Application

Numerical Differentiation

Numerical Differentiation

- Is the process where derivatives are replaced by discrete corresponding forms, known as finite-difference approximations.
- The use of finite differences transforms an ordinary differential equation into an algebraic equation, which will be easier to solve.
- The numerical analysis that deals with the discretization of derivatives is known as Finite Difference Calculus.

Taylor's Series Expansion

- Using Taylor's Series Expansions, the value of a function in a point close to x_i can be estimated.

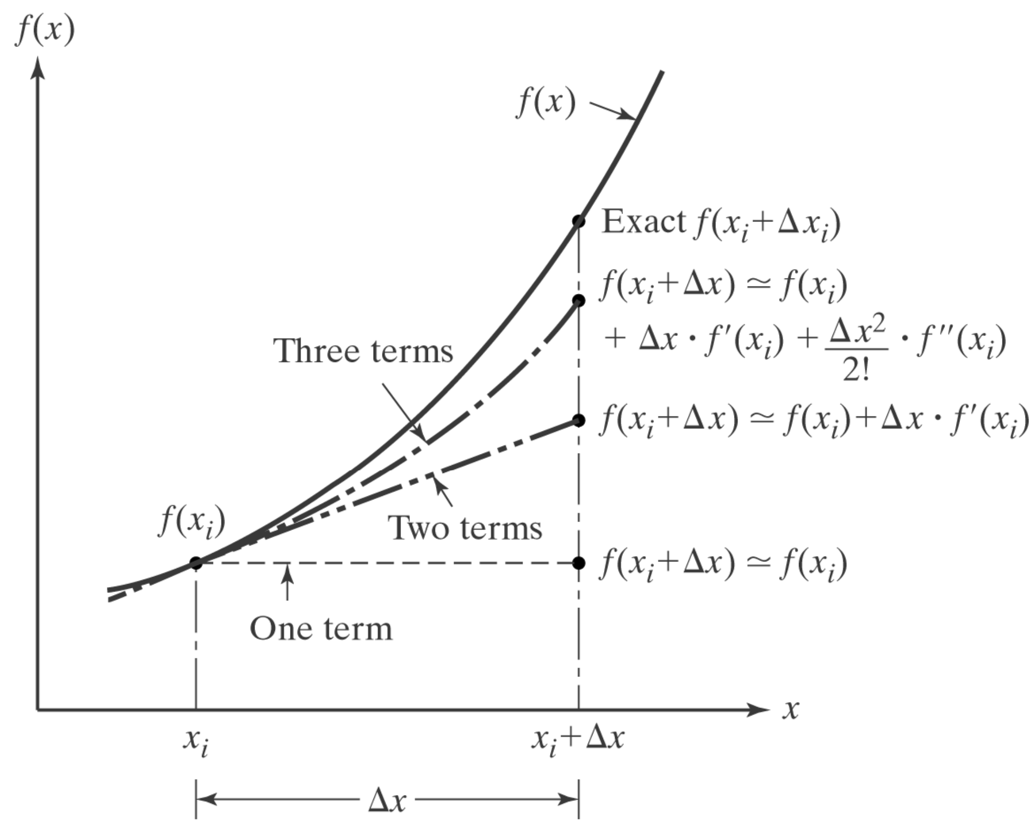


Figure 7.4
Approximations using different number of terms.

Finite-Difference Approximations of First Derivatives

- Forward Difference Approximation

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

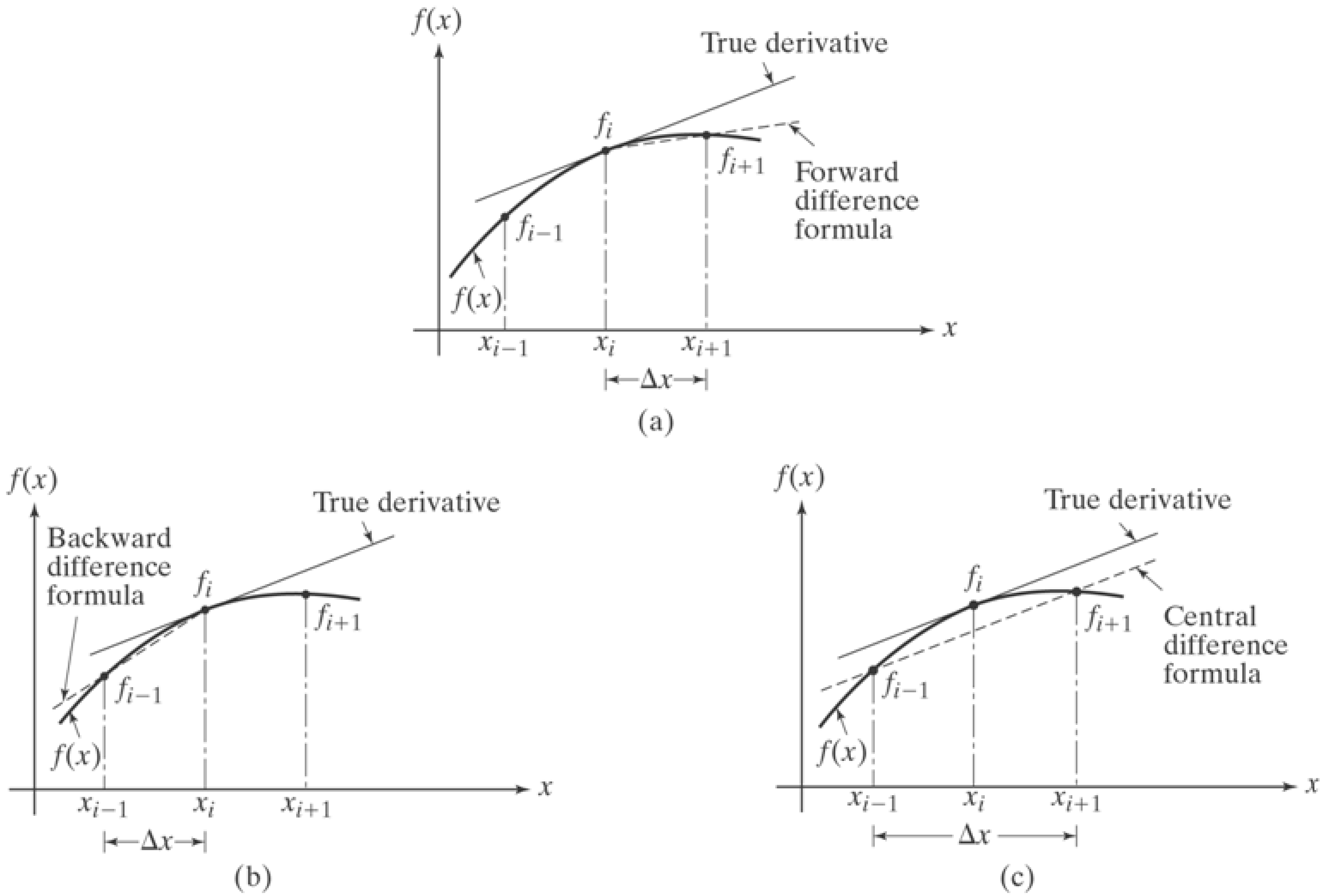
- Backward Difference Approximation

$$f'(x_i) = \frac{f(x_i) - f(x_i - \Delta x)}{\Delta x}$$

- Central Difference Approximation

$$f'(x_i) = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2\Delta x}$$

First Derivative Approximations



Finite-Difference Approximations of Second Derivatives

- Forward Difference Approximation

$$f''(x_i) = \frac{f_{i+2} - 2f_{i+1} + f_i}{\Delta x^2}$$

- Backward Difference Approximation

$$f''(x_i) = \frac{f_i - 2f_{i-1} + f_{i-2}}{\Delta x^2}$$

- Central Difference Approximation

$$f''(x_i) = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2}$$

Common finite-difference approximations

Table 7.1 Common finite-difference formulas.

Type of approximation	Formula	Truncation error
Forward differences	$f_i' = (f_{i+1} - f_i)/(\Delta x)$	$O(\Delta x)$
	$f_i'' = (f_{i+2} - 2f_{i+1} + f_i)/(\Delta x)^2$	
	$f_i''' = (f_{i+3} - 3f_{i+2} + 3f_{i+1} - f_i)/(\Delta x)^3$	
	$f_i'''' = (f_{i+4} - 4f_{i+3} + 6f_{i+2} - 4f_{i+1} + f_i)/(\Delta x)^4$	
Backward differences	$f_i' = (f_i - f_{i-1})/(\Delta x)$	$O(\Delta x)$
	$f_i'' = (f_i - 2f_{i-1} + f_{i-2})/(\Delta x)^2$	
	$f_i''' = (f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3})/(\Delta x)^3$	
	$f_i'''' = (f_i - 4f_{i-1} + 6f_{i-2} - 4f_{i-3} + f_{i-4})/(\Delta x)^4$	
Central differences	$f_i' = (f_{i+1} - f_{i-1})/(2\,\Delta x)$	$O(\Delta x^2)$
	$f_i'' = (f_{i+1} - 2f_i + f_{i-1})/(\Delta x)^2$	
	$f_i''' = (f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2})/(2(\Delta x)^3)$	
	$f_i'''' = (f_{i+2} - 4f_{i+1} + 6f_i - 4f_{i-1} + f_{i-2})/(\Delta x)^4$	

Higher Accuracy Finite Difference approximations

Table 7.2 Higher order finite-difference formulas.

Type of formula	Formula	Truncation error
Forward differences	$f_i' = (-f_{i+2} + 4f_{i+1} - 3f_i)/(2(\Delta x))$	$O(\Delta x)^2$
	$f_i'' = (-f_{i+3} + 4f_{i+2} - 5f_{i+1} + 2f_i)/(\Delta x)^2$	
	$f_i''' = (-3f_{i+4} + 14f_{i+3} - 24f_{i+2} + 18f_{i+1} - 5f_i)/(2(\Delta x)^3)$	
	$f_i'''' = (-2f_{i+5} + 11f_{i+4} - 24f_{i+3} + 26f_{i+2} - 14f_{i+1} + 3f_i)/(\Delta x)^4$	
Backward differences	$f_i' = (3f_i - 4f_{i-1} + f_{i-2})/(2(\Delta x))$	$O(\Delta x^2)$
	$f_i'' = (2f_i - 5f_{i-1} + 4f_{i-2} - f_{i-3})/(\Delta x)^2$	
	$f_i''' = (5f_i - 18f_{i-1} + 24f_{i-2} - 14f_{i-3} + 3f_{i-4})/(2(\Delta x)^3)$	
	$f_i'''' = (3f_i - 14f_{i-1} + 26f_{i-2} - 24f_{i-3} + 11f_{i-4} - 2f_{i-5})/(\Delta x)^4$	
Central differences	$f_i' = (-f_{i+2} + 8f_{i+1} - 8f_{i-1} + f_{i-2})/(12(\Delta x))$	$O(\Delta x^4)$
	$f_i'' = (-f_{i+2} + 16f_{i+1} - 30f_i + 16f_{i-1} - f_{i-2})/(12(\Delta x)^2)$	
	$f_i''' = (-f_{i+3} + 8f_{i+2} - 13f_{i+1} + 13f_{i-1} - 8f_{i-2} + f_{i-3})/(8(\Delta x)^3)$	
	$f_i'''' = (-f_{i+3} + 12f_{i+2} - 39f_{i+1} + 56f_i - 39f_{i-1} + 12f_{i-2} - f_{i-3})/(6(\Delta x)^4)$	

Remarks about Finite Difference approximations

1. In all the finite-difference formulas, the sum of all the coefficients of the function values (f_i) appearing in the numerator can be seen to be zero. This implies that the derivative becomes zero if *f(x)* is a constant.

2. The accuracy of the computed derivatives can be improved either by using a smaller step size (Δx) or by using a higher accuracy formulas.