In this course we consider mostly vector spaces with infinite dimension

the set $x_1 \dots x_n$ span the set $W \subset V$
if $\forall x \in W$

$$x = \alpha x_1 + \dots + x_n$$

$\forall x \in V$ $x = \alpha_1 x_1 \dots + \alpha_n x_n$, $\alpha_1 \dots \alpha_n$ unique
$(x_1 \dots x_n \dots)$ is called a basis

Vector subspaces

A nonempty subset V' of V is called a vector subspace of V if it is itself a vector space with respect to the addition and scalar multiplication.

In other words $x \in V'$, $y \in V'$ $\Longrightarrow$ $\alpha x + \beta y \in V'$ $\forall \alpha, \beta$

Every vector space V has two trivial subspaces.
$\{0\}$, V itself.

A vector subspace $\neq V$ having at least $\neq 0$ as element is called eigen space

Examples:

1 - V a vector space, $x \in V$ $x \neq 0$.
Set generated by x. $V_1 = \{ y = \lambda x \}$ is a vector subspace V_1

$y_1 \in V_1 \Longrightarrow y_1 = \lambda_1 x$
$y_2 \in V_1 \Longrightarrow y_2 = \lambda_2 x$

$$\alpha y_1 + \beta y_2 = \alpha \lambda_1 x + \beta \lambda_2 x = (\alpha \lambda_1 + \beta \lambda_2) x$$

Sub Space with dimension 1 $= \lambda x$

2. Vector Space of continuous function in the segment $[a, b]$
$C[a, b]$, $P[a, b]$, set of polynomials $P[a, b]$ is a vector subspace of C

Space l_2 sequences $x_1 \dots x_n$. such that

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty$$

c_0 . set of convergent sequences

l_2 is a subspace of c_0

C set of convergent sequences

c_0 is a subspace of C

m : set of bounded sequences.

c is a subspace of m

Definition Two vector spaces V and V^* are isomorphic if there is a one to one correspondence, compatibles with the defined operation. In ~~another~~ ~~on~~ word

$$\begin{array}{ccc} V & & V^* \\ x & \longleftrightarrow & x^* \\ y & \longleftrightarrow & y^* \end{array}$$

$$x+y \longleftrightarrow x^* + y^*$$

and $\qquad \lambda x \longleftrightarrow \lambda x^*$

Two isomorphic vector spaces can be considered as the same space.

Example : $\mathbb{R}^n \quad x = (\alpha_1, \alpha_2 \ldots\ldots \alpha_n) \in \mathbb{R}$

$\mathcal{P}$: Polynomial with degree $n-1$

$P(x) \in \mathcal{P} \qquad P(x) = a_0 + a_1 x + a_2 x^2 \ldots + a_{n-1} x^{n-1}$

~~consider~~ Consider.

~~$(\alpha_1 \ldots\ldots \alpha_n) \longrightarrow$~~

$a = (\alpha_0 \ldots\ldots \alpha_{n-1}) \in \mathbb{R}^n \longrightarrow a^* = \alpha_0 + \alpha_1 x \ldots\ldots \alpha_{n-1} x^{n-1}$

$b = (\beta_0, \ldots \beta_{n-1}) \in \mathbb{R}^n \longrightarrow b^* = \beta_0 + \beta_1 x \ldots\ldots \beta_{n-1} x^{n-1}$

$a+b = (\alpha_0 + \beta_0, \ldots \alpha_{n-1} + \beta_{n-1}) \longrightarrow (a+b)^* = (\alpha_0+\beta_0) + (\alpha_1+\beta_1)x \ldots (\alpha_{n-1}+\beta_{n-1})x^{n-1}$

$$(a+b)^* = \alpha_0 + \beta_0 + \alpha_1 x + \beta_1 x \ldots + \alpha_{n-1} x^{n-1} + \beta_{n-1} x^{n-1}$$

$$= (\alpha_0 + \alpha_1 x \ldots + \alpha_{n-1} x^{n-1}) + (\beta_0 + \beta_1 x \ldots + \beta_{n-1} x^{n-1})$$

$$= a^* + b^*$$

$$\lambda a = (\lambda\alpha_0, \lambda\alpha_1 \ldots \lambda\alpha_{n-1})$$

$$(\lambda a)^* = \lambda\alpha_0 + \lambda\alpha_1 x \ldots \lambda\alpha_{n-1} x^{n-1}$$

$$= \lambda(\alpha_0 + \alpha_1 x \ldots + \alpha_{n-1} x^{n-1})$$

$$= \lambda a^*$$

Linear functional

function $f : V \longrightarrow \mathbb{R}\ (\mathbb{C})$

$f(x+y) = f(x) + f(y)$ additive

homogeneous if

$f(\alpha x) = \alpha f(x)$

if $f(\alpha x) = \bar{\alpha} f(x)$. semi-homogeneous

An additive ~~functional~~ is and homogeneous functional is called linear functional

An additive and semi-homogeneous functional is called ~~sr~~ semi-linear.

<u>Examples</u>:

1) $\mathbb{R}^n$ $x = (x_1 \ldots x_n)$ $a = (a_1 \ldots a_n)$

$$f(x) = \sum_{i=1}^{n} a_i x_i$$ is a linear functional on $\mathbb{R}^n$

2) x a function $x(t)$

$$I[x] = \int_a^b x(t)\,dt$$ linear functional

3) $y_0(t)$ a continuous function on $[a,b]$, $x(t) \in C[a,b]$

$$F[x] = \int_a^b x(t)\, y_0(t)\, dt$$

$$F[x+y] = \int_a^b [x(t) + y(t)]\, y_0(t)\, dt = \int_a^b x(t)\, y_0(t)\, dt + \int_a^b y(t)\, y_0(t)\, dt$$

$$= F[x] + F[y]$$

4) $x \in C[a,b]$

$$\delta_{t_0}[x] = x(t_0)$$

$$\delta_{t_0}(x) = \int_a^b x(t)\, \delta(t - t_0)\, dt$$

δ : Dirac function

5). $x = (x_1, \dots \dots x_n \dots)$ sequence $\in \ell_2$

$$x \in \ell_2 \Leftrightarrow \sum_{n=1}^{\infty} |x_n|^2 < \infty$$

$$f_k(x) \longrightarrow x_k$$

Let f be a linear functional. $\{x \in V, f(x) = 0\}$
Ker f.

Normed Space

V a vector Space.
V is a normed space if there is a correspondence

$$V \longrightarrow \mathbb{R}^+$$
$$x \longrightarrow \|x\| \geq 0$$

such that

1) $\|x\| = 0$ if and only if $x = 0$
2) $\|\alpha x\| = |\alpha| \, \|x\|$
3) $\|x+y\| \leq \|x\| + \|y\|$

Complete Spaces.

Cauchy Sequence: $\forall \varepsilon > 0$, $\exists N(\varepsilon)$, such that
$\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N(\varepsilon)$

A normed Space is complete if every Cauchy sequence is ~~convergente~~

A complete normed space is called Banach Space.
B-Space

Examples of normed spaces.

1) $\mathbb{R}$. $\|x\| = |x|$

$|x| \geq 0$
1) $|x| = 0 \iff x = 0$
2) $|\alpha x| = |\alpha| \, |x|$
3) $|x+y| \leq |x| + |y|$

2) $x = (x_1, x_2, \ldots x_n)$

$$\|x\|_2 = \sqrt{\sum_{k=1}^{n} x_k^2}$$

We can define now the distance between two vectors:

$$d(x, y) = \|x - y\|$$

More generally. $x = (x_1, x_2 \dots x_n)$

$$\|x\|_p = \begin{cases} \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p} & 1 \le p < \infty \\ \max_{1 \le k \le n} |x_k| & p = \infty \end{cases}$$

The most used norms are $\| \|_1$, $\| \|_2$, $\| \|_\infty$

for a continuous function

$$\|x\|_p = \begin{cases} \left(\int_{-\infty}^{\infty} |x(t)|^p dt \right)^{1/p} & 1 \le p < \infty \\ \max |x(t)| & p = \infty \end{cases}$$

$\mathbb{R}^2$: $x = (1, 1)$

$\|x\|_1 = 1 + 1 = 2$

$\|x\|_2 = \sqrt{1+1} = \sqrt{2}$

$\|x\|_\infty = \max(1, 1) = 1$

<u>Euclidian Spaces</u>

Inner product (scalar product)

V a vector space

$V \times V \longrightarrow \mathbb{R}$ ~~($\mathbb{C}$)~~

$x, y \in V \longrightarrow \langle x, y \rangle \in \mathbb{R}$

Inner product has the following properties

1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for all $x, y \in V$

$\bar{a}$: complex conjugate of a

2) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$

3) $\langle x, x\rangle \geq 0$ $\quad \langle x, x\rangle = 0$ if only if $x = 0$

A vector space ~~E~~ with an ~~an~~ inner product (scalar product) is called. Euclidian space.

~~Before~~

Cauchy - Schwarz ~~ineq~~ inequality

$\cancel{|\langle x,y\rangle| \leq \|x\|}$

Natural norm induced by the inner product

$$\|x\| = \langle x, x\rangle^{1/2}$$

Prove that ~~is~~ it defines a norm.

1) $\|x\| = \langle x, x\rangle^{1/2} \geq 0$

2) $\|x\| = \langle x, x\rangle^{1/2} = 0 \iff x = 0$

3) $\|\alpha x\| = \langle \lambda x, \lambda x\rangle^{1/2} = (\lambda^2 \langle x, x\rangle)^{1/2} = |\lambda| \langle x, x\rangle^{1/2} = |\lambda| \, \|x\|$

$$\|x+y\| = \langle x+y, x+y\rangle^{1/2}$$

$$= \left[\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle\right]^{1/2}$$

$$= \left[\|x\|^2 + 2\langle x, y\rangle + \|y\|^2\right]^{1/2}$$

$$\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

$$\|x+y\|^2 \leq (\|x\| + \|y\|)^2$$

$$\|x+y\|^2 \leq \|x\|^2 + \|y\|^2 + 2\|x\| . \|y\|$$

This is true. if $\quad |\langle x, y\rangle| \leq \|x\| . \|y\|$

$|\langle x, y\rangle| \leq \|x\| \, \|y\|$ Cauchy-Schwartz inequality

Prove Cauchy-Schwartz inequality

$$\varphi(\lambda) = \langle \lambda x + y, \lambda x + y \rangle \geq 0 \quad \text{for any } \lambda$$

$$= \lambda^2 \langle x, x \rangle + \langle y, y \rangle + 2\lambda \langle x, y \rangle$$

$$= \lambda^2 \|x\|^2 + 2\lambda \langle x, y \rangle + \|y\|^2$$

quadratic equation with λ must be positive for any λ

$\Delta \leq 0$

$$\Delta = 4 \langle x, y \rangle^2 - 4\|x\|^2 \|y\|^2 \leq 0$$

$$\langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$$

$$|\langle x, y \rangle| \leq \|x\| \, \|y\|$$

Before we continue with more definitions. let's give few examples

l : set of all complex-valued time sequences

$x = (\dots (-1), x(0), \dots x(n) \dots$

is a linear space :

$(x + y)(n) = x(n) + y(n)$

$(\alpha x)(n) = \alpha x(n)$

$\mathcal{L}$ set of all continuous-time signals

$(x+y)(t) = x(t) + y(t)$

$(\alpha x)(t) = \alpha x(t)$

let's see the meaning of the norms $\| \|_1$, $\| \|_2$, $\| \|_\infty$ for the signals.

Discrete :

$\|x\|_\infty = \sup_{n \in \mathbb{Z}} |x(n)|$

Continuous

$\|x\|_\infty = \sup_{t \in \mathbb{R}} |x(t)|$

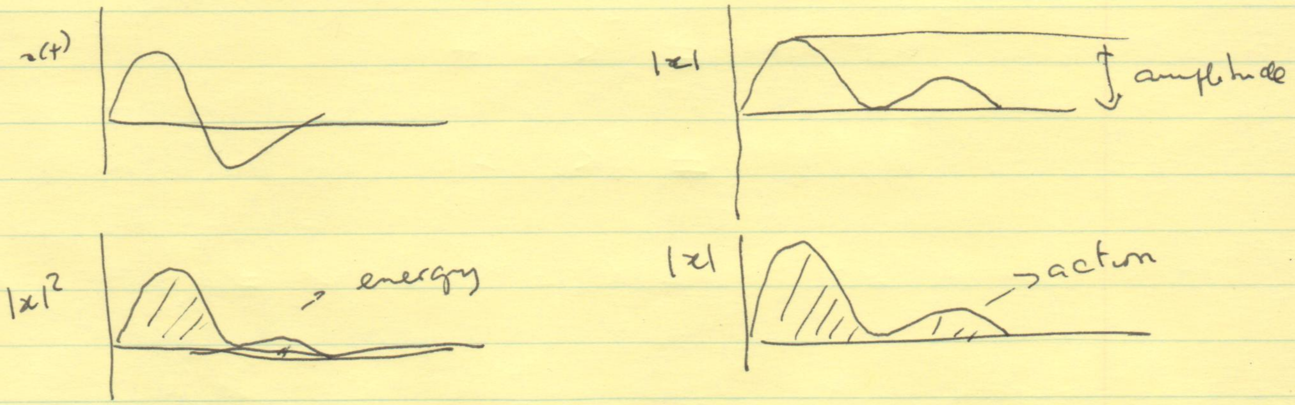
It is the amplitude. It is the largest magnitude the signal assumes

$$\|x\|_2^2 = \sum_{n=-\infty}^{+\infty} |x(n)|^2 \qquad \|x\|_2^2 = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

This is referred to as the energy of the signal

$$\|x\|_1 = \sum_{n=-\infty}^{+\infty} |x(n)| \qquad \|x\|_1 = \int_{-\infty}^{+\infty} |x(t)| dt$$

This called the action of the signal



$$l_\infty = \{ x \in l \quad \|x\|_\infty < +\infty \}$$

set of finite-amplitude ~~seque~~ time sequences.

$$\mathcal{L}_\infty = \{ x \in \mathcal{L} \quad \|x\|_\infty < \infty \}$$

set of finite amplitude continuous time signals

$$l_2 = \{ x \in l \quad \|x\|_2 < +\infty \}$$

set of all finite-energy times sequences

$$\mathcal{L}_2 : \{ x \in \mathcal{L} \quad \|x\|_2 < \infty \}$$

set of all finite energy continuous time signals

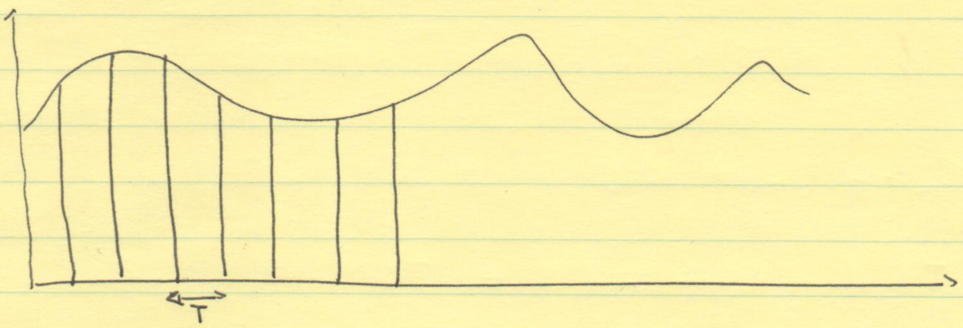
$$l_1 = \{ x \in l \quad \|x\|_1 < +\infty \}$$

set of all finite-action time sequence

$$\mathcal{L}_1 = \{ x \in \mathcal{L} \quad \|x\|_1 < +\infty \}$$

set of all finite-action continuous time signals

Norms of sampled signals



The p-norm of a sampled signal x given on the discrete time axis $\mathbb{Z}(T)$ is defined as

$$\|x\|_p = \begin{cases} \left(T \sum_{t \in \mathbb{Z}(T)} |x(t)|^p \right)^{1/p} & \text{for } 1 \leq p < \infty \\ \sup_{t \in \mathbb{Z}(T)} |x(t)| & \text{for } p = \infty \end{cases}$$

In particular

$$\|x\|_\infty = \sup_{t \in \mathbb{Z}(T)} |x(t)| \qquad \text{Amplitude}$$

$$\|x\|_2^2 = T \sum_{t \in \mathbb{Z}(T)} |x(t)|^2 \qquad \text{energy}$$

$$\|x\|_1 = T \sum_{t \in \mathbb{Z}(T)} |x(t)| \qquad \text{action}$$

The reason for including the factor T in the definition of the energy and action is that the resulting expressions are approximating sums for the integrals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt \quad \text{and} \quad \int_{-\infty}^{+\infty} |x(t)| dt$$

that determine the energy and action of the underlying continuous-time signals. The approximation improves as T decreases.

If the sampling intervals T equals 1, the sampled signal x may be viewed as a time sequence.

The sets

$$l_\infty(T) = \{x \in l(T), \|x\|_\infty < \infty\}$$

$$l_2(T) = \{x \in l(T), \|x\|_2 < \infty\}$$

$$l_1(T) = \{x \in l(T), \|x\|_1 < \infty\}$$

of all finite-amplitude, finite energy, and finite action sampled signals on time axis $\mathbb{Z}(T)$, respectively are normed spaces.

Examples: Inner products on $\mathbb{R}^n$ and $\mathbb{C}^n$

a) $\mathbb{R}^n$

$$x = (x_1 \; \ldots \; x_n)$$

$$y = (y_1 \; \ldots \; y_n)$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

b) $\mathbb{C}^n$ $\quad x = (x_1 \; \ldots \; x_n)$

$$y = (y_1 \; \ldots \; y_n)$$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i \overline{y_i}$$

The overbar denote the complex conjugate. The introduction of the conjugate guarantees the inner product $\langle x, x \rangle$ of any element x of $\mathbb{C}^N$ with itself to be real and positive

$$\langle x, x \rangle = \sum_{i=1}^{n} x_i \overline{x_i} = \sum_{n=1}^{n} |x_i|^2 = \|x\|_2^2$$

Inner Products on signal spaces

$$\langle x, y \rangle = \sum x(n) \overline{y(n)} \qquad \langle x, y \rangle = \int_{-\infty}^{+\infty} x(t) \overline{y(t)} \, dt$$

$$\langle x, x \rangle = \sum_n |x(n)|^2 \qquad \langle x, x \rangle = \int_{-\infty}^{+\infty} |x(t)|^2 \, dt$$

From the inner product, one not only can define the length of a vector (norm) but also the angle.

$$\cos \varphi = \frac{\langle x, y \rangle}{\|x\| \, \|y\|}$$

From Cauchy-Schwarz inequality

$$|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$$

$$-\|x\| \cdot \|y\| \leq \langle x, y \rangle \leq \|x\| \cdot \|y\|$$

$$-1 \leq \frac{\langle x, y \rangle}{\|x\| \, \|y\|} \leq 1$$

This formula defines ~~a cos~~ an angle $0 \leq \varphi \leq \pi$

if $x \neq 0$, $y \neq 0$ $\langle x, y \rangle = 0 \Rightarrow \cos \varphi = 0$

$\varphi = \frac{\pi}{2}$

~~xy~~ x and y are said to be ~~o~~ orthogonal

A system of non zero vectors $\{x_\alpha\}$ is orthogonal if

$$\langle x_\alpha, x_\beta \rangle = 0 \quad \text{for} \quad \alpha \neq \beta$$

If the vectors $\{x_\alpha\}$ are orthogonal, then they are linearly independent.

$$a_1 x_1 + a_2 x_2 \cdots + a_n x_n = 0$$

$\langle x_i, a_1x_1 + a_2x_2 \dots a_i x_i \quad + a_n x_n \rangle = a_1 \langle x_i, x_1 \rangle + a_2 \langle x_i, x_2 \rangle$
$+ \dots \quad a_i \langle x_i, x_i \rangle \quad \dots + a_n \langle x_i, x_n \rangle = a_i \langle x_i, x_i \rangle = 0$
$\Longrightarrow a_i = 0$, repeat for $i = 1 \to n$

If An orthogonal system $\{x_\alpha\}$ is complete it = an orthogonal basis.
Complete means. it may span the entire space
If the norm of x_α is 1 it is an orthonormal basis

$$\langle x_\alpha, x_\beta \rangle = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

the system is orthonormal.
If $\{x_\alpha\}$ is an orthogonal system $\to \left\{ \frac{x_\alpha}{\|x_\alpha\|} \right\}$ is an orthonormal system

Examples.
$\mathbb{R}^n$

1) $x = (x_1, x_2 \dots, x_n)$

$$\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$$

$\mathbb{R}^n$ is an Euclidian space

$e_1 = (1, 0 \dots 0)$
$e_2 = (0, 1, 0 \dots 0)$
$\vdots$
$e_n = (0, 0 \dots, 1)$

form an orthonormal basis

2) l_2

$$x = (x_1, x_2 \quad \dots \quad x_n), \quad \sum_{i=1}^{\infty} x_i^2 < \infty$$

with inner product

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i$$

$\langle x, y \rangle$ is finite $\quad \left(\sum_{i=1}^{\infty} x_i y_i \right)^2 \leq \sum_{i=1}^{\infty} \|x_0\|^2 \cdot \sum |y_i|^2 < \infty$

a base for l_2 would be

$$e_1 = (1, 0 \dots \dots \dots)$$
$$e_2 = (0, 1 \dots \dots \dots)$$
$$e_3 = (0, 0, 1, \dots \dots)$$
$$\vdots$$

this system is obviously orthonormal. Let's prove it is complete

Any $x \in l_2$ can be a linear combination of $e_1, e_2 \dots e_n \dots$

$x = (x_1, x_2 \quad x_n \dots) \in l_2$

$x^{(n)} = (x_1, x_2 \dots x_n, 0, 0, \dots)$

$x^{(n)} = \sum_{i=1}^{n} x_i e_i$

$$\|x^{(n)} - x\| \to 0 \quad n \to +\infty$$

Orthogonalization theorem.

$f_1, f_2, \dots f_n$ is a system of linearly independent vectors in V. There exist a system of vectors $\varphi_1, \varphi_2 \dots \varphi_n \dots$ such that

1) $\varphi_1, \varphi_2 \dots \varphi_n \dots$ is orthonormal
2) each element φ_n is a linear combination of $f_1 \dots f_n$

$$\varphi_n = a_{n1} f_1 \dots a_{nn} f_n \qquad \text{where } a_{nn} \neq 0$$

no

no

$$\varphi_2 = \frac{\psi_2}{\|\psi_2\|} = 2\sqrt{3}\left(t - \tfrac{1}{2}\right)$$

$$\psi_3 = t^2 - \cancel{\langle t^2, 1\rangle 1} - \langle t^2, 1\rangle 1 - \langle t^2, 2\sqrt{3}(t - \tfrac{1}{2})\rangle\, 2\sqrt{3}(t - \tfrac{1}{2})$$

$$\psi_3 = t^2 - \langle t^2, 1\rangle 1 - 12 \langle t^2, (t - \tfrac{1}{2})\rangle (t - \tfrac{1}{2})$$

$$\psi_3 = t^2 - \tfrac{1}{3} - (t - \tfrac{1}{2})$$

$$\psi_3 = t^2 - t + \tfrac{1}{6}$$

$$\varphi_3 = \frac{\psi_3}{\|\psi_3\|}$$

$$\|\psi_3\|^2 = \int_0^1 \left(t^2 - t + \tfrac{1}{6}\right)^2 dt = \frac{1}{180}$$

$$\|\psi_3\| = \frac{1}{\sqrt{180}} = \frac{1}{6\sqrt{5}}$$

$$\varphi_3 = 6\sqrt{5}\left(t^2 - t + \tfrac{1}{6}\right)$$

$\{\varphi_1, \varphi_2, \varphi_3\}$ is an orthonormal ~~basis~~ system

~~Bessel Inequality~~

Hilbert Space

Definition: ~~An Euclidian space~~ A complete Euclidian space with infinite-dimension is called an Hilbert Space

An Hilbert Space H ~~has~~ satisfies the following ~~properties~~ conditions

1) H is an Euclidian Space (Vector space with inner product)

2) H is complete in the sens of the norms.

3) H is has an of infinite dimension that is for every n one can find n linearly independent vectors

4) H is separable. that is there exists inside H countable and dense space in H

Riesz - Fischer Theorem   Let be $\{\varphi_n\}$ an orthonormal system of in an complete Euclidean space V ~~et~~ and let

$c_1, c_2 \ldots c_n \ldots$ such that

$$\sum_{k=1}^{\infty} c_k^2 < \infty$$

Then there exists $f \in V$ such that

$$c_k = \langle f, \varphi_k \rangle$$

and

$$\sum_{k=1}^{\infty} c_k^2 = \langle f, f \rangle = \|f\|^2$$

Hilbert space and isomorphism

l_2 , H have the same properties

Isomorphism between Hilbert spaces

$$x \in H \qquad x^* \in H^*$$
$$y \in H \qquad y^* \in H^*$$
$$x \longrightarrow x^* \ , \ y \longrightarrow y^*$$
$$x+y \longrightarrow x^*+y^*$$

$$\langle x, y \rangle \longleftrightarrow \langle x^*, y^* \rangle$$

All Hilbert spaces are isomorphics between themselves

let show that H is isomorphic to ℓ_2

Assume there is an orthonormal basis $\{\varphi_n\}$ . . . . .

to every $f \in H \longrightarrow$ sequence $c_1 \ldots c_n \ldots$ as Fourier coefficients of f

$$f = \sum_{n=1}^{\infty} c_k \varphi_k$$

$$\Rightarrow \sum_{n=1}^{\infty} c_k^2 < +\infty \quad \Rightarrow \quad (c_1 \ldots c_n) \in \ell_2$$

From Riesz-Fischer Theorem for every $(c_1 \ldots c_n \ldots)$ $\in \ell_2$ there exist f such that $\langle f, \varphi_k \rangle = c_k$

So there is a one-to-one correspondence between. H and ℓ_2

$$f \longleftrightarrow (c_1 \ldots c_n)$$
$$g \longleftrightarrow (d_1 \ldots d_n)$$

$$f+g \longleftrightarrow (c_1+d_1, \ldots c_n+d_n, \ldots)$$

$$\alpha f \longleftrightarrow (\alpha c_1, \ldots \alpha c_n)$$

Let's see if $\langle f, g \rangle \rightarrow \sum_{n=1}^{\infty} c_n d_n$

$$\langle f,f\rangle = \|f\|_2^2 = \sum_{n=1}^{\infty} c_n^2 \quad , \quad \langle g,g\rangle = \|g\|_2^2 = \sum_{n=1}^{\infty} d_n^2$$

Parseval Theorem. (Energy conservation

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \sum_{n=1}^{\infty} |c_n|^2$$

$$\langle f+g, f+g\rangle = \langle f,f\rangle + 2\langle f,g\rangle + \langle g,g\rangle$$

$$= \sum_{n=1}^{\infty} (c_n + d_n)^2 = \sum_{n=1}^{\infty} c_n^2 + 2c_n d_n \; d_n^2$$

$$= \sum_{n=1}^{\infty} c_n^2 + \sum_{n=1}^{\infty} d_n^2 + 2\sum_{n=1}^{\infty} c_n d_n$$

$$\langle f,g\rangle = \sum_{n=1}^{\infty} c_n d_n$$

<u>Hilbert subspaces</u>

$L \subset H \qquad f \in L \; , \; g \in L \quad , \; \alpha f + \beta g \in L$

and $\quad \lim_{n\to+\infty} f_n \in L$ if $f_1 \ldots f_n \ldots \in L$

$\longrightarrow \quad \Rightarrow L$ is a closed space

Then L is an Hilbert subspace

<u>Examples</u>

1) let $h \in H$, $G = \{ g \in H , \; \langle g,h\rangle = 0 \}$

G : set of all elements orthogonal to h.

$g \in G$
$k \in G \qquad \alpha g + \beta k \overset{?}{\in} G$

$$\langle \alpha g + \beta k ; h\rangle = \langle \alpha g, h\rangle + \langle \beta k, h\rangle$$

$$= \alpha \langle g,h\rangle + \beta\langle k,h\rangle \doteq 0$$

since $g \in G$, $k \in G \Rightarrow \langle g,h\rangle = \langle k,h\rangle = 0$

$\Rightarrow \langle \alpha g + \beta k, h\rangle = 0 \quad \alpha g + \beta k \in G$

$g_n \in G \qquad \langle g_n, h\rangle = 0$

$\lim_{n\to+\infty} \langle g_n, h\rangle = 0 \Rightarrow \lim g_n \in G$

2) $(x_1, x_2 \dots x_n \dots) \in l_2$

$L \subset l_2$. $(x_1 \dots x_n \dots)$ ~~$x_{2p} = 0$~~ $x_1 = x_2$

verify

3) $(x_1, x_2 \dots x_n)$

$L \subset l_2$ .. $(x_1 \dots x_n)$ $x_{2p} = 0$

In Every ~~Hilbert~~ subspace M of an Hilbert Space H, there is an orthonormal system $\{\varphi_n\}$ which span M

$$H = M \oplus M^\perp$$

$M^\perp$ is the orthogonal supplementary of M.

For every $f \in H$ $\quad f = h + h'$ $\quad h \in M$, $h' \in M^\perp$

~~H~~ H is the direct sum of ~~H~~ M and $M^\perp$

$$H = M_1 \oplus M_2 \oplus \dots \oplus M_n \dots$$

M_i and M_j are orthogonal. $i \neq j$

$\forall f \in H$ $\quad f = f_1 + f_2 \dots + f_n \dots$

$f_1 \in M_1$, $f_2 \in M_2$ - - - -