2.53 An electric motor (Figure P2.53) has an eccentric mass of 10 kg (10% of the total mass) and is set on two identical springs ($k = 3200$ /m). The motor runs at 1750 rpm, and the mass eccentricity is 100 mm from the center. The springs are mounted 250 mm apart with the motor shaft in the center. Neglect damping and determine the amplitude of the vertical vibration.

Solution:
Given $m_0 = 10$ kg, $m = 100$ kg, $k = 2\text{x}3.2$ N/mm, ; $e = 0.1$ m

$$\omega_r = 1750\frac{\text{rev}}{\text{min}}(\frac{\text{min}}{60\,\text{sec}}\frac{2\pi\ \text{rad}}{\text{rev}}) = 183.26\frac{\text{rad}}{\text{s}}\ \text{rad/s}$$

Vertical vibration:

$$\omega_n = \sqrt{\frac{2(3.2)(1000)}{100}} = 8\ \text{rad/s}$$

$$r = \frac{\omega_r}{\omega_n} = \frac{183.3}{8} = 22.9$$

From equation (2.84)

$$X = e\frac{m_0}{m}\frac{r^2}{|1-r^2|} == 0.01\ \text{m}$$

2.54 Consider a system with rotating unbalance as illustrated in Figure P2.53. Suppose the deflection at 1750 rpm is measured to be 0.05 m and the damping ratio is measured to be $\zeta = 0.1$. The out-of-balance mass is estimated to be 10%. Locate the unbalanced mass by computing *e*.

Solution: Given: $X = 0.05$ m, $\zeta = 0.1$, $m_e = 0.1m$, and from the solution to problem 2.53 the frequency ratio is calculated to be $r = 22.9$. Solving the rotating unbalance Equation (2.84) for *e* yields:

$$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow e = \frac{mX}{m_0}\frac{\sqrt{(1-r^2)^2+(2\zeta r)^2}}{r^2} = \underline{0.499\ \text{m}}$$

This sort of calculation can be introduced to discuss the application of machinery diagnostics if time permits. Machinery diagnostics deals with determining the location and extend of damage from measurements of the response and input.

Rotating Unbalance

2.55 A fan of 45 kg has an unbalance that creates a harmonic force. A spring-damper system is designed to minimize the force transmitted to the base of the fan. A damper is used having a damping ratio of $\zeta = 0.2$. Calculate the required spring stiffness so that only 10% of the force is transmitted to the ground when the fan is running at 10,000 rpm. T Review

Solution: The equation of motion of the fan is

$$m\ddot{x} + c\dot{x} + kx = m_0 e\omega^2 \sin\cancel{(\omega t + \theta)}\ \sin\omega_r t$$

The steady state solution as given by equation (2.84) is

$$x(t) = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \psi)$$

where r is the standard frequency ratio. The force transmitted to the ground is

$$F(t) = kx + c\dot{x} = \frac{m_0 e}{m}\frac{kr^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin\omega t + \frac{m_0 e}{m}\frac{c\omega r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos\omega t$$

Taking the magnitude of this quantity, the magnitude of the force transmitted becomes

$$F_{tr,max,RU} = \frac{m_0 e}{m}\frac{r^2\sqrt{k^2+c^2\omega^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = m_0 e\omega_f^2\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

From equation (2.81) the magnitude of the force generated by the rotating mass F_r is

$$F_r = m_0 e\omega^2$$

$F_{trans,max,RU}$ must be $0.1\ m_0 e\omega_f^2$ from prob. statement

The limitation stated in the problem is that $F = 0.1F_r$, or

$$m_0 e\omega^2\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 0.1 m_0 e\omega^2$$

Setting $\zeta = 0.2$ and solving for r yields:

$$r^4 - 17.84r^2 - 99 = 0$$

which yields only one positive solution for r^2, which is

$$r^2 = 22.28 = \frac{\omega^2}{k/m} \Rightarrow \frac{k}{m} = \left(\frac{10000\times 2\pi}{60}\right)^2\frac{1}{22.28}$$

$$\Rightarrow k = 45\left(\frac{10000\times 2\pi}{60}\right)^2\frac{1}{22.28} = 2.21\times 10^6\ \text{N/m}$$

$$1+(2\zeta r)^2 = .01\left[(1-r^2)^2+(2\zeta r)^2\right]$$

$$\frac{(1+4\zeta^2 r^2)}{.01} = 100(1+4\zeta^2 r^2) = 1 - 2r^2 + r^4 + 4\zeta^2 r^2$$

$$0 = r^4 + \left[-2 - 396\zeta^2\right]r^2 - 99$$

$$0 = r^4 - 17.84r^2 - 99$$

$$r^2 = \frac{17.84 \pm \sqrt{(17.84)^2 - 4(1)(-99)}}{2} = \frac{17.84 \pm 26.73}{2}$$

$$= 22.28 \text{ or } -4.445$$

not possible

$$\therefore r = \sqrt{22.28} = 4.72 = \frac{\omega_f}{\omega_n}$$

$$\omega_f = \frac{2\pi}{60}f_t = \frac{2\pi}{60}\times 10{,}000\ \text{rpm} = 1047.2\ \text{rad/s}$$

$$\therefore \omega_n = \omega_f/r = 221.86\ \text{rad/s}$$

$$k = m\omega_n^2 = 2.215\times 10^6\ \text{N/m}$$

The time t', for which $x = 0$ when the curve crosses the axis, is given by

$$t' = \frac{\ln(-C_2/C_1)}{\beta - \alpha} = \frac{\ln(-C_2/C_1)}{2\omega\sqrt{\zeta^2 - 1}} \tag{3-21}$$

The time t^*, at which the maximum point on the curve occurs, obtained by setting $\dot{x} = 0$, is defined by $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$

$$t^* = \frac{\ln(-s_2 C_2/s_1 C_1)}{-(s_2 - s_1)} = \frac{\ln(-s_2 C_2/s_1 C_1)}{2\omega\sqrt{\zeta^2 - 1}} \tag{3-22}$$

$s_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$

$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$

$\zeta > 1$

EXAMPLE 3-1 A weight of 9 lb is supported by a spring having a constant of 2 lb/in. and a dashpot having a damping constant of 1.3 lb sec/in. The mass has an initial displacement of 4 in. and an initial velocity of zero. Express the relation that governs the motion of the mass.

$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$

SOLUTION

$K = 2\frac{lb}{in} \times 12\frac{in}{ft} = 24\frac{lb}{ft}$

$c = 1.3\frac{lb\text{-}s}{in} \times \frac{12in}{ft} = 15.6\frac{lb\text{-}s}{ft}$

$m = \frac{W}{g} = \frac{9}{32.2} = .2795\,slug$

$g = 32.2\frac{ft}{s^2} = 386\frac{in}{s^2}$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2 \times 386}{9}} = 9.26 \text{ rad/sec}$$

$$c_c = 2m\omega_n = 2 \times \frac{9}{386} \times 9.26 = 0.432 \text{ lb sec/in.} \quad \left(5.176\,\frac{lb\text{-}s}{ft}\right)$$

$$\zeta = \frac{c}{c_c} = \frac{1.3}{0.432} = 3.01 = \frac{15.6}{5.176}$$

$$\sqrt{\zeta^2 - 1} = 2.84$$

$$s_1 = -(\zeta - \sqrt{\zeta^2 - 1})\omega_n = -(3.01 - 2.84) \times 9.26 = -1.57$$

$$s_2 = -(\zeta + \sqrt{\zeta^2 - 1})\omega_n = -(3.01 + 2.84) \times 9.26 = -54.2$$

From Eq. 3-20,

$$C_1 + C_2 = 4 \qquad x(t{=}0) = x_0$$

$$1.57C_1 + 54.2C_2 = 0 \qquad \dot{x}(t{=}0) = v_0$$

Solving simultaneously yields

$$C_1 = 4.12$$

$$C_2 = -0.119$$

Then the motion relation becomes

$$x = 4.12e^{-1.57t} - 0.119e^{-54.2t}$$

3-6. CRITICALLY DAMPED MOTION

For the case of critical damping, corresponding to $\zeta = 1$, the motion is governed by Eq. 3-17. This relation is a product of the linear function $(A + Bt)$ and the decaying exponential $e^{-\omega t}$. Separate curves for these parts

(10)

The values of constants A and B can be determined from these. The time t', for which $x = 0$ at the crossing point, is defined by

$$t' = -\frac{C_1}{C_2} \tag{3-24}$$

The time t^*, for which x maximum occurs, obtained by setting $\dot{x} = 0$, is given by

$$t^* = \frac{1}{\omega} - \frac{C_1}{C_2} \tag{3-25}$$

$\zeta = 1$

EXAMPLE 3-2 A mass of 20 kg is supported by an elastic member having a modulus of 320 N/m, and the system is critically damped. For initial conditions of zero displacement and velocity of 3 m/s, determine the maximum displacement and the displacement at $t = 0.5$, 1, and 2 sec.

SOLUTION Based on the initial conditions, the constants C_1 and C_2 are determined as

$$C_1 = 0 \qquad C_2 = \dot{x}_0$$

$(C_1 + C_2 t)e^{-\omega_n t} = x(t)$

$[C_2 + (C_1 + C_2 t)(-\omega_n)]e^{-\omega_n t} = \dot{x}(t)$

and the solution (Eq. 3-17) becomes

$$x = \dot{x}_0 t e^{-\omega t}$$

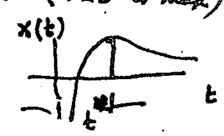
with

$$\dot{x}_0 = 3 \text{ m/s}$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{320}{20}} = 4 \text{ rad/s}$$

The time at which the maximum displacement occurs is (v=0 at max)

$$t^* = \frac{1}{\omega} - \frac{C_1}{C_2} = \frac{1}{\omega} = \frac{1}{4} = 0.25 \text{ s}$$



and

$$x_{max} = \dot{x}_0 t^* e^{-\omega t^*} = 3 \times 0.25 e^{-4 \times 0.25} = 0.2759 \text{ m} = 27.59 \text{ cm}$$

$$x_{t=0.5} = 3 \times 0.5 e^{-4 \times 0.5} = 0.2030 \text{ m} = 20.30 \text{ cm}$$

$$x_{t=1} = 3 \times 1 e^{-4 \times 1} = 0.054\,95 \text{ m} = 5.495 \text{ cm}$$

$$x_{t=2} = 3 \times 2 e^{-4 \times 2} = 0.002\,013 \text{ m} = 0.2013 \text{ cm}$$

3-7. MOTION FOR BELOW-CRITICAL DAMPING

Damping corresponding to $\zeta < 1$ is referred to as subcritical or below-critical damping, and the system is said to be *underdamped*. For this condition, the solution is specified by Eq. 3-15. The motion is of harmonic form, 11

Examination of Eqs. 3-32 through 3-34 reveals that in every instance the time between an $x = 0$ point and the next x_{max} point is less than the time between this x_{max} point and the following $x = 0$ point. Similarly, the time between an $x = 0$ point and the next x_{min} point is less than the time between this x_{min} point and the following $x = 0$ point. The described unequal time intervals occur because the damping aids the spring in stopping the outward motion but opposes the spring during the inward motion.

The foregoing is made clearer by the curve for Eq. 3-26, shown in Fig. 3-6. It should be noted that the period τ, as defined by the time between successive points of the same kind (for example, maximum points), is given by

$$\omega_d \tau = 2\pi$$

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega} \qquad (3\text{-}35)$$

From Eqs. 3-30 and 3-26 the curve which goes through the maximum points is found to have the equation

$$x = \sqrt{1-\zeta^2}\,Xe^{-\zeta\omega t} \qquad (3\text{-}36)$$

Likewise, by substituting Eq. 3-31 into Eq. 3-26, the curve through the minimum points is expressed by

$$x = -\sqrt{1-\zeta^2}\,Xe^{-\zeta\omega t} \qquad (3\text{-}37)$$

$\zeta < 1$

EXAMPLE 3-3 A mass of 3.174 kg is supported by a spring having a modulus of 700 N/m and a dashpot having a damping constant of 14.18 N · s/m. Write the equation that governs the motion of the mass.

SOLUTION

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{700}{3.174}} = 14.85 \text{ rad/sec}$$

$$c_c = 2m\omega = 2 \times 3.174 \times 14.85 = 94.27 \text{ N} \cdot \text{s/m}$$

$$\zeta = \frac{14.18}{94.27} = 0.15$$

$$\sqrt{1-\zeta^2} = \sqrt{1-(0.15)^2} = 0.989$$

$$\omega_d = 0.989 \times 14.85 = 14.7 = \omega_n\sqrt{1-\zeta^2} \text{ rad/s}$$

$$\zeta\omega_n = 0.15 \times 14.85 = 2.23 \text{ rad/s}$$

Then the equation of motion is

$$x = Xe^{-2.23t} \sin(14.7t + \phi)$$

log decrement $\delta = 2\pi\zeta$ (since $\zeta < .3$)

$\cong .942 = \ln\left(\frac{x_0}{x_1}\right)$ or $\ln\left(\frac{x_1}{x_2}\right)$ where x_0, x_1, x_2 are max. of 3 consecutive cycles