HW Assignment Solution for EML 4806 CH 5

Problem 2:

From exercise 3.3 we have:

$$ {}^0_3T = \begin{bmatrix} C_1C_{23} & -C_1S_{23} & S_1 & L_1C_1 + L_2C_1C_2 \\ S_1C_{23} & -S_1S_{23} & -C_1 & L_1S_1 + L_2S_1C_2 \\ S_{23} & C_{23} & 0 & L_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

and:

$$ {}^3_4T = \begin{bmatrix} 1 & 0 & 0 & L_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; \quad {}^0_4T = {}^0_3T\,{}^3_4T $$

we could then find ${}^0J(\underline{\theta})$ quite easily by differentiating ${}^0P_{YORG}$. Finally, ${}^4J(\underline{\theta})$ can be calculated as ${}^4_0R^0J(\underline{\theta})$. This might be tedious, so lets try "standard" velocity propagation as done in the text:

$$ {}^1W_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} {}^1V_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

$$ {}^2W_2 = {}^2_1R\,{}^1W_1 + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot\theta_2 \end{bmatrix} $$

$$ {}^2W_2 = \begin{bmatrix} S_2\dot\theta_1 \\ C_2\dot\theta_1 \\ \dot\theta_2 \end{bmatrix} {}^2V_2 = {}^2_1R({}^1V_1 + {}^1W_1 \times {}^1P_2) $$

$$ {}^2V_2 = \begin{bmatrix} C_2 & 0 & S_2 \\ -S_2 & 0 & C_2 \\ 0 & -1 & 0 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ L_1\dot\theta_1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 0 \\ -L_1\dot\theta_1 \end{bmatrix} $$

$$^3W_3 = {}^3_2R\,{}^2W_2 + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix} = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} S_2\dot{\theta}_1 \\ C_2\dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3 \end{bmatrix}$$

$$^3W_3 = \begin{bmatrix} S_{23}\dot{\theta}_1 \\ C_{23}\dot{\theta}_1 \\ \dot{\theta}_2 + \dot{\theta}_3 \end{bmatrix} {}^3V_3 = {}^3_2R({}^2V_2 + {}^2W_2 \times {}^2P_3)$$

$$^3V_3 = \begin{bmatrix} C_3 & S_3 & 0 \\ -S_3 & C_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 \\ 0 \\ -L_1\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ L_2\dot{\theta}_2 \\ -L_2C_2\dot{\theta}_1 \end{bmatrix} \right)$$

$$^3V_3 = \begin{bmatrix} S_3L_2\dot{\theta}_2 \\ C_3L_2\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix} {}^4W_4 = {}^3W_3$$

$$^4V_4 = {}^4_3R({}^3V_3 + {}^3W_3 \times {}^3P_y) = {}^3V_3 + {}^3W_3 \times {}^3P_4$$

$$= \begin{bmatrix} S_2L_2\dot{\theta}_2 \\ C_3L_2\dot{\theta}_2 \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ L_3(\dot{\theta}_2 + \dot{\theta}_3 \\ -L_3C_{23}\dot{\theta}_1 \end{bmatrix}$$

$$= \begin{bmatrix} S_3L_2\dot{\theta}_2 \\ C_3L_2\dot{\theta}_2 - L_3(\dot{\theta}_2 + \dot{\theta}_3) \\ -L_1\dot{\theta}_1 - L_2C_2\dot{\theta}_1 - L_3C_{23}\dot{\theta}_1 \end{bmatrix}$$

$$\therefore {}^4J(\underline{\theta}) = \begin{bmatrix} 0 & S_3L_2 & 0 \\ 0 & C_3L_2 + L_3 & L_3 \\ (-L_1 - L_2C_2 - L_3C_{23}) & 0 & 0 \end{bmatrix}$$

Problem 13:

$$\underline{\tau} = {}^oJ^T(\underline{\theta})\,{}^o\underline{F}$$

$$\underline{\tau} = \begin{bmatrix} -L_1S_1 - L_2S_{12} & L_1C_1 + L_2C_{12} \\ -L_2S_{12} & L_2C_{12} \end{bmatrix} \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

$$\tau_1 = -10S_1L_1 - 10L_2S_{12}$$

$$\tau_2 = -10L_2S_{12}$$

Problem 15:

The kinematics can be done easily to obtain:

$$^0P_{\text{YORG}} = \begin{bmatrix} (d_2 + L_2 + L_3)S_1 \\ -(d_2 + L_2 + L_3)C_1 \\ 0 \end{bmatrix}$$

$$^0V = {}^0J\dot{\theta}$$

$$^0J = \begin{bmatrix} \frac{\partial^0 P_{\text{YORGX}}}{\partial\theta_1} & \frac{\partial^0 P_{\text{YORGX}}}{\partial\theta_2} & \frac{\partial^0 P_{\text{YORGX}}}{\partial\theta_3} \\ \frac{\partial^0 P_{\text{YORGY}}}{\partial\theta_1} & \frac{\partial^0 P_{\text{YORGY}}}{\partial\theta_2} & \frac{\partial^0 P_{\text{YORGY}}}{\partial\theta_3} \\ \frac{\partial^0 P_{\text{YORGZ}}}{\partial\theta_1} & \frac{\partial^0 P_{\text{YORGZ}}}{\partial\theta_2} & \frac{\partial^0 P_{\text{YORGZ}}}{\partial\theta_3} \end{bmatrix}$$

So,

$$^0J = \begin{bmatrix} (d_2 + L_2 + L_3)C_1 & S_1 & 0 \\ (d_2 + L_2 + L_3)S_1 & -C_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Use $^3P = ^3P_{\text{heel}} = [0 \quad -50 \quad 0]^{\mathsf{T}}$. Since

$$^0_3R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have

$$^0P_{3ORG} = \begin{bmatrix} 900 - 200 \\ 0 + 400 + 50 \\ 0 \end{bmatrix} = \begin{bmatrix} 700 \\ 450 \\ 0 \end{bmatrix}$$

Now, starting with (4.6), one follows a method of §4.4 to get $\Theta = [52.6^\circ \quad -45.1^\circ \quad -7.56^\circ]^{\mathsf{T}}$. (The other solution would not be feasible for the human knee joint.)

Problem 26:

Find an epression for the joint velocities using $\dot{\Theta} = {}^0J^{-1}\,{}^3v$.

$$^3J^{-1} = \frac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 & 0 \\ -l_1 c_2 - l_2 & l_1 s_2 \end{bmatrix}$$

$$^3v = {}^3_0R\,{}^0v = {}^0_3R^{\mathsf{T}}\,{}^0v = \begin{bmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} c_{12} \\ -s_{12} \\ 0 \end{bmatrix}$$

Using these two factors yields

$$\dot{\theta}_1 = \frac{c_{12}}{l_1 s_2}$$

$$\dot{\theta}_2 = \frac{-l_1 c_{12} c_2 - l_2 c_{12} - l_1 s_{12} s_2}{l_1 l_2 s_2} = \frac{-l_2 c_{12} - l_1(c_{12} c_2 + s_{12} s_2)}{l_1 l_2 s_2} = -\frac{c_{12}}{l_1 s_2} - \frac{c_1}{l_2 s_2},$$

which is the same result as in Example 5.5, showing that "as the arm stretches out toward $\theta_2 = 0$, both joint rates go to infinity."

Problem 27:

The transformation matrix

$$^0_2T = \begin{bmatrix} c_1 & s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_1 & 0 & s_1 & -s_1 d_2 \\ s_1 & 0 & c_1 & c_1 d_2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

provides

$$^0P_{2ORG} = \begin{bmatrix} -s_1 d_2 \\ c_1 d_2 \\ 0 \end{bmatrix}.$$

Hence

$$^0J = \begin{bmatrix} \frac{\partial\,^0P_{2ORGX}}{\partial\theta_1} & \frac{\partial\,^0P_{2ORGX}}{\partial d_2} \\ \frac{\partial\,^0P_{2ORGY}}{\partial\theta_1} & \frac{\partial\,^0P_{2ORGY}}{\partial d_2} \end{bmatrix} = \begin{bmatrix} -c_1 d_2 & -s_1 \\ -s_1 d_2 & c_1 \end{bmatrix}.$$

Since $\det({}^0J) = -d_2(c_1^2 + s_1^2)$, the manipulator is in singularity when $d_2 = 0$; there are some velocities it cannot provide in this configuration.

Problem 29:

(5.13)

$$\begin{aligned}\boldsymbol{\tau} &= {}^{0}J^{\mathsf{T}}(\theta)\,{}^{0}\boldsymbol{F} \\ &= \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & l_1 c_1 + l_2 c_{12} \\ -l_2 s_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} 5 \\ 3 \end{bmatrix} \\ \tau_1 &= l_1(3c_1 - 5s_1) + l_2(3c_{12} - 5s_{12}) \\ \tau_2 &= l_2(3c_{12} - s_{12})\end{aligned}$$