Problem Solutions – Chapter 3

Problem 3.1.1 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \leq 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] + P[X = -1/2] - P[X = 3/4] \tag{3}$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] = F_X(3/4) - F_X(-1/2) = 5/8 \tag{4}$$

(c)

$$P[|X| \leq 1/2] = P[-1/2 \leq X \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] \tag{5}$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P[X \leq a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

Problem 3.1.2 Solution

The CDF of V was given to be

$$F_V(v) = \begin{cases} 0 & v < -5 \\ c(v+5)^2 & -5 \leq v < 7 \\ 1 & v \geq 7 \end{cases} \tag{1}$$

(a) For V to be a continuous random variable, $F_V(v)$ must be a continuous function. This occurs if we choose c such that $F_V(v)$ doesn't have a discontinuity at $v = 7$. We meet this requirement if $c(7+5)^2 = 1$. This implies $c = 1/144$.

(b)
$$P[V > 4] = 1 - P[V \leq 4] = 1 - F_V(4) = 1 - 81/144 = 63/144 \tag{2}$$

(c)
$$P[-3 < V \leq 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144 \tag{3}$$

(d) Since $0 \leq F_V(v) \leq 1$ and since $F_V(v)$ is a nondecreasing function, it must be that $-5 \leq a \leq 7$. In this range,
$$P[V > a] = 1 - F_V(a) = 1 - (a+5)^2/144 = 2/3 \tag{4}$$
The unique solution in the range $-5 \leq a \leq 7$ is $a = 4\sqrt{3} - 5 = 1.928$.

Problem 3.1.3 Solution

In this problem, the CDF of W is
$$F_W(w) = \begin{cases} 0 & w < -5 \\ (w+5)/8 & -5 \leq w < -3 \\ 1/4 & -3 \leq w < 3 \\ 1/4 + 3(w-3)/8 & 3 \leq w < 5 \\ 1 & w \geq 5. \end{cases} \tag{1}$$

Each question can be answered directly from this CDF.

(a)
$$P[W \leq 4] = F_W(4) = 1/4 + 3/8 = 5/8. \tag{2}$$

(b)
$$P[-2 < W \leq 2] = F_W(2) - F_W(-2) = 1/4 - 1/4 = 0. \tag{3}$$

(c)
$$P[W > 0] = 1 - P[W \leq 0] = 1 - F_W(0) = 3/4 \tag{4}$$

(d) By inspection of $F_W(w)$, we observe that $P[W \leq a] = F_W(a) = 1/2$ for a in the range $3 \leq a \leq 5$. In this range,
$$F_W(a) = 1/4 + 3(a-3)/8 = 1/2 \tag{5}$$
This implies $a = 11/3$.

Problem 3.1.4 Solution

(a) By definition, $\lceil nx \rceil$ is the smallest integer that is greater than or equal to nx. This implies $nx \leq \lceil nx \rceil \leq nx + 1$.

(b) By part (a),
$$\frac{nx}{n} \leq \frac{\lceil nx \rceil}{n} \leq \frac{nx+1}{n} \tag{1}$$
That is,
$$x \leq \frac{\lceil nx \rceil}{n} \leq x + \frac{1}{n} \tag{2}$$
This implies
$$x \leq \lim_{n\to\infty} \frac{\lceil nx \rceil}{n} \leq \lim_{n\to\infty} x + \frac{1}{n} = x \tag{3}$$

(c) In the same way, $\lfloor nx \rfloor$ is the largest integer that is less than or equal to nx. This implies $nx - 1 \le \lfloor nx \rfloor \le nx$. It follows that

$$\frac{nx-1}{n} \le \frac{\lfloor nx \rfloor}{n} \le \frac{nx}{n} \tag{4}$$

That is,

$$x - \frac{1}{n} \le \frac{\lfloor nx \rfloor}{n} \le x \tag{5}$$

This implies

$$\lim_{n\to\infty} x - \frac{1}{n} = x \le \lim_{n\to\infty} \frac{\lceil nx \rceil}{n} \le x \tag{6}$$

Problem 3.2.1 Solution

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\,dx = 2c = 1 \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \le X \le 1] = \int_0^1 \frac{x}{2}\,dx = 1/4$

(c) $P[-1/2 \le X \le 1/2] = \int_0^{1/2} \frac{x}{2}\,dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\,dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{3}$$

Problem 3.2.2 Solution

From the CDF, we can find the PDF by direct differentiation. The CDF and correponding PDF are

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad f_X(x) = \begin{cases} 1/2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 3.2.3 Solution

We find the PDF by taking the derivative of $F_U(u)$ on each piece that $F_U(u)$ is defined. The CDF and corresponding PDF of U are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{1}$$

Problem 3.2.4 Solution

For $x < 0$, $F_X(x) = 0$. For $x \geq 0$,

$$F_X(x) = \int_0^x f_X(y)\, dy \tag{1}$$

$$= \int_0^x a^2 y e^{-a^2y^2/2}\, dy \tag{2}$$

$$= -e^{-a^2y^2/2}\Big|_0^x = 1 - e^{-a^2x^2/2} \tag{3}$$

A complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-a^2x^2/2} & x \geq 0 \end{cases} \tag{4}$$

Problem 3.2.5 Solution

$$f_X(x) = \begin{cases} ax^2 + bx & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

First, we note that a and b must be chosen such that the above PDF integrates to 1.

$$\int_0^1 (ax^2 + bx)\, dx = a/3 + b/2 = 1 \tag{2}$$

Hence, $b = 2 - 2a/3$ and our PDF becomes

$$f_X(x) = x(ax + 2 - 2a/3) \tag{3}$$

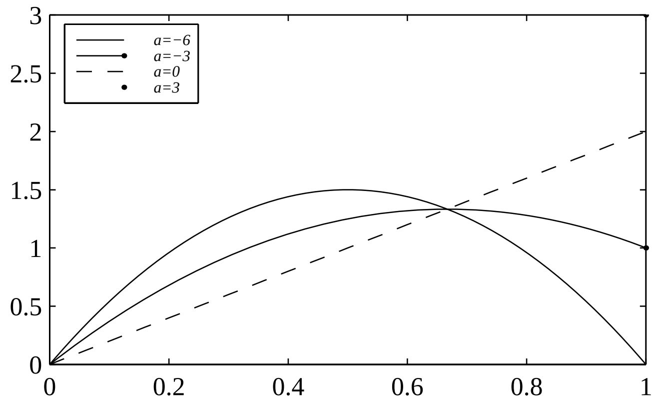
For the PDF to be non-negative for $x \in [0, 1]$, we must have $ax + 2 - 2a/3 \geq 0$ for all $x \in [0, 1]$. This requirement can be written as

$$a(2/3 - x) \leq 2 \qquad (0 \leq x \leq 1) \tag{4}$$

For $x = 2/3$, the requirement holds for all a. However, the problem is tricky because we must consider the cases $0 \leq x < 2/3$ and $2/3 < x \leq 1$ separately because of the sign change of the inequality. When $0 \leq x < 2/3$, we have $2/3 - x > 0$ and the requirement is most stringent at $x = 0$ where we require $2a/3 \leq 2$ or $a \leq 3$. When $2/3 < x \leq 1$, we can write the constraint as $a(x - 2/3) \geq -2$. In this case, the constraint is most stringent at $x = 1$, where we must have $a/3 \geq -2$ or $a \geq -6$. Thus a complete expression for our requirements are

$$-6 \leq a \leq 3 \qquad b = 2 - 2a/3 \tag{5}$$

As we see in the following plot, the shape of the PDF $f_X(x)$ varies greatly with the value of a.



Problem 3.3.1 Solution

$$f_X(x) = \begin{cases} 1/4 & -1 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We recognize that X is a uniform random variable from [-1,3].

(a) $E[X] = 1$ and $\text{Var}[X] = \frac{(3+1)^2}{12} = 4/3$.

(b) The new random variable Y is defined as $Y = h(X) = X^2$. Therefore

$$h(E[X]) = h(1) = 1 \tag{2}$$

and

$$E[h(X)] = E\left[X^2\right] = \text{Var}[X] + E[X]^2 = 4/3 + 1 = 7/3 \tag{3}$$

(c) Finally

$$E[Y] = E[h(X)] = E\left[X^2\right] = 7/3 \tag{4}$$

$$\text{Var}[Y] = E\left[X^4\right] - E\left[X^2\right]^2 = \int_{-1}^{3} \frac{x^4}{4}\,dx - \frac{49}{9} = \frac{61}{5} - \frac{49}{9} \tag{5}$$

Problem 3.3.2 Solution

(a) Since the PDF is uniform over [1,9]

$$E[X] = \frac{1+9}{2} = 5 \qquad \text{Var}[X] = \frac{(9-1)^2}{12} = \frac{16}{3} \tag{1}$$

(b) Define $h(X) = 1/\sqrt{X}$ then

$$h(E[X]) = 1/\sqrt{5} \tag{2}$$

$$E[h(X)] = \int_1^9 \frac{x^{-1/2}}{8}\,dx = 1/2 \tag{3}$$

(c)

$$E[Y] = E[h(X)] = 1/2 \tag{4}$$

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = \int_1^9 \frac{x^{-1}}{8}\,dx - E[X]^2 = \frac{\ln 9}{8} - 1/4 \tag{5}$$

Problem 3.3.3 Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x/2 & 0 \le x < 2 \\ 1 & x \ge 2 \end{cases} \tag{1}$$

(a) To find $E[X]$, we first find the PDF by differentiating the above CDF.

$$f_X(x) = \begin{cases} 1/2 & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The expected value is then

$$E[X] = \int_0^2 \frac{x}{2}\,dx = 1 \tag{3}$$

(b)

$$E\left[X^2\right] = \int_0^2 \frac{x^2}{2}\,dx = 8/3 \tag{4}$$

$$\text{Var}[X] = E\left[X^2\right] - E[X]^2 = 8/3 - 1 = 5/3 \tag{5}$$

Problem 3.3.4 Solution

We can find the expected value of X by direct integration of the given PDF.

$$f_Y(y) = \begin{cases} y/2 & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expectation is

$$E[Y] = \int_0^2 \frac{y^2}{2}\,dy = 4/3 \tag{2}$$

To find the variance, we first find the second moment

$$E\left[Y^2\right] = \int_0^2 \frac{y^3}{2}\,dy = 2. \tag{3}$$

The variance is then $\text{Var}[Y] = E[Y^2] - E[Y]^2 = 2 - (4/3)^2 = 2/9$.

Problem 3.3.5 Solution

The CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < -1 \\ (y+1)/2 & -1 \le y < 1 \\ 1 & y \ge 1 \end{cases} \tag{1}$$

(a) We can find the expected value of Y by first differentiating the above CDF to find the PDF

$$f_Y(y) = \begin{cases} 1/2 & -1 \leq y \leq 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

It follows that

$$E[Y] = \int_{-1}^{1} y/2 \, dy = 0. \tag{3}$$

(b)

$$E\left[Y^2\right] = \int_{-1}^{1} \frac{y^2}{2} \, dy = 1/3 \tag{4}$$

$$\text{Var}[Y] = E\left[Y^2\right] - E[Y]^2 = 1/3 - 0 = 1/3 \tag{5}$$

Problem 3.3.6 Solution

To evaluate the moments of V, we need the PDF $f_V(v)$, which we find by taking the derivative of the CDF $F_V(v)$. The CDF and corresponding PDF of V are

$$F_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)^2/144 & -5 \leq v < 7 \\ 1 & v \geq 7 \end{cases} \qquad f_V(v) = \begin{cases} 0 & v < -5 \\ (v+5)/72 & -5 \leq v < 7 \\ 0 & v \geq 7 \end{cases} \tag{1}$$

(a) The expected value of V is

$$E[V] = \int_{-\infty}^{\infty} v f_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^2 + 5v) \, dv \tag{2}$$

$$= \frac{1}{72} \left(\frac{v^3}{3} + \frac{5v^2}{2} \right) \Bigg|_{-5}^{7} = \frac{1}{72} \left(\frac{343}{3} + \frac{245}{2} + \frac{125}{3} - \frac{125}{2} \right) = 3 \tag{3}$$

(b) To find the variance, we first find the second moment

$$E\left[V^2\right] = \int_{-\infty}^{\infty} v^2 f_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^3 + 5v^2) \, dv \tag{4}$$

$$= \frac{1}{72} \left(\frac{v^4}{4} + \frac{5v^3}{3} \right) \Bigg|_{-5}^{7} = 6719/432 = 15.55 \tag{5}$$

The variance is $\text{Var}[V] = E[V^2] - (E[V])^2 = 2831/432 = 6.55$.

(c) The third moment of V is

$$E\left[V^3\right] = \int_{-\infty}^{\infty} v^3 f_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^4 + 5v^3) \, dv \tag{6}$$

$$= \frac{1}{72} \left(\frac{v^5}{5} + \frac{5v^4}{4} \right) \Bigg|_{-5}^{7} = 86.2 \tag{7}$$

Problem 3.3.7 Solution

To find the moments, we first find the PDF of U by taking the derivative of $F_U(u)$. The CDF and corresponding PDF are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{1}$$

(a) The expected value of U is

$$E[U] = \int_{-\infty}^{\infty} u f_U(u)\, du = \int_{-5}^{-3} \frac{u}{8}\, du + \int_{3}^{5} \frac{3u}{8}\, du \tag{2}$$

$$= \left.\frac{u^2}{16}\right|_{-5}^{-3} + \left.\frac{3u^2}{16}\right|_{3}^{5} = 2 \tag{3}$$

(b) The second moment of U is

$$E\left[U^2\right] \int_{-\infty}^{\infty} u^2 f_U(u)\, du = \int_{-5}^{-3} \frac{u^2}{8}\, du + \int_{3}^{5} \frac{3u^2}{8}\, du \tag{4}$$

$$= \left.\frac{u^3}{24}\right|_{-5}^{-3} + \left.\frac{u^3}{8}\right|_{3}^{5} = 49/3 \tag{5}$$

The variance of U is $\text{Var}[U] = E[U^2] - (E[U])^2 = 37/3$.

(c) Note that $2^U = e^{(\ln 2)U}$. This implies that

$$\int 2^u\, du = \int e^{(\ln 2)u}\, du = \frac{1}{\ln 2} e^{(\ln 2)u} = \frac{2^u}{\ln 2} \tag{6}$$

The expected value of 2^U is then

$$E\left[2^U\right] = \int_{-\infty}^{\infty} 2^u f_U(u)\, du = \int_{-5}^{-3} \frac{2^u}{8}\, du + \int_{3}^{5} \frac{3 \cdot 2^u}{8}\, du \tag{7}$$

$$= \left.\frac{2^u}{8\ln 2}\right|_{-5}^{-3} + \left.\frac{3 \cdot 2^u}{8\ln 2}\right|_{3}^{5} = \frac{2307}{256\ln 2} = 13.001 \tag{8}$$

Problem 3.3.8 Solution

The Pareto (α, μ) random variable has PDF

$$f_X(x) = \begin{cases} (\alpha/\mu)(x/\mu)^{-(\alpha+1)} & x \ge \mu \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The nth moment is

$$E[X^n] = \int_{\mu}^{\infty} x^n \frac{\alpha}{\mu} \left(\frac{x}{\mu}\right)^{-(\alpha+1)} dx = \mu^n \int_{\mu}^{\infty} \frac{\alpha}{\mu} \left(\frac{x}{\mu}\right)^{-(\alpha-n+1)} dx \tag{2}$$

With the variable substitution $y = x/\mu$, we obtain

$$E[X^n] = \alpha\mu^n \int_1^\infty y^{-(\alpha-n+1)}\,dy \tag{3}$$

We see that $E[X^n] < \infty$ if and only if $\alpha - n + 1 > 1$, or, equivalently, $n < \alpha$. In this case,

$$E[X^n] = \frac{\alpha\mu^n}{-(\alpha-n+1)+1} y^{-(\alpha-n+1)+1}\bigg|_{y=1}^{y=\infty} \tag{4}$$

$$= \frac{-\alpha\mu^n}{\alpha-n} y^{-(\alpha-n)}\bigg|_{y=1}^{y=\infty} = \frac{\alpha\mu^n}{\alpha-n} \tag{5}$$

Problem 3.4.1 Solution

The reflected power Y has an exponential ($\lambda = 1/P_0$) PDF. From Theorem 3.8, $E[Y] = P_0$. The probability that an aircraft is correctly identified is

$$P[Y > P_0] = \int_{P_0}^\infty \frac{1}{P_0} e^{-y/P_0}\,dy = e^{-1}. \tag{1}$$

Fortunately, real radar systems offer better performance.

Problem 3.4.2 Solution

From Appendix A, we observe that an exponential PDF Y with parameter $\lambda > 0$ has PDF

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In addition, the mean and variance of Y are

$$E[Y] = \frac{1}{\lambda} \qquad \text{Var}[Y] = \frac{1}{\lambda^2} \tag{2}$$

(a) Since $\text{Var}[Y] = 25$, we must have $\lambda = 1/5$.

(b) The expected value of Y is $E[Y] = 1/\lambda = 5$.

(c)

$$P[Y > 5] = \int_5^\infty f_Y(y)\,dy = -e^{-y/5}\Big|_5^\infty = e^{-1} \tag{3}$$

Problem 3.4.3 Solution

From Appendix A, an Erlang random variable X with parameters $\lambda > 0$ and n has PDF

$$f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x}/(n-1)! & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In addition, the mean and variance of X are

$$E[X] = \frac{n}{\lambda} \qquad \text{Var}[X] = \frac{n}{\lambda^2} \tag{2}$$

(a) Since $\lambda = 1/3$ and $E[X] = n/\lambda = 15$, we must have $n = 5$.

(b) Substituting the parameters $n = 5$ and $\lambda = 1/3$ into the given PDF, we obtain

$$f_X(x) = \begin{cases} (1/3)^5 x^4 e^{-x/3}/24 & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(c) From above, we know that $\text{Var}[X] = n/\lambda^2 = 45$.

Problem 3.4.4 Solution

Since Y is an Erlang random variable with parameters $\lambda = 2$ and $n = 2$, we find in Appendix A that

$$f_Y(y) = \begin{cases} 4ye^{-2y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Appendix A tells us that $E[Y] = n/\lambda = 1$.

(b) Appendix A also tells us that $\text{Var}[Y] = n/\lambda^2 = 1/2$.

(c) The probability that $1/2 \leq Y < 3/2$ is

$$P[1/2 \leq Y < 3/2] = \int_{1/2}^{3/2} f_Y(y)\, dy = \int_{1/2}^{3/2} 4ye^{-2y}\, dy \tag{2}$$

This integral is easily completed using the integration by parts formula $\int u\, dv = uv - \int v\, du$ with

$$u = 2y \qquad dv = 2e^{-2y}$$
$$du = 2dy \qquad v = -e^{-2y}$$

Making these substitutions, we obtain

$$P[1/2 \leq Y < 3/2] = -2ye^{-2y}\Big|_{1/2}^{3/2} + \int_{1/2}^{3/2} 2e^{-2y}\, dy \tag{3}$$
$$= 2e^{-1} - 4e^{-3} = 0.537 \tag{4}$$

Problem 3.4.5 Solution

(a) The PDF of a continuous uniform $(-5, 5)$ random variable is

$$f_X(x) = \begin{cases} 1/10 & -5 \leq x \leq 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) For $x < -5$, $F_X(x) = 0$. For $x \geq 5$, $F_X(x) = 1$. For $-5 \leq x \leq 5$, the CDF is

$$F_X(x) = \int_{-5}^{x} f_X(\tau)\, d\tau = \frac{x+5}{10} \tag{2}$$

The complete expression for the CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -5 \\ (x+5)/10 & 5 \leq x \leq 5 \\ 1 & x > 5 \end{cases} \tag{3}$$

(c) The expected value of X is

$$\int_{-5}^{5} \frac{x}{10}\, dx = \frac{x^2}{20}\Big|_{-5}^{5} = 0 \tag{4}$$

Another way to obtain this answer is to use Theorem 3.6 which says the expected value of X is $E[X] = (5 + -5)/2 = 0$.

(d) The fifth moment of X is

$$\int_{-5}^{5} \frac{x^5}{10}\,dx = \left.\frac{x^6}{60}\right|_{-5}^{5} = 0 \tag{5}$$

(e) The expected value of e^X is

$$\int_{-5}^{5} \frac{e^x}{10}\,dx = \left.\frac{e^x}{10}\right|_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.84 \tag{6}$$

Problem 3.4.6 Solution

We know that X has a uniform PDF over $[a, b)$ and has mean $\mu_X = 7$ and variance $\text{Var}[X] = 3$. All that is left to do is determine the values of the constants a and b, to complete the model of the uniform PDF.

$$E[X] = \frac{a+b}{2} = 7 \qquad \text{Var}[X] = \frac{(b-a)^2}{12} = 3 \tag{1}$$

Since we assume $b > a$, this implies

$$a + b = 14 \qquad b - a = 6 \tag{2}$$

Solving these two equations, we arrive at

$$a = 4 \qquad b = 10 \tag{3}$$

And the resulting PDF of X is,

$$f_X(x) = \begin{cases} 1/6 & 4 \le x \le 10 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 3.4.7 Solution

Given that

$$f_X(x) = \begin{cases} (1/2)e^{-x/2} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a)

$$P[1 \le X \le 2] = \int_1^2 (1/2)e^{-x/2}\,dx = e^{-1/2} - e^{-1} = 0.2387 \tag{2}$$

(b) The CDF of X may be be expressed as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ \int_0^x (1/2)e^{-x/2}\,d\tau & x \ge 0 \end{cases} = \begin{cases} 0 & x < 0 \\ 1 - e^{-x/2} & x \ge 0 \end{cases} \tag{3}$$

(c) X is an exponential random variable with parameter $a = 1/2$. By Theorem 3.8, the expected value of X is $E[X] = 1/a = 2$.

(d) By Theorem 3.8, the variance of X is $\text{Var}[X] = 1/a^2 = 4$.

Problem 3.4.8 Solution
Given the uniform PDF

$$f_U(u) = \begin{cases} 1/(b-a) & a \le u \le b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The mean of U can be found by integrating

$$E[U] = \int_a^b u/(b-a)\,du = \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2} \tag{2}$$

Where we factored $(b^2-a^2) = (b-a)(b+a)$. The variance of U can also be found by finding $E[U^2]$.

$$E\left[U^2\right] = \int_a^b u^2/(b-a)\,du = \frac{(b^3-a^3)}{3(b-a)} \tag{3}$$

Therefore the variance is

$$\text{Var}[U] = \frac{(b^3-a^3)}{3(b-a)} - \left(\frac{b+a}{2}\right)^2 = \frac{(b-a)^2}{12} \tag{4}$$

Problem 3.4.9 Solution
Let X denote the holding time of a call. The PDF of X is

$$f_X(x) = \begin{cases} (1/\tau)e^{-x/\tau} & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We will use $C_A(X)$ and $C_B(X)$ to denote the cost of a call under the two plans. From the problem statement, we note that $C_A(X) = 10X$ so that $E[C_A(X)] = 10E[X] = 10\tau$. On the other hand

$$C_B(X) = 99 + 10(X-20)^+ \tag{2}$$

where $y^+ = y$ if $y \ge 0$; otherwise $y^+ = 0$ for $y < 0$. Thus,

$$\begin{aligned} E[C_B(X)] &= E\left[99 + 10(X-20)^+\right] && (3)\\ &= 99 + 10E\left[(X-20)^+\right] && (4)\\ &= 99 + 10E\left[(X-20)^+ | X \le 20\right] P[X \le 20] \\ &\qquad + 10E\left[(X-20)^+ | X > 20\right] P[X > 20] && (5) \end{aligned}$$

Given $X \le 20$, $(X-20)^+ = 0$. Thus $E[(X-20)^+|X \le 20] = 0$ and

$$E[C_B(X)] = 99 + 10E[(X-20)|X > 20]\,P[X > 20] \tag{6}$$

Finally, we observe that $P[X > 20] = e^{-20/\tau}$ and that

$$E[(X-20)|X > 20] = \tau \tag{7}$$

since given $X \ge 20$, $X-20$ has a PDF identical to X by the memoryless property of the exponential random variable. Thus,

$$E[C_B(X)] = 99 + 10\tau e^{-20/\tau} \tag{8}$$

Some numeric comparisons show that $E[C_B(X)] \le E[C_A(X)]$ if $\tau > 12.34$ minutes. That is, the flat price for the first 20 minutes is a good deal only if your average phone call is sufficiently long.

Problem 3.4.10 Solution

The integral I_1 is

$$I_1 = \int_0^\infty \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^\infty = 1 \tag{1}$$

For $n > 1$, we have

$$I_n = \int_0^\infty \underbrace{\frac{\lambda^{n-1}x^{n-1}}{(n-1)!}}_{u} \underbrace{\lambda e^{-\lambda x}\,dt}_{dv} \tag{2}$$

We define u and dv as shown above in order to use the integration by parts formula $\int u\,dv = uv - \int v\,du$. Since

$$du = \frac{\lambda^{n-1}x^{n-1}}{(n-2)!}\,dx \qquad v = -e^{-\lambda x} \tag{3}$$

we can write

$$I_n = uv|_0^\infty - \int_0^\infty v\,du \tag{4}$$

$$= -\frac{\lambda^{n-1}x^{n-1}}{(n-1)!}e^{-\lambda x}\Big|_0^\infty + \int_0^\infty \frac{\lambda^{n-1}x^{n-1}}{(n-2)!}e^{-\lambda x}\,dx = 0 + I_{n-1} \tag{5}$$

Hence, $I_n = 1$ for all $n \geq 1$.

Problem 3.4.11 Solution

For an Erlang (n, λ) random variable X, the kth moment is

$$E\left[X^k\right] = \int_0^\infty x^k f_X(x)\,dt \tag{1}$$

$$= \int_0^\infty \frac{\lambda^n x^{n+k-1}}{(n-1)!}e^{-\lambda x}\,dt = \frac{(n+k-1)!}{\lambda^k(n-1)!}\underbrace{\int_0^\infty \frac{\lambda^{n+k}x^{n+k-1}}{(n+k-1)!}e^{-\lambda t}\,dt}_{1} \tag{2}$$

The above marked integral equals 1 since it is the integral of an Erlang PDF with parameters λ and $n+k$ over all possible values. Hence,

$$E\left[X^k\right] = \frac{(n+k-1)!}{\lambda^k(n-1)!} \tag{3}$$

This implies that the first and second moments are

$$E[X] = \frac{n!}{(n-1)!\lambda} = \frac{n}{\lambda} \qquad E\left[X^2\right] = \frac{(n+1)!}{\lambda^2(n-1)!} = \frac{(n+1)n}{\lambda^2} \tag{4}$$

It follows that the variance of X is n/λ^2.

Problem 3.4.12 Solution

In this problem, we prove Theorem 3.11 which says that for $x \geq 0$, the CDF of an Erlang (n, λ) random variable X_n satisfies

$$F_{X_n}(x) = 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k e^{-\lambda x}}{k!}. \tag{1}$$

We do this in two steps. First, we derive a relationship between $F_{X_n}(x)$ and $F_{X_{n-1}}(x)$. Second, we use that relationship to prove the theorem by induction.

(a) By Definition 3.7, the CDF of Erlang (n, λ) random variable X_n is

$$F_{X_n}(x) = \int_{-\infty}^{x} f_{X_n}(t)\, dt = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt. \tag{2}$$

(b) To use integration by parts, we define

$$u = \frac{t^{n-1}}{(n-1)!} \qquad dv = \lambda^n e^{-\lambda t}\, dt \tag{3}$$

$$du = \frac{t^{n-2}}{(n-2)!} \qquad v = -\lambda^{n-1} e^{-\lambda t} \tag{4}$$

Thus, using the integration by parts formula $\int u\, dv = uv - \int v\, du$, we have

$$F_{X_n}(x) = \int_0^x \frac{\lambda^n t^{n-1} e^{-\lambda t}}{(n-1)!}\, dt = -\frac{\lambda^{n-1} t^{n-1} e^{-\lambda t}}{(n-1)!}\Big|_0^x + \int_0^x \frac{\lambda^{n-1} t^{n-2} e^{-\lambda t}}{(n-2)!}\, dt \tag{5}$$

$$= -\frac{\lambda^{n-1} x^{n-1} e^{-\lambda x}}{(n-1)!} + F_{X_{n-1}}(x) \tag{6}$$

(c) Now we do proof by induction. For $n = 1$, the Erlang (n, λ) random variable X_1 is simply an exponential random variable. Hence for $x \geq 0$, $F_{X_1}(x) = 1 - e^{-\lambda x}$. Now we suppose the claim is true for $F_{X_{n-1}}(x)$ so that

$$F_{X_{n-1}}(x) = 1 - \sum_{k=0}^{n-2} \frac{(\lambda x)^k e^{-\lambda x}}{k!}. \tag{7}$$

Using the result of part (a), we can write

$$F_{X_n}(x) = F_{X_{n-1}}(x) - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \tag{8}$$

$$= 1 - \sum_{k=0}^{n-2} \frac{(\lambda x)^k e^{-\lambda x}}{k!} - \frac{(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \tag{9}$$

which proves the claim.

Problem 3.4.13 Solution

For $n = 1$, we have the fact $E[X] = 1/\lambda$ that is given in the problem statement. Now we assume that $E[X^{n-1}] = (n-1)!/\lambda^{n-1}$. To complete the proof, we show that this implies that $E[X^n] = n!/\lambda^n$. Specifically, we write

$$E[X^n] = \int_0 x^n \lambda e^{-\lambda x}\, dx \tag{1}$$

Now we use the integration by parts formula $\int u\, dv = uv - \int v\, du$ with $u = x^n$ and $dv = \lambda e^{-\lambda x}\, dx$. This implies $du = nx^{n-1}\, dx$ and $v = -e^{-\lambda x}$ so that

$$E[X^n] = -x^n e^{-\lambda x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-\lambda x}\, dx \tag{2}$$

$$= 0 + \frac{n}{\lambda} \int_0^\infty x^{n-1} \lambda e^{-\lambda x}\, dx \tag{3}$$

$$= \frac{n}{\lambda} E\left[X^{n-1}\right] \tag{4}$$

By our induction hyothesis, $E[X^{n-1}] = (n-1)!/\lambda^{n-1}$ which implies

$$E[X^n] = n!/\lambda^n \tag{5}$$

Problem 3.4.14 Solution

(a) Since $f_X(x) \geq 0$ and $x \geq r$ over the entire integral, we can write

$$\int_r^\infty x f_X(x)\, dx \geq \int_r^\infty r f_X(x)\, dx = rP[X > r] \tag{1}$$

(b) We can write the expected value of X in the form

$$E[X] = \int_0^r x f_X(x)\, dx + \int_r^\infty x f_X(x)\, dx \tag{2}$$

Hence,

$$rP[X > r] \leq \int_r^\infty x f_X(x)\, dx = E[X] - \int_0^r x f_X(x)\, dx \tag{3}$$

Allowing r to approach infinity yields

$$\lim_{r\to\infty} rP[X > r] \leq E[X] - \lim_{r\to\infty} \int_0^r x f_X(x)\, dx = E[X] - E[X] = 0 \tag{4}$$

Since $rP[X > r] \geq 0$ for all $r \geq 0$, we must have $\lim_{r\to\infty} rP[X > r] = 0$.

(c) We can use the integration by parts formula $\int u\, dv = uv - \int v\, du$ by defining $u = 1 - F_X(x)$ and $dv = dx$. This yields

$$\int_0^\infty [1 - F_X(x)]\, dx = x[1 - F_X(x)]|_0^\infty + \int_0^\infty x f_X(x)\, dx \tag{5}$$

By applying part (a), we now observe that

$$x[1 - F_X(x)]|_0^\infty = \lim_{r\to\infty} r[1 - F_X(r)] - 0 = \lim_{r\to\infty} rP[X > r] \tag{6}$$

By part (b), $\lim_{r\to\infty} rP[X > r] = 0$ and this implies $x[1 - F_X(x)]|_0^\infty = 0$. Thus,

$$\int_0^\infty [1 - F_X(x)]\, dx = \int_0^\infty x f_X(x)\, dx = E[X] \tag{7}$$

Problem 3.5.1 Solution

Given that the peak temperature, T, is a Gaussian random variable with mean 85 and standard deviation 10 we can use the fact that $F_T(t) = \Phi((t - \mu_T)/\sigma_T)$ and Table 3.1 on page 123 to evaluate the following

$$P[T > 100] = 1 - P[T \leq 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100 - 85}{10}\right) \tag{1}$$

$$= 1 - \Phi(1.5) = 1 - 0.933 = 0.066 \tag{2}$$

$$P[T < 60] = \Phi\left(\frac{60 - 85}{10}\right) = \Phi(-2.5) \tag{3}$$

$$= 1 - \Phi(2.5) = 1 - .993 = 0.007 \tag{4}$$

$$P[70 \leq T \leq 100] = F_T(100) - F_T(70) \tag{5}$$

$$= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866 \tag{6}$$

Problem 3.5.2 Solution

The standard normal Gaussian random variable Z has mean $\mu = 0$ and variance $\sigma^2 = 1$. Making these substitutions in Definition 3.8 yields

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{1}$$

Problem 3.5.3 Solution

X is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$P[|X| \leq 10] = 0.1 \tag{1}$$

We can find the variance Var$[X]$ by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P[-10 \leq X \leq 10] = F_X(10) - F_X(-10) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1 \tag{2}$$

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table 3.1 for the Gaussian CDF, we find that $10/\sigma_X = 0.15$ or $\sigma_X = 66.6$.

Problem 3.5.4 Solution

Repeating Definition 3.11,

$$Q(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-u^2/2}\, du \tag{1}$$

Making the substitution $x = u/\sqrt{2}$, we have

$$Q(z) = \frac{1}{\sqrt{\pi}} \int_{z/\sqrt{2}}^\infty e^{-x^2}\, dx = \frac{1}{2}\text{erfc}\left(\frac{z}{\sqrt{2}}\right) \tag{2}$$

Problem 3.5.5 Solution

Moving to Antarctica, we find that the temperature, T is still Gaussian but with variance 225. We also know that with probability 1/2, T exceeds 10 degrees. First we would like to find the mean temperature, and we do so by looking at the second fact.

$$P[T > 10] = 1 - P[T \leq 10] = 1 - \Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{1}$$

By looking at the table we find that if $\Phi(\Gamma) = 1/2$, then $\Gamma = 0$. Therefore,

$$\Phi\left(\frac{10 - \mu_T}{15}\right) = 1/2 \tag{2}$$

implies that $(10 - \mu_T)/15 = 0$ or $\mu_T = 10$. Now we have a Gaussian T with mean 10 and standard deviation 15. So we are prepared to answer the following problems.

$$
\begin{aligned}
P[T > 32] &= 1 - P[T \leq 32] = 1 - \Phi\left(\frac{32 - 10}{15}\right) && (3)\\
&= 1 - \Phi(1.45) = 1 - 0.926 = 0.074 && (4)\\
P[T < 0] &= F_T(0) = \Phi\left(\frac{0 - 10}{15}\right) && (5)\\
&= \Phi(-2/3) = 1 - \Phi(2/3) && (6)\\
&= 1 - \Phi(0.67) = 1 - 0.749 = 0.251 && (7)\\
P[T > 60] &= 1 - P[T \leq 60] = 1 - F_T(60) && (8)\\
&= 1 - \Phi\left(\frac{60 - 10}{15}\right) = 1 - \Phi(10/3) && (9)\\
&= Q(3.33) = 4.34 \cdot 10^{-4} && (10)
\end{aligned}
$$

Problem 3.5.6 Solution

In this problem, we use Theorem 3.14 and the tables for the Φ and Q functions to answer the questions. Since $E[Y_{20}] = 40(20) = 800$ and $\text{Var}[Y_{20}] = 100(20) = 2000$, we can write

$$
\begin{aligned}
P[Y_{20} > 1000] &= P\left[\frac{Y_{20} - 800}{\sqrt{2000}} > \frac{1000 - 800}{\sqrt{2000}}\right] && (1)\\
&= P\left[Z > \frac{200}{20\sqrt{5}}\right] = Q(4.47) = 3.91 \times 10^{-6} && (2)
\end{aligned}
$$

The second part is a little trickier. Since $E[Y_{25}] = 1000$, we know that the prof will spend around \$1000 in roughly 25 years. However, to be certain with probability 0.99 that the prof spends \$1000 will require more than 25 years. In particular, we know that

$$P[Y_n > 1000] = P\left[\frac{Y_n - 40n}{\sqrt{100n}} > \frac{1000 - 40n}{\sqrt{100n}}\right] = 1 - \Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 0.99 \quad (3)$$

Hence, we must find n such that

$$\Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 0.01 \quad (4)$$

Recall that $\Phi(x) = 0.01$ for a negative value of x. This is consistent with our earlier observation that we would need $n > 25$ corresponding to $100 - 4n < 0$. Thus, we use the identity $\Phi(x) = 1 - \Phi(-x)$ to write

$$\Phi\left(\frac{100 - 4n}{\sqrt{n}}\right) = 1 - \Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.01 \quad (5)$$

Equivalently, we have

$$\Phi\left(\frac{4n - 100}{\sqrt{n}}\right) = 0.99 \quad (6)$$

From the table of the Φ function, we have that $(4n - 100)/\sqrt{n} = 2.33$, or

$$(n - 25)^2 = (0.58)^2 n = 0.3393n. \quad (7)$$

Solving this quadratic yields $n = 28.09$. Hence, only after 28 years are we 99 percent sure that the prof will have spent \$1000. Note that a second root of the quadratic yields $n = 22.25$. This root is not a valid solution to our problem. Mathematically, it is a solution of our quadratic in which we choose the negative root of $\sqrt{n}$. This would correspond to assuming the standard deviation of Y_n is negative.

Problem 3.5.7 Solution

We are given that there are 100,000,000 men in the United States and 23,000 of them are at least 7 feet tall, and the heights of U.S men are independent Gaussian random variables with mean $5'10''$.

(a) Let H denote the height in inches of a U.S male. To find σ_X, we look at the fact that the probability that $P[H \geq 84]$ is the number of men who are at least 7 feet tall divided by the total number of men (the frequency interpretation of probability). Since we measure H in inches, we have

$$P[H \geq 84] = \frac{23{,}000}{100{,}000{,}000} = \Phi\left(\frac{70-84}{\sigma_X}\right) = 0.00023 \tag{1}$$

Since $\Phi(-x) = 1 - \Phi(x) = Q(x)$,

$$Q(14/\sigma_X) = 2.3 \cdot 10^{-4} \tag{2}$$

From Table 3.2, this implies $14/\sigma_X = 3.5$ or $\sigma_X = 4$.

(b) The probability that a randomly chosen man is at least 8 feet tall is

$$P[H \geq 96] = Q\left(\frac{96-70}{4}\right) = Q(6.5) \tag{3}$$

Unfortunately, Table 3.2 doesn't include $Q(6.5)$, although it should be apparent that the probability is very small. In fact, $Q(6.5) = 4.0 \times 10^{-11}$.

(c) First we need to find the probability that a man is at least 7'6".

$$P[H \geq 90] = Q\left(\frac{90-70}{4}\right) = Q(5) \approx 3 \cdot 10^{-7} = \beta \tag{4}$$

Although Table 3.2 stops at $Q(4.99)$, if you're curious, the exact value is $Q(5) = 2.87 \cdot 10^{-7}$.

Now we can begin to find the probability that no man is at least 7'6". This can be modeled as 100,000,000 repetitions of a Bernoulli trial with parameter $1-\beta$. The probability that no man is at least 7'6" is

$$(1-\beta)^{100{,}000{,}000} = 9.4 \times 10^{-14} \tag{5}$$

(d) The expected value of N is just the number of trials multiplied by the probability that a man is at least 7'6".

$$E[N] = 100{,}000{,}000 \cdot \beta = 30 \tag{6}$$

Problem 3.5.8 Solution

This problem is in the wrong section since the erf($\cdot$) function is defined later on in Section 3.9 as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2}\, du. \tag{1}$$

(a) Since Y is Gaussian $(0, 1/\sqrt{2})$, Y has variance 1/2 and

$$f_Y(y) = \frac{1}{\sqrt{2\pi(1/2)}} e^{-y^2/[2(1/2)]} = \frac{1}{\sqrt{\pi}} e^{-y^2}. \tag{2}$$

For $y \geq 0$, $F_Y(y) = \int_{-\infty}^{y} f_Y(u)\,du = 1/2 + \int_0^y f_Y(u)\,du$. Substituting $f_Y(u)$ yields

$$F_Y(y) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^y e^{-u^2}\,du = \frac{1}{2} + \operatorname{erf}(y). \tag{3}$$

(b) Since Y is Gaussian $(0, 1/\sqrt{2})$, $Z = \sqrt{2}Y$ is Gaussian with expected value $E[Z] = \sqrt{2}E[Y] = 0$ and variance $\operatorname{Var}[Z] = 2\operatorname{Var}[Y] = 1$. Thus Z is Gaussian $(0, 1)$ and

$$\Phi(z) = F_Z(z) = P\left[\sqrt{2}Y \leq z\right] = P\left[Y \leq \frac{z}{\sqrt{2}}\right] = F_Y\left(\frac{z}{\sqrt{2}}\right) = \frac{1}{2} + \operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) \tag{4}$$

Problem 3.5.9 Solution

First we note that since W has an $N[\mu, \sigma^2]$ distribution, the integral we wish to evaluate is

$$I = \int_{-\infty}^{\infty} f_W(w)\,dw = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-(w-\mu)^2/2\sigma^2}\,dw \tag{1}$$

(a) Using the substitution $x = (w - \mu)/\sigma$, we have $dx = dw/\sigma$ and

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx \tag{2}$$

(b) When we write I^2 as the product of integrals, we use y to denote the other variable of integration so that

$$I^2 = \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\,dx\right)\left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\,dy\right) \tag{3}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)/2}\,dx\,dy \tag{4}$$

(c) By changing to polar coordinates, $x^2 + y^2 = r^2$ and $dx\,dy = r\,dr\,d\theta$ so that

$$I^2 = \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\infty} e^{-r^2/2} r\,dr\,d\theta \tag{5}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} -e^{-r^2/2}\Big|_0^{\infty}\,d\theta = \frac{1}{2\pi} \int_0^{2\pi} d\theta = 1 \tag{6}$$

Problem 3.5.10 Solution

This problem is mostly calculus and only a little probability. From the problem statement, the SNR Y is an exponential $(1/\gamma)$ random variable with PDF

$$f_Y(y) = \begin{cases} (1/\gamma)e^{-y/\gamma} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Thus, from the problem statement, the BER is

$$\overline{P}_e = E\left[P_e(Y)\right] = \int_{-\infty}^{\infty} Q(\sqrt{2y}) f_Y(y)\, dy = \int_0^{\infty} Q(\sqrt{2y}) \frac{y}{\gamma} e^{-y/\gamma}\, dy \tag{2}$$

Like most integrals with exponential factors, its a good idea to try integration by parts. Before doing so, we recall that if X is a Gaussian $(0,1)$ random variable with CDF $F_X(x)$, then

$$Q(x) = 1 - F_X(x). \tag{3}$$

It follows that $Q(x)$ has derivative

$$Q'(x) = \frac{dQ(x)}{dx} = -\frac{dF_X(x)}{dx} = -f_X(x) = -\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{4}$$

To solve the integral, we use the integration by parts formula $\int_a^b u\, dv = uv|_a^b - \int_a^b v\, du$, where

$$u = Q(\sqrt{2y}) \qquad dv = \frac{1}{\gamma} e^{-y/\gamma}\, dy \tag{5}$$

$$du = Q'(\sqrt{2y}) \frac{1}{\sqrt{2y}} = -\frac{e^{-y}}{2\sqrt{\pi y}} \qquad v = -e^{-y/\gamma} \tag{6}$$

From integration by parts, it follows that

$$\overline{P}_e = uv|_0^{\infty} - \int_0^{\infty} v\, du = -Q(\sqrt{2y}) e^{-y/\gamma}\Big|_0^{\infty} - \int_0^{\infty} \frac{1}{\sqrt{y}} e^{-y[1+(1/\gamma)]}\, dy \tag{7}$$

$$= 0 + Q(0)e^{-0} - \frac{1}{2\sqrt{\pi}} \int_0^{\infty} y^{-1/2} e^{-y/\bar{\gamma}}\, dy \tag{8}$$

where $\bar{\gamma} = \gamma/(1+\gamma)$. Next, recalling that $Q(0) = 1/2$ and making the substitution $t = y/\bar{\gamma}$, we obtain

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \int_0^{\infty} t^{-1/2} e^{-t}\, dt \tag{9}$$

From Math Fact B.11, we see that the remaining integral is the $\Gamma(z)$ function evaluated $z = 1/2$. Since $\Gamma(1/2) = \sqrt{\pi}$,

$$\overline{P}_e = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{\bar{\gamma}}{\pi}} \Gamma(1/2) = \frac{1}{2}\left[1 - \sqrt{\bar{\gamma}}\right] = \frac{1}{2}\left[1 - \sqrt{\frac{\gamma}{1+\gamma}}\right] \tag{10}$$

Problem 3.6.1 Solution

(a) Using the given CDF

$$P[X < -1] = F_X\left(-1^-\right) = 0 \tag{1}$$

$$P[X \leq -1] = F_X(-1) = -1/3 + 1/3 = 0 \tag{2}$$

Where $F_X(-1^-)$ denotes the limiting value of the CDF found by approaching -1 from the left. Likewise, $F_X(-1^+)$ is interpreted to be the value of the CDF found by approaching -1 from the right. We notice that these two probabilities are the same and therefore the probability that X is exactly -1 is zero.

(b)

$$P[X < 0] = F_X(0^-) = 1/3 \quad (3)$$
$$P[X \le 0] = F_X(0) = 2/3 \quad (4)$$

Here we see that there is a discrete jump at $X = 0$. Approached from the left the CDF yields a value of 1/3 but approached from the right the value is 2/3. This means that there is a non-zero probability that $X = 0$, in fact that probability is the difference of the two values.

$$P[X = 0] = P[X \le 0] - P[X < 0] = 2/3 - 1/3 = 1/3 \quad (5)$$

(c)

$$P[0 < X \le 1] = F_X(1) - F_X(0^+) = 1 - 2/3 = 1/3 \quad (6)$$
$$P[0 \le X \le 1] = F_X(1) - F_X(0^-) = 1 - 1/3 = 2/3 \quad (7)$$

The difference in the last two probabilities above is that the first was concerned with the probability that X was strictly greater then 0, and the second with the probability that X was greater than or equal to zero. Since the the second probability is a larger set (it includes the probability that $X = 0$) it should always be greater than or equal to the first probability. The two differ by the probability that $X = 0$, and this difference is non-zero only when the random variable exhibits a discrete jump in the CDF.

Problem 3.6.2 Solution

Similar to the previous problem we find

(a)

$$P[X < -1] = F_X(-1^-) = 0 \qquad P[X \le -1] = F_X(-1) = 1/4 \quad (1)$$

Here we notice the discontinuity of value 1/4 at $x = -1$.

(b)

$$P[X < 0] = F_X(0^-) = 1/2 \qquad P[X \le 0] = F_X(0) = 1/2 \quad (2)$$

Since there is no discontinuity at $x = 0$, $F_X(0^-) = F_X(0^+) = F_X(0)$.

(c)

$$P[X > 1] = 1 - P[X \le 1] = 1 - F_X(1) = 0 \quad (3)$$
$$P[X \ge 1] = 1 - P[X < 1] = 1 - F_X(1^-) = 1 - 3/4 = 1/4 \quad (4)$$

Again we notice a discontinuity of size 1/4, here occurring at $x = 1$.

Problem 3.6.3 Solution

(a) By taking the derivative of the CDF $F_X(x)$ given in Problem 3.6.2, we obtain the PDF

$$f_X(x) = \begin{cases} \frac{\delta(x+1)}{4} + 1/4 + \frac{\delta(x-1)}{4} & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b) The first moment of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx \quad (2)$$

$$= x/4|_{x=-1} + x^2/8\big|_{-1}^{1} + x/4|_{x=1} = -1/4 + 0 + 1/4 = 0. \quad (3)$$

(c) The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \quad (4)$$

$$= x^2/4\big|_{x=-1} + x^3/12\big|_{-1}^{1} + x^2/4\big|_{x=1} = 1/4 + 1/6 + 1/4 = 2/3. \quad (5)$$

Since $E[X] = 0$, $\text{Var}[X] = E[X^2] = 2/3$.

Problem 3.6.4 Solution

The PMF of a Bernoulli random variable with mean p is

$$P_X(x) = \begin{cases} 1-p & x = 0 \\ p & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The corresponding PDF of this discrete random variable is

$$f_X(x) = (1-p)\delta(x) + p\delta(x-1) \quad (2)$$

Problem 3.6.5 Solution

The PMF of a geometric random variable with mean $1/p$ is

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The corresponding PDF is

$$f_X(x) = p\delta(x-1) + p(1-p)\delta(x-2) + \cdots \quad (2)$$

$$= \sum_{j=1}^{\infty} p(1-p)^{j-1}\delta(x-j) \quad (3)$$

Problem 3.6.6 Solution

(a) Since the conversation time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. The conversation time W is zero iff either the phone is busy, no one answers, or if the conversation time X of a completed call is zero. Let A be the event that the call is answered. Note that the event A^c implies $W = 0$. For $w \geq 0$,

$$F_W(w) = P[A^c] + P[A]\, F_{W|A}(w) = (1/2) + (1/2)F_X(w) \quad (1)$$

Thus the complete CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2)F_X(w) & w \geq 0 \end{cases} \quad (2)$$

(b) By taking the derivative of $F_W(w)$, the PDF of W is

$$f_W(w) = \begin{cases} (1/2)\delta(w) + (1/2)f_X(w) & \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Next, we keep in mind that since X must be nonnegative, $f_X(x) = 0$ for $x < 0$. Hence,

$$f_W(w) = (1/2)\delta(w) + (1/2)f_X(w) \quad (4)$$

(c) From the PDF $f_W(w)$, calculating the moments is straightforward.

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} w f_X(w)\, dw = E[X]/2 \quad (5)$$

The second moment is

$$E[W^2] = \int_{-\infty}^{\infty} w^2 f_W(w)\, dw = (1/2)\int_{-\infty}^{\infty} w^2 f_X(w)\, dw = E[X^2]/2 \quad (6)$$

The variance of W is

$$\begin{aligned} \text{Var}[W] = E[W^2] - (E[W])^2 &= E[X^2]/2 - (E[X]/2)^2 \quad (7)\\ &= (1/2)\,\text{Var}[X] + (E[X])^2/4 \quad (8)\end{aligned}$$

Problem 3.6.7 Solution

The professor is on time 80 percent of the time and when he is late his arrival time is uniformly distributed between 0 and 300 seconds. The PDF of T, is

$$f_T(t) = \begin{cases} 0.8\delta(t-0) + \frac{0.2}{300} & 0 \le t \le 300 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The CDF can be found be integrating

$$F_T(t) = \begin{cases} 0 & t < -1 \\ 0.8 + \frac{0.2t}{300} & 0 \le t < 300 \\ 1 & t \ge 300 \end{cases} \quad (2)$$

Problem 3.6.8 Solution

Let G denote the event that the throw is good, that is, no foul occurs. The CDF of D obeys

$$F_D(y) = P[D \le y|G]\,P[G] + P[D \le y|G^c]\,P[G^c] \quad (1)$$

Given the event G,

$$P[D \le y|G] = P[X \le y - 60] = 1 - e^{-(y-60)/10} \quad (y \ge 60) \quad (2)$$

Of course, for $y < 60$, $P[D \le y|G] = 0$. From the problem statement, if the throw is a foul, then $D = 0$. This implies

$$P[D \le y|G^c] = u(y) \quad (3)$$

where $u(\cdot)$ denotes the unit step function. Since $P[G] = 0.7$, we can write

$$F_D(y) = P[G]\,P[D \le y|G] + P[G^c]\,P[D \le y|G^c] \tag{4}$$

$$= \begin{cases} 0.3u(y) & y < 60 \\ 0.3 + 0.7(1 - e^{-(y-60)/10}) & y \ge 60 \end{cases} \tag{5}$$

Another way to write this CDF is

$$F_D(y) = 0.3u(y) + 0.7u(y-60)(1 - e^{-(y-60)/10}) \tag{6}$$

However, when we take the derivative, either expression for the CDF will yield the PDF. However, taking the derivative of the first expression perhaps may be simpler:

$$f_D(y) = \begin{cases} 0.3\delta(y) & y < 60 \\ 0.07e^{-(y-60)/10} & y \ge 60 \end{cases} \tag{7}$$

Taking the derivative of the second expression for the CDF is a little tricky because of the product of the exponential and the step function. However, applying the usual rule for the differentation of a product does give the correct answer:

$$f_D(y) = 0.3\delta(y) + 0.7\delta(y-60)(1 - e^{-(y-60)/10}) + 0.07u(y-60)e^{-(y-60)/10} \tag{8}$$

$$= 0.3\delta(y) + 0.07u(y-60)e^{-(y-60)/10} \tag{9}$$

The middle term $\delta(y-60)(1 - e^{-(y-60)/10})$ dropped out because at $y = 60$, $e^{-(y-60)/10} = 1$.

Problem 3.6.9 Solution

The professor is on time and lectures the full 80 minutes with probability 0.7. In terms of math,

$$P[T = 80] = 0.7. \tag{1}$$

Likewise when the professor is more than 5 minutes late, the students leave and a 0 minute lecture is observed. Since he is late 30% of the time and given that he is late, his arrival is uniformly distributed between 0 and 10 minutes, the probability that there is no lecture is

$$P[T = 0] = (0.3)(0.5) = 0.15 \tag{2}$$

The only other possible lecture durations are uniformly distributed between 75 and 80 minutes, because the students will not wait longer then 5 minutes, and that probability must add to a total of $1 - 0.7 - 0.15 = 0.15$. So the PDF of T can be written as

$$f_T(t) = \begin{cases} 0.15\delta(t) & t = 0 \\ 0.03 & 75 \le 7 < 80 \\ 0.7\delta(t-80) & t = 80 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 3.7.1 Solution

Since $0 \le X \le 1$, $Y = X^2$ satisfies $0 \le Y \le 1$. We can conclude that $F_Y(y) = 0$ for $y < 0$ and that $F_Y(y) = 1$ for $y \ge 1$. For $0 \le y < 1$,

$$F_Y(y) = P\left[X^2 \le y\right] = P[X \le \sqrt{y}] \tag{1}$$

Since $f_X(x) = 1$ for $0 \le x \le 1$, we see that for $0 \le y < 1$,

$$P[X \le \sqrt{y}] = \int_0^{\sqrt{y}} dx = \sqrt{y} \tag{2}$$

Hence, the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ \sqrt{y} & 0 \le y < 1 \\ 1 & y \ge 1 \end{cases} \tag{3}$$

By taking the derivative of the CDF, we obtain the PDF

$$f_Y(y) = \begin{cases} 1/(2\sqrt{y}) & 0 \le y < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 3.7.2 Solution

Since $Y = \sqrt{X}$, the fact that X is nonegative and that we asume the squre root is always positive implies $F_Y(y) = 0$ for $y < 0$. In addition, for $y \ge 0$, we can find the CDF of Y by writing

$$F_Y(y) = P[Y \le y] = P\left[\sqrt{X} \le y\right] = P\left[X \le y^2\right] = F_X\left(y^2\right) \tag{1}$$

For $x \ge 0$, $F_X(x) = 1 - e^{-\lambda x}$. Thus,

$$F_Y(y) = \begin{cases} 1 - e^{-\lambda y^2} & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

By taking the derivative with respect to y, it follows that the PDF of Y is

$$f_Y(y) = \begin{cases} 2\lambda y e^{-\lambda y^2} & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

In comparing this result to the Rayleigh PDF given in Appendix A, we observe that Y is a Rayleigh (a) random variable with $a = \sqrt{2\lambda}$.

Problem 3.7.3 Solution

Since X is non-negative, $W = X^2$ is also non-negative. Hence for $w < 0$, $f_W(w) = 0$. For $w \ge 0$,

$$\begin{aligned} F_W(w) &= P[W \le w] = P\left[X^2 \le w\right] &(1) \\ &= P[X \le w] &(2) \\ &= 1 - e^{-\lambda\sqrt{w}} &(3) \end{aligned}$$

Taking the derivative with respect to w yields $f_W(w) = \lambda e^{-\lambda\sqrt{w}}/(2\sqrt{w})$. The complete expression for the PDF is

$$f_W(w) = \begin{cases} \frac{\lambda e^{-\lambda\sqrt{w}}}{2\sqrt{w}} & w \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 3.7.4 Solution

From Problem 3.6.1, random variable X has CDF

$$F_X(x) = \begin{cases} 0 & x < -1 \\ x/3 + 1/3 & -1 \leq x < 0 \\ x/3 + 2/3 & 0 \leq x < 1 \\ 1 & 1 \leq x \end{cases} \tag{1}$$

(a) We can find the CDF of Y, $F_Y(y)$ by noting that Y can only take on two possible values, 0 and 100. And the probability that Y takes on these two values depends on the probability that $X < 0$ and $X \geq 0$, respectively. Therefore

$$F_Y(y) = P[Y \leq y] = \begin{cases} 0 & y < 0 \\ P[X < 0] & 0 \leq y < 100 \\ 1 & y \geq 100 \end{cases} \tag{2}$$

The probabilities concerned with X can be found from the given CDF $F_X(x)$. This is the general strategy for solving problems of this type: to express the CDF of Y in terms of the CDF of X. Since $P[X < 0] = F_X(0^-) = 1/3$, the CDF of Y is

$$F_Y(y) = P[Y \leq y] = \begin{cases} 0 & y < 0 \\ 1/3 & 0 \leq y < 100 \\ 1 & y \geq 100 \end{cases} \tag{3}$$

(b) The CDF $F_Y(y)$ has jumps of 1/3 at $y = 0$ and 2/3 at $y = 100$. The corresponding PDF of Y is

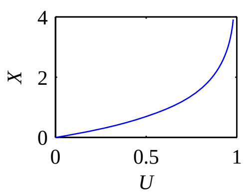
$$f_Y(y) = \delta(y)/3 + 2\delta(y - 100)/3 \tag{4}$$

(c) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = 0 \cdot \frac{1}{3} + 100 \cdot \frac{2}{3} = 66.66 \tag{5}$$

Problem 3.7.5 Solution

Before solving for the PDF, it is helpful to have a sketch of the function $X = -\ln(1 - U)$.



(a) From the sketch, we observe that X will be nonnegative. Hence $F_X(x) = 0$ for $x < 0$. Since U has a uniform distribution on $[0, 1]$, for $0 \leq u \leq 1$, $P[U \leq u] = u$. We use this fact to find the CDF of X. For $x \geq 0$,

$$F_X(x) = P[-\ln(1 - U) \leq x] = P\left[1 - U \geq e^{-x}\right] = P\left[U \leq 1 - e^{-x}\right] \tag{1}$$

For $x \geq 0$, $0 \leq 1 - e^{-x} \leq 1$ and so

$$F_X(x) = F_U\left(1 - e^{-x}\right) = 1 - e^{-x} \tag{2}$$

The complete CDF can be written as

$$F_X(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases} \tag{3}$$

(b) By taking the derivative, the PDF is

$$f_X(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Thus, X has an exponential PDF. In fact, since most computer languages provide uniform $[0,1]$ random numbers, the procedure outlined in this problem provides a way to generate exponential random variables from uniform random variables.

(c) Since X is an exponential random variable with parameter $a = 1$, $E[X] = 1$.

Problem 3.7.6 Solution

We wish to find a transformation that takes a uniformly distributed random variable on [0,1] to the following PDF for Y.

$$f_Y(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We begin by realizing that in this case the CDF of Y must be

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ y^3 & 0 \leq y \leq 1 \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

Therefore, for $0 \leq y \leq 1$,

$$P[Y \leq y] = P[g(X) \leq y] = y^3 \tag{3}$$

Thus, using $g(X) = X^{1/3}$, we see that for $0 \leq y \leq 1$,

$$P[g(X) \leq y] = P\left[X^{1/3} \leq y\right] = P\left[X \leq y^3\right] = y^3 \tag{4}$$

which is the desired answer.

Problem 3.7.7 Solution

Since the microphone voltage V is uniformly distributed between -1 and 1 volts, V has PDF and CDF

$$f_V(v) = \begin{cases} 1/2 & -1 \leq v \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_V(v) = \begin{cases} 0 & v < -1 \\ (v+1)/2 & -1 \leq v \leq 1 \\ 1 & v > 1 \end{cases} \tag{1}$$

The voltage is processed by a limiter whose output magnitude is given by below

$$L = \begin{cases} |V| & |V| \leq 0.5 \\ 0.5 & \text{otherwise} \end{cases} \tag{2}$$

(a)

$$P[L=0.5] = P[|V| \geq 0.5] = P[V \geq 0.5] + P[V \leq -0.5] \quad (3)$$
$$= 1 - F_V(0.5) + F_V(-0.5) \quad (4)$$
$$= 1 - 1.5/2 + 0.5/2 = 1/2 \quad (5)$$

(b) For $0 \leq l \leq 0.5$,

$$F_L(l) = P[|V| \leq l] = P[-l \leq v \leq l] = F_V(l) - F_V(-l) \quad (6)$$
$$= 1/2(l+1) - 1/2(-l+1) = l \quad (7)$$

So the CDF of L is

$$F_L(l) = \begin{cases} 0 & l < 0 \\ l & 0 \leq l < 0.5 \\ 1 & l \geq 0.5 \end{cases} \quad (8)$$

(c) By taking the derivative of $F_L(l)$, the PDF of L is

$$f_L(l) = \begin{cases} 1 + (0.5)\delta(l-0.5) & 0 \leq l \leq 0.5 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

The expected value of L is

$$E[L] = \int_{-\infty}^{\infty} l f_L(l)\, dl = \int_0^{0.5} l\, dl + 0.5 \int_0^{0.5} l(0.5)\delta(l-0.5)\, dl = 0.375 \quad (10)$$

Problem 3.7.8 Solution

Let X denote the position of the pointer and Y denote the area within the arc defined by the stopping position of the pointer.

(a) If the disc has radius r, then the area of the disc is πr^2. Since the circumference of the disc is 1 and X is measured around the circumference, $Y = \pi r^2 X$. For example, when $X = 1$, the shaded area is the whole disc and $Y = \pi r^2$. Similarly, if $X = 1/2$, then $Y = \pi r^2/2$ is half the area of the disc. Since the disc has circumference 1, $r = 1/(2\pi)$ and

$$Y = \pi r^2 X = \frac{X}{4\pi} \quad (1)$$

(b) The CDF of Y can be expressed as

$$F_Y(y) = P[Y \leq y] = P\left[\frac{X}{4\pi} \leq y\right] = P[X \leq 4\pi y] = F_X(4\pi y) \quad (2)$$

Therefore the CDF is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 4\pi y & 0 \leq y \leq \frac{1}{4\pi} \\ 1 & y \geq \frac{1}{4\pi} \end{cases} \quad (3)$$

(c) By taking the derivative of the CDF, the PDF of Y is

$$f_Y(y) = \begin{cases} 4\pi & 0 \leq y \leq \frac{1}{4\pi} \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(d) The expected value of Y is $E[Y] = \int_0^{1/(4\pi)} 4\pi y\, dy = 1/(8\pi)$.

Problem 3.7.9 Solution

The uniform $(0, 2)$ random variable U has PDF and CDF

$$f_U(u) = \begin{cases} 1/2 & 0 \le u \le 2, \\ 0 & \text{otherwise,} \end{cases} \qquad F_U(u) = \begin{cases} 0 & u < 0, \\ u/2 & 0 \le u < 2, \\ 1 & u > 2. \end{cases} \tag{1}$$

The uniform random variable U is subjected to the following clipper.

$$W = g(U) = \begin{cases} U & U \le 1 \\ 1 & U > 1 \end{cases} \tag{2}$$

To find the CDF of the output of the clipper, W, we remember that $W = U$ for $0 \le U \le 1$ while $W = 1$ for $1 \le U \le 2$. First, this implies W is nonnegative, i.e., $F_W(w) = 0$ for $w < 0$. Furthermore, for $0 \le w \le 1$,

$$F_W(w) = P[W \le w] = P[U \le w] = F_U(w) = w/2 \tag{3}$$

Lastly, we observe that it is always true that $W \le 1$. This implies $F_W(w) = 1$ for $w \ge 1$. Therefore the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w/2 & 0 \le w < 1 \\ 1 & w \ge 1 \end{cases} \tag{4}$$

From the jump in the CDF at $w = 1$, we see that $P[W = 1] = 1/2$. The corresponding PDF can be found by taking the derivative and using the delta function to model the discontinuity.

$$f_W(w) = \begin{cases} 1/2 + (1/2)\delta(w-1) & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

The expected value of W is

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_0^1 w[1/2 + (1/2)\delta(w-1)]\, dw \tag{6}$$

$$= 1/4 + 1/2 = 3/4. \tag{7}$$

Problem 3.7.10 Solution

Given the following function of random variable X,

$$Y = g(X) = \begin{cases} 10 & X < 0 \\ -10 & X \ge 0 \end{cases} \tag{1}$$

we follow the same procedure as in Problem 3.7.4. We attempt to express the CDF of Y in terms of the CDF of X. We know that Y is always less than -10. We also know that $-10 \le Y < 10$ when $X \ge 0$, and finally, that $Y = 10$ when $X < 0$. Therefore

$$F_Y(y) = P[Y \le y] = \begin{cases} 0 & y < -10 \\ P[X \ge 0] = 1 - F_X(0) & -10 \le y < 10 \\ 1 & y \ge 10 \end{cases} \tag{2}$$

Problem 3.7.11 Solution

The PDF of U is

$$f_U(u) = \begin{cases} 1/2 & -1 \le u \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since $W \ge 0$, we see that $F_W(w) = 0$ for $w < 0$. Next, we observe that the rectifier output W is a mixed random variable since

$$P[W = 0] = P[U < 0] = \int_{-1}^{0} f_U(u)\, du = 1/2 \tag{2}$$

The above facts imply that

$$F_W(0) = P[W \le 0] = P[W = 0] = 1/2 \tag{3}$$

Next, we note that for $0 < w < 1$,

$$F_W(w) = P[U \le w] = \int_{-1}^{w} f_U(u)\, du = (w+1)/2 \tag{4}$$

Finally, $U \le 1$ implies $W \le 1$, which implies $F_W(w) = 1$ for $w \ge 1$. Hence, the complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ (w+1)/2 & 0 \le w \le 1 \\ 1 & w > 1 \end{cases} \tag{5}$$

By taking the derivative of the CDF, we find the PDF of W; however, we must keep in mind that the discontinuity in the CDF at $w = 0$ yields a corresponding impulse in the PDF.

$$f_W(w) = \begin{cases} (\delta(w) + 1)/2 & 0 \le w \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

From the PDF, we can calculate the expected value

$$E[W] = \int_0^1 w(\delta(w) + 1)/2\, dw = 0 + \int_0^1 (w/2)\, dw = 1/4 \tag{7}$$

Perhaps an easier way to find the expected value is to use Theorem 2.10. In this case,

$$E[W] = \int_{-\infty}^{\infty} g(u) f_W(w)\, du = \int_0^1 u(1/2)\, du = 1/4 \tag{8}$$

As we expect, both approaches give the same answer.

Problem 3.7.12 Solution

Theorem 3.19 states that for a constant $a > 0$, $Y = aX$ has CDF and PDF

$$F_Y(y) = F_X(y/a) \qquad\qquad f_Y(y) = \frac{1}{a} f_X(y/a) \tag{1}$$

(a) If X is uniform (b, c), then $Y = aX$ has PDF

$$f_Y(y) = \frac{1}{a} f_X(y/a) = \begin{cases} \frac{1}{a(c-b)} & b \le y/a \le c \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{ac-ab} & ab \le y \le ac \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Thus Y has the PDF of a uniform (ab, ac) random variable.

(b) Using Theorem 3.19, the PDF of $Y = aX$ is

$$f_Y(y) = \frac{1}{a} f_X(y/a) = \begin{cases} \frac{\lambda}{a} e^{-\lambda(y/a)} & y/a \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$= \begin{cases} (\lambda/a) e^{-(\lambda/a)y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Hence Y is an exponential (λ/a) exponential random variable.

(c) Using Theorem 3.19, the PDF of $Y = aX$ is

$$f_Y(y) = \frac{1}{a} f_X(y/a) = \begin{cases} \frac{\lambda^n (y/a)^{n-1} e^{-\lambda(y/a)}}{a(n-1)!} & y/a \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$= \begin{cases} \frac{(\lambda/a)^n y^{n-1} e^{-(\lambda/a)y}}{(n-1)!} & y \geq 0, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

which is an Erlang (n, λ) PDF.

(d) If X is a Gaussian (μ, σ) random variable, then $Y = aX$ has PDF

$$f_Y(y) = f_X(y/a) = \frac{1}{a\sqrt{2\pi\sigma^2}} e^{-((y/a)-\mu)^2/2\sigma^2} \tag{7}$$

$$= \frac{1}{\sqrt{2\pi a^2\sigma^2}} e^{-(y-a\mu)^2/2(a^2\sigma^2)} \tag{8}$$

$$\tag{9}$$

Thus Y is a Gaussian random variable with expected value $E[Y] = a\mu$ and $\text{Var}[Y] = a^2\sigma^2$. That is, Y is a Gaussian $(a\mu, a\sigma)$ random variable.

Problem 3.7.13 Solution

If X has a uniform distribution from 0 to 1 then the PDF and corresponding CDF of X are

$$f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_X(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1}$$

For $b - a > 0$, we can find the CDF of the function $Y = a + (b-a)X$

$$F_Y(y) = P[Y \leq y] = P[a + (b-a)X \leq y] \tag{2}$$

$$= P\left[X \leq \frac{y-a}{b-a}\right] \tag{3}$$

$$= F_X\left(\frac{y-a}{b-a}\right) = \frac{y-a}{b-a} \tag{4}$$

Therefore the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < a \\ \frac{y-a}{b-a} & a \leq y \leq b \\ 1 & y \geq b \end{cases} \tag{5}$$

By differentiating with respect to y we arrive at the PDF

$$f_Y(y) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

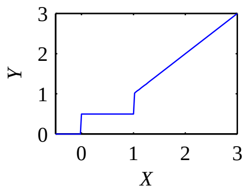
which we recognize as the PDF of a uniform (a, b) random variable.

Problem 3.7.14 Solution

Since $X = F^{-1}(U)$, it is desirable that the function $F^{-1}(u)$ exist for all $0 \le u \le 1$. However, for the continuous uniform random variable U, $P[U = 0] = P[U = 1] = 0$. Thus, it is a zero probability event that $F^{-1}(U)$ will be evaluated at $U = 0$ or $U = 1$. Asa result, it doesn't matter whether $F^{-1}(u)$ exists at $u = 0$ or $u = 1$.

Problem 3.7.15 Solution

The relationship between X and Y is shown in the following figure:



(a) Note that $Y = 1/2$ if and only if $0 \le X \le 1$. Thus,

$$P[Y = 1/2] = P[0 \le X \le 1] = \int_0^1 f_X(x)\, dx = \int_0^1 (x/2)\, dx = 1/4 \qquad (1)$$

(b) Since $Y \ge 1/2$, we can conclude that $F_Y(y) = 0$ for $y < 1/2$. Also, $F_Y(1/2) = P[Y = 1/2] = 1/4$. Similarly, for $1/2 < y \le 1$,

$$F_Y(y) = P[0 \le X \le 1] = P[Y = 1/2] = 1/4 \qquad (2)$$

Next, for $1 < y \le 2$,

$$F_Y(y) = P[X \le y] = \int_0^y f_X(x)\, dx = y^2/4 \qquad (3)$$

Lastly, since $Y \le 2$, $F_Y(y) = 1$ for $y \ge 2$. The complete expression of the CDF is

$$F_Y(y) = \begin{cases} 0 & y < 1/2 \\ 1/4 & 1/2 \le y \le 1 \\ y^2/4 & 1 < y < 2 \\ 1 & y \ge 2 \end{cases} \qquad (4)$$

Problem 3.7.16 Solution

We can prove the assertion by considering the cases where $a > 0$ and $a < 0$, respectively. For the case where $a > 0$ we have

$$F_Y(y) = P[Y \le y] = P\left[X \le \frac{y-b}{a}\right] = F_X\left(\frac{y-b}{a}\right) \qquad (1)$$

Therefore by taking the derivative we find that

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a > 0 \qquad (2)$$

Similarly for the case when $a < 0$ we have

$$F_Y(y) = P[Y \leq y] = P\left[X \geq \frac{y-b}{a}\right] = 1 - F_X\left(\frac{y-b}{a}\right) \tag{3}$$

And by taking the derivative, we find that for negative a,

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a}\right) \qquad a < 0 \tag{4}$$

A valid expression for both positive and negative a is

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right) \tag{5}$$

Therefore the assertion is proved.

Problem 3.7.17 Solution

Understanding this claim may be harder than completing the proof. Since $0 \leq F(x) \leq 1$, we know that $0 \leq U \leq 1$. This implies $F_U(u) = 0$ for $u < 0$ and $F_U(u) = 1$ for $u \geq 1$. Moreover, since $F(x)$ is an increasing function, we can write for $0 \leq u \leq 1$,

$$F_U(u) = P[F(X) \leq u] = P\left[X \leq F^{-1}(u)\right] = F_X\left(F^{-1}(u)\right) \tag{1}$$

Since $F_X(x) = F(x)$, we have for $0 \leq u \leq 1$,

$$F_U(u) = F(F^{-1}(u)) = u \tag{2}$$

Hence the complete CDF of U is

$$F_U(u) = \begin{cases} 0 & u < 0 \\ u & 0 \leq u < 1 \\ 1 & u \geq 1 \end{cases} \tag{3}$$

That is, U is a uniform $[0, 1]$ random variable.

Problem 3.7.18 Solution

(a) Given $F_X(x)$ is a continuous function, there exists x_0 such that $F_X(x_0) = u$. For each value of u, the corresponding x_0 is unique. To see this, suppose there were also x_1 such that $F_X(x_1) = u$. Without loss of generality, we can assume $x_1 > x_0$ since otherwise we could exchange the points x_0 and x_1. Since $F_X(x_0) = F_X(x_1) = u$, the fact that $F_X(x)$ is nondecreasing implies $F_X(x) = u$ for all $x \in [x_0, x_1]$, i.e., $F_X(x)$ is flat over the interval $[x_0, x_1]$, which contradicts the assumption that $F_X(x)$ has no flat intervals. Thus, for any $u \in (0, 1)$, there is a unique x_0 such that $F_X(x) = u$. Moreiver, the same x_0 is the minimum of all x' such that $F_X(x') \geq u$. The uniqueness of x_0 such that $F_X(x)x_0 = u$ permits us to define $\tilde{F}(u) = x_0 = F_X^{-1}(u)$.

(b) In this part, we are given that $F_X(x)$ has a jump discontinuity at x_0. That is, there exists $u_0^- = F_X(x_0^-)$ and $u_0^+ = F_X(x_0^+)$ with $u_0^- < u_0^+$. Consider any u in the interval $[u_0^-, u_0^+]$. Since $F_X(x_0) = F_X(x_0^+)$ and $F_X(x)$ is nondecreasing,

$$F_X(x) \geq F_X(x_0) = u_0^+, \qquad x \geq x_0. \tag{1}$$

Moreover,

$$F_X(x) < F_X\left(x_0^-\right) = u_0^-, \qquad x < x_0. \tag{2}$$

Thus for any u satisfying $u_o^- \le u \le u_0^+$, $F_X(x) < u$ for $x < x_0$ and $F_X(x) \ge u$ for $x \ge x_0$. Thus, $\tilde{F}(u) = \min\{x|F_X(x) \ge u\} = x_0$.

(c) We note that the first two parts of this problem were just designed to show the properties of $\tilde{F}(u)$. First, we observe that

$$P\left[\hat{X} \le x\right] = P\left[\tilde{F}(U) \le x\right] = P\left[\min\left\{x'|F_X\left(x'\right) \ge U\right\} \le x\right]. \tag{3}$$

To prove the claim, we define, for any x, the events

$$A: \quad \min\left\{x'|F_X\left(x'\right) \ge U\right\} \le x, \tag{4}$$
$$B: \quad U \le F_X(x). \tag{5}$$

Note that $P[A] = P[\hat{X} \le x]$. In addition, $P[B] = P[U \le F_X(x)] = F_X(x)$ since $P[U \le u] = u$ for any $u \in [0,1]$.

We will show that the events A and B are the same. This fact implies

$$P\left[\hat{X} \le x\right] = P[A] = P[B] = P[U \le F_X(x)] = F_X(x). \tag{6}$$

All that remains is to show A and B are the same. As always, we need to show that $A \subset B$ and that $B \subset A$.

- To show $A \subset B$, suppose A is true and $\min\{x'|F_X(x') \ge U\} \le x$. This implies there exists $x_0 \le x$ such that $F_X(x_0) \ge U$. Since $x_0 \le x$, it follows from $F_X(x)$ being nondecreasing that $F_X(x_0) \le F_X(x)$. We can thus conclude that

$$U \le F_X(x_0) \le F_X(x). \tag{7}$$

  That is, event B is true.

- To show $B \subset A$, we suppose event B is true so that $U \le F_X(x)$. We define the set

$$L = \left\{x'|F_X\left(x'\right) \ge U\right\}. \tag{8}$$

  We note $x \in L$. It follows that the minimum element $\min\{x'|x' \in L\} \le x$. That is,

$$\min\left\{x'|F_X\left(x'\right) \ge U\right\} \le x, \tag{9}$$

  which is simply event A.

Problem 3.8.1 Solution

The PDF of X is

$$f_X(x) = \begin{cases} 1/10 & -5 \le x \le 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event B has probability

$$P[B] = P[-3 \le X \le 3] = \int_{-3}^{3} \frac{1}{10}\,dx = \frac{3}{5} \tag{2}$$

From Definition 3.15, the conditional PDF of X given B is

$$f_{X|B}(x) = \begin{cases} f_X(x)/P[B] & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/6 & |x| \le 3 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

(b) Given B, we see that X has a uniform PDF over $[a, b]$ with $a = -3$ and $b = 3$. From Theorem 3.6, the conditional expected value of X is $E[X|B] = (a + b)/2 = 0$.

(c) From Theorem 3.6, the conditional variance of X is $\text{Var}[X|B] = (b - a)^2/12 = 3$.

Problem 3.8.2 Solution

From Definition 3.6, the PDF of Y is

$$f_Y(y) = \begin{cases} (1/5)e^{-y/5} & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The event A has probability

$$P[A] = P[Y < 2] = \int_0^2 (1/5)e^{-y/5}\,dy = -e^{-y/5}\Big|_0^2 = 1 - e^{-2/5} \tag{2}$$

From Definition 3.15, the conditional PDF of Y given A is

$$f_{Y|A}(y) = \begin{cases} f_Y(y)/P[A] & x \in A \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$= \begin{cases} (1/5)e^{-y/5}/(1 - e^{-2/5}) & 0 \le y < 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) The conditional expected value of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} y f_{Y|A}(y)\,dy = \frac{1/5}{1 - e^{-2/5}} \int_0^2 y e^{-y/5}\,dy \tag{5}$$

Using the integration by parts formula $\int u\,dv = uv - \int v\,du$ with $u = y$ and $dv = e^{-y/5}\,dy$ yields

$$E[Y|A] = \frac{1/5}{1 - e^{-2/5}} \left(-5ye^{-y/5}\Big|_0^2 + \int_0^2 5e^{-y/5}\,dy \right) \tag{6}$$

$$= \frac{1/5}{1 - e^{-2/5}} \left(-10e^{-2/5} - 25e^{-y/5}\Big|_0^2 \right) \tag{7}$$

$$= \frac{5 - 7e^{-2/5}}{1 - e^{-2/5}} \tag{8}$$

Problem 3.8.3 Solution

The condition *right side of the circle* is $R = [0, 1/2]$. Using the PDF in Example 3.5, we have

$$P[R] = \int_0^{1/2} f_Y(y)\, dy = \int_0^{1/2} 3y^2\, dy = 1/8 \tag{1}$$

Therefore, the conditional PDF of Y given event R is

$$f_{Y|R}(y) = \begin{cases} 24y^2 & 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The conditional expected value and mean square value are

$$E[Y|R] = \int_{-\infty}^{\infty} y f_{Y|R}(y)\, dy = \int_0^{1/2} 24y^3\, dy = 3/8 \text{ meter} \tag{3}$$

$$E\left[Y^2|R\right] = \int_{-\infty}^{\infty} y^2 f_{Y|R}(y)\, dy = \int_0^{1/2} 24y^4\, dy = 3/20 \text{ m}^2 \tag{4}$$

The conditional variance is

$$\text{Var}[Y|R] = E\left[Y^2|R\right] - (E[Y|R])^2 = \frac{3}{20} - \left(\frac{3}{8}\right)^2 = 3/320 \text{ m}^2 \tag{5}$$

The conditional standard deviation is $\sigma_{Y|R} = \sqrt{\text{Var}[Y|R]} = 0.0968$ meters.

Problem 3.8.4 Solution

From Definition 3.8, the PDF of W is

$$f_W(w) = \frac{1}{\sqrt{32\pi}} e^{-w^2/32} \tag{1}$$

(a) Since W has expected value $\mu = 0$, $f_W(w)$ is symmetric about $w = 0$. Hence $P[C] = P[W > 0] = 1/2$. From Definition 3.15, the conditional PDF of W given C is

$$f_{W|C}(w) = \begin{cases} f_W(w)/P[C] & w \in C \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2e^{-w^2/32}/\sqrt{32\pi} & w > 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The conditional expected value of W given C is

$$E[W|C] = \int_{-\infty}^{\infty} w f_{W|C}(w)\, dw = \frac{2}{4\sqrt{2\pi}} \int_0^{\infty} w e^{-w^2/32}\, dw \tag{3}$$

Making the substitution $v = w^2/32$, we obtain

$$E[W|C] = \frac{32}{\sqrt{32\pi}} \int_0^{\infty} e^{-v}\, dv = \frac{32}{\sqrt{32\pi}} \tag{4}$$

(c) The conditional second moment of W is

$$E\left[W^2|C\right] = \int_{-\infty}^{\infty} w^2 f_{W|C}(w)\, dw = 2\int_0^{\infty} w^2 f_W(w)\, dw \tag{5}$$

We observe that $w^2 f_W(w)$ is an even function. Hence

$$E\left[W^2|C\right] = 2\int_0^\infty w^2 f_W(w)\, dw \tag{6}$$

$$= \int_{-\infty}^\infty w^2 f_W(w)\, dw = E\left[W^2\right] = \sigma^2 = 16 \tag{7}$$

Lastly, the conditional variance of W given C is

$$\text{Var}[W|C] = E\left[W^2|C\right] - (E[W|C])^2 = 16 - 32/\pi = 5.81 \tag{8}$$

Problem 3.8.5 Solution

(a) We first find the conditional PDF of T. The PDF of T is

$$f_T(t) = \begin{cases} 100e^{-100t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The conditioning event has probability

$$P[T > 0.02] = \int_{0.02}^\infty f_T(t)\, dt = -e^{-100t}\Big|_{0.02}^\infty = e^{-2} \tag{2}$$

From Definition 3.15, the conditional PDF of T is

$$f_{T|T>0.02}(t) = \begin{cases} \frac{f_T(t)}{P[T>0.02]} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 100e^{-100(t-0.02)} & t \geq 0.02 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional expected value of T is

$$E[T|T > 0.02] = \int_{0.02}^\infty t(100)e^{-100(t-0.02)}\, dt \tag{4}$$

The substitution $\tau = t - 0.02$ yields

$$E[T|T > 0.02] = \int_0^\infty (\tau + 0.02)(100)e^{-100\tau}\, d\tau \tag{5}$$

$$= \int_0^\infty (\tau + 0.02) f_T(\tau)\, d\tau = E[T + 0.02] = 0.03 \tag{6}$$

(b) The conditional second moment of T is

$$E\left[T^2|T > 0.02\right] = \int_{0.02}^\infty t^2(100)e^{-100(t-0.02)}\, dt \tag{7}$$

The substitution $\tau = t - 0.02$ yields

$$E\left[T^2|T > 0.02\right] = \int_0^\infty (\tau + 0.02)^2(100)e^{-100\tau}\, d\tau \tag{8}$$

$$= \int_0^\infty (\tau + 0.02)^2 f_T(\tau)\, d\tau \tag{9}$$

$$= E\left[(T + 0.02)^2\right] \tag{10}$$

Now we can calculate the conditional variance.

$$\begin{aligned}
\text{Var}[T|T > 0.02] &= E\left[T^2|T > 0.02\right] - (E\left[T|T > 0.02\right])^2 && (11)\\
&= E\left[(T + 0.02)^2\right] - (E\left[T + 0.02\right])^2 && (12)\\
&= \text{Var}[T + 0.02] && (13)\\
&= \text{Var}[T] = 0.01 && (14)
\end{aligned}$$

Problem 3.8.6 Solution

(a) In Problem 3.6.8, we found that the PDF of D is

$$f_D(y) = \begin{cases} 0.3\delta(y) & y < 60 \\ 0.07e^{-(y-60)/10} & y \geq 60 \end{cases} \quad (1)$$

First, we observe that $D > 0$ if the throw is good so that $P[D > 0] = 0.7$. A second way to find this probability is

$$P[D > 0] = \int_{0^+}^{\infty} f_D(y)\, dy = 0.7 \quad (2)$$

From Definition 3.15, we can write

$$f_{D|D>0}(y) = \begin{cases} \frac{f_D(y)}{P[D>0]} & y > 0 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1/10)e^{-(y-60)/10} & y \geq 60 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

(b) If instead we learn that $D \leq 70$, we can calculate the conditional PDF by first calculating

$$\begin{aligned}
P[D \leq 70] &= \int_0^{70} f_D(y)\, dy && (4)\\
&= \int_0^{60} 0.3\delta(y)\, dy + \int_{60}^{70} 0.07e^{-(y-60)/10}\, dy && (5)\\
&= 0.3 + -0.7e^{-(y-60)/10}\Big|_{60}^{70} = 1 - 0.7e^{-1} && (6)
\end{aligned}$$

The conditional PDF is

$$\begin{aligned}
f_{D|D\leq 70}(y) &= \begin{cases} \frac{f_D(y)}{P[D\leq 70]} & y \leq 70 \\ 0 & \text{otherwise} \end{cases} && (7)\\
&= \begin{cases} \frac{0.3}{1-0.7e^{-1}}\delta(y) & 0 \leq y < 60 \\ \frac{0.07}{1-0.7e^{-1}}e^{-(y-60)/10} & 60 \leq y \leq 70 \\ 0 & \text{otherwise} \end{cases} && (8)
\end{aligned}$$

Problem 3.8.7 Solution

(a) Given that a person is healthy, X is a Gaussian ($\mu = 90, \sigma = 20$) random variable. Thus,

$$f_{X|H}(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2} = \frac{1}{20\sqrt{2\pi}}e^{-(x-90)^2/800} \quad (1)$$

(b) Given the event H, we use the conditional PDF $f_{X|H}(x)$ to calculate the required probabilities

$$P\left[T^+|H\right] = P\left[X \geq 140|H\right] = P\left[X - 90 \geq 50|H\right] \tag{2}$$

$$= P\left[\frac{X-90}{20} \geq 2.5|H\right] = 1 - \Phi(2.5) = 0.006 \tag{3}$$

Similarly,

$$P\left[T^-|H\right] = P\left[X \leq 110|H\right] = P\left[X - 90 \leq 20|H\right] \tag{4}$$

$$= P\left[\frac{X-90}{20} \leq 1|H\right] = \Phi(1) = 0.841 \tag{5}$$

(c) Using Bayes Theorem, we have

$$P\left[H|T^-\right] = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-\right]} = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-|D\right]P\left[D\right] + P\left[T^-|H\right]P\left[H\right]} \tag{6}$$

In the denominator, we need to calculate

$$P\left[T^-|D\right] = P\left[X \leq 110|D\right] = P\left[X - 160 \leq -50|D\right] \tag{7}$$

$$= P\left[\frac{X-160}{40} \leq -1.25|D\right] \tag{8}$$

$$= \Phi(-1.25) = 1 - \Phi(1.25) = 0.106 \tag{9}$$

Thus,

$$P\left[H|T^-\right] = \frac{P\left[T^-|H\right]P\left[H\right]}{P\left[T^-|D\right]P\left[D\right] + P\left[T^-|H\right]P\left[H\right]} \tag{10}$$

$$= \frac{0.841(0.9)}{0.106(0.1) + 0.841(0.9)} = 0.986 \tag{11}$$

(d) Since T^-, T^0, and T^+ are mutually exclusive and collectively exhaustive,

$$P\left[T^0|H\right] = 1 - P\left[T^-|H\right] - P\left[T^+|H\right] = 1 - 0.841 - 0.006 = 0.153 \tag{12}$$

We say that a test is a failure if the result is T^0. Thus, given the event H, each test has conditional failure probability of $q = 0.153$, or success probability $p = 1 - q = 0.847$. Given H, the number of trials N until a success is a geometric (p) random variable with PMF

$$P_{N|H}(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Problem 3.8.8 Solution

(a) The event B_i that $Y = \Delta/2 + i\Delta$ occurs if and only if $i\Delta \leq X < (i+1)\Delta$. In particular, since X has the uniform $(-r/2, r/2)$ PDF

$$f_X(x) = \begin{cases} 1/r & -r/2 \leq x < r/2, \\ 0 & \text{otherwise,} \end{cases} \tag{1}$$

we observe that

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{1}{r}\, dx = \frac{\Delta}{r} \tag{2}$$

In addition, the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} f_X(x)/P[B] & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/\Delta & i\Delta \leq x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

It follows that given B_i, $Z = X - Y = X - \Delta/2 - i\Delta$, which is a uniform $(-\Delta/2, \Delta/2)$ random variable. That is,

$$f_{Z|B_i}(z) = \begin{cases} 1/\Delta & -\Delta/2 \leq z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) We observe that $f_{Z|B_i}(z)$ is the same for every i. Thus, we can write

$$f_Z(z) = \sum_i P[B_i]\, f_{Z|B_i}(z) = f_{Z|B_0}(z) \sum_i P[B_i] = f_{Z|B_0}(z) \tag{5}$$

Thus, Z is a uniform $(-\Delta/2, \Delta/2)$ random variable. From the definition of a uniform (a, b) random variable, Z has mean and variance

$$E[Z] = 0, \qquad \text{Var}[Z] = \frac{(\Delta/2 - (-\Delta/2))^2}{12} = \frac{\Delta^2}{12}. \tag{6}$$

Problem 3.8.9 Solution

For this problem, almost any non-uniform random variable X will yield a non-uniform random variable Z. For example, suppose X has the "triangular" PDF

$$f_X(x) = \begin{cases} 8x/r^2 & 0 \leq x \leq r/2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

In this case, the event B_i that $Y = i\Delta + \Delta/2$ occurs if and only if $i\Delta \leq X < (i+1)\Delta$. Thus

$$P[B_i] = \int_{i\Delta}^{(i+1)\Delta} \frac{8x}{r^2}\, dx = \frac{8\Delta(i\Delta + \Delta/2)}{r^2} \tag{2}$$

It follows that the conditional PDF of X given B_i is

$$f_{X|B_i}(x) = \begin{cases} \frac{f_X(x)}{P[B_i]} & x \in B_i \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{x}{\Delta(i\Delta + \Delta/2)} & i\Delta \leq x < (i+1)\Delta \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Given event B_i, $Y = i\Delta + \Delta/2$, so that $Z = X - Y = X - i\Delta - \Delta/2$. This implies

$$f_{Z|B_i}(z) = f_{X|B_i}(z + i\Delta + \Delta/2) = \begin{cases} \frac{z + i\Delta + \Delta/2}{\Delta(i\Delta + \Delta/2)} & -\Delta/2 \leq z < \Delta/2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We observe that the PDF of Z depends on which event B_i occurs. Moreover, $f_{Z|B_i}(z)$ is non-uniform for all B_i.

Problem 3.9.1 Solution

Taking the derivative of the CDF $F_Y(y)$ in Quiz 3.1, we obtain

$$f_Y(y) = \begin{cases} 1/4 & 0 \le y \le 4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We see that Y is a uniform $(0,4)$ random variable. By Theorem 3.20, if X is a uniform $(0,1)$ random variable, then $Y = 4X$ is a uniform $(0,4)$ random variable. Using `rand` as MATLAB's uniform $(0,1)$ random variable, the program `quiz31rv` is essentially a one line program:

```
function y=quiz31rv(m)
%Usage y=quiz31rv(m)
%Returns the vector y holding m
%samples of the uniform (0,4) random
%variable Y of Quiz 3.1
y=4*rand(m,1);
```

Problem 3.9.2 Solution

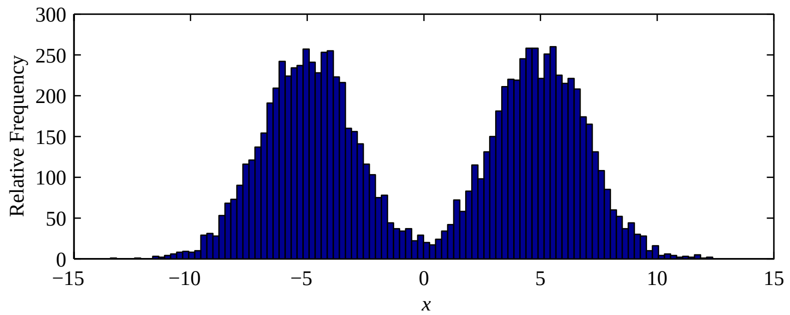
The modem receiver voltage is genrated by taking a ± 5 voltage representing data, and adding to it a Gaussian $(0,2)$ noise variable. Although siuations in which two random variables are added together are not analyzed until Chapter 4, generating samples of the receiver voltage is easy in MATLAB. Here is the code:

```
function x=modemrv(m);
%Usage: x=modemrv(m)
%generates m samples of X, the modem
%receiver voltage in Exampe 3.32.
%X=+-5 + N where N is Gaussian (0,2)
sb=[-5; 5]; pb=[0.5; 0.5];
b=finiterv(sb,pb,m);
noise=gaussrv(0,2,m);
x=b+noise;
```

The commands

```
x=modemrv(10000); hist(x,100);
```

generate 10,000 sample of the modem receiver voltage and plots the relative frequencies using 100 bins. Here is an example plot:



As expected, the result is qualitatively similar ("hills" around $X = -5$ and $X = 5$) to the sketch in Figure 3.3.

Problem 3.9.3 Solution

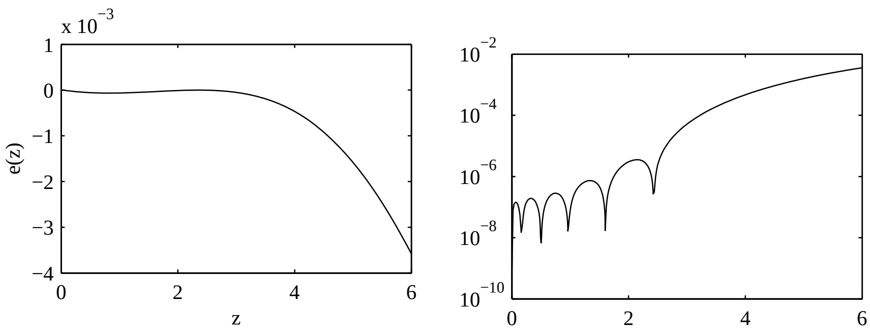
The code for $\hat{Q}(z)$ is the MATLAB function

```
function p=qapprox(z);
%approximation  to the Gaussian
% (0,1) complementary CDF Q(z)
t=1./(1.0+(0.231641888.*z(:)));
a=[0.127414796; -0.142248368; 0.7107068705; ...
        -0.7265760135; 0.5307027145];
p=([t t.^2 t.^3 t.^4 t.^5]*a).*exp(-(z(:).^2)/2);
```

This code generates two plots of the relative error $e(z)$ as a function of z:

```
z=0:0.02:6;
q=1.0-phi(z(:));
qhat=qapprox(z);
e=(q-qhat)./q;
plot(z,e); figure;
semilogy(z,abs(e));
```

Here are the output figures of `qtest.m`:



The left side plot graphs $e(z)$ versus z. It appears that the $e(z) = 0$ for $z \le 3$. In fact, $e(z)$ is nonzero over that range, but the relative error is so small that it isn't visible in comparison to $e(6) \approx -3.5 \times 10^{-3}$. To see the error for small z, the right hand graph plots $|e(z)|$ versus z in log scale where we observe very small relative errors on the order of 10^{-7}.

Problem 3.9.4 Solution

By Theorem 3.9, if X is an exponential (λ) random variable, then $K = \lceil X \rceil$ is a geometric (p) random variable with $p = 1 - e^{-\lambda}$. Thus, given p, we can write $\lambda = -\ln(1-p)$ and $\lceil X \rceil$ is a geometric (p) random variable. Here is the MATLAB function that implements this technique:

```
function k=georv(p,m);
lambda= -log(1-p);
k=ceil(exponentialrv(lambda,m));
```

To compare this technique with that use in `geometricrv.m`, we first examine the code for `exponentialrv.m`:

```
function x=exponentialrv(lambda,m)
x=-(1/lambda)*log(1-rand(m,1));
```

To analyze how $m = 1$ random sample is generated, let $R =$ `rand(1,1)`. In terms of mathematics, `exponentialrv(lambda,1)` generates the random variable

$$X = -\frac{\ln(1-R)}{\lambda} \tag{1}$$

For $\lambda = -\ln(1-p)$, we have that

$$K = \lceil X \rceil = \left\lceil \frac{\ln(1-R)}{\ln(1-p)} \right\rceil \tag{2}$$

This is precisely the same function implemented by `geometricrv.m`. In short, the two methods for generating geometric (p) random samples are one in the same.

Problem 3.9.5 Solution

Given $0 \leq u \leq 1$, we need to find the "inverse" function that finds the value of w satisfying $u = F_W(w)$. The problem is that for $u = 1/4$, any w in the interval $[-3, 3]$ satisfies $F_W(w) = 1/4$. However, in terms of generating samples of random variable W, this doesn't matter. For a uniform $(0, 1)$ random variable U, $P[U = 1/4] = 0$. Thus we can choose any $w \in [-3, 3]$. In particular, we define the inverse CDF as

$$w = F_W^{-1}(u) = \begin{cases} 8u - 5 & 0 \leq u \leq 1/4 \\ (8u + 7)/3 & 1/4 < u \leq 1 \end{cases} \tag{1}$$

Note that because $0 \leq F_W(w) \leq 1$, the inverse $F_W^{-1}(u)$ is defined only for $0 \leq u \leq 1$. Careful inspection will show that $u = (w + 5)/8$ for $-5 \leq w < -3$ and that $u = 1/4 + 3(w - 3)/8$ for $-3 \leq w \leq 5$. Thus, for a uniform $(0, 1)$ random variable U, the function $W = F_W^{-1}(U)$ produces a random variable with CDF $F_W(w)$. To implement this solution in MATLAB, we define

```
function w=iwcdf(u);
w=((u>=0).*(u <= 0.25).*(8*u-5))+...
    ((u > 0.25).*(u<=1).*((8*u+7)/3));
```

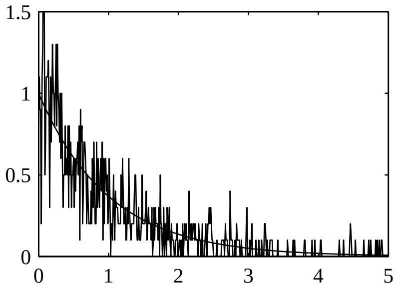
so that the MATLAB code `W=icdfrv(@iwcdf,m)` generates m samples of random variable W.

Problem 3.9.6 Solution

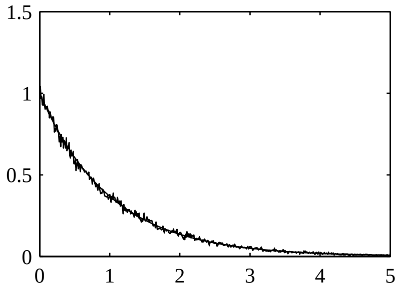
(a) To test the exponential random variables, the following code

```
function exponentialtest(lambda,n)
delta=0.01;
x=exponentialrv(lambda,n);
xr=(0:delta:(5.0/lambda))';
fxsample=(histc(x,xr)/(n*delta));
fx=exponentialpdf(lambda,xr);
plot(xr,fx,xr,fxsample);
```

generates n samples of an exponential λ random variable and plots the relative frequency $n_i/(n\Delta)$ against the corresponding exponential PDF. Note that the `histc` function generates a histogram using `xr` to define the edges of the bins. Two representative plots for $n = 1{,}000$ and $n = 100{,}000$ samples appear in the following figure:



`exponentialtest(1,1000)`



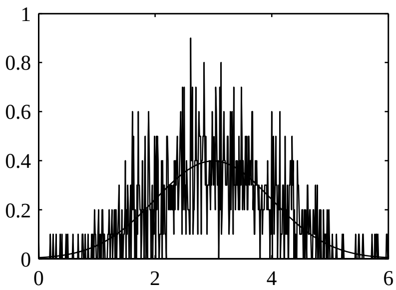
`exponentialtest(1,100000)`

For $n = 1{,}000$, the jaggedness of the relative frequency occurs because δ is sufficiently small that the number of sample of X in each bin $i\Delta < X \leq (i{+}1)\Delta$ is fairly small. For $n = 100{,}000$, the greater smoothness of the curve demonstrates how the relative frequency is becoming a better approximation to the actual PDF.

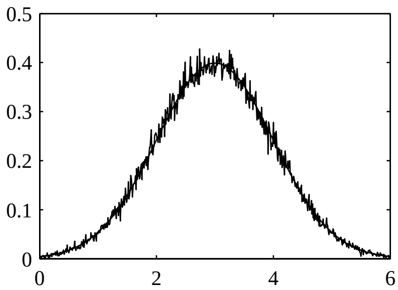
(b) Similar results hold for Gaussian random variables. The following code generates the same comparison between the Gaussian PDF and the relative frequency of n samples.

```
function gausstest(mu,sigma2,n)
delta=0.01;
x=gaussrv(mu,sigma2,n);
xr=(0:delta:(mu+(3*sqrt(sigma2))))';
fxsample=(histc(x,xr)/(n*delta));
fx=gausspdf(mu,sigma2,xr);
plot(xr,fx,xr,fxsample);
```

Here are two typical plots produced by `gaussiantest.m`:



`gausstest(3,1,1000)`



`gausstest(3,1,100000)`

Problem 3.9.7 Solution

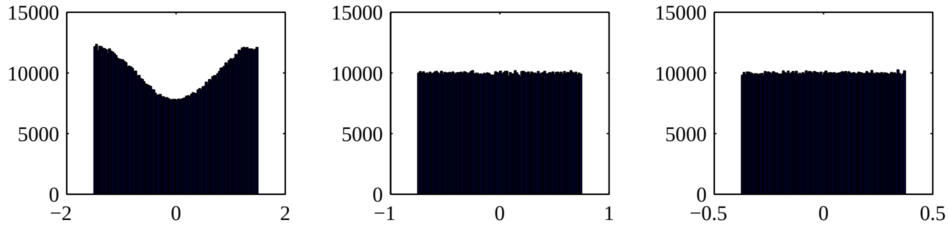
First we need to build a uniform $(-r/2, r/2)$ b-bit quantizer. The function `uquantize` does this.

```
function y=uquantize(r,b,x)
%uniform (-r/2,r/2) b bit quantizer
n=2^b;
delta=r/n;
x=min(x,(r-delta/2)/2);
x=max(x,-(r-delta/2)/2);
y=(delta/2)+delta*floor(x/delta);
```

Note that if $|x| > r/2$, then x is truncated so that the quantizer output has maximum amplitude. Next, we generate Gaussian samples, quantize them and record the errors:

```
function stdev=quantizegauss(r,b,m)
x=gaussrv(0,1,m);
x=x((x<=r/2)&(x>=-r/2));
y=uquantize(r,b,x);
z=x-y;
hist(z,100);
stdev=sqrt(sum(z.^2)/length(z));
```

For a Gaussian random variable X, $P[|X| > r/2] > 0$ for any value of r. When we generate enough Gaussian samples, we will always see some quantization errors due to the finite $(-r/2, r/2)$ range. To focus our attention on the effect of b bit quantization, `quantizegauss.m` eliminates Gaussian samples outside the range $(-r/2, r/2)$. Here are outputs of `quantizegauss` for $b = 1, 2, 3$ bits.



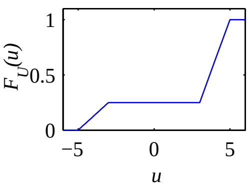
$b = 1$ $\quad$ $b = 2$ $\quad$ $b = 3$

It is obvious that for $b = 1$ bit quantization, the error is decidely not uniform. However, it appears that the error is uniform for $b = 2$ and $b = 3$. You can verify that uniform errors is a reasonable model for larger values of b.

Problem 3.9.8 Solution

To solve this problem, we want to use Theorem 3.22. One complication is that in the theorem, U denotes the uniform random variable while X is the derived random variable. In this problem, we are using U for the random variable we want to derive. As a result, we will use Theorem 3.22 with the roles of X and U reversed. Given U with CDF $F_U(u) = F(u)$, we need to find the inverse functon $F^{-1}(x) = F_U^{-1}(x)$ so that for a uniform $(0, 1)$ random variable X, $U = F^{-1}(X)$.

Recall that random variable U defined in Problem 3.3.7 has CDF



$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \le u < -3 \\ 1/4 & -3 \le u < 3 \\ 1/4 + 3(u-3)/8 & 3 \le u < 5 \\ 1 & u \ge 5. \end{cases} \tag{1}$$

At $x = 1/4$, there are multiple values of u such that $F_U(u) = 1/4$. However, except for $x = 1/4$, the inverse $F_U^{-1}(x)$ is well defined over $0 < x < 1$. At $x = 1/4$, we can arbitrarily define a value for $F_U^{-1}(1/4)$ because when we produce sample values of $F_U^{-1}(X)$, the event $X = 1/4$ has probability zero. To generate the inverse CDF, given a value of x, $0 < x < 1$, we ave to find the value of u such that $x = F_U(u)$. From the CDF we see that

$$0 \le x \le \frac{1}{4} \qquad \Rightarrow x = \frac{u+5}{8} \tag{2}$$

$$\frac{1}{4} < x \le 1 \qquad \Rightarrow x = \frac{1}{4} + \frac{3}{8}(u-3) \tag{3}$$

$$\tag{4}$$

These conditions can be inverted to express u as a function of x.

$$u = F^{-1}(x) = \begin{cases} 8x - 5 & 0 \le x \le 1/4 \\ (8x+7)/3 & 1/4 < x \le 1 \end{cases} \tag{5}$$

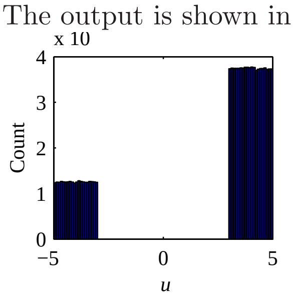
In particular, when X is a uniform $(0, 1)$ random variable, $U = F^{-1}(X)$ will generate samples of the rndom variable U. A MATLAB program to implement this solution is now straightforward:

```
function u=urv(m)
%Usage: u=urv(m)
%Generates m samples of the random
%variable U defined in Problem 3.3.7
x=rand(m,1);
u=(x<=1/4).*(8*x-5);
u=u+(x>1/4).*(8*x+7)/3;
```

To see that this generates the correct output, we can generate a histogram of a million sample values of U using the commands

```
u=urv(1000000); hist(u,100);
```

The output is shown in the following graph, alongside the corresponding PDF of U.



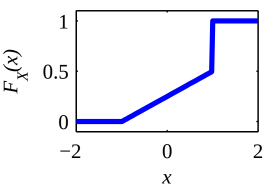
$$f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \le u < -3 \\ 0 & -3 \le u < 3 \\ 3/8 & 3 \le u < 5 \\ 0 & u \ge 5. \end{cases} \tag{6}$$

Note that the scaling constant 10^4 on the histogram plot comes from the fact that the histogram was generated using 10^6 sample points and 100 bins. The width of each bin is $\Delta = 10/100 = 0.1$. Consider a bin of idth Δ centered at u_0. A sample value of U would fall in that bin with probability $f_U(u_0)\Delta$. Given that we generate $m = 10^6$ samples, we would expect about $mf_U(u_0)\Delta = 10^5 f_U(u_0)$ samples in each bin. For $-5 < u_0 < -3$, we would expect to see about 1.25×10^4 samples in each bin. For $3 < u_0 < 5$, we would expect to see about 3.75×10^4 samples in each bin. As can be seen, these conclusions are consistent with the histogam data.

Finally, we comment that if you generate histograms for a range of values of m, the number of samples, you will see that the histograms will become more and more similar to a scaled version of the PDF. This gives the (false) impression that any bin centered on u_0 has a number of samples increasingly close to $mf_U(u_0)\Delta$. Because the histpgram is always the same height, what is actually happening is that the vertical axis is effectively scaled by $1/m$ and the height of a histogram bar is proportional to *the fraction* of m samples that land in that bin. We will see in Chapter 7 that the fraction of samples in a bin does converge to the probability of a sample being in that bin as the number of samples m goes to infinity.

Problem 3.9.9 Solution

From Quiz 3.6, random variable X has CDF The CDF of X is



$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/4 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases} \tag{1}$$

Following the procedure outlined in Problem 3.7.18, we define for $0 < u \leq 1$,

$$\tilde{F}(u) = \min \{x | F_X(x) \geq u\}. \tag{2}$$

We observe that if $0 < u < 1/4$, then we can choose x so that $F_X(x) = u$. In this case, $(x+1)/4 = u$, or equivalently, $x = 4u - 1$. For $1/4 \leq u \leq 1$, the minimum x that satisfies $F_X(x) \geq u$ is $x = 1$. These facts imply

$$\tilde{F}(u) = \begin{cases} 4u - 1 & 0 < u < 1/4 \\ 1 & 1/4 \leq u \leq 1 \end{cases} \tag{3}$$

It follows that if U is a uniform $(0, 1)$ random variable, then $\tilde{F}(U)$ has the same CDF as X. This is trivial to implement in MATLAB.

```
function x=quiz36rv(m)
%Usage x=quiz36rv(m)
%Returns the vector x holding m samples
%of the random variable X of Quiz 3.6
u=rand(m,1);
x=((4*u-1).*(u< 0.25))+(1.0*(u>=0.25));
```