Soil Dynamics

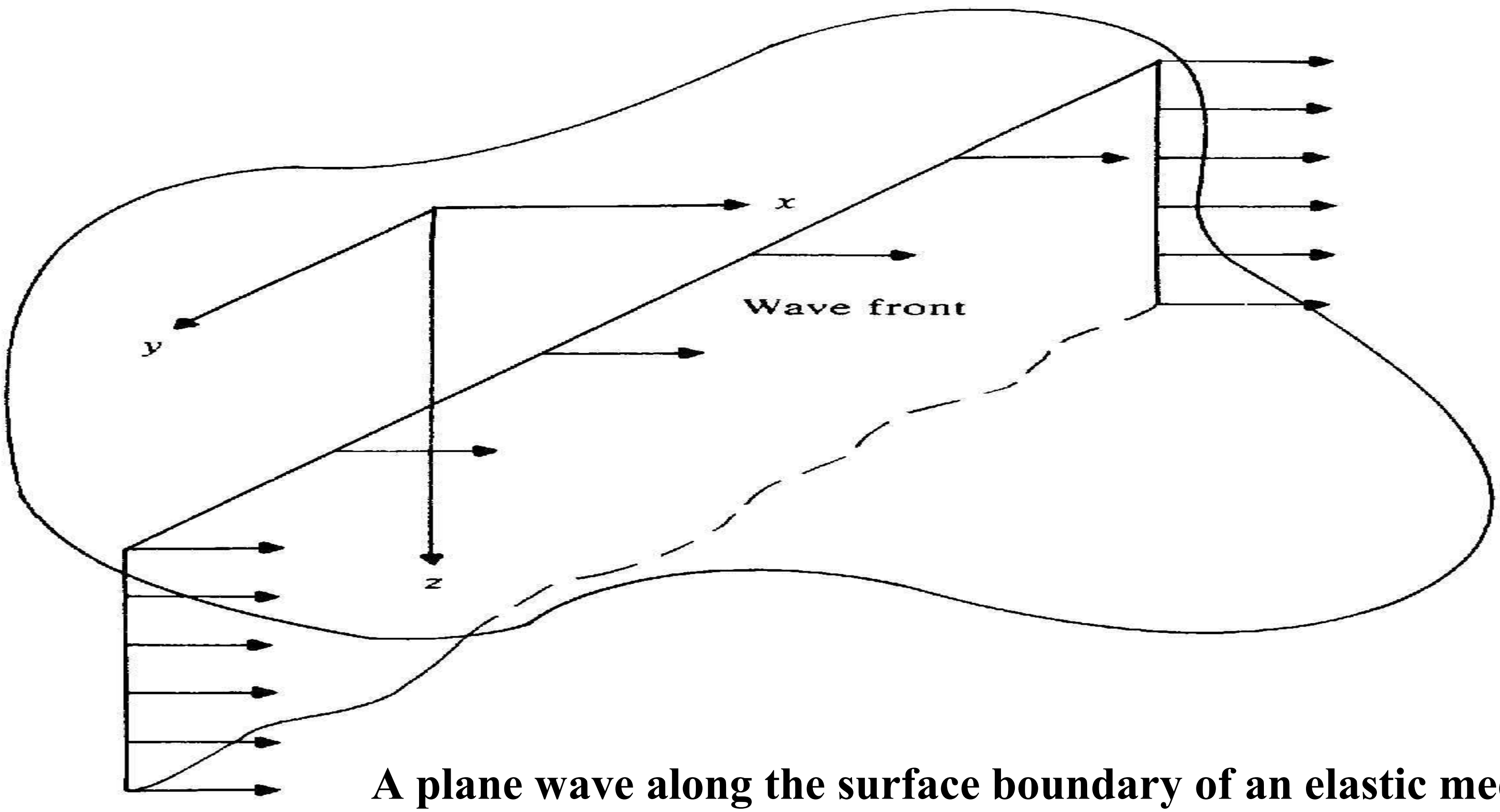
Lecture 07

Stress Waves in a Semi-Infinite Media

Rayleigh waves.

The stress waves we studied in Lecture #6 dealt with waves *within bodies* that are infinite, elastic and isotropic.

There is another type of wave, called the *Rayleigh wave*, that can only exist near the *boundary* of an elastic half-space. This wave was first discovered by Lord Rayleigh in the year 1885 for an elastic half-space as shown below.



A plane wave along the surface boundary of an elastic media.

The $x-y$ plane is the surface of the half-space. Let u and w be the displacements in the x and z directions, independent of y. Therefore,

$$u=\frac{\partial\phi}{\partial x}+\frac{\partial\psi}{\partial z}\quad \text{and}\quad w=\frac{\partial\phi}{\partial z}-\frac{\partial\psi}{\partial x}\qquad (1)$$

where ϕ and ψ are two potential functions.

The dilation $\bar{\varepsilon}$ can be defined as,

$$\bar{\varepsilon}=\varepsilon_x+\varepsilon_y+\varepsilon_z=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

$$\bar{\varepsilon}=\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\psi}{\partial x\partial z}\right)+(0)+\left(\frac{\partial^2\phi}{\partial z^2}-\frac{\partial^2\psi}{\partial x\partial z}\right)=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial z^2}=\nabla^2\phi\qquad (2)$$

and the rotation in the $x-z$ plane is given by,

$$2\bar{\omega}_y=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial z^2}=\nabla^2\psi$$

Replace (1) and (2) into the compression wave differential equation,

$$\rho\frac{\partial^2 u}{\partial t^2}=(\lambda+G)\frac{\partial\bar{\varepsilon}}{\partial x}+G\nabla^2 u$$

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \right) = (\lambda + G)\frac{\partial}{\partial x}\left(\nabla^2 \phi\right) + G\nabla^2 \left(\frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \right)$$

or

$$\rho \frac{\partial}{\partial x}\left(\frac{\partial^2 \phi}{\partial t^2} \right) + \rho \frac{\partial}{\partial z}\left(\frac{\partial^2 \psi}{\partial t^2} \right) = (\lambda + 2G)\frac{\partial}{\partial x}\left(\nabla^2 \phi\right) + G\frac{\partial}{\partial z}\left(\nabla^2 \psi\right) \quad (3)$$

and

$$\rho \frac{\partial}{\partial z}\left(\frac{\partial^2 \phi}{\partial t^2} \right) - \rho \frac{\partial}{\partial x}\left(\frac{\partial^2 \psi}{\partial t^2} \right) = (\lambda + 2G)\frac{\partial}{\partial z}\left(\nabla^2 \phi\right) - G\frac{\partial}{\partial x}\left(\nabla^2 \psi\right) \quad (4)$$

Equations (3) and (4) will be satisfied if ,

$$\rho\left(\frac{\partial^2 \phi}{\partial t^2} \right) = (\lambda + 2G)\left(\nabla^2 \phi\right) \quad \text{or} \quad \frac{\partial^2 \phi}{\partial t^2} = \left(\frac{\lambda + 2G}{\rho} \right)\nabla^2 \phi = v_P^2 \nabla^2 \phi$$

and

$$\rho\left(\frac{\partial^2 \psi}{\partial t^2} \right) = G\nabla^2 \psi \quad \text{or} \quad \frac{\partial^2 \psi}{\partial t^2} = \frac{G}{\rho}\nabla^2 \psi = v_S^2 \nabla^2 \psi$$

For a sinusoidal wave traveling in the positive x – direction, the solutions,

$$\phi = F(z)\exp\left[i\left(wt - fx\right)\right] \quad \text{and} \quad \psi = G(z)\exp\left[i\left(wt - fx\right)\right]$$

where F(z) and G(z) are functions of depth, and

$$f = \frac{2\pi}{wavelength} \quad \text{and} \quad i = \sqrt{-1}$$

or

$$wavelength = \frac{2\pi}{f} = \frac{\text{velocity of the wave}}{\omega / 2\pi} = \frac{v_R}{\omega / 2\pi}$$

where v_R is the Rayleigh wave velocity.

Substituting the solutions of ϕ and ψ into the differential equations,

and defining the ratio $V = \frac{v_R}{v_S}$ yields,

$$\boxed{V^6 - 8V^4 - \left(16\alpha^2 - 24\right)V^2 - 16\left(1 - \alpha^2\right) = 0}$$

where $\alpha = \sqrt{\frac{v_S^2}{v_P^2}}$

ν	$V = v_r/v_s$
0.25	0.919
0.29	0.926
0.33	0.933
0.4	0.943
0.5	0.955

The displacement of Rayleigh waves.

From,

$$u = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial z} \quad and \quad w = \frac{\partial \phi}{\partial z} - \frac{\partial \psi}{\partial x}$$

substitute into,

$$\phi = \left(A_1 e^{-qz}\right)\left[e^{i(wt - fx)}\right] \quad and \quad \psi = \left(B_1 e^{-sz}\right)\left[e^{i(wt - fx)}\right] \quad yields,$$

$$u = -\left(ifA_1 e^{-qz} + B_1 s e^{-sz}\right)\left[e^{i(wt - fx)}\right]$$

$$w = -\left(A_1 q e^{-qz} - B_1 ife^{-sz}\right)\left[e^{i(wt - fx)}\right]$$

However,

$$B_1 = -\frac{2iA_1 fq}{\left(s^2 + f^2\right)} \quad therefore,$$

$$u = A_1 fi\left(-e^{-qz} + \frac{2qs}{\left(s^2 + f^2\right)} e^{-sz}\right)\left[e^{i(wt - fx)}\right] \quad and$$

$$w = A_1 q\left(-e^{-qz} + \frac{2f^2}{\left(s^2 + f^2\right)} e^{-sz}\right)\left[e^{i(wt - fx)}\right]$$

The rate of attenuation of the displacement along the x − axis with depth z,

$$U = -e^{-qz} + \frac{2qs}{s^2 + f^2} e^{-sz} = -e^{-(q/f)(fz)} + \left[\frac{2(q/f)(s/f)}{s^2/f^2 + 1} \right] e^{-(s/f)(fz)}$$

$$W = -e^{-qz} + \frac{2f^2}{s^2 + f^2} e^{-sz} = -e^{-(q/f)(fz)} + \frac{2}{s^2/f^2 + 1} e^{-(s/f)(fz)}$$

where

$$q^2 = f^2 - \frac{\omega^2}{v_P^2} \quad or \quad \frac{q^2}{f^2} = 1 - \frac{\omega^2}{f^2 v_P^2} = 1 - \alpha^2 V^2$$

$$and \quad \frac{s^2}{f^2} = 1 - \frac{\omega^2}{f^2 v_S^2} = 1 - \frac{v_R^2}{v_S^2} = 1 - V^2$$

Example 1.

If a soil has a Poisson ratio $\nu = 0.25$, $E = 8$ ksi and $\rho = 1.6 \times 10^{-4}$ lb-s²/in⁴, determine its Rayleigh wave velocity.

$$v_S = \sqrt{\frac{E}{2\rho(1+\mu)}} = \sqrt{\frac{8{,}000\ lb/in^2}{2x1.6x10^{-4}\ lb-s^2/in^4\,(1+0.25)}} = 4{,}472\ in/s = 373\ ft/s$$

$$V^6 - 8V^4 - \left(16\alpha^2 - 24\right)V^2 - 16\left(1-\alpha^2\right) = 0$$

$$but \quad \alpha = \sqrt{\frac{v_S^2}{v_P^2}} \quad or \quad \alpha = \frac{v_S}{v_P} = \sqrt{\frac{E}{\rho\,2(1+\mu)}}\sqrt{\frac{\rho(1+\mu)(1-2\mu)}{E(1-\mu)}} = \sqrt{\frac{(1-2\mu)}{2(1-\mu)}}$$

$$\alpha^2 = \frac{1-2\mu}{2-2\mu} = \frac{1-0.5}{2-0.5} = \frac{1}{3}$$

$$\therefore \quad V^6 - 8V^4 - \left(\frac{16}{3} - 24\right)V^2 - 16\left(1 - \frac{1}{3}\right) = 0$$

which yields three roots, $V^2 = 4,\quad V^2 = 2 + \frac{2}{\sqrt{3}}$ *and* $V^2 = 2 - \frac{2}{\sqrt{3}}$

For $V^2 = 4$

$\frac{s^2}{f^2} = 1 - V^2 = 1 - 4 = -3$ *so that* $\frac{s}{f}$ *is imaginary; same for* $V^2 = 2 + \frac{2}{\sqrt{3}}$

Therefore, using $V^2 = 2 - \frac{2}{\sqrt{3}} \quad \therefore \quad V = 0.9194$

but $V = \frac{v_R}{v_S} = 0.9194 \quad \therefore \quad v_R = (0.9194)(373\ ft/s) = \underline{343\ ft/s}$

Example 2.

For the same soil of *Example 1*, what is the rate of attenuation of *U* and *W* in that soil as a function of the depth *z* and the frequency of vibration *f*?

$$\textit{Since } V = 0.9194$$

$$\frac{q^2}{f^2} = 1 - \alpha^2 V^2 = 1 - \left(\frac{1-2\nu}{2-2\nu}\right) V^2 = 1 - \left(\frac{1-0.5}{2-0.5}\right)(0.9194) = 0.7182$$

$$\textit{or} \quad \frac{q}{f} = 0.8475 \quad \textit{and}$$

$$\frac{s^2}{f^2} = 1 - V^2 = 1 - (0.9194)^2 = 0.1547$$

$$\textit{or} \quad \frac{s}{f} = 0.3933 \quad \textit{but}$$

$$U = -e^{-(q/f)(fz)} + \left[\frac{2(q/f)(s/f)}{s^2/f^2+1}\right] e^{-(s/f)(fz)} = -e^{(-0.8475\, fz)} + (0.5773)\, e^{(-0.3933\, fz)}$$

$$W = -e^{-(q/f)(fz)} + \frac{2}{s^2/f^2+1} e^{-(s/f)(fz)} = -e^{(-0.8475\, fz)} + (1.7321)\, e^{(-0.3933\, fz)}$$

From the two equations of U and W the following is evident:

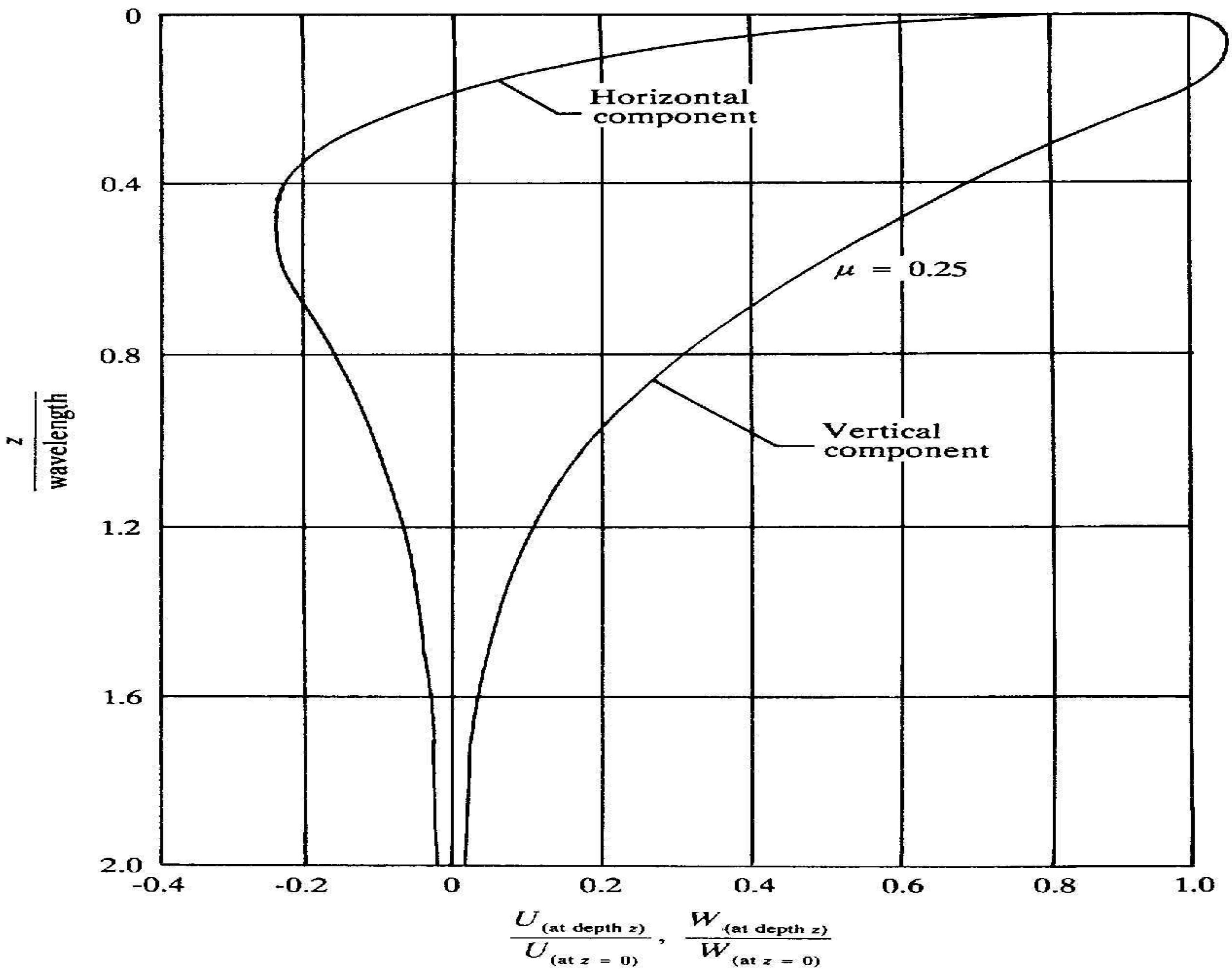
The magnitude of U decreases rapidly with increasing value of fz.
When $fz = 1.21$ the value of $U = 0$ and there is no motion parallel to the surface.
Since $f = 2\pi$ / wavelength, therefore
when $z = 1.21 / f = 1.21$ (wavelength) / $2\pi = 0.1926$ (wavelength) the value $U = 0$.

At greater depth, U becomes finite but of opposite sign (that is, the vibration takes place in opposite phase).

The magnitude of W first increases with fz and reaches a maximum value at $z = 0.076$ (wavelength) that is, $fz = 0.4775$, and then decreases with depth.

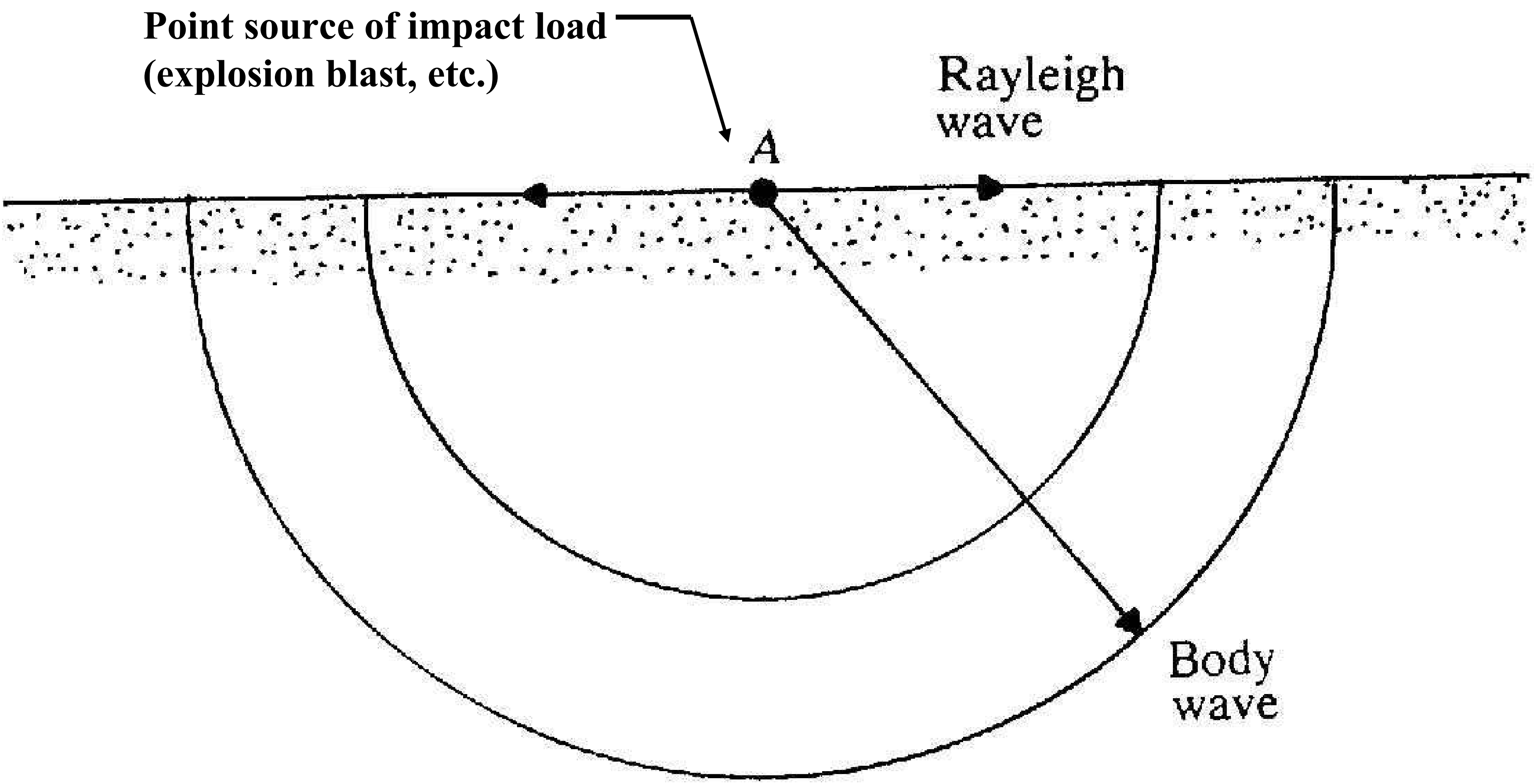
The next slide shows a non-dimensional plot of the variation of the amplitude of the vertical and horizontal components of the Rayleigh waves with depth.

The equations for U and W found in the previous slide show that the path of a particle in the medium is an ellipse with its major axis normal to the surface.

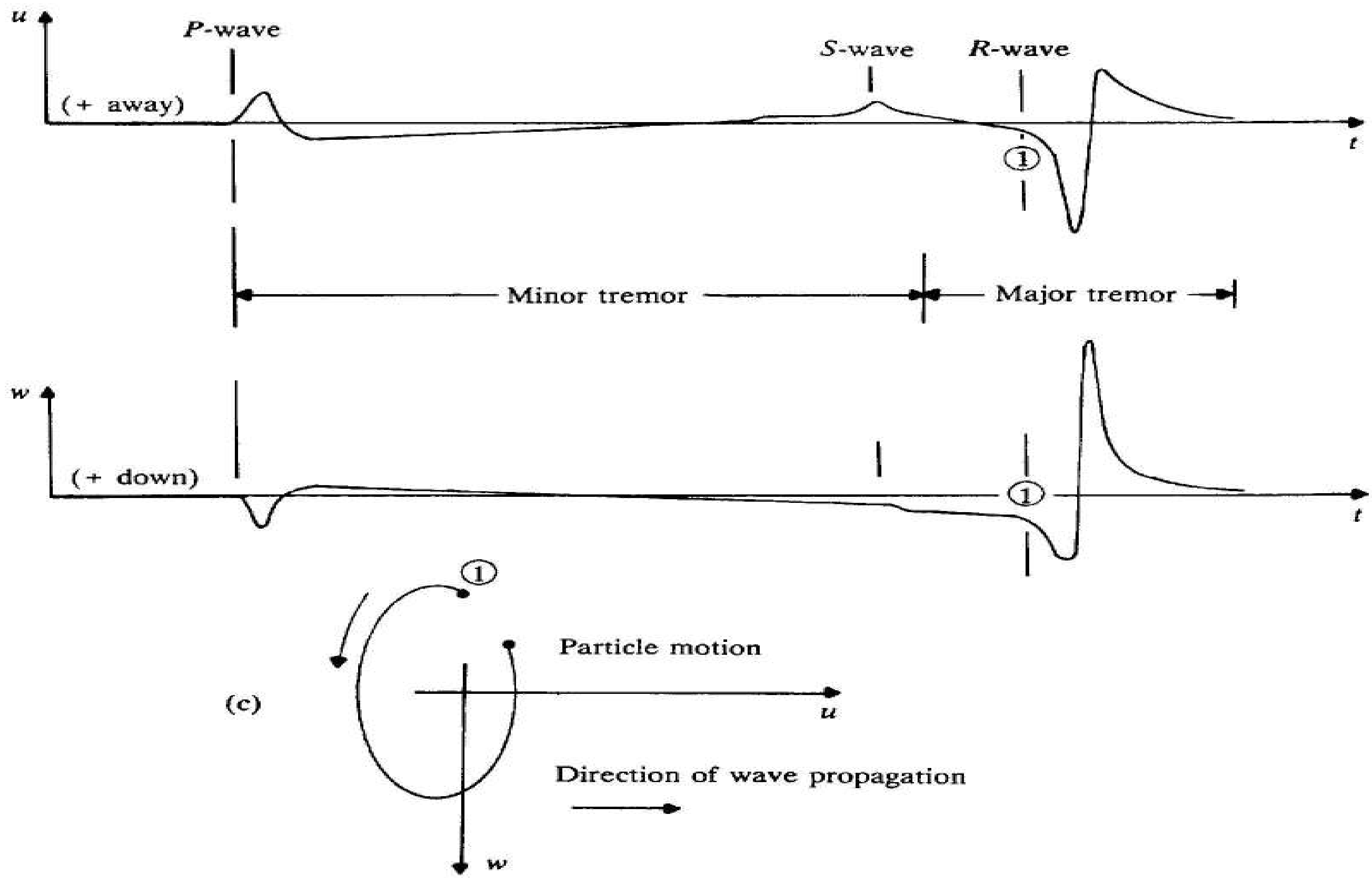

0
0.4
0.8
1.2
1.6
2.0
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1.0
Horizontal component
Vertical component
μ = 0.25
z / wavelength
U (at depth z) / U (at z = 0), W (at depth z) / W (at z = 0)

The attenuation of the amplitude of elastic waves with distance.

Consider a impulse force applied to the surface of a body as shown below. The body waves travel into the medium with *hemispherical wave fronts*. The Rayleigh waves will propagate outward along a *cylindrical wave front*.



At some point away from "A" on the previous figure the ground disturbance will be as shown below. The *P*-waves are the fastest, followed by the *S*-waves, and finally the *R*-waves which will produce the greatest amount of disturbance. The amplitude of disturbance gradually decreases with distance.



Notice that in figure (c) is shown the particle motion due to the Rayleigh wave from figures (a) and (b). This motion is called a retrograde ellipse.

As the body waves spread out along a hemispheric wave front, the energy is distributed over an area that increases with the square of the radius from the source. In other words,

$$\textit{Energy } E \propto \frac{1}{r^2} \quad \textit{but the wave amplitude} \propto \sqrt{E} \propto \sqrt{\frac{1}{r^2}} \propto \frac{1}{r}$$

For Rayleigh waves the amplitude is proportional to 1/√r, and therefore the attenuation of the amplitude of the Rayleigh waves is slower than that for body waves.

This loss of the amplitude of the waves due to the spreading out is called *geometrical damping*. In addition to geometrical damping there is the absorption of the energy by the soil itself to produce *material damping*. Bornitz proposed the following equation in 1931 to account for both types of damping,

$$\overline{w_n} = \overline{w_1}\sqrt{\frac{r_1}{r_2}}\, exp\left[-\beta\left(r_n - r_1\right)\right] \quad \textit{where } \overline{w_n} \textit{ and } \overline{w_1} \textit{ are the vertical amplitudes}$$

$$\textit{of the Rayleigh waves at distances } r_n \textit{ and } r_1 \textit{ and } \beta \textit{ is the absorption coefficient}.$$

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