Soil Dynamics

Lecture 04

Systems with Two-Degrees of Freedom

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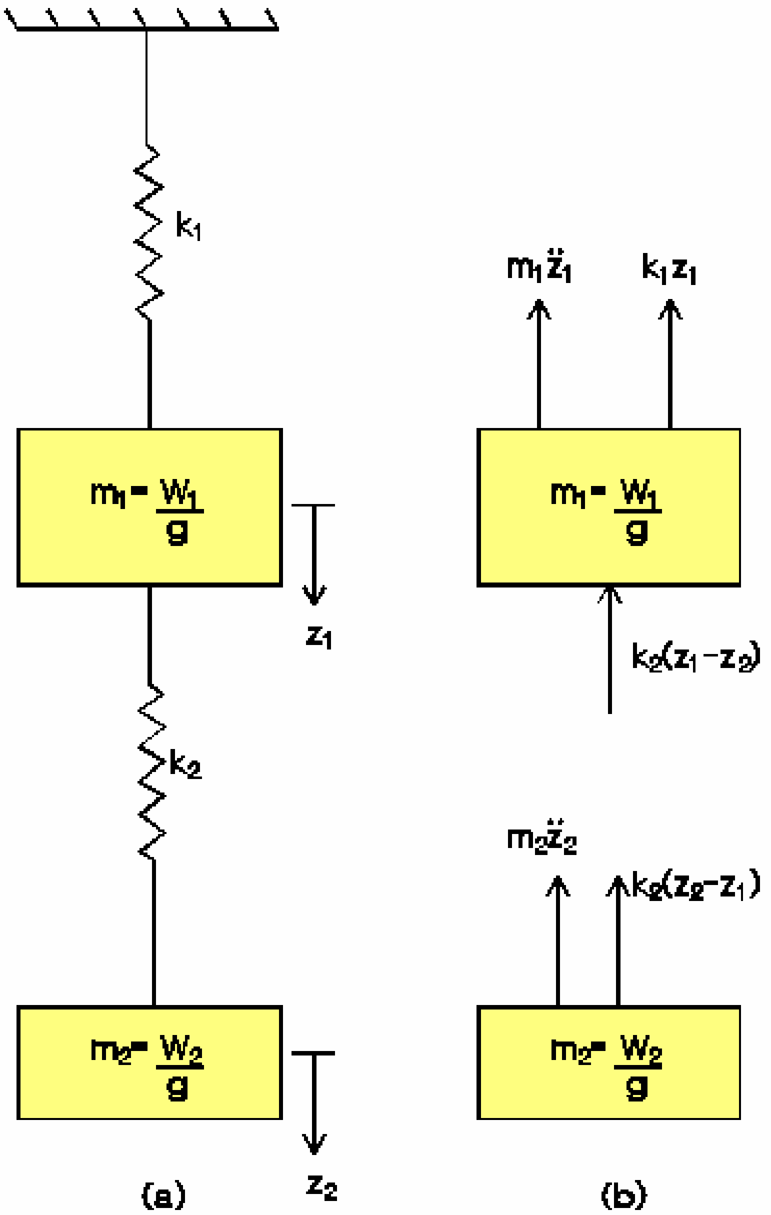
The figure at right shows a simple non-damped mass and spring system with two-degrees of freedom.

This simple system can be excited into vibration in two different ways:

(1) A sinusoidal force is applied to the mass *m_1*, thereby resulting in a forced vibration of the system, or

(2) The system is set to vibrate by an impact force on the mass *m_2*.



The calculation of the system's natural frequency. **Consider the free-body diagram on the previous slide. The differential equations of motion are,**

$$m_1\ddot{z}_1 + k_1 z_1 + k_2(z_1 - z_2) = 0 \quad and$$

$$m_2\ddot{z}_2 \qquad\qquad + k_2(z_2 - z_1) = 0$$

Let $z_1 = A\sin\omega_n t$ and $z_2 = B\sin\omega_n t$

Backsubstituting these solutions into the basic differential equations,

$$A(k_1 + k_2 - m_1\omega_n^2) - k_2 B = 0 \quad and$$

$$-Ak_2 + (k_2 - m_2\omega_n^2)B = 0$$

Since A and B are not zero, the non-trivial solution is,

$$(k_1 + k_2 - m_1\omega_n^2)(k_2 - m_2\omega_n^2) = k_2^2$$

or,

$$\omega_n^4 - \left(\frac{k_1 m_1 + k_2 m_2 + k_2 m_1}{m_1 m_2}\right)\omega_n^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

The equation for the natural frequency of the system ω_n can be simplified by setting $\eta = \dfrac{m_2}{m_1}$

and $\omega_{nl_1} = \sqrt{\dfrac{k_1}{m_1 + m_2}}$ ***and*** $\omega_{nl_2} = \sqrt{\dfrac{k_2}{m_2}}$

which yields,

$$\boxed{\omega_n^4 - (1+\eta)(\omega_{nl_1}^2 + \omega_{nl_2}^2)\omega_n^2 + (1+\eta)(\omega_{nl_1}^2)(\omega_{nl_2}^2) = 0}$$

Case 1: The amplitude of vibration for a force on mass m_1.

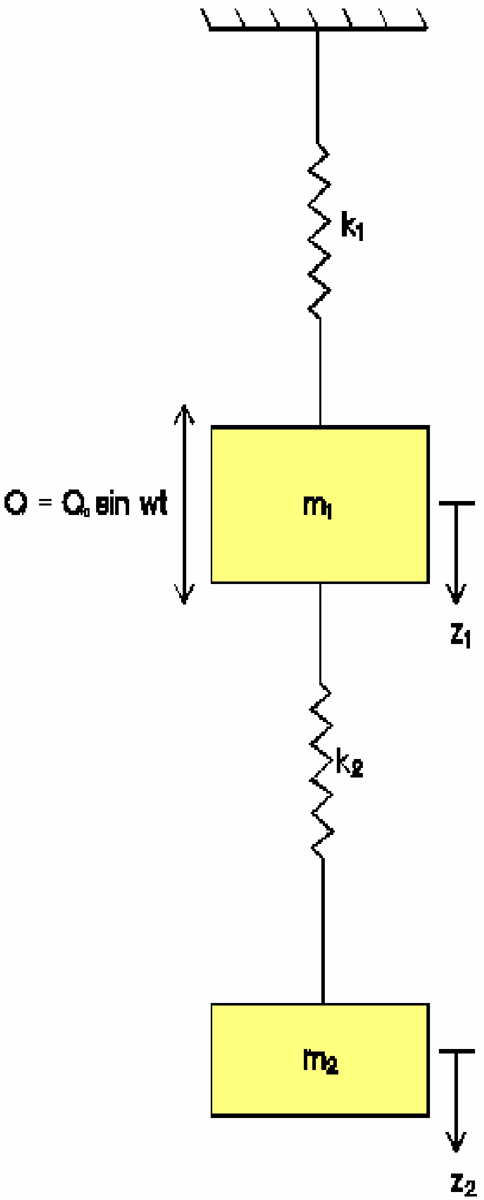
Consider the case when a vibration is induced on the system through a force acting upon the mass *m_1*. The differential equations of motion are now,

$$m_1\ddot{z}_1 + k_1 z_1 + k_2(z_1 - z_2) = Q_0 \sin\omega t \quad and$$
$$m_2\ddot{z}_2 + k_2(z_2 - z_1) = 0$$

$$Let \quad z_1 = A_1 \sin\omega t \quad and \quad z_2 = A_2 \sin\omega t$$

Substituting back into the d.e.s,

$$A_1(-m_1\omega^2 + k_1 + k_2) - A_2 k_2 = Q_0$$
$$A_2(k_2 - m_2\omega^2) - A_1 k_2 = 0$$



From these two equations, we obtain the two coefficients,

$$A_1 = \frac{Q_0(\omega_{nl_2}^2 - \omega^2)}{m_1\Delta\left(\omega^2\right)} \quad \text{and} \quad A_2 = \frac{Q_0(\omega_{nl_2}^2)}{m_1\Delta\left(\omega^2\right)}$$

where $\Delta\left(\omega^2\right) = \omega^4 - (1+\eta)\left(\omega_{nl_1}^2 + \omega_{nl_2}^2\right)\omega^2 + (1+\eta)(\omega_{nl_1}^2)(\omega_{nl_2}^2)$

Notice that if $A_1 = 0$ then $\omega_{nl_2} = \omega$

The main system is formed by m_1 and k_1, whereas the auxiliary system is formed by m_2 and k_2. Thus, the vibration of the main system can be reduced or even eliminated by the auxiliary system if its natural frequency $\omega_{nl_2} = \omega$. The auxiliary system is therefore a vibration absorber.

Case 2: An impact force upon mass m_2. **This case can be modeled by assuming that the vibration is induced via an initial velocity v_0 to the mass m_2. Hence,**

$$z_1 = C_1 \sin \omega_{n_1} t + C_2 \sin \omega_{n_2} t$$

$$z_2 = D_1 \sin \omega_{n_1} t + D_2 \sin \omega_{n_2} t$$

$$\text{At time } t = 0 \quad z_1 = z_2 = 0 \quad \text{and} \quad \dot{z}_1 = 0 \quad \text{and} \quad \dot{z}_2 = v_0$$

Substituting,

$$z_1 = \frac{\left(\omega_{nl_2}^2 - \omega_{n_1}^2\right)\left(\omega_{nl_2}^2 - \omega_{n_2}^2\right)}{\omega_{nl_2}^2\left(\omega_{n_1}^2 - \omega_{n_2}^2\right)}\left(\frac{\sin \omega_{n_1} t}{\omega_{n_1}} - \frac{\sin \omega_{n_2} t}{\omega_{n_2}}\right) v_0 \quad \text{and}$$

$$z_2 = \frac{1}{\left(\omega_{n_1}^2 - \omega_{n_2}^2\right)}\left[\frac{\left(\omega_{nl_2}^2 - \omega_{n_2}^2\right)\sin \omega_{n_1} t}{\omega_{n_1}} - \frac{\left(\omega_{nl_2}^2 - \omega_{n_1}^2\right)\sin \omega_{n_2} t}{\omega_{n_2}}\right] v_0$$

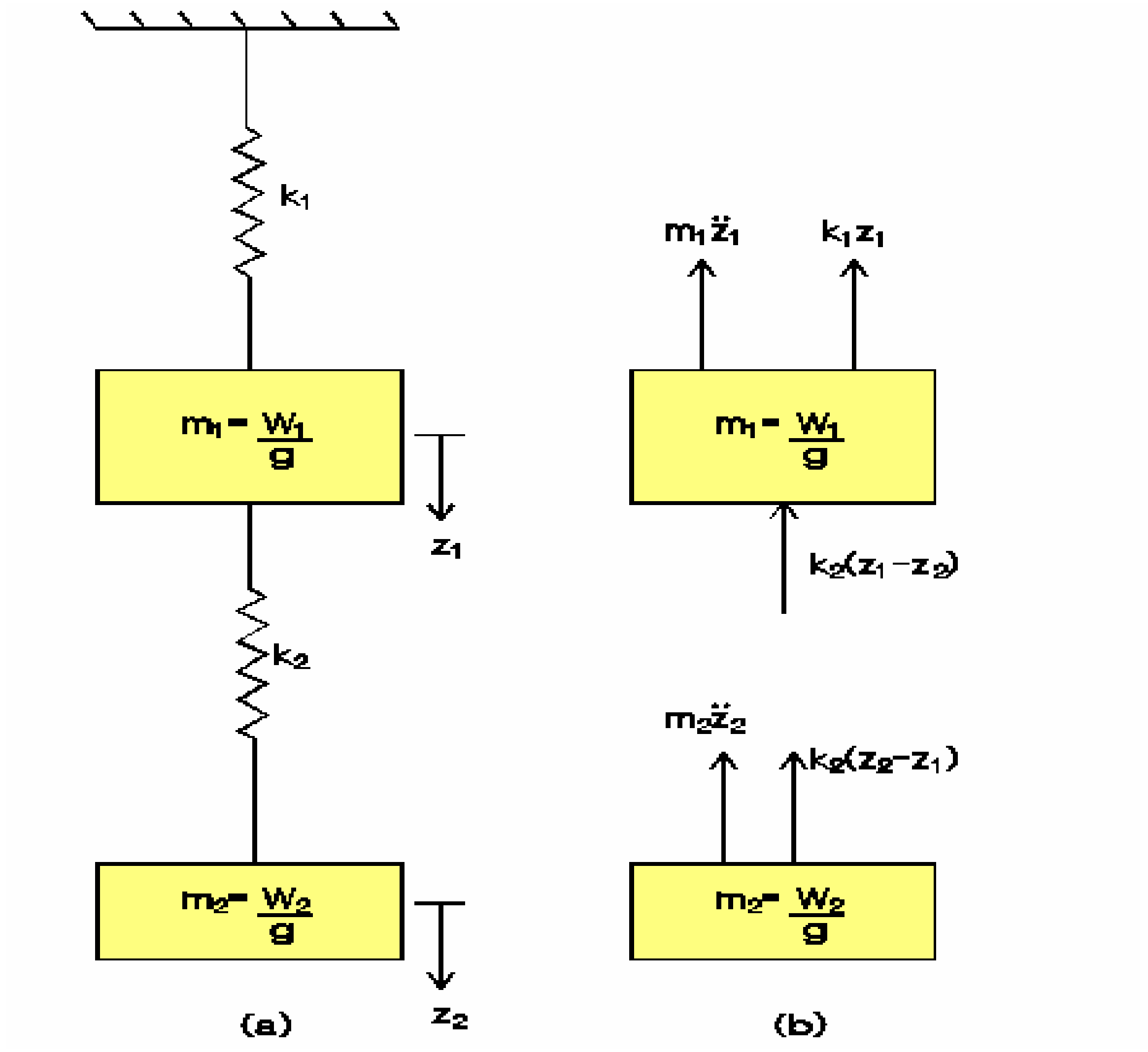
therefore, the amplitudes Z_1 and Z_2 of the masses m_1 and m_2 are,

$$\boxed{Z_1 = \frac{\left(\omega_{nl_2}^2 - \omega_{n_1}^2\right)\left(\omega_{nl_2}^2 - \omega_{n_2}^2\right)}{\omega_{nl_2}^2\left(\omega_{n_1}^2 - \omega_{n_2}^2\right)\omega_{n_2}} v_0} \quad \text{and}$$

$$\boxed{Z_2 = \frac{\left(\omega_{nl_2}^2 - \omega_{n_1}^2\right)}{\left(\omega_{n_1}^2 - \omega_{n_2}^2\right)\omega_{n_2}} v_0}$$

Example 1.

Calculate the natural frequencies of the system ω_{n1} and ω_{n2} shown below, if W_1 = 25 lb and W_2 = 5 lb, and the spring constants are k_1 = 100 lb/in and k_2 = 50 lb/in.



The equation for the natural frequencies is,

$$\boxed{\omega_n^4-(1+\eta)(\omega_{nl_1}^2+\omega_{nl_2}^2)\omega_n^2+(1+\eta)(\omega_{nl_1}^2)(\omega_{nl_2}^2)=0}$$

where,

$$\eta=\frac{m_2}{m_1}=\frac{W_2}{W_1}=\frac{5}{25}=0.20$$

$$\omega_{nl_1}=\sqrt{\frac{k_1}{m_1+m_2}}=\sqrt{\frac{(100)(32.2)(12)}{(25+5)}}=36\ radians/s$$

$$\omega_{nl_2}=\sqrt{\frac{k_2}{m_2}}=\sqrt{\frac{(50)(32.2)(12)}{(5)}}=62\ radians/s$$

Therefore,

$$\omega_n^4-(1+\eta)(\omega_{nl_1}^2+\omega_{nl_2}^2)\omega_n^2+(1+\eta)(\omega_{nl_1}^2)(\omega_{nl_2}^2)=0$$

$$\omega_n^4-(1+0.2)\left[(36)^2+(62)^2\right]\omega_n^2+(1+0.2)(36)^2(62)^2=0$$

$$\omega_n^4-6{,}140\,\omega_n^2+5{,}900{,}000=0$$

$$\omega_{n_{1,2}}^2=\frac{6{,}140\pm\sqrt{(6{,}140)^2-4(5{,}900{,}000)}}{2}$$

or $\underline{\omega_{n_1}=34\ rad/s\ \textbf{and}\ \omega_{n_2}=70\ rad/s}$

Example 2.

Calculate the amplitudes of vibration Z_1 and Z_2 for the two masses of *Example 1*, if a vibratory force $Q = 10 \sin\omega t$ (lb) is applied to the mass m_1 with a frequency ω =78.54 radians/s.

This is case #1, where a vibration is induced on m_1*. We seek the values of* A_1 *and* A_2*,*

$$\Delta(\omega^2) = \omega^4 - (1+\eta)(\omega_{nl_1}^2 + \omega_{nl_2}^2)\omega^2 + (1+\eta)(\omega_{nl_1}^2)(\omega_{nl_2}^2)$$

$$= (78.54)^4 - (1+0.2)\left[(36)^2 + (62)^2\right](78.54)^2 + (1+0.2)(36)^2(62)^2$$

$$= 6{,}000{,}000$$

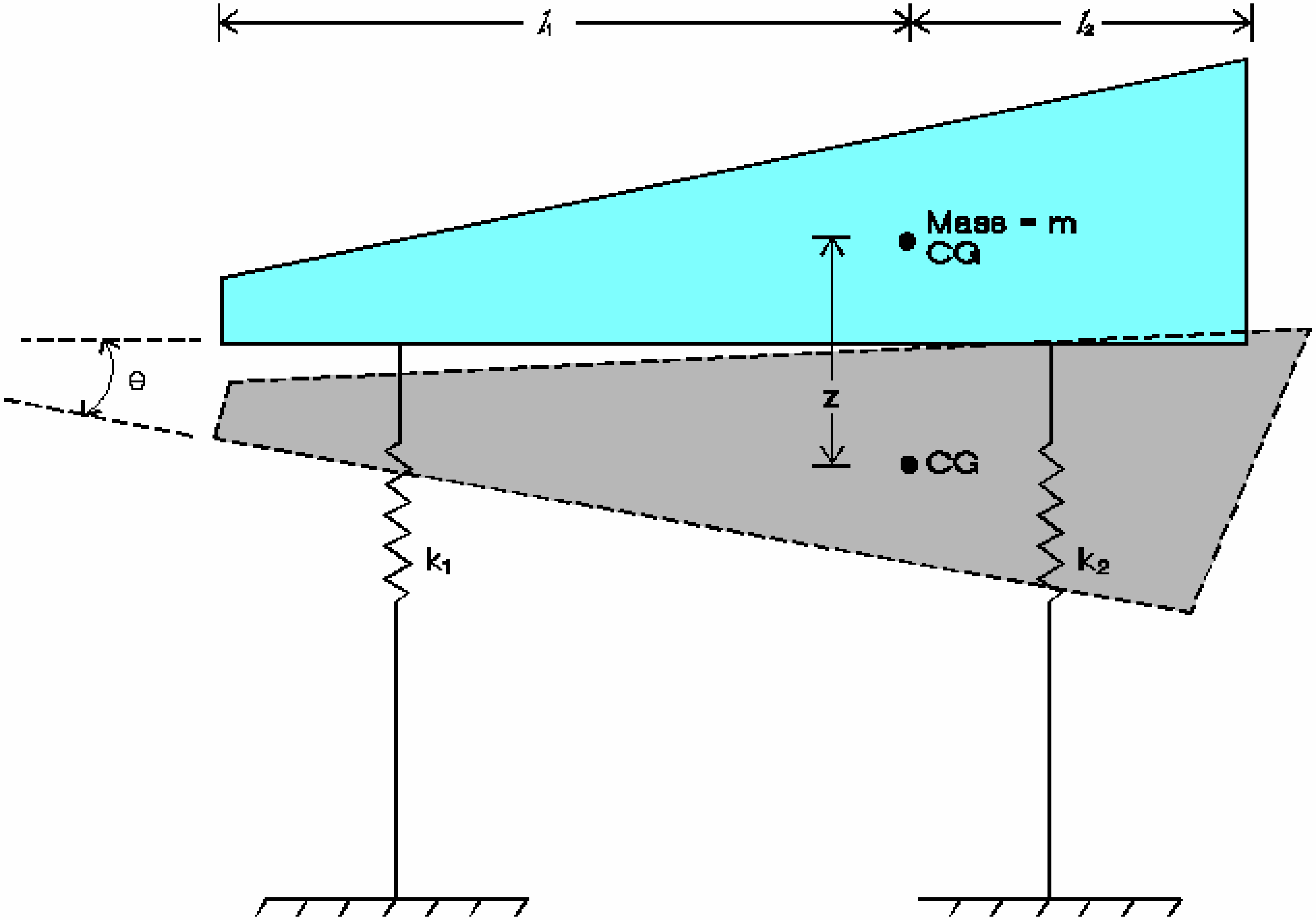
From these two equations, we obtain the coefficients,

$$A_1 = \frac{Q_0(\omega_{nl_2}^2 - \omega^2)}{m_1\Delta(\omega^2)} = \frac{(10)\left[(62)^2 - (78.54)^2\right]}{\left(\frac{25}{32.2(12)}\right)(6x10^6)} = \underline{-0.06\ inches}$$

$$A_2 = \frac{Q_0(\omega_{nl_2}^2)}{m_1\Delta(\omega^2)} = \frac{(10)(62)^2}{\left(\frac{25}{32.2(12)}\right)(6x10^6)} = \underline{0.10\ inches}$$

A free vibration system of coupled translation and rotation.

The 2-D of freedom system shown below will experience both *translation and rotation*. The two differential equations of motion of the mass *m* are,



$$m\ddot{z} + k_1(z - l_1\theta) + k_2(z + l_2\theta) = 0$$

$$mr^2\ddot{\theta} - l_1k_1(z - l_1\theta) + l_2k_2(z + l_2\theta) = 0$$

where

θ *is the angle of rotation of the mass m,*

$$\ddot{\theta} = \frac{d^2\theta}{dt^2}$$

r *is the radius of gyration of the body about the center of gravity,*

$mr^2 = J$ *is the mass moment of inertia about the center of gravity.*

Let,

$$k_1 + k_2 = k_z \quad \text{and} \quad l_1^2k_1 + l_2^2k_2 = k_\theta$$

Therefore,

$$m\ddot{z} + k_z z + \left(l_2k_2 - l_1k_1\right)\theta = 0$$

$$mr^2\ddot{\theta} + k_\theta\theta + \left(l_2k_2 - l_1k_1\right)z = 0$$

Notice that if $l_1 k_1 = l_2 k_2$ *then the* $m\ddot{z}$ *equation is independent of* θ.

Similarly, the $mr^2\ddot{\theta}$ *equation is independent of* z.

Therefore, the two motions, translation and rotation exist independent of each other,

$$m\ddot{z} + k_z z = 0$$

$$mr^2\ddot{\theta} + k_\theta \theta = 0$$

The natural circular frequency ω_{nz} *of translation is,*

$$\omega_{nz} = \sqrt{\frac{k_z}{m}}$$

and the natural circular frequency of rotation $\omega_{n\theta}$ *is,*

$$\omega_{n\theta} = \sqrt{\frac{k_\theta}{mr^2}}$$

However, if $l_1k_1 \neq l_2k_2$ the equations are coupled, and can be solved,

$$\frac{k_z}{m} = E_1 \quad and \quad \frac{l_2k_2 - l_1k_1}{m} = E_2 \quad and \quad \frac{k_\theta}{m} = E_2$$

So, the coupled differential equations are,

$$\ddot{z} + E_1 z + E_2 \theta = 0$$

$$\ddot{\theta} + \left(\frac{E_3}{r^2}\right)\theta + \left(\frac{E_2}{r^2}\right)z = 0$$

Let the solutions be,

$$z = Z\cos\omega_n t \quad and \quad \theta = \Theta\cos\omega_n t$$

from whence the general equations of motion are,

$$z = Z_1\cos\omega_{n1}t + Z_2\cos\omega_{n2}t \quad and \quad \theta = \Theta_1\cos\omega_{n1}t + \Theta_2\cos\omega_{n2}t$$

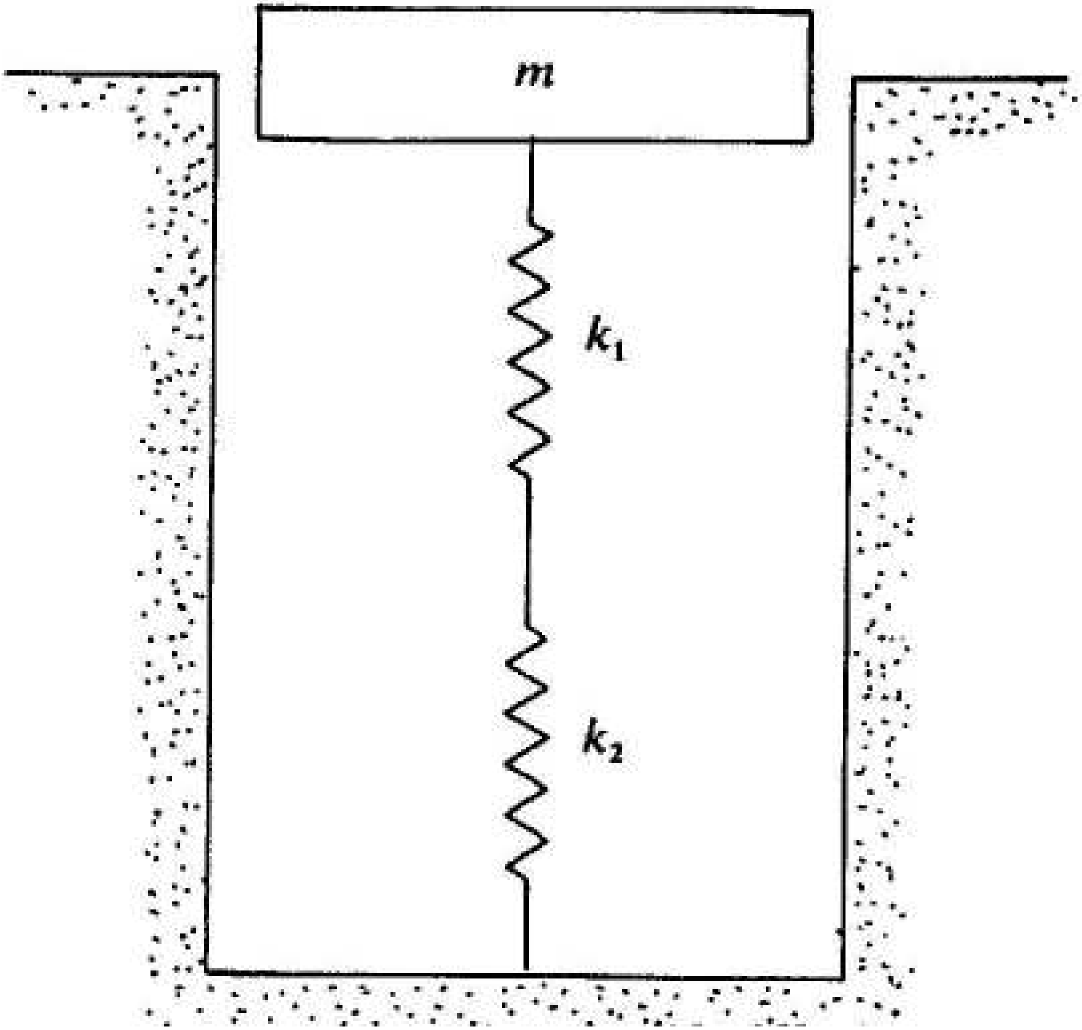
where the amplitude ratios are,

$$\frac{Z_1}{\Theta_1} = -\frac{E_2}{E_1 - \omega_{n_1}^2} = \frac{-\left(E_3/r^2 - \omega_{n_1}^2\right)}{E_2/r^2} \quad and$$

$$\frac{Z_2}{\Theta_2} = -\frac{E_2}{E_1 - \omega_{n_2}^2} = \frac{-\left(E_3/r^2 - \omega_{n_2}^2\right)}{E_2/r^2}$$

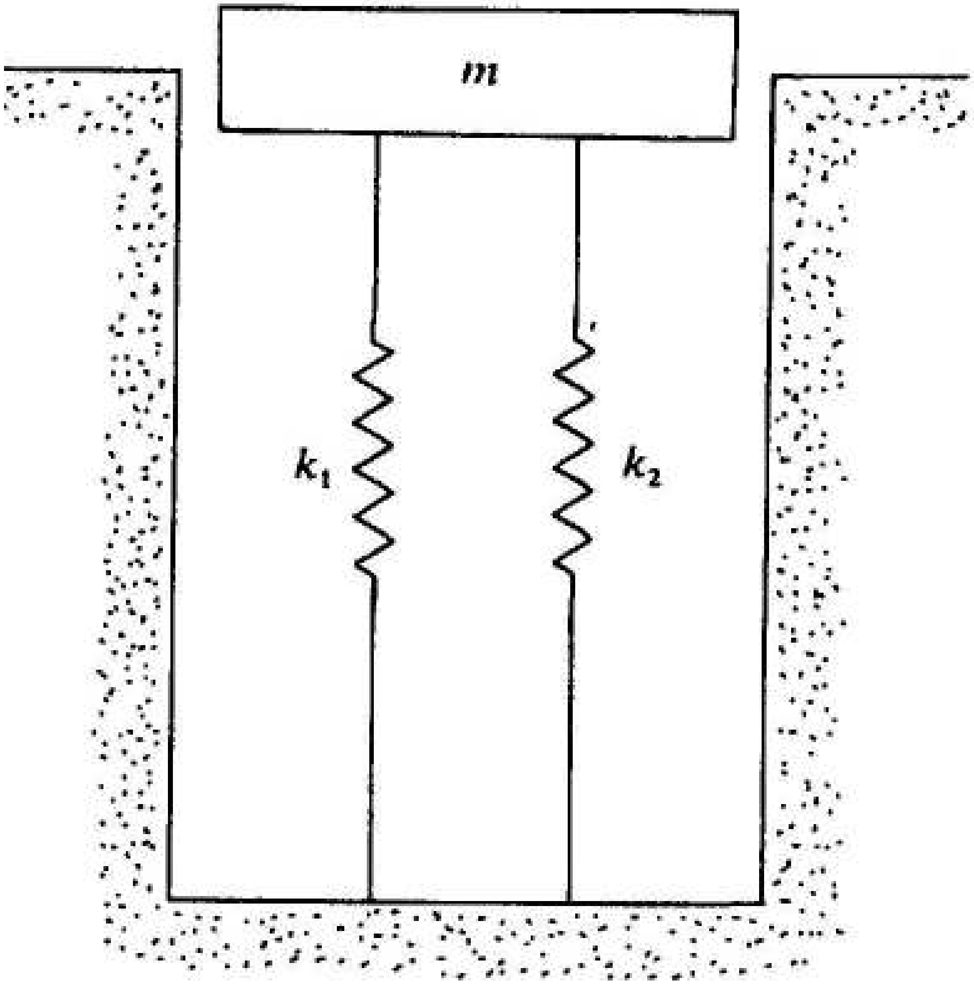
Homework Problem #1.

Determine the natural frequency of the undamped free vibration of the system shown below.



Homework Problem #2.

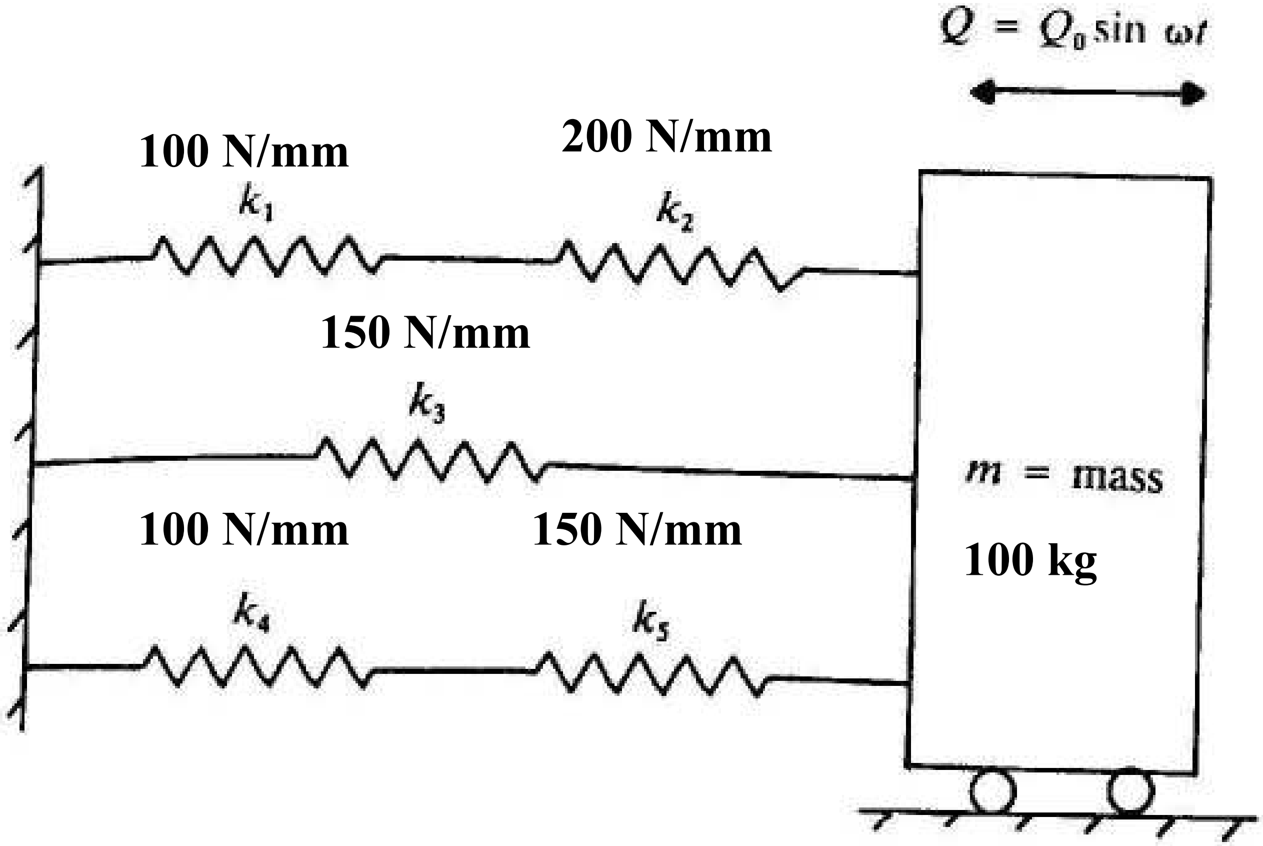
Determine the natural frequency of the undamped free vibration of the system shown below.



Homework Problem #3.

Determine the natural frequency and the period of the system shown below.

Q=50(N)sinωt where ω=47 rad/s



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