Mechanical Vibrations

FUNDAMENTALS OF VIBRATION

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Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 1: Fundamentals of Vibration

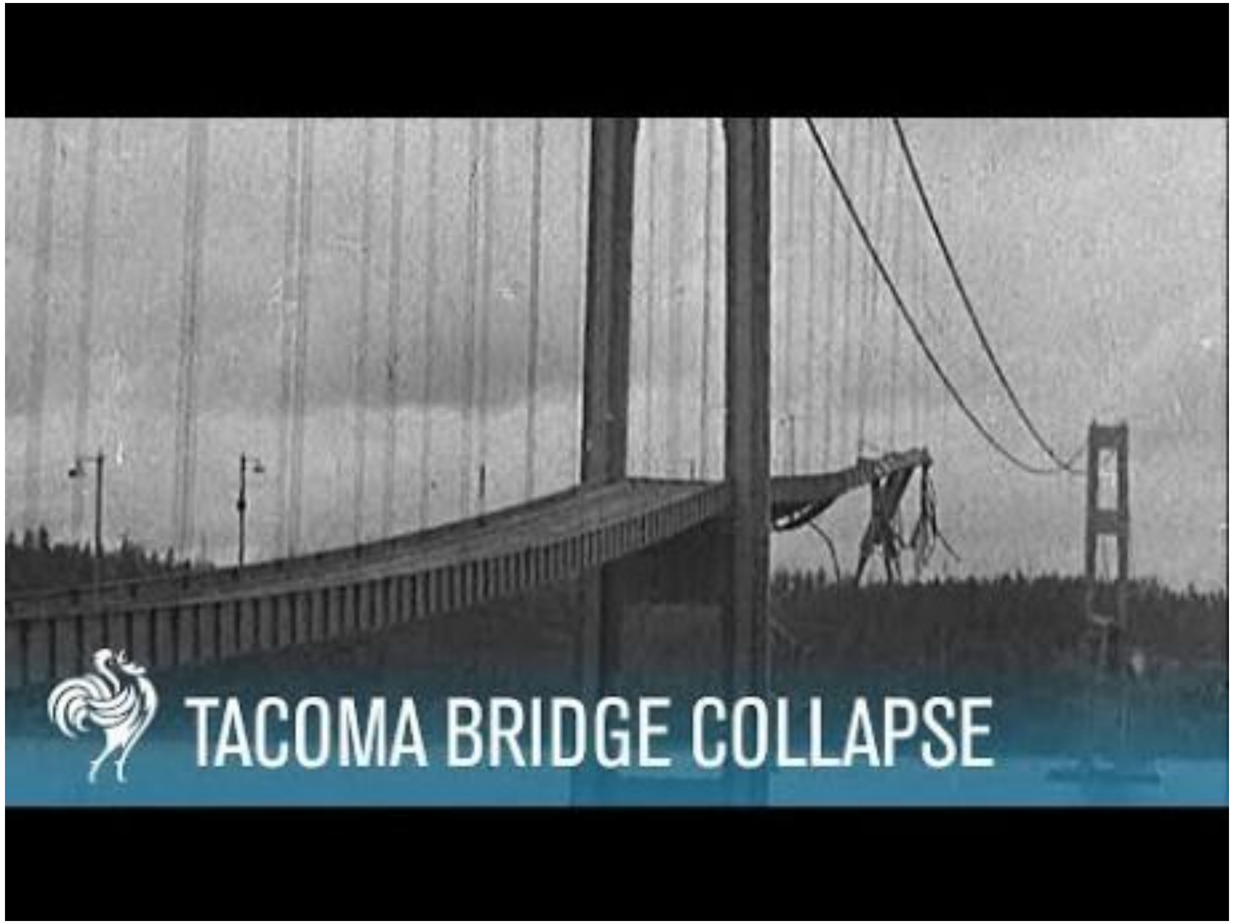
Learning Objectives

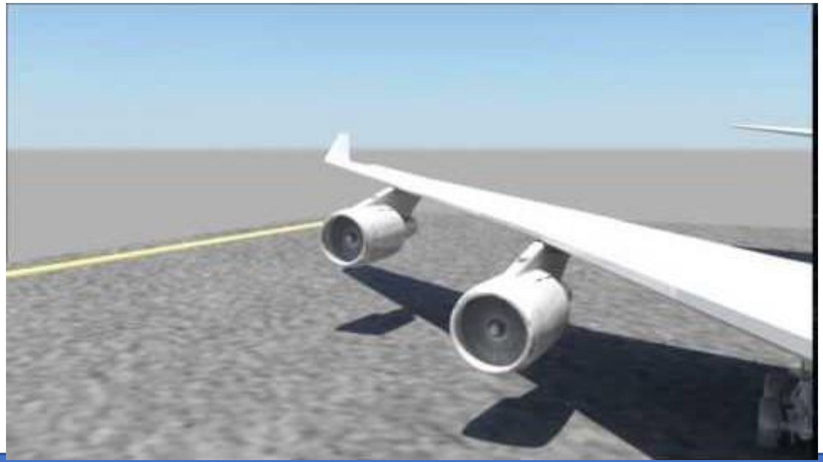
- Recognize the importance of studying Vibration
- Describe a brief the history of vibration
- Understand the definition of Vibration
- State the process of modeling systems
- Determine the Degrees of Freedom (DOF) of a system
- Identify the different types of Mechanical Vibrations
- Compute equivalent values for Spring elements, Mass elements and Damping elements
- Characterize harmonic motion and the different possible representation
- Add and subtract harmonic motions
- Conduct Fourier series expansion of given periodic functions

Importance of studying Vibration

- All systems that have mass and any type of flexible components are vibrating system.
- Examples are many:
  - We hear because our eardrums vibrate
  - Human speech requires the oscillatory motion of larynges
  - In machines, vibration can loosen fasteners such as nuts.
  - In balance in machine can cause problem to the machine itself or surrounding machines or environment.
  - Periodic forces bring dynamic responses that can cause fatigue in materials
  - The phenomenon known as Resonance leads to excessive deflections and failure.
  - The vibration and noise generated by engines causes annoyance to people and, sometimes, damage to property.

Importance of studying Vibration



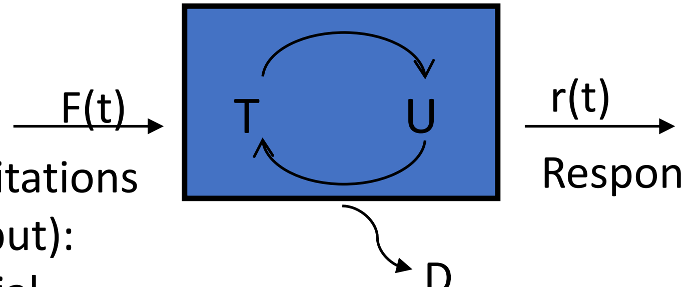


Brief history

- People became interested in vibration when they created the first musical instruments (as long as 4000 B.C.).
- **Pythagoras** (582 – 507 B.C) is considered the fisrt person to investigate musical sounds.
- **Galileo Galilei** (1564-1642) is considered to be the founder of modern experimental science, he conduct experiments on the simple pendulum, describing the dependence of the frequency of vibration and the length.
- **Robert Hooke** (1635–1703) also conducted experiments to find a relation between the pitch and frequency of vibration of a string.
- **Joseph Sauveur** (1653–1716) coined the word “acoustics” for the science of sound.
- **Sir Isaac Newton** (1642–1727) his law of motion is routinely used to derive the equations of motion of a vibrating body.
- **Brook Taylor** (1685–1731), obtained the natural frequency of vibration observed by Galilei and Mersenne.
- **Daniel Bernoulli** (1700–1782), **Jean D’Alembert** (1717–1783), and **Leonard Euler** (1707–1783)., introduced partial derivatives in the equations of motion.
- **J. B. J. Fourier** (1768–1830) contributed on the development of the theory of vibrations and led to the possibility of expressing any arbitrary function using the principle of superposition.
- **Joseph Lagrange** (1736–1813) presented the analytical solution of the vibrating string.
- **Charles Coulomb** did both theoretical and experimental studies in 1784 on the torsional oscillations of a metal cylinder suspended by a wire. He also contributed in the modeling of dry friction.
- **E. F. F. Chladni** (1756–1824) developed the method of placing sand on a vibrating plate to find its mode shapes.
- **Simeon Poisson** (1781–1840) study vibration of a rectangular flexible membrane.
- **Lord Baron Rayleigh** (1842 – 1919) Among the many contributions, he develop the method of finding the fundamental frequency of vibration of a conservative system by making use of the principle of conservation of energy.

Definition of Vibration

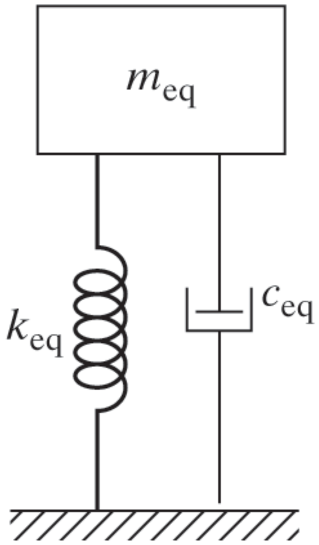
- Any motion that repeats itself after an interval of time.
- A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

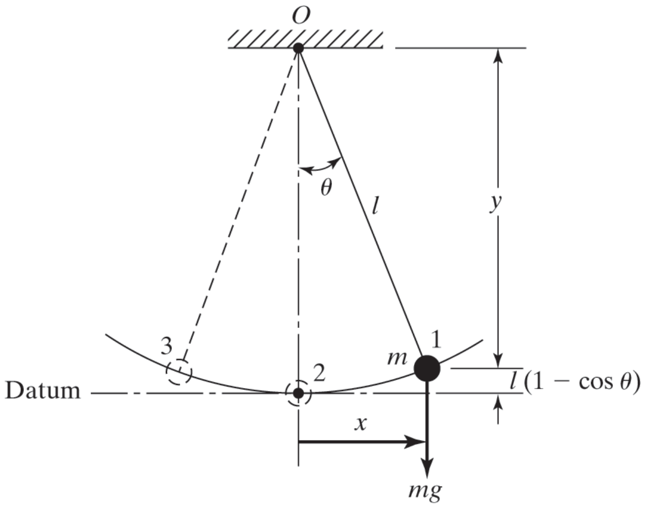


Excitations (input): Initial conditions of external force

Responses (output)

Energy dissipation





Modeling systems

- All mechanical and structural systems can be modeled as mass-spring-damper systems

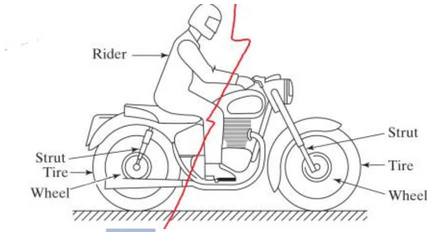
Real system

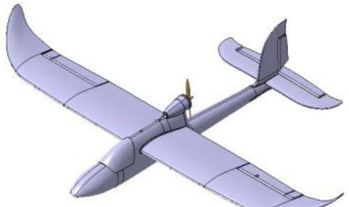
Mechanical Model

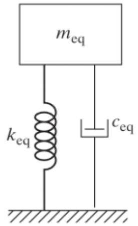
Mathematical Model

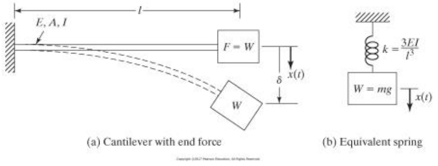
Solution

Analysis









$$m\ddot{x} + c\dot{x} + kx = 0$$

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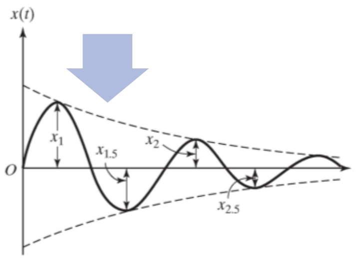
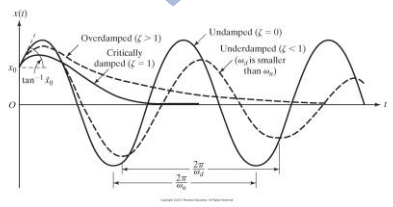




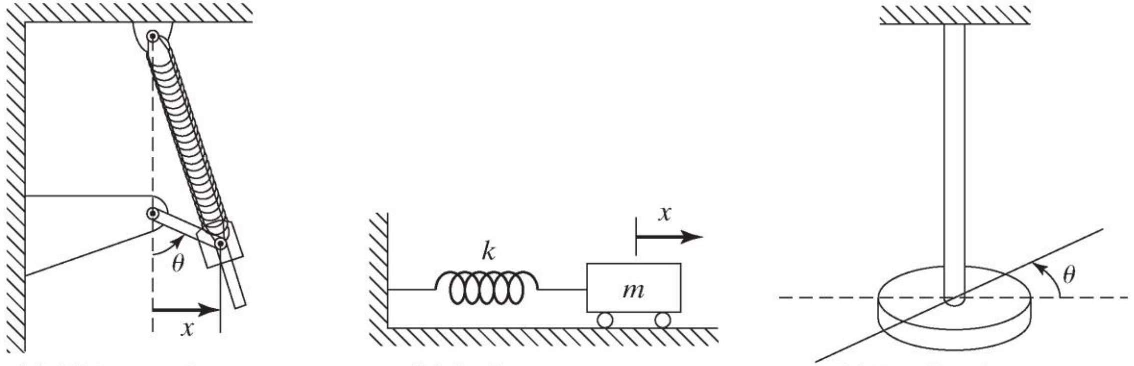
Respuesta análisis del sistema para que la respuesta sea coherente.

Respuesta análisis del sistema para que la respuesta sea coherente.

Degrees of freedom

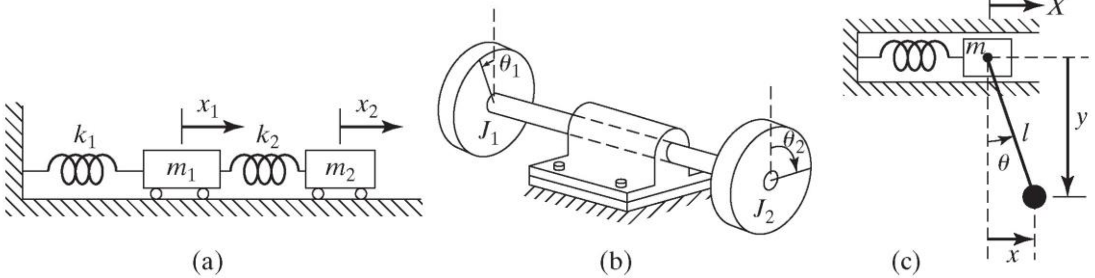
The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time

Single DoF systems:



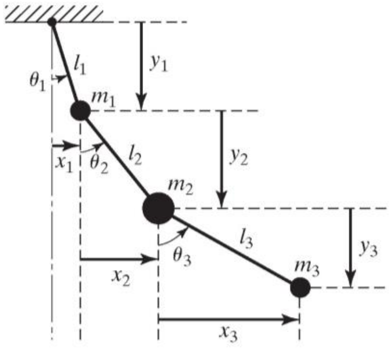
(a) Slider-crank-spring mechanism (b) Spring-mass system (c) Torsional system

Two DOF System



(a) (b) (c)

Three DoF systems:

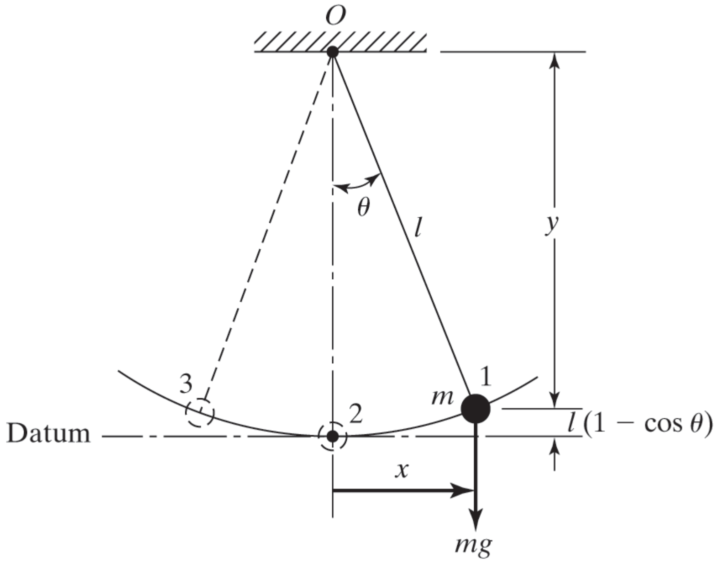


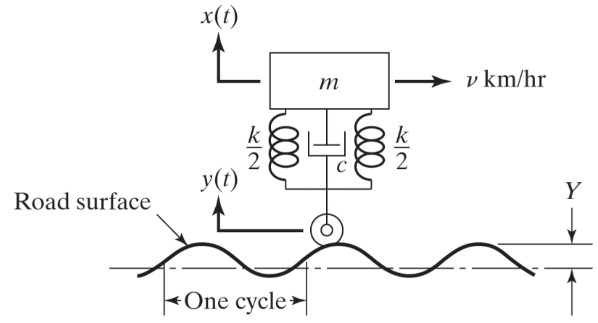
(b)

Classification of Vibration

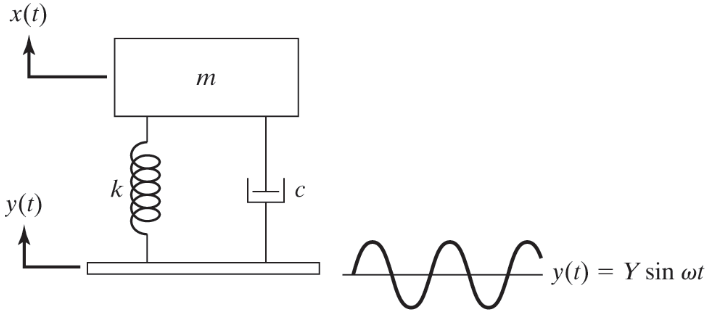
- Free Vibration: When a system, after an initial disturbance, is left to vibrate on its own. No external force acts on the system. The system oscillates at its natural frequency. Example: a pendulum.

- Forced Vibration: When a system is subjected to an external force (often, a repeating type of force). The oscillation that arises in machines such as diesel engines is an example of forced vibration.





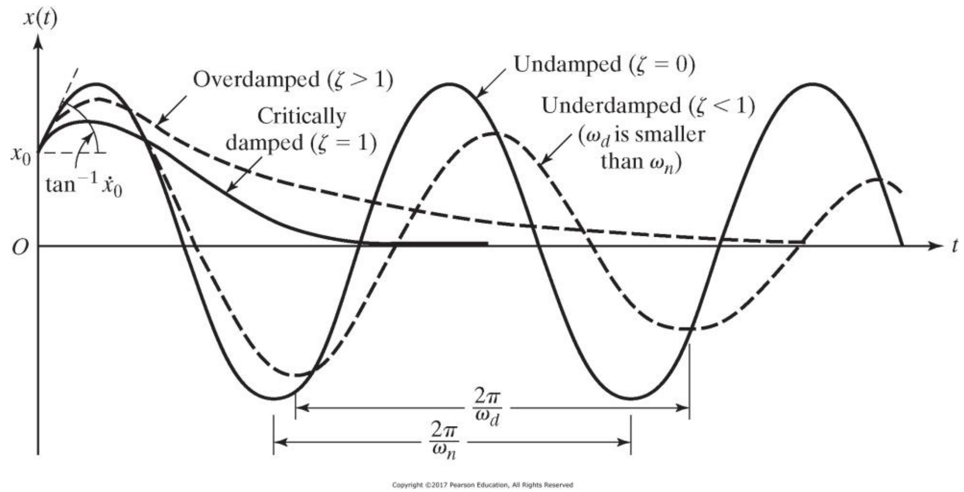
(a)



(b)

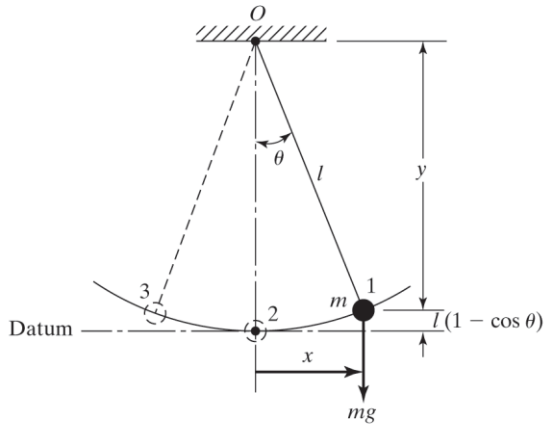
Types of response

- Undamped and damped Vibration

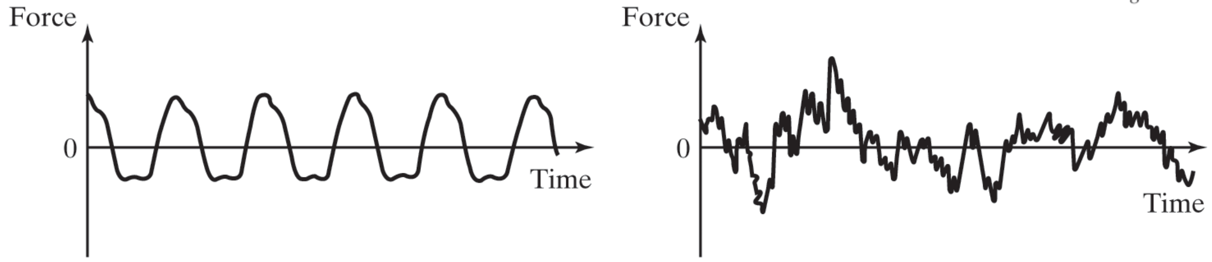


- Linear of nonlinear Vibration

$$l\ddot{\theta} + mgsin\theta = 0$$



- Deterministic ond Random Vibration



(a) A deterministic (periodic) excitation

(b) A random excitation

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SPRING, MASS and DAMPING ELEMENTS

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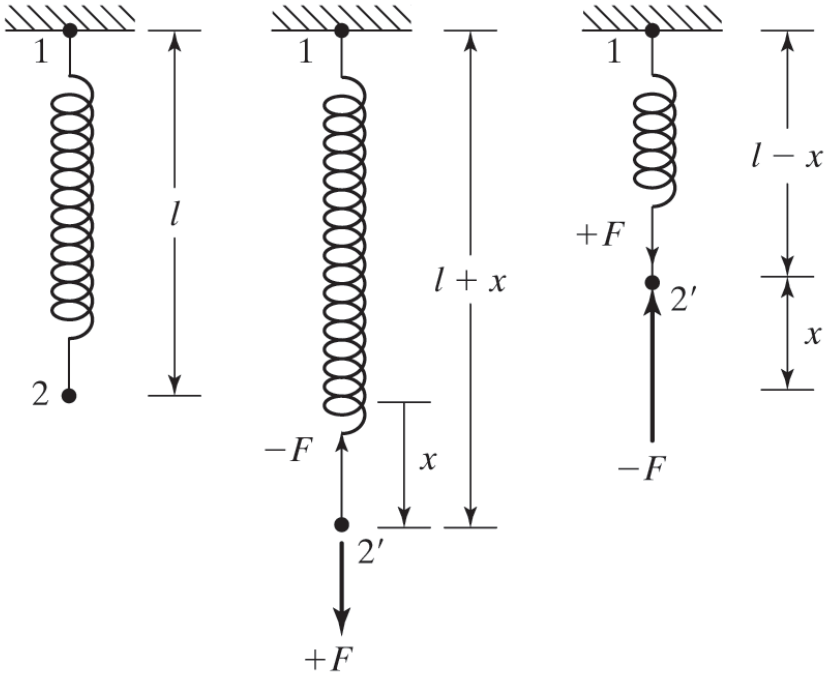
Chapter 1: Fundamentals of Vibration

Spring Elements



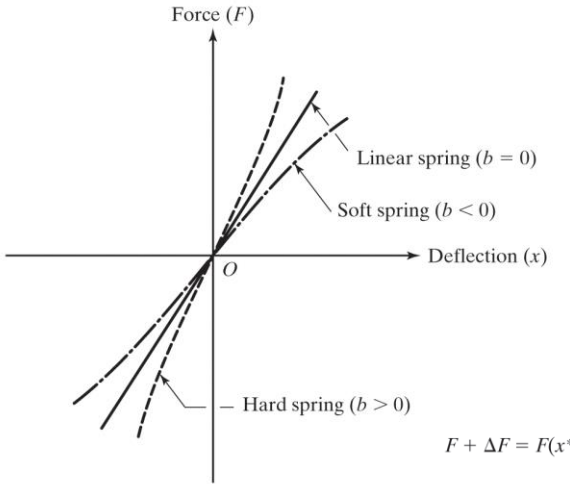
The force related to a elongation or reduction in length opposes to the displacement of the end of the spring and is given by :

$$F = kx$$

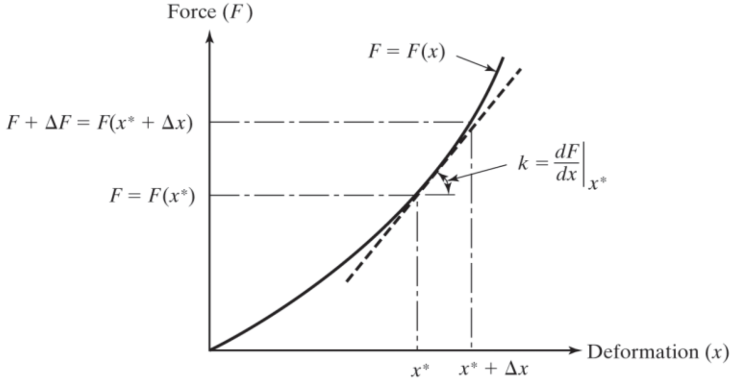
The work done (U) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

$$U = \frac{1}{2} k x^2$$

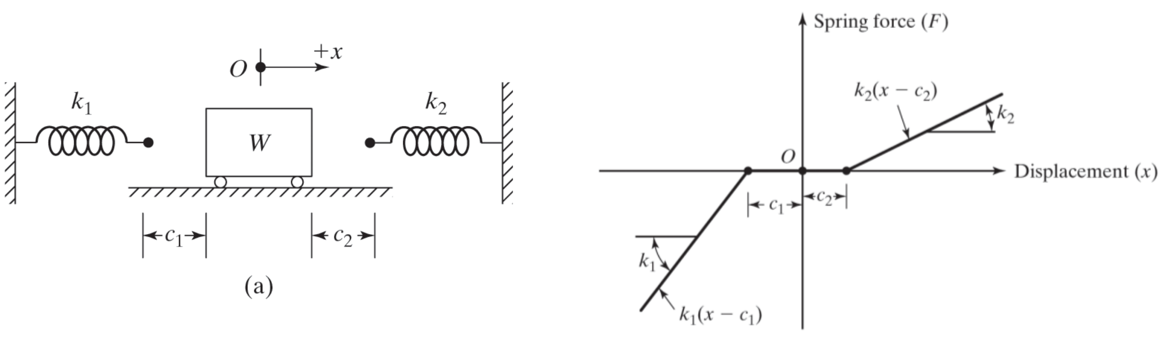
Non-linear spring element:



$$F = ax + bx^3$$



$$k = \left.\frac{dF}{dx}\right|_{x^*}$$



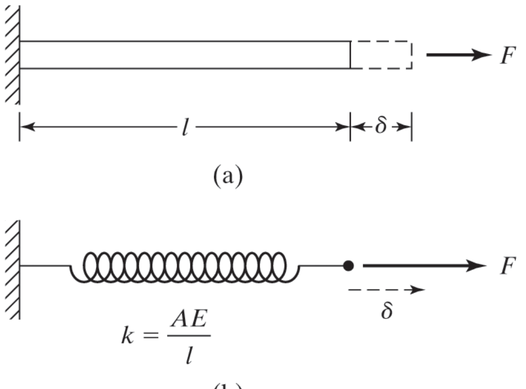
(a)

(b)

Example: Spring constant of a rod

- Find the equivalent spring constant of a uniform rod of length *l*, cross-sectional area *A*, Area Moment of Inertia *I* and Young's modulus *E*

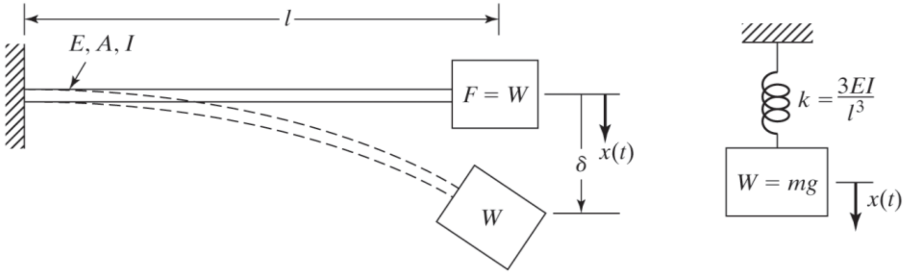
1. Subjected to an axial tensile (or compressive) force *F*



$$\delta = \frac{\delta}{l} l = \varepsilon l = \frac{Fl}{AE}$$

$$k = \frac{Force}{deflexion} = \frac{F}{\delta} = \frac{AE}{l}$$

2. Cantilever bean subjected to a transversal load at the free end.



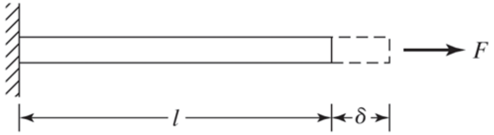
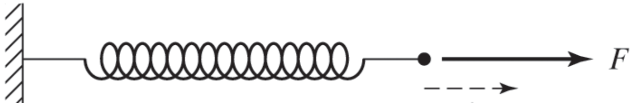
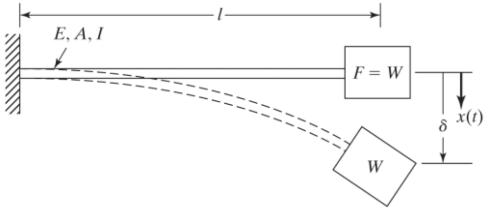
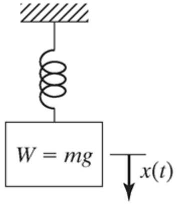
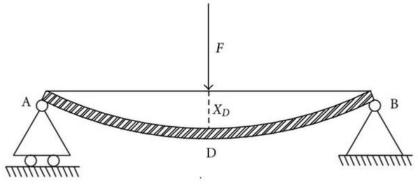
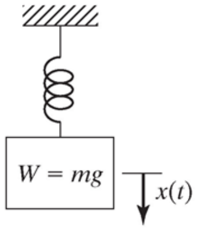
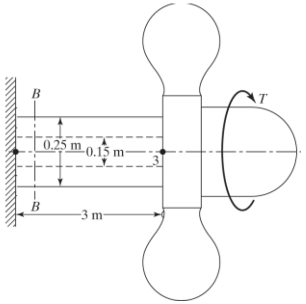
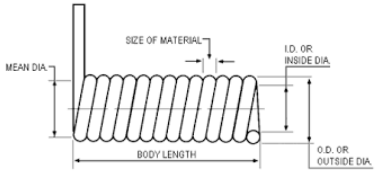
(a) Cantilever with end force

(b) Equivalent spring

$$\delta = \frac{Wl^3}{3EI}$$

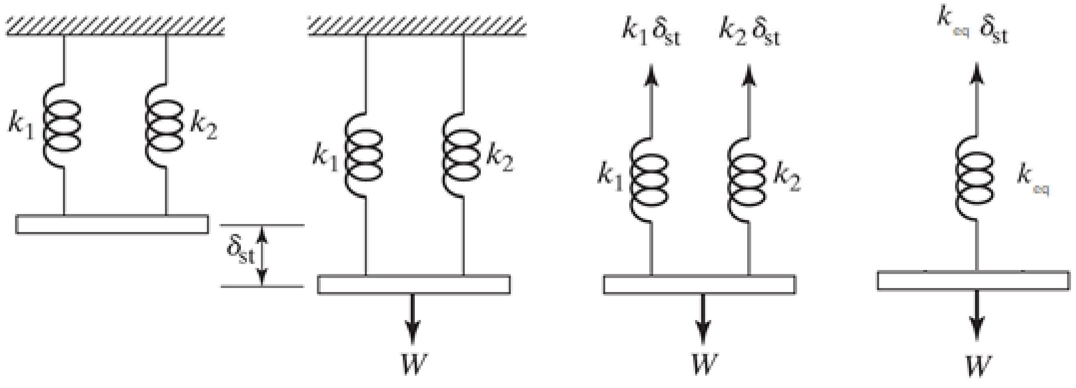
$$k = \frac{Force}{deflexion} = \frac{W}{\delta} = \frac{3EI}{l^3}$$

Equivalent spring constants

	Original system	Equivalent model	Spring Constant of a
Rod under axial load			$k = \frac{AE}{l}$
Cantilever beam with end force			$k = \frac{3EI}{l^3}$
Simple support beam with load in the middle			$k = \frac{48EI}{l^3}$
Propeller Shaft subjected to a torsional moment			$k = \frac{GJ}{l}$ $\quad J = \frac{\pi(D^4 - d^4)}{32}$

Combination of Springs

Springs in Parallel



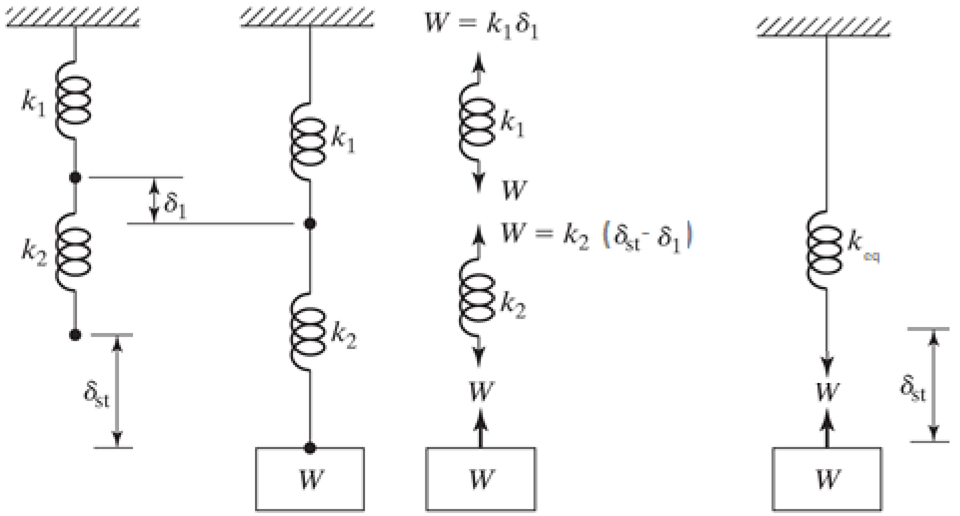
W= $k_1\delta_{st} + k_2\delta_{st}$

W= $(k_1+k_2)\delta_{st}$

W= $(k_{eq})\delta_{st}$

$$\boxed{k_{eq} = (k_1+k_2)}$$

Springs in Series



W= $k_1\delta_1 = k_2(\delta_{st}-\delta_1) = k_{eq}\delta_{st}$

$$\delta_1 = \frac{W}{k_1} \qquad k_2\left(\delta_{st} - \frac{W}{k_1}\right) = W \Rightarrow \delta_{st} = \frac{W}{k_2} + \frac{W}{k_1}$$

$$\delta_{st} = \frac{W}{k_{eq}}$$

$$\boxed{\frac{1}{k_{eq}} = \left(\frac{1}{k_1} + \frac{1}{k_2}\right)}$$

Mass or Inertia Elements

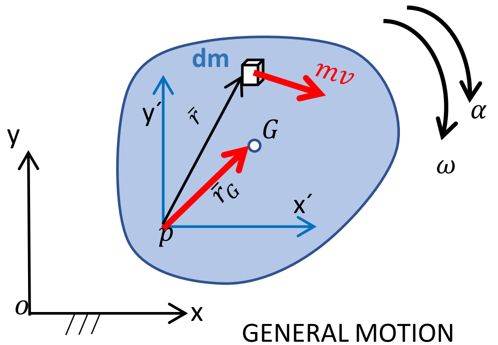
- The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes.
- For single DoF system for a simple analysis, we can replace several masses by a single equivalent mass.
- The key step is to choose properly a parameter that will describe the motion of the system and express all other parameters in term of the chosen one.
- Calculate the kinetic energy of the system and make it equal to the kinetic energy of the equivalent system

Kinetic energy

In general for one rigid body the kinetic energy can be calculated as

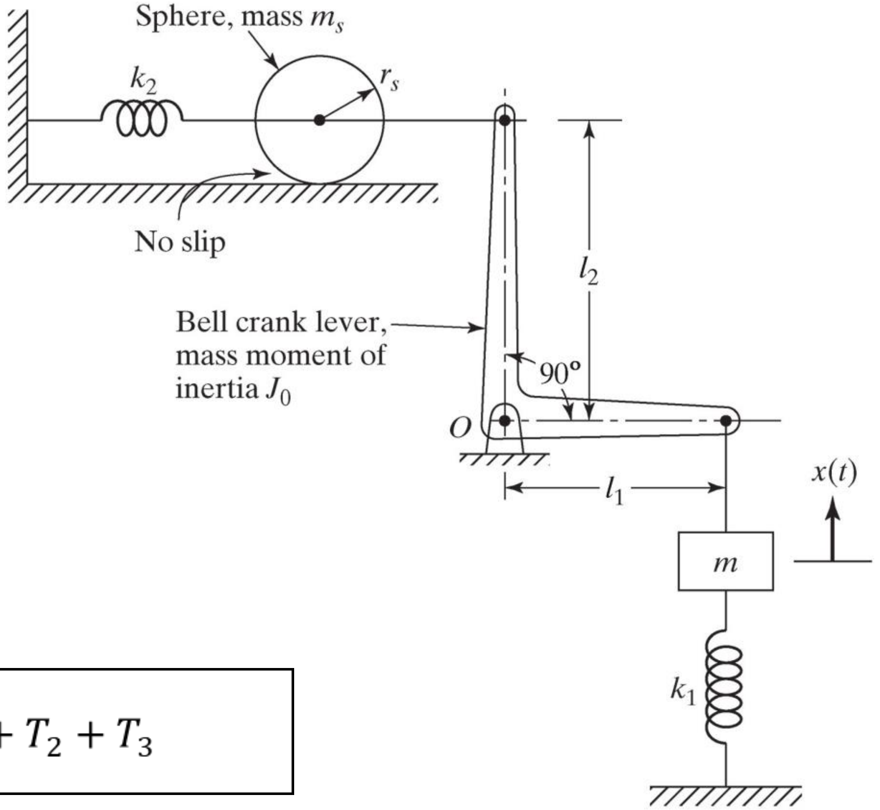
$$\boxed{T = \frac{1}{2} m (\bar{v}_p)^2 + \frac{1}{2} I_{pzz} (\omega_z)^2 + \bar{v}_p \cdot (\bar{\omega}_z \times m \bar{r}_G)}$$

$$\bar{v} = \bar{v}_p + \bar{\omega}_z \times \bar{r}$$



GENERAL MOTION

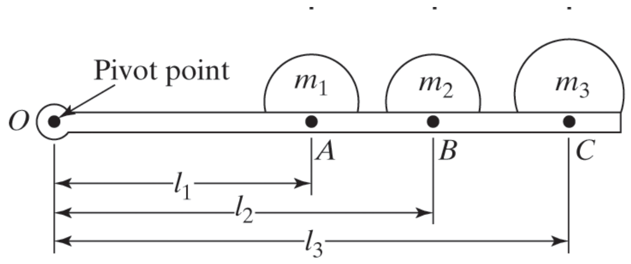
For a system with several rigid bodies is the sum of the kinetic energy of each body:



$$\boxed{T = T_1 + T_2 + T_3}$$

3

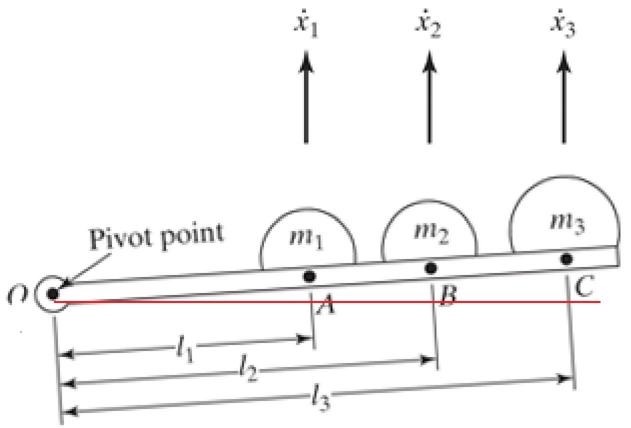
Example 1: Translational masses by a rigid bar



$$T = \frac{1}{2}m_1(\dot{x}_1)^2 + \frac{1}{2}m_2(\dot{x}_2)^2 + \frac{1}{2}m_3(\dot{x}_3)^2$$

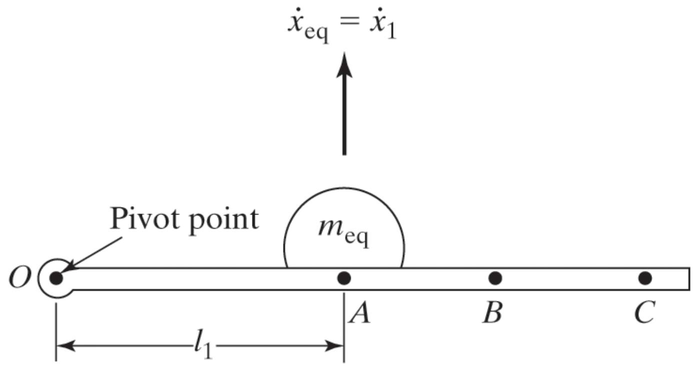
$$\boxed{\dot{x}_{eq} = \dot{x}_1}$$

- Let's choose $\dot{x}_1$ as the parameter that will describe the motion of the system and find the equivalent kinetic energy



$$T = \frac{1}{2}m_1(\dot{x}_1)^2 + \frac{1}{2}m_2\left(\frac{l_2}{l_1}\,\dot{x}_1\right)^2 + \frac{1}{2}m_3\left(\frac{l_3}{l_1}\,\dot{x}_1\right)^2$$

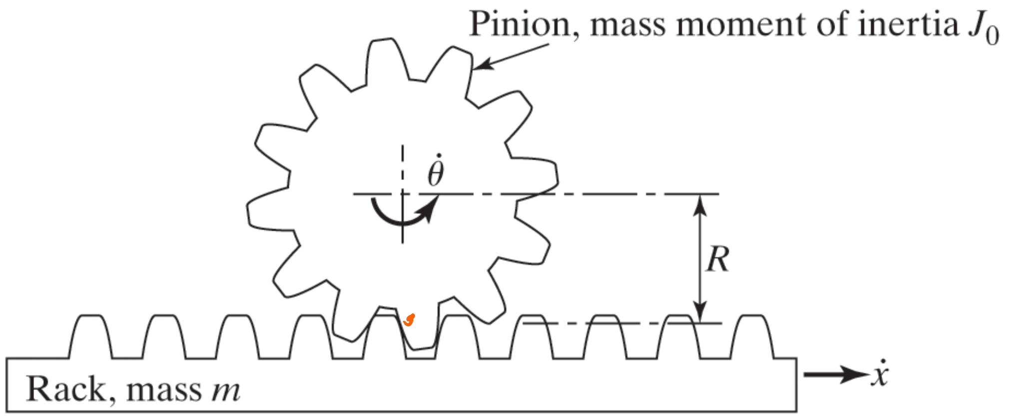
$$T = \frac{1}{2}\left[m_1 + m_2\left(\frac{l_2}{l_1}\right)^2 + m_3\left(\frac{l_3}{l_1}\right)^2\right](\dot{x}_1)^2$$



$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1 \qquad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1$$

$$m_{eq} = m_1 + m_2\left(\frac{l_2}{l_1}\right)^2 + m_3\left(\frac{l_3}{l_1}\right)^2$$

Example 2: Translational and rotational masses coupled together



Since the system is rotating without slipping and the gear is fix at its center $\dot{\theta} = \dot{x}/R$

$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}J_o(\dot{\theta})^2$$

- If we choose as the parameter that will describe the motion of the system $\dot{x}_{eq} = \dot{x}$

$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}J_o\left(\frac{\dot{x}}{R}\right)^2$$

$$T = \frac{1}{2}\left[m + \frac{J_o}{R^2}\right](\dot{x})^2$$

$$m_{eq} = m + \frac{J_o}{R^2}$$

- If we choose as the parameter that will describe the motion of the system $\dot{x}_{eq} = \dot{\theta}$

$$T = \frac{1}{2}m(\dot{x})^2 + \frac{1}{2}J_o(\dot{\theta})^2$$

$$T = \frac{1}{2}m(\dot{\theta}R)^2 + \frac{1}{2}J_o(\dot{\theta})^2$$

$$T = \frac{1}{2}[mR^2 + J_o](\dot{\theta})^2$$

$$m_{eq} = mR^2 + J_o$$

Damping element

- In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. Damping force exists only if there is relative velocity between the two ends of the damper.
- We will consider three types of damping:
  - Viscous Damping: the damping force is proportional to the velocity of the vibrating body.
  - Coulomb or Dry-Friction Damping: damping force is constant in magnitude but opposite in direction to that of the motion
  - Material or Solid or Hysteretic Damping: due to friction between the internal planes, which slip or slide as the deformations take place

In general viscous Dampers

Non-linear damper element:

$$F = cv$$

Non-linear damper element:

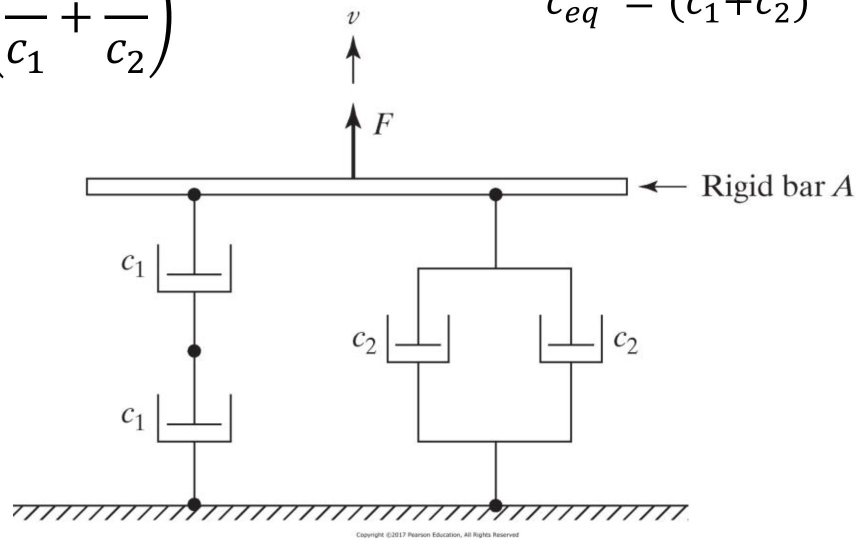
$$c = \left.\frac{dF}{d\dot{x}}\right|_{x^*}$$

Dampers in Series

$$\frac{1}{c_{eq}} = \left(\frac{1}{c_1} + \frac{1}{c_2}\right)$$

Dampers in Parallel

$$c_{eq} = (c_1 + c_2)$$



Damping constant of parallel plates separated by viscous fluid.

- According to Newton's laws of viscous flow:

$$\tau = \mu \frac{du}{dy}$$

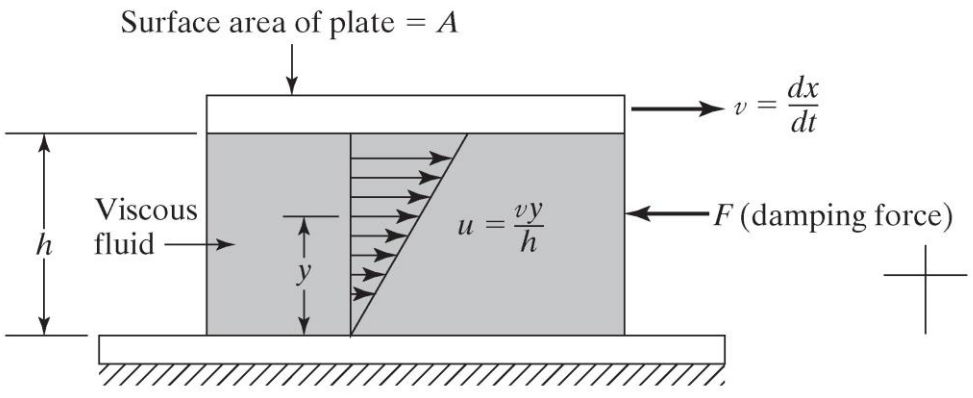
- Gradient of the velocity: $$\frac{du}{dy} = \frac{v}{h}$$

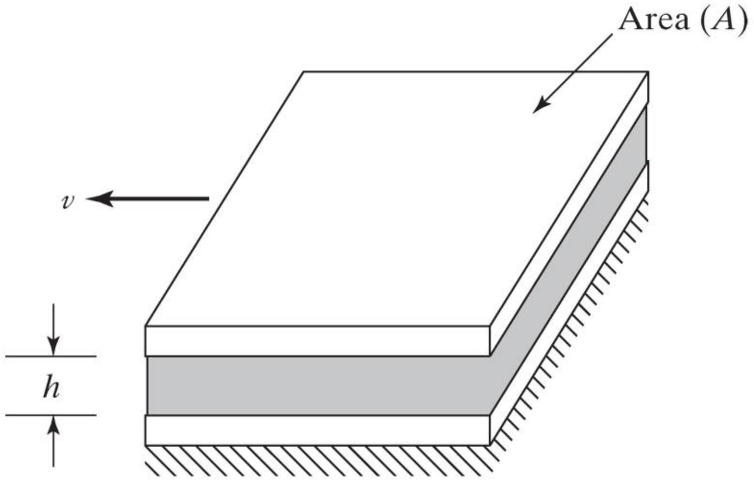
- If A is the surface area at the moving plate, and expressing the force in term of the damping constant:

$$F = \tau A = \frac{\mu A v}{h}$$

$$F = cv = \frac{\mu A}{h} v$$

$$\boxed{c = \frac{\mu A}{h}}$$



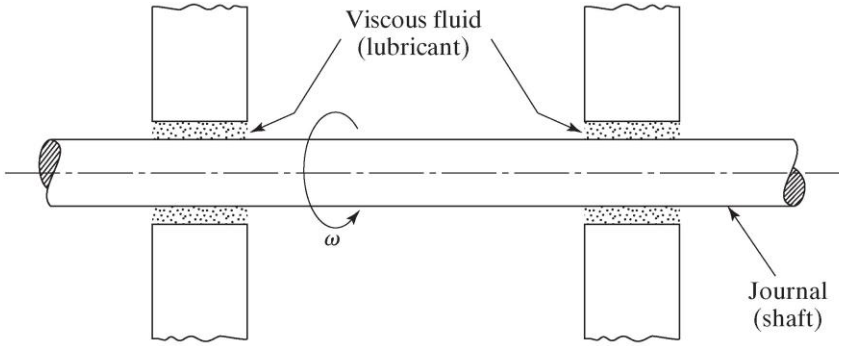


Damping constant of a journal bearing

- According to Newton's laws of viscous flow, u is the radial velocity, and v the tangential velocity of the shaft:

$$\tau = \mu \frac{du}{dr}$$

- Gradient of the velocity: $\dfrac{du}{dr} = \dfrac{v}{d} = \dfrac{\omega R}{d}$



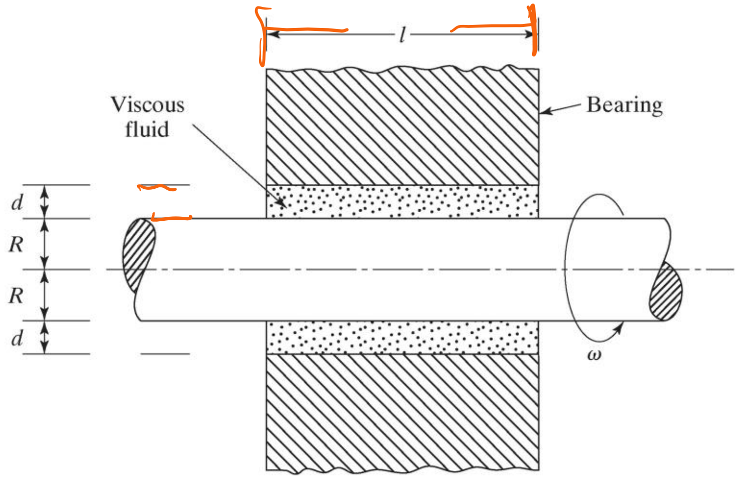
(a)

If A is the surface area at the moving shaft $(2\pi R l)$, and expressing the torque T=$F\,R$ in term of the damping constant:

$$T = (\tau A)R = \frac{\mu(\omega R)(2\pi R l)R}{d}$$

$$T = c\omega$$

$$\boxed{c = \frac{\mu 2\pi R^3 l}{d}}$$



(b)

Mechanical Vibrations

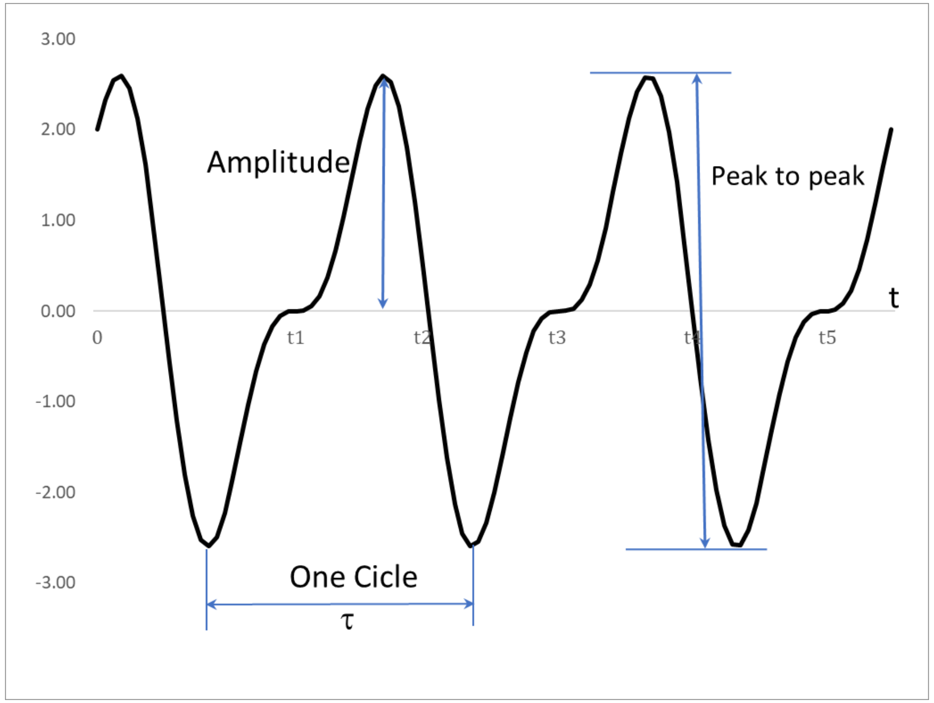
Harmonic Motion

Prof. Carmen Muller-Karger, PhD
Florida International University

Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 1: Fundamentals of Vibration

Harmonic Motion or Periodic Motion, Definitions and Terminology

Harmonic motion: Motion is repeated after equal intervals of time



Cycle: The movement from one position, going to the other direction and returning to the same position

Amplitude: The maximum displacement of a vibrating body from its equilibrium position. A

Period of oscillation: The time taken to complete one cycle of motion, is denoted by τ.

$$\tau = \frac{2\pi}{\omega}$$

Frequency of oscillation: The number of cycles per unit time

$$f = \frac{1}{\tau} = \frac{\omega}{2\pi}$$

Circular frequency of oscillation: The number of cycles per unit time.

$$\omega = \frac{2\pi}{\tau}$$

Harmonic Motion or Periodic Motion, Definitions and Terminology

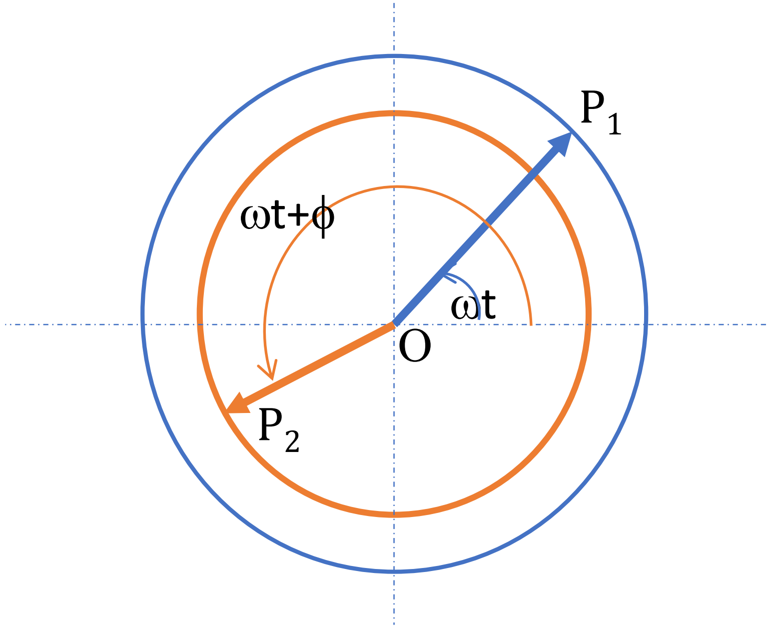
Synchronous motion : have the same frequency or angular velocity, ω. Need not have the same amplitude, and they need not attain their maximum values at the same time.

Phase angle: means that the maximum of the second vector would occur ϕ radians earlier than that of the first vector.

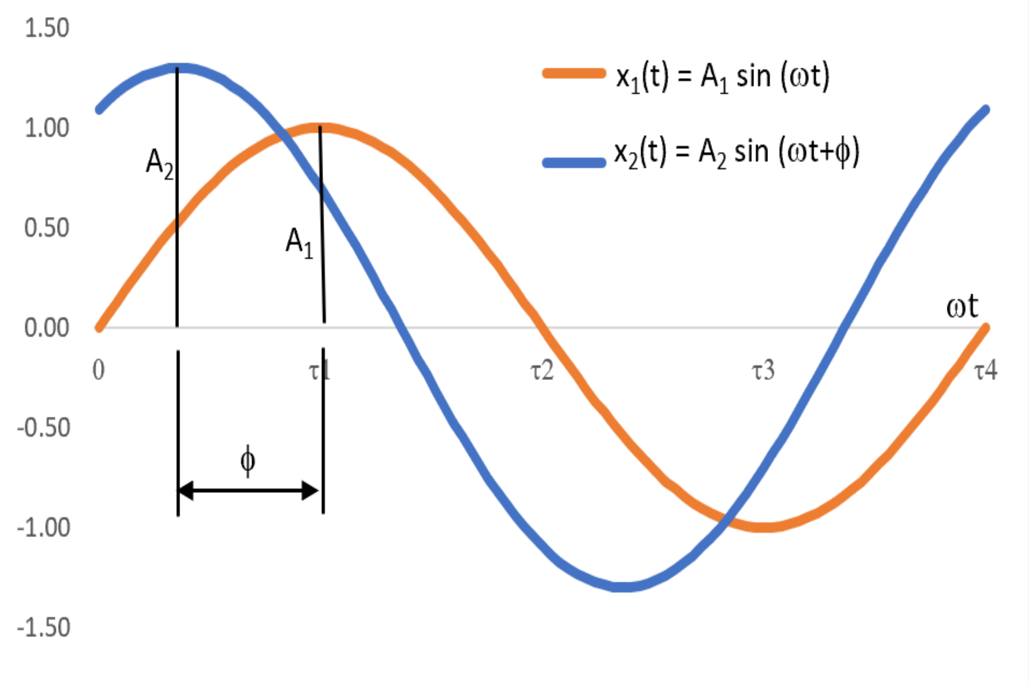
Displacement: $x(t) = \sin(\omega t)$

Velocity: $\dot{x}(t) = -\omega cos(\omega t)$

Acceleration: $\ddot{x}(t) = -\omega^2 \sin(\omega t)$



Vectorial representation



Complex-number representation

- Any vector in the *xy*-plane can be represented as a complex number:

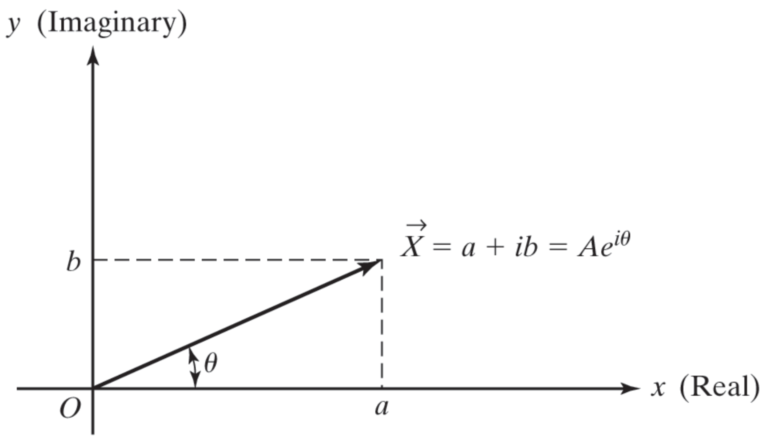
$$\bar{X} = \mathrm{a} + \mathrm{ib} = A[\cos(\omega t) + i \sin(\omega t)]$$

- The magnitude $\mathrm{A} = \sqrt{a^2 + b^2}$
- Using Euler form

$$\bar{X} = \mathrm{A}e^{i\omega t} = \mathrm{A}e^{i\theta} = A[\cos(\omega t) + i \sin(\omega t)]$$

or

$$\bar{X} = \mathrm{A}e^{-i\omega t} = \mathrm{A}e^{-i\theta} = A[\cos(\omega t) - i \sin(\omega t)]$$



Displacement: $\bar{X} = \mathrm{A}e^{i\omega t}$

Velocity: $\dot{\bar{X}} = \dfrac{d\bar{X}}{dt} = \dfrac{d(\mathrm{A}e^{i\omega t})}{dt} = i\omega \mathrm{A}e^{i\omega t}$

Acceleration: $\ddot{\bar{X}} = \dfrac{d^2\bar{X}}{dt^2} = \dfrac{d^2(\mathrm{A}e^{i\omega t})}{dt^2} = -\omega^2 \mathrm{A}e^{i\omega t}$

Expansion by series

$$\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots$$

$$\sin(\theta) = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots$$

For very small angles:

$\cos(\theta) \approx 1$

$\sin(\theta) \approx \theta$

Adding harmonic motion

- Recall trigonometry identities:

$$\sin(a+b) = \sin(a)\, cos(b) + \cos(a)\, sin(b)$$

$$\cos(a+b) = cos(a)\, cos(b) - \sin(a)\, sin(b)$$

- Adding harmonic motions:

$$x_1(t) = A_1 \cos(\omega t)$$

$$x_2\,(t) = A_2\, sin(\omega t)$$

$$x_t(t) = x_1(t) + x_2(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t) = Acos\,(\omega t - \alpha)$$

$$x_t(t) = Acos\,(\omega t - \alpha) = \mathrm{Acos}(\alpha)\cos(\omega t) + \mathrm{Asin}(\alpha)\sin(\omega t)$$

$$A_1 = \mathrm{Acos}(\alpha)$$

$$A_2 = \mathrm{Asin}(\alpha)$$

$$A = \sqrt{A_1^2 + A_2^2} = \sqrt{(\mathrm{Acos}(\alpha))^2 + (\mathrm{Asin}(\alpha))^2}$$

$$\alpha = tan^{-1}\left(\frac{A_2}{A_1}\right)$$

- Two different ways of write a harmonic motion:

$$\boldsymbol{x_t(t) = A_1 \cos(\omega t) + A_2 \sin(\omega t)}$$

or

$$\boldsymbol{x_t(t) = Acos\,(\omega t - \alpha)}$$

$$\boldsymbol{A = \sqrt{A_1^2 + A_2^2}}$$

$$\boldsymbol{\alpha = tan^{-1}\left(\frac{A_2}{A_1}\right)}$$

Adding harmonic motion

- Recall trigonometry identities:

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$$

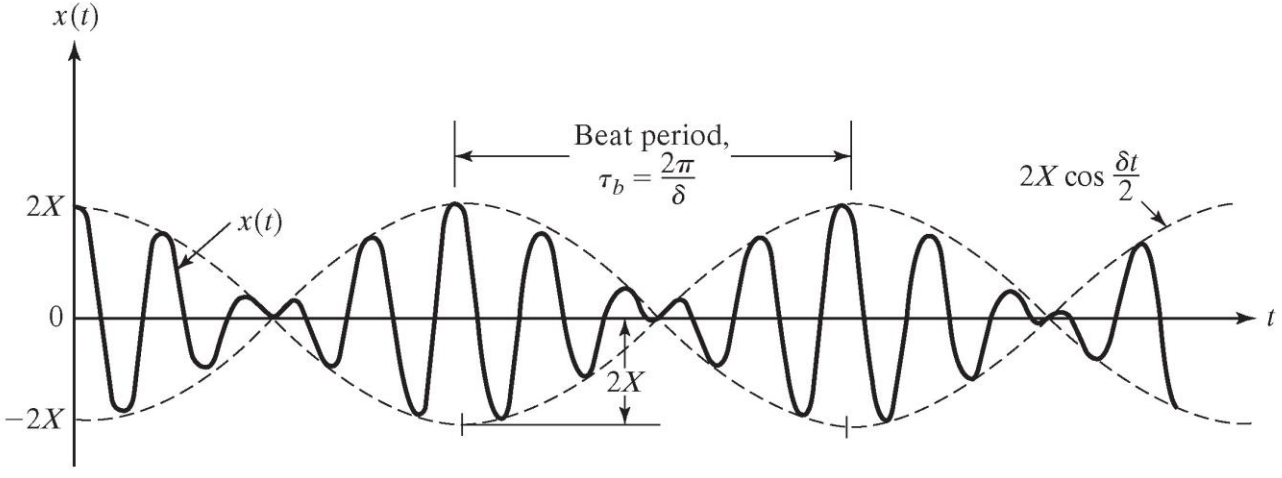
$$\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$$

Phenomenon Beats: occurs when adding two harmonic motion with frequencies close too one another the resultant motion

$$x_1(t) = X\cos(\omega)t \qquad x_2(t) = X\cos(\omega+\delta)t$$

$$x(t) = X\cos(\omega)t + X\cos(\omega+\delta)t$$

$$x(t) = X\cos\left(\frac{\delta}{2}\right)t\ \cos\left(\omega+\frac{\delta}{2}\right)t$$



Beats frequency is twice the frequency of the term $X\cos\left(\frac{\delta}{2}\right)t$ since two peaks pass in each cycle.

Harmonic Analysis: Fourier Series

Any periodic function of time can be represented by Fourier series as an infinite sum of sine and cosine terms

$$x(t) = \frac{a_0}{2} + a_1\, cos(\omega t) + a_2\, cos(2\omega t) + \cdots b_1\, sin(\omega t) + b_2\, sin(2\omega t) + \cdots$$

$$\boxed{x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\, cos(n\omega t) + b_n\, sin(n\omega t)\right)}$$

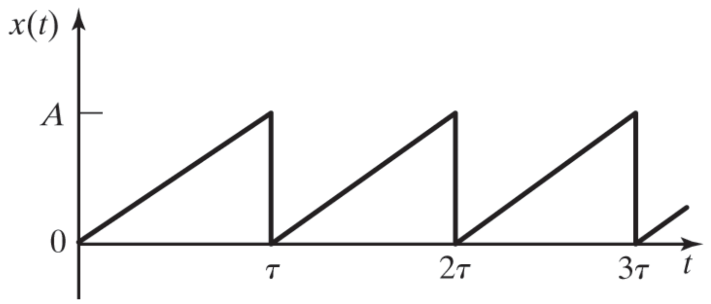
where ω=2π/τ is the fundamental frequency.

To determine the coefficients , we multiply by *cos(nωt)* and sin(nωt), respectively, and integrate over one period τ=2π/ω—for example, from 0 to 2π/ω.

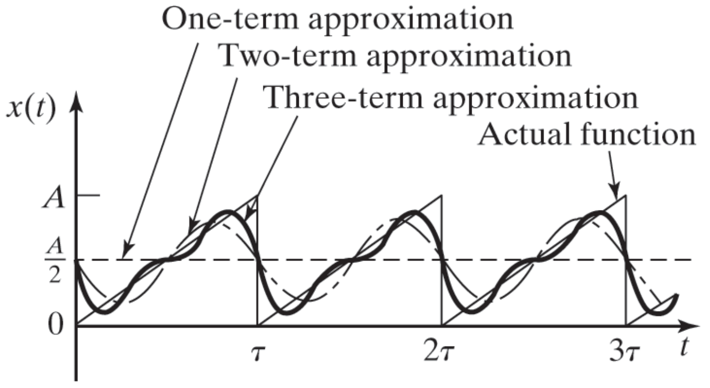
$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)dt = \frac{2}{\tau}\int_0^{\tau} x(t)dt$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\, cos(n\omega t)\, dt = \frac{2}{\tau}\int_0^{\tau} x(t)\, cos(n\omega t)\, dt$$

$$b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)\, sin(n\omega t)\, dt = \frac{2}{\tau}\int_0^{\tau} x(t)\, sin(n\omega t)\, dt$$



(a)



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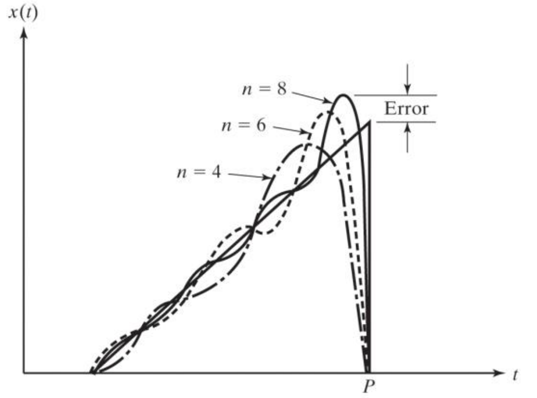


Harmonic Analysis: Fourier Series

Fourier series can also be represented by the sum of sine terms only or cosine terms only.

$$x(t) = d_0 + d_1\, cos(\omega t - \varphi_1) + d_2\, cos(2\omega t - \varphi_1) + \cdots$$

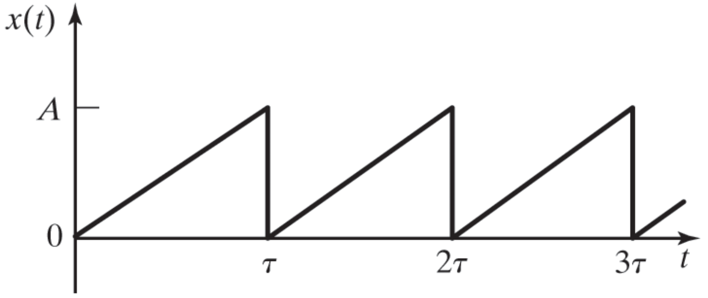
$$\boxed{x(t) = d_0 + \sum_{n=1}^{\infty} (d_n\, cos(n\omega t - \varphi_n))}$$

Where:

$$d_0 = \frac{a_0}{2}$$

$$d_n = \sqrt{a_n^2 + b_n^2}$$

$$\varphi_n = tan^{-1}\left(\frac{b_n}{a_n}\right)$$



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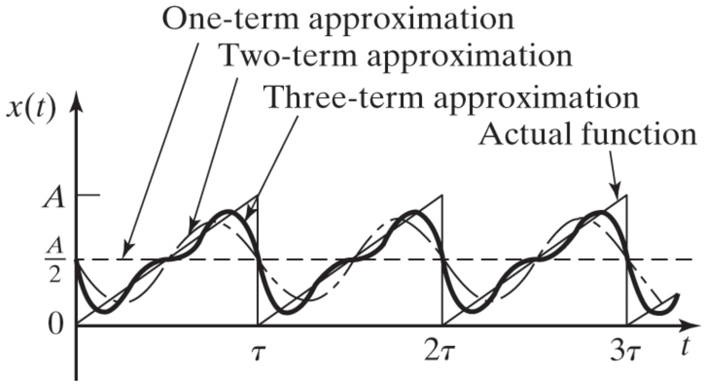

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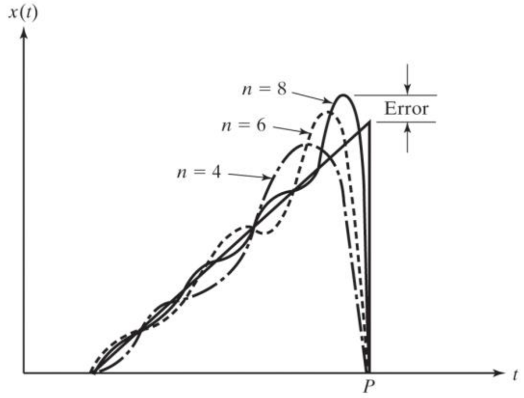

Fourier Series in complex numbers

Since:

$$e^{i\omega t} = \cos(\omega t) + i sin\,(\omega t)$$

$$e^{-i\omega t} = \cos(\omega t) - i sin\,(\omega t)$$

$$\Rightarrow$$

$$\cos(\omega t) = \frac{e^{i\omega t}+e^{-i\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{i\omega t}-e^{-i\omega t}}{2}$$

The Fourier Series can be written as :

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\left(\frac{e^{in\omega t}+e^{-in\omega t}}{2}\right) + b_n\left(\frac{e^{in\omega t}-e^{-in\omega t}}{2}\right)\right) = e^{i\omega 0}\left(\frac{a_0}{2} - i\frac{b_0}{2}\right) + \sum_{n=1}^{\infty}\left(e^{in\omega t}\left(\frac{a_n - ib_n}{2}\right) + e^{-i\omega t}\left(\frac{a_n + ib_n}{2}\right)\right)$$

With:

$$c_n = \frac{a_n - ib_n}{2} \qquad c_{-n} = \frac{a_n + ib_n}{2} \qquad b_0 = 0$$

The Fourier Series can be written in a very compact form :

$$\boxed{x(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$

with $\quad c_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} x(t)[cos(n\omega t) - sin(n\omega t)]dt = \frac{1}{\tau}\int_0^{\tau} x(t)e^{-in\omega t}dt$

Time and frequency domain representations

- The Fourier series expansion permits the description of any periodic function using either a time-domain or a frequency-domain representation.

- Note that the amplitudes d_n and the phase angles φ_n corresponding to the frequencies ω_n can be used in place of the amplitudes a_n and b_n for representation in the frequency domain.