Chapter 9

Reservoir and Stream Flow Routing

9.1 ROUTING

Figure 9.1.1 illustrates how stream flow increases as the *variable source area* extends into the drainage basin. The variable source area is the area of the watershed that is actually contributing flow to the stream at any point. The variable source area expands during rainfall and contracts thereafter.

Flow routing is the procedure to determine the time and magnitude of flow (i.e., the flow hydrograph) at a point on a watercourse from known or assumed hydrographs at one or more points upstream. If the flow is a flood, the procedure is specifically known as flood routing. Routing by lumped system methods is called *hydrologic (lumped) routing*, and routing by distributed systems methods is called *hydraulic (distributed) routing*.

For hydrologic routing, input $I(t)$, output $Q(t)$, and storage $S(t)$ as functions of time are related by the continuity equation (3.3.10)

$$\frac{dS}{dt} = I(t) - Q(t) \tag{9.1.1}$$

Even if an inflow hydrograph $I(t)$ is known, equation (9.1.1) cannot be solved directly to obtain the outflow hydrograph $Q(t)$, because, both Q and S are unknown. A second relationship, or storage function, is required to relate S, I, and Q; coupling the storage function with the continuity equations provides a solvable combination of two equations and two unknowns.

The specific form of the storage function depends on the nature of the system being analyzed. In reservoir routing by the level pool method (Section 9.2), storage is a nonlinear function of Q, $S = f(Q)$, and the function $f(Q)$ is determined by relating reservoir storage and outflow to reservoir water level. In the Muskingum method (Section 9.3) for flow routing in channels, storage is linearly related to I and Q.

The effect of storage is to redistribute the hydrograph by shifting the centroid of the inflow hydrograph to the position of the outflow hydrograph in a *time of redistribution*. In very long channels, the entire flood wave also travels a considerable distance, and the centroid of its hydrograph may then be shifted by a time period longer than the time of redistribution. This additional time may be considered the *time of translation*. The total time of flood movement between the centroids of the inflow and outflow hydrographs is equal to the sum of the time of redistribution and the time of translation. The process of redistribution modifies the shape of the hydrograph, while translation changes its position.

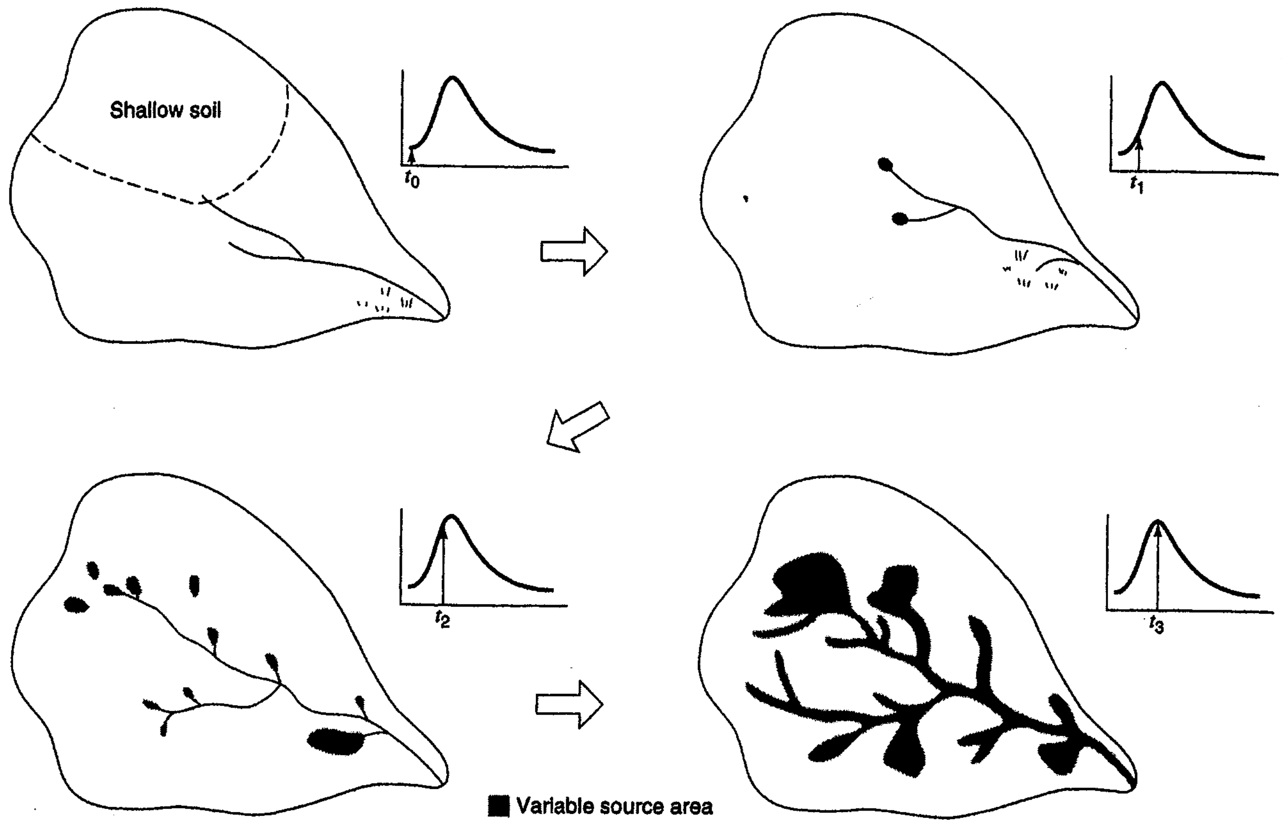


Figure 9.1.1 The small arrows in the hydrographs show how streamflow increases as the variable source extends into swamps, shallow soils, and ephemeral channels. The process reverses as streamflow declines (from Hewlett (1982)).

9.2 HYDROLOGIC RESERVOIR ROUTING

Level pool routing is a procedure for calculating the outflow hydrograph from a reservoir assuming a horizontal water surface, given its inflow hydrograph and storage-outflow characteristics. Equation (9.1.1) can be expressed in the infinite-difference form to express the change in storage over a time interval (see Figure 9.2.1) as

$$S_{j+1} - S_j = \frac{I_j + I_{j+1}}{2}\Delta t - \frac{Q_j + Q_{j+1}}{2}\Delta t \tag{9.2.1}$$

The inflow values at the beginning and end of the jth time interval are I_j and I_{j+1}, respectively, and the corresponding values of the outflow are Q_j and Q_{j+1}. The values of I_j and I_{j+1} are prespecified. The values of Q_j and S_j are known at the jth time interval from calculations for the previous time interval. Hence, equation (9.2.1) contains two unknowns, Q_{j+1} and S_{j+1}, which are isolated by multiplying (9.2.1) through by $2/\Delta t$, and rearranging the result to produce:

$$\left[\frac{2S_{j+1}}{\Delta t} + Q_{j+1}\right] = (I_j + I_{j+1}) + \left[\frac{2S_j}{\Delta t} - Q_j\right] \tag{9.2.2}$$

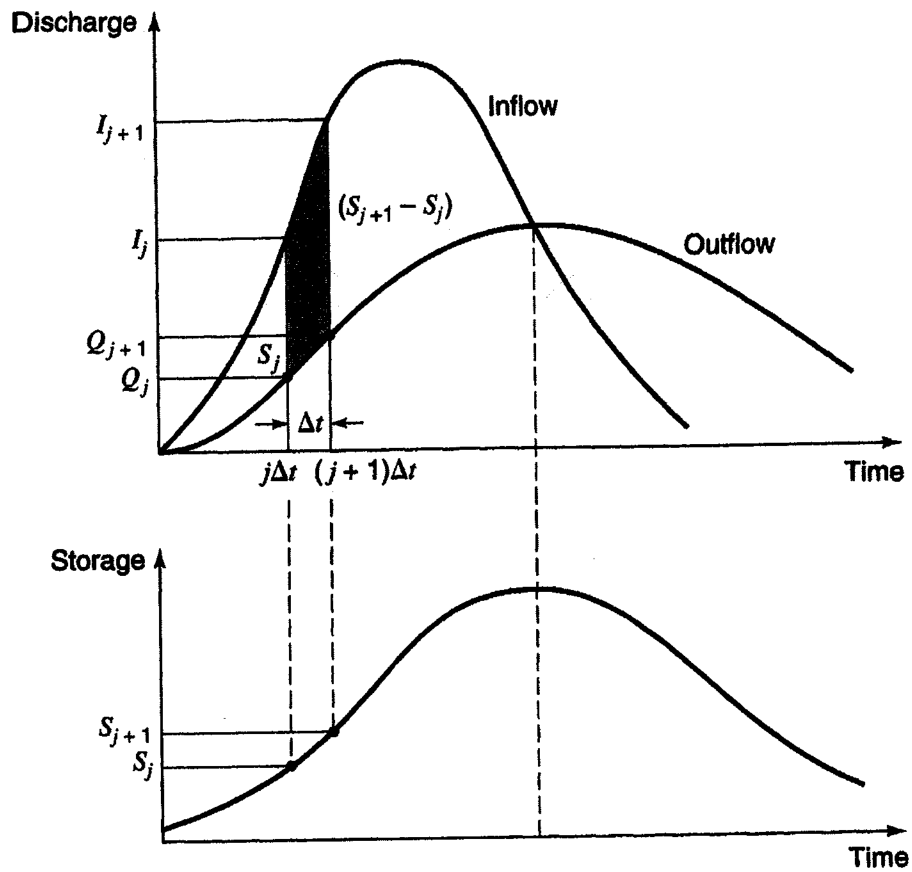


Figure 9.2.1 Change of storage during a routing period Δt.

In order to calculate the outflow Q_{j+1}, a storage-outflow function relating $2S/\Delta t + Q$ and Q is needed. The method for developing this function using elevation-storage and elevation-outflow relationships is shown in Figure 9.2.2. The relationship between water surface elevation and reservoir storage can be derived by planimetering topographic maps or from field surveys. The elevation-discharge relation is derived from hydraulic equations relating head and discharge for various types of spillways and outlet works. (See Chapter 17.) The value of Δt is taken as the time interval of the inflow hydrograph. For a given value of water surface elevation, the values of storage S and discharge Q are determined (parts (*a*) and (*b*) of Figure 9.2.2), and then the value of $2S/\Delta t + Q$ is calculated and plotted on the horizontal axis of a graph with the value of the outflow Q on the vertical axis (part (*c*) of Figure 9.2.2).

In routing the flow through time interval j, all terms on the right side of equation (9.2.2) are known, and so the value of $2S_{j+1}/\Delta t + Q_{j+1}$ can be computed. The corresponding value of Q_{j+1} can be determined from the storage-outflow function $2S/\Delta t + Q$ versus Q, either graphically or by linear interpolation of tabular values. To set up the data required for the next time interval, the value of $(2S_{j+1}/\Delta t - Q_{j+1})$ is calculated using

$$\left[\frac{2S_{j+1}}{\Delta t} - Q_{j+1}\right] = \left[\frac{2S_{j+1}}{\Delta t} + Q_{j+1}\right] - 2Q_{j+1} \tag{9.2.3}$$

The computation is then repeated for subsequent routing periods.

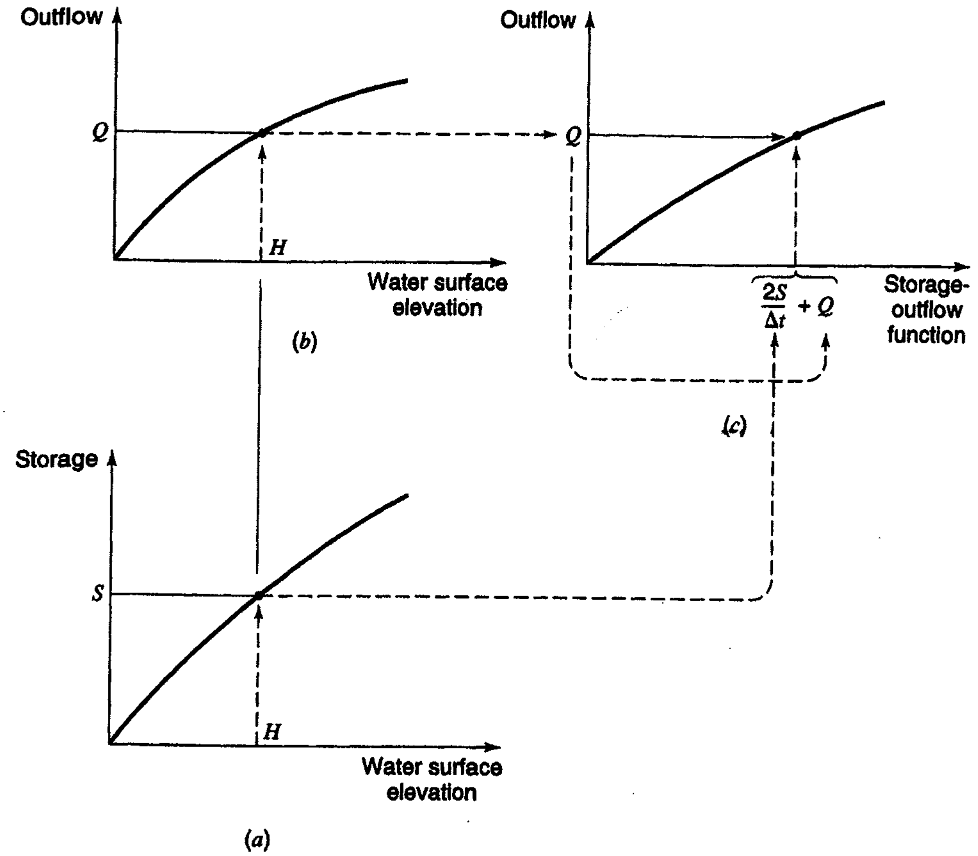


Figure 9.2.2 Development of the storage-outflow function for level pool routing on the basis of storage-elevation-outflow curves (from Chow et al. (1988)).

EXAMPLE 9.2.1

Consider a 2-acre stormwater detention basin with vertical walls. The triangular inflow hydrograph increases linearly from zero to a peak of 60 cfs at 60 min and then decreases linearly to a zero discharge at 180 min. Route the inflow hydrograph through the detention basin using the head-discharge relationship for the 5-ft diameter pipe spillway in columns (1) and (2) of Table 9.2.1. The pipe is located at the bottom of the basin. Assuming the basin is initially empty, use the level pool routing procedure with a 10-min time interval to determine the maximum depth in the detention basin.

SOLUTION

The inflow hydrograph and the head-discharge (columns (1) and (2)) and discharge-storage (columns (2) and (3)) relationships are used to determine the routing relationship in Table 9.2.1. A routing interval of 10 min is used to determine the routing relationship $2S/\Delta t + Q$ vs. Q, which is columns (2) and (4) in Table 9.2.1. The routing computations are presented in Table 9.2.2. These computations are carried out using equation (9.2.3). For the first time interval, $S_1 = Q_1 = 0$ because the reservoir is empty at $t = 0$; then $(2S_1/\Delta t - Q_1) = 0$. The value of the storage-outflow function at the end of the time interval is

$$\left[\frac{2S_2}{\Delta t} + Q_2\right] = (I_1 + I_2) + \left[\frac{2S_1}{\Delta t} - Q_1\right] = (0 + 10) + 0 = 10$$

The value of Q_2 is determined using linear interpolation, so that

$$Q_2 = 0 + \frac{(3-0)}{(148.2-0)}(10-0) = 0.2 \text{ cfs}$$

Table 9.2.1 Elevation-Discharge-Storage Data for Example 9.2.1

1 Head H (ft)	2 Discharge Q (cfs)	3 Storage S (ft^3)	4 $\frac{2S}{\Delta t}+Q$ (cfs)
0.0	0	0	0.00
0.5	3	43,500	148.20
1.0	8	87,120	298.40
1.5	17	130,680	452.60
2.0	30	174,240	610.80
2.5	43	217,800	769.00
3.0	60	261,360	931.20
3.5	78	304,920	1094.40
4.0	97	348,480	1258.60
4.5	117	392,040	1423.80
5.0	137	435,600	1589.00

Table 9.2.2 Routing of Flow Through Detention Reservoir by the Level Pool Method (Example 9.2.1)

Time t (min)	Inflow I_j (cfs)	I_j+I_{j+1} (cfs)	$\frac{2S_j}{\Delta t}-Q_j$ (cfs)	$\frac{2S_{j+1}}{\Delta t}+Q_{j+1}$ (cfs)	Outflow (cfs)
0.00	0.00				0.00
10.00	10.00	10.00	0.00	10.00	0.20
20.00	20.00	30.00	9.60	39.60	0.80
30.00	30.00	50.00	37.99	87.99	1.78
40.00	40.00	70.00	84.43	154.43	3.21
50.00	50.00	90.00	148.01	238.01	5.99
60.00	60.00	110.00	226.04	336.04	10.20
70.00	55.00	115.00	315.64	430.64	15.72
80.00	50.00	105.00	399.21	504.21	21.24
90.00	45.00	95.00	461.72	556.72	25.56
100.00	40.00	85.00	505.61	590.61	28.34
110.00	35.00	75.00	533.93	608.93	29.85
120.00	30.00	65.00	549.24	614.24	30.28
130.00	25.00	55.00	553.67	608.67	29.83
140.00	20.00	45.00	549.02	594.02	28.62
150.00	15.00	35.00	536.78	571.78	26.79
160.00	10.00	25.00	518.19	543.19	24.44
170.00	5.00	15.00	494.30	509.30	21.66
180.00	0.00	5.00	465.98	470.98	18.51
190.00	0.00	0.00	433.96	433.96	15.91
200.00	0.00	0.00	402.14	402.14	14.05
210.00	0.00	0.00	374.03	374.03	12.41
220.00	0.00	0.00	349.20	349.20	10.97
230.00	0.00	0.00	327.27	327.27	9.69
240.00	0.00	0.00	307.90	307.90	8.55

With $Q_1 = 0.2$, then $2S_2/\Delta t - Q_2$ for the next iteration is

$$\left[\frac{2S_2}{\Delta t} - Q_2\right] = \left[\frac{2S_2}{\Delta t} + Q_2\right] - 2Q_2 = 10 - 2(0.2) = 9.6 \text{ cfs}$$

The computation now proceeds to the next time interval. Refer to Table 9.2.2 for the remaining computations.

9.3 HYDROLOGIC RIVER ROUTING

The *Muskingum method* is a commonly used hydrologic routing method that is based upon a variable discharge-storage relationship. This method models the storage volume of flooding in a river channel by a combination of wedge and prism storage (Figure 9.3.1). During the advance of a flood wave, inflow exceeds outflow, producing a wedge of storage. During the recession, outflow exceeds inflow, resulting in a negative wedge. In addition, there is a prism of storage that is formed by a volume of constant cross-section along the length of prismatic channel.

Assuming that the cross-sectional area of the flood flow is directly proportional to the discharge at the section, the *volume of prism storage* is equal to KQ, where K is a proportionality coefficient (approximate as the travel time through the reach), and the *volume of wedge storage* is equal to $KX(I-Q)$, where X is a weighting factor having the range $0 \leq X \leq 0.5$. The total storage is defined as the sum of two components,

$$S = KQ + KX(I - Q) \tag{9.3.1}$$

which can be rearranged to give the storage function for the Muskingum method

$$S = K[XI + (I - X)Q] \tag{9.3.2}$$

and represents a linear model for routing flow in streams.

The value of X depends on the shape of the modeled wedge storage. The value of X ranges from 0 for reservoir-type storage to 0.5 for a full wedge. When $X = 0$, there is no wedge and hence no backwater; this is the case for a level-pool reservoir. In natural streams, X is between 0 and 0.3, with a mean value near 0.2. Great accuracy in determining X may not be necessary because the results of the method are relatively insensitive to the value of this parameter. The parameter K is the time of travel of the flood wave through the channel reach. For hydrologic routing, the values of K and X are assumed to be specified and constant throughout the range of flow.

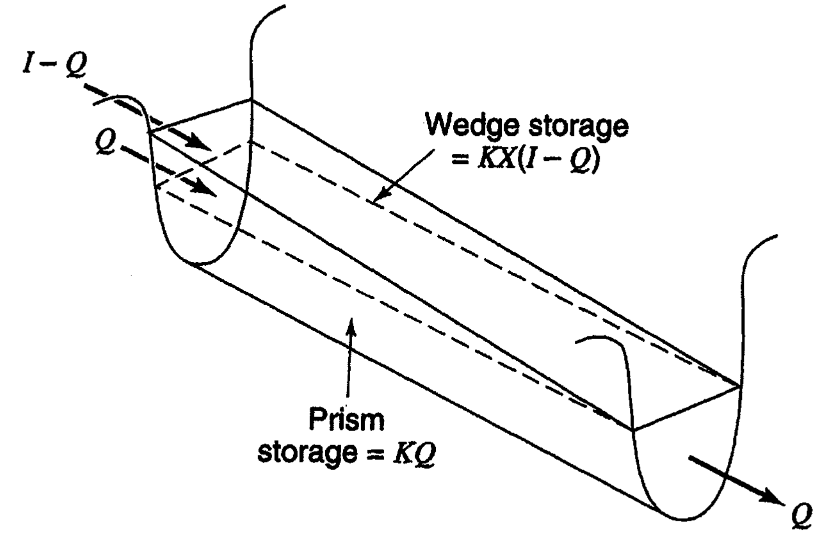


Figure 9.3.1 Prism and wedge storages in a channel reach.

The values of storage at time j and $j+1$ can be written, respectively, as

$$S_j = K[XI_j + (1-X)Q_j] \tag{9.3.3}$$

$$S_{j+1} = K[XI_{j+1} + (1-X)Q_{j+1}] \tag{9.3.4}$$

Using equations (9.3.3) and (9.3.4), the change in storage over time interval Δt is

$$S_{j+1} - S_j = K\{[XI_{j+1} + (1-X)Q_{j+1}] - [XI_j + (1-X)Q_j]\} \tag{9.3.5}$$

The change in storage can also be expressed using equation (9.2.1). Combining equations (9.3.5) and (9.2.1) and simplifying gives

$$Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j \tag{9.3.6}$$

which is the routing equation for the Muskingum method, where

$$C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t} \tag{9.3.7}$$

$$C_2 = \frac{\Delta t + 2KX}{2K(1-X) + \Delta t} \tag{9.3.8}$$

$$C_3 = \frac{2K(1-X) - \Delta t}{2K(1-X) + \Delta t} \tag{9.3.9}$$

Note that $C_1 + C_2 + C_3 = 1$.

The routing procedure can be repeated for several sub-reaches (N_{steps}) so that the total travel time through the reach is K. To insure that the method is computationally stable and accurate, the U.S. Army Corps of Engineers (1990) uses the following criterion to determine the number of routing reaches:

$$\frac{1}{2(1-X)} \le \frac{K}{N_{\text{steps}}\Delta t} \le \frac{1}{2X} \tag{9.3.10}$$

If observed inflow and outflow hydrographs are available for a river reach, the values of K and X can be determined. Assuming various values of X and using known values of the inflow and outflow, successive values of the numerator and denominator of the following expression for K, derived from equations (9.3.5) and (9.2.1), can be computed using

$$K = \frac{0.5\Delta t[(I_{j+1} + I_j) - (Q_{j+1} + Q_j)]}{X(I_{j+1} - I_j) + (1-X)(Q_{j+1} - Q_j)} \tag{9.3.11}$$

The computed values of the numerator (storage) and denominator (weighted discharges) are plotted for each time interval, with the numerator on the vertical axis and the denominator on the horizontal axis. This usually produces a graph in the form of a loop, as shown in Figure 9.3.2. The value of X that produces a loop closest to a single line is taken to be the correct value for the reach, and K, according to equation (9.3.11), is equal to the slope of the line. Since K is the time required for the incremental flood wave to traverse the reach, its value may also be estimated as the observed time of travel of peak flow through the reach.

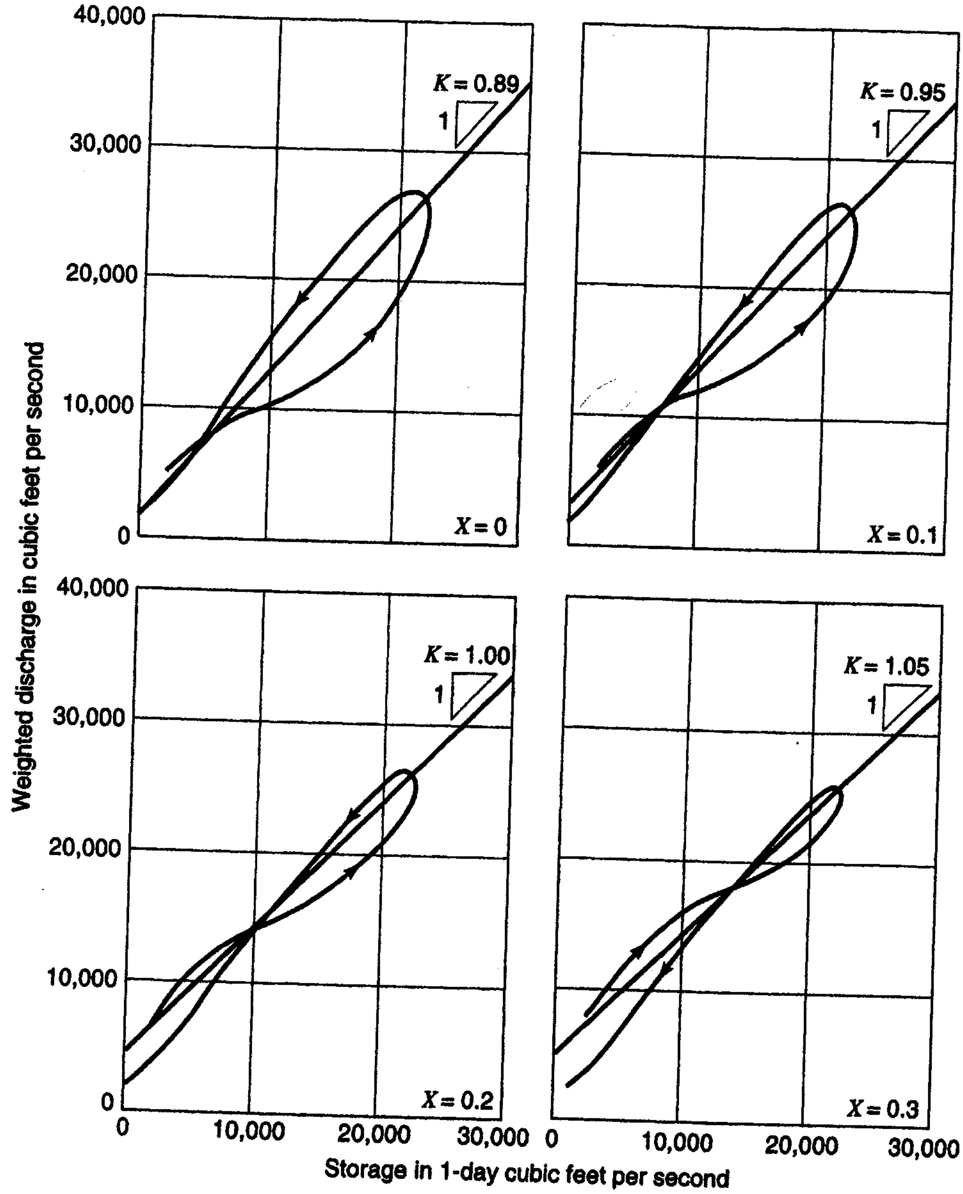


Figure 9.3.2 Typical valley storage curves (after Cudworth (1989)).

EXAMPLE 9.3.1

The objective of this example is to determine K and X for the Muskingum routing method using the February 26 to March 4, 1929 data on the Tuscarawas River from Dover to Newcomerstown. This example is taken from the U.S. Army Corps of Engineers (1960) as used in Cudworth (1989). Columns (2) and (3) in Table 9.3.1 are the inflow and outflow hydrographs for the reach. The numerator and denominator of equation (9.3.11) were computed (for each time period) using four values of $X = 0, 0.1, 0.2$, and 0.3. The accumulated numerators are in column (9) and the accumulated denominators (weighted discharges) are in columns (11), (13), (15), and (17). In Figure 9.3.2, the accumulated numerator (storages) from column (9) are plotted against the corresponding accumulated denominator (weighted discharges) for each of the four X values. According to Figure 9.3.2, the best fit (linear relationship) appears to be for $X = 0.2$, which has a resulting $K = 1.0$. To perform a routing, K should equal Δt, so that if $\Delta t = 0.5$ day, as in this case, the reach should be subdivided into two equal reaches ($N_{\text{steps}} = 2$) and the value of K should be 0.5 day for each reach.

Table 9.3.1 Determination of Coefficients K and X for the Muskingum Routing Method. Tuscarawas River, Muskingum Basin, Ohio Reach from Dover to Newcomerstown, February 26 to March 4, 1929

									Values of D and ΣD for assumed values of X							
									$X = 0$		$X = 0.1$		$X = 0.2$		$X = 0.3$	
(1) Date $\Delta t = 0.5$ day	(2) In-flow[1], ft^3/s	(3) Out-flow[2], ft^3/s	(4) $I_2 + I_1$, ft^3/s	(5) $O_2 + O_1$, ft^3/s	(6) $I_2 - I_1$, ft^3/s	(7) $O_2 - O_1$, ft^3/s	(8) [3]N	(9) ΣN	[4]D (10)	ΣD (11)	D (12)	ΣD (13)	D (14)	ΣD (15)	D (16)	ΣD (17)
2-26-29 a.m.	2200	2000	16,700	9000	12,300	5000	1900		5000		5700		6500		7200	
p.m.	14,500	7000	42,900	18,700	13,900	4700	6100	1900	4700	5000	5600	5700	6500	6500	7500	7200
2-27-29 a.m.	28,400	11,700	60,200	28,200	3400	4800	8000	8000	4800	9700	4600	11,300	4500	13,000	4300	14,700
p.m.	31,800	16,500	61,500	40,500	−2100	7500	5200	16,000	7500	14,500	6700	15,900	5600	17,500	4600	19,000
2-28-29 a.m.	29,700	24,000	55,000	53,100	−4400	5100	500	21,200	5100	22,000	4100	22,600	3200	23,100	2300	23,600
p.m.	25,300	29,100	45,700	57,500	−4900	−700	−2900	21,700	−700	27,100	−1100	26,700	−1500	26,300	−2000	25,900
3-01-29 a.m.	20,400	28,400	36,700	52,200	−4100	−4600	−3900	18,800	−4600	26,400	−4600	25,600	−4500	24,800	−4400	23,900
p.m.	16,300	23,800	28,900	43,200	−3700	−4400	−3600	14,900	−4400	21,800	−4300	21,000	−4300	20,300	−4200	19,500
3-02-29 a.m.	12,600	19,400	21,900	34,700	−3300	−4100	−3200	11,300	−4100	17,400	−4000	16,700	−3900	16,000	−3900	15,300
p.m.	9300	15,300	16,000	26,500	−2600	−4100	−2500	8100	−4100	13,300	−4000	12,700	−3800	12,100	−3600	11,400
3-03-29 a.m.	6700	11,200	11,700	19,400	−1700	−3000	−1900	5500	−3000	9200	−2800	8700	−2800	8300	−2600	7800
p.m.	5000	8200	9100	14,600	−900	−1800	−1400	3600	−1800	6200	−1700	5900	−1600	5500	−1600	5200
3-04-29 a.m.	4100	6400	7700	11,600	−500	−1200	−1000	2200	−1200	4400	−1200	4200	−1100	3900	−900	3600
p.m.	3600	5200	6000	9800	−1200	−600	−1000	1200	−600	3200	−600	3000	−700	2800	−800	2700
3-05-29 a.m.	2400	4600	—	—	—	—	—	200	—	2600	—	2400	—	2100	—	1900

[1]Inflow to reach was adjusted to equal volume of outflow.
[2]Outflow is the hydrograph at Newcomerstown.
[3]Numerator, N, is $\Delta t/2$, column (4) − column (5).
[4]Denominator, D, is column (7) + X [column (6) - column (7)].
Note: From plottings of column (9) versus columns (11), (13), (15), and (17), the plot giving the best fit is considered to define K and X.

$$K = \frac{\text{Numerator, } N}{\text{Denominator, } D} = \frac{0.5\Delta t[(I_2 + I_1) - (O_2 + O_1)]}{X(I_2 - I_1) + (1 - X)(O_2 - O_1)}$$

Source: Cudworth (1989).

EXAMPLE 9.3.2 Route the inflow hydrograph below using the Muskingum method; $\Delta t = 1$ hr, $X = 0.2$, $K = 0.7$ hr.

Time (hr)	0	1	2	3	4	5	6	7
Inflow (cfs)	0	800	2000	4200	5200	4400	3200	2500
Time (hr)	8	9	10	11	12	13		
Inflow (cfs)	2000	1500	1000	700	400	0		

$$C_1 = \frac{1.0 - 2(0.7)(0.2)}{2(0.7)(1-0.2)+1.0} = 0.3396$$

$$C_2 = \frac{1.0 + 2(0.7)(0.2)}{2(0.7)(1-0.2)+1.0} = 0.6038$$

$$C_3 = \frac{2(0.7)(1-0.2) - 1.0}{2(0.7)(1-0.2)+1.0} = 0.0566$$

(Adapted from Masch (1984).)

Check to see if $C_1 + C_2 + C_3 = 1$:

$0.3396 + 0.6038 + 0.0566 = 1$

Using equation (9.3.6) with $I_1 = 0$ cfs, $I_2 = 800$ cfs, and $Q_1 = 0$ cfs, compute Q_2 at $t = 1$ hr:

$$\begin{aligned} Q_2 &= C_1 I_2 + C_2 I_1 + C_3 Q_1 \\ &= (0.3396)(800) + 0.6038(0) + 0.0566(0) \\ &= 272 \text{ cfs } (7.7 \text{ m}^3/\text{s}) \end{aligned}$$

Next compute Q_3 at $t = 2$ hr :

$$\begin{aligned} Q_3 &= C_1 I_3 + C_2 I_2 + C_3 Q_2 \\ &= (0.3396)(2000) + 0.6038(800) + 0.0566(272) \\ &= 1{,}178 \text{ cfs } (33 \text{ m}^3/\text{s}) \end{aligned}$$

The remaining computations result in

Time (hr)	0	1	2	3	4	5	6	7
Q (cfs)	0	272	1178	2701	4455	4886	4020	3009
Time (hr)	8	9	10	11	12	13	14	15
Q (cfs)	2359	1851	1350	918	610	276	16	1

9.4 HYDRAULIC (DISTRIBUTED) ROUTING

Distributed routing or *hydraulic routing*, also referred to as *unsteady flow routing*, is based upon the one-dimensional unsteady flow equations referred to as the *Saint-Venant equations*. The hydrologic river routing and the hydrologic reservoir routing procedures presented previously are lumped procedures and compute flow rate as a function of time alone at a downstream location. Hydraulic (distributed) flow routings allow computation of the flow rate and water surface elevation (or depth) as a function of both space (location) and time. The Saint-Venant equations are presented in Table 9.4.1 in both the *velocity-depth (nonconservation) form* and the *discharge-area (conservation) form*.

The momentum equation contains terms for the physical processes that govern the flow momentum. These terms are: the *local acceleration term*, which describes the change in momentum due to the change in velocity over time, the *convective acceleration term*, which describes the change in momentum due to change in velocity along the channel, the *pressure force term*,

Table 9.4.1 Summary of the Saint-Venant Equations*

Continuity equation

Conservation form

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = 0$$

Nonconservation form

$$V\frac{\partial y}{\partial x} + \frac{\partial V}{\partial x} - \frac{\partial y}{\partial t} = 0$$

Momentum equation

Conservation form

$$\frac{1}{A}\frac{\partial Q}{\partial t} + \frac{1}{A}\frac{\partial}{\partial x}\left(\frac{Q^2}{A}\right) + g\frac{\partial y}{\partial x} - g(S_0 - S_f) = 0$$

Local acceleration term	Convective acceleration term	Pressure force term	Gravity force term	Friction force term

Nonconservation form (unit with element)

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial y}{\partial x} - g(S_0 - S_f) = 0$$

Kinematic wave (gravity and friction force terms)
Diffusion wave (pressure, gravity and friction force terms)
Dynamic wave (all terms)

*Neglecting lateral inflow, wind shear, and eddy losses, and assuming $\beta = 1$.

x = longitudinal distance along the channel or river, t = time, A = cross-sectional area of flow, h = water surface elevation, S_f = friction slope, S_0 = channel bottom slope, g = acceleration due to gravity, V = velocity of flow, and y = depth of flow.

proportional to the change in the water depth along the channel, the gravity force term, proportional to the bed slope S_0, and the friction force term, proportional to the friction slope S_f. The local and convective acceleration terms represent the effect of inertial forces on the flow.

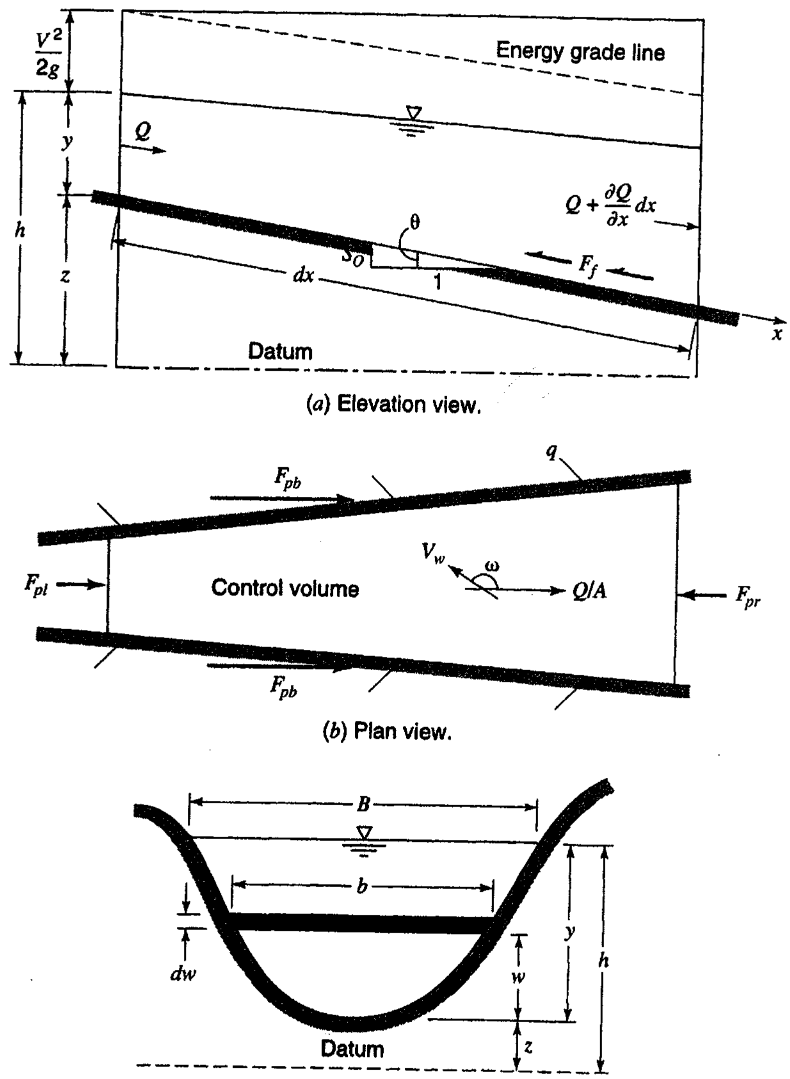
Alternative distributed flow routing models are produced by using the full continuity equation while eliminating some terms of the momentum equation (refer to Table 9.4.1). The simplest distributed model is the *kinematic wave model*, which neglects the local acceleration, convective acceleration, and pressure terms in the momentum equation; that is, it assumes that $S_0 = S_f$ and the friction and gravity forces balance each other. The *diffusion wave model* neglects the local and convective acceleration terms but incorporates the pressure term. The *dynamic wave model* considers all the acceleration and pressure terms in the momentum equation.

The momentum equation can also be written in forms that take into account whether the flow is steady or unsteady, and uniform or nonuniform, as illustrated in Table 9.4.1. In the continuity equation, $\partial A/\partial t = 0$ for a steady flow, and the lateral inflow q is zero for a uniform flow.

9.4.1 Unsteady Flow Equations: Continuity Equation

The *continuity equation* for an unsteady variable-density flow through a control volume can be written as in equation (3.3.1):

$$0 = \frac{d}{dt}\int_{CV} \rho\, d\forall + \int_{CS} \rho \mathbf{V}\cdot d\mathbf{A} \qquad (9.4.1)$$



(*a*) Elevation view.

(*b*) Plan view.

(*c*) Cross-section.

Figure 9.4.1 An elemental reach of channel for derivation of Saint-Venant equations.

Consider an elemental control volume of length dx in a channel. Figure 9.4.1 shows three views of the control volume: (*a*) an elevation view from the side, (*b*) a plan view from above, and (*c*) a channel cross-section. The inflow to the control volume is the sum of the flow Q entering the control volume at the upstream end of the channel and the lateral inflow q entering the control volume as a distributed flow along the side of the channel. The dimensions of q are those of flow per unit length of channel, so the rate of lateral inflow is qdx and the mass inflow rate is

$$\int_{\text{inlet}} \rho \mathbf{V} \cdot \mathbf{dA} = -\rho(Q + qdx) \tag{9.4.2}$$

This is negative because inflows are considered negative in the control volume approach (Reynolds transport theorem). The mass outflow from the control volume is

$$\int_{\text{outlet}} \rho \mathbf{V} \cdot \mathbf{dA} = \rho\left(Q + \frac{\partial Q}{\partial x} dx\right) \tag{9.4.3}$$

where $\partial Q/\partial x$ is the rate of change of channel flow with distance. The volume of the channel element is Adx, where A is the average cross-sectional area, so the rate of change of mass stored within the control volume is

$$\frac{d}{dt}\int_{CV}\rho\, d\forall = \frac{\partial(\rho A dx)}{\partial t} \tag{9.4.4}$$

where the partial derivative is used because the control volume is defined to be fixed in size (though the water level may vary within it). The net outflow of mass from the control volume is found by substituting equations (9.4.2)–(9.4.4) into (9.4.1):

$$\frac{\partial(\rho A dx)}{dt} - \rho(Q + q dx) + \rho\left(Q + \frac{\partial Q}{\partial x}dx\right) = 0 \tag{9.4.5}$$

Assuming the fluid density ρ is constant, equation (9.4.5) is simplified by dividing through by ρdx and rearranging to produce the *conservation form* of the continuity equation,

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} - q = 0 \tag{9.4.6}$$

which is applicable at a channel cross-section. This equation is valid for a *prismatic* or a *non-prismatic* channel; a prismatic channel is one in which the cross-sectional shape does not vary along the channel and the bed slope is constant.

For some methods of solving the Saint-Venant equations, the *nonconservation form* of the continuity equation is used, in which the average flow velocity V is a dependent variable, instead of Q. This form of the continuity equation can be derived for a unit width of flow within the channel, neglecting lateral inflow, as follows. For a unit width of flow, $A = y \times 1 = y$ and $Q = VA = Vy$. Substituting into equation (9.4.6) yields

$$\frac{\partial(Vy)}{\partial x} + \frac{\partial y}{\partial t} = 0 \tag{9.4.7}$$

or

$$V\frac{\partial y}{\partial x} + y\frac{\partial V}{\partial x} + \frac{\partial y}{\partial t} = 0 \tag{9.4.8}$$

9.4.2 Momentum Equation

Newton's second law is written in the form of Reynolds transport theorem as in equation (3.5.5):

$$\sum \mathbf{F} = \frac{d}{dt}\int_{CV} \mathbf{V}\rho\, d\forall + \sum_{CS} \mathbf{V}\rho\mathbf{V}\cdot d\mathbf{A} \tag{9.4.9}$$

This states that the sum of the forces applied is equal to the rate of change of momentum stored within the control volume plus the net outflow of momentum across the control surface. This equation, in the form $\sum F = 0$, was applied to steady uniform flow in an open channel in Chapter 5. Here, unsteady nonuniform flow is considered.

Forces. There are five forces acting on the control volume:

$$\sum F = F_g + F_f + F_e + F_p \tag{9.4.10}$$

where F_g is the *gravity force* along the channel due to the weight of the water in the control volume, F_f is the *friction force* along the bottom and sides of the control volume, F_e is the *contraction/*

expansion force produced by abrupt changes in the channel cross-section, and F_p is the *unbalanced pressure force* (see Figure 9.4.1). Each of these four forces is evaluated in the following paragraphs.

Gravity. The volume of fluid in the control volume is Adx and its weight is $\rho gAdx$. For a small angle of channel inclination θ, $S_0 \approx \sin\theta$ and the gravity force is given by

$$F_g = \rho gAdx \sin\theta \approx \rho gAS_0 dx \tag{9.4.11}$$

where the channel bottom slope S_0 equals $-\partial z/\partial x$.

Friction. Frictional forces created by the shear stress along the bottom and sides of the control volume are given by $-\tau_0 Pdx$, where $\tau_0 = \gamma RS_f = \rho g(A/P)S_f$ is the bed shear stress and P is the wetted perimeter. Hence the friction force is written as

$$F_f = -\rho gAS_f dx \tag{9.4.12}$$

where the friction slope S_f is derived from resistance equations such as Manning's equation.

Contraction/expansion. Abrupt contractions or expansions of the channel cause energy losses through eddy motion. Such losses are similar to minor losses in a pipe system. The magnitude of eddy losses is related to the change in velocity head $V^2/2g = (Q/A)^2/2g$ through the length of channel causing the losses. The drag forces creating these eddy losses are given by

$$F_e = -\rho gAS_e dx \tag{9.4.13}$$

where S_e is the eddy loss slope

$$S_e = \frac{K_e}{2g}\frac{\partial(Q/A)^2}{\partial x} \tag{9.4.14}$$

in which K_e is the nondimensional expansion or contraction coefficient, negative for channel expansion (where $\partial(Q/A)^2/\partial x$ is negative) and positive for channel contractions.

Pressure. Referring to Figure 9.4.1, the unbalanced pressure force is the resultant of the hydrostatic force on the each side of the control volume. Chow et al. (1988) provide a detailed derivation of the pressure force F_p as simply

$$F_p = \rho gA\frac{\partial y}{\partial x}dx \tag{9.4.15}$$

The sum of the forces in equation (9.4.10) can be expressed, after substituting equations (9.4.11), (9.4.12), (9.4.13), and (9.4.15), as

$$\sum F = \rho AS_0 dx - \rho gAS_f dx - \rho gAS_e dx - \rho gA\frac{\partial y}{\partial x}dx \tag{9.4.16}$$

Momentum. The two momentum terms on the right-hand side of equation (9.4.9) represent the rate of change of storage of momentum in the control volume, and the net outflow of momentum across the control surface, respectively.

Net momentum outflow. The mass inflow rate to the control volume (equation (9.4.2)) is $-\rho(Q+qdx)$, representing both stream inflow and lateral inflow. The corresponding momentum is computed by multiplying the two mass inflow rates by their respective velocity and a *momentum correction factor* β:

$$\int_{\text{inlet}} \mathbf{V}\rho\mathbf{V}\,d\mathbf{A} = -\rho(\beta VQ + \beta v_x q dx) \tag{9.4.17}$$

where $-\rho\beta VQ$ is the momentum entering from the upstream end of the channel, and $-\rho\beta v_x qdx$ is the momentum entering the main channel with the lateral inflow, which has a velocity v_x in the x

direction. The term β is known as the *momentum coefficient* or *Boussinesq coefficient*; it accounts for the nonuniform distribution of velocity at a channel cross-section in computing the momentum. The value of β is given by

$$\beta = \frac{1}{V^2 A}\int v^2 dA \tag{9.4.18}$$

where v is the velocity through a small element of area dA in the channel cross-section. The value of β ranges from 1.01 for straight prismatic channels to 1.33 for river valleys with floodplains (Chow, 1959; Henderson, 1966).

The momentum leaving the control volume is

$$\int_{\text{outlet}} \mathbf{V}\rho\mathbf{V}\,\mathbf{dA} = \rho\left[\beta VQ + \frac{\partial(\beta VQ)}{\partial x}dx\right] \tag{9.4.19}$$

The net outflow of momentum across the control surface is the sum of equations (9.4.17) and (9.4.19):

$$\int_{CS} \mathbf{V}\rho\mathbf{V}\,\mathbf{dA} = -\rho(\beta VQ + \beta v_x q dx) + \rho\left[\beta VQ + \frac{\partial(\beta VQ)}{\partial x}dx\right] = -\rho\left[\beta v_x q - \frac{\partial(\beta VQ)}{\partial x}\right]dx \tag{9.4.20}$$

Momentum storage. The time rate of change of momentum stored in the control volume is found by using the fact that the volume of the elemental channel is Adx, so its momentum is $\rho A dx V$, or $\rho Q dx$, and then

$$\frac{d}{dt}\int_{CV} \mathbf{V}\rho d\forall = \rho\frac{\partial Q}{\partial x}dx \tag{9.4.21}$$

After substituting the force terms from equation (9.4.16) and the momentum terms from equations (9.4.20) and (9.4.21) into the momentum equation (9.4.9), it reads

$$\rho g A S_0 dx - \rho g A S_f dx - \rho g A S_e dx - \rho g A\frac{\partial y}{\partial x}dx = -\rho\left[\beta v_x q - \frac{\partial(\beta VQ)}{\partial x}\right]dx + \rho\frac{\partial Q}{\partial t}dx \tag{9.4.22}$$

Dividing through by ρdx, replacing V with Q/A, and rearranging produces the conservation form of the momentum equation:

$$\frac{\partial Q}{\partial t} + \frac{\partial(\beta Q^2/A)}{\partial t} + gA\left(\frac{\partial y}{\partial x} - S_0 + S_f + S_e\right) - \beta q v_x = 0 \tag{9.4.23}$$

The depth y in equation (9.4.23) can be replaced by the water surface elevation h, using

$$h = y + z \tag{9.4.24}$$

where z is the elevation of the channel bottom above a datum such as mean sea level. The derivative of equation (9.4.24) with respect to the longitudinal distance x along the channel is

$$\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} + \frac{\partial z}{\partial x} \tag{9.4.25}$$

but $\partial z/\partial x = -S_0$, so

$$\frac{\partial h}{\partial x} = \frac{\partial y}{\partial x} - S_0 \tag{9.4.26}$$

The momentum equation can now be expressed in terms of h by using equation (9.4.26) in (9.4.23):

$$\frac{\partial Q}{\partial t} + \frac{\partial(\beta Q^2/A)}{\partial x} + gA\left(\frac{\partial h}{\partial x} + S_f + S_e\right) - \beta q v_x = 0 \qquad (9.4.27)$$

The Saint-Venant equations, (9.4.6) for continuity and (9.4.27) for momentum, are the governing equations for one-dimensional, unsteady flow in an open channel. The use of the terms S_f and S_e in equation (9.4.27), which represent the rate of energy loss as the flow passes through the channel, illustrates the close relationship between energy and momentum considerations in describing the flow. Strelkoff (1969) showed that the momentum equation for the Saint-Venant equations can also be derived from energy principles, rather than by using Newton's second law as presented here.

The nonconservation form of the momentum equation can be derived in a similar manner to the nonconservation form of the continuity equation. Neglecting eddy losses, wind shear effect, and lateral inflow, the nonconservation form of the momentum equation for a unit width in the flow is

$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\left(\frac{\partial y}{\partial x} - S_0 + S_f\right) = 0 \qquad (9.4.28)$$

9.5 KINEMATIC WAVE MODEL FOR CHANNELS

In Section 8.9, a kinematic wave overland flow runoff model was presented. This is an implicit nonlinear kinematic model that is used in the KINEROS model. This section presents a general discussion of the kinematic wave followed by a brief description of the very simplest linear models, such as those found in the U.S. Army Corps of Engineers HEC-HMS and HEC-1, and the more complicated models such as the KINEROS model (Woolhiser et al., 1990).

Kinematic waves govern flow when inertial and pressure forces are not important. Dynamic waves govern flow when these forces are important, as in the movement of a large flood wave in a wide river. In a kinematic wave, the gravity and friction forces are balanced, so the flow does not accelerate appreciably.

For a kinematic wave, the energy grade line is parallel to the channel bottom and the flow is steady and uniform ($S_0 = S_f$) within the differential length, while for a dynamic wave the energy grade line and water surface elevation are not parallel to the bed, even within a differential element.

9.5.1 Kinematic Wave Equations

A *wave* is a variation in a flow, such as a change in flow rate or water surface elevation, and the *wave celerity* is the velocity with which this variation travels along the channel. The celerity depends on the type of wave being considered and may be quite different from the water velocity. For a kinematic wave, the acceleration and pressure terms in the momentum equation are negligible, so the wave motion is described principally by the equation of continuity. The name kinematic is thus applicable, as *kinematics* refers to the study of motion exclusive of the influence of mass and force; in *dynamics* these quantities are included.

The kinematic wave model is defined by the following equations.
Continuity:

$$\frac{\partial Q}{\partial x} + \frac{\partial A}{\partial t} = q(x, t) \qquad (9.5.1)$$

Momentum:

$$S_0 = S_f \qquad (9.5.2)$$

where $q(x, t)$ is the net lateral inflow per unit length of channel.

The momentum equation can also be expressed in the form

$$A = \alpha Q^{\beta} \tag{9.5.3}$$

For example, Manning's equation written with $S_0 = S_f$ and $R = A/P$ is

$$Q = \frac{1.49 S_0^{1/2}}{n P^{2/3}} A^{5/3} \tag{9.5.4}$$

which can be solved for A as

$$A = \left(\frac{n P^{2/3}}{1.49\sqrt{S_0}} \right)^{3/5} Q^{3/5} \tag{9.5.5}$$

so $\alpha = [nP^{2/3}/(1.49\sqrt{S_0})]^{0.6}$ and $\beta = 0.6$ in this case.

Equation (9.5.1) contains two dependent variables, A and Q, but A can be eliminated by differentiating equation (9.5.3):

$$\frac{\partial A}{\partial t} = \alpha \beta Q^{\beta - 1} \left(\frac{\partial Q}{\partial t} \right) \tag{9.5.6}$$

and substituting for $\partial A/\partial t$ in equation (9.5.1) to give

$$\frac{\partial Q}{\partial x} + \alpha \beta Q^{\beta - 1} \left(\frac{\partial Q}{\partial t} \right) = q \tag{9.5.7}$$

Alternatively, the momentum equation could be expressed as

$$Q = aA^{B} \tag{9.5.8}$$

where a and B are defined using Manning's equation. Using

$$\frac{\partial Q}{\partial x} = \frac{dQ}{dA}\frac{\partial A}{\partial x} \tag{9.5.9}$$

the governing equation is

$$\frac{\partial A}{\partial t} + \frac{dQ}{dA}\frac{\partial A}{\partial x} = q \tag{9.5.10}$$

where dQ/dA is determined by differentiating equation (9.5.8):

$$\frac{dQ}{dA} = aBA^{B-1} \tag{9.5.11}$$

and substituting in equation (9.5.10):

$$\frac{\partial A}{\partial t} + aBA^{B-1}\frac{\partial A}{\partial x} = q \tag{9.5.12}$$

The kinematic wave equation (9.5.7) has Q as the dependent variable and the kinematic wave equation (9.5.12) has A as the dependent variable. First consider equation (9.5.7), by taking the logarithm of (9.5.3):

$$\ln A = \ln \alpha + \beta \ln Q \tag{9.5.13}$$

and differentiating

$$\frac{dQ}{Q} = \frac{1}{\beta}\left(\frac{dA}{A} \right) \tag{9.5.14}$$

This defines the relationship between relative errors dA/A and dQ/Q. For Manning's equation $\beta < 1$ so that the discharge estimation error would be magnified by the ratio $1/\beta$ if A were the dependent variable instead of Q.

Next consider equation (9.5.12); by taking the logarithm of (9.5.8):

$$\ln Q = \ln a + B \ln A$$

$$\frac{dA}{A} = \frac{1}{B}\left(\frac{dQ}{Q}\right) \tag{9.5.15}$$

or

$$\frac{dQ}{Q} = B\left(\frac{dA}{A}\right) \tag{9.5.16}$$

In this case $\beta > 1$, so that the discharge estimation error would be decreased by B if A were the dependent variable instead of Q. In summary, if we use equation (9.5.3) as the form of the momentum equation, then Q is the dependent variable with equation (9.5.7) being the governing equation; if we use equation (9.5.8) as the form of the momentum equation, then A is the dependent variable with equation (9.5.12) being the governing equation.

9.5.2 U.S. Army Corps of Engineers Kinematic Wave Model for Overland Flow and Channel Routing

The HEC-1 (HEC-HMS) computer program actually has two forms of the kinematic wave. The first is based upon equation (9.5.12) where an explicit finite difference form is used (refer to Figures 9.5.1 and 8.9.2):

$$\frac{\partial A}{\partial t} = \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} \tag{9.5.17}$$

$$\frac{\partial A}{\partial x} = \frac{A_{i+1}^{j} - A_{i}^{j}}{\Delta x} \tag{9.5.18}$$

and

$$A = \frac{A_{i+1}^{j} + A_{i}^{j}}{2} \tag{9.5.19}$$

$$q = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \tag{9.5.20}$$

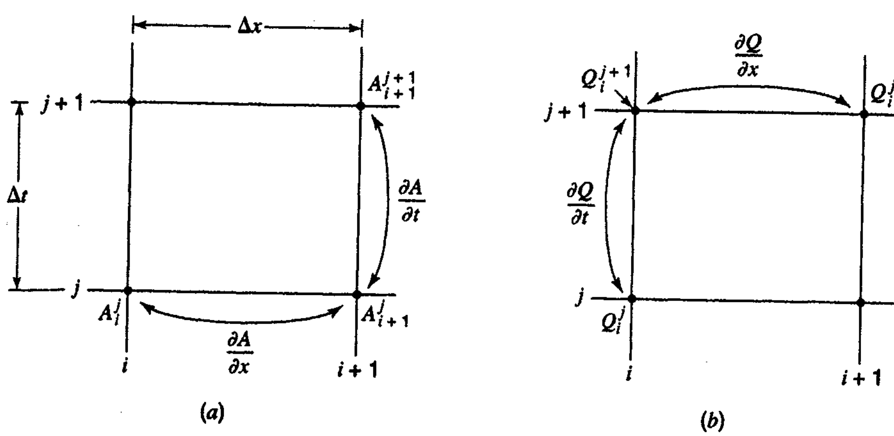


Figure 9.5.1 Finite difference forms. (*a*) HEC-1 "standard form;" (*b*) HEC-1 "conservation form."

Substituting these finite-difference approximations into equation (9.5.12) gives

$$\frac{1}{\Delta t}(A_{i+1}^{j+1} - A_{i+1}^{j}) + aB\left[\frac{A_{i+1}^{j} + A_{i}^{j}}{2}\right]^{B-1}\left[\frac{A_{i+1}^{j} - A_{i}^{j}}{\Delta x}\right] = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} \tag{9.5.21}$$

The only unknown in equation (9.5.21) is A_{i+1}^{j+1}, so

$$A_{i+1}^{j+1} = A_{i+1}^{j} - aB\left(\frac{\Delta t}{\Delta x}\right)\left[\frac{A_{i+1}^{j} + A_{i}^{j}}{2}\right]^{B-1}(A_{i+1}^{j} - A_{i,j}) + (q_{i+1}^{j+1} + q_{i+1}^{j})\frac{\Delta t}{2} \tag{9.5.22}$$

After computing A_{i+1}^{j+1} at each grid along a time line going from upstream to downstream (see Figure 8.9.2), compute the flow using equation (9.5.8):

$$Q_{i+1}^{j+1} = a\left(A_{i+1}^{j+1}\right)^{B} \tag{9.5.23}$$

The HEC-1 model uses the above kinematic wave model as long as a stability factor $R < 1$ (Alley and Smith, 1987), defined by

$$R = \frac{a}{q\Delta x}\left[\left(q\Delta t + A_{i}^{j}\right)^{B} - \left(A_{i}^{j}\right)^{B}\right] \text{ for } q > 0 \tag{9.5.24a}$$

$$R = aB(A_{i}^{j})^{B-1}\frac{\Delta t}{\Delta x} \text{ for } q = 0 \tag{9.5.24b}$$

Otherwise the form of equation (9.5.1) is used, where (see Figure 9.5.1)

$$\frac{\partial Q}{\partial x} = \frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} \tag{9.5.25}$$

$$\frac{\partial A}{\partial t} = \frac{A_{i}^{j+1} - A_{i}^{j}}{\Delta t} \tag{9.5.26}$$

so

$$\frac{Q_{i+1}^{j+1} - Q_{i}^{j+1}}{\Delta x} + \frac{A_{i}^{j+1} - A_{i}^{j}}{\Delta x} = q \tag{9.5.27}$$

Solving for the only unknown Q_{i+1}^{j+1} yields

$$Q_{i+1}^{j+1} = Q_{i}^{j+1} + q\Delta x - \frac{\Delta x}{\Delta t}(A_{i}^{j+1} - A_{i}^{j}) \tag{9.5.28}$$

Then solve for A_{i+1}^{j+1} using equation (9.5.23):

$$A_{i+1}^{j+1} = \left(\frac{1}{a}Q_{i+1}^{j+1}\right)^{1/B} \tag{9.5.29}$$

The *initial condition* (values of A and Q at time 0 along the grid, referring to Figure 8.9.2) are computed assuming uniform flow or nonuniform flow for an initial discharge. The *upstream boundary* is the inflow hydrograph from which Q is obtained.

The kinematic wave schemes used in the HEC-1 (HEC-HMS) model are very simplified. Chow, et al. (1988) presented both linear and nonlinear kinematic wave schemes based upon the equation (9.5.7) formulation. An example of a more desirable kinematic wave formulation is that by Woolhiser et al. (1990) presented in the next subsection.

9.5.3 KINEROS Channel Flow Routing Model

The KINEROS channel routing model uses the equation (9.5.10) form of the kinematic wave equation (Woolhiser et al., 1990):

$$\frac{\partial A}{\partial t} + \frac{dQ}{dA}\frac{\partial A}{\partial x} = q(x,\ t) \qquad (9.5.10)$$

where $q(x, t)$ is the net lateral inflow per unit length of channel. The derivatives are approximated using an implicit scheme in which the spatial and temporal derivatives are, respectively,

$$\frac{\partial A}{\partial x} = \theta\frac{A_{i+1}^{j+1} - A_i^{j+1}}{\Delta x} + (1-\theta)\frac{A_{i+1}^{j} - A_i^{j}}{\Delta x} \qquad (9.5.30)$$

$$\frac{dQ}{dA}\frac{\partial A}{\partial x} = \theta\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j+1} - A_i^{j+1}}{\Delta x}\right) + (1-\theta)\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j} - A_i^{j}}{\Delta x}\right) \qquad (9.5.31)$$

and

$$\frac{\partial A}{\partial t} = \frac{1}{2}\left[\frac{A_i^{j+1} - A_i^{j}}{\Delta t} + \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t}\right] \qquad (9.5.32)$$

or

$$\frac{\partial A}{\partial t} = \frac{A_i^{j+1} + A_{i+1}^{j+1} - A_i^{j} - A_{i+1}^{j}}{2\Delta t} \qquad (9.5.33)$$

Substituting equations (9.5.31) and (9.5.33) into (9.5.10), we have

$$\frac{A_{i+1}^{j+1} - A_{i+1}^{j} + A_i^{j+1} - A_i^{j}}{2\Delta t} + \left\{\theta\left[\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j+1} - A_i^{j+1}}{\Delta x}\right)\right] + (1-\theta)\left[\left(\frac{dQ}{dA}\right)^{j+1}\left(\frac{A_{i+1}^{j} - A_i^{j}}{\Delta x}\right)\right]\right\}$$
$$= \frac{1}{2}(q_{i+1}^{j+1} + q_i^{j+1} + q_{i+1}^{j} + q_i^{j}) \qquad (9.5.34)$$

The only unknown in this equation is A_{i+1}^{j+1}, which must be solved for numerically by use of an iterative scheme such as the Newton-Raphson method (see Appendix A).

Woolhiser et al. (1990) use the following relationship between channel discharge and cross-sectional area, which embodies the kinematic wave assumption:

$$Q = \alpha R^{m-1} A \qquad (9.5.35)$$

where R is the hydraulic radius and $\alpha = 1.49S^{1/2}/n$ and $m = 5/3$ for Manning's equation.

9.5.4 Kinematic Wave Celerity

Kinematic waves result from changes in Q. An increment in flow dQ can be written as

$$dQ = \frac{\partial Q}{\partial x}dx + \frac{\partial Q}{\partial t}dt \qquad (9.5.36)$$

Dividing through by dx and rearranging produces:

$$\frac{\partial Q}{\partial x} + \frac{dt}{dx}\frac{\partial Q}{\partial t} = \frac{dQ}{dx} \qquad (9.5.37)$$

Equations (9.5.7) and (9.5.37) are identical if

$$\frac{dQ}{dx} = q \tag{9.5.38}$$

and

$$\frac{dx}{dt} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{9.5.39}$$

Differentiating equation (9.5.3) and rearranging gives

$$\frac{dQ}{dA} = \frac{1}{\alpha\beta Q^{\beta-1}} \tag{9.5.40}$$

and by comparing equations (9.5.39) and (9.5.40), it can be seen that

$$\frac{dx}{dt} = \frac{dQ}{dA} \tag{9.5.41}$$

or

$$c_k = \frac{dx}{dt} = \frac{dQ}{dA} \tag{9.5.42}$$

where c_k is the kinematic wave celerity. This implies that an observer moving at a velocity $dx/dt = c_k$ with the flow would see the flow rate increasing at a rate of $dQ/dx = q$. If $q = 0$, the observer would see a constant discharge. Equations (9.5.38) and (9.5.42) are the *characteristic equations* for a kinematic wave, two ordinary differential equations that are mathematically equivalent to the governing continuity and momentum equations.

The kinematic wave celerity can also be expressed in terms of the depth y as

$$c_k = \frac{1}{B}\frac{dQ}{dy} \tag{9.5.43}$$

where $dA = Bdy$.

Both kinematic and dynamic wave motion are present in natural flood waves. In many cases the channel slope dominates in the momentum equation; therefore, most of a flood wave moves as a kinematic wave. Lighthill and Whitham (1955) proved that the velocity of the main part of a natural flood wave approximates that of a kinematic wave. If the other momentum terms $\left(\partial V/\partial t,\ V(\partial V/\partial x) \text{ and } (1/g)\partial y/\partial x\right)$ are not negligible, then a dynamic wave front exists that can propagate both upstream and downstream from the main body of the flood wave.

9.6 MUSKINGUM–CUNGE MODEL

Cunge (1969) proposed a variation of the kinematic wave method based upon the Muskingum method (see Chapter 8). With the grid shown in Figure 9.6.1, the unknown discharge Q_{i+1}^{j+1} can be expressed using the Muskingum equation ($Q_{j+1} = C_1 I_{j+1} + C_2 I_j + C_3 Q_j$):

$$Q_{i+1}^{j+1} = C_1 Q_i^{j+1} + C_2 Q_i^j + C_3 Q_{i+1}^j \tag{9.6.1}$$

where $Q_{i+1}^{j+1} = Q_{j+1}$; $Q_i^{j+1} = I_{j+1}$; $Q_i^j = I_j$; and $Q_{i+1}^j = Q_j$. The Muskingum coefficients are

$$C_1 = \frac{\Delta t - 2KX}{2K(1-X) + \Delta t} \tag{9.6.2}$$

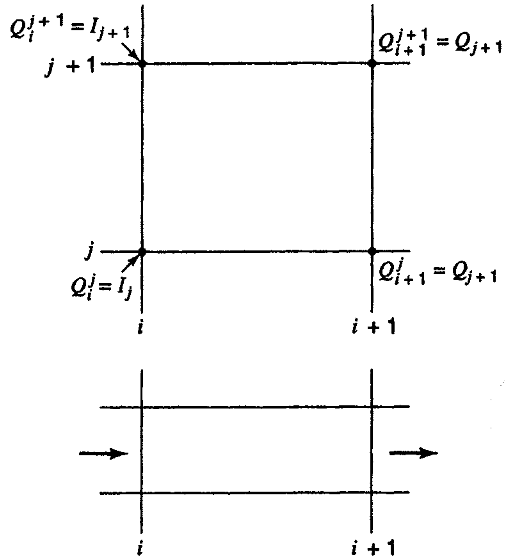


Figure 9.6.1 Finite-difference grid for the Muskingum–Cunge method.

$$C_2 = \frac{\Delta t + 2KX}{2K(1-X) + \Delta t} \tag{9.6.3}$$

$$C_3 = \frac{2K(1-X) - \Delta t}{2K(1-X) + \Delta t} \tag{9.6.4}$$

Cunge (1969) showed that when K and Δt are considered constant, equation (9.6.1) is an approximate solution of the kinematic wave. He further demonstrated that equation (9.6.1) can be considered an approximation of a modified diffusion equation if

$$K = \frac{\Delta x}{c_k} = \frac{\Delta x}{dQ/dA} \tag{9.6.5}$$

and

$$X = \frac{1}{2}\left(1 - \frac{Q}{Bc_k S_0 \Delta x}\right) \tag{9.6.6}$$

where c_k is the celerity corresponding to Q and B, and B is the width of the water surface. The value of $\Delta x/(dQ/dA)$ in equation (9.6.5) represents the time propagation of a given discharge along a channel reach of length Δx. Numerical stability requires $0 \leq X \leq 1/2$. The solution procedure is basically the same as the kinematic wave.

9.7 IMPLICIT DYNAMIC WAVE MODEL

The conservation form of the Saint-Venant equations is used because this form provides the versatility required to simulate a wide range of flows from gradual long-duration flood waves in rivers to abrupt waves similar to those caused by a dam failure. The equations are developed from equations (9.4.6) and (9.4.25) as follows.

Weighted four-point finite-difference approximations given by equations (9.7.1)–(9.7.3) are used for dynamic routing with the Saint-Venant equations. The spatial derivatives $\partial Q/\partial x$ and $\partial h/\partial x$ are estimated between adjacent time lines:

$$\frac{\partial Q}{\partial x} = \theta \frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x_i} \tag{9.7.1}$$

$$\frac{\partial h}{\partial x} = \theta \frac{h_{i+1}^{j+1} - h_i^{j+1}}{\Delta x_i} + (1-\theta)\frac{h_{i+1}^{j} - h_i^{j}}{\Delta x_i} \tag{9.7.2}$$

and the time derivatives are:

$$\frac{\partial (A+A_0)}{\partial t} = \frac{(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}}{2\Delta t_j} \tag{9.7.3}$$

$$\frac{\partial Q}{\partial t} = \frac{Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^{j} - Q_{i+1}^{j}}{2\Delta t_j} \tag{9.7.4}$$

The nonderivative terms, such as q and A, are estimated between adjacent time lines, using:

$$q = \theta \frac{q_i^{j+1} + q_{i+1}^{j+1}}{2} + (1-\theta)\frac{q_i^{j} + q_{i+1}^{j}}{2} = \theta \bar{q}_i^{j+1} + (1-\theta)\bar{q}_i^{j} \tag{9.7.5}$$

$$A = \theta\left[\frac{A_i^{j+1} + A_{i+1}^{j+1}}{2}\right] + (1-\theta)\left[\frac{A_i^{j} + A_{i+1}^{j}}{2}\right] = \theta \bar{A}_i^{j+1} + (1-\theta)\bar{A}_i^{j} \tag{9.7.6}$$

where $\bar{q}_i$ and $\bar{A}_i$ indicate the lateral flow and cross-sectional area averaged over the reach Δx_i.

The finite-difference form of the continuity equation is produced by substituting equations (9.7.1), (9.7.3), and (9.7.5) into (9.4.6):

$$\begin{aligned}&\theta\left(\frac{Q_{i+1}^{j+1} - Q_i^{j+1}}{\Delta x_i} - \bar{q}_i^{j+1}\right) + (1-\theta)\left(\frac{Q_{i+1}^{j} - Q_i^{j}}{\Delta x_1} - \bar{q}_i^{j}\right)\\ &+ \frac{(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}}{2\Delta t_j} = 0\end{aligned} \tag{9.7.7}$$

Similarly, the momentum equation (9.4.27) is written in finite-difference form as:

$$\begin{aligned}&\frac{Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^{j} - Q_{i+1}^{j}}{2\Delta t_j}\\ &+ \theta\left[\frac{(\beta Q^2/A)_i^{j+1} - (\beta Q^2/A)_{i+1}^{j+1}}{\Delta x_i} + g\bar{A}_i^{j+1}\left(\frac{h_{i+1}^{j+1} - h_i^{j+1}}{\Delta x_i} + (\bar{S}_f)_i^{j+1} + (\bar{S}_e)_i^{j+1}\right) - (\overline{\beta q v_x})_i^{j+1}\right]\\ &+ (1-\theta)\left[\frac{(\beta Q^2/A)_{i+1}^{j} - (\beta Q^2/A)_{i+1}^{j}}{\Delta x_i} + g\bar{A}_i^{j}\left(\frac{h_{i+1}^{j} - h_i^{j}}{\Delta x_i} + (\bar{S}_f)_i^{j} + (\bar{S}_e)_i^{j}\right) - (\overline{\beta q v_x})_i^{j}\right] = 0\end{aligned} \tag{9.7.8}$$

The four-point finite-difference form of the continuity equation can be further modified by multiplying equation (9.7.7) by Δx_i to obtain

$$\begin{aligned}&\theta\left(Q_{i+1}^{j+1} - Q_i^{j+1} - \bar{q}_i^{j+1}\Delta x_i\right) + (1-\theta)\left(Q_{i+1}^{j} - Q_i^{j} - \bar{q}_i^{j}\Delta x_i\right)\\ &+ \frac{\Delta x_i}{2\Delta t_i}\left[(A+A_0)_i^{j+1} + (A+A_0)_{i+1}^{j+1} - (A+A_0)_i^{j} - (A+A_0)_{i+1}^{j}\right] = 0\end{aligned} \tag{9.7.9}$$

Similarly, the momentum equation can be modified by multiplying by Δx_i to obtain

$$\frac{\Delta x_i}{2\Delta t_j}\left(Q_i^{j+1} + Q_{i+1}^{j+1} - Q_i^j - Q_{i+1}^j\right)$$

$$+\theta\left\{\left(\frac{\beta Q^2}{A}\right)_{i+1}^{j+1} - \left(\frac{\beta Q^2}{A}\right)_i^{j+1} + g\overline{A}_i^{j+1}\left[h_{i+1}^{j+1} - h_i^{j+1} + \left(\overline{S}_f\right)_i^{j+1} + \left(\overline{S}_e\right)_i^{j+1}\Delta x_i\right] - \left(\overline{\beta q v_x}\right)_i^{j+1}\Delta x_i\right\}$$

$$+(1-\theta)\left\{\left(\frac{\beta Q^2}{A}\right)_{i+1}^{j} - \left(\frac{\beta Q^2}{A}\right)_i^{j} + g\overline{A}_i^{j}\left[h_{i+1}^{j} - h_i^{j} + \left(\overline{S}_f\right)_i^{j}\Delta x_i + \left(\overline{S}_e\right)_i^{j}\Delta x_i\right] - \left(\overline{\beta q v_x}\right)_i^{j}\Delta x_i\right\} = 0$$

(9.7.10)

where the average values (marked with an overbar) over a reach are defined as

$$\overline{\beta}_i = \frac{\beta_i + \beta_{i+1}}{2} \qquad (9.7.11)$$

$$\overline{A}_i = \frac{A_i + A_{i+1}}{2} \qquad (9.7.12)$$

$$\overline{B}_i = \frac{B_i + B_{i+1}}{2} \qquad (9.7.13)$$

$$\overline{Q}_i = \frac{Q_i + Q_{i+1}}{2} \qquad (9.7.14)$$

Also,

$$\overline{R}_i = \overline{A}_i/\overline{B}_i \qquad (9.7.15)$$

for use in Manning's equation. Manning's equation may be solved for S_f and written in the form shown below, where the $|Q|Q$ has magnitude Q^2 and sign positive or negative depending on whether the flow is downstream or upstream, respectively:

$$\left(\overline{S}_f\right)_i = \frac{\overline{n}_i^2|\overline{Q}_i|\overline{Q}_i}{2.208\overline{A}_i^2\overline{R}_i^{4/3}} \qquad (9.7.16)$$

The minor headlosses arising from contraction and expansion of the channel are proportional to the difference between the squares of the downstream and upstream velocities, with a contraction/expansion loss coefficient K_e:

$$\left(\overline{S}_e\right)_i = \frac{(K_e)_i}{2g\Delta x_i}\left[\left(\frac{Q}{A}\right)_{i+1}^2 - \left(\frac{Q}{A}\right)_i^2\right] \qquad (9.7.17)$$

The terms having superscript j in equations (9.7.9) and (9.7.10) are known either from initial conditions or from a solution of the Saint-Venant equations for a previous time line. The terms g, Δx_i, β_i, K_e, C_w, and V_w are known and must be specified independently of the solution. The unknown terms are Q_i^{j+1}, Q_{i+1}^{j+1}, h_{i+1}^{j+1}, A_i^{j+1}, $A_{i+1}^{j+1}B_i^{j+1}$, and B_{i+1}^{j+1}. However, all the terms can be expressed as functions of the unknowns Q_i^{j+1}, Q_{i+1}^{j+1}, h_i^{j+1}, and h_{i+1}^{j+1}, so there are actually four unknowns. The unknowns are raised to powers other than unity, so equations (9.7.9) and (9.7.10) are nonlinear equations.

The continuity and momentum equations are considered at each of the N-1 rectangular grids shown in Figure 9.7.1 between the upstream boundary at $i = 1$ and the downstream boundary at

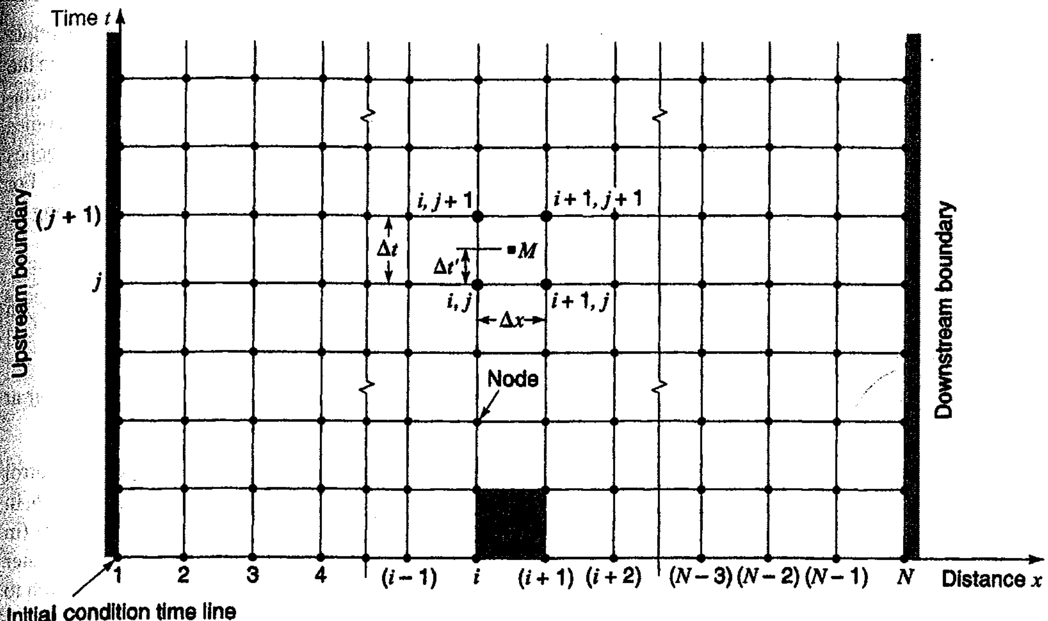


Figure 9.7.1 The x-t solution plane. The finite-difference forms of the Saint–Venant equations are solved at a discrete number of points (values of the independent variables x and t) arranged to form the rectangular grid shown. Lines parallel to the time axis represent locations along the channel, and those parallel to the distance axis represent times (from Fread (1974)).

$i = N$. This yields $2N$-2 equations. There are two unknowns at each of the N grid points (Q and h), so there are $2N$ unknowns in all. The two additional equations required to complete the solution are supplied by the upstream and downstream boundary conditions. The upstream boundary condition is usually specified as a known inflow hydrograph, while the downstream boundary condition can be specified as a known stage hydrograph, a known discharge hydrograph, or a known relationship between stage and discharge, such as a rating curve. The U.S. National Weather Service FLDWAV model (hsp.nws.noaa.gov/oh/hrl/rvmech) uses the above to describe the implicit dynamic wave model formulation.

PROBLEMS

9.1.1 Consider a river segment with the surface area of 10 km^2. For a given flood event, the measured time variation of inflow rate (called inflow hydrograph) at the upstream section of the river segment and the outflow hydrograph at the downstream section are shown in Figure P9.1.1. Assume that the initial storage of water in the river segment is 15 mm in depth.

(a) Determine the time at which the change in storage of the river segment is increasing, decreasing, and at its maximum.
(b) Calculate the storage change (in mm) in the river segment during the time periods of [0, 4 hr], and [6, 8 hr].
(c) Determine the amount of water (in mm) that is 'lost' or 'gained' in the river segment over the time period of 12 hours.
(d) What is the storage volume (in mm) at the end of the twelfth hour?

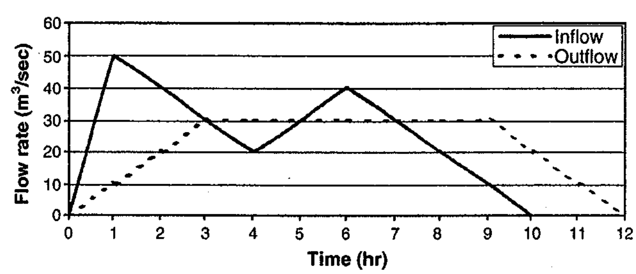


Figure P9.1.1

9.1.2 Consider a river segment with the surface area of 5 km^2. For a given flood event, the measured time variation of inflow rate (called inflow hydrograph, in m^3/sec) at the upstream section of the river segment and the outflow hydrograph at the downstream section are shown in Figure P9.1.1. Assume that the initial storage of water in the river segment is 10 mm in depth.

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