} \), where \( n \) is the sample size.

Ideally, we would like to plot the center line at \( \mu \) and 3\( \sigma \) control limits at \( \mu \pm 3\sigma_X \).

However the values of the mean and the standard deviation are unknown and have to be estimated from the data.
**X-bar Chart**

In an X-bar chart, when is used to estimate, the center line and the $3\sigma$ upper and lower control limits are given by

- $3\sigma$ upper limit = $\bar{X} + A_2 \bar{R}$
- Center line = $\bar{X}$
- $3\sigma$ lower limit = $\bar{X} - A_2 \bar{R}$

Where $\bar{X}$ is the average of the sample means.

The value $A_2$ depends on the sample size. Values are tabulated in table A.8

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**Comments**

- There is an out of control signal on the X-bar chart.
- When there is an out of control signal, the process must be stopped and the special causes remedied.
- Even when a process is in control, the process must be continually monitored, since new special causes are bound to crop up from time to time and will need to be detected and corrected.
- Note that while control charts can detect the presence of a special cause, they cannot determine its nature, nor how to correct it.
- It is necessary for the process engineer to have a good understanding of the process, so that special causes detected by control charts can be diagnosed and corrected.
Summary of Steps for Control Chart

The steps in using the $R$ chart and $\bar{X}$ chart are

1. Choose rational subgroups.
2. Compute the $R$ chart.
3. Determine the special causes for any out-of-control points.
4. Recompute the $R$ chart, omitting samples that resulted in out-of-control points.
5. Once the $R$ chart indicates a state of control, compute the $\bar{X}$ chart.
6. If the $\bar{X}$ chart indicates that the process is not in control, identify and correct any special causes.
7. Continue to monitor $\bar{X}$ and $R$. 
Control Chart Performance

- There is a close connection between control charts and hypothesis tests.
- The null hypothesis is that the process is in a state of control.
- A point plotting outside the $3\sigma$ control limits presents evidence against the null hypothesis.
- Occasionally a point will plot outside of the control limits when the process is in control. This is called a false alarm.
- If no points plot outside of the control limits and the process is not in control, then we say that we have failure to detect.
- It is desirable for these errors to occur as infrequently as possible. The frequency with which these errors occur is called the average run length (ARL).

ARL

Definition: The average run length (ARL) is the number of samples that will be observed, on average, before a point plots outside the control limits. If $p$ is the probability that any given point plots outside the control limits, then

$$ARL = \frac{1}{p}$$  \hspace{1cm} (10.1)

Note: We would like the ARL to be large when the process is in control, and small when the process is out of control.
Example 10.4

A process has mean $\mu = 3$ and standard deviation $\sigma = 1$. Samples of size $n = 4$ are taken. If a special cause shifts the process mean to a value of 3.5, find the ARL.

The Western Electric Rules

Any one of the following conditions is evidence that a process is out of control:

1. Any point plotting outside the $3\sigma$ limits.
2. Two out of three consecutive points plotting above the upper $2\sigma$ limits or two out of three consecutive points plotting below the lower $2\sigma$ limits.
3. Four out of five consecutive points plotting above the upper $1\sigma$ limits or four out of five consecutive points plotting below the lower $1\sigma$ limits.
4. Eight consecutive points plotting on the same side of the center line.
The Western Electric Rules

### S Chart

- This chart can be used in place of the $R$ chart.
- In an $S$ chart, the center line and the $3\sigma$ upper and lower control limits are given by

  $$3\sigma \text{ upper limit} = B_4 \bar{s}$$
  $$\text{Center line} = \bar{s}$$
  $$3\sigma \text{ lower limit} = B_3 \bar{s}$$

- The values $B_3$ and $B_4$ depend on the sample size. Values are tabulated in Table A.8.
Example of S and R charts

This is an example of the $S$ chart and $X$ chart.

Here is the process is in a state of statistical control.

Which is better to use, R or S chart?

- Both the $R$ and $S$ chart have the same purpose: to estimate the process standard deviation and to determine whether it is in control.
- It seems more natural to estimate the process standard deviation $s$ than with the range, $R$.
- In fact, when the population is normal, $s$ is a more precise estimate of the process standard deviation than $R$, because it has smaller uncertainty.
- It turns out that the improvement in the precision obtained with $s$ as opposed to $R$ increases as the sample size increases.
More of $R$ Versus $S$ Charts

- It follows that the $S$ chart is a better choice, especially for larger sample sizes.
- Sometimes it is necessary to define rational subgroups in such a way that each sample can contain only one value. It is impossible to compute a sample range or sample standard deviation for a sample of size 1 so some other method is needed to construct control charts.

Samples of Size 1

- Sometimes it is necessary to define rational subgroups in such a way that each sample can contain only one value.
- In this case, we use the CUSUM chart.
Section 10.3: Control Charts for Attributes

- The \( p \) chart is used when the quality characteristic being measured on each unit has only two possible values, usually “defective” and “not defective.”
- In each sample, the proportion of defectives is calculated; these sample proportions are then plotted.
- Let \( p \) be the probability that a given unit is defective.
- If the process is in control, this probability is constant over time.
- Let \( k \) be the number of samples.
- We will assume that all samples are the same size, and we will denote this size by \( n \).

\[ \begin{align*}
\text{Center line} & = \bar{p} \\
3\sigma \text{ upper limit} & = \bar{p} + 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}} \\
3\sigma \text{ lower limit} & = \bar{p} - 3 \sqrt{\frac{\bar{p}(1-\bar{p})}{n}}
\end{align*} \]

where \( \bar{p} = \frac{\sum_{i=1}^{k} \hat{p}_i}{k} \)

These control limits will be valid if \( n\bar{p} > 10 \).
Example of a $p$ chart

**TABLE 10.2 Number and proportion defective, for Example 10.9**

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<thead>
<tr>
<th>Sample</th>
<th>Number Defective</th>
<th>Proportion Defective ($\hat{p}$)</th>
<th>Sample</th>
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<th>Proportion Defective ($\hat{p}$)</th>
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Example of a $p$ chart

![p chart](chart.png)

- **$\overline{p}$** = 0.0509
- **UCL** = 0.0803
- **LCL** = 0.0214
Signals

- A point plotted above the upper control limits indicates that the percent defective has increased, so action must be taken to identify and remove the special cause.
- A point plotted below the lower control limits indicates that the percent defective has decreased. The special cause needs to be identified, but the action should be to make it continue, so that the proportion of defective can decrease permanently.

Set-Up for the $c$ Chart

- The $c$ chart is used when the quality measurement is a count of the number of defects, or flaws, in a given unit.
- A unit may be a single item, or it may be a group of items large enough so that the expected number of flaws is sufficiently large.
- Use of the $c$ chart requires that the number of defects follow a Poisson distribution.
- Assume that $k$ units are sampled, and let $c_i$ denote the total number of defects in the $i^{th}$ unit.
- Let $\lambda$ denote the mean total number of flaws per unit.
- Then $c_i \sim \text{Poisson}(\lambda)$.
- If the process is in control, the value of $\lambda$ is constant over time.
The c Chart

In a c chart, the center line and the $3\sigma$ upper and lower control limits are given by

$3\sigma$ upper limit = $\bar{c} + 3\sqrt{c}$

Center line = $\bar{c}$

$3\sigma$ lower limit = $\bar{c} - 3\sqrt{c}$

Where $\bar{c} = \frac{\sum_{i=1}^{k} c_i}{k}$

These control limits will be valid if $\bar{c} > 10$.

Example of a c chart

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<th>Sample</th>
<th>Number of Flaws (c)</th>
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Section 10.4: The CUSUM Chart

- We have discussed how to detect a shift in the process mean.
- Unless a shift is fairly large, however, it may be some more time before a point plots outside the $3\sigma$ control limits.
- The Western Electric rules provide one method for reducing the ARL.
- CUSUM charts provide another.
- Imagine that a process mean shifts upward slightly.
- Then there will be a tendency for points to plot above the center line.
- If we add deviations from the center line as we go along, and plot the cumulative sums, the points will drift upward and will exceed a control limit sooner than they would in an $\bar{X}$ chart.
More on the CUSUM

- We assume that we have \( m \) samples of size \( n \), with sample means.
- To begin, a target value, \( \mu \) must be specified for the process mean. We also need an estimate of the standard deviations of the sample means.
- Even a rough guess of the standard deviation can produce good results, so the CUSUM procedure can be useful with \( n = 1 \).
- Two constants, usually called \( k \) and \( h \), must be specified. Larger values for these constants result in longer average run lengths, and thus fewer alarms, but also result in longer waiting times to discover that a process is out of control.
- The values \( k = 0.5 \) and \( h = 4 \) or \( 5 \) are often used, because they provide a reasonably long ARL when the process is in control but still have fairly good power to detect a shift in the mean.

CUSUM Chart
Set-Up for CUSUM Chart

In a CUSUM chart, two cumulative sums, SH and SL, are plotted.

The initial values are $SH_0 = SL_0 = 0$. For $i \geq 1$,

$$SH_i = \max\{0, \bar{X}_i - \mu - k\sigma_{\bar{X}} + SH_{i-1}\}$$

$$SL_i = \min\{0, \bar{X}_i - \mu + k\sigma_{\bar{X}} + SL_{i-1}\}$$

The constants $k$ and $h$ are specified. Good results are often obtained for the values $k = 0.5$ and $h = 4$ or 5.

If for any $i$, $SH_i > h\sigma_{\bar{X}}$ or $SL_i < -h\sigma_{\bar{X}}$, the process is judged to be out of control.

Section 10.5: Process Capability

If a quality characteristic from a process in a state of control is normally distributed, then the process mean $\hat{\mu}$ and standard deviation $\hat{\sigma}$ can be estimated from the control chart as follows:

$$\hat{\mu} = \bar{X}$$

$$\hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{\bar{s}}{c_4}$$

The values of $d_2$ and $c_4$ depend on the sample size. Values are tabulated in Table A.8.
To be fit for use, a quality characteristic must fall between a lower specification limit (LSL) and an upper specification limit (USL).

Sometimes there is only one specification limit. For example, suppose that the strength of a wire has a minimum that is acceptable but it cannot be too strong.

The specification limits are determined by design requirements.

They are not the control limits found on control charts.

We will assume that the process mean falls between the LSL and the USL.

The index $C_{pk}$ describes the capability of the process as it is.

The index $C_p$ describes the potential capability of the process.

The index $C_{pk}$ is equal either to $\frac{\mu - LSL}{3\sigma}$ or $\frac{USL - \mu}{3\sigma}$ whichever is less.

By convention, the min. acceptable value for $C_{pk}$ is 1.

That is, a process is considered minimally capable if the process mean is three standard deviations from the nearest specification limit.

A $C_{pk}$ value of 1.33, indicating that the process mean is four standard deviations from the nearest specification limit, is generally considered good.
Example 10.11

The design specifications for a piston rod used in an automatic transmission call for the rod length to be between 71.4 and 72.8 mm. The process is monitored with an $\bar{X}$ chart and an $S$ chart, using sample of size $n = 5$. These show that the process appears to be in control. Compute the value of $C_{pk}$. Is this process capability acceptable?

$\bar{X} = 71.8 \text{ mm} \quad \bar{s} = 0.20 \text{ mm}$

Example 10.12

Assume that it is possible to adjust the process mean to any desired value. To what value should it be set to maximize the value of $C_{pk}$? What will the value of $C_{pk}$ be?
The Index $C_p$

- The capability that can potentially be achieved by shifting the process mean to the midpoint between the upper and lower specification limits is called the **process capability index**, denoted $C_p$.
- If the process mean is at the midpoint between LSL and USL, then the distance from the mean to either specification limit is equal to one-half the distance between the specification limits.
- It follows that $C_p = \frac{USL-LSL}{6\hat{\sigma}}$.
- This measures the greatest capability that the process can achieve without reducing the process standard deviation.

Example 10.13

Specifications for the output voltage of a certain electric circuit are 48 to 52 V. The process is in control with $\hat{\sigma} = 0.482 V$.

Compute the process capability index $C_p$. 
One-Sided Tolerances

- Some characteristics have only one specification limit. For example, strengths usually have a lower specification limit but no upper limit, since for most applications a part cannot be too strong.
- The analog of $C_{pk}$ when there is only a lower specification limit is the lower capability index $C_{pl}$; when there is only an upper limit, it is the upper capability index $C_{pu}$.

$$C_{pl} = \frac{\hat{\mu} - LSL}{3\hat{\sigma}} \quad \text{and} \quad C_{pu} = \frac{USL - \hat{\mu}}{3\hat{\sigma}}$$

- There is no analog for $C_{pk}$ for processes with only one specification limit.

Summary

- $\bar{X}$ and $R$ charts
- $\bar{X}$ and $s$ charts
- $p$ chart
- $c$ chart
- CUSUM chart
- Process capability