REVIEW OF UPW BASICS

Example: $\hat{x}$-polarized UPW traveling in $+\hat{z}$ direction

$$\bar{E} = \hat{x} E_o \cos(\omega t - kz)$$

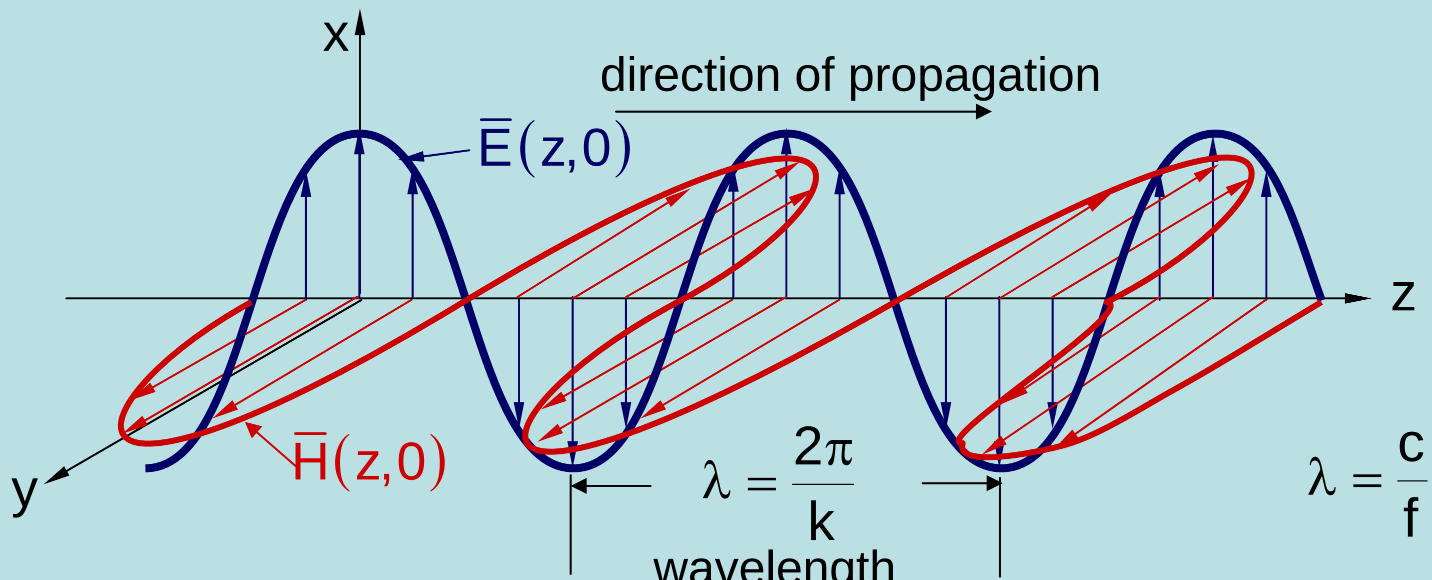
$$\bar{H} = \hat{y} \frac{E_o}{\eta_o} \cos(\omega t - kz)$$

$\bar{E} \times \bar{H}$: $\hat{z}$ direction of propagation

$$\underline{\bar{E}}(z) = \hat{x} \underline{E}_o e^{-jkz}$$

$$\underline{\bar{H}}(z) = \hat{y} \frac{\underline{E}_o}{\eta_o} e^{-jkz}$$

$$c = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = 3\text{x}10^8 \, \text{m/s} \qquad \omega(\text{rads/s}) = 2\pi f \qquad k(\text{rads/m}) = \frac{2\pi}{\lambda} = \frac{\omega}{c} \qquad \eta_o = \sqrt{\frac{\mu_o}{\varepsilon_0}}$$



HOW DO WAVES CONVEY POWER, ENERGY?

Recall: $\bar{E}\ [\text{V/m}] \bullet \bar{J}\ [\text{A/m}^2] = P_d\ [\text{W/m}^3]$ But $\bar{E} \perp \bar{H}$

Manipulate Ampere's law to get $\bar{E} \bullet \bar{J}$

$\bar{E}\cdot(\nabla\times\bar{H} = \bar{J} + \frac{\partial\bar{D}}{\partial t})$ For symmetry, compute $\bar{H}\cdot(\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t})$

$$\underbrace{\bar{H}\cdot\left(\nabla\times\bar{E}\right) - \bar{E}\cdot\left(\nabla\times\bar{H}\right)}_{\text{Vector Identity}} = -\bar{H}\cdot(\frac{\partial\bar{B}}{\partial t}) - \bar{E}\cdot(\bar{J} + \cdot\frac{\partial\bar{D}}{\partial t})$$

$$\boxed{\nabla\cdot\left(\bar{E}\times\bar{H}\right) = -\bar{H}\cdot\frac{\partial\bar{B}}{\partial t} - \bar{E}\cdot\bar{J} - \bar{E}\cdot\frac{\partial\bar{D}}{\partial t}\quad(\text{W/m}^3)}$$

This is $\boxed{\text{Poynting's Theorem}}$ What does it mean?

POYNTING THEOREM

Poynting's Theorem: $\nabla\cdot\left(\overline{E}\times\overline{H}\right) = -\overline{H}\cdot\dfrac{\partial\overline{B}}{\partial t} - \overline{E}\cdot\dfrac{\partial\overline{D}}{\partial t} - \overline{E}\cdot\overline{J}$

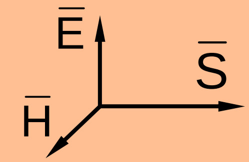
$\overline{B}=\mu\overline{H} \qquad \overline{D}=\varepsilon\overline{E}$

$$\boxed{\underbrace{\nabla\cdot\left(\overline{E}\times\overline{H}\right)}_{\substack{\text{Poynting}\\ \text{vector,}\\ \overline{S}\ [\mathrm{W/m^2}]}} = -\frac{d}{dt}\underbrace{\left(\frac{1}{2}\mu\left|\overline{H}\right|^2\right)}_{\substack{\text{Stored magnetic}\\ \text{energy density,}\\ W_m}} - \frac{d}{dt}\underbrace{\left(\frac{1}{2}\varepsilon\left|\overline{E}\right|^2\right)}_{\substack{\text{Stored electric}\\ \text{energy density,}\\ W_e}} - \underbrace{\left(\overline{E}\cdot\overline{J}\right)}_{\substack{\text{Power density}\\ \text{dissipated/m}^3,\\ W_d}}\ (\mathrm{W/m^3})}$$

Energy is conserved!

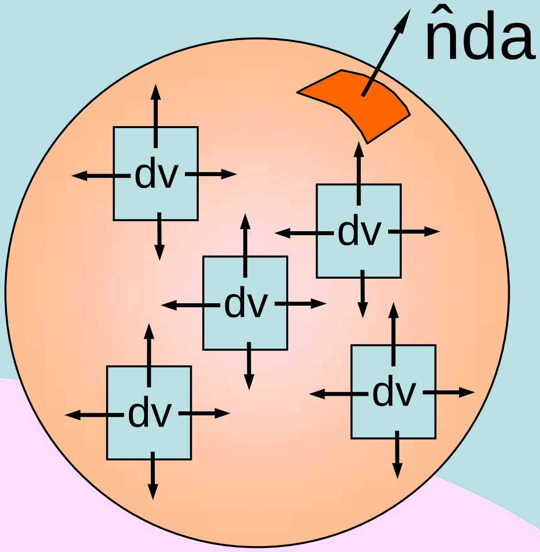
Note : $\dfrac{d}{dt}\left(\dfrac{1}{2}\mu\left|\overline{H}\right|^2\right)=\mu\left|\overline{H}\right|\dfrac{d\left|\overline{H}\right|}{dt}=\overline{H}\bullet\dfrac{d\overline{B}}{dt}$

<u>Poynting Vector</u> $\overline{S}=\overline{E}\times\overline{H}\ (\mathrm{W/m^2})\quad\left(\dfrac{\text{volts}}{\text{m}}\cdot\dfrac{\text{amps}}{\text{m}}=\dfrac{\text{watts}}{\text{m}^2}\right)$



INTEGRAL POYNTING THEOREM

Use: $\oint_S \bar{A}\cdot\hat{n}\, da = \int_V \nabla\cdot\bar{A}\, dv$



Gauss's Theorem (not Gauss's Law)

Therefore:

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\, da = \int_V \nabla\cdot(\bar{E}\times\bar{H})\, dv$$

$$= \int_V \left[-\frac{d}{dt}\left(\frac{1}{2}\mu|\bar{H}|^2\right) - \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2\right) - (\bar{E}\cdot\bar{J})\right] dv$$

$$\oint_S (\bar{E}\times\bar{H})\cdot\hat{n}\, da = -\int_V \frac{d}{dt}\left(\frac{1}{2}\varepsilon|\bar{E}|^2 + \frac{1}{2}\mu|\bar{H}|^2\right) dv - \int_V \bar{E}\cdot\bar{J}\, dv$$

Power emerging = released stored energy - dissipation [W]

The Poynting vector $\triangleq$ $\bar{S} = \bar{E}\times\bar{H}$ gives both the magnitude of the power density (intensity) and the direction of its flow.

UNIFORM PLANE WAVE EXAMPLE

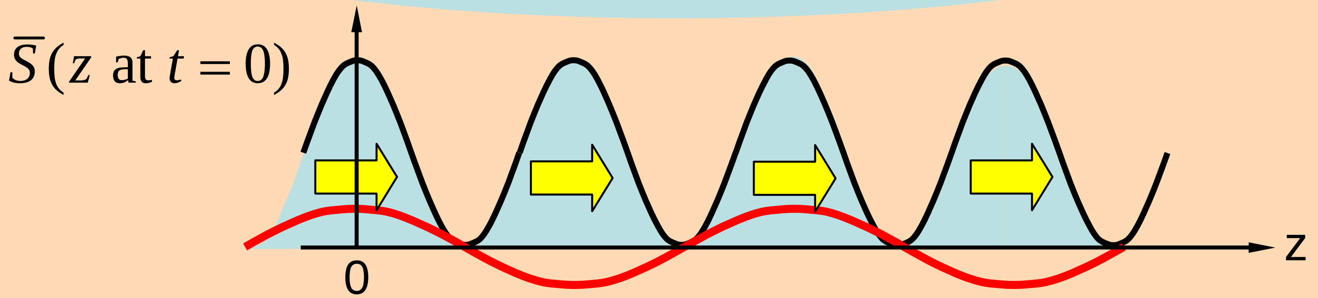
$$\bar{E} = \hat{x} E_o \cos(\omega t - kz)$$

$$W_e = \frac{1}{2}\varepsilon_o E_o^2 \cos^2(\omega t - kz)$$

$$\bar{H} = \hat{y}\left(\frac{E_o}{\eta_o}\right)\cos(\omega t - kz)$$

$$W_m = \frac{1}{2}\frac{\mu_o}{\eta_o^2} E_o^2 \cos^2(\omega t - kz)$$

$$\bar{S}(t) = \bar{E} \times \bar{H} = \hat{z}\left(\frac{E_o^2}{\eta_o}\right)\cos^2(\omega t - kz) \quad (\mathrm{W/m^2})$$



$$\bar{S} = \hat{z}\frac{E_o^2}{\eta_0}\cos^2(\omega t - kz) \Rightarrow \boxed{\langle\bar{S}\rangle = \hat{z}\frac{1}{2}\frac{E_o^2}{\eta_o} = \mathrm{I}(\theta, \phi, r) \ \ [\mathrm{w/m^2}]}$$

The time average $\langle\bar{S}(r,\theta,\phi)\rangle$ is “intensity” [W/m²]

COMPLEX NOTATION – POYNTING VECTOR

Defining a meaningful $\underline{\bar{S}}$ and relating it to $\bar{S}$ is not obvious.
Let's work backwards to find the time average $\langle \bar{S} \rangle$ and then $\underline{\bar{S}}$

$$\bar{S}(t) = \bar{E} \times \bar{H} = \mathrm{Re}\left[\underline{\bar{E}} \cdot e^{j\omega t}\right] \times \mathrm{Re}\left[\underline{\bar{H}} \cdot e^{j\omega t}\right]$$

$$= [\bar{E}_r \cos(\omega t) - \bar{E}_i \sin(\omega t)] \times [\bar{H}_r \cos(\omega t) - \bar{H}_i \sin(\omega t)]$$

$$\Rightarrow \langle \bar{S}(t) \rangle = \frac{1}{2}\left[\left(\underline{\bar{E}}_r \times \underline{\bar{H}}_r\right) + \left(\underline{\bar{E}}_i \times \underline{\bar{H}}_i\right)\right]$$

$$= \frac{1}{2} R_e \underbrace{\left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right)}_{\underline{\bar{S}} \text{ (by definition}} \quad \left[= \frac{1}{2} R_e\{(E_r + jE_i) \times (H_r - jH_i)\}\right]$$

Thus, we can define $\boxed{\langle \bar{S} \rangle = \frac{1}{2}\mathrm{Re}\left(\underline{\bar{E}} \times \underline{\bar{H}}^*\right)}$ and $\boxed{\underline{\bar{S}} = \underline{\bar{E}} \times \underline{\bar{H}}^*}$

Recall: $\underline{\bar{E}} = \bar{E}_r + j\bar{E}_i \qquad \underline{\bar{H}} = \bar{H}_r + j\bar{H}_i \qquad e^{j\omega t} = \cos\omega t + j\sin\omega t$

UPW REFLECTED BY PERFECT CONDUCTOR

$$\bar{E} = \hat{x}E_+ \cos(\omega t - kz) + \hat{x}E_- \cos(\omega t + kz)$$
$$= 0 \ \text{ at } \ z = 0 \text{ (perfect conductor)}$$

Forward plus reflected wave

$\Rightarrow E_- = -E_+$ (Solving for unknown reflection)

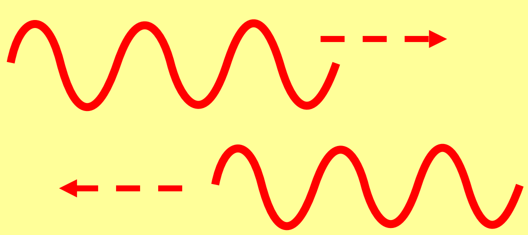
$\Rightarrow \bar{E} = \hat{x}2E_+ \sin\omega t \cdot \sin kz$ Standing waves, oscillate without moving

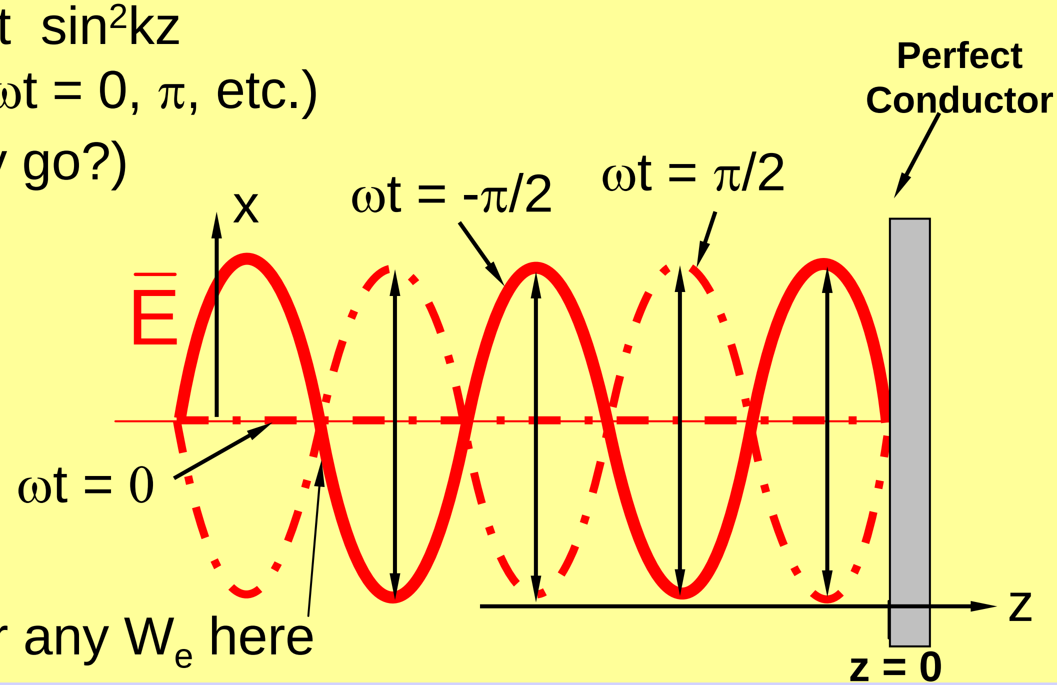
(recall: $\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2} \bullet \sin\frac{\alpha-\beta}{2}$)

W_e [J/m^3] = $2\,\varepsilon E^2 \sin^2\omega t \ \sin^2 kz$

E = 0 every half cycle ($\omega t = 0, \pi$, etc.)

(Where does the energy go?)





STANDING WAVE EXAMPLE - CONTINUED

(It's in the H field!)

$$\overline{E} = \hat{x}\,[E_+ \cos(\omega t - kz) + E_- \cos(\omega t + kz)]$$

$$\overline{H} = \hat{y}\,[\frac{E_+}{\eta_o}\cos(\omega t - kz) - \frac{E_-}{\eta_o} y \cos(\omega t + kz)]$$

Note

$$E_- = -E_+ \Rightarrow \quad \overline{H} = \hat{y}\frac{2E_+}{\eta_o}\cos\omega t \cdot \cos kz$$

$$W_m[\mathrm{J/m^3}] = \frac{1}{2}\mu_o\left(\frac{2E_+}{\eta_o}\right)^2 \cos^2\omega t \cos^2 kz = 2\varepsilon E_+^2 \cos^2\omega t \cos^2 kz$$

$$= 0 \text{ when } \omega t = \pi/2,\ 3\pi/2, \text{ etc.})$$

$$\overline{S} = \overline{E}\times\overline{H} = \hat{z}\ 4\frac{E_+^2}{\eta_o}\cos\omega t \sin\omega t \cdot \cos kz \sin kz$$

$$= \hat{z}\frac{E_+^2}{\eta_o}\cdot \sin 2kz \cdot \sin 2\omega t$$

$$\Rightarrow \langle \overline{S} \rangle = 0 = \frac{1}{2}\mathrm{Re}\left(\underline{\overline{E}} \times \underline{\overline{H}}^*\right)$$