Dimensional Analysis and Similitude


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Isabella Lake Dam Hydraulic model, CA

https://www.youtube.com/watch?v=aDhd88lWtbc
6.1 Introduction

- **Dimensional analysis** is used to keep the required experimental studies to a minimum.
  - Based off **dimensional homogeneity** [all terms in an equation should have the same dimension.]

\[
\frac{P_1}{\gamma z_1} + \frac{V_1^2}{2g} + z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + z_2
\]

Bernoulli’s equation:
Dimension of each term is length

\[
\frac{P_1}{\gamma z_1} + \frac{V_1^2}{2g z_1} + 1 = \left(\frac{P_2}{\gamma z_2} + \frac{V_2^2}{2g z_2} + 1\right)\frac{z_2}{z_1}
\]

Bernoulli’s equation in this form: Each term is dimensionless
Similitude is the study of predicting prototype conditions from model observations.

- Uses dimensionless parameters obtained in dimensional analysis.

Two approaches can be used in dimensional analysis:

- **Buckingham π-theorem**: Theorem that organizes steps to ensure dimensional homogeneity.
- Extract dimensionless parameters from the differential equations and boundary conditions.
6.2 Dimensional Analysis

6.2.1 Motivation

In the interest of saving time and money in the study of fluid flows, the fewest possible combinations of parameters should be utilized.

For pressure drop across a slider valve above:

- We can assume that it depends on pipe mean velocity $V$, fluid density $\rho$, fluid viscosity $\mu$, pipe diameter $d$, and gap height $h$

$$\Delta p = f(V, \rho, \mu, d, h)$$
Could fix all parameters except velocity and find pressure dependence on average velocity.
- Repeat with changing diameter, etc.

![Flow around a slider valve](image_url)

**Fig. 6.2** Flow around a slider valve.

![Pressure drop versus velocity curves](image_url)

**Fig. 6.3** Pressure drop versus velocity curves: (a) $\rho, \mu, h$ fixed; (b) $\rho, \mu, d$ fixed.
The equation could be rewritten in terms of dimensionless parameters as:

\[ \Delta p = f(V, \rho, \mu, d, h) \]

- The equation could be rewritten in terms of dimensionless parameters as:

\[ \frac{\Delta p}{\rho V^2} = f\left(\frac{V_p d}{\mu}, \frac{h}{d}\right) \]
6.2.2 Review of Dimensions

- All quantities have a combination of dimensions of length, time, mass, and force by Newton’s Second Law:

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Symbol</th>
<th>Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>$l$</td>
<td>$L$</td>
</tr>
<tr>
<td>Time</td>
<td>$t$</td>
<td>$T$</td>
</tr>
<tr>
<td>Mass</td>
<td>$m$</td>
<td>$M$</td>
</tr>
<tr>
<td>Force</td>
<td>$F$</td>
<td>$ML/T^2$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$V$</td>
<td>$L/T$</td>
</tr>
<tr>
<td>Acceleration</td>
<td>$a$</td>
<td>$L/T^2$</td>
</tr>
<tr>
<td>Frequency</td>
<td>$\omega$</td>
<td>$T^{-1}$</td>
</tr>
<tr>
<td>Gravity</td>
<td>$g$</td>
<td>$L/T^2$</td>
</tr>
<tr>
<td>Area</td>
<td>$A$</td>
<td>$L^2$</td>
</tr>
<tr>
<td>Flow rate</td>
<td>$Q$</td>
<td>$L^3/T$</td>
</tr>
<tr>
<td>Mass flux</td>
<td>$\dot{m}$</td>
<td>$M/T$</td>
</tr>
<tr>
<td>Pressure</td>
<td>$p$</td>
<td>$M/LT^2$</td>
</tr>
<tr>
<td>Stress</td>
<td>$\tau$</td>
<td>$M/LT^2$</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>$M/L^3$</td>
</tr>
<tr>
<td>Specific weight</td>
<td>$\gamma$</td>
<td>$M/L^2T^2$</td>
</tr>
<tr>
<td>Viscosity</td>
<td>$\mu$</td>
<td>$M/LT$</td>
</tr>
<tr>
<td>Kinematic viscosity</td>
<td>$\nu$</td>
<td>$L^2/T$</td>
</tr>
<tr>
<td>Work</td>
<td>$W$</td>
<td>$ML^2/T^2$</td>
</tr>
<tr>
<td>Power, heat flux</td>
<td>$\dot{W}, \dot{Q}$</td>
<td>$ML^2/T^3$</td>
</tr>
<tr>
<td>Surface tension</td>
<td>$\sigma$</td>
<td>$M/T^2$</td>
</tr>
<tr>
<td>Bulk modulus</td>
<td>$B$</td>
<td>$M/LT^2$</td>
</tr>
</tbody>
</table>
6.2.3 Buckingham π-Theorem

- In any problem, a dependent variable \( x_1 \) is expressed in terms of independent variables, i.e., \( x_1 = f(x_2, x_3, x_4, \ldots, x_n) \) [\( n \): Number of variables]

π-Terms

- The Buckingham π –theorem states that \((n-m)\) dimensionless groups of variables, called π –terms, can be related by

\[
\Pi_1 = f_1(\Pi_1, \Pi_2, \ldots, \Pi_{n-m})
\]

- \( m \): Number of basic dimensions included in the variables.
- \( \pi_1 \): includes the dependent variable; remaining π-terms include only independent variables.
- For a successful dimensional analysis, a dimension must occur at least twice or not at all.
Example: The flow rate $Q$ in an open channel depends on the hydraulic radius $R$, the cross-sectional area $A$, the wall roughness height $e$, gravity $g$, and the slope $S$. Relate $Q$ to the other variables using

(a) the $M$-$L$-$T$ system

(b) the $F$-$L$-$T$ system.

$$Q = f(R, A, e, g, S)$$

$[Q] = \frac{L^3}{T^2}$, $[R] = \frac{L^2}{L} = L$

$[A] = \frac{L^2}{T}$, $[e] = L$, $[S] = \frac{L}{T^2}$

$[S] = 1$

# basic dimensions $= 2$ ($L$, $T$) $[m=2]$ $\Rightarrow e$

$$\Pi_1 = f_1(\Pi_2, \Pi_3, \ldots \Pi_4)$$

$n - m$
\[ \Pi_1 = Q R^2 g \]
\[ \Pi_2 = A R^2 g \]
\[ \Pi_3 = C R^2 g \]
\[ \Pi_4 = S R^2 g \]

**Repeating variables:**
\[ [g], [g] \]

\[ L^3 T^{-1} a_1 (L T^{-2})^{b_1} = 0 \]
\[ 3 + a_1 + b_1 = 0 \]
\[ -1 - 2b_1 = 0 \Rightarrow b_1 = -\frac{1}{2} \]
\[ a_1 = -\frac{5}{2} \]

**\( \Pi_2 \)**
\[ z a_2 (L T^{-2})^{b_2} = 0 \]
\[ 2 + a_2 + b_2 = 0 \]
\[ -2b_2 = 0 \Rightarrow b_2 = 0 \]
\[ a_2 = -2 \]

**\( \Pi_3 \)**
\[ a_3 (L T^{-2})^{b_3} = 0 \]
\[ b_3 = 0 \]
\[ a_3 = -1 \]
\[
\begin{align*}
\pi_4 & = 0 \\
\pi_4 & = 0 \\
\pi_1 & = \frac{Q}{\sqrt{g_{25}}} \\
\pi_2 & = \frac{A}{R^2} \\
\pi_3 & = \frac{e}{R} \\
\pi_4 & = S
\end{align*}
\]

\[
\frac{Q}{\sqrt{g_{25}}} = f\left(\frac{A}{R^2}, \frac{e}{R}, S\right)
\]
6.2.4 Common Dimensionless Parameters

- Each dimensionless number can be written as a ratio of two forces.

\[ \phi = \frac{m}{V} \rightarrow m = \rho l^3 \]

- Euler number
- Pressure force
- Inertial force

\[ F_P = \text{pressure force} = \Delta pA \sim \Delta p l^2 \]

\[ F_I = \text{inertial force} = mV \frac{dV}{ds} \sim \rho l^3 V^2 \frac{V}{l} = \rho l^2 V^2 \]

\[ F_\mu = \text{viscous force} = \tau A = \mu \frac{du}{dy} A \sim \mu \frac{V}{l} l^2 = \mu lV \]

\[ F_g = \text{gravity force} = mg \sim \rho l^3 g \]

\[ F_B = \text{compressibility force} = BA \sim \rho \frac{dp}{d\rho} l^2 = \rho c^2 l^2 \]

\[ F_\omega = \text{centrifugal force} = mr \omega^2 \sim \rho l^3 l \omega^2 = \rho l^4 \omega^2 \]

\[ F_\sigma = \text{surface tension force} = \sigma l \]

- Reynolds number
- Inertial force
- Viscous force

- Franne number
- Inertial force
- Gravity force

- Mach number
- Inertial force
- Compressibility force

- Strouhal number
- Centrifugal force

- Weber number
- Inertial force
- Surface tension force
### 6.2.4 Common Dimensionless Parameters (Cont.)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Expression</th>
<th>Flow situations where parameter is important</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euler number</td>
<td>( \frac{\Delta p}{\rho V^2} )</td>
<td>Flows in which pressure drop is significant: most flow situations</td>
</tr>
<tr>
<td>Reynolds number</td>
<td>( \frac{\rho lV}{\mu} )</td>
<td>Flows that are influenced by viscous effects: internal flows, boundary layer flows</td>
</tr>
<tr>
<td>Froude number</td>
<td>( \frac{V}{\sqrt{lg}} )</td>
<td>Flows that are influenced by gravity: primarily free surface flows</td>
</tr>
<tr>
<td>Mach number</td>
<td>( \frac{V}{c} )</td>
<td>Compressibility is important in these flows, usually if ( V &gt; 0.3 c )</td>
</tr>
<tr>
<td>Strouhal number</td>
<td>( \frac{l\omega}{V} )</td>
<td>Flow with an unsteady component that repeats itself periodically</td>
</tr>
<tr>
<td>Weber number</td>
<td>( \frac{V^2l\rho}{\sigma} )</td>
<td>Surface tension influences the flow; flow with an interface may be such a flow</td>
</tr>
</tbody>
</table>
6.3 Similitude

6.3.1 General Information

• Study of predicting prototype conditions from model observations.

• If a model study has to be performed:
  • Need a quantity measured on the model (subscript $m$) to predict an associated quantity on the prototype (subscript $p$).
  • This needs dynamic similarity between the model and prototype.
  • Forces which act on corresponding masses in the model flow and prototype flow are in the same ratio throughout the entire flow field.
6.3.1 General Information (Cont.)

- If inertial forces, pressure forces, viscous forces, and gravity forces are present:

\[
\frac{(F_I)_m}{(F_p)_m} = \frac{(F_I)_p}{(F_p)_p} = \frac{(F_u)_m}{(F_u)_p} = \frac{(F_g)_m}{(F_g)_p}
\]

Due to dynamic similarity at corresponding points in the flow fields.

These can be rearranged as

\[
\frac{(F_I)_m}{(F_p)_m} = \frac{(F_I)_p}{(F_p)_p} \quad \frac{(F_I)_m}{(F_u)_m} = \frac{(F_I)_p}{(F_u)_p} \quad \frac{(F_I)_m}{F_g_m} = \frac{(F_I)_p}{F_g_p}
\]

\[
Eu_m = Eu_p \quad Re_m = Re_p \quad Fr_m = Fr_p
\]

**Euler number**

**Reynolds**
6.3.1 General Information (Cont.)

- **Kinematic Similarity**: Velocity ratio is a constant between all corresponding points in the flow fields.
  - Streamline pattern around the model is the same as that around the prototype except for a scale factor.
    \[
    \frac{V_m}{V_p} = \text{constant}
    \]

- **Geometric Similarity**: Length ratio is a constant between all corresponding points in the flow fields.
  - Model has the same shape as the prototype.
    \[
    \frac{L_m}{L_p} = \text{constant}
    \]
To ensure complete similarity between model and prototype:

- Geometric similarity must be satisfied.  \( \frac{L_m}{L_p} = \text{constant} \)
- Mass ratio of corresponding fluid elements is a constant.
- Dimensionless numbers in model and prototype should be equal.
6.3.2 Confined Flows

- A confined flow is a flow that has no free surface (liquid-gas surface) or interface (two different liquids).
- Can only move within a specific region (e.g., internal flows in pipes).
- Isn’t influenced by gravity or surface tension.
- Dominant effect is viscosity in incompressible confined flows.
- Relevant forces are pressure, inertial, and viscous forces.
  - Dynamic similarity is obtained if the ratios between the model and the prototype are the same.

- **Hence, only the Reynolds number is the dominant dimensionless parameter.**
  - If compressibility effects are significant, Mach number would become important.
6.3.3 Free-Surface Flows

- A free-surface flow is a flow where part of the boundary involves a pressure boundary condition.
  - E.g., Flows over weirs and dams, flows in channels, flows with two fluids separated by an interface, etc.
- Gravity controls the location and motion of the free surface.
- Viscous effects are significant.
- Requires the Froude number similitude

\[ \text{Fr}_m = \text{Fr}_p \]
\[ \text{Fr} = \frac{V}{\sqrt{gA/T}} \]

T: Length of free surface width
Example: A 1:5 scale model of a large pump is used to test a proposed change. The prototype pump produces a pressure rise of 600 kPa at a mass flux of 800 kg/s. Determine the mass flux to be used in the model and the expected pressure rise.

(a) Water at the same temperature is used in both model and prototype.
(b) The water in the model study is at 30°C and the water in the prototype is at 15°C.

\[ \frac{L_p}{L_m} = 5 \]

Pipe flow: \( \frac{V_m L_m}{V_m} = \frac{V_p L_p}{V_p} \rightarrow \frac{V_m}{V_p} = \left( \frac{V_m}{V_p} \right) \frac{L_p}{L_m} \)
\[ \frac{V_m}{V_p} = 5 \quad \ldots \quad 0 \]

\[ \frac{m_m}{800 \text{ kg/s}} = \left( \frac{1}{5} \right)^2 \cdot 5 \]

\[ \Delta P_m \sim \frac{P_m}{V_m} V_m^2 \]

\[ \Delta P_p \sim \frac{P_p}{V_p} V_p^2 \]

\[ \frac{\Delta P_m}{\Delta P_p} = \frac{P_m}{P_p} \left( \frac{V_m}{V_p} \right)^2 \rightarrow \Delta P_m = \Delta P_p \left[ \frac{V_m}{V_p} \right]^2 \]

\[ \Delta P_m = \frac{600 \times 25}{15000} = 15,000 \text{ kPa} \]

Mass flux \( = P L^2 \cdot V \)

\[ m_m = \frac{P_m L_m^2 V_m}{P_p L_p^2 V_p} \Rightarrow m_m = 160 \text{ kg/s} \]
b) \( V_m = 0.8 \times 10^{-6} \text{ m}^2/\text{s} \)

\( V_p = 1.14 \times 10^{-6} \text{ m}^2/\text{s} \)

\[
\frac{V_m \cdot L_m}{V_p} = \frac{V_p}{V_p} \cdot \frac{L_p}{L_m}
\]

\[
\frac{V_m}{V_p} = \frac{0.8 \times 5}{1.14} = 3.51
\]

\[
\frac{m_m}{m_p} = \frac{P_m L_m^2 V_m}{P_p L_p^3 V_p} = \left(\frac{1}{5}\right)^2 \cdot 3.51
\]

\[
m_m = 800 \times \frac{1}{25} \times 3.51 = 112.3 \text{ kg}
\]

\[
\Delta P_m = 600 \text{ KPa} \times 3.51^2 = 7392 \text{ KPa}
\]