Florida International University

Department of Electrical and Computer Engineering

Digital Filters

Design of an IIR Filter using the Bilinear Transformation Method

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Objective

This paper shows the steps to design an IIR filter using the Bilinear Transformation Technique. The design of a specific LP filter is presented along with a MATLAB® program developed to perform the calculations and verify the results.

Problem

Design and realize a digital LP filter using the Bilinear Transformation method to satisfy the following specifications:

1) monothonic stopband and passband
2) –3.01dB cutoff frequency of 0.5π radians
3) Magnitude down at least 20dB at 0.75π radians

Solution

1) A Butterworth LP filter accomplishes the monotonic stopband and passband requirement

Step 1. Pre-Warp the frequencies:

Using T=1sec and $\theta_1 = 0.5\pi$ and $\theta_1 = 0.75\pi$ the pre-warped frequencies are:

$$\Omega_1 = \tan(\theta_1 / 2) = \tan(0.5\pi / 2) = 1.0$$
$$\Omega_2 = \tan(\theta_2 / 2) = \tan(0.75\pi / 2) = 2.4141$$

Step 2. Design an analog LP filter that satisfies the following criteria

$$k1 = 20\log\left|H(j\Omega_1)\right| \geq -3.01dB$$
$$k2 = 20\log\left|H(j\Omega_2)\right| \geq -20dB$$

The Butterworth Filter Order, *n*, can be calculated as follows:

$$n = \frac{\log_{10}\left[(10^{-k1/10} - 1)/(10^{-k2/10} - 1)\right]}{2\log_{10}(\Omega_1 / \Omega_2)}$$

$$n = 2.6 \approx 3$$

Step 3. Obtain *H*(*s*) from the Butterworth Table for *n*=3:

$$H(s) = \frac{1}{(p^2 + p + 1)(p + 1)}$$ which is the normalized polynomial. Since $\Omega_1 = 1.0$ we don't have to denormalize the polynomial.

Lets convert H(s) to the equivalent H(z):

$$H(s) = \frac{1}{p^3 + 2p^2 + 2p + 1}$$

$$H(z) = H(s)|_{s=\frac{z-1}{z+1}} = \frac{1}{\left(\frac{z-1}{z+1}\right)^3 + 2\left(\frac{z-1}{z+1}\right)^2 + 2\left(\frac{z-1}{z+1}\right) + 1}$$

$$H(z) = \frac{(z+1)^3}{(z-1)^3 + 2(z-1)^2(z+1) + 2(z-1)(z+1)^2 + (z+1)^3}$$

$$H(z) = \frac{z^3 + 3z^2 + 3z + 1}{(z^3 - 3z^2 + 3z - 1) + (2z^3 - 2z^2 - 2z + 2) + 2(z^3 + z^2 - z - 1) + (z^3 + 3z^2 + 3z + 1)}$$

$$H(z) = \frac{z^3 + 3z^2 + 3z + 1}{6z^3 + 2z}$$

$$H(z) = \frac{\frac{z^3 + 3z^2 + 3z + 1}{z^3}}{\frac{6z^3 + 2z}{z^3}}$$

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{6 + 2z^{-2}}$$

$$\frac{Y(z)}{X(z)} = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{6 + 2z^{-2}}$$

$$Y(z)(6 + 2z^{-2}) = (1 + 3z^{-1} + 3z^{-2} + z^{-3})X(z)$$

Applying the Inverse Z-Transform on both sides of the equation we obtain:

$$6y[n] + 2y[n-2] = x[n] + 3x[n-1] + 3x[n-2] + x[n-3]$$

And the **Difference Equation** of the **IIR filter** is:

$$y[n] = \frac{1}{6}x[n] + \frac{1}{2}x[n-1] + \frac{1}{2}x[n-2] + \frac{1}{6}x[n-3] - \frac{1}{3}y[n-2]$$

Frequency Amplitude Response

The Magnitude Response is obtained from the Transfer Function H(z) defined by:

$$H(z) = \frac{1 + 3z^{-1} + 3z^{-2} + z^{-3}}{6 + 2z^{-2}}$$

The magnitude response is calculated replacing $z = e^{j\theta}$ in the above equation, and using

$e^{j\theta} = \cos(\theta) + j\sin(\theta)$ we have:

$$H(e^{j\theta}) = \frac{1 + 3(\cos(\theta) - j\sin(\theta)) + 3(\cos(2\theta) - j\sin(2\theta)) + (\cos(3\theta) - j\sin(3\theta))}{6 + 2(\cos(2\theta) - j\sin(2\theta))}$$

$$H(e^{j\theta}) = \frac{1 + 3\cos(\theta) - 3j\sin(\theta) + 3\cos(2\theta) - 3j\sin(2\theta) + \cos(3\theta) - j\sin(3\theta)}{6 + 2\cos(2\theta) - 2j\sin(2\theta)}$$

$$H(e^{j\theta}) = \frac{(1 + 3\cos(\theta) + 3\cos(2\theta) + \cos(3\theta)) - j(3\sin(\theta) + 3\sin(2\theta) + \sin(3\theta))}{(6 + 2\cos(2\theta)) - j(2\sin(2\theta))}$$

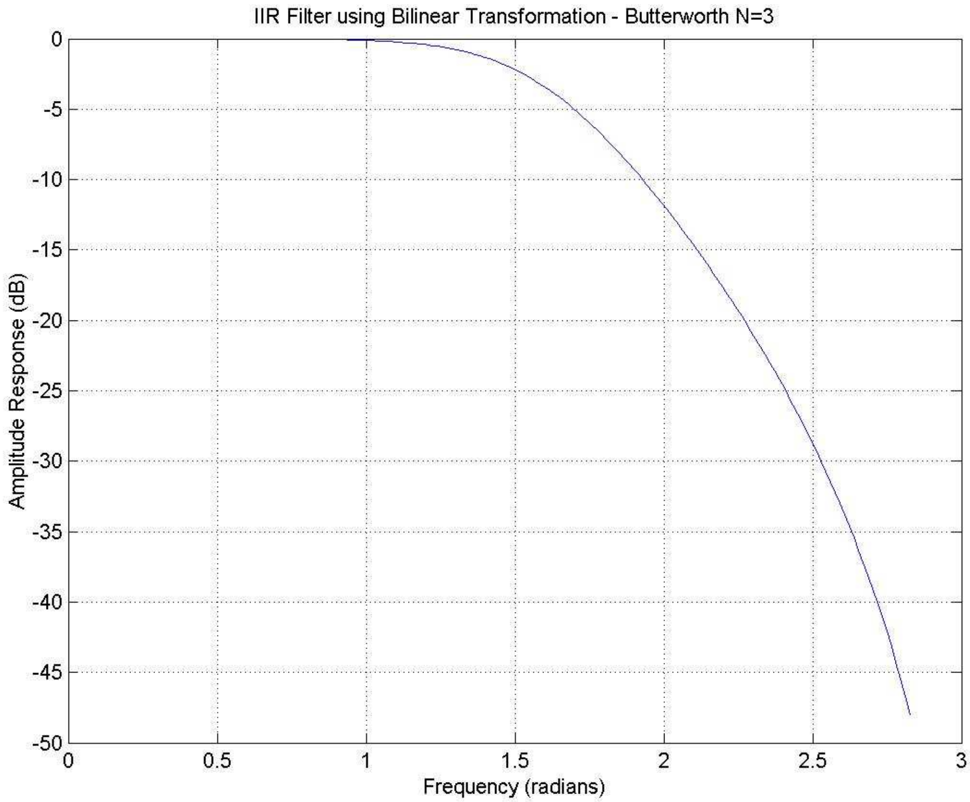
$$\left|H(e^{j\theta})\right| = \frac{\sqrt{(1 + 3\cos(\theta) + 3\cos(2\theta) + \cos(3\theta))^2 + (3\sin(\theta) + 3\sin(2\theta) + \sin(3\theta))^2}}{\sqrt{(6 + 2\cos(2\theta))^2 + (2\sin(2\theta))^2}}$$

The previous equation was implemented using the MATLAB® program in Appendix A. Figure 1 shows the Frequency Amplitude Response of the IIR filter. The attenuation at frequency 0.5π radians is –3.01dB and the attenuation at frequency 0.75π radians is –22.98dB. The performance of the designed filter exceeds the requirements.

CONCLUSIONS

The Bilinear Transformation Method was successfully used to design and realize an IIR LP Butherworth filter. The method was explained in great detail to allow the interested reader to apply it to his or her own design.

APPENDIX A: Frequency Magnitude Response Plot and MATLAB program




```
%*****************************************************************
% Florida International University
% College of Electrical Engineering
% Program Description:
% -------------------
% To design and realize an IIR Filter using the Bilinear
% transformation. The Butterworth LP analog filter has n=3
% The Transfer Function is:
%
% H(z) = (1 + 3z^-1 + 3z^-2 + z^-3)/(6 + 2z^-2)
%
% Author: Pablo Gomez
% Date: October 22, 2001
%*****************************************************************
theta=0;
delta=pi/200;
i=1;
while theta < 0.9*pi
   Num = sqrt((1+3*cos(theta)+3*cos(2*theta)+cos(3*theta))^2 +
(3*sin(theta)+3*sin(2*theta)+sin(3*theta))^2);
```

```
    Den = sqrt((6+2*cos(2*theta))^2 + (2*sin(2*theta))^2);
    H(i)= 20*log10(abs(Num/Den));
    w(i)=theta;
    theta=theta+delta;
    i=i+1;
end

%Plot the Frequency Response:
plot(w,H);
grid on;
title('IIR Filter using Bilinear Transformation - Butterworth N=3');
ylabel('Amplitude Response (dB)');
xlabel('Frequency (radians)');
%End of Program
```