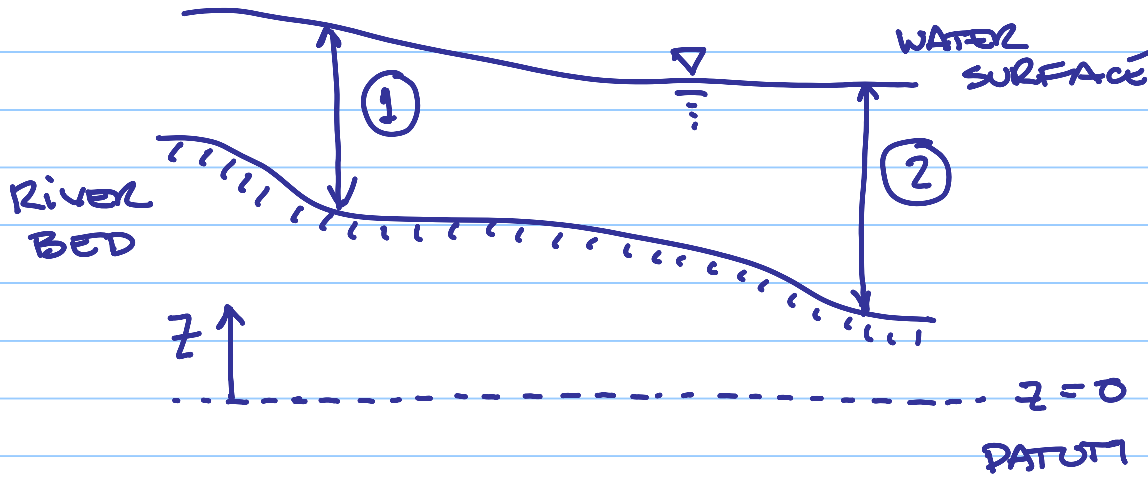
SOLUTION OF NONLINEAR EQUATIONS

EXAMPLE APPLICATION:

WATER FLOW IN A RIVER OF VARYING CROSS-SECTION



WATER FLOW IN THE RIVER IS DRIVEN BY BERNOULLI'S EQUATION

$$\underbrace{\frac{P_1}{\rho g}}_{\text{WATER DEPHT @ 1}} + \underbrace{z_1}_{\text{ELEVATION OF RIVER BED @ 1}} + \underbrace{\frac{V_1^2}{2g}}_{\text{KINETIC ENERGY @ 1}} = \frac{P_2}{\rho g} + z_2 + \frac{V_2^2}{2g}$$

SO:

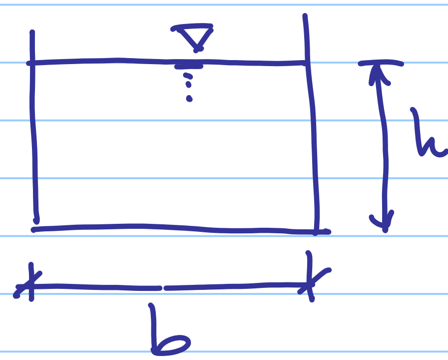
$$h_1 + z_1 + \frac{V_1^2}{2g} = h_2 + z_2 + \frac{V_2^2}{2g}$$

THE VELOCITY AND HEIGHT OF FLOW ARE RELATED THROUGH MASS CONSERVATION FOR THE WATER, IE.

$$A_1 V_1 = A_2 V_2 = \underbrace{Q}_{\text{VOLUME PER UNIT TIME}}$$

ASSUMING RECTANGULAR CROSS SECTION

$$A = b \cdot h$$



REPLACING THIS IN BERNOULLI'S EQN. YIELDS:

$$h_1 + z_1 + \frac{Q^2}{2g h_1^2 b_1^2} = h_2 + z_2 + \frac{Q^2}{2g h_2^2 b_2^2}$$

ALLOWS TO CALCULATE WATER DEPTH IN ONE LOCATION GIVEN ITS VALUE IN ANOTHER LOCATION, SO YOU CAN CALCULATE h_2 GIVEN h_1, VELOCITIES ARE CALCULATED AS WELL.

LET'S USE FOR EXAMPLE:

$Q = 5\ m^3/s$ $\qquad h_2 = ?$

$h_1 = 2m$ $\qquad V_2 = ?$

$b_1 = 2.5\ m$ $\qquad b_2 = 1\ m$

$Z_1 = 10\ m$ $\qquad Z_2 = 7m$

REPLACING VALUES:

$$2 + 10 + \frac{5^2}{2 \times 9.8 \times 2^2 \times 2.5^2} = h_2 + 7 + \frac{5^2}{2 \times 9.8 \times h_2^2 \cdot 1^2}$$

OR

$$\underbrace{h_2^3 - 5.051\, h_2^2 + 1.2755 = 0}_{\text{ALLOWS TO FIND } h_2}$$

SOLUTION OF THIS EQUATION CAN NOT BE OBTAIN IN ONE SINGLE STEP, AN ITERATIVE PROCEDURE IS REQUIRED

WE'LL LOOK AT SOME NUMERICAL METHODS FOR THE SOLUTION.

① SHOOTING METHOD

THIS METHOD USES A SERIES OF SUCCESSIVE GUESSES TO "CONVERGE" ON THE SOLUTION SOUGHT.

GUESS (h_2)	$f(h_2)$ (SHOULD BE ZERO)
1	-2.7755
2	-10.93 → WRONG DIRECTION
0	1.2755 → RIGHT DIRECTION
0.5	0.1378
0.7	-0.8565
0.55	-0.086
0.54	-0.04
0.53	0.0056
0.535	-0.017
$\boxed{0.531}$	$\boxed{0.001}$

KEY QUESTIONS: HOW GOOD IS GOOD ENOUGH?

OTHER SOLUTIONS?

$h_2 = 0.531\ m$

$h_2 = 5\ m$

$h_2 = -0.48\ m$ (NO POSSIBLE)

② Direct Substitution Method

In this method, the eqn. $f(x) = 0$ is rewritten in the form:

$$x = g(x)$$

And an iterative procedure is adopted using the relation

$$x_{i+1} = g(x_i) \qquad i = 1, 2, 3 \ldots\ldots$$

The iterative process can be stopped when

$$|x_{i+1} - x_i| \leq \varepsilon$$

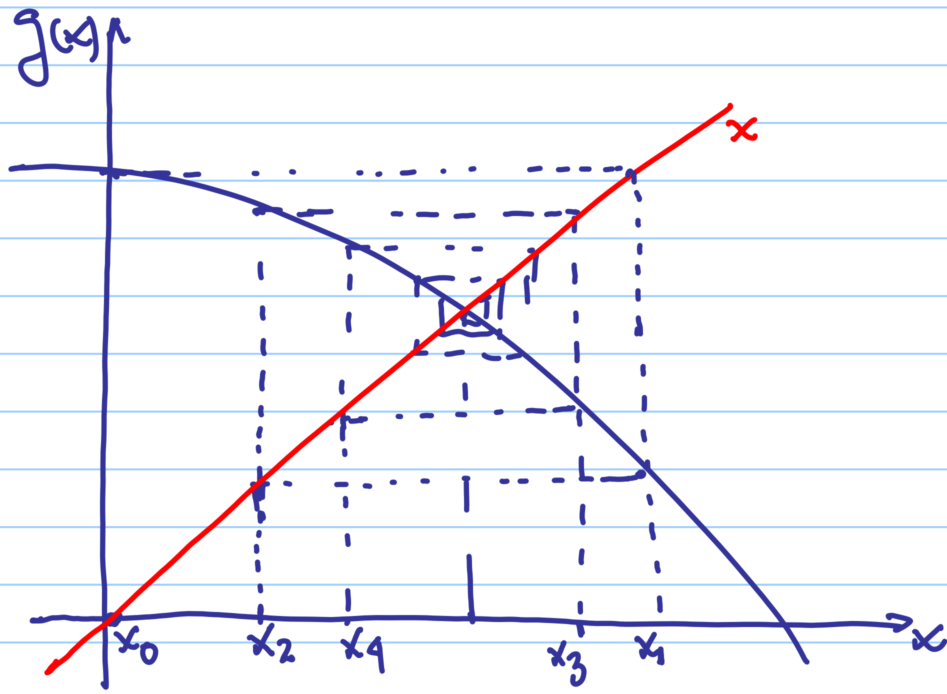
where ε is a small number on the order of 10^{-3} to 10^{-6}

The method is very simple, however, it may not always converge. The condition for convergency is:

$$|g'(x)| < 1$$

which means:

$$-1 < g(x_0) < 1$$



FOR OUR EXAMPLE :

$$h_2^3 - 5.051\,h_2^2 + 1.2755 = 0$$

$$h_2 = \sqrt{\frac{h_2^3 + 1.2755}{5.051}}$$

GUESS (h_2)	$g(h_2)$ (SHOULD BE EQUAL TO h_2)
1	GO TO EXCEL FILE NON-LINEAR EQUATIONS!

③ NEWTON-RAPHSON METHOD

THIS IS THE MOST POWERFUL METHOD USED FOR FINDING THE ROOT OF THE EQUATION :

SOLUTION POINT

FUNCTION $f(x) = 0$

THE NEWTON METHOD CAN BE DERIVED BY CONSIDERING THE CONCEPT OF FIRST DERIVATIVE :

(TRUE ONLY IF $x - x_0$ IS VERY SMALL!)

$$f'(x_0) = \frac{f(x) - f(x_0)}{x - x_0}$$

IN ORDER TO FIND THE ROOT OF $f(x) = 0$, WE SET :

$$f'(x_0)\cdot(x - x_0) + f(x_0) = 0$$

THEN $$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

THE ITERATIVE PROCEDURE CAN BE GENERALIZED AS :

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} \qquad i = 1, 2, 3 \ldots\ldots$$

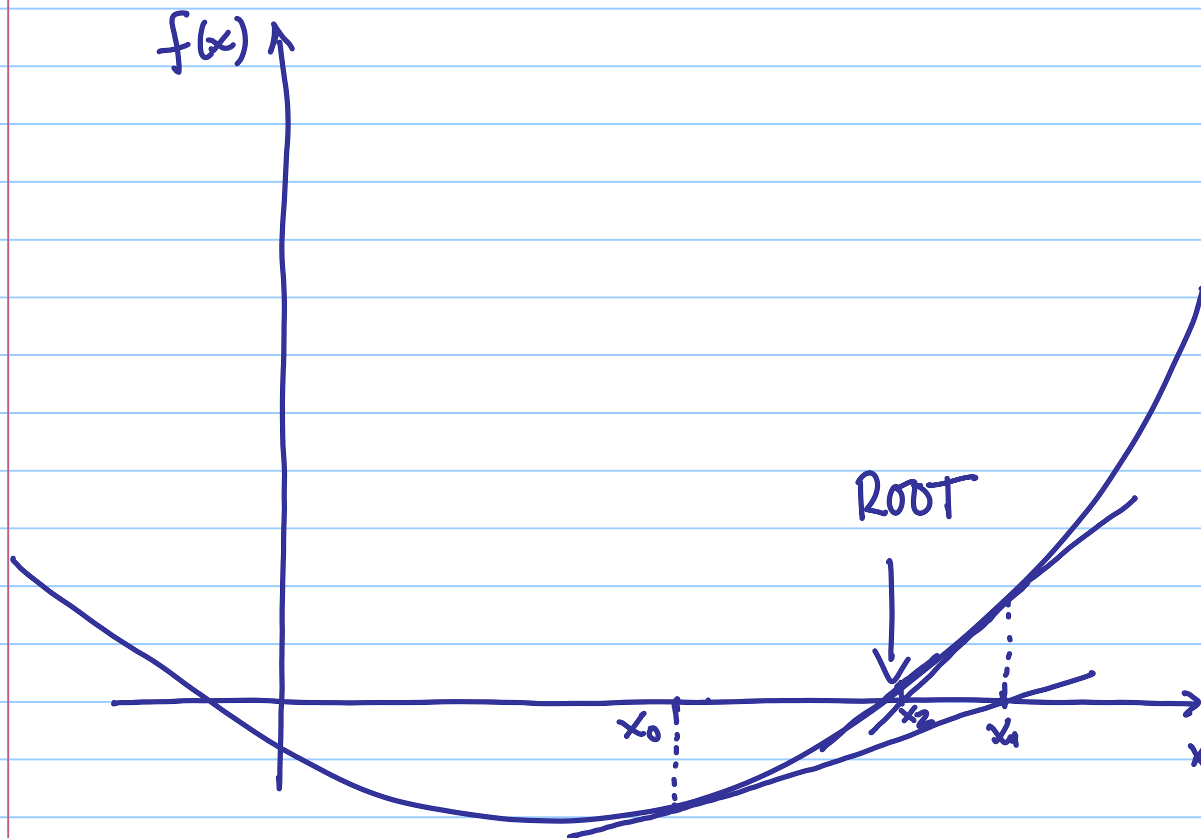
THE PROCEDURE STOPS WHEN :

$$|x_{i+1} - x_i| < \varepsilon$$

OR $$|f(x_i)| < \varepsilon$$

WHERE ε IS VERY SMALL, SO THE CONCEPT OF DERIVATIVE IS TRUE !

Ex :



FOR OUR EXAMPLE OF FLUID MECHANICS $x = h_2$, THEN

$$f(h_2) = h_2^3 - 5.051\, h_2^2 + 1.2755$$

OR

$$f(x) = x^3 - 5.051x^2 + 1.2755$$

IN ORDER TO APPLY NEWTON-RAPHSON WE NEED $f'(x)$

$$f'(x) = 3x^2 - 10.102x$$

WE START THE METHOD WITH A GUESSING VALUE, WHICH IS x_0

x_0	$f(x_0)$	$f'(x_0)$	x	ERROR $(x-x_0)$
1	...	...	...	SMALL? IF NOT, MAKE $x_0 = x$ AND CONTINUE

REMARKABLY, THE NEWTON-RAPHSON METHOD DOES VERY WELL, EVEN WITH POOR GUESSING

GO TO EXCEL FILE NON-LINEAR EQUATIONS!

4 SECANT METHOD

NEWTON-RAPHSON METHOD REQUIRES THE DERIVATIVE OF THE FUNCTION

$$f'(x) = \frac{df}{dx}$$

IN MANY CASES, THE DERIVATIVE OF $f(x)$ CAN BE FOUND EASILY.

HOWEVER, IN SOME PROBLEMS, THE DIFFERENTIATION OF $f(x)$ MAY BE COMPLICATED. IN THESE CASES THE DERIVATIVE OF $f(x)$ COULD BE OBTAINED NUMERICALLY, BY USING TWO CONSECUTIVE VALUES OF $f(x)$.

NEWTON'S METHOD :

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

APPROXIMATE :

$$f'(x_i) \approx \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

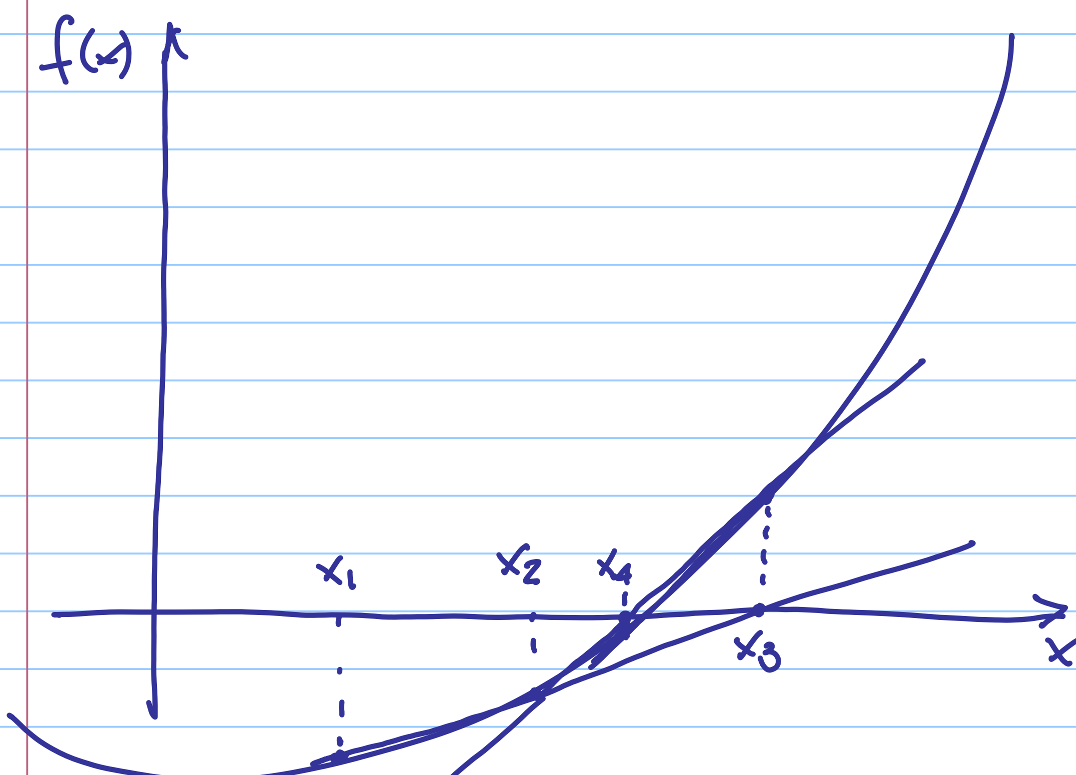
AND SUBSTITUTE IN EQUATION ABOVE

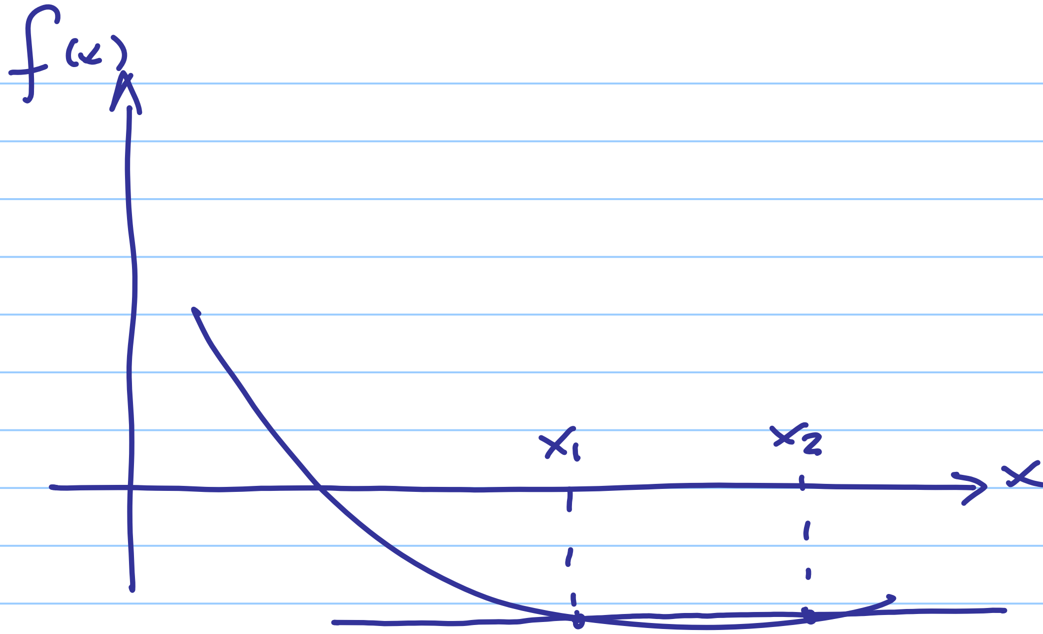
$$x_{i+1} = x_i - \frac{f(x_i)\,(x_i - x_{i-1})}{f(x_i) - f(x_{i-1})}$$

NOTICE THAT TWO INITIAL GUESSES ARE REQUIRED TO START THE ITERATIVE PROCESS, THEN THE ITERATIVE PROCESS CAN BE STOPPED WHENEVER

$$|f(x_{i+1})| \leq \varepsilon$$

OR $\quad |x_{i+1} - x_i| \leq \varepsilon$





PROBLEM... $f(x_1) \approx f(x_2)$

2.14

Ex. FIND THE ROOT OF THE FOLLOWING EQUATION :

$$f(x) = \frac{1.5x}{(1+x^2)^2} - 0.65\,\text{Tan}^{-1}\left(\frac{1}{x}\right) + \frac{0.65x}{(1+x^2)}$$

$$f'(x) = ?$$

USE AS INITIAL GUESSES

$x_1 = 0$, $x_2 = 0.5$

AND TOLERANCE $\varepsilon = 10^{-5}$

EXCEL $\Rightarrow$ SECANT METHOD.

5 Modified Newton-Raphson

Determination of multiple roots.

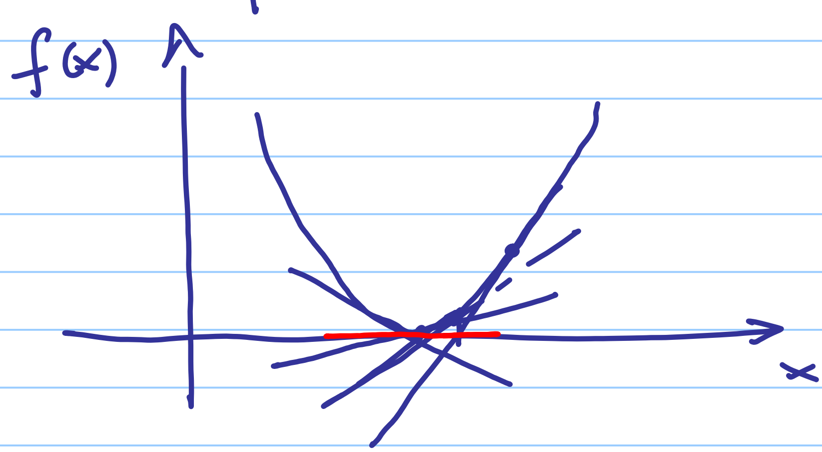
A function $f(x)$ is said to have multiple roots when it can be expressed as:

$$f(x) = (x - x^*)^p \, q(x)$$

ROOT OF MULTIPLICITY 'p'

Then, $f(x^*) = f'(x^*) = f''(x^*) = 0$

Since $f(x)$ and derivatives are zero at x^*, the Newton-Raphson and Secant methods, which contain derivatives in the denominator), cannot be apply conveniently to find the root.



HOWEVER, A SLIGHTLY MODIFIED FORMULA HAS BEEN SHOWN TO IMPROVE CONVERGENCY:

$$x_{i+1} = x_i - p\,\frac{f(x_i)}{f'(x_i)}$$

WHERE p = MULTIPLICITY OF THE ROOT (USUALLY UNKNOWN !! ☹)

ANOTHER APPROACH HAS BEEN SUGGESTED CONSIDERING:

$$g(x) = \frac{f(x)}{f'(x)}$$

IT CAN BE SHOWN THAT $g(x)$ HAS THE SAME ROOTS AS $f(x)$, SO LOOKING FOR THE ROOTS OF $g(x)$ IS THE SAME AS FOR $f(x)$.

$$g'(x) = \frac{f'(x)\,f'(x) - f''(x)\,f(x)}{\left(f'(x)\right)^2}$$

MNR :

$$x_{i+1} = x_i - \frac{g(x_i)}{g'(x_i)}$$

NEWTON RAPHSON METHOD IS SUPPOSED TO HAVE QUADRATIC CONVERGENCE.

$$e_{i+1} \propto e_i^2$$

WHEN MULTIPLE ROOTS ARE PRESENT CONVERGENCE BECOME LINEAR (NO GOOD)

USING MNR CONVERGENCE BECOMES QUADRATIC

CHECK !