More on BCs

When we look at heat type problems boundary conditions on heat flow are of the Neuman type since $q$, the heat is related to the normal derivative to the boundary (see Ch 1 of Trim).

Look at exercise 1.2 problem 2 in Trim.
If this were a 3-D problem, then the governing equation would be given by

$$ \frac{\partial^2 T}{\partial x^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} = \frac{\alpha}{r} \frac{\partial T}{\partial t} $$

Where the heat flow rate in the radial direction $\frac{\partial T}{\partial r}=0$ means that the radial surface is insulated ($q=0$). The last three terms on the left hand side of the equation are the laplacian written in $r$ and $\theta$.

However, the problem statement for problem 2 only asks to solve the problem as a 1-D heat conduction problem so
And the temperature $T$ here is the averaged temperature over the cross-sectional area. Since we only have 2 derivatives in $x$ and one in $T$, we need two BCs in at $x=0$ and at $x=L$ and one IC namely $f(x)$.

In the case where the left-hand boundary at $x=0$ is insulated instead of having the temperature specified then the left-hand BS is replaced by $\partial T/\partial x = 0$. Supposing the RHS BC is changed from a constant to a ramp type condition

In the case where you have two different types of BCs, you may have to determine a particular (steady state) solution
We begin the separation of variables solution methodology

The equation on the bottom right is in error and it should be the Laplacian of $u(x,y) = 0$
Natural Freqs of Vibration.

If \( u(x,y,t) = \hat{X}(x) \hat{Y}(y) T(t) \)

S.S. Vibration of a Membrane

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0
\]

\[
u = u(x,y) = \hat{X}(x) \hat{Y}(y)
\]

\[
\hat{X}'' Y + \hat{Y}'' X = 0
\]

\[
\frac{\partial^2 u}{\partial x^2} = \hat{X}'' Y
\]

\[
\frac{\partial^2 u}{\partial y^2} = \hat{Y}'' X
\]
Application of SOV to determine derivatives with respect to $x$ and $y$

\[
\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0
\]

\[
\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = -z
\]

\[
\frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = -z
\]

\[
\frac{\partial^2 z}{\partial x^2} = -x^2 \Rightarrow \frac{\partial^2 z}{\partial x^2} + x^2 = 0
\]

\[
\frac{\partial^2 z}{\partial y^2} = x^2 \Rightarrow \frac{\partial^2 z}{\partial y^2} - x^2 = 0
\]
In the case of a simply supported membrane where \( u = 0 \) all around

These boundary conditions translate to
Since $u(x,y)=X(x)Y(y)$ then these boundary conditions translate to

Since $X(0)Y(y)=0$ for all $y$, then that can only happen when $X(0)=0$. 
Since $\lambda$ depends on $n$, then $X(x)$ also depends on $n$ and so must $u(x,y)$. Since there are an infinite number of solutions and this is a linear problem, then adding the individual solutions is also a solution to the problem.