Chapter 6

Control Charts for Attributes

Introduction

- It is not always possible or practical to use measurement data
  - Number of non-conforming parts for a given time period
  - Clerical operations
- The objective is to continually reduce the number of non-conforming units.
- Control charts for attributes might be used in conjunction with measurement charts. They should be used alone only when there is no other choice.

Terminology

- Fraction of non-conforming units (ANSI standard)
  - Fraction or percentage of…
  - non-conforming, or defective, or rejected
- Non-conformity
  - Defect

6.1 Charts for Non-conforming Units

- Charts based on fraction of non-conforming units: p-chart
- Charts based on number of non-conforming units: np-chart
- Assumptions using control limits at $\mu \pm 3\sigma$:
  - Normal approximation to the binomial distribution is adequate
  - non-conformities occur independently
- Extrabinomial variation (overdispersion) can occur when the independency assumption was violated

6.1 Charts for Non-conforming Units

- Let X be the number of non-conforming unit in a sample of size n, and $\hat{p} = X/n$ denote the proportion of such units,

$$\frac{X - np}{\sqrt{np(1-p)}} \quad (6.1)$$

and

$$\frac{\hat{p} - p}{\sqrt{\frac{p(1-p)}{n}}} \quad (6.2)$$

Will be approximately distributed as N(μ=0, σ=1) when n is at least moderately large and p does not differ greatly from 0.5.

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6.1.1 np-Chart

- Control limits for np-chart are $\mathrm{np} \pm 3\sqrt{np(1-p)}$
- Rule-of-thumb: np>5; n(1-p)>5 (6.3)
- Example: for n=400, p=.10; UCL=58, LCL=22
- Binomial: P(X>58)=0.0017146; P(X<22)=0.0004383
- The adequacy of the approximation depends primarily on p.
- If p=.09, P(X<22)=0.00352185, ARL=284
- If p=.08, P(X<22)=0.02166257, ARL=46
- LCL is not very sensitive to quality improvement
- UCL has meaning only for defensive purpose
- LCL will be 0 if p<9/(9+n); or n<9(1-p)/p

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6.1.2 p-Chart

- Control limits for np-chart are $p \pm 3\sqrt{\frac{p(1-p)}{n}}$ (6.4)
- A scaled version of np-chart

6.1.3 Stage 1 and Stage 2 Use of p-Charts and np-Charts

- Need to estimate p in order to determine the sample size
  - If p~.01, n should be 900 or larger to ensure LCL
- An estimate of p would be obtained as

$$\bar{p} = \frac{total\ number\ of\ non-conforming\ units}{total\ number\ inspected}$$

  - $\bar{p} = \frac{318}{30{,}000} = .0106$
- 3-sigma limits for np-chart

$$n\bar{p} \pm 3\sqrt{n\bar{p}\,(1-\bar{p})} = 10.6 \pm 9.715$$

- UCL=20.315, LCL=.885

Table 6.1 No. of Non-conforming Transistors out of 1000 Inspected

Day	No. of non-conf.	Day	No. of non-conf.	Day	No. of non-conf.
1	7	11	9	21	13
2	5	12	13	22	7
3	11	13	8	23	9
4	13	14	11	24	12
5	9	15	12	25	8
6	12	16	10	26	14
7	10	17	9	27	12
8	10	18	12	28	12
9	6	19	14	29	11
10	14	20	12	30	13

Figure 6.1 p-chart

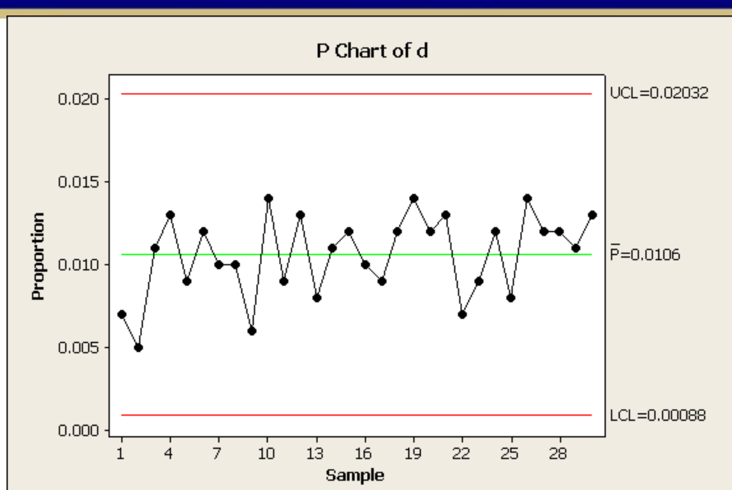
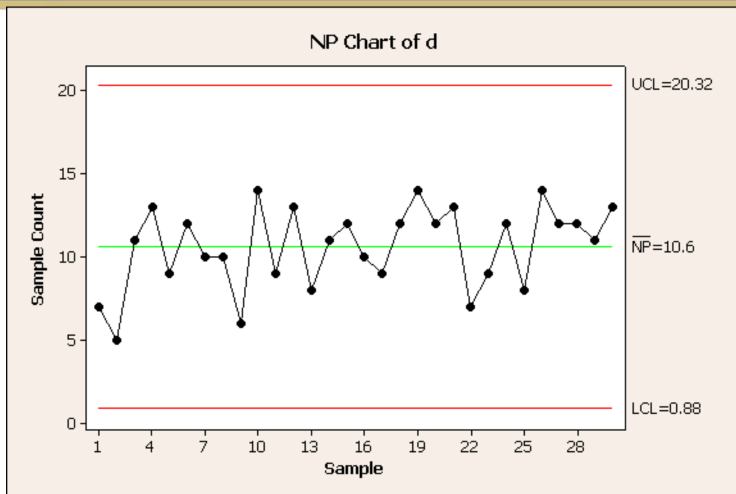


Figure 6.2 np-chart



6.1.4 Alternative Approaches

- Alternatives to the use of 3-sigma limits: (since the LCL generally too small)
  - Arcsin Transformation
  - Q-Chart
  - Regression-based Limits
  - ARL-Unbiased Charts

6.1.4.1 Arcsin Transformation

$$y = sin^{-1}\sqrt{\frac{x + {}^{3}/_{8}}{n + {}^{3}/_{4}}} \quad (6.5)$$

Will be approximately normal with mean $sin^{-1}\sqrt{p}$ and variance $\frac{1}{4n}$

For each sample of size n, one would plot the value of y on a control chart with the midline at $sin^{-1}\sqrt{p}$ and the control limits:

$$sin^{-1}\sqrt{p} \pm 3\sqrt{\frac{1}{4n}} = sin^{-1}\sqrt{p} \pm \frac{3}{2\sqrt{n}} \quad (6.6)$$

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6.1.4.1 Arcsin Transformation: Example

When n=400 and p=.10, the control limits would be:

$$sin^{-1}\sqrt{.10} \pm \frac{3}{2\sqrt{400}} = .32175 \pm .075$$

UCL=.39675 (X=59.29), LCL=.24675 (X=23.53)
Bino(X≥60)=0.001052825; Bino(X≤23)=0.00167994

3-sigma limits: UCL=58, LCL=22
Bino(X≥59)=0.001714566; Bino(X≤21)=0.000438333

Table 6.2, Table 6.3 Comparison of np-chart based on 3-sigma and arcsin transformation

Table 6.4 Minimum sample size necessary for LCL to exist

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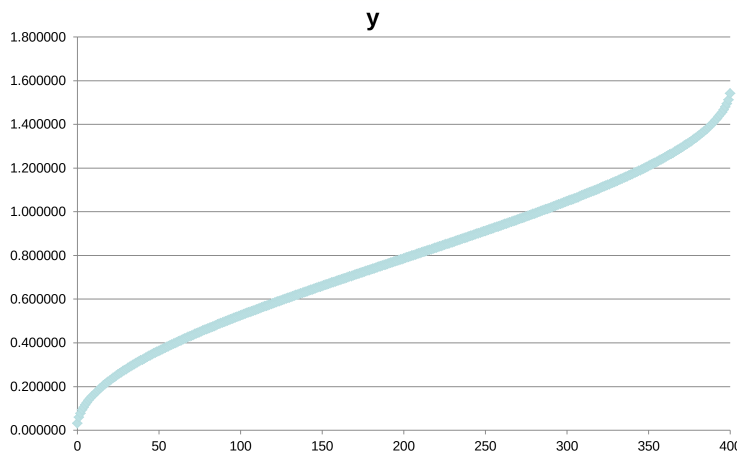
6.1.4.1 Arcsin Transformation: Example



6.1.4.2 Q-Chart

- Let $u_i = B(x_i; n, p)$ denote $P(X \leq x_i)$
- If 3-sigma limits are used, the statistics $Q_i = \Phi^{-1}(u_i)$ are plotted against control limits of ± 3, where Φ denotes the normal cumulative distribution function (cdf) and p is known.
- Q-charts have LTA less than .00135, and UTA greater than .00135.
- Arcsin approach gives a better approximation to the LTA.

6.1.4.3 Regression-based Limits

- Regression-based Limits for an np-chart

$$UCL = 0.6195 + 1.00523np + 2.983\sqrt{np}$$
$$LCL = 2.9529 + 1.01956np - 3.2729\sqrt{np} \qquad (6.7)$$

- The objective is to minimize

$$\left|\frac{1}{LTA} - \frac{1}{.00135}\right| + \left|\frac{1}{UTA} - \frac{1}{.00135}\right|$$

- Will produce the optimal limits when p~.01

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6.1.4.4 ARL-Unbiased Charts

- Control limits are such that the in-control ARL is larger than any of the parameter-change ARLs
- Problem with skewed distributions

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6.1.5 Using Software to Obtain Probability Limits for p- and np-Charts

- INVCDF probibility; (In Minitab)
  - Possible distributions and their parameters are
  - bernoulli p = k
  - binomial n = k p = k
  - poisson mu=k
  - normal [mu=k [sigma=k]]
  - uniform [a=k b=k]
  - t df=k
  - f df1=k df2=k
  - chisquare df=k

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6.1.6 Variable Sample Size

- The variable limits for the p-chart

$$\bar{p} \pm 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n_i}} \qquad (6.8)$$

- Standardized p-chart,

$$Z_i = \frac{\hat{p}_i - p}{\sqrt{\frac{p(1-p)}{n_i}}}$$

with UCL=3, LCL=-3

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6.1.7 Charts Based on the Geometric and Negative Binomial Distributions

- The use of p-chart when p is extremely small is inadvisable
- It is preferable to plot the number of plotted points until k non-conforming units are observed.
- If k=1, it is geometric distribution

$$LCL = 1 + \frac{\log(.99865)}{\log(1-p)}$$

$$UCL = \frac{\log(.00135)}{\log(1-p)}$$

- However, these limits are not ARL-unbiased.

6.1.8 Overdispersion

- The actual variance of X (or $\hat{p}$) is greater than the variance obtained using the binomial distribution.
- Causes of overdispersion:
  - Non-constant p:

$$np(1-p) + n(n-1){\sigma_p}^2$$

  - Autocorrelation

6.2 Charts for Non-conformities

- A unit of production can have one or more non-conformities without being labeled a non-conforming unit.
- non-conformities can occur in non-manufacturing applications

6.2.1 c-Chart

- C-chart can be used to control the number of non-conformities per sample of inspection units.
- Control limits for c-chart are $\bar{c} \pm 3\sqrt{\bar{c}}$ (6.9)
  - Poisson distribution
  - Adequacy of normal approximation to Poisson
    - $\bar{c}$ should be at least 5
  - No LCL when $5 \leq \bar{c} \leq 9$

Table 6.5 Non-conformity Data

Bolt No.	No. of non-conf.	Bolt No.	No. of non-conf.	Bolt No.	No. of non-conf.
1	9	3	10	4	7
2	15	4	12	5	9
3	11	5	4	1	1
4	8	1	3	2	5
5	17	2	7	3	8
1	11	3	2		
2	5	4	3		
3	11	1	3		
1	13	2	6		
2	7	3	2		

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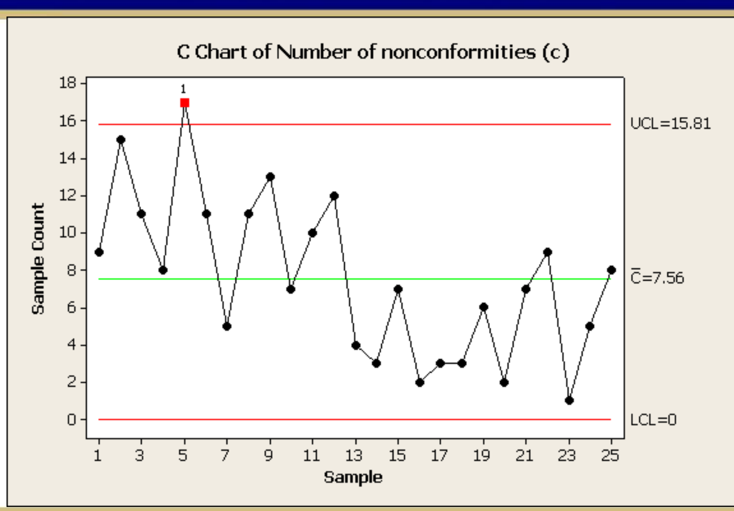
Figure 6.3 c-chart



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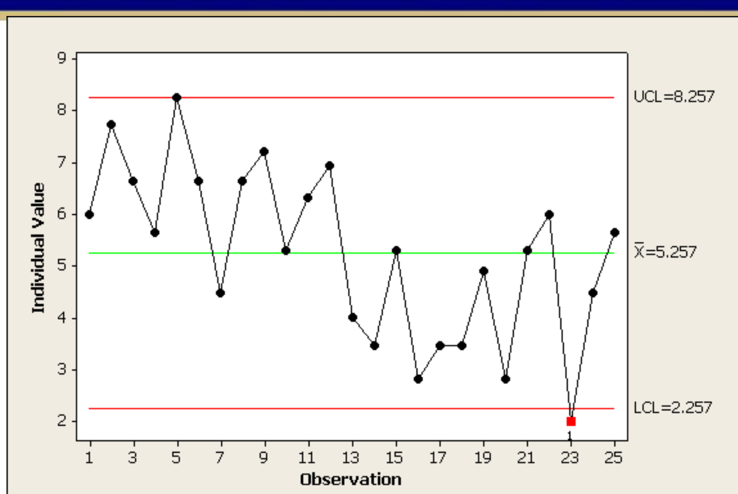
6.2.2 Transforming Poisson Data

Transformation	Mean, Variance	Control Limits
$y = 2\sqrt{c}$	$2\sqrt{\lambda}$, 1	$\bar{y} \pm 3$
$y_1 = 2\sqrt{c + {}^{3}/_{8}}$		$\overline{y_1} \pm 3$
$y_2 = \sqrt{c} + \sqrt{c+1}$		$\overline{y_2} \pm 3$

c-chart with Transformation ($y = \sqrt{c}$)



c-chart with Transformation $(y_2 = \sqrt{c + {}^3/_8})$

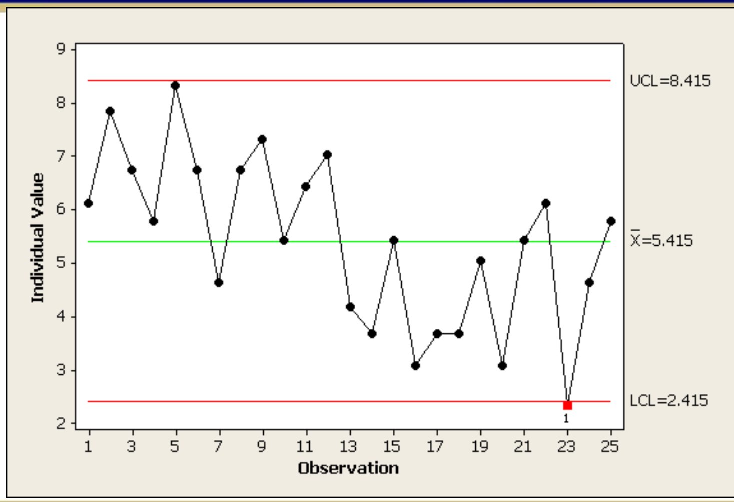
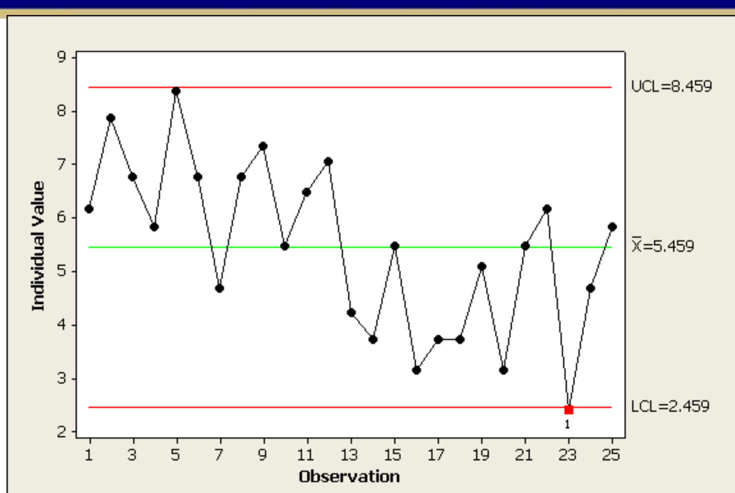


Figure 6.4 c-chart with Transformation $(y_2 = \sqrt{c} + \sqrt{c+1})$



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6.2.4 Optimal c-Chart Limits $\lambda = 5(1)50$

λ	UCL	LCL
5	12	1
6	14	1
7	16	1
8	17	2
9	19	2
10	20	3
11	22	3
12	23	4
13	24	4
14	26	5
15	27	6
20	34	9
25	41	12
30	47	16

6.2.4 Regression-based Limits

- Regression-based Limits for a c-chart

$$UCL = 0.6195 + 1.00523\lambda + 2.983\sqrt{\lambda}$$
$$LCL = 2.9529 + 1.01956\lambda - 3.2729\sqrt{\lambda} \quad (6.10)$$

- For $\lambda = 5(.25)50$
- LCL exists when $\lambda > 4.07$
- For the previous example, $\lambda = 7.56$, LCL=1.66, UCL=16.42
- Regression-based limits are superior to the transformation limits

6.2.5 Using Software to Obtain Probability Limits for c-Charts

- INVCDF probibility; (In Minitab)
  - Possible distributions and their parameters are
  - bernoulli p = k
  - binomial n = k p = k
  - poisson mu=k
  - normal [mu=k [sigma=k]]
  - uniform [a=k b=k]
  - t df=k
  - f df1=k df2=k
  - chisquare df=k

6.2.6 u-Chart

- Used when the area of opportunity for the occurrence of non-conformities does not remain constant.
- The control limits for the u-chart, $u = c/n$

$$\bar{u} \pm 3\sqrt{\frac{\bar{u}}{n}} \quad where\ \bar{u} = {}^{\bar{c}}/_{n} \qquad (6.11)$$

- Use n_i instead of n with variable sample size
- Alternatively, use $\bar{n}$ if individual sample size differs from the average no more than 25%

6.2.6 u-Chart with Transformation

- For constant sample size,

$$\frac{\bar{y}}{n} \pm \frac{3}{n} \quad where\ \bar{u} = \bar{y}/n$$

- For variable sample size,

$$\frac{\sum y}{\sum n_i} \pm \frac{3}{n_i}$$

6.2.6.1 Regression-based Limits for u-chart

- Let the UCL for a c-chart be represented by $\bar{c} + k_1\sqrt{\bar{c}}$ and LCL be represented by $\bar{c} - k_2\sqrt{\bar{c}}$
- Solve for k_1 and k_2
- For variable n_i the control limits for the u-chart would be

$$UCL = \bar{u} + k_1\sqrt{\bar{u}/n_i}$$
$$LCL = \bar{u} - k_2\sqrt{\bar{u}/n_i}$$

6.2.6.1 Regression-based Limits for u-chart Example

- $\bar{c} = \frac{1264}{200} = 6.32$
- $UCL = 0.6195 + 1.00523\bar{c} + 2.983\sqrt{\bar{c}} = 14.4717$
- $LCL = 2.9529 + 1.01956\bar{c} - 3.2729\sqrt{\bar{c}} = 1.1686$
- Let UCL $= \bar{c} + k_1\sqrt{\bar{c}}$; $k_1 = 3.2428$
- Let $LCL = \bar{c} - k_2\sqrt{\bar{c}}$; $k_2 = 2.0491$
- The control limits for the u-chart would be

$$UCL = \bar{u} + k_1\sqrt{\bar{u}/n_i} = \frac{6.32}{100} + 3.2428\sqrt{\frac{6.32/100}{100}} = .1447$$

$$LCL = \bar{u} - k_2\sqrt{\bar{u}/n_i} = \frac{6.32}{100} - 2.0491\sqrt{\frac{6.32/100}{100}} = .0119$$

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6.2.7 Overdispersion

- If overdispersion is found to exist, the negative binomial distribution may be a suitable model.

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6.2.8 D-Chart

- C-chart can be used to chart a single type of non-conformity, or to chart the sum of different types of non-conformities with equal weight
- If different weights (demerits) can be assigned, use D-charts
- Assuming 3 different types of non-conformities (c_1, c_2, c_3) with weights (w_1, w_2, w_3), then D represents the number of demerits $D = w_1 c_1 + w_2 c_2 + w_2 c_3$
- Assuming $w_i \geq 1$, D will not have a Poisson distribution

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6.2.8 D-Chart

- For k different types of independent non-conformities

$$Var(D) = \sum_{i=1}^{k} {w_i}^2 \lambda_i$$

Where λ_i is the mean of c_i, and estimated by $\bar{c}_i$

- Let c_{ij} represent the number of non-conformities of type i in inspection unit j, then $\bar{c}_i = \sum_{j=1}^{n} c_{ij}/n$
- The 3-sigma control limits for D-chart are

$$\bar{D} \pm 3\sqrt{\sum_{i=1}^{k} {w_i}^2 \bar{c}_i} \quad where\ \bar{D} = \frac{\sum_{j=1}^{n} \sum_{i=1}^{k} w_i c_{ij}}{n} = \frac{\sum_{j=1}^{n} D_j}{n}$$

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6.2.8 Du-Chart for Variable Units

- If each sample contains more than 1 inspection unit and it is desired to chart the number of demerits per inspection unit, then the counterpart of u-chart would be produced.

$$D_u = \sum_{i=1}^{k} w_i u_i$$

- $u_i = c_i / n_l$ is the number of non-conformities of type i per inspection unit in a sample that contains n_l such units
- If m samples are available

$$\bar{D}_u = \sum_{i=1}^{k} w_i \bar{u}_i \qquad where\ \ \bar{u}_i = \frac{\sum_{l=1}^{m} c_{il}}{\sum_{l=1}^{m} n_l}$$

6.2.8 Du-Chart for Variable Units

- For k different types of independent non-conformities

$$Var(D_u) = \frac{1}{{n_l}^2} \sum_{i=1}^{k} {w_i}^2 \lambda_i$$

Where λ_i is the mean of c_i, and estimated by $\bar{c}_i$

$$\bar{c}_i = \frac{\sum_{l=1}^{m} n_l c_{il}}{\sum_{l=1}^{m} n_l}$$

- The 3-sigma control limits for D-chart are

$$\bar{D}_u \pm \frac{3}{n_l} \sqrt{\sum_{i=1}^{k} {w_i}^2 \bar{c}_i}$$