Since one of you asked about what happens when your eigenvalues are complex, I have worked out an example so that you can see how it is done.

Given the following matrix

\[
A = \begin{bmatrix}
5 & 3 \\
-7 & -2
\end{bmatrix}
\]

in the equation \( \mathbf{z} = A \mathbf{z} \), let’s look at what happens when the eigenvalues are not real.

\[
A - \lambda I = \begin{bmatrix}
5 - \lambda & 3 \\
-7 & -2 - \lambda
\end{bmatrix}
\]

and \( \det(A - \lambda I) = 0 \) gives \( (5 - \lambda)(-2 - \lambda) + 21 = 0 \) or \(-10 + 2\lambda - 5\lambda + \lambda^2 + 21 = \lambda^2 - 3\lambda + 11 = 0 \). Because the coefficients of the characteristic equation, \( \lambda^2 - 3\lambda + 11 = 0 \), are real, then if your roots are complex they must come in conjugate pairs.

Now the solution to this is

\[
\lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(1)(11)}}{2} \quad \text{or} \quad \lambda = 1.5 \pm 2.958 \, i
\]

Let’s start by assuming that \( \lambda_1 = 1.5 + 2.958 \, i \)

\[
(A - \lambda_1 I) \mathbf{y}_1 = \begin{bmatrix}
5 - \lambda & 3 \\
-7 & -2 - \lambda
\end{bmatrix} \begin{bmatrix}
3.5 - 2.958 \, i \\
-3.5 - 2.958 \, i
\end{bmatrix} \begin{bmatrix}
y_{11} \\
y_{12}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

we only need one of the equations. Let’s choose the first one

\[
(3.5 - 2.958 \, i) \cdot y_{11} + 3y_{12} = 0 \quad \text{Choose } y_{11} = 1 \quad \text{Then } y_{12} = \frac{-3.5 - 2.958 \, i}{3}
\]

In my original email there was an error in both \((A - \lambda_1 I)\) and \((A - \lambda_2 I)\)

Next, let’s assume that \( \lambda_2 = 1.5 - 2.958 \, i \)

\[
(A - \lambda_2 I) \mathbf{y}_2 = \begin{bmatrix}
5 - \lambda & 3 \\
-7 & -2 - \lambda
\end{bmatrix} \begin{bmatrix}
3.5 + 2.958 \, i \\
-3.5 + 2.958 \, i
\end{bmatrix} \begin{bmatrix}
y_{21} \\
y_{22}
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

we only need one of the equations. Let’s choose the first one

\[
(3.5 + 2.958 \, i) \cdot y_{21} + 3y_{22} = 0 \quad \text{Choose } y_{21} = 1 \quad \text{Then } y_{22} = \frac{-3.5 + 2.958 \, i}{3}
\]

So that

\[
\mathbf{z}_1 = \mathbf{y}_1 e^{\lambda_1 t} = \left(\frac{1}{3}\right) e^{(1.5 + 2.958 \, i) t}
\]

and

\[
\mathbf{z}_2 = \mathbf{y}_2 e^{\lambda_2 t} = \left(\frac{1}{3}\right) e^{(1.5 - 2.958 \, i) t}
\]
Now note that $\zeta_1$ and $\zeta_2$ are complex conjugates of each other so that we will have to find the real and imaginary part of either $\zeta_1$ or $\zeta_2$.

So let us take $\zeta_1$ and find its real and imaginary part:

$$\zeta_1 = \left(-\frac{3.5 - 2.958i}{3}\right)e^{(1.5 + 2.958i)t} = \left(-\frac{3.5 - 2.958i}{3}\right)e^{1.5t}\left[\cos 2.958t + i \sin 2.958t\right]$$

$$= e^{1.5t}\left\{\begin{array}{l}
\cos 2.958t + i \sin 2.958t \\
(-3.5 \cos 2.958t - 2.958 \sin 2.958t) / 3 + i(-3.5 \sin 2.958t + 2.958 \cos 2.958t) / 3
\end{array}\right\}$$

Now take the real and imaginary part of $\zeta_1$, namely:

$$e^{1.5t}\left(\begin{array}{c}
\cos 2.958t \\
(-3.5 \cos 2.958t - 2.958 \sin 2.958t) / 3
\end{array}\right) \text{ and }$$

$$e^{1.5t}\left(\begin{array}{c}
\sin 2.958t \\
(-3.5 \sin 2.958t + 2.958 \cos 2.958t) / 3
\end{array}\right)$$

These then become the $\zeta_1$ and $\zeta_2$ for this problem. That is that

$$\zeta_1 = e^{1.5t}\left(\begin{array}{c}
\cos 2.958t \\
(-3.5 \cos 2.958t - 2.958 \sin 2.958t) / 3
\end{array}\right)$$

and

$$\zeta_2 = e^{1.5t}\left(\begin{array}{c}
\sin 2.958t \\
(-3.5 \sin 2.958t + 2.958 \cos 2.958t) / 3
\end{array}\right)$$

Now define the matrix $\Psi$, whose columns are $[\zeta_1 \ \zeta_2] = e^{1.5t}\left[\begin{array}{cc}
\cos 2.958t & \sin 2.958t \\
(-3.5 \cos 2.958t - 2.958 \sin 2.958t) / 3 & (-3.5 \sin 2.958t + 2.958 \cos 2.958t) / 3
\end{array}\right]$.

Note that the determinant of this matrix is $e^{1.5t}(2.958/3)$ which is never zero, so that the matrix $\Psi^{-1}$ exists and is equal to:

$$e^{1.5t}\left[\begin{array}{cc}
(-3.5 \sin 2.958t + 2.958 \cos 2.958t) / 3 & -\sin 2.958t \\
-(3.5 \cos 2.958t - 2.958 \sin 2.958t) / 3 & \cos 2.958t
\end{array}\right] / e^{1.5t}(2.958/3) \text{ or }$$
\[
\psi^{-1} = 1.0142 \begin{bmatrix}
-3.5 \sin 2.958t + 2.958 \cos 2.958t / 3 & - \sin 2.958t \\
3.5 \cos 2.958t + 2.958 \sin 2.958t / 3 & \cos 2.958t
\end{bmatrix}
\]

Hopefully, you now understand the process of getting the homogeneous solution to
\[
\dot{\zeta} = A \zeta
\]
when the roots of the characteristic equation are complex.