

Frames and Riesz Bases in Hilbert Space.

A. Nonorthogonal Bases in Finite Dimensions.

Definition 1 A finite collection of elements, $\{x_i\}_{i=1}^n$ in a linear space L is linearly independent if any collection of scalars $\{\alpha_i\}_{i=1}^n$ satisfying $\sum_{i=1}^n \alpha_i x_i = 0$ must all be zero. Otherwise, the collection $\{x_i\}_{i=1}^n$ is linearly dependent.

A linear space is n -dimensional if there exist n linearly independent elements in L but every set of $n + 1$ elements is linearly dependent. If there is a set of n linearly independent elements for every n , then L is infinite-dimensional.

A set of n linearly independent elements in an n -dimensional linear space is a basis for that space.

Remarks.

1. If $\{v_k\}_{k=1}^n$ is a basis for the n -dimensional linear space then every $x \in L$ can be written uniquely as $x = \sum_k \alpha_k v_k$. Every such linear space has a basis and any n -dimensional linear space L (over \mathbf{R} or \mathbf{C}) is isomorphic to \mathbf{R}^n (or to \mathbf{C}^n). Also by means of this isomorphism, any such L can be equipped with an inner product, so that we can talk about *orthogonality* of such vectors. From now on we will treat L as though it were \mathbf{R}^n or \mathbf{C}^n .
2. Any finite-dimensional linear space L with an inner product has an orthonormal basis. This can be obtained from any basis using the Gram-Schmidt process.
3. Given two bases for L , $\{v_k\}_{k=1}^n$ and $\{w_k\}_{k=1}^n$ there is a unique $n \times n$ matrix T with the property that if $\alpha = (\alpha_1, \dots, \alpha_n)$ contains the expansion coefficients of x in the first basis, then $T\alpha$ contains the expansion coefficients of x in the second. T is called the *change-of-basis* matrix.
4. Any basis $\{v_k\}_{k=1}^n$ for \mathbf{R}^n is the image under an invertible linear transformation of an orthonormal basis.

B. Riesz Bases in Hilbert Spaces.

Definition 2 A collection of vectors $\{x_k\}_k$ in a Hilbert space H is a Riesz basis for H if it is the image of an orthonormal basis for H under an invertible linear transformation. In other words, if there is an orthonormal basis $\{e_k\}$ for H and an invertible transformation T such that $Te_k = x_k$ for all k .

Theorem 1 Let $\{x_k\}$ be a collection of vectors in a Hilbert space H .

- (a) If $\{x_k\}$ is a Riesz basis for H then there is a unique collection $\{y_k\}$ such that $\langle x_k, y_k \rangle = \delta_k$, that is such that $\{y_k\}$ is biorthogonal to $\{x_k\}$. In this case $\{y_k\}$ is also a Riesz basis.
- (b) If $\{x_k\}$ is a Riesz basis for H then there are constants $0 \leq A \leq B$ such that for all $x \in H$, $A\|x\|^2 \leq \sum_k |\langle x, x_k \rangle|^2 \leq B\|x\|^2$. This inequality is called the frame inequality.

- (c) $\{x_k\}$ is a Riesz basis for H if and only if there are constants $0 \leq A \leq B$ such that for all finite sequences $\{\alpha_k\}$, $A \sum_k |\alpha_k|^2 \leq \left\| \sum_k \alpha_k x_k \right\|^2 \leq B \sum_k |\alpha_k|^2$.
- (d) If $\{x_k\}$ is a Riesz basis for H then for each $x \in H$ there is a unique collection of scalars $\{\alpha_k\}$ such that $x = \sum_k \alpha_k x_k$ and $\sum_k |\alpha_k|^2 < \infty$.

Remarks.

1. Note that if $\{x_k\}$ were an orthonormal basis then (a) would be obvious (just take $y_k = x_k$) and (b) would hold with $A = B = 1$ (Plancherel's formula). In fact we have seen that if $\{x_k\}$ satisfies the frame inequality with $A = B = 1$ and if $\|x_k\| = 1$ for all k , then $\{x_k\}$ is an orthonormal basis for H .
2. A different way to characterize some of these properties is to think of two operators associated to a Riesz basis $\{x_k\}$. The first is the *analysis operator* $T: H \rightarrow l^2$ given by $T(x) = \{\langle x, x_k \rangle\}_k$. That $T(x) \in l^2$ (and further that T is a bounded linear operator) follows from the frame inequality. The second is the *synthesis operator* from $l^2 \rightarrow H$ given by $\{\alpha_k\} \mapsto \sum_k \alpha_k x_k$. Since the synthesis operator is the adjoint of T , we will just denote it by T^* .
3. In this language, $\{x_k\}$ is a Riesz basis if and only if T is a bounded linear bijection from H onto l^2 . In other words there is a one to one correspondence between sequences of the form $\{\langle x, x_k \rangle\}_k$ and sequences in l^2 . In other words, every l^2 sequence gets "hit" by something in H through the analysis operator. In still other words, this statement is equivalent to (c) above.

Definition 3 Let H be an infinite-dimensional Hilbert space. An infinite collection $\{x_k\}$ of vectors in H is (finitely) linearly independent if every finite subset of $\{x_k\}$ is linearly independent. It is ω -linearly independent if a sequence $\{\alpha_k\}$ such that $\sum_k \alpha_k x_k$ converges in the norm of H to 0 must be identically zero.

Lemma 1 If $\{x_k\}$ is ω -linearly independent then it is linearly independent. However a sequence can be finitely linearly independent without being ω -linearly independent.

Remark.

If $\{x_k\}$ is an orthonormal basis, then it is ω -linearly independent. If $\{x_k\}$ is a Riesz basis, then it is ω -linearly independent. Both of these facts follow from the assertion that an orthonormal or Riesz basis has a biorthogonal sequence.

Theorem 2 A sequence $\{x_k\}$ in a Hilbert space H is a Riesz basis for H if and only if $\{x_k\}$ satisfies the frame condition and is ω -linearly independent.

C. Frames in Hilbert Spaces.

Definition 4 A sequence $\{x_k\}$ in a Hilbert space H is a frame if there exist numbers $A, B > 0$ such that for all $x \in H$ we have

$$A\|x\|^2 \leq \sum_k |\langle x, x_n \rangle|^2 \leq B\|x\|^2.$$

The numbers A, B are called the frame bounds. The frame is tight if $A = B$. The frame is exact if it ceases to be a frame whenever any single element is deleted from the sequence.

Remarks.

1. From the Plancherel formula we see that every orthonormal basis is a tight exact frame with $A = B = 1$. For orthonormal bases, the Plancherel formula is equivalent to the basis property, which gives a decomposition of the Hilbert space. The weakened form of the Plancherel formula satisfied by frames also gives a decomposition, although the representations need not be unique.

2. A frame is a complete set since if $x \in H$ satisfies $\langle x, x_n \rangle = 0$ for all n , then $A\|x\|^2 \leq \sum |\langle x, x_n \rangle|^2 = 0$, so $x = 0$.

3. A tight frame need not be exact and vice versa. For example let $\{e_n\}_{n=1}^\infty$ be an orthonormal basis for H . Then

- (a) $\{e_1, e_1, e_2, e_2, e_3, e_3, \dots\}$ is a tight inexact frame with bounds $A = B = 2$, but is not an orthonormal basis, although it contains one.
- (b) $\{e_1, e_2/2, e_3/3, \dots\}$ is a complete orthogonal sequence, but not a frame.
- (c) $\{e_1, e_2/\sqrt{2}, e_2/\sqrt{2}, e_3/\sqrt{3}, e_3/\sqrt{3}, e_3/\sqrt{3}, \dots\}$ is a tight frame with bounds $A = B = 1$ but is not an exact sequence, and no nonredundant subsequence is a frame.
- (d) $\{2e_1, e_2, e_3, \dots\}$ is an exact frame with bounds $A = 1, B = 2$ and is not a tight frame.

Theorem 3 Given a sequence $\{x_n\}$ in a Hilbert space H , the following two statements are equivalent:

- (a) $\{x_n\}$ is a frame with bounds A, B .
- (b) $Sx = \sum \langle x, x_n \rangle x_n$ is a bounded linear operator with $AI \leq S \leq BI$, called the frame operator for $\{x_n\}$.

Corollary 1 (a) S is invertible and $B^{-1}I \leq S^{-1} \leq A^{-1}I$.

- (b) $\{S^{-1}x_n\}$ is a frame with bounds B^{-1}, A^{-1} , called the dual frame of $\{x_n\}$.
- (c) Every $x \in H$ can be written $x = \sum \langle x, S^{-1}x_n \rangle x_n = \sum \langle x, x_n \rangle S^{-1}x_n$.

Theorem 4 Given a frame $\{x_n\}$ and given $x \in H$ let $a_n = \langle x, S^{-1}x_n \rangle$, so $x = \sum a_n x_n$. If it is possible to find other scalars c_n such that $x = \sum c_n x_n$ then $\sum |c_n|^2 = \sum |a_n|^2 + \sum |a_n - c_n|^2$.

Theorem 5 The removal of a vector from a frame leaves either a frame or an incomplete set. In particular, if for a given m , $\langle x_m, S^{-1}x_m \rangle \neq 1$ then $\{x_n\}_{n \neq m}$ is a frame; and if $\langle x_m, S^{-1}x_m \rangle = 1$ then $\{x_n\}_{n \neq m}$ is not complete in H .

Corollary 2 If $\{x_n\}$ is an exact frame, then $\{x_n\}$ and $\{S^{-1}x_n\}$ are biorthogonal, i.e., $\langle x_m, S^{-1}x_n \rangle = \delta_{mn}$.

Theorem 6 A sequence $\{x_n\}$ in a Hilbert space H is an exact frame for H if and only if it is a Riesz basis for H .

D. Example: Nonharmonic Fourier Series.

One illustrative way to generate examples of Riesz bases and frames is as ‘‘perturbations’’ of orthonormal bases. One classic result in this direction is the following

Theorem 7 (Paley-Wiener) Let $\{e_k\}$ be an orthonormal basis for the Hilbert space H and suppose that $\{x_k\}$ is a sequence in H with the property that for some $0 \leq \lambda < 1$,

$$\left\| \sum_k \alpha_k (e_k - x_k) \right\| \leq \lambda \left(\sum_k |\alpha_k|^2 \right)^{1/2}$$

for every finite sequence $\{\alpha_k\}$. Then $\{x_k\}$ is a Riesz basis for H .

The proof of this theorem uses the following important result from operator theory.

Lemma 2 A bounded linear operator T on a Hilbert space is invertible whenever $\|I - T\| < 1$.

One application of the theorem of Paley and Wiener is to the problem of *nonharmonic Fourier series*. We know that the collection $\{e^{2\pi i n t}\}_{n \in \mathbf{Z}}$ is an orthonormal basis for $L^2[0, 1]$. What are the basis properties of collections of the form $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbf{Z}}$ where $\{\lambda_n\}$ is a sequence of real or complex numbers?

The following result follows directly from the above theorem of Paley and Wiener.

Theorem 8 There is an $\epsilon > 0$ such that whenever $|\lambda_n - n| < \epsilon$ then $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbf{Z}}$ is a Riesz basis for $L^2[0, 1]$.

An interesting question is: What is the largest value of ϵ for which the above theorem is valid? The answer is $\epsilon = 1/4$. This result is known as the Kadec 1/4 theorem. Implicit in this solution is the statement that the theorem fails if $\epsilon \geq 1/4$. In fact the following is true.

Theorem 9 The collection $\{e^{\pm 2\pi i (n-1/4)t}\}_{n=1}^{\infty}$ is complete in $L^2[0, 1]$.

This implies that the collection $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbf{Z}}$ where $\lambda_n = n - 1/4$ if $n > 0$, $n + 1/4$ if $n < 0$ and 0 if $n = 0$ is not a Riesz basis for $L^2[0, 1]$.

In terms of frames of exponentials, the following result is due to Duffin and Schaeffer.

Theorem 10 Suppose there are constants $L > 0$ and $\epsilon > 0$ such that (1) for all n , $|\lambda_n - n| < L$, and (2) for all $n \neq m$, $|\lambda_n - \lambda_m| \geq \epsilon$. Then the collection $\{e^{2\pi i \lambda_n t}\}_{n \in \mathbf{Z}}$ is a frame for $L^2[0, \gamma]$ for all $0 \leq \gamma < 1$.