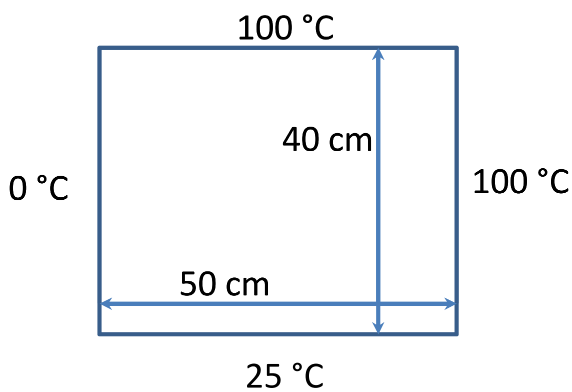
Conduction Heat Transfer (pg. 236 textbook)

The differencial equation:

$$\frac{\partial T}{\partial t} = \frac{k}{\rho C_P}\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

Describes energy conduction through a two-dimensional region, and can be applied to a surface exposed to various boundary temperatures.

For example, we might want to know the temperature distribution in a 50-cm x 40-cm metal plate esposed to boiling water (100 °C) along two edges, ice water (0 °C) on one edge, and room temperature (25 °C) along another edge, as shown in the figure:



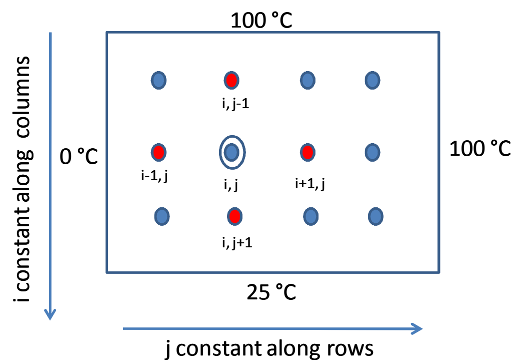
Assuming steady state, the equation simplifies to a form known as Laplace's Equation:

$$0 = \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right]$$

In order to solve this equation, the partial derivatives can be approximated by using finite differences (numerical method), to produce:

$$0 = \left[\frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{\Delta x^2} \right] + \left[\frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{\Delta y^2} \right]$$

Where subscript *i, j* represents a point on the plate at which temperature is $\boldsymbol{T_{i,j}}$

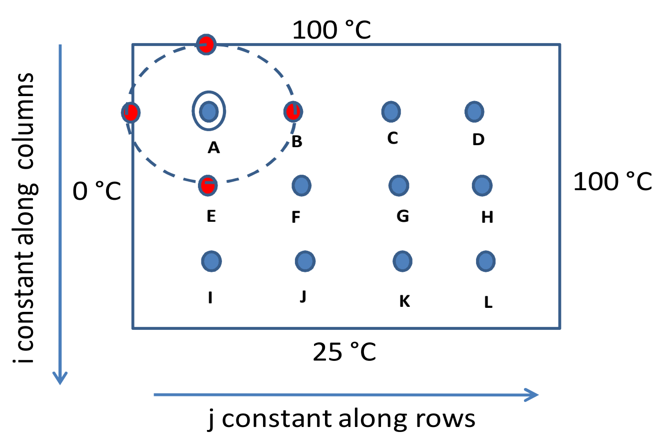


If we choose to make Δx=Δy, we get a simple result:

$$4T_{i,j} = \left[T_{i+1,j} + T_{i-1,j} \right] + \left[T_{i,j+1} + T_{i,j-1} \right]$$

This equation says that the sum of temperatures at the four points around any central point, is equal to four times the temperature at the central point. We can apply this equation at each interior point to develop a system of equations that, when solved simultaneously, will yield the temperatures at each point.

To help see how this is done, let's assign each interior point a letter designation and show the four point surrounding point A with a circle, as show in the following figure:



Applying the general equation at point A, yields:

$$4T_A = 0 + 100 + T_B + T_E$$

Applying the general equation at point B, yields:

$$4T_B = T_A + 100 + T_C + T_F$$

By continuing to apply the general equation to the rest of the points, we generate 12 equations (one for each interior node):

$$
\begin{aligned}
4T_A &= 0 + 100 + T_B + T_E \\
4T_B &= T_A + 100 + T_C + T_F \\
4T_C &= T_B + 100 + T_D + T_G \\
4T_D &= T_C + 100 + 100 + T_H \\
4T_E &= 0 + T_A + T_F + T_I \\
4T_F &= T_E + T_B + T_G + T_J \\
4T_G &= T_F + T_C + T_H + T_K \\
4T_H &= T_G + T_D + 100 + T_L \\
4T_I &= 0 + T_E + T_J + 25 \\
4T_J &= T_I + T_F + T_K + 25 \\
4T_K &= T_J + T_G + T_L + 25 \\
4T_L &= T_K + T_H + 100 + 25
\end{aligned}
$$

In matrix form, these equations can be written as:

$$[Coeff][T]=[rhs]$$

$$[Coeff]=\begin{bmatrix} -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & -4 \end{bmatrix} \quad [T]=\begin{bmatrix} T_A \\ T_B \\ T_C \\ T_D \\ T_E \\ T_F \\ T_G \\ T_H \\ T_I \\ T_J \\ T_K \\ T_L \end{bmatrix} \quad [rhs]=\begin{bmatrix} -100 \\ -100 \\ -100 \\ -200 \\ 0 \\ 0 \\ 0 \\ -100 \\ -25 \\ -25 \\ -25 \\ -125 \end{bmatrix}$$

Now, solve this system of equations to find the tempeture distribution of the metal plate:

[Coeff]	A	B	C	D	E	F	G	H	I	J	K	L	[rhs]
A	-4	1	0	0	1	0	0	0	0	0	0	0	-100
B	1	-4	1	0	0	1	0	0	0	0	0	0	-100
C	0	1	-4	1	0	0	1	0	0	0	0	0	-100
D	0	0	1	-4	0	0	0	1	0	0	0	0	-200
E	1	0	0	0	-4	1	0	0	1	0	0	0	0
F	0	1	0	0	1	-4	1	0	0	1	0	0	0
G	0	0	1	0	0	1	-4	1	0	0	1	0	0
H	0	0	0	1	0	0	1	-4	0	0	0	1	-100
I	0	0	0	0	1	0	0	0	-4	1	0	0	-25
J	0	0	0	0	0	1	0	0	1	-4	1	0	-25
K	0	0	0	0	0	0	1	0	0	1	-4	1	-25
L	0	0	0	0	0	0	0	1	0	0	1	-4	-125

Procedure:

1) Calculate the determinant of [coeff] to ensure that this system of equations has a solution (use MDETERM)
2)Calculate the inverse of [coeff], using MINVERSE function
3) The vector of temperatures can be found by multiplying $[coeff]^{-1}$*[rhs] (use MMULT function)