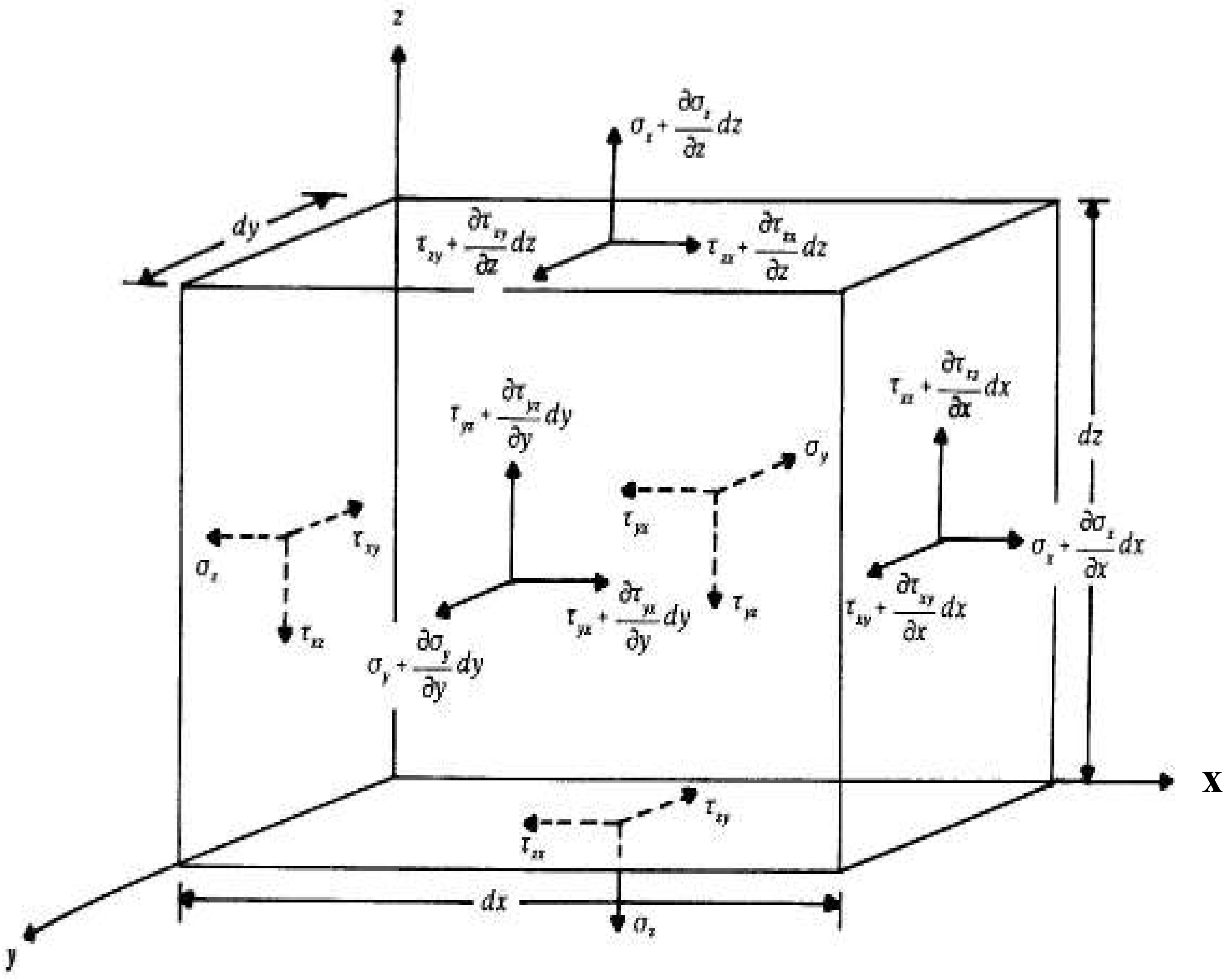
Soil Dynamics

Lecture 06

Stress Waves in Infinite Media

The equation of motion of a stress wave in an elastic medium.

Consider an element of an elastic medium, as shown below, with all the possible stresses on each of its six faces.



The equation of motion can be found through a summation of the forces along all three axes, and using Newton's second law (*F* = *ma*),

$$\textit{Consider the displacement } u \textit{ in the } x-\textit{direction},$$

$$\left[\left(\sigma_x + \frac{\partial \sigma_x}{\partial x} dx\right) - \sigma_x\right] dydz + \left[\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz\right) - \tau_{zx}\right] dxdy + \left[\left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy\right) - \tau_{yx}\right] dxdz$$

$$= \rho(dx)(dy)(dz)\frac{\partial^2 u}{\partial t^2}$$

$$\textit{Simplifying, and expanding to all three axes},$$

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} = \rho \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} = \rho \frac{\partial^2 v}{\partial t^2}$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = \rho \frac{\partial^2 w}{\partial t^2}$$

Compression stress waves (P-waves, or Primary waves or Dilatational waves).

The stress wave of motion in the x-direction was developed on the previous slide,

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

Recall that,

$$\tau_{yx} = G\gamma_{yx} \quad \textit{and} \quad \tau_{zx} = G\gamma_{zx} \quad \textit{and} \quad \sigma_x = \lambda\bar{\varepsilon} + 2G\varepsilon_x \quad \textit{therefore,}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda\bar{\varepsilon} + 2G\varepsilon_x\right) + \frac{\partial}{\partial y}\left(G\gamma_{yx}\right) + \frac{\partial}{\partial z}\left(G\gamma_{zx}\right)$$

and again recall that,

$$\gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \textit{and} \quad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

$$\rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\lambda\bar{\varepsilon} + 2G\varepsilon_x\right) + G\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + G\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

Simplifying,

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2} = \lambda \frac{\partial \overline{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{x}} + \boldsymbol{G}\left(\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \boldsymbol{v}}{\partial \boldsymbol{x} \partial \boldsymbol{y}} + \frac{\partial^2 \boldsymbol{w}}{\partial \boldsymbol{x} \partial \boldsymbol{z}} + \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{y}^2} + \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{z}^2} \right)$$

but

$$\frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{x}^2} + \frac{\partial^2 \boldsymbol{v}}{\partial \boldsymbol{x} \partial \boldsymbol{y}} + \frac{\partial^2 \boldsymbol{w}}{\partial \boldsymbol{x} \partial \boldsymbol{z}} = \frac{\partial \overline{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{x}}$$

therefore, simplifying and extending to all three axes,

$$\rho \frac{\partial^2 \boldsymbol{u}}{\partial \boldsymbol{t}^2} = (\lambda + \boldsymbol{G}) \frac{\partial \overline{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{x}} + \boldsymbol{G} \nabla^2 \boldsymbol{u} \quad \textit{where} \quad \nabla^2 = \frac{\partial^2}{\partial \boldsymbol{x}^2} + \frac{\partial^2}{\partial \boldsymbol{y}^2} + \frac{\partial^2}{\partial \boldsymbol{z}^2}$$

$$\rho \frac{\partial^2 \boldsymbol{v}}{\partial \boldsymbol{t}^2} = (\lambda + \boldsymbol{G}) \frac{\partial \overline{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{y}} + \boldsymbol{G} \nabla^2 \boldsymbol{v}$$

gradient squared or "del" squared

$$\rho \frac{\partial^2 \boldsymbol{w}}{\partial \boldsymbol{t}^2} = (\lambda + \boldsymbol{G}) \frac{\partial \overline{\boldsymbol{\varepsilon}}}{\partial \boldsymbol{z}} + \boldsymbol{G} \nabla^2 \boldsymbol{w}$$

Differentiating these three differential equations w / rt x, y and z and adding,

$$\rho\frac{\partial^2}{\partial t^2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=(\lambda+G)\left(\frac{\partial^2\bar{\varepsilon}}{\partial x^2}+\frac{\partial^2\bar{\varepsilon}}{\partial y^2}+\frac{\partial^2\bar{\varepsilon}}{\partial z^2}\right)+G\nabla^2\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$

or

$$\rho\frac{\partial^2\bar{\varepsilon}}{\partial t^2}=(\lambda+G)\nabla^2\bar{\varepsilon}+G\nabla^2\bar{\varepsilon}=(\lambda+2G)\nabla^2\bar{\varepsilon}$$

$$\boxed{\frac{\partial^2\bar{\varepsilon}}{\partial t^2}=\frac{\lambda+2G}{\rho}\nabla^2\bar{\varepsilon}=v_P^2\nabla^2\bar{\varepsilon}}\quad \textbf{\textit{where}}\quad \boxed{v_P=\sqrt{\frac{\lambda+2G}{\rho}}}$$

Notice that v_p is the primary wave velocity (also known as the *__compression wave__*, or the *__dilatational wave__*, or the *__P-wave__*).

Compare v_p with v_c (the longitudinal compression wave along a rod) equal to,

$$v_c=\sqrt{\frac{E}{\rho}}\quad \textit{which is obviously smaller than } v_P=\sqrt{\frac{\lambda+2G}{\rho}}$$

$$\left(\textit{remember that } \lambda=\frac{\nu E}{(1+\nu)(1-2\nu)}\right)$$

Distortional waves (S-waves, or Shear waves).

If we differentiate with respect to y and z (instead of x, as before) we obtain,

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial y}\right) = (\lambda + G)\left(\frac{\partial^2 \bar{\varepsilon}}{\partial y \partial z}\right) + G\nabla^2\left(\frac{\partial w}{\partial y}\right)$$

and

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial v}{\partial z}\right) = (\lambda + G)\left(\frac{\partial^2 \bar{\varepsilon}}{\partial z \partial y}\right) + G\nabla^2\left(\frac{\partial v}{\partial z}\right)$$

Subtracting the second from the first,

$$\rho \frac{\partial^2}{\partial t^2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = G\nabla^2\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$$

But we showed in Lecture #5 that $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 2\bar{\omega}_x$

therefore $\rho \frac{\partial^2 \bar{\omega}_x}{\partial t^2} = G\nabla^2 \bar{\omega}_x$

Therefore,

$$\boxed{\frac{\partial^2 \overline{\omega}_x}{\partial t^2} = \frac{G}{\rho}\nabla^2 \overline{\omega}_x = v_s^2 \nabla^2 \overline{\omega}_x} \quad \textit{where} \quad \boxed{v_s = \sqrt{\frac{G}{\rho}}}$$

This is the equation that represents the equation for the distorsional waves and their velocity for propagation is v_S which is known as the shear wave or S – wave.

Similarly,

$$\boxed{\frac{\partial^2 \overline{\omega}_y}{\partial t^2} = v_s^2 \nabla^2 \overline{\omega}_y}$$

$$\boxed{\frac{\partial^2 \overline{\omega}_z}{\partial t^2} = v_s^2 \nabla^2 \overline{\omega}_z}$$

Thus far, we have derived the equations of motion for primary (or P-waves) and shear (or S-waves). We have also found that they travel at different velocities.

$$v_P = \sqrt{\frac{\lambda + 2G}{\rho}} \quad \textit{however} \quad \lambda = \frac{\nu E}{(1+\nu)(1-\nu)} \quad \textit{and} \quad G = \frac{E}{2(1+\nu)}$$

therefore,

$$\boxed{v_P = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}}}$$

Similarly,

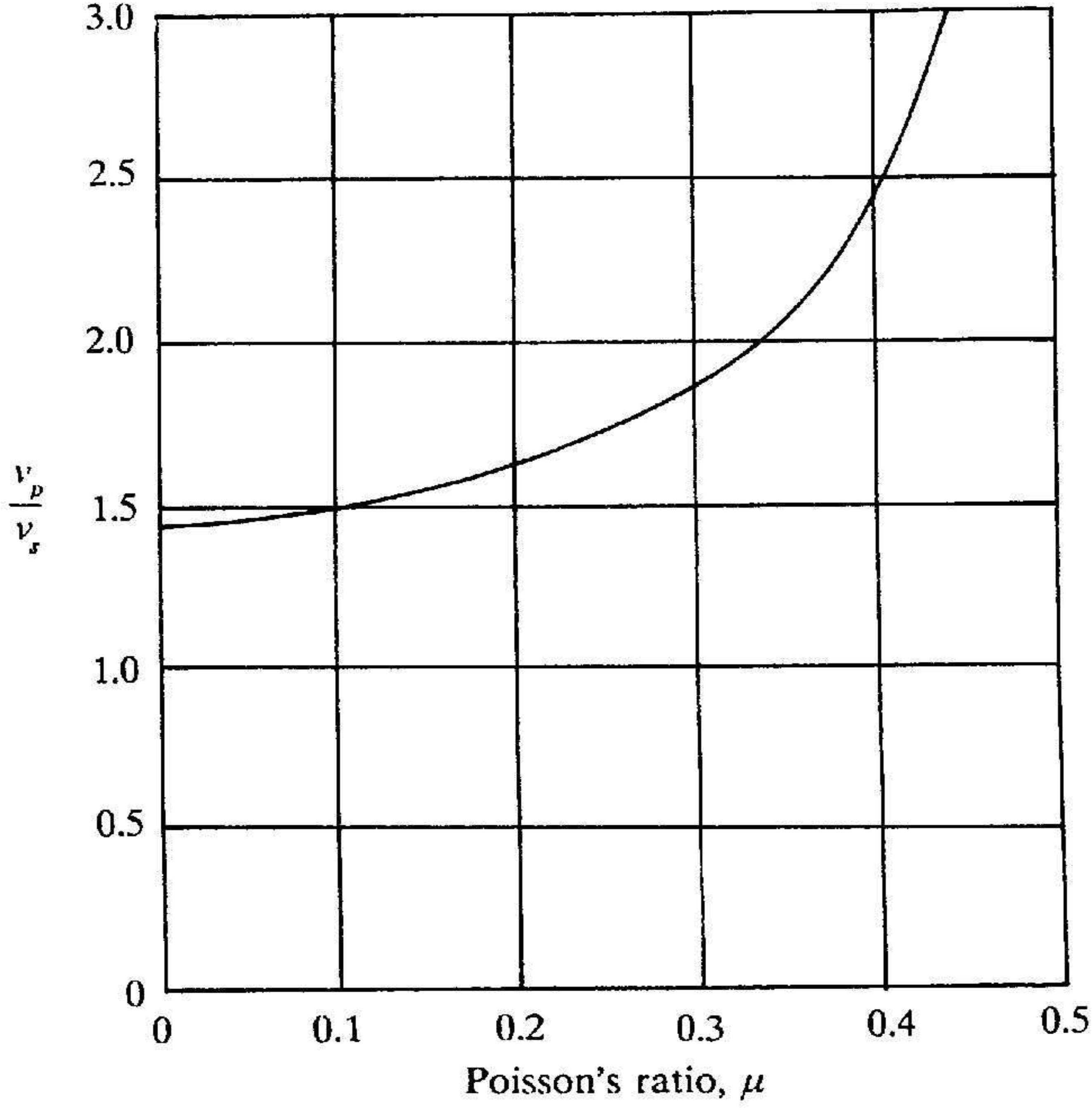
$$\boxed{v_s = \sqrt{\frac{G}{\rho}} = \sqrt{\frac{E}{\rho\, 2(1+\nu)}}}$$

Combining both v_P and v_s,

$$\frac{v_P}{v_s} = \sqrt{\frac{2(1-\nu)}{(1-2\nu)}}$$

The next slide shows the result of plotting this ratio.

Note that for all values of ν, the ratio v_P / v_S is always greater than unity.


3.0
2.5
2.0
1.5
1.0
0.5
0
$\frac{v_p}{v_s}$
0
0.1
0.2
0.3
0.4
0.5
Poisson's ratio, μ

Typical Values of v_p and v_s

Soil type	Compressive wave velocity, v_p (ft/s)	Shear wave velocity, v_s (ft/s)
Fine sand	1,000	300–500
Dense sand	1,500	750
Gravel	2,500	600–750
Moist clay	4,000–4,500	500
Granite	13,000–18,000	7,000–11,000
Sandstone	4,500–14,000	2,000–7,000

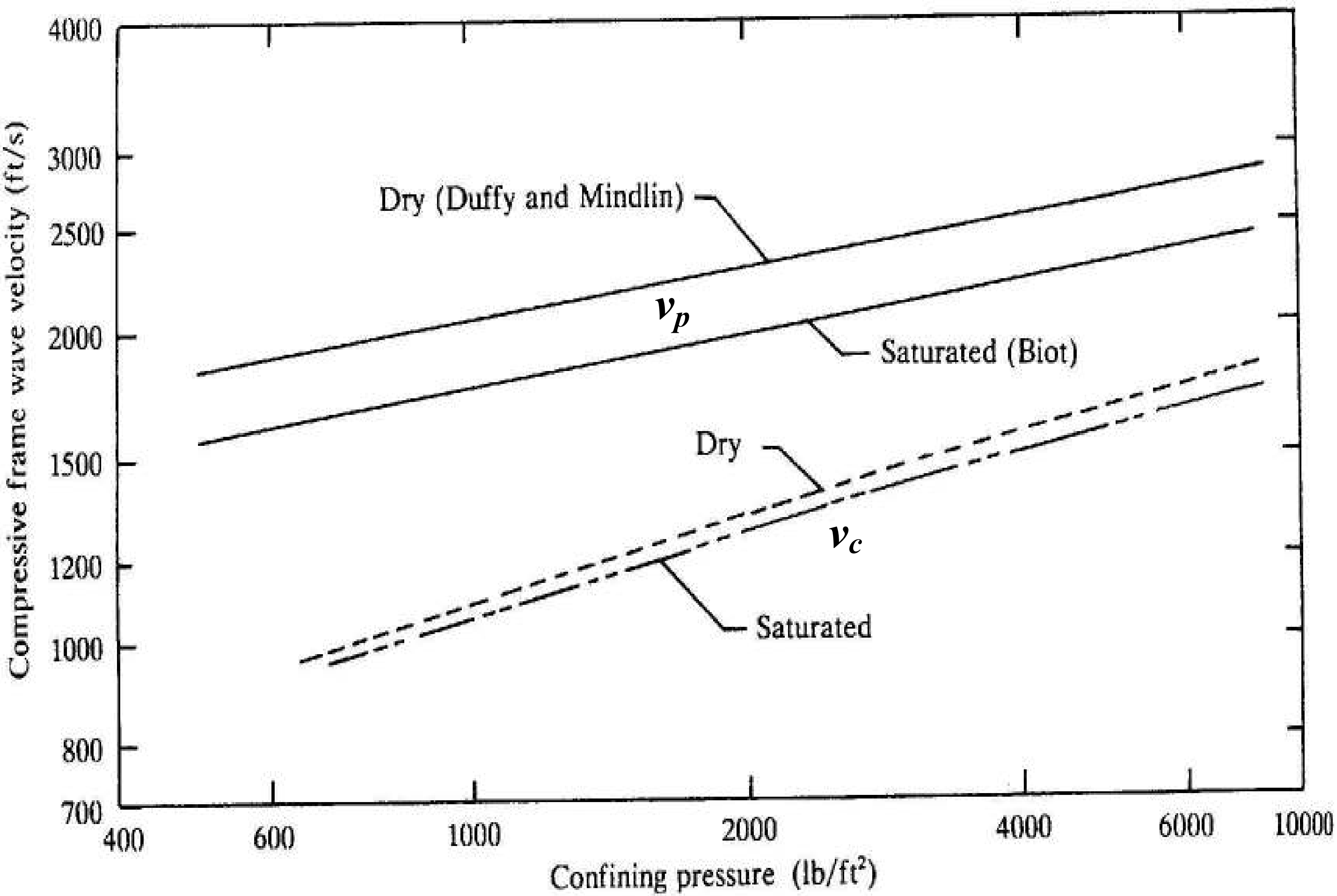
Biot (1956) studied the effects of wave propagation through saturated soils (that is, through the skeleton of solids with the pores filled with water).

This study showed that there were two compressive waves and one shear wave through the saturated media. The two compressive waves were construed to be *fluid* waves (that is, transmitted through the fluid) and a *frame* wave (transmitted through the skeleton of solid particles). Obviously, the shear wave can not flow through the water which has zero shear capacity. Therefore, the shear wave is solely dependent on the properties of the soil skeleton.

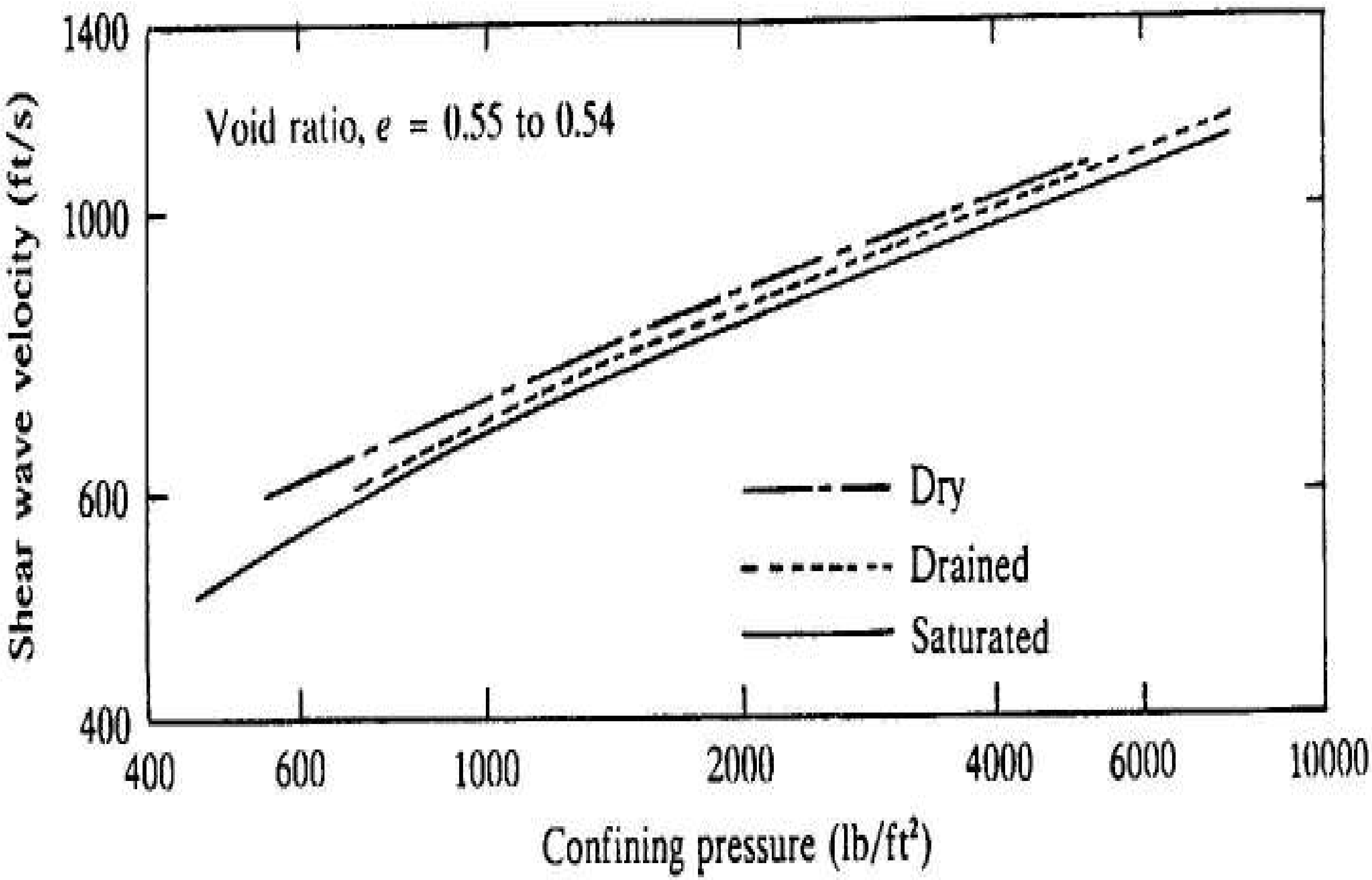
The next slide shows Biot's theory prediction of the compressive frame wave velocities in dry and saturated sands, performed by Hardin and Richart in 1963 with quartz sands. In addition, as a comparison, are the plots for the experimental longitudinal wave velocities for dry and saturated Ottawa sands. At equal confining pressures, the difference of the wave velocities between dry and saturated samples is negligible. This small difference may due to the unit weight of the soil.

The velocity of the compression waves v_w through water is and is about 4,800 ft/s, and where B_w is the bulk modulus of water and ρ_w is the density of water.

$$v_w = \sqrt{\frac{B_w}{\rho_w}}$$



Compressive frame wave velocity (ft/s)
4000
3000
2500
2000
1500
1200
1000
800
700
Dry (Duffy and Mindlin)
v_p
Saturated (Biot)
Dry
v_c
Saturated
400
600
1000
2000
4000
6000
10000
Confining pressure (lb/ft^2)

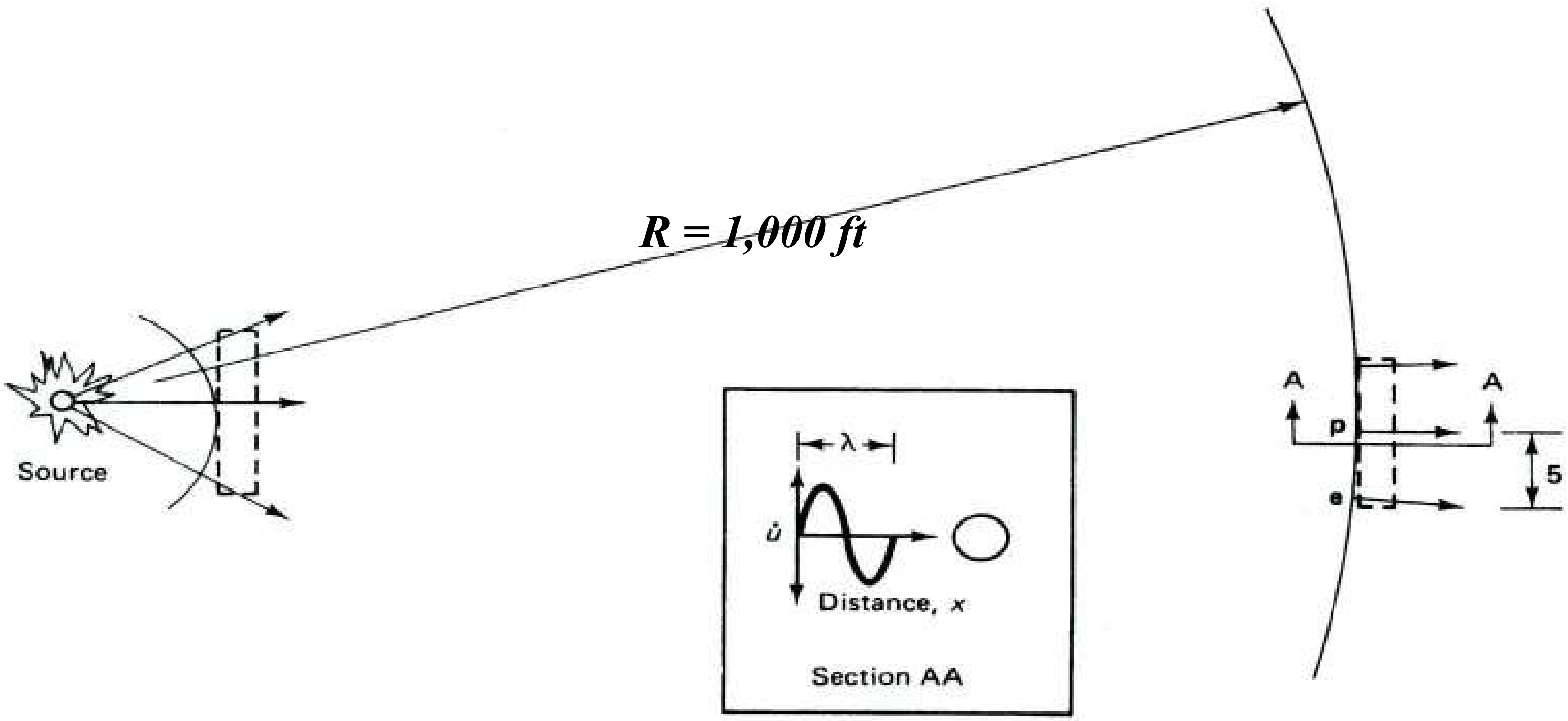


1400
1000
600
400
Shear wave velocity (ft/s)
Void ratio, e = 0.55 to 0.54
Dry
Drained
Saturated
400
600
1000
2000
4000
6000
10000
Confining pressure (lb/ft²)

Example #1.

Calculate the propagation velocities for the *P*, *S* and *R* waves in the figure shown below. The letters mark the time of arrival. What are the maximum strains if the maximum *ů* = *0.5 in/s* for the *R* wave at 1,000 feet and the particle velocities are 0.17, 0.25 and 0.50 in/s respectively?

At *R* = 1,000 ft *P* arrives at 0.13 sec
**   *S* arrives at 0.29 sec**
**   *R* arrives at 0.71 sec**



(a) The propagation velocity is the ratio of the distance from the blast (1,000 feet) divided by the time of arrival of the first part of each particular wave, 0.13, 0.29 and 0.71 seconds for the compressive (*P*-wave), the shear (*S*-wave) and the Rayleigh (*R*-wave) respectively.

Therefore, the *propagation velocities are 7,692 ft/s, 3,448 ft/s and 1,408 ft/s* respectively.

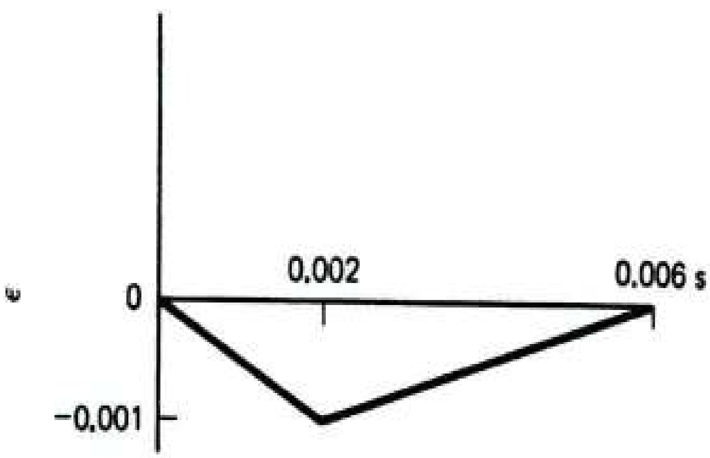
(b) Since longitudinal strains are the ratio of the particle and propagation velocities (see slide #15) for the compressive and Rayleigh waves, and the ratio of the particle to twice the propagation velocity for the shear wave, the strains can be found by dividing the appropriate particle velocities by the appropriate propagation velocities.

The particle velocities *ů* are 0.17 in/s, 0.25 in/s and 0.5 in/s for the *P*, *S* and *R*-waves, respectively.

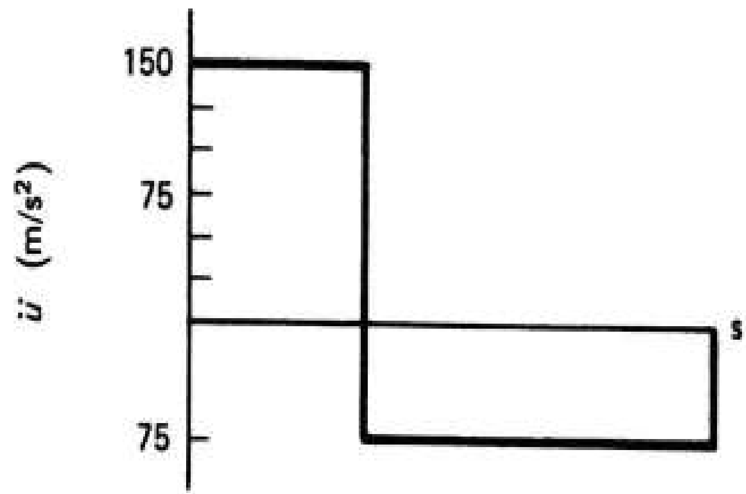
The *resulting strains are 1.8, 3.0 and 30 (in/in x 10^{-6})* respectively.

Example #2.

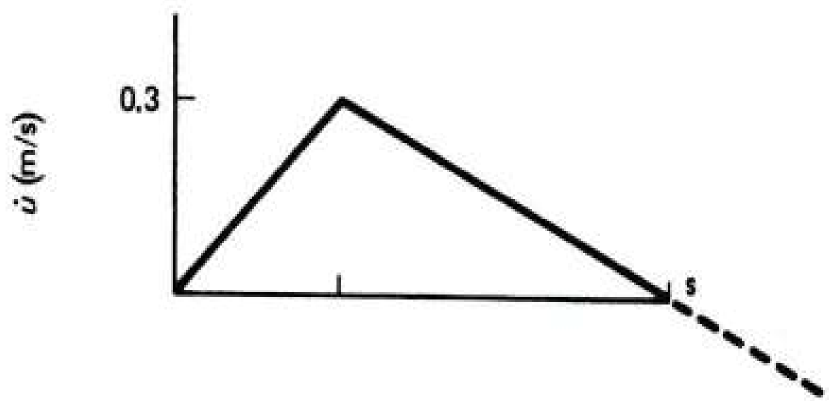
Calculate the transient compressive stress for the wave shown in the figure below, if the elastic modulus of the soil is 8 ksi when its density is ρ = 1.76 g/cm^3; assume ν = 0.2.



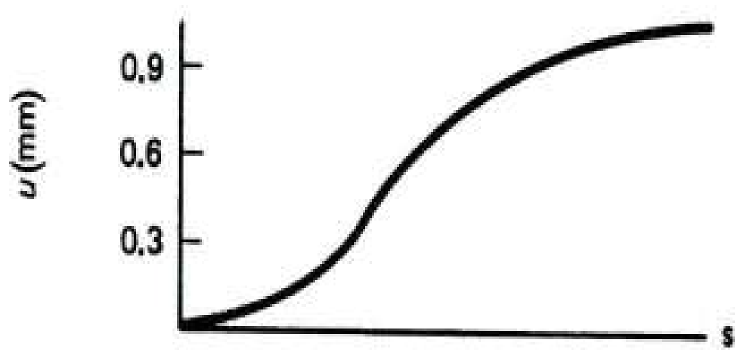
Strain ε in in/in versus time in sec.



Acceleration $\bar{\ddot{u}}$ in in/s2 versus time.



Particle velocity ů in in/s versus time.



Displacement *u* in inches versus time.

Convert ρ from metric to English units,

$$\rho = \frac{(1.76\ g_m / cm^3)(2.54\ cm)^3(1\ ft/12\ in)}{(453.6\ g_m / 1\ lb_m)(1\ in)^3(32.2\ ft/s^2)(lb_m / lb_f)} = 1.6x10^{-4}\ lb_f - s^2/in^4$$

The primary (P − wave) and shear (S − wave) velocities are,

$$v_P = \sqrt{\frac{E(1-\nu)}{\rho(1+\nu)(1-2\nu)}} = \sqrt{\frac{8{,}000\ lb/in^2\,(1-0.2)}{1.6x10^{-4}\ lb - s^2/in^4\,(1+0.2)(1-2x0.2)}}$$

$$= 7{,}453\ in/s = 621\ ft/s$$

The maximum particle velocity $\dot{u} = 0.3\ in/s$, *therefore,*

$$\sigma = \frac{\dot{u}E}{v_P} = \frac{(0.3\ in/s)(8{,}000\ psi)}{(7{,}453\ in/s)} = \underline{0.32\ psi}$$

References.

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Das, B., "Principles of Soil Dynamics", PWS-Kent Publishing Co., Boston, 1993;

Richart F.E., Hall J.R., Woods R.D., "Vibrations of Soils and Foundations", Prentice-Hall Inc., New Jersey, 1970;

Humar J.L., "Dynamics of Structures", Prentice-Hall, New Jersey, 1990;

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