Problem Solutions – Chapter 4

Problem 4.1.1 Solution

(a) The probability $P[X \leq 2, Y \leq 3]$ can be found be evaluating the joint CDF $F_{X,Y}(x, y)$ at $x = 2$ and $y = 3$. This yields

$$P[X \leq 2, Y \leq 3] = F_{X,Y}(2, 3) = (1 - e^{-2})(1 - e^{-3}) \tag{1}$$

(b) To find the marginal CDF of X, $F_X(x)$, we simply evaluate the joint CDF at $y = \infty$.

$$F_X(x) = F_{X,Y}(x, \infty) = \begin{cases} 1 - e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(c) Likewise for the marginal CDF of Y, we evaluate the joint CDF at $X = \infty$.

$$F_Y(y) = F_{X,Y}(\infty, y) = \begin{cases} 1 - e^{-y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.1.2 Solution

(a) Because the probability that any random variable is less than $-\infty$ is zero, we have

$$F_{X,Y}(x, -\infty) = P[X \leq x, Y \leq -\infty] \leq P[Y \leq -\infty] = 0 \tag{1}$$

(b) The probability that any random variable is less than infinity is always one.

$$F_{X,Y}(x, \infty) = P[X \leq x, Y \leq \infty] = P[X \leq x] = F_X(x) \tag{2}$$

(c) Although $P[Y \leq \infty] = 1$, $P[X \leq -\infty] = 0$. Therefore the following is true.

$$F_{X,Y}(-\infty, \infty) = P[X \leq -\infty, Y \leq \infty] \leq P[X \leq -\infty] = 0 \tag{3}$$

(d) Part (d) follows the same logic as that of part (a).

$$F_{X,Y}(-\infty, y) = P[X \leq -\infty, Y \leq y] \leq P[X \leq -\infty] = 0 \tag{4}$$

(e) Analogous to Part (b), we find that

$$F_{X,Y}(\infty, y) = P[X \leq \infty, Y \leq y] = P[Y \leq y] = F_Y(y) \tag{5}$$

Problem 4.1.3 Solution

We wish to find $P[x_1 \leq X \leq x_2]$ or $P[y_1 \leq Y \leq y_2]$. We define events $A = \{y_1 \leq Y \leq y_2\}$ and $B = \{y_1 \leq Y \leq y_2\}$ so that

$$P[A \cup B] = P[A] + P[B] - P[AB] \tag{1}$$

Keep in mind that the intersection of events A and B are all the outcomes such that both A and B occur, specifically, $AB = \{x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2\}$. It follows that

$$\begin{aligned} P[A \cup B] = P[x_1 \leq X \leq x_2] + P[y_1 \leq Y \leq y_2] \\ - P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2]. \end{aligned} \tag{2}$$

By Theorem 4.5,

$$\begin{aligned} &P[x_1 \leq X \leq x_2, y_1 \leq Y \leq y_2] \\ &\quad = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1). \end{aligned} \tag{3}$$

Expressed in terms of the marginal and joint CDFs,

$$P[A \cup B] = F_X(x_2) - F_X(x_1) + F_Y(y_2) - F_Y(y_1) \tag{4}$$

$$- F_{X,Y}(x_2, y_2) + F_{X,Y}(x_2, y_1) + F_{X,Y}(x_1, y_2) - F_{X,Y}(x_1, y_1) \tag{5}$$

Problem 4.1.4 Solution

Its easy to show that the properties of Theorem 4.1 are satisfied. However, those properties are necessary but not sufficient to show $F(x, y)$ is a CDF. To convince ourselves that $F(x, y)$ is a valid CDF, we show that for all $x_1 \leq x_2$ and $y_1 \leq y_2$,

$$P[x_1 < X_1 \leq x_2, y_1 < Y \leq y_2] \geq 0 \tag{1}$$

In this case, for $x_1 \leq x_2$ and $y_1 \leq y_2$, Theorem 4.5 yields

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F(x_2, y_2) - F(x_1, y_2) - F(x_2, y_1) + F(x_1, y_1) \tag{2}$$

$$= F_X(x_2) F_Y(y_2) - F_X(x_1) F_Y(y_2) \tag{3}$$

$$- F_X(x_2) F_Y(y_1) + F_X(x_1) F_Y(y_1) \tag{4}$$

$$= [F_X(x_2) - F_X(x_1)][F_Y(y_2) - F_Y(y_1)] \tag{5}$$

$$\geq 0 \tag{6}$$

Hence, $F_X(x)F_Y(y)$ is a valid joint CDF.

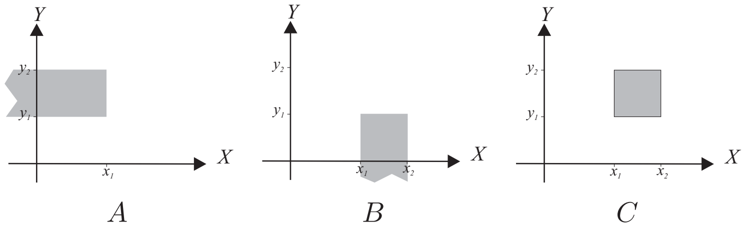
Problem 4.1.5 Solution

In this problem, we prove Theorem 4.5 which states

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) \tag{1}$$

$$- F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1) \tag{2}$$

(a) The events A, B, and C are



(3)

(b) In terms of the joint CDF $F_{X,Y}(x,y)$, we can write

$$P[A] = F_{X,Y}(x_1,y_2) - F_{X,Y}(x_1,y_1) \quad (4)$$
$$P[B] = F_{X,Y}(x_2,y_1) - F_{X,Y}(x_1,y_1) \quad (5)$$
$$P[A \cup B \cup C] = F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_1) \quad (6)$$

(c) Since A, B, and C are mutually exclusive,

$$P[A \cup B \cup C] = P[A] + P[B] + P[C] \quad (7)$$

However, since we want to express

$$P[C] = P[x_1 < X \le x_2, y_1 < Y \le y_2] \quad (8)$$

in terms of the joint CDF $F_{X,Y}(x,y)$, we write

$$\begin{aligned} P[C] &= P[A \cup B \cup C] - P[A] - P[B] & (9) \\ &= F_{X,Y}(x_2,y_2) - F_{X,Y}(x_1,y_2) - F_{X,Y}(x_2,y_1) + F_{X,Y}(x_1,y_1) & (10) \end{aligned}$$

which completes the proof of the theorem.

Problem 4.1.6 Solution

The given function is

$$F_{X,Y}(x,y) = \begin{cases} 1 - e^{-(x+y)} & x, y \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

First, we find the CDF $F_X(x)$ and $F_Y(y)$.

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 1 & x \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$
$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 1 & y \ge 0 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Hence, for any $x \ge 0$ or $y \ge 0$,

$$P[X > x] = 0 \qquad P[Y > y] = 0 \quad (4)$$

For $x \ge 0$ and $y \ge 0$, this implies

$$P[\{X > x\} \cup \{Y > y\}] \le P[X > x] + P[Y > y] = 0 \quad (5)$$

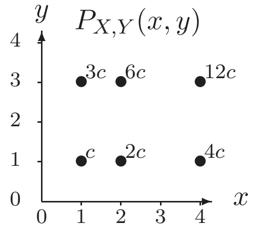
However,

$$P[\{X > x\} \cup \{Y > y\}] = 1 - P[X \le x, Y \le y] = 1 - (1 - e^{-(x+y)}) = e^{-(x+y)} \quad (6)$$

Thus, we have the contradiction that $e^{-(x+y)} \le 0$ for all $x, y \ge 0$. We can conclude that the given function is not a valid CDF.

Problem 4.2.1 Solution

In this problem, it is helpful to label points with nonzero probability on the X, Y plane:



(a) We must choose c so the PMF sums to one:

$$\sum_{x=1,2,4} \sum_{y=1,3} P_{X,Y}(x,y) = c \sum_{x=1,2,4} x \sum_{y=1,3} y \tag{1}$$

$$= c\,[1(1+3) + 2(1+3) + 4(1+3)] = 28c \tag{2}$$

Thus $c = 1/28$.

(b) The event $\{Y < X\}$ has probability

$$P[Y < X] = \sum_{x=1,2,4} \sum_{y<x} P_{X,Y}(x,y) = \frac{1(0) + 2(1) + 4(1+3)}{28} = \frac{18}{28} \tag{3}$$

(c) The event $\{Y > X\}$ has probability

$$P[Y > X] = \sum_{x=1,2,4} \sum_{y>x} P_{X,Y}(x,y) = \frac{1(3) + 2(3) + 4(0)}{28} = \frac{9}{28} \tag{4}$$

(d) There are two ways to solve this part. The direct way is to calculate

$$P[Y = X] = \sum_{x=1,2,4} \sum_{y=x} P_{X,Y}(x,y) = \frac{1(1) + 2(0)}{28} = \frac{1}{28} \tag{5}$$

The indirect way is to use the previous results and the observation that

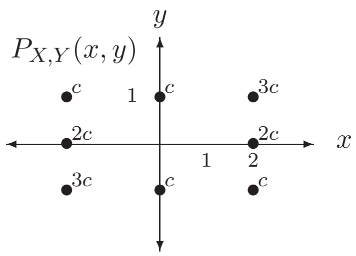
$$P[Y = X] = 1 - P[Y < X] - P[Y > X] = (1 - 18/28 - 9/28) = 1/28 \tag{6}$$

(e)

$$P[Y = 3] = \sum_{x=1,2,4} P_{X,Y}(x,3) = \frac{(1)(3) + (2)(3) + (4)(3)}{28} = \frac{21}{28} = \frac{3}{4} \tag{7}$$

Problem 4.2.2 Solution

On the X, Y plane, the joint PMF is



(a) To find c, we sum the PMF over all possible values of X and Y. We choose c so the sum equals one.

$$\sum_x \sum_y P_{X,Y}(x,y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c\,|x+y| = 6c + 2c + 6c = 14c \tag{1}$$

Thus $c = 1/14$.

(b)

$$P[Y < X] = P_{X,Y}(0,-1) + P_{X,Y}(2,-1) + P_{X,Y}(2,0) + P_{X,Y}(2,1) \tag{2}$$
$$= c + c + 2c + 3c = 7c = 1/2 \tag{3}$$

(c)

$$P[Y > X] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1) + P_{X,Y}(0,1) \tag{4}$$
$$= 3c + 2c + c + c = 7c = 1/2 \tag{5}$$

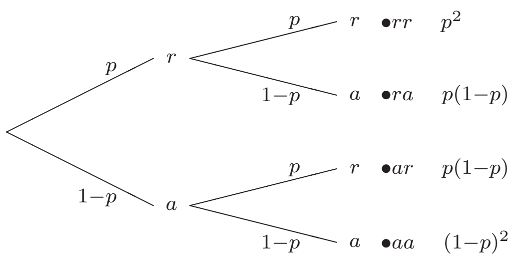
(d) From the sketch of $P_{X,Y}(x,y)$ given above, $P[X = Y] = 0$.

(e)

$$P[X < 1] = P_{X,Y}(-2,-1) + P_{X,Y}(-2,0) + P_{X,Y}(-2,1)$$
$$+ P_{X,Y}(0,-1) + P_{X,Y}(0,1) \tag{6}$$
$$= 8c = 8/14. \tag{7}$$

Problem 4.2.3 Solution

Let r (reject) and a (accept) denote the result of each test. There are four possible outcomes: rr, ra, ar, aa. The sample tree is



Now we construct a table that maps the sample outcomes to values of X and Y.

outcome	$P[\cdot]$	X	Y
rr	p^2	1	1
ra	$p(1-p)$	1	0
ar	$p(1-p)$	0	1
aa	$(1-p)^2$	0	0

(1)

This table is esentially the joint PMF $P_{X,Y}(x,y)$.

$$P_{X,Y}(x,y) = \begin{cases} p^2 & x=1, y=1 \\ p(1-p) & x=0, y=1 \\ p(1-p) & x=1, y=0 \\ (1-p)^2 & x=0, y=0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.2.4 Solution

The sample space is the set $S = \{hh, ht, th, tt\}$ and each sample point has probability 1/4. Each sample outcome specifies the values of X and Y as given in the following table

outcome	X	Y
hh	0	1
ht	1	0
th	1	1
tt	2	0

(1)

The joint PMF can represented by the table

$P_{X,Y}(x,y)$	$y=0$	$y=1$
$x=0$	0	1/4
$x=1$	1/4	1/4
$x=2$	1/4	0

(2)

Problem 4.2.5 Solution

As the problem statement says, reasonable arguments can be made for the labels being X and Y or x and y. As we see in the arguments below, the lowercase choice of the text is somewhat arbitrary.

- *Lowercase axis labels:* For the lowercase labels, we observe that we are depicting the masses associated with the joint PMF $P_{X,Y}(x,y)$ whose arguments are x and y. Since the PMF function is defined in terms of x and y, the axis labels should be x and y.

- *Uppercase axis labels:* On the other hand, we are depicting the possible outcomes (labeled with their respective probabilities) of the pair of random variables X and Y. The corresponding axis labels should be X and Y just as in Figure 4.2. The fact that we have labeled the possible outcomes by their probabilities is irrelevant. Further, since the expression for the PMF $P_{X,Y}(x,y)$ given in the figure could just as well have been written $P_{X,Y}(\cdot,\cdot)$, it is clear that the lowercase x and y are not what matter.

Problem 4.2.6 Solution

As the problem statement indicates, $Y = y < n$ if and only if

A: the first y tests are acceptable, and

B: test $y+1$ is a rejection.

Thus $P[Y = y] = P[AB]$. Note that $Y \leq X$ since the number of acceptable tests before the first failure cannot exceed the number of acceptable circuits. Moreover, given the occurrence of AB, the event $X = x < n$ occurs if and only if there are $x - y$ acceptable circuits in the remaining $n - y - 1$ tests. Since events A, B and C depend on disjoint sets of tests, they are independent events. Thus, for $0 \leq y \leq x < n$,

$$\begin{aligned}
P_{X,Y}(x, y) = P[X = x, Y = y] &= P[ABC] && (1)\\
&= P[A]\,P[B]\,P[C] && (2)\\
&= \underbrace{p^y}_{P[A]}\underbrace{(1-p)}_{P[B]}\underbrace{\binom{n-y-1}{x-y} p^{x-y}(1-p)^{n-y-1-(x-y)}}_{P[C]} && (3)\\
&= \binom{n-y-1}{x-y} p^x (1-p)^{n-x} && (4)
\end{aligned}$$

The case $y = x = n$ occurs when all n tests are acceptable and thus $P_{X,Y}(n, n) = p^n$.

Problem 4.2.7 Solution

The joint PMF of X and K is $P_{K,X}(k, x) = P[K = k, X = x]$, which is the probability that $K = k$ and $X = x$. This means that both events must be satisfied. The approach we use is similar to that used in finding the Pascal PMF in Example 2.15. Since X can take on only the two values 0 and 1, let's consider each in turn. When $X = 0$ that means that a rejection occurred on the last test and that the other $k - 1$ rejections must have occurred in the previous $n - 1$ tests. Thus,

$$P_{K,X}(k, 0) = \binom{n-1}{k-1}(1-p)^{k-1} p^{n-1-(k-1)}(1-p) \qquad k = 1, \ldots, n \tag{1}$$

When $X = 1$ the last test was acceptable and therefore we know that the $K = k \leq n-1$ tails must have occurred in the previous $n - 1$ tests. In this case,

$$P_{K,X}(k, 1) = \binom{n-1}{k}(1-p)^k p^{n-1-k} p \qquad k = 0, \ldots, n-1 \tag{2}$$

We can combine these cases into a single complete expression for the joint PMF.

$$P_{K,X}(k, x) = \begin{cases} \binom{n-1}{k-1}(1-p)^k p^{n-k} & x = 0, k = 1, 2, \ldots, n \\ \binom{n-1}{k}(1-p)^k p^{n-k} & x = 1, k = 0, 1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.2.8 Solution

Each circuit test produces an acceptable circuit with probability p. Let K denote the number of rejected circuits that occur in n tests and X is the number of acceptable circuits before the first reject. The joint PMF, $P_{K,X}(k, x) = P[K = k, X = x]$ can be found by realizing that $\{K = k, X = x\}$ occurs if and only if the following events occur:

A The first x tests must be acceptable.

B Test $x+1$ must be a rejection since otherwise we would have $x+1$ acceptable at the beginnning.

C The remaining $n-x-1$ tests must contain $k-1$ rejections.

Since the events A, B and C are independent, the joint PMF for $x+k \leq r$, $x \geq 0$ and $k \geq 0$ is

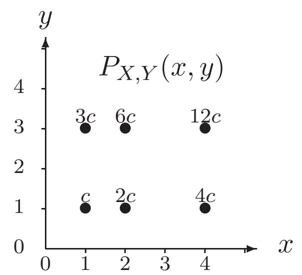
$$P_{K,X}(k,x) = \underbrace{p^x}_{P[A]} \underbrace{(1-p)}_{P[B]} \underbrace{\binom{n-x-1}{k-1}(1-p)^{k-1}p^{n-x-1-(k-1)}}_{P[C]} \tag{1}$$

After simplifying, a complete expression for the joint PMF is

$$P_{K,X}(k,x) = \begin{cases} \binom{n-x-1}{k-1}p^{n-k}(1-p)^k & x+k \leq n, x \geq 0, k \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.3.1 Solution

On the X, Y plane, the joint PMF $P_{X,Y}(x,y)$ is



By choosing $c = 1/28$, the PMF sums to one.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=1,3} P_{X,Y}(x,y) = \begin{cases} 4/28 & x=1 \\ 8/28 & x=2 \\ 16/28 & x=4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_Y(y) = \sum_{x=1,2,4} P_{X,Y}(x,y) = \begin{cases} 7/28 & y=1 \\ 21/28 & y=3 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=1,2,4} xP_X(x) = (4/28) + 2(8/28) + 4(16/28) = 3 \tag{3}$$

$$E[Y] = \sum_{y=1,3} yP_Y(y) = 7/28 + 3(21/28) = 5/2 \tag{4}$$

(c) The second moments are

$$E\left[X^2\right] = \sum_{x=1,2,4} x P_X(x) = 1^2(4/28) + 2^2(8/28) + 4^2(16/28) = 73/7 \qquad (5)$$

$$E\left[Y^2\right] = \sum_{y=1,3} y P_Y(y) = 1^2(7/28) + 3^2(21/28) = 7 \qquad (6)$$

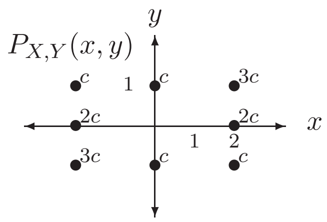
The variances are

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 10/7 \qquad \text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 3/4 \qquad (7)$$

The standard deviations are $\sigma_X = \sqrt{10/7}$ and $\sigma_Y = \sqrt{3/4}$.

Problem 4.3.2 Solution

On the X, Y plane, the joint PMF is



The PMF sums to one when $c = 1/14$.

(a) The marginal PMFs of X and Y are

$$P_X(x) = \sum_{y=-1,0,1} P_{X,Y}(x,y) = \begin{cases} 6/14 & x = -2, 2 \\ 2/14 & x = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

$$P_Y(y) = \sum_{x=-2,0,2} P_{X,Y}(x,y) = \begin{cases} 5/14 & y = -1, 1 \\ 4/14 & y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

(b) The expected values of X and Y are

$$E[X] = \sum_{x=-2,0,2} x P_X(x) = -2(6/14) + 2(6/14) = 0 \qquad (3)$$

$$E[Y] = \sum_{y=-1,0,1} y P_Y(y) = -1(5/14) + 1(5/14) = 0 \qquad (4)$$

(c) Since X and Y both have zero mean, the variances are

$$\text{Var}[X] = E\left[X^2\right] = \sum_{x=-2,0,2} x^2 P_X(x) = (-2)^2(6/14) + 2^2(6/14) = 24/7 \qquad (5)$$

$$\text{Var}[Y] = E\left[Y^2\right] = \sum_{y=-1,0,1} y^2 P_Y(y) = (-1)^2(5/14) + 1^2(5/14) = 5/7 \qquad (6)$$

The standard deviations are $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.

Problem 4.3.3 Solution

We recognize that the given joint PMF is written as the product of two marginal PMFs $P_N(n)$ and $P_K(k)$ where

$$P_N(n) = \sum_{k=0}^{100} P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{n!} & n = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$P_K(k) = \sum_{n=0}^{\infty} P_{N,K}(n,k) = \begin{cases} \binom{100}{k} p^k (1-p)^{100-k} & k = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.3.4 Solution

The joint PMF of N, K is

$$P_{N,K}(n,k) = \begin{cases} (1-p)^{n-1} p/n & k = 1, 2, \ldots, n \\ & n = 1, 2 \ldots \\ o & \text{otherwise} \end{cases} \tag{1}$$

For $n \geq 1$, the marginal PMF of N is

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n,k) = \sum_{k=1}^{n} (1-p)^{n-1} p/n = (1-p)^{n-1} p \tag{2}$$

The marginal PMF of K is found by summing $P_{N,K}(n,k)$ over all possible N. Note that if $K = k$, then $N \geq k$. Thus,

$$P_K(k) = \sum_{n=k}^{\infty} \frac{1}{n} (1-p)^{n-1} p \tag{3}$$

Unfortunately, this sum cannot be simplified.

Problem 4.3.5 Solution

For $n = 0, 1, \ldots$, the marginal PMF of N is

$$P_N(n) = \sum_k P_{N,K}(n,k) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!} \tag{1}$$

For $k = 0, 1, \ldots$, the marginal PMF of K is

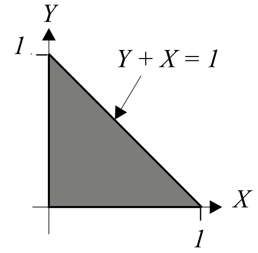
$$P_K(k) = \sum_{n=k}^{\infty} \frac{100^n e^{-100}}{(n+1)!} = \frac{1}{100} \sum_{n=k}^{\infty} \frac{100^{n+1} e^{-100}}{(n+1)!} \tag{2}$$

$$= \frac{1}{100} \sum_{n=k}^{\infty} P_N(n+1) \tag{3}$$

$$= P[N > k] / 100 \tag{4}$$

Problem 4.4.1 Solution

(a) The joint PDF of X and Y is



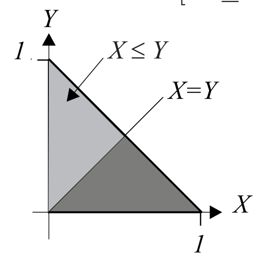
$$f_{X,Y}(x,y) = \begin{cases} c & x + y \leq 1, x, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

To find the constant c we integrate over the region shown. This gives

$$\int_0^1 \int_0^{1-x} c\, dy\, dx = cx - \frac{cx}{2}\Big|_0^1 = \frac{c}{2} = 1 \quad (2)$$

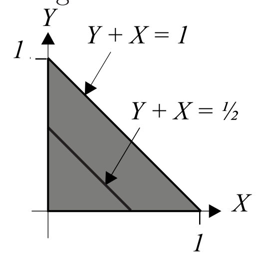
Therefore $c = 2$.

(b) To find the $P[X \leq Y]$ we look to integrate over the area indicated by the graph



$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} dy\, dx \quad (3)$$
$$= \int_0^{1/2} (2 - 4x)\, dx \quad (4)$$
$$= 1/2 \quad (5)$$

(c) The probability $P[X + Y \leq 1/2]$ can be seen in the figure. Here we can set up the following integrals



$$P[X + Y \leq 1/2] = \int_0^{1/2} \int_0^{1/2-x} 2\, dy\, dx \quad (6)$$
$$= \int_0^{1/2} (1 - 2x)\, dx \quad (7)$$
$$= 1/2 - 1/4 = 1/4 \quad (8)$$

Problem 4.4.2 Solution

Given the joint PDF

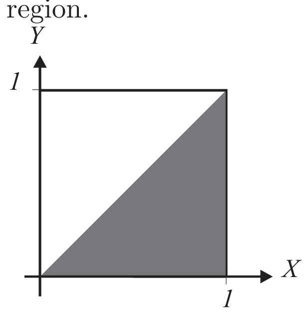
$$f_{X,Y}(x,y) = \begin{cases} cxy^2 & 0 \leq x, y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) To find the constant c integrate $f_{X,Y}(x,y)$ over the all possible values of X and Y to get

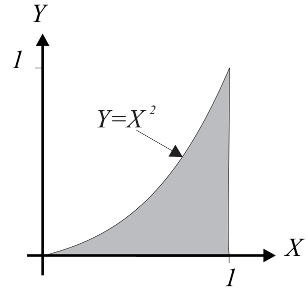
$$1 = \int_0^1 \int_0^1 cxy^2\, dx\, dy = c/6 \quad (2)$$

Therefore $c = 6$.

(b) The probability $P[X \geq Y]$ is the integral of the joint PDF $f_{X,Y}(x, y)$ over the indicated shaded region.



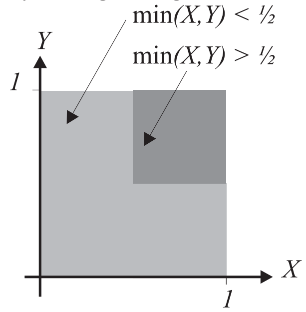
$$P[X \geq Y] = \int_0^1 \int_0^x 6xy^2 \, dy \, dx \tag{3}$$
$$= \int_0^1 2x^4 \, dx \tag{4}$$
$$= 2/5 \tag{5}$$



Similarly, to find $P[Y \leq X^2]$ we can integrate over the region shown in the figure.

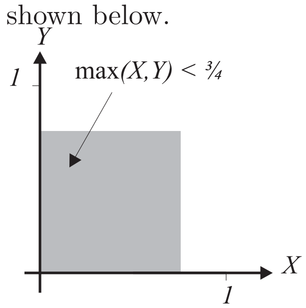
$$P\left[Y \leq X^2\right] = \int_0^1 \int_0^{x^2} 6xy^2 \, dy \, dx \tag{6}$$
$$= 1/4 \tag{7}$$

(c) Here we can choose to either integrate $f_{X,Y}(x, y)$ over the lighter shaded region, which would require the evaluation of two integrals, or we can perform one integral over the darker region by recognizing



$$P[\min(X, Y) \leq 1/2] = 1 - P[\min(X, Y) > 1/2] \tag{8}$$
$$= 1 - \int_{1/2}^1 \int_{1/2}^1 6xy^2 \, dx \, dy \tag{9}$$
$$= 1 - \int_{1/2}^1 \frac{9y^2}{4} \, dy = \frac{11}{32} \tag{10}$$

(d) The probability $P[\max(X, Y) \leq 3/4]$ can be found be integrating over the shaded region shown below.



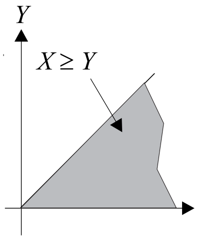
$$P[\max(X, Y) \leq 3/4] = P[X \leq 3/4, Y \leq 3/4] \tag{11}$$
$$= \int_0^{\frac{3}{4}} \int_0^{\frac{3}{4}} 6xy^2 \, dx \, dy \tag{12}$$
$$= \left(x^2 \Big|_0^{3/4} \right) \left(y^3 \Big|_0^{3/4} \right) \tag{13}$$
$$= (3/4)^5 = 0.237 \tag{14}$$

Problem 4.4.3 Solution

The joint PDF of X and Y is

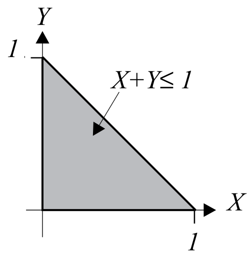
$$f_{X,Y}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x \geq 0, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) The probability that $X \geq Y$ is:



$$P[X \geq Y] = \int_0^\infty \int_0^x 6e^{-(2x+3y)}\,dy\,dx \tag{2}$$
$$= \int_0^\infty 2e^{-2x}\left(-e^{-3y}\Big|_{y=0}^{y=x}\right) dx \tag{3}$$
$$= \int_0^\infty [2e^{-2x} - 2e^{-5x}]\,dx = 3/5 \tag{4}$$

The $P[X+Y \leq 1]$ is found by integrating over the region where $X+Y \leq 1$



$$P[X+Y \leq 1] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\,dy\,dx \tag{5}$$
$$= \int_0^1 2e^{-2x}\left[-e^{-3y}\Big|_{y=0}^{y=1-x}\right] dx \tag{6}$$
$$= \int_0^1 2e^{-2x}\left[1 - e^{-3(1-x)}\right] dx \tag{7}$$
$$= -e^{-2x} - 2e^{x-3}\Big|_0^1 \tag{8}$$
$$= 1 + 2e^{-3} - 3e^{-2} \tag{9}$$

(b) The event $\min(X,Y) \geq 1$ is the same as the event $\{X \geq 1, Y \geq 1\}$. Thus,

$$P[\min(X,Y) \geq 1] = \int_1^\infty \int_1^\infty 6e^{-(2x+3y)}\,dy\,dx = e^{-(2+3)} \tag{10}$$

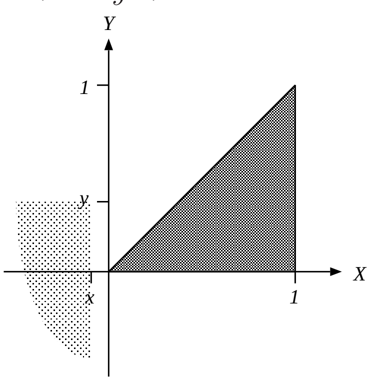
(c) The event $\max(X,Y) \leq 1$ is the same as the event $\{X \leq 1, Y \leq 1\}$ so that

$$P[\max(X,Y) \leq 1] = \int_0^1 \int_0^1 6e^{-(2x+3y)}\,dy\,dx = (1-e^{-2})(1-e^{-3}) \tag{11}$$

Problem 4.4.4 Solution

The only difference between this problem and Example 4.5 is that in this problem we must integrate the joint PDF over the regions to find the probabilities. Just as in Example 4.5, there are five cases. We will use variable u and v as dummy variables for x and y.

- $x < 0$ or $y < 0$

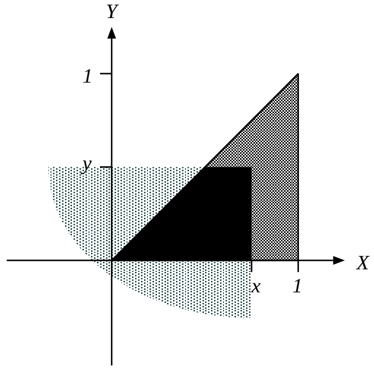


In this case, the region of integration doesn't overlap the region of nonzero probability and

$$F_{X,Y}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv = 0 \tag{1}$$

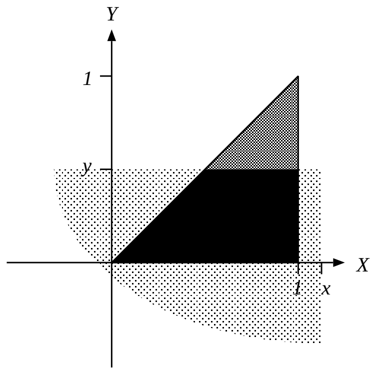
- $0 < y \leq x \leq 1$

  In this case, the region where the integral has a nonzero contribution is



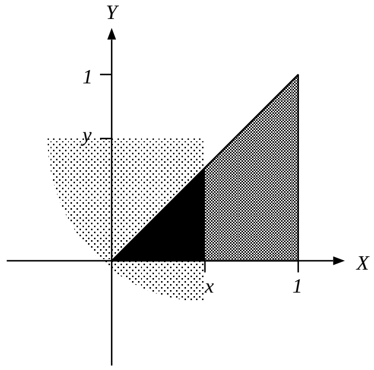
$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\; dy\, dx \tag{2}$$
$$= \int_{0}^{y}\int_{v}^{x} 8uv\, du\, dv \tag{3}$$
$$= \int_{0}^{y} 4(x^2 - v^2)v\, dv \tag{4}$$
$$= 2x^2v^2 - v^4\Big|_{v=0}^{v=y} = 2x^2y^2 - y^4 \tag{5}$$

- $0 < x \leq y$ and $0 \leq x \leq 1$



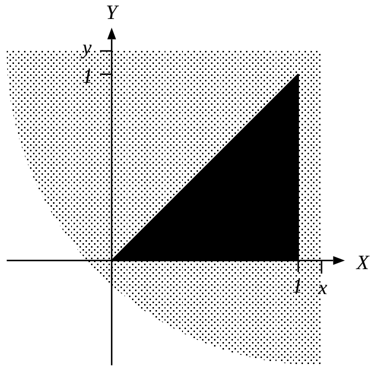
$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\; dv\, du \tag{6}$$
$$= \int_{0}^{x}\int_{0}^{u} 8uv\, dv\, du \tag{7}$$
$$= \int_{0}^{x} 4u^3\, du = x^4 \tag{8}$$

- $0 < y \leq 1$ and $x \geq 1$



$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\; dv\, du \tag{9}$$
$$= \int_{0}^{y}\int_{v}^{1} 8uv\, du\, dv \tag{10}$$
$$= \int_{0}^{y} 4v(1 - v^2)\, dv \tag{11}$$
$$= 2y^2 - y^4 \tag{12}$$

- $x \geq 1$ and $y \geq 1$



In this case, the region of integration completely covers the region of nonzero probability and

$$F_{X,Y}(x,y) = \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv \tag{13}$$

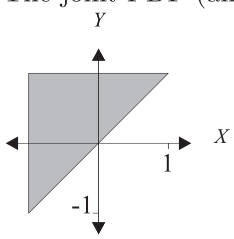
$$= 1 \tag{14}$$

The complete answer for the joint CDF is

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ 2x^2y^2 - y^4 & 0 < y \le x \le 1 \\ x^4 & 0 \le x \le y, 0 \le x \le 1 \\ 2y^2 - y^4 & 0 \le y \le 1, x \ge 1 \\ 1 & x \ge 1, y \ge 1 \end{cases} \tag{15}$$

Problem 4.5.1 Solution

(a) The joint PDF (and the corresponding region of nonzero probability) are



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b)

$$P[X > 0] = \int_0^1 \int_x^1 \frac{1}{2}\,dy\,dx = \int_0^1 \frac{1-x}{2}\,dx = 1/4 \tag{2}$$

This result can be deduced by geometry. The shaded triangle of the X, Y plane corresponding to the event $X > 0$ is $1/4$ of the total shaded area.

(c) For $x > 1$ or $x < -1$, $f_X(x) = 0$. For $-1 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_x^1 \frac{1}{2}\,dy = (1-x)/2. \tag{3}$$

The complete expression for the marginal PDF is

$$f_X(x) = \begin{cases} (1-x)/2 & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

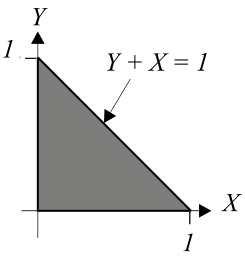
(d) From the marginal PDF $f_X(x)$, the expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \frac{1}{2}\int_{-1}^{1} x(1-x)\, dx \tag{5}$$

$$= \left.\frac{x^2}{4} - \frac{x^3}{6}\right|_{-1}^{1} = -\frac{1}{3}. \tag{6}$$

Problem 4.5.2 Solution

$$f_{X,Y}(x,y) = \begin{cases} 2 & x+y \leq 1, x,y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$



Using the figure to the left we can find the marginal PDFs by integrating over the appropriate regions.

$$f_X(x) = \int_0^{1-x} 2\, dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Likewise for $f_Y(y)$:

$$f_Y(y) = \int_0^{1-y} 2\, dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.5.3 Solution

Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & 0 \leq x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The marginal PDF of X is

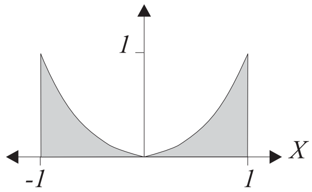
$$f_X(x) = 2\int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(b) Similarly, for random variable Y,

$$f_Y(y) = 2\int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\, dx = \begin{cases} \frac{2\sqrt{r^2-y^2}}{\pi r^2} & -r \leq y \leq r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Problem 4.5.4 Solution

The joint PDF of X and Y and the region of nonzero probability are



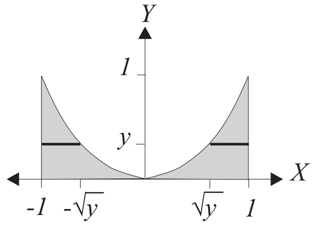
$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \leq x \leq 1, 0 \leq y \leq x^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the appropriate marginal PDFs by integrating the joint PDF.

(a) The marginal PDF of X is

$$f_X(x) = \int_0^{x^2} \frac{5x^2}{2}\,dy = \begin{cases} 5x^4/2 & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) Note that $f_Y(y) = 0$ for $y > 1$ or $y < 0$. For $0 \le y \le 1$,



$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx \tag{3}$$

$$= \int_{-1}^{-\sqrt{y}} \frac{5x^2}{2}\,dx + \int_{\sqrt{y}}^{1} \frac{5x^2}{2}\,dx \tag{4}$$

$$= 5(1 - y^{3/2})/3 \tag{5}$$

The complete expression for the marginal CDF of Y is

$$f_Y(y) = \begin{cases} 5(1-y^{3/2})/3 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 4.5.5 Solution

In this problem, the joint PDF is

$$f_{X,Y}(x,y) = \begin{cases} 2\,|xy|\,/r^4 & 0 \le x^2 + y^2 \le r^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since $|xy| = |x||y|$, for $-r \le x \le r$, we can write

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \frac{2\,|x|}{r^4} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} |y|\,dy \tag{2}$$

Since $|y|$ is symmetric about the origin, we can simplify the integral to

$$f_X(x) = \frac{4\,|x|}{r^4} \int_0^{\sqrt{r^2-x^2}} y\,dy = \frac{2\,|x|}{r^4} y^2 \Big|_0^{\sqrt{r^2-x^2}} = \frac{2\,|x|\,(r^2-x^2)}{r^4} \tag{3}$$

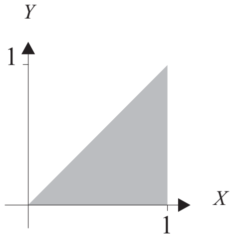
Note that for $|x| > r$, $f_X(x) = 0$. Hence the complete expression for the PDF of X is

$$f_X(x) = \begin{cases} \frac{2|x|(r^2-x^2)}{r^4} & -r \le x \le r \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) Note that the joint PDF is symmetric in x and y so that $f_Y(y) = f_X(y)$.

Problem 4.5.6 Solution

(a) The joint PDF of X and Y and the region of nonzero probability are



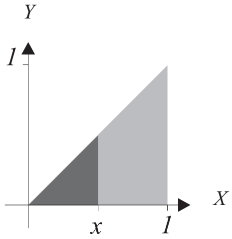
$$f_{X,Y}(x,y) = \begin{cases} cy & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) To find the value of the constant, c, we integrate the joint PDF over all x and y.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_0^1\int_0^x cy\,dy\,dx = \int_0^1 \frac{cx^2}{2}\,dx = \left.\frac{cx^3}{6}\right|_0^1 = \frac{c}{6}. \tag{2}$$

Thus $c = 6$.

(c) We can find the CDF $F_X(x) = P[X \le x]$ by integrating the joint PDF over the event $X \le x$. For $x < 0$, $F_X(x) = 0$. For $x > 1$, $F_X(x) = 1$. For $0 \le x \le 1$,



$$F_X(x) = \iint_{x' \le x} f_{X,Y}(x',y')\,dy'\,dx' \tag{3}$$

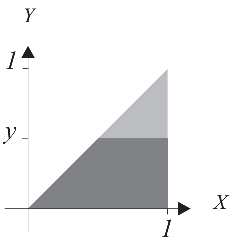
$$= \int_0^x \int_0^{x'} 6y'\,dy'\,dx' \tag{4}$$

$$= \int_0^x 3(x')^2\,dx' = x^3. \tag{5}$$

The complete expression for the joint CDF is

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases} \tag{6}$$

(d) Similarly, we find the CDF of Y by integrating $f_{X,Y}(x,y)$ over the event $Y \le y$. For $y < 0$, $F_Y(y) = 0$ and for $y > 1$, $F_Y(y) = 1$. For $0 \le y \le 1$,



$$F_Y(y) = \iint_{y' \le y} f_{X,Y}(x',y')\,dy'\,dx' \tag{7}$$

$$= \int_0^y \int_{y'}^1 6y'\,dx'\,dy' \tag{8}$$

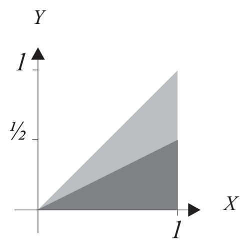
$$= \int_0^y 6y'(1-y')\,dy' \tag{9}$$

$$= 3(y')^2 - 2(y')^3\big|_0^y = 3y^2 - 2y^3. \tag{10}$$

The complete expression for the CDF of Y is

$$F_Y(y) = \begin{cases} 0 & y < 0 \\ 3y^2 - 2y^3 & 0 \le y \le 1 \\ 1 & y > 1 \end{cases} \tag{11}$$

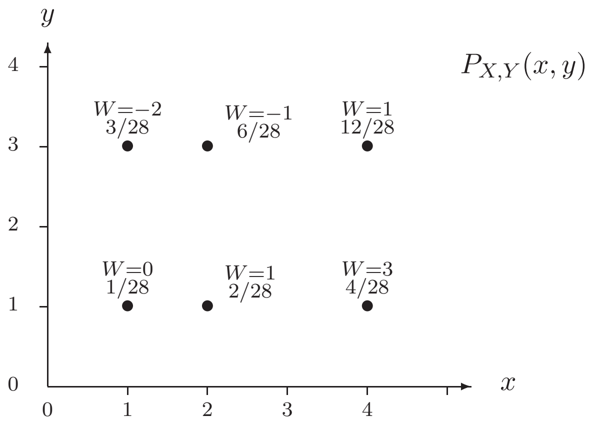
(e) To find $P[Y \leq X/2]$, we integrate the joint PDF $f_{X,Y}(x,y)$ over the region $y \leq x/2$.



$$P[Y \leq X/2] = \int_0^1 \int_0^{x/2} 6y\,dy\,dx \tag{12}$$
$$= \int_0^1 3y^2\Big|_0^{x/2}\,dx \tag{13}$$
$$= \int_0^1 \frac{3x^2}{4}\,dx = 1/4 \tag{14}$$

Problem 4.6.1 Solution

In this problem, it is helpful to label possible points X, Y along with the corresponding values of $W = X - Y$. From the statement of Problem 4.6.1,



(a) To find the PMF of W, we simply add the probabilities associated with each possible value of W:

$$P_W(-2) = P_{X,Y}(1,3) = 3/28 \qquad P_W(-1) = P_{X,Y}(2,3) = 6/28 \tag{1}$$
$$P_W(0) = P_{X,Y}(1,1) = 1/28 \qquad P_W(1) = P_{X,Y}(2,1) + P_{X,Y}(4,3) \tag{2}$$
$$P_W(3) = P_{X,Y}(4,1) = 4/28 \qquad = 14/28 \tag{3}$$

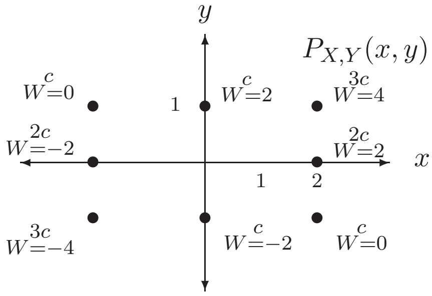
For all other values of w, $P_W(w) = 0$.

(b) The expected value of W is

$$E[W] = \sum_w w P_W(w) \tag{4}$$
$$= -2(3/28) + -1(6/28) + 0(1/28) + 1(14/28) + 3(4/28) = 1/2 \tag{5}$$

(c) $P[W > 0] = P_W(1) + P_W(3) = 18/28$.

Problem 4.6.2 Solution



In Problem 4.2.2, the joint PMF $P_{X,Y}(x,y)$ is given in terms of the parameter c. For this problem, we first need to find c. Before doing so, it is convenient to label each possible X, Y point with the corresponding value of $W = X + 2Y$.

To find c, we sum the PMF over all possible values of X and Y. We choose c so the sum equals one.

$$\sum_x \sum_y P_{X,Y}(x,y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c\,|x+y| \tag{1}$$

$$= 6c + 2c + 6c = 14c \tag{2}$$

Thus $c = 1/14$. Now we can solve the actual problem.

(a) From the above graph, we can calculate the probability of each possible value of w.

$$P_W(-4) = P_{X,Y}(-2,-1) = 3c \tag{3}$$

$$P_W(-2) = P_{X,Y}(-2,0) + P_{X,Y}(0,-1) = 3c \tag{4}$$

$$P_W(0) = P_{X,Y}(-2,1) + P_{X,Y}(2,-1) = 2c \tag{5}$$

$$P_W(2) = P_{X,Y}(0,1) + P_{X,Y}(2,0) = 3c \tag{6}$$

$$P_W(4) = P_{X,Y}(2,1) = 3c \tag{7}$$

With $c = 1/14$, we can summarize the PMF as

$$P_W(w) = \begin{cases} 3/14 & w = -4,-2,2,4 \\ 2/14 & w = 0 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

(b) The expected value is now straightforward:

$$E[W] = \frac{3}{14}(-4 + -2 + 2 + 4) + \frac{2}{14}0 = 0. \tag{9}$$

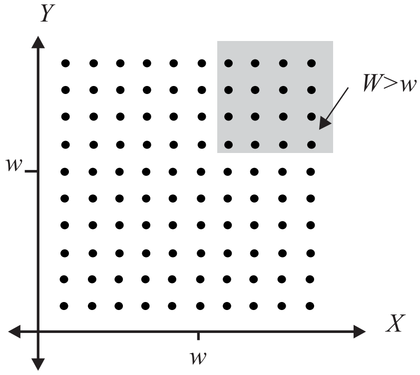
(c) Lastly, $P[W > 0] = P_W(2) + P_W(4) = 3/7$.

Problem 4.6.3 Solution

We observe that when $X = x$, we must have $Y = w - x$ in order for $W = w$. That is,

$$P_W(w) = \sum_{x=-\infty}^{\infty} P[X = x, Y = w - x] = \sum_{x=-\infty}^{\infty} P_{X,Y}(x, w - x) \tag{1}$$

Problem 4.6.4 Solution



The x, y pairs with nonzero probability are shown in the figure. For $w = 0, 1, \ldots, 10$, we observe that

$$\begin{aligned} P[W > w] &= P[\min(X, Y) > w] && (1)\\ &= P[X > w, Y > w] && (2)\\ &= 0.01(10 - w)^2 && (3) \end{aligned}$$

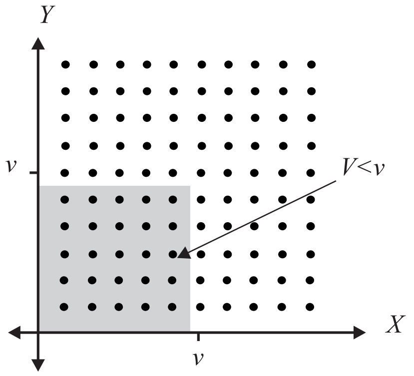
To find the PMF of W, we observe that for $w = 1, \ldots, 10$,

$$\begin{aligned} P_W(w) &= P[W > w - 1] - P[W > w] && (4)\\ &= 0.01[(10 - w - 1)^2 - (10 - w)^2] = 0.01(21 - 2w) && (5) \end{aligned}$$

The complete expression for the PMF of W is

$$P_W(w) = \begin{cases} 0.01(21 - 2w) & w = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases} \qquad (6)$$

Problem 4.6.5 Solution



The x, y pairs with nonzero probability are shown in the figure. For $v = 1, \ldots, 11$, we observe that

$$\begin{aligned} P[V < v] &= P[\max(X, Y) < v] && (1)\\ &= P[X < v, Y < v] && (2)\\ &= 0.01(v - 1)^2 && (3) \end{aligned}$$

To find the PMF of V, we observe that for $v = 1, \ldots, 10$,

$$\begin{aligned} P_V(v) &= P[V < v + 1] - P[V < v] && (4)\\ &= 0.01[v^2 - (v - 1)^2] && (5)\\ &= 0.01(2v - 1) && (6) \end{aligned}$$

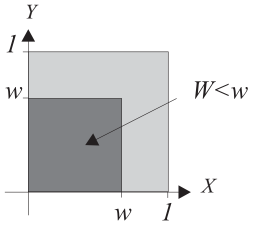
The complete expression for the PMF of V is

$$P_V(v) = \begin{cases} 0.01(2v - 1) & v = 1, 2, \ldots, 10 \\ 0 & \text{otherwise} \end{cases} \qquad (7)$$

Problem 4.6.6 Solution

(a) The minimum value of W is $W = 0$, which occurs when $X = 0$ and $Y = 0$. The maximum value of W is $W = 1$, which occurs when $X = 1$ or $Y = 1$. The range of W is $S_W = \{w|0 \leq w \leq 1\}$.

(b) For $0 \leq w \leq 1$, the CDF of W is



$$F_W(w) = P[\max(X,Y) \leq w] \quad (1)$$
$$= P[X \leq w, Y \leq w] \quad (2)$$
$$= \int_0^w \int_0^w f_{X,Y}(x,y)\, dy\, dx \quad (3)$$

Substituting $f_{X,Y}(x,y) = x + y$ yields

$$F_W(w) = \int_0^w \int_0^w (x+y)\, dy\, dx \quad (4)$$
$$= \int_0^w \left(xy + \left.\frac{y^2}{2}\right|_{y=0}^{y=w} \right) dx = \int_0^w (wx + w^2/2)\, dx = w^3 \quad (5)$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w^3 & 0 \leq w \leq 1 \\ 1 & \text{otherwise} \end{cases} \quad (6)$$

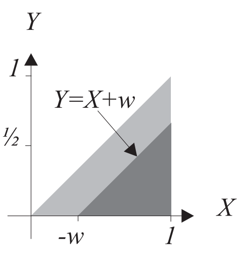
The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 3w^2 & 0 \leq w \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Problem 4.6.7 Solution

(a) Since the joint PDF $f_{X,Y}(x,y)$ is nonzero only for $0 \leq y \leq x \leq 1$, we observe that $W = Y - X \leq 0$ since $Y \leq X$. In addition, the most negative value of W occurs when $Y = 0$ and $X = 1$ and $W = -1$. Hence the range of W is $S_W = \{w| -1 \leq w \leq 0\}$.

(b) For $w < -1$, $F_W(w) = 0$. For $w > 0$, $F_W(w) = 1$. For $-1 \leq w \leq 0$, the CDF of W is



$$F_W(w) = P[Y - X \leq w] \quad (1)$$
$$= \int_{-w}^1 \int_0^{x+w} 6y\, dy\, dx \quad (2)$$
$$= \int_{-w}^1 3(x+w)^2\, dx \quad (3)$$
$$= (x+w)^3\Big|_{-w}^1 = (1+w)^3 \quad (4)$$

Therefore, the complete CDF of W is

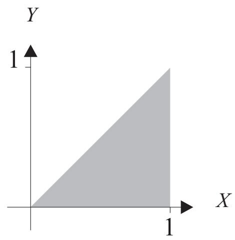
$$F_W(w) = \begin{cases} 0 & w < -1 \\ (1+w)^3 & -1 \le w \le 0 \\ 1 & w > 0 \end{cases} \tag{5}$$

By taking the derivative of $f_W(w)$ with respect to w, we obtain the PDF

$$f_W(w) = \begin{cases} 3(w+1)^2 & -1 \le w \le 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 4.6.8 Solution

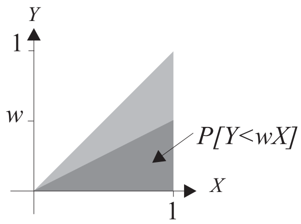
Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since X and Y are both nonnegative, $W = Y/X \ge 0$. Since $Y \le X$, $W = Y/X \le 1$. Note that $W = 0$ can occur if $Y = 0$. Thus the range of W is $S_W = \{w | 0 \le w \le 1\}$.

(b) For $0 \le w \le 1$, the CDF of W is



$$F_W(w) = P[Y/X \le w] = P[Y \le wX] = w \tag{2}$$

The complete expression for the CDF is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ w & 0 \le w < 1 \\ 1 & w \ge 1 \end{cases} \tag{3}$$

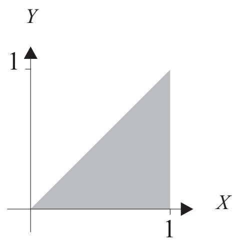
By taking the derivative of the CDF, we find that the PDF of W is

$$f_W(w) = \begin{cases} 1 & 0 \le w < 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

We see that W has a uniform PDF over $[0, 1]$. Thus $E[W] = 1/2$.

Problem 4.6.9 Solution

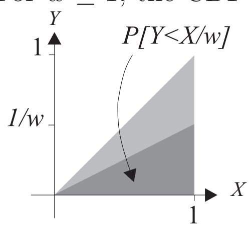
Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since $f_{X,Y}(x,y) = 0$ for $y > x$, we can conclude that $Y \le X$ and that $W = X/Y \ge 1$. Since Y can be arbitrarily small but positive, W can be arbitrarily large. Hence the range of W is $S_W = \{w | w \ge 1\}$.

(b) For $w \ge 1$, the CDF of W is



$$F_W(w) = P[X/Y \le w] \quad (2)$$
$$= 1 - P[X/Y > w] \quad (3)$$
$$= 1 - P[Y < X/w] \quad (4)$$
$$= 1 - 1/w \quad (5)$$

Note that we have used the fact that $P[Y < X/w]$ equals 1/2 times the area of the corresponding triangle. The complete CDF is

$$F_W(w) = \begin{cases} 0 & w < 1 \\ 1 - 1/w & w \ge 1 \end{cases} \quad (6)$$

The PDF of W is found by differentiating the CDF.

$$f_W(w) = \frac{dF_W(w)}{dw} = \begin{cases} 1/w^2 & w \ge 1 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

To find the expected value $E[W]$, we write

$$E[W] = \int_{-\infty}^{\infty} w f_W(w)\, dw = \int_1^{\infty} \frac{dw}{w}. \quad (8)$$

However, the integral diverges and $E[W]$ is undefined.

Problem 4.6.10 Solution

The position of the mobile phone is equally likely to be anywhere in the area of a circle with radius 16 km. Let X and Y denote the position of the mobile. Since we are given that the cell has a radius of 4 km, we will measure X and Y in kilometers. Assuming the base station is at the origin of the X, Y plane, the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{16\pi} & x^2 + y^2 \le 16 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Since the mobile's radial distance from the base station is $R = \sqrt{X^2 + Y^2}$, the CDF of R is

$$F_R(r) = P[R \le r] = P\left[X^2 + Y^2 \le r^2\right] \quad (2)$$

By changing to polar coordinates, we see that for $0 \le r \le 4$ km,

$$F_R(r) = \int_0^{2\pi} \int_0^r \frac{r'}{16\pi}\, dr'\, d\theta' = r^2/16 \quad (3)$$

So

$$F_R(r) = \begin{cases} 0 & r < 0 \\ r^2/16 & 0 \le r < 4 \\ 1 & r \ge 4 \end{cases} \quad (4)$$

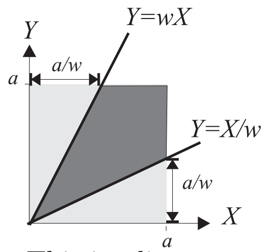
Then by taking the derivative with respect to r we arrive at the PDF

$$f_R(r) = \begin{cases} r/8 & 0 \le r \le 4 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

Problem 4.6.11 Solution

Following the hint, we observe that either $Y \geq X$ or $X \geq Y$, or, equivalently, $(Y/X) \geq 1$ or $(X/Y) \geq 1$. Hence, $W \geq 1$. To find the CDF $F_W(w)$, we know that $F_W(w) = 0$ for $w < 1$. For $w \geq 1$, we solve

$$\begin{aligned} F_W(w) &= P[\max[(X/Y), (Y/X)] \leq w] \\ &= P[(X/Y) \leq w, (Y/X) \leq w] \\ &= P[Y \geq X/w, Y \leq wX] \\ &= P[X/w \leq Y \leq wX] \end{aligned}$$



We note that in the middle of the above steps, nonnegativity of X and Y was essential. We can depict the given set $\{X/w \leq Y \leq wX\}$ as the dark region on the X, Y plane. Because the PDF is uniform over the square, it is easier to use geometry to calculate the probability. In particular, each of the lighter triangles that are not part of the region of interest has area $a^2/2w$.

This implies

$$P[X/w \leq Y \leq wX] = 1 - \frac{a^2/2w + a^2/2w}{a^2} = 1 - \frac{1}{w} \tag{1}$$

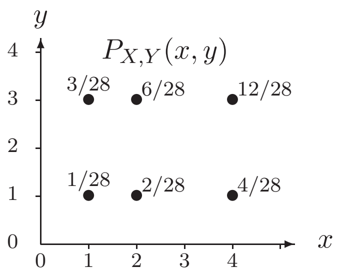
The final expression for the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < 1 \\ 1 - 1/w & w \geq 1 \end{cases} \tag{2}$$

By taking the derivative, we obtain the PDF

$$f_W(w) = \begin{cases} 0 & w < 1 \\ 1/w^2 & w \geq 1 \end{cases} \tag{3}$$

Problem 4.7.1 Solution



In Problem 4.2.1, we found the joint PMF $P_{X,Y}(x, y)$ as shown. Also the expected values and variances were

$$E[X] = 3 \qquad \text{Var}[X] = 10/7 \tag{1}$$

$$E[Y] = 5/2 \qquad \text{Var}[Y] = 3/4 \tag{2}$$

We use these results now to solve this problem.

(a) Random variable $W = Y/X$ has expected value

$$E[Y/X] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{y}{x} P_{X,Y}(x, y) \tag{3}$$

$$= \frac{1}{1}\frac{1}{28} + \frac{3}{1}\frac{3}{28} + \frac{1}{2}\frac{2}{28} + \frac{3}{2}\frac{6}{28} + \frac{1}{4}\frac{4}{28} + \frac{3}{4}\frac{12}{28} = 15/14 \tag{4}$$

(b) The correlation of X and Y is

$$r_{X,Y} = \sum_{x=1,2,4} \sum_{y=1,3} xy P_{X,Y}(x,y) \tag{5}$$
$$= \frac{1\cdot 1\cdot 1}{28} + \frac{1\cdot 3\cdot 3}{28} + \frac{2\cdot 1\cdot 2}{28} + \frac{2\cdot 3\cdot 6}{28} + \frac{4\cdot 1\cdot 4}{28} + \frac{4\cdot 3\cdot 12}{28} \tag{6}$$
$$= 210/28 = 15/2 \tag{7}$$

Recognizing that $P_{X,Y}(x,y) = xy/28$ yields the faster calculation

$$r_{X,Y} = E[XY] = \sum_{x=1,2,4} \sum_{y=1,3} \frac{(xy)^2}{28} \tag{8}$$
$$= \frac{1}{28} \sum_{x=1,2,4} x^2 \sum_{y=1,3} y^2 \tag{9}$$
$$= \frac{1}{28}(1+2^2+4^2)(1^2+3^2) = \frac{210}{28} = \frac{15}{2} \tag{10}$$

(c) The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{15}{2} - 3\frac{5}{2} = 0 \tag{11}$$

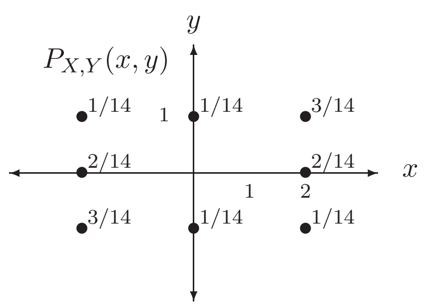
(d) Since X and Y have zero covariance, the correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = 0. \tag{12}$$

(e) Since X and Y are uncorrelated, the variance of $X+Y$ is

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = \frac{61}{28}. \tag{13}$$

Problem 4.7.2 Solution



In Problem 4.2.1, we found the joint PMF $P_{X,Y}(x,y)$ shown here. The expected values and variances were found to be

$$E[X] = 0 \qquad \text{Var}[X] = 24/7 \tag{1}$$
$$E[Y] = 0 \qquad \text{Var}[Y] = 5/7 \tag{2}$$

We need these results to solve this problem.

(a) Random variable $W = 2^{XY}$ has expected value

$$E\left[2^{XY}\right] = \sum_{x=-2,0,2} \sum_{y=-1,0,1} 2^{xy} P_{X,Y}(x,y) \tag{3}$$

$$= 2^{-2(-1)}\frac{3}{14} + 2^{-2(0)}\frac{2}{14} + 2^{-2(1)}\frac{1}{14} + 2^{0(-1)}\frac{1}{14} + 2^{0(1)}\frac{1}{14} \tag{4}$$

$$+ 2^{2(-1)}\frac{1}{14} + 2^{2(0)}\frac{2}{14} + 2^{2(1)}\frac{3}{14} \tag{5}$$

$$= 61/28 \tag{6}$$

(b) The correlation of X and Y is

$$r_{X,Y} = \sum_{x=-2,0,2} \sum_{y=-1,0,1} xy P_{X,Y}(x,y) \tag{7}$$

$$= \frac{-2(-1)(3)}{14} + \frac{-2(0)(2)}{14} + \frac{-2(1)(1)}{14} + \frac{2(-1)(1)}{14} + \frac{2(0)(2)}{14} + \frac{2(1)(3)}{14} \tag{8}$$

$$= 4/7 \tag{9}$$

(c) The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 4/7 \tag{10}$$

(d) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X,Y]}{\sqrt{\text{Var}[X]\,\text{Var}[Y]}} = \frac{2}{\sqrt{30}} \tag{11}$$

(e) By Theorem 4.16,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] \tag{12}$$

$$= \frac{24}{7} + \frac{5}{7} + 2\frac{4}{7} = \frac{37}{7}. \tag{13}$$

Problem 4.7.3 Solution

In the solution to Quiz 4.3, the joint PMF and the marginal PMFs are

$P_{H,B}(h,b)$	$b=0$	$b=2$	$b=4$	$P_H(h)$
$h=-1$	0	0.4	0.2	0.6
$h=0$	0.1	0	0.1	0.2
$h=1$	0.1	0.1	0	0.2
$P_B(b)$	0.2	0.5	0.3	

(1)

From the joint PMF, the correlation coefficient is

$$r_{H,B} = E[HB] = \sum_{h=-1}^{1} \sum_{b=0,2,4} hb P_{H,B}(h,b) \tag{2}$$

$$= -1(2)(0.4) + 1(2)(0.1) + -1(4)(0.2) + 1(4)(0) \tag{3}$$

$$= -1.4 \tag{4}$$

since only terms in which both h and b are nonzero make a contribution. Using the marginal PMFs, the expected values of X and Y are

$$E[H] = \sum_{h=-1}^{1} hP_H(h) = -1(0.6) + 0(0.2) + 1(0.2) = -0.2 \quad (5)$$

$$E[B] = \sum_{b=0,2,4} bP_B(b) = 0(0.2) + 2(0.5) + 4(0.3) = 2.2 \quad (6)$$

The covariance is

$$\text{Cov}[H,B] = E[HB] - E[H]E[B] = -1.4 - (-0.2)(2.2) = -0.96 \quad (7)$$

Problem 4.7.4 Solution

From the joint PMF, $P_X(x)Y$, found in Example 4.13, we can find the marginal PMF for X or Y by summing over the columns or rows of the joint PMF.

$$P_Y(y) = \begin{cases} 25/48 & y=1 \\ 13/48 & y=2 \\ 7/48 & y=3 \\ 3/48 & y=4 \\ 0 & \text{otherwise} \end{cases} \qquad P_X(x) = \begin{cases} 1/4 & x=1,2,3,4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The expected values are

$$E[Y] = \sum_{y=1}^{4} yP_Y(y) = 1\frac{25}{48} + 2\frac{13}{48} + 3\frac{7}{48} + 4\frac{3}{48} = 7/4 \quad (2)$$

$$E[X] = \sum_{x=1}^{4} xP_X(x) = \frac{1}{4}(1+2+3+4) = 5/2 \quad (3)$$

(b) To find the variances, we first find the second moments.

$$E\left[Y^2\right] = \sum_{y=1}^{4} y^2 P_Y(y) = 1^2\frac{25}{48} + 2^2\frac{13}{48} + 3^2\frac{7}{48} + 4^2\frac{3}{48} = 47/12 \quad (4)$$

$$E\left[X^2\right] = \sum_{x=1}^{4} x^2 P_X(x) = \frac{1}{4}\left(1^2+2^2+3^2+4^2\right) = 15/2 \quad (5)$$

Now the variances are

$$\text{Var}[Y] = E\left[Y^2\right] - (E[Y])^2 = 47/12 - (7/4)^2 = 41/48 \quad (6)$$

$$\text{Var}[X] = E\left[X^2\right] - (E[X])^2 = 15/2 - (5/2)^2 = 5/4 \quad (7)$$

(c) To find the correlation, we evaluate the product XY over all values of X and Y. Specifically,

$$r_{X,Y} = E[XY] = \sum_{x=1}^{4}\sum_{y=1}^{x} xyP_{X,Y}(x,y) \tag{8}$$
$$= \frac{1}{4}+\frac{2}{8}+\frac{3}{12}+\frac{4}{16}+\frac{4}{8}+\frac{6}{12}+\frac{8}{16}+\frac{9}{12}+\frac{12}{16}+\frac{16}{16} \tag{9}$$
$$= 5 \tag{10}$$

(d) The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 5 - (7/4)(10/4) = 10/16 \tag{11}$$

(e) The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[W,V]}{\sqrt{\text{Var}[W]\,\text{Var}[V]}} = \frac{10/16}{\sqrt{(41/48)(5/4)}} \approx 0.605 \tag{12}$$

Problem 4.7.5 Solution

For integers $0 \leq x \leq 5$, the marginal PMF of X is

$$P_X(x) = \sum_y P_{X,Y}(x,y) = \sum_{y=0}^{x}(1/21) = \frac{x+1}{21} \tag{1}$$

Similarly, for integers $0 \leq y \leq 5$, the marginal PMF of Y is

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \sum_{x=y}^{5}(1/21) = \frac{6-y}{21} \tag{2}$$

The complete expressions for the marginal PMFs are

$$P_X(x) = \begin{cases} (x+1)/21 & x = 0,\ldots,5 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$
$$P_Y(y) = \begin{cases} (6-y)/21 & y = 0,\ldots,5 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The expected values are

$$E[X] = \sum_{x=0}^{5} x\frac{x+1}{21} = \frac{70}{21} = \frac{10}{3} \qquad E[Y] = \sum_{y=0}^{5} y\frac{6-y}{21} = \frac{35}{21} = \frac{5}{3} \tag{5}$$

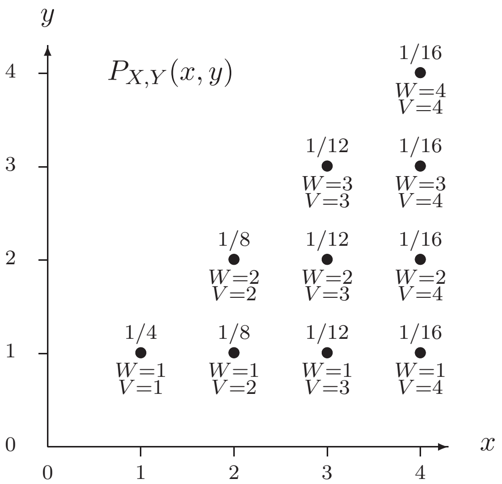
To find the covariance, we first find the correlation

$$E[XY] = \sum_{x=0}^{5}\sum_{y=0}^{x}\frac{xy}{21} = \frac{1}{21}\sum_{x=1}^{5} x\sum_{y=1}^{x} y = \frac{1}{42}\sum_{x=1}^{5} x^2(x+1) = \frac{280}{42} = \frac{20}{3} \tag{6}$$

The covariance of X and Y is

$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = \frac{20}{3} - \frac{50}{9} = \frac{10}{9} \tag{7}$$

Problem 4.7.6 Solution



To solve this problem, we identify the values of $W = \min(X, Y)$ and $V = \max(X, Y)$ for each possible pair x, y. Here we observe that $W = Y$ and $V = X$. This is a result of the underlying experiment in that given $X = x$, each $Y \in \{1, 2, \ldots, x\}$ is equally likely. Hence $Y \le X$. This implies $\min(X, Y) = Y$ and $\max(X, Y) = X$.

Using the results from Problem 4.7.4, we have the following answers.

(a) The expected values are

$$E[W] = E[Y] = 7/4 \qquad E[V] = E[X] = 5/2 \tag{1}$$

(b) The variances are

$$\text{Var}[W] = \text{Var}[Y] = 41/48 \qquad \text{Var}[V] = \text{Var}[X] = 5/4 \tag{2}$$

(c) The correlation is

$$r_{W,V} = E[WV] = E[XY] = r_{X,Y} = 5 \tag{3}$$

(d) The covariance of W and V is

$$\text{Cov}[W, V] = \text{Cov}[X, Y] = 10/16 \tag{4}$$

(e) The correlation coefficient is

$$\rho_{W,V} = \rho_{X,Y} = \frac{10/16}{\sqrt{(41/48)(5/4)}} \approx 0.605 \tag{5}$$

Problem 4.7.7 Solution

First, we observe that Y has mean $\mu_Y = a\mu_X + b$ and variance $\text{Var}[Y] = a^2 \text{Var}[X]$. The covariance of X and Y is

$$\begin{aligned}
\text{Cov}[X, Y] &= E[(X - \mu_X)(aX + b - a\mu_X - b)] && (1)\\
&= aE\left[(X - \mu_X)^2\right] && (2)\\
&= a\,\text{Var}[X] && (3)
\end{aligned}$$

The correlation coefficient is

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sqrt{\text{Var}[X]}\sqrt{\text{Var}[Y]}} = \frac{a\,\text{Var}[X]}{\sqrt{\text{Var}[X]}\sqrt{a^2\,\text{Var}[X]}} = \frac{a}{|a|} \tag{4}$$

When $a > 0$, $\rho_{X,Y} = 1$. When $a < 0$, $\rho_{X,Y} = 1$.

Problem 4.7.8 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Before calculating moments, we first find the marginal PDFs of X and Y. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^2 \frac{x+y}{3}\, dy = \left.\frac{xy}{3} + \frac{y^2}{6}\right|_{y=0}^{y=2} = \frac{2x+2}{3} \tag{2}$$

For $0 \le y \le 2$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_0^1 \left(\frac{x}{3} + \frac{y}{3}\right) dx = \left.\frac{x^2}{6} + \frac{xy}{3}\right|_{x=0}^{x=1} = \frac{2y+1}{6} \tag{3}$$

Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} \frac{2x+2}{3} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{2y+1}{6} & 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) The expected value of X is

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 x\frac{2x+2}{3}\, dx = \left.\frac{2x^3}{9} + \frac{x^2}{3}\right|_0^1 = \frac{5}{9} \tag{5}$$

The second moment of X is

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 x^2\frac{2x+2}{3}\, dx = \left.\frac{x^4}{6} + \frac{2x^3}{9}\right|_0^1 = \frac{7}{18} \tag{6}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 7/18 - (5/9)^2 = 13/162$.

(b) The expected value of Y is

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^2 y\frac{2y+1}{6}\, dy = \left.\frac{y^2}{12} + \frac{y^3}{9}\right|_0^2 = \frac{11}{9} \tag{7}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^2 y^2\frac{2y+1}{6}\, dy = \left.\frac{y^3}{18} + \frac{y^4}{12}\right|_0^2 = \frac{16}{9} \tag{8}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/81$.

(c) The correlation of X and Y is

$$E[XY] = \iint xy f_{X,Y}(x,y)\, dx\, dy \tag{9}$$

$$= \int_0^1 \int_0^2 xy\left(\frac{x+y}{3}\right) dy\, dx \tag{10}$$

$$= \int_0^1 \left(\left.\frac{x^2y^2}{6} + \frac{xy^3}{9}\right|_{y=0}^{y=2}\right) dx \tag{11}$$

$$= \int_0^1 \left(\frac{2x^2}{3} + \frac{8x}{9}\right) dx = \left.\frac{2x^3}{9} + \frac{4x^2}{9}\right|_0^1 = \frac{2}{3} \tag{12}$$

The covariance is $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = -1/81$.

(d) The expected value of X and Y is

$$E[X+Y] = E[X] + E[Y] = 5/9 + 11/9 = 16/9 \tag{13}$$

(e) By Theorem 4.15,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} \tag{14}$$

Problem 4.7.9 Solution

(a) The first moment of X is

$$E[X] = \int_0^1 \int_0^1 4x^2 y\,dy\,dx = \int_0^1 2x^2\,dx = \frac{2}{3} \tag{1}$$

The second moment of X is

$$E\left[X^2\right] = \int_0^1 \int_0^1 4x^3 y\,dy\,dx = \int_0^1 2x^3\,dx = \frac{1}{2} \tag{2}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/2 - (2/3)^2 = 1/18$.

(b) The mean of Y is

$$E[Y] = \int_0^1 \int_0^1 4xy^2\,dy\,dx = \int_0^1 \frac{4x}{3}\,dx = \frac{2}{3} \tag{3}$$

The second moment of Y is

$$E\left[Y^2\right] = \int_0^1 \int_0^1 4xy^3\,dy\,dx = \int_0^1 x\,dx = \frac{1}{2} \tag{4}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/2 - (2/3)^2 = 1/18$.

(c) To find the covariance, we first find the correlation

$$E[XY] = \int_0^1 \int_0^1 4x^2y^2\,dy\,dx = \int_0^1 \frac{4x^2}{3}\,dx = \frac{4}{9} \tag{5}$$

The covariance is thus

$$\text{Cov}[X,Y] = E[XY] - E[X]\,E[Y] = \frac{4}{9} - \left(\frac{2}{3}\right)^2 = 0 \tag{6}$$

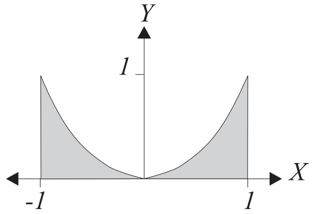
(d) $E[X+Y] = E[X] + E[Y] = 2/3 + 2/3 = 4/3$.

(e) By Theorem 4.15, the variance of $X+Y$ is

$$\text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1/18 + 1/18 + 0 = 1/9 \tag{7}$$

Problem 4.7.10 Solution

The joint PDF of X and Y and the region of nonzero probability are



$$f_{X,Y}(x,y) = \begin{cases} 5x^2/2 & -1 \le x \le 1, 0 \le y \le x^2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The first moment of X is

$$E[X] = \int_{-1}^{1}\int_{0}^{x^2} x\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^5}{2}\,dx = \left.\frac{5x^6}{12}\right|_{-1}^{1} = 0 \tag{2}$$

Since $E[X] = 0$, the variance of X and the second moment are both

$$\text{Var}[X] = E\left[X^2\right] = \int_{-1}^{1}\int_{0}^{x^2} x^2\frac{5x^2}{2}\,dy\,dx = \left.\frac{5x^7}{14}\right|_{-1}^{1} = \frac{10}{14} \tag{3}$$

(b) The first and second moments of Y are

$$E[Y] = \int_{-1}^{1}\int_{0}^{x^2} y\frac{5x^2}{2}\,dy\,dx = \frac{5}{14} \tag{4}$$

$$E\left[Y^2\right] = \int_{-1}^{1}\int_{0} x^2y^2\frac{5x^2}{2}\,dy\,dx = \frac{5}{26} \tag{5}$$

Therefore, $\text{Var}[Y] = 5/26 - (5/14)^2 = .0576$.

(c) Since $E[X] = 0$, $\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = E[XY]$. Thus,

$$\text{Cov}[X,Y] = E[XY] = \int_{1}^{1}\int_{0}^{x^2} xy\frac{5x^2}{2}\,dy\,dx = \int_{-1}^{1}\frac{5x^7}{4}\,dx = 0 \tag{6}$$

(d) The expected value of the sum $X+Y$ is

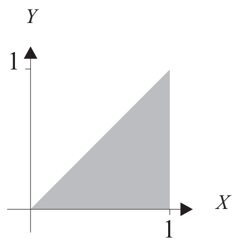
$$E[X+Y] = E[X] + E[Y] = \frac{5}{14} \tag{7}$$

(e) By Theorem 4.15, the variance of $X+Y$ is

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 5/7 + 0.0576 = 0.7719 \tag{8}$$

Problem 4.7.11 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Before finding moments, it is helpful to first find the marginal PDFs. For $0 \le x \le 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_0^x 2\, dy = 2x \tag{2}$$

Note that $f_X(x) = 0$ for $x < 0$ or $x > 1$. For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \int_y^1 2\, dx = 2(1-y) \tag{3}$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. Complete expressions for the marginal PDFs are

$$f_X(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \quad f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(a) The first two moments of X are

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3 \tag{5}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx = \int_0^1 2x^3\, dx = 1/2 \tag{6}$$

The variance of X is $\text{Var}[X] = E[X^2] - (E[X])^2 = 1/2 - 4/9 = 1/18$.

(b) The expected value and second moment of Y are

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy = \int_0^1 2y(1-y)\, dy = y^2 - \left.\frac{2y^3}{3}\right|_0^1 = 1/3 \tag{7}$$

$$E[Y^2] = \int_{-\infty}^{\infty} y^2 f_Y(y)\, dy = \int_0^1 2y^2(1-y)\, dy = \frac{2y^3}{3} - \left.\frac{y^4}{2}\right|_0^1 = 1/6 \tag{8}$$

The variance of Y is $\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1/6 - 1/9 = 1/18$.

(c) Before finding the covariance, we find the correlation

$$E[XY] = \int_0^1 \int_0^x 2xy\, dy\, dx = \int_0^1 x^3\, dx = 1/4 \tag{9}$$

The covariance is

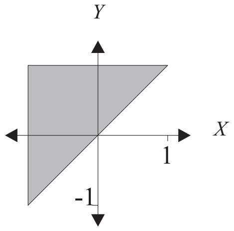
$$\text{Cov}[X,Y] = E[XY] - E[X]E[Y] = 1/36. \tag{10}$$

(d) $E[X+Y] = E[X] + E[Y] = 2/3 + 1/3 = 1$

(e) By Theorem 4.15,

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\,\text{Cov}[X,Y] = 1/6. \tag{11}$$

Problem 4.7.12 Solution



Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The region of possible pairs (x, y) is shown with the joint PDF. The rest of this problem is just calculus.

$$E[XY] = \int_{-1}^{1}\int_{x}^{1} \frac{xy}{2}\,dy\,dx = \frac{1}{4}\int_{-1}^{1} x(1-x^2)\,dx = \left.\frac{x^2}{8} - \frac{x^4}{16}\right|_{-1}^{1} = 0 \tag{2}$$

$$E\left[e^{X+Y}\right] = \int_{-1}^{1}\int_{x}^{1} \frac{1}{2} e^x e^y\,dy\,dx \tag{3}$$

$$= \frac{1}{2}\int_{-1}^{1} e^x(e^1 - e^x)\,dx \tag{4}$$

$$= \left.\frac{1}{2}e^{1+x} - \frac{1}{4}e^{2x}\right|_{-1}^{1} = \frac{e^2}{4} + \frac{e^{-2}}{4} - \frac{1}{2} \tag{5}$$

Problem 4.7.13 Solution

For this problem, calculating the marginal PMF of K is not easy. However, the marginal PMF of N is easy to find. For $n = 1, 2, \ldots$,

$$P_N(n) = \sum_{k=1}^{n} \frac{(1-p)^{n-1}p}{n} = (1-p)^{n-1}p \tag{1}$$

That is, N has a geometric PMF. From Appendix A, we note that

$$E[N] = \frac{1}{p} \qquad \text{Var}[N] = \frac{1-p}{p^2} \tag{2}$$

We can use these facts to find the second moment of N.

$$E\left[N^2\right] = \text{Var}[N] + (E[N])^2 = \frac{2-p}{p^2} \tag{3}$$

Now we can calculate the moments of K.

$$E[K] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} k\frac{(1-p)^{n-1}p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n}\sum_{k=1}^{n} k \tag{4}$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E[K] = \sum_{n=1}^{\infty} \frac{n+1}{2}(1-p)^{n-1}p = E\left[\frac{N+1}{2}\right] = \frac{1}{2p} + \frac{1}{2} \tag{5}$$

We now can calculate the sum of the moments.

$$E[N+K] = E[N] + E[K] = \frac{3}{2p} + \frac{1}{2} \tag{6}$$

The second moment of K is

$$E\left[K^2\right] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} k^2 \frac{(1-p)^{n-1}p}{n} = \sum_{n=1}^{\infty} \frac{(1-p)^{n-1}p}{n} \sum_{k=1}^{n} k^2 \tag{7}$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$, we obtain

$$E\left[K^2\right] = \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{6}(1-p)^{n-1}p = E\left[\frac{(N+1)(2N+1)}{6}\right] \tag{8}$$

Applying the values of $E[N]$ and $E[N^2]$ found above, we find that

$$E\left[K^2\right] = \frac{E\left[N^2\right]}{3} + \frac{E\left[N\right]}{2} + \frac{1}{6} = \frac{2}{3p^2} + \frac{1}{6p} + \frac{1}{6} \tag{9}$$

Thus, we can calculate the variance of K.

$$\text{Var}[K] = E\left[K^2\right] - (E\left[K\right])^2 = \frac{5}{12p^2} - \frac{1}{3p} + \frac{5}{12} \tag{10}$$

To find the correlation of N and K,

$$E\left[NK\right] = \sum_{n=1}^{\infty}\sum_{k=1}^{n} nk \frac{(1-p)^{n-1}p}{n} = \sum_{n=1}^{\infty}(1-p)^{n-1}p \sum_{k=1}^{n} k \tag{11}$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

$$E\left[NK\right] = \sum_{n=1}^{\infty} \frac{n(n+1)}{2}(1-p)^{n-1}p = E\left[\frac{N(N+1)}{2}\right] = \frac{1}{p^2} \tag{12}$$

Finally, the covariance is

$$\text{Cov}\left[N,K\right] = E\left[NK\right] - E\left[N\right]E\left[K\right] = \frac{1}{2p^2} - \frac{1}{2p} \tag{13}$$

Problem 4.8.1 Solution

The event A occurs iff $X > 5$ and $Y > 5$ and has probability

$$P\left[A\right] = P\left[X > 5, Y > 5\right] = \sum_{x=6}^{10}\sum_{y=6}^{10} 0.01 = 0.25 \tag{1}$$

From Theorem 4.19,

$$P_{X,Y|A}\left(x,y\right) = \begin{cases} \frac{P_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$= \begin{cases} 0.04 & x = 6,\ldots,10; y = 6,\ldots,20 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.8.2 Solution
The event B occurs iff $X \leq 5$ and $Y \leq 5$ and has probability

$$P[B] = P[X \leq 5, Y \leq 5] = \sum_{x=1}^{5} \sum_{y=1}^{5} 0.01 = 0.25 \tag{1}$$

From Theorem 4.19,

$$P_{X,Y|B}(x, y) = \begin{cases} \frac{P_{X,Y}(x,y)}{P[B]} & (x, y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$= \begin{cases} 0.04 & x = 1, \ldots, 5; y = 1, \ldots, 5 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.8.3 Solution
Given the event $A = \{X + Y \leq 1\}$, we wish to find $f_{X,Y|A}(x, y)$. First we find

$$P[A] = \int_0^1 \int_0^{1-x} 6e^{-(2x+3y)}\, dy\, dx = 1 - 3e^{-2} + 2e^{-3} \tag{1}$$

So then

$$f_{X,Y|A}(x, y) = \begin{cases} \frac{6e^{-(2x+3y)}}{1-3e^{-2}+2e^{-3}} & x + y \leq 1, x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 4.8.4 Solution
First we observe that for $n = 1, 2, \ldots$, the marginal PMF of N satisfies

$$P_N(n) = \sum_{k=1}^{n} P_{N,K}(n, k) = (1-p)^{n-1} p \sum_{k=1}^{n} \frac{1}{n} = (1-p)^{n-1} p \tag{1}$$

Thus, the event B has probability

$$P[B] = \sum_{n=10}^{\infty} P_N(n) = (1-p)^9 p[1 + (1-p) + (1-p)^2 + \cdots] = (1-p)^9 \tag{2}$$

From Theorem 4.19,

$$P_{N,K|B}(n, k) = \begin{cases} \frac{P_{N,K}(n,k)}{P[B]} & n, k \in B \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$= \begin{cases} (1-p)^{n-10} p/n & n = 10, 11, \ldots; k = 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The conditional PMF $P_{N|B}(n|b)$ could be found directly from $P_N(n)$ using Theorem 2.17. However, we can also find it just by summing the conditional joint PMF.

$$P_{N|B}(n) = \sum_{k=1}^{n} P_{N,K|B}(n, k) = \begin{cases} (1-p)^{n-10} p & n = 10, 11, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

From the conditional PMF $P_{N|B}(n)$, we can calculate directly the conditional moments of N given B. Instead, however, we observe that given B, $N' = N - 9$ has a geometric PMF with mean $1/p$. That is, for $n = 1, 2, \ldots$,

$$P_{N'|B}(n) = P[N = n + 9|B] = P_{N|B}(n + 9) = (1 - p)^{n-1} p \tag{6}$$

Hence, given B, $N = N' + 9$ and we can calculate the conditional expectations

$$E[N|B] = E\left[N' + 9|B\right] = E\left[N'|B\right] + 9 = 1/p + 9 \tag{7}$$

$$\text{Var}[N|B] = \text{Var}[N' + 9|B] = \text{Var}[N'|B] = (1 - p)/p^2 \tag{8}$$

Note that further along in the problem we will need $E[N^2|B]$ which we now calculate.

$$E\left[N^2|B\right] = \text{Var}[N|B] + (E[N|B])^2 \tag{9}$$

$$= \frac{2}{p^2} + \frac{17}{p} + 81 \tag{10}$$

For the conditional moments of K, we work directly with the conditional PMF $P_{N,K|B}(n, k)$.

$$E[K|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} k \frac{(1 - p)^{n-10} p}{n} = \sum_{n=10}^{\infty} \frac{(1 - p)^{n-10} p}{n} \sum_{k=1}^{n} k \tag{11}$$

Since $\sum_{k=1}^{n} k = n(n + 1)/2$,

$$E[K|B] = \sum_{n=1}^{\infty} \frac{n + 1}{2} (1 - p)^{n-1} p = \frac{1}{2} E[N + 1|B] = \frac{1}{2p} + 5 \tag{12}$$

We now can calculate the conditional expectation of the sum.

$$E[N + K|B] = E[N|B] + E[K|B] = 1/p + 9 + 1/(2p) + 5 = \frac{3}{2p} + 14 \tag{13}$$

The conditional second moment of K is

$$E\left[K^2|B\right] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} k^2 \frac{(1 - p)^{n-10} p}{n} = \sum_{n=10}^{\infty} \frac{(1 - p)^{n-10} p}{n} \sum_{k=1}^{n} k^2 \tag{14}$$

Using the identity $\sum_{k=1}^{n} k^2 = n(n + 1)(2n + 1)/6$, we obtain

$$E\left[K^2|B\right] = \sum_{n=10}^{\infty} \frac{(n + 1)(2n + 1)}{6} (1 - p)^{n-10} p = \frac{1}{6} E[(N + 1)(2N + 1)|B] \tag{15}$$

Applying the values of $E[N|B]$ and $E[N^2|B]$ found above, we find that

$$E\left[K^2|B\right] = \frac{E\left[N^2|B\right]}{3} + \frac{E[N|B]}{2} + \frac{1}{6} = \frac{2}{3p^2} + \frac{37}{6p} + 31\frac{2}{3} \tag{16}$$

Thus, we can calculate the conditional variance of K.

$$\text{Var}[K|B] = E\left[K^2|B\right] - (E[K|B])^2 = \frac{5}{12p^2} - \frac{7}{6p} + 6\frac{2}{3} \tag{17}$$

To find the conditional correlation of N and K,

$$E[NK|B] = \sum_{n=10}^{\infty} \sum_{k=1}^{n} nk \frac{(1-p)^{n-10}p}{n} = \sum_{n=10}^{\infty} (1-p)^{n-1} p \sum_{k=1}^{n} k \tag{18}$$

Since $\sum_{k=1}^{n} k = n(n+1)/2$,

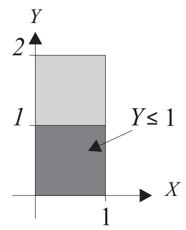
$$E[NK|B] = \sum_{n=10}^{\infty} \frac{n(n+1)}{2} (1-p)^{n-10} p = \frac{1}{2} E[N(N+1)|B] = \frac{1}{p^2} + \frac{9}{p} + 45 \tag{19}$$

Problem 4.8.5 Solution

The joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} (x+y)/3 & 0 \le x \le 1, 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability that $Y \le 1$ is



$$P[A] = P[Y \le 1] = \iint_{y \le 1} f_{X,Y}(x,y)\, dx\, dy \tag{2}$$

$$= \int_0^1 \int_0^1 \frac{x+y}{3}\, dy\, dx \tag{3}$$

$$= \int_0^1 \left(\frac{xy}{3} + \frac{y^2}{6} \Big|_{y=0}^{y=1} \right) dx \tag{4}$$

$$= \int_0^1 \frac{2x+1}{6}\, dx = \frac{x^2}{6} + \frac{x}{6} \Big|_0^1 = \frac{1}{3} \tag{5}$$

(b) By Definition 4.10, the conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} x+y & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

From $f_{X,Y|A}(x,y)$, we find the conditional marginal PDF $f_{X|A}(x)$. For $0 \le x \le 1$,

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dy = \int_0^1 (x+y)\, dy = xy + \frac{y^2}{2} \Big|_{y=0}^{y=1} = x + \frac{1}{2} \tag{7}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} x + 1/2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

For $0 \le y \le 1$, the conditional marginal PDF of Y is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dx = \int_0^1 (x+y)\, dx = \frac{x^2}{2} + xy \Big|_{x=0}^{x=1} = y + 1/2 \tag{9}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} y + 1/2 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Problem 4.8.6 Solution
Random variables X and Y have joint PDF

$$f_{X,Y}(x,y) = \begin{cases} (4x+2y)/3 & 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The probability of event $A = \{Y \le 1/2\}$ is

$$P[A] = \iint_{y \le 1/2} f_{X,Y}(x,y)\, dy\, dx = \int_0^1 \int_0^{1/2} \frac{4x+2y}{3}\, dy\, dx. \tag{2}$$

With some calculus,

$$P[A] = \int_0^1 \left. \frac{4xy+y^2}{3} \right|_{y=0}^{y=1/2} dx = \int_0^1 \frac{2x+1/4}{3}\, dx = \left. \frac{x^2}{3} + \frac{x}{12} \right|_0^1 = \frac{5}{12}. \tag{3}$$

(b) The conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$= \begin{cases} 8(2x+y)/5 & 0 \le x \le 1, 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

For $0 \le x \le 1$, the PDF of X given A is

$$f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dy = \frac{8}{5} \int_0^{1/2} (2x+y)\, dy \tag{6}$$

$$= \frac{8}{5} \left. \left(2xy + \frac{y^2}{2} \right) \right|_{y=0}^{y=1/2} = \frac{8x+1}{5} \tag{7}$$

The complete expression is

$$f_{X|A}(x) = \begin{cases} (8x+1)/5 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

For $0 \le y \le 1/2$, the conditional marginal PDF of Y given A is

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\, dx = \frac{8}{5} \int_0^1 (2x+y)\, dx \tag{9}$$

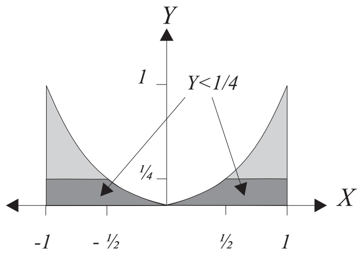
$$= \left. \frac{8x^2+8xy}{5} \right|_{x=0}^{x=1} = \frac{8y+8}{5} \tag{10}$$

The complete expression is

$$f_{Y|A}(y) = \begin{cases} (8y+8)/5 & 0 \le y \le 1/2 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

Problem 4.8.7 Solution

(a) The event $A = \{Y \le 1/4\}$ has probability



$$P[A] = 2\int_0^{1/2}\int_0^{x^2} \frac{5x^2}{2}\,dy\,dx + 2\int_{1/2}^{1}\int_0^{1/4} \frac{5x^2}{2}\,dy\,dx \tag{1}$$

$$= \int_0^{1/2} 5x^4\,dx + \int_{1/2}^{1} \frac{5x^2}{4}\,dx \tag{2}$$

$$= x^5\Big|_0^{1/2} + 5x^3/12\Big|_{1/2}^{1} \tag{3}$$

$$= 19/48 \tag{4}$$

This implies

$$f_{X,Y|A}(x,y) = \begin{cases} f_{X,Y}(x,y)/P[A] & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$= \begin{cases} 120x^2/19 & -1 \le x \le 1, 0 \le y \le x^2, y \le 1/4 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

(b)

$$f_{Y|A}(y) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dx = 2\int_{\sqrt{y}}^{1} \frac{120x^2}{19}\,dx = \begin{cases} \frac{80}{19}(1 - y^{3/2}) & 0 \le y \le 1/4 \\ 0 & \text{otherwise} \end{cases} \tag{7}$$

(c) The conditional expectation of Y given A is

$$E[Y|A] = \int_0^{1/4} y\frac{80}{19}(1 - y^{3/2})\,dy = \frac{80}{19}\left(\frac{y^2}{2} - \frac{2y^{7/2}}{7}\right)\Bigg|_0^{1/4} = \frac{65}{532} \tag{8}$$

(d) To find $f_{X|A}(x)$, we can write $f_{X|A}(x) = \int_{-\infty}^{\infty} f_{X,Y|A}(x,y)\,dy$. However, when we substitute $f_{X,Y|A}(x,y)$, the limits will depend on the value of x. When $|x| \le 1/2$,

$$f_{X|A}(x) = \int_0^{x^2} \frac{120x^2}{19}\,dy = \frac{120x^4}{19} \tag{9}$$

When $-1 \le x \le -1/2$ or $1/2 \le x \le 1$,

$$f_{X|A}(x) = \int_0^{1/4} \frac{120x^2}{19}\,dy = \frac{30x^2}{19} \tag{10}$$

The complete expression for the conditional PDF of X given A is

$$f_{X|A}(x) = \begin{cases} 30x^2/19 & -1 \le x \le -1/2 \\ 120x^4/19 & -1/2 \le x \le 1/2 \\ 30x^2/19 & 1/2 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

(e) The conditional mean of X given A is

$$E[X|A] = \int_{-1}^{-1/2} \frac{30x^3}{19}\,dx + \int_{-1/2}^{1/2} \frac{120x^5}{19}\,dx + \int_{1/2}^{1} \frac{30x^3}{19}\,dx = 0 \tag{12}$$

Problem 4.9.1 Solution

The main part of this problem is just interpreting the problem statement. No calculations are necessary. Since a trip is equally likely to last 2, 3 or 4 days,

$$P_D(d) = \begin{cases} 1/3 & d = 2,3,4 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Given a trip lasts d days, the weight change is equally likely to be any value between $-d$ and d pounds. Thus,

$$P_{W|D}(w|d) = \begin{cases} 1/(2d+1) & w = -d, -d+1, \ldots, d \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The joint PMF is simply

$$P_{D,W}(d,w) = P_{W|D}(w|d)\,P_D(d) \tag{3}$$

$$= \begin{cases} 1/(6d+3) & d = 2,3,4; w = -d, \ldots, d \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 4.9.2 Solution

We can make a table of the possible outcomes and the corresponding values of W and Y

outcome	$P[\cdot]$	W	Y
hh	p^2	0	2
ht	$p(1-p)$	1	1
th	$p(1-p)$	-1	1
tt	$(1-p)^2$	0	0

(1)

In the following table, we write the joint PMF $P_{W,Y}(w,y)$ along with the marginal PMFs $P_Y(y)$ and $P_W(w)$.

$P_{W,Y}(w,y)$	$w=-1$	$w=0$	$w=1$	$P_Y(y)$
$y=0$	0	$(1-p)^2$	0	$(1-p)^2$
$y=1$	$p(1-p)$	0	$p(1-p)$	$2p(1-p)$
$y=2$	0	p^2	0	p^2
$P_W(w)$	$p(1-p)$	$1-2p+2p^2$	$p(1-p)$	

(2)

Using the definition $P_{W|Y}(w|y) = P_{W,Y}(w,y)/P_Y(y)$, we can find the conditional PMFs of W given Y.

$$P_{W|Y}(w|0) = \begin{cases} 1 & w = 0 \\ 0 & \text{otherwise} \end{cases} \qquad P_{W|Y}(w|1) = \begin{cases} 1/2 & w = -1, 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$P_{W|Y}(w|2) = \begin{cases} 1 & w = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Similarly, the conditional PMFs of Y given W are

$$P_{Y|W}(y|-1) = \begin{cases} 1 & y = 1 \\ 0 & \text{otherwise} \end{cases} \qquad P_{Y|W}(y|0) = \begin{cases} \frac{(1-p)^2}{1-2p+2p^2} & y = 0 \\ \frac{p^2}{1-2p+2p^2} & y = 2 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

$$P_{Y|W}(y|1) = \begin{cases} 1 & y = 1 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

Problem 4.9.3 Solution

$$f_{X,Y}(x,y) = \begin{cases} (x+y) & 0 \le x, y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The conditional PDF $f_{X|Y}(x|y)$ is defined for all y such that $0 \le y \le 1$. For $0 \le y \le 1$,

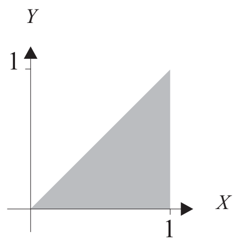
$$f_{X|Y}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{(x+y)}{\int_0^1 (x+y)\,dy} = \begin{cases} \frac{(x+y)}{x+1/2} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

(b) The conditional PDF $f_{Y|X}(y|x)$ is defined for all values of x in the interval $[0,1]$. For $0 \le x \le 1$,

$$f_{Y|X}(y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{(x+y)}{\int_0^1 (x+y)\,dx} = \begin{cases} \frac{(x+y)}{y+1/2} & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Problem 4.9.4 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $0 \le y \le 1$,

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_y^1 2\,dx = 2(1-y) \tag{2}$$

Also, for $y < 0$ or $y > 1$, $f_Y(y) = 0$. The complete expression for the marginal PDF is

$$f_Y(y) = \begin{cases} 2(1-y) & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

By Theorem 4.24, the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1-y} & y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

That is, since $Y \leq X \leq 1$, X is uniform over $[y, 1]$ when $Y = y$. The conditional expectation of X given $Y = y$ can be calculated as

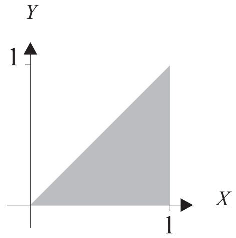
$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \tag{5}$$

$$= \int_{y}^{1} \frac{x}{1-y}\, dx = \left. \frac{x^2}{2(1-y)} \right|_{y}^{1} = \frac{1+y}{2} \tag{6}$$

In fact, since we know that the conditional PDF of X is uniform over $[y, 1]$ when $Y = y$, it wasn't really necessary to perform the calculation.

Problem 4.9.5 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 2 & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For $0 \leq x \leq 1$, the marginal PDF for X satisfies

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy = \int_{0}^{x} 2\, dy = 2x \tag{2}$$

Note that $f_X(x) = 0$ for $x < 0$ or $x > 1$. Hence the complete expression for the marginal PDF of X is

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

The conditional PDF of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/x & 0 \leq y \leq x \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Given $X = x$, Y has a uniform PDF over $[0, x]$ and thus has conditional expected value $E[Y|X = x] = x/2$. Another way to obtain this result is to calculate $\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\, dy$.

Problem 4.9.6 Solution

We are told in the problem statement that if we know r, the number of feet a student sits from the blackboard, then we also know that that student's grade is a Gaussian random variable with mean $80 - r$ and standard deviation r. This is exactly

$$f_{X|R}(x|r) = \frac{1}{\sqrt{2\pi r^2}} e^{-(x-[80-r])^2/2r^2} \tag{1}$$

Problem 4.9.7 Solution

(a) First we observe that A takes on the values $S_A = \{-1, 1\}$ while B takes on values from $S_B = \{0, 1\}$. To construct a table describing $P_{A,B}(a, b)$ we build a table for all possible values of pairs (A, B). The general form of the entries is

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & P_{B|A}(0|-1)\,P_A(-1) & P_{B|A}(1|-1)\,P_A(-1) \\ a=1 & P_{B|A}(0|1)\,P_A(1) & P_{B|A}(1|1)\,P_A(1) \end{array} \tag{1}$$

Now we fill in the entries using the conditional PMFs $P_{B|A}(b|a)$ and the marginal PMF $P_A(a)$. This yields

$$\begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & (1/3)(1/3) & (2/3)(1/3) \\ a=1 & (1/2)(2/3) & (1/2)(2/3) \end{array} \text{ which simplifies to } \begin{array}{c|cc} P_{A,B}(a,b) & b=0 & b=1 \\ \hline a=-1 & 1/9 & 2/9 \\ a=1 & 1/3 & 1/3 \end{array} \tag{2}$$

(b) Since $P_A(1) = P_{A,B}(1, 0) + P_{A,B}(1, 1) = 2/3$,

$$P_{B|A}(b|1) = \frac{P_{A,B}(1,b)}{P_A(1)} = \begin{cases} 1/2 & b = 0, 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

If $A = 1$, the conditional expectation of B is

$$E[B|A=1] = \sum_{b=0}^{1} b P_{B|A}(b|1) = P_{B|A}(1|1) = 1/2. \tag{4}$$

(c) Before finding the conditional PMF $P_{A|B}(a|1)$, we first sum the columns of the joint PMF table to find

$$P_B(b) = \begin{cases} 4/9 & b = 0 \\ 5/9 & b = 1 \end{cases} \tag{5}$$

The conditional PMF of A given $B = 1$ is

$$P_{A|B}(a|1) = \frac{P_{A,B}(a,1)}{P_B(1)} = \begin{cases} 2/5 & a = -1 \\ 3/5 & a = 1 \end{cases} \tag{6}$$

(d) Now that we have the conditional PMF $P_{A|B}(a|1)$, calculating conditional expectations is easy.

$$E[A|B=1] = \sum_{a=-1,1} a P_{A|B}(a|1) = -1(2/5) + (3/5) = 1/5 \tag{7}$$

$$E\left[A^2|B=1\right] = \sum_{a=-1,1} a^2 P_{A|B}(a|1) = 2/5 + 3/5 = 1 \tag{8}$$

The conditional variance is then

$$\text{Var}[A|B=1] = E\left[A^2|B=1\right] - (E[A|B=1])^2 = 1 - (1/5)^2 = 24/25 \tag{9}$$

(e) To calculate the covariance, we need

$$E[A] = \sum_{a=-1,1} aP_A(a) = -1(1/3) + 1(2/3) = 1/3 \tag{10}$$

$$E[B] = \sum_{b=0}^{1} bP_B(b) = 0(4/9) + 1(5/9) = 5/9 \tag{11}$$

$$E[AB] = \sum_{a=-1,1} \sum_{b=0}^{1} abP_{A,B}(a,b) \tag{12}$$

$$= -1(0)(1/9) + -1(1)(2/9) + 1(0)(1/3) + 1(1)(1/3) = 1/9 \tag{13}$$

The covariance is just

$$\text{Cov}[A,B] = E[AB] - E[A]E[B] = 1/9 - (1/3)(5/9) = -2/27 \tag{14}$$

Problem 4.9.8 Solution

First we need to find the conditional expectations

$$E[B|A=-1] = \sum_{b=0}^{1} bP_{B|A}(b|-1) = 0(1/3) + 1(2/3) = 2/3 \tag{1}$$

$$E[B|A=1] = \sum_{b=0}^{1} bP_{B|A}(b|1) = 0(1/2) + 1(1/2) = 1/2 \tag{2}$$

Keep in mind that $E[B|A]$ is a random variable that is a function of A. that is we can write

$$E[B|A] = g(A) = \begin{cases} 2/3 & A=-1 \\ 1/2 & A=1 \end{cases} \tag{3}$$

We see that the range of U is $S_U = \{1/2, 2/3\}$. In particular,

$$P_U(1/2) = P_A(1) = 2/3 \qquad P_U(2/3) = P_A(-1) = 1/3 \tag{4}$$

The complete PMF of U is

$$P_U(u) = \begin{cases} 2/3 & u = 1/2 \\ 1/3 & u = 2/3 \end{cases} \tag{5}$$

Note that

$$E[E[B|A]] = E[U] = \sum_u uP_U(u) = (1/2)(2/3) + (2/3)(1/3) = 5/9 \tag{6}$$

You can check that $E[U] = E[B]$.

Problem 4.9.9 Solution

Random variables N and K have the joint PMF

$$P_{N,K}(n,k) = \begin{cases} \frac{100^n e^{-100}}{(n+1)!} & k = 0,1,\ldots,n; \\ & n = 0,1,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can find the marginal PMF for N by summing over all possible K. For $n \geq 0$,

$$P_N(n) = \sum_{k=0}^{n} \frac{100^n e^{-100}}{(n+1)!} = \frac{100^n e^{-100}}{n!} \tag{2}$$

We see that N has a Poisson PMF with expected value 100. For $n \geq 0$, the conditional PMF of K given $N = n$ is

$$P_{K|N}(k|n) = \frac{P_{N,K}(n,k)}{P_N(n)} = \begin{cases} 1/(n+1) & k = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

That is, given $N = n$, K has a discrete uniform PMF over $\{0, 1, \ldots, n\}$. Thus,

$$E[K|N = n] = \sum_{k=0}^{n} k/(n+1) = n/2 \tag{4}$$

We can conclude that $E[K|N] = N/2$. Thus, by Theorem 4.25,

$$E[K] = E[E[K|N]] = E[N/2] = 50. \tag{5}$$

Problem 4.9.10 Solution

This problem is fairly easy when we use conditional PMF's. In particular, given that $N = n$ pizzas were sold before noon, each of those pizzas has mushrooms with probability 1/3. The conditional PMF of M given N is the binomial distribution

$$P_{M|N}(m|n) = \begin{cases} \binom{n}{m}(1/3)^m(2/3)^{n-m} & m = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The other fact we know is that for each of the 100 pizzas sold, the pizza is sold before noon with probability 1/2. Hence, N has the binomial PMF

$$P_N(n) = \begin{cases} \binom{100}{n}(1/2)^n(1/2)^{100-n} & n = 0, 1, \ldots, 100 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

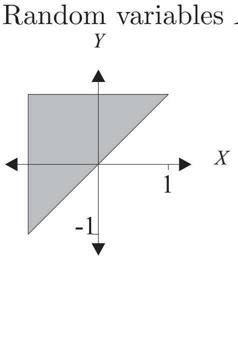
The joint PMF of N and M is for integers m, n,

$$P_{M,N}(m,n) = P_{M|N}(m|n) P_N(n) \tag{3}$$

$$= \begin{cases} \binom{n}{m}\binom{100}{n}(1/3)^m(2/3)^{n-m}(1/2)^{100} & 0 \leq m \leq n \leq 100 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Problem 4.9.11 Solution

Random variables X and Y have joint PDF



$$f_{X,Y}(x,y) = \begin{cases} 1/2 & -1 \leq x \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) For $-1 \leq y \leq 1$, the marginal PDF of Y is

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dx = \frac{1}{2}\int_{-1}^{y} dx = (y+1)/2 \qquad (2)$$

The complete expression for the marginal PDF of Y is

$$f_Y(y) = \begin{cases} (y+1)/2 & -1 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

(b) The conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{1+y} & -1 \leq x \leq y \\ 0 & \text{otherwise} \end{cases} \qquad (4)$$

(c) Given $Y = y$, the conditional PDF of X is uniform over $[-1, y]$. Hence the conditional expected value is $E[X|Y = y] = (y-1)/2$.

Problem 4.9.12 Solution

We are given that the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \begin{cases} 1/(\pi r^2) & 0 \leq x^2 + y^2 \leq r^2 \\ 0 & \text{otherwise} \end{cases} \qquad (1)$$

(a) The marginal PDF of X is

$$f_X(x) = 2\int_0^{\sqrt{r^2-x^2}} \frac{1}{\pi r^2}\, dy = \begin{cases} \frac{2\sqrt{r^2-x^2}}{\pi r^2} & -r \leq x \leq r \\ 0 & \text{otherwise} \end{cases} \qquad (2)$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} 1/(2\sqrt{r^2-x^2}) & y^2 \leq r^2 - x^2 \\ 0 & \text{otherwise} \end{cases} \qquad (3)$$

(b) Given $X = x$, we observe that over the interval $[-\sqrt{r^2-x^2}, \sqrt{r^2-x^2}]$, Y has a uniform PDF. Since the conditional PDF $f_{Y|X}(y|x)$ is symmetric about $y = 0$,

$$E[Y|X = x] = 0 \qquad (4)$$

Problem 4.9.13 Solution

The key to solving this problem is to find the joint PMF of M and N. Note that $N \geq M$. For $n > m$, the joint event $\{M = m, N = n\}$ has probability

$$\begin{aligned} P[M = m, N = n] &= P[\overbrace{dd\cdots d}^{\substack{m-1 \\ \text{calls}}}v\, \overbrace{dd\cdots d}^{\substack{n-m-1 \\ \text{calls}}}\, v] && (1) \\ &= (1-p)^{m-1}p(1-p)^{n-m-1}p && (2) \\ &= (1-p)^{n-2}p^2 && (3) \end{aligned}$$

A complete expression for the joint PMF of M and N is

$$P_{M,N}(m,n) = \begin{cases} (1-p)^{n-2}p^2 & m = 1,2,\ldots,n-1;\ n = m+1, m+2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The marginal PMF of N satisfies

$$P_N(n) = \sum_{m=1}^{n-1}(1-p)^{n-2}p^2 = (n-1)(1-p)^{n-2}p^2, \qquad n = 2,3,\ldots \tag{5}$$

Similarly, for $m = 1, 2, \ldots$, the marginal PMF of M satisfies

$$P_M(m) = \sum_{n=m+1}^{\infty}(1-p)^{n-2}p^2 \tag{6}$$

$$= p^2[(1-p)^{m-1} + (1-p)^m + \cdots] \tag{7}$$

$$= (1-p)^{m-1}p \tag{8}$$

The complete expressions for the marginal PMF's are

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

$$P_N(n) = \begin{cases} (n-1)(1-p)^{n-2}p^2 & n = 2,3,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

Not surprisingly, if we view each voice call as a successful Bernoulli trial, M has a geometric PMF since it is the number of trials up to and including the first success. Also, N has a Pascal PMF since it is the number of trials required to see 2 successes. The conditional PMF's are now easy to find.

$$P_{N|M}(n|m) = \frac{P_{M,N}(m,n)}{P_M(m)} = \begin{cases} (1-p)^{n-m-1}p & n = m+1, m+2,\ldots \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

The interpretation of the conditional PMF of N given M is that given $M = m$, $N = m + N'$ where N' has a geometric PMF with mean $1/p$. The conditional PMF of M given N is

$$P_{M|N}(m|n) = \frac{P_{M,N}(m,n)}{P_N(n)} = \begin{cases} 1/(n-1) & m = 1,\ldots,n-1 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

Given that call $N = n$ was the second voice call, the first voice call is equally likely to occur in any of the previous $n-1$ calls.

Problem 4.9.14 Solution

(a) The number of buses, N, must be greater than zero. Also, the number of minutes that pass cannot be less than the number of buses. Thus, $P[N = n, T = t] > 0$ for integers n, t satisfying $1 \le n \le t$.

(b) First, we find the joint PMF of N and T by carefully considering the possible sample paths. In particular, $P_{N,T}(n,t) = P[ABC] = P[A]P[B]P[C]$ where the events A, B and C are

$$A = \{n-1 \text{ buses arrive in the first } t-1 \text{ minutes}\} \quad (1)$$

$$B = \{\text{none of the first } n-1 \text{ buses are boarded}\} \quad (2)$$

$$C = \{\text{at time } t \text{ a bus arrives and is boarded}\} \quad (3)$$

These events are independent since each trial to board a bus is independent of when the buses arrive. These events have probabilities

$$P[A] = \binom{t-1}{n-1} p^{n-1}(1-p)^{t-1-(n-1)} \quad (4)$$

$$P[B] = (1-q)^{n-1} \quad (5)$$

$$P[C] = pq \quad (6)$$

Consequently, the joint PMF of N and T is

$$P_{N,T}(n,t) = \begin{cases} \binom{t-1}{n-1} p^{n-1}(1-p)^{t-n}(1-q)^{n-1}pq & n \geq 1, t \geq n \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

(c) It is possible to find the marginal PMF's by summing the joint PMF. However, it is much easier to obtain the marginal PMFs by consideration of the experiment. Specifically, when a bus arrives, it is boarded with probability q. Moreover, the experiment ends when a bus is boarded. By viewing whether each arriving bus is boarded as an independent trial, N is the number of trials until the first success. Thus, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-q)^{n-1}q & n = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

To find the PMF of T, suppose we regard each minute as an independent trial in which a success occurs if a bus arrives and that bus is boarded. In this case, the success probability is pq and T is the number of minutes up to and including the first success. The PMF of T is also geometric.

$$P_T(t) = \begin{cases} (1-pq)^{t-1}pq & t = 1,2,\ldots \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

(d) Once we have the marginal PMFs, the conditional PMFs are easy to find.

$$P_{N|T}(n|t) = \frac{P_{N,T}(n,t)}{P_T(t)} = \begin{cases} \binom{t-1}{n-1} \left(\frac{p(1-q)}{1-pq}\right)^{n-1} \left(\frac{1-p}{1-pq}\right)^{t-1-(n-1)} & n = 1,2,\ldots,t \\ 0 & \text{otherwise} \end{cases} \quad (10)$$

That is, given you depart at time $T = t$, the number of buses that arrive during minutes $1,\ldots,t-1$ has a binomial PMF since in each minute a bus arrives with probability p. Similarly, the conditional PMF of T given N is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \binom{t-1}{n-1} p^n(1-p)^{t-n} & t = n, n+1,\ldots \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

This result can be explained. Given that you board bus $N = n$, the time T when you leave is the time for n buses to arrive. If we view each bus arrival as a success of an independent trial, the time for n buses to arrive has the above Pascal PMF.

Problem 4.9.15 Solution

If you construct a tree describing what type of call (if any) that arrived in any 1 millisecond period, it will be apparent that a fax call arrives with probability $\alpha = pqr$ or no fax arrives with probability $1-\alpha$. That is, whether a fax message arrives each millisecond is a Bernoulli trial with success probability α. Thus, the time required for the first success has the geometric PMF

$$P_T(t) = \begin{cases} (1-\alpha)^{t-1}\alpha & t = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Note that N is the number of trials required to observe 100 successes. Moreover, the number of trials needed to observe 100 successes is $N = T + N'$ where N' is the number of trials needed to observe successes 2 through 100. Since N' is just the number of trials needed to observe 99 successes, it has the Pascal $(k = 99, p)$ PMF

$$P_{N'}(n) = \binom{n-1}{98} \alpha^{99} (1-\alpha)^{n-99}. \quad (2)$$

Since the trials needed to generate successes 2 though 100 are independent of the trials that yield the first success, N' and T are independent. Hence

$$P_{N|T}(n|t) = P_{N'|T}(n-t|t) = P_{N'}(n-t). \quad (3)$$

Applying the PMF of N' found above, we have

$$P_{N|T}(n|t) = \binom{n-t-1}{98} \alpha^{99} (1-\alpha)^{n-t-99}. \quad (4)$$

Finally the joint PMF of N and T is

$$P_{N,T}(n,t) = P_{N|T}(n|t)\, P_T(t) \quad (5)$$

$$= \begin{cases} \binom{n-t-1}{98} \alpha^{100} (1-\alpha)^{n-100} & t = 1, 2, \ldots; n = 99+t, 100+t, \ldots \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

This solution can also be found a consideration of the sample sequence of Bernoulli trials in which we either observe or do not observe a fax message.

To find the conditional PMF $P_{T|N}(t|n)$, we first must recognize that N is simply the number of trials needed to observe 100 successes and thus has the Pascal PMF

$$P_N(n) = \binom{n-1}{99} \alpha^{100} (1-\alpha)^{n-100} \quad (7)$$

Hence for any integer $n \geq 100$, the conditional PMF is

$$P_{T|N}(t|n) = \frac{P_{N,T}(n,t)}{P_N(n)} = \begin{cases} \frac{\binom{n-t-1}{98}}{\binom{n-1}{99}} & t = 1, 2, \ldots, n-99 \\ 0 & \text{otherwise.} \end{cases} \quad (8)$$

Problem 4.10.1 Solution

Flip a fair coin 100 times and let X be the number of heads in the first 75 flips and Y be the number of heads in the last 25 flips. We know that X and Y are independent and can find their PMFs easily.

$$P_X(x) = \binom{75}{x} (1/2)^{75} \qquad P_Y(y) = \binom{25}{y} (1/2)^{25} \quad (1)$$

The joint PMF of X and N can be expressed as the product of the marginal PMFs because we know that X and Y are independent.

$$P_{X,Y}(x,y) = \binom{75}{x}\binom{25}{y}(1/2)^{100} \tag{2}$$

Problem 4.10.2 Solution

Using the following probability model

$$P_X(k) = P_Y(k) = \begin{cases} 3/4 & k = 0 \\ 1/4 & k = 20 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

We can calculate the requested moments.

$$E[X] = 3/4 \cdot 0 + 1/4 \cdot 20 = 5 \tag{2}$$

$$\text{Var}[X] = 3/4 \cdot (0-5)^2 + 1/4 \cdot (20-5)^2 = 75 \tag{3}$$

$$E[X+Y] = E[X] + E[X] = 2E[X] = 10 \tag{4}$$

Since X and Y are independent, Theorem 4.27 yields

$$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] = 2\,\text{Var}[X] = 150 \tag{5}$$

Since X and Y are independent, $P_{X,Y}(x,y) = P_X(x)P_Y(y)$ and

$$E\left[XY2^{XY}\right] = \sum_{x=0,20}\sum_{y=0,20} XY2^{XY} P_{X,Y}(x,y) = (20)(20)2^{20(20)} P_X(20)\, P_Y(20) \tag{6}$$

$$= 2.75 \times 10^{12} \tag{7}$$

Problem 4.10.3 Solution

(a) Normally, checking independence requires the marginal PMFs. However, in this problem, the zeroes in the table of the joint PMF $P_{X,Y}(x,y)$ allows us to verify very quickly that X and Y are dependent. In particular, $P_X(-1) = 1/4$ and $P_Y(1) = 14/48$ but

$$P_{X,Y}(-1,1) = 0 \neq P_X(-1)\, P_Y(1) \tag{1}$$

(b) To fill in the tree diagram, we need the marginal PMF $P_X(x)$ and the conditional PMFs $P_{Y|X}(y|x)$. By summing the rows on the table for the joint PMF, we obtain

$P_{X,Y}(x,y)$	$y=-1$	$y=0$	$y=1$	$P_X(x)$
$x=-1$	3/16	1/16	0	1/4
$x=0$	1/6	1/6	1/6	1/2
$x=1$	0	1/8	1/8	1/4

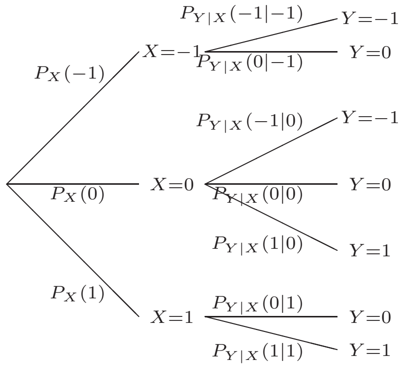
(2)

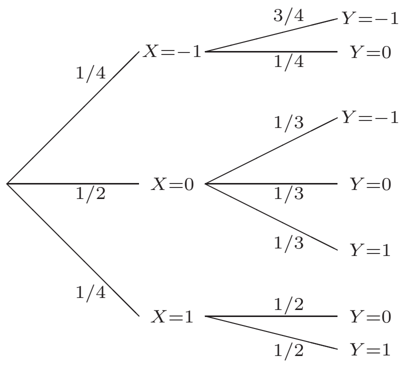
Now we use the conditional PMF $P_{Y|X}(y|x) = P_{X,Y}(x,y)/P_X(x)$ to write

$$P_{Y|X}(y|-1) = \begin{cases} 3/4 & y = -1 \\ 1/4 & y = 0 \\ 0 & \text{otherwise} \end{cases} \qquad P_{Y|X}(y|0) = \begin{cases} 1/3 & y = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$P_{Y|X}(y|1) = \begin{cases} 1/2 & y = 0, 1 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

Now we can us these probabilities to label the tree. The generic solution and the specific solution with the exact values are





Problem 4.10.4 Solution

In the solution to Problem 4.9.10, we found that the conditional PMF of M given N is

$$P_{M|N}(m|n) = \begin{cases} \binom{n}{m}(1/3)^m(2/3)^{n-m} & m = 0, 1, \ldots, n \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since $P_{M|N}(m|n)$ depends on the event $N = n$, we see that M and N are dependent.

Problem 4.10.5 Solution

We can solve this problem for the general case when the probability of heads is p. For the fair coin, $p = 1/2$. Viewing each flip as a Bernoulli trial in which heads is a success, the number of flips until heads is the number of trials needed for the first success which has the geometric PMF

$$P_{X_1}(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Similarly, no matter how large X_1 may be, the number of *additional* flips for the second heads is the same experiment as the number of flips needed for the first occurrence of heads. That is, $P_{X_2}(x) = P_{X_1}(x)$. Moreover, the flips needed to generate the second occurrence of heads are independent of the flips that yield the first heads. Hence, it should be apparent that X_1 and X_2 are independent and

$$P_{X_1,X_2}(x_1, x_2) = P_{X_1}(x_1)\, P_{X_2}(x_2) = \begin{cases} (1-p)^{x_1+x_2-2}p^2 & x_1 = 1, 2, \ldots; x_2 = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

However, if this independence is not obvious, it can be derived by examination of the sample path. When $x_1 \geq 1$ and $x_2 \geq 1$, the event $\{X_1 = x_1, X_2 = x_2\}$ occurs iff we observe the sample sequence

$$\underbrace{tt\cdots t}_{x_1 - 1 \text{ times}} \quad h \quad \underbrace{tt\cdots t}_{x_2 - 1 \text{ times}} \quad h \tag{3}$$

The above sample sequence has probability $(1-p)^{x_1-1}p(1-p)^{x_2-1}p$ which in fact equals $P_{X_1,X_2}(x_1, x_2)$ given earlier.

Problem 4.10.6 Solution

We will solve this problem when the probability of heads is p. For the fair coin, $p = 1/2$. The number X_1 of flips until the first heads and the number X_2 of additional flips for the second heads both have the geometric PMF

$$P_{X_1}(x) = P_{X_2}(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Thus, $E[X_i] = 1/p$ and $\text{Var}[X_i] = (1-p)/p^2$. By Theorem 4.14,

$$E[Y] = E[X_1] - E[X_2] = 0 \tag{2}$$

Since X_1 and X_2 are independent, Theorem 4.27 says

$$\text{Var}[Y] = \text{Var}[X_1] + \text{Var}[-X_2] = \text{Var}[X_1] + \text{Var}[X_2] = \frac{2(1-p)}{p^2} \tag{3}$$

Problem 4.10.7 Solution

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} \frac{1}{3}e^{-x/3} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{2}e^{-y/2} & y \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) To calculate $P[X > Y]$, we use the joint PDF $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

$$\begin{aligned} P[X > Y] &= \iint_{x>y} f_X(x) f_Y(y)\, dx\, dy && (2) \\ &= \int_0^\infty \frac{1}{2}e^{-y/2} \int_y^\infty \frac{1}{3}e^{-x/3}\, dx\, dy && (3) \\ &= \int_0^\infty \frac{1}{2}e^{-y/2}e^{-y/3}\, dy && (4) \\ &= \int_0^\infty \frac{1}{2}e^{-(1/2+1/3)y}\, dy = \frac{1/2}{1/2 + 2/3} = \frac{3}{7} && (5) \end{aligned}$$

(b) Since X and Y are exponential random variables with parameters $\lambda_X = 1/3$ and $\lambda_Y = 1/2$, Appendix A tells us that $E[X] = 1/\lambda_X = 3$ and $E[Y] = 1/\lambda_Y = 2$. Since X and Y are independent, the correlation is $E[XY] = E[X]E[Y] = 6$.

(c) Since X and Y are independent, $\text{Cov}[X, Y] = 0$.

Problem 4.10.8 Solution

(a) Since $E[-X_2] = -E[X_2]$, we can use Theorem 4.13 to write

$$\begin{aligned} E[X_1 - X_2] = E[X_1 + (-X_2)] &= E[X_1] + E[-X_2] && (1) \\ &= E[X_1] - E[X_2] && (2) \\ &= 0 && (3) \end{aligned}$$

(b) By Theorem 3.5(f), $\text{Var}[-X_2] = (-1)^2 \text{Var}[X_2] = \text{Var}[X_2]$. Since X_1 and X_2 are independent, Theorem 4.27(a) says that

$$\begin{aligned} \text{Var}[X_1 - X_2] &= \text{Var}[X_1 + (-X_2)] && (4)\\ &= \text{Var}[X_1] + \text{Var}[-X_2] && (5)\\ &= 2\,\text{Var}[X] && (6)\end{aligned}$$

Problem 4.10.9 Solution

Since X and Y are take on only integer values, $W = X + Y$ is integer valued as well. Thus for an integer w,

$$P_W(w) = P[W = w] = P[X + Y = w]. \tag{1}$$

Suppose $X = k$, then $W = w$ if and only if $Y = w - k$. To find all ways that $X + Y = w$, we must consider each possible integer k such that $X = k$. Thus

$$P_W(w) = \sum_{k=-\infty}^{\infty} P[X = k, Y = w - k] = \sum_{k=-\infty}^{\infty} P_{X,Y}(k, w - k). \tag{2}$$

Since X and Y are independent, $P_{X,Y}(k, w - k) = P_X(k)P_Y(w - k)$. It follows that for any integer w,

$$P_W(w) = \sum_{k=-\infty}^{\infty} P_X(k)\, P_Y(w - k). \tag{3}$$

Problem 4.10.10 Solution

The key to this problem is understanding that “short order” and “long order” are synonyms for $N = 1$ and $N = 2$. Similarly, “vanilla”, “chocolate”, and “strawberry” correspond to the events $D = 20$, $D = 100$ and $D = 300$.

(a) The following table is given in the problem statement.

	vanilla	choc.	strawberry
short order	0.2	0.2	0.2
long order	0.1	0.2	0.1

This table can be translated directly into the joint PMF of N and D.

$P_{N,D}(n,d)$	$d = 20$	$d = 100$	$d = 300$
$n = 1$	0.2	0.2	0.2
$n = 2$	0.1	0.2	0.1

(1)

(b) We find the marginal PMF $P_D(d)$ by summing the columns of the joint PMF. This yields

$$P_D(d) = \begin{cases} 0.3 & d = 20, \\ 0.4 & d = 100, \\ 0.3 & d = 300, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(c) To find the conditional PMF $P_{D|N}(d|2)$, we first need to find the probability of the conditioning event

$$P_N(2) = P_{N,D}(2,20) + P_{N,D}(2,100) + P_{N,D}(2,300) = 0.4 \tag{3}$$

The conditional PMF of N D given $N = 2$ is

$$P_{D|N}(d|2) = \frac{P_{N,D}(2,d)}{P_N(2)} = \begin{cases} 1/4 & d = 20 \\ 1/2 & d = 100 \\ 1/4 & d = 300 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The conditional expectation of D given $N = 2$ is

$$E[D|N=2] = \sum_d d P_{D|N}(d|2) = 20(1/4) + 100(1/2) + 300(1/4) = 130 \tag{5}$$

(e) To check independence, we could calculate the marginal PMFs of N and D. In this case, however, it is simpler to observe that $P_D(d) \neq P_{D|N}(d|2)$. Hence N and D are dependent.

(f) In terms of N and D, the cost (in cents) of a fax is $C = ND$. The expected value of C is

$$E[C] = \sum_{n,d} nd P_{N,D}(n,d) \tag{6}$$

$$= 1(20)(0.2) + 1(100)(0.2) + 1(300)(0.2) \tag{7}$$

$$+ 2(20)(0.3) + 2(100)(0.4) + 2(300)(0.3) = 356 \tag{8}$$

Problem 4.10.11 Solution

The key to this problem is understanding that "Factory Q" and "Factory R" are synonyms for $M = 60$ and $M = 180$. Similarly, "small", "medium", and "large" orders correspond to the events $B = 1$, $B = 2$ and $B = 3$.

(a) The following table given in the problem statement

	Factory Q	Factory R
small order	0.3	0.2
medium order	0.1	0.2
large order	0.1	0.1

can be translated into the following joint PMF for B and M.

$P_{B,M}(b,m)$	$m = 60$	$m = 180$
$b = 1$	0.3	0.2
$b = 2$	0.1	0.2
$b = 3$	0.1	0.1

(1)

(b) Before we find $E[B]$, it will prove helpful to find the marginal PMFs $P_B(b)$ and $P_M(m)$. These can be found from the row and column sums of the table of the joint PMF

$P_{B,M}(b,m)$	$m=60$	$m=180$	$P_B(b)$
$b=1$	0.3	0.2	0.5
$b=2$	0.1	0.2	0.3
$b=3$	0.1	0.1	0.2
$P_M(m)$	0.5	0.5	

(2)

The expected number of boxes is

$$E[B] = \sum_b b P_B(b) = 1(0.5) + 2(0.3) + 3(0.2) = 1.7 \tag{3}$$

(c) From the marginal PMF of B, we know that $P_B(2) = 0.3$. The conditional PMF of M given $B = 2$ is

$$P_{M|B}(m|2) = \frac{P_{B,M}(2,m)}{P_B(2)} = \begin{cases} 1/3 & m=60 \\ 2/3 & m=180 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(d) The conditional expectation of M given $B = 2$ is

$$E[M|B=2] = \sum_m m P_{M|B}(m|2) = 60(1/3) + 180(2/3) = 140 \tag{5}$$

(e) From the marginal PMFs we calculated in the table of part (b), we can conclude that B and M are not independent. since $P_{B,M}(1,60) \neq P_B(1)P_M(m)60$.

(f) In terms of M and B, the cost (in cents) of sending a shipment is $C = BM$. The expected value of C is

$$E[C] = \sum_{b,m} bm P_{B,M}(b,m) \tag{6}$$

$$= 1(60)(0.3) + 2(60)(0.1) + 3(60)(0.1) \tag{7}$$

$$+ 1(180)(0.2) + 2(180)(0.2) + 3(180)(0.1) = 210 \tag{8}$$

Problem 4.10.12 Solution

Random variables X_1 and X_2 are iiid with PDF

$$f_X(x) = \begin{cases} x/2 & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

(a) Since X_1 and X_2 are identically distributed they will share the same CDF $F_X(x)$.

$$F_X(x) = \int_0^x f_X(x')\, dx' = \begin{cases} 0 & x \le 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x \ge 2 \end{cases} \tag{2}$$

(b) Since X_1 and X_2 are independent, we can say that

$$P[X_1 \leq 1, X_2 \leq 1] = P[X_1 \leq 1] P[X_2 \leq 1] = F_{X_1}(1) F_{X_2}(1) = [F_X(1)]^2 = \frac{1}{16} \quad (3)$$

(c) For $W = \max(X_1, X_2)$,

$$F_W(1) = P[\max(X_1, X_2) \leq 1] = P[X_1 \leq 1, X_2 \leq 1] \quad (4)$$

Since X_1 and X_2 are independent,

$$F_W(1) = P[X_1 \leq 1] P[X_2 \leq 1] = [F_X(1)]^2 = 1/16 \quad (5)$$

(d)

$$F_W(w) = P[\max(X_1, X_2) \leq w] = P[X_1 \leq w, X_2 \leq w] \quad (6)$$

Since X_1 and X_2 are independent,

$$F_W(w) = P[X_1 \leq w] P[X_2 \leq w] = [F_X(w)]^2 = \begin{cases} 0 & w \leq 0 \\ w^4/16 & 0 \leq w \leq 2 \\ 1 & w \geq 2 \end{cases} \quad (7)$$

Problem 4.10.13 Solution

X and Y are independent random variables with PDFs

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \qquad f_Y(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For the event $A = \{X > Y\}$, this problem asks us to calculate the conditional expectations $E[X|A]$ and $E[Y|A]$. We will do this using the conditional joint PDF $f_{X,Y|A}(x,y)$. Since X and Y are independent, it is tempting to argue that the event $X > Y$ does not alter the probability model for X and Y. Unfortunately, this is not the case. When we learn that $X > Y$, it increases the probability that X is large and Y is small. We will see this when we compare the conditional expectations $E[X|A]$ and $E[Y|A]$ to $E[X]$ and $E[Y]$.

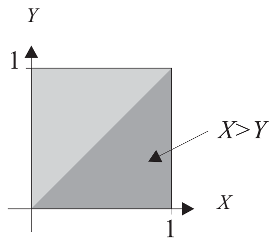
(a) We can calculate the unconditional expectations, $E[X]$ and $E[Y]$, using the marginal PDFs $f_X(x)$ and $f_Y(y)$.

$$E[X] = \int_{-\infty}^{\infty} f_X(x)\, dx = \int_0^1 2x^2\, dx = 2/3 \quad (2)$$

$$E[Y] = \int_{-\infty}^{\infty} f_Y(y)\, dy = \int_0^1 3y^3\, dy = 3/4 \quad (3)$$

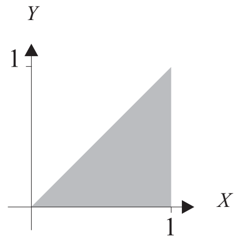
(b) First, we need to calculate the conditional joint PDF *ipdf*$X, Y|Ax, y$. The first step is to write down the joint PDF of X and Y:

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 6xy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$



The event A has probability

$$P[A] = \iint_{x>y} f_{X,Y}(x,y)\, dy\, dx \tag{5}$$
$$= \int_0^1 \int_0^x 6xy^2\, dy\, dx \tag{6}$$
$$= \int_0^1 2x^4\, dx = 2/5 \tag{7}$$



The conditional joint PDF of X and Y given A is

$$f_{X,Y|A}(x,y) = \begin{cases} \frac{f_{X,Y}(x,y)}{P[A]} & (x,y) \in A \\ 0 & \text{otherwise} \end{cases} \tag{8}$$
$$= \begin{cases} 15xy^2 & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} \tag{9}$$

The triangular region of nonzero probability is a signal that given A, X and Y are no longer independent. The conditional expected value of X given A is

$$E[X|A] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f_{X,Y|A}(x,y|a)\, x, y\, dy\, dx \tag{10}$$
$$= 15 \int_0^1 x^2 \int_0^x y^2\, dy\, dx \tag{11}$$
$$= 5 \int_0^1 x^5\, dx = 5/6. \tag{12}$$

The conditional expected value of Y given A is

$$E[Y|A] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f_{X,Y|A}(x,y)\, dy\, dx \tag{13}$$
$$= 15 \int_0^1 x \int_0^x y^3\, dy\, dx \tag{14}$$
$$= \frac{15}{4} \int_0^1 x^5\, dx = 5/8. \tag{15}$$

We see that $E[X|A] > E[X]$ while $E[Y|A] < E[Y]$. That is, learning $X > Y$ gives us a clue that X may be larger than usual while Y may be smaller than usual.

Problem 4.10.14 Solution

This problem is quite straightforward. From Theorem 4.4, we can find the joint PDF of X and Y is

$$f_{X,Y}(x,y) = \frac{\partial^2 [F_X(x) F_Y(y)]}{\partial x\, \partial y} = \frac{\partial [f_X(x) F_Y(y)]}{\partial y} = f_X(x) f_Y(y) \tag{1}$$

Hence, $F_{X,Y}(x,y) = F_X(x)F_Y(y)$ implies that X and Y are independent.

If X and Y are independent, then

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) \tag{2}$$

By Definition 4.3,

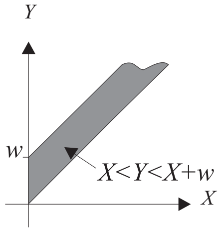
$$
\begin{aligned}
F_{X,Y}(x,y) &= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du &(3)\\
&= \left(\int_{-\infty}^{x} f_X(u)\,du\right)\left(\int_{-\infty}^{y} f_Y(v)\,dv\right) &(4)\\
&= F_X(x)\,F_X(x) &(5)
\end{aligned}
$$

Problem 4.10.15 Solution

Random variables X and Y have joint PDF

$$
f_{X,Y}(x,y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x \le y \\ 0 & \text{otherwise} \end{cases} \qquad (1)
$$

For $W = Y - X$ we can find $f_W(w)$ by integrating over the region indicated in the figure below to get $F_W(w)$ then taking the derivative with respect to w. Since $Y \ge X$, $W = Y - X$ is nonnegative. Hence $F_W(w) = 0$ for $w < 0$. For $w \ge 0$,



$$
\begin{aligned}
F_W(w) = 1 - P[W > w] &= 1 - P[Y > X + w] &(2)\\
&= 1 - \int_0^{\infty}\int_{x+w}^{\infty} \lambda^2 e^{-\lambda y}\,dy\,dx &(3)\\
&= 1 - e^{-\lambda w} &(4)
\end{aligned}
$$

The complete expressions for the joint CDF and corresponding joint PDF are

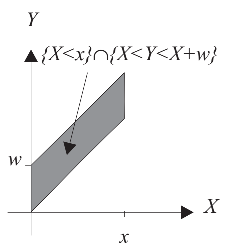
$$
F_W(w) = \begin{cases} 0 & w < 0 \\ 1 - e^{-\lambda w} & w \ge 0 \end{cases} \qquad f_W(w) = \begin{cases} 0 & w < 0 \\ \lambda e^{-\lambda w} & w \ge 0 \end{cases} \qquad (5)
$$

Problem 4.10.16 Solution

(a) To find if W and X are independent, we must be able to factor the joint density function $f_{X,W}(x,w)$ into the product $f_X(x)f_W(w)$ of marginal density functions. To verify this, we must find the joint PDF of X and W. First we find the joint CDF.

$$
\begin{aligned}
F_{X,W}(x,w) &= P[X \le x, W \le w] &(1)\\
&= P[X \le x, Y - X \le w] = P[X \le x, Y \le X + w] &(2)
\end{aligned}
$$

Since $Y \ge X$, the CDF of W satisfies $F_{X,W}(x,w) = P[X \le x, X \le Y \le X + w]$. Thus, for $x \ge 0$ and $w \ge 0$,



$$
\begin{aligned}
F_{X,W}(x,w) &= \int_0^x \int_{x'}^{x'+w} \lambda^2 e^{-\lambda y}\,dy\,dx' &(3)\\
&= \int_0^x \left(-\lambda e^{-\lambda y}\Big|_{x'}^{x'+w}\right) dx' &(4)\\
&= \int_0^x \left(-\lambda e^{-\lambda(x'+w)} + \lambda e^{-\lambda x'}\right) dx' &(5)\\
&= e^{-\lambda(x'+w)} - e^{-\lambda x'}\Big|_0^x &(6)\\
&= (1 - e^{-\lambda x})(1 - e^{-\lambda w}) &(7)
\end{aligned}
$$

We see that $F_{X,W}(x,w) = F_X(x)F_W(w)$. Moreover, by applying Theorem 4.4,

$$f_{X,W}(x,w) = \frac{\partial^2 F_{X,W}(x,w)}{\partial x\,\partial w} = \lambda e^{-\lambda x}\lambda e^{-\lambda w} = f_X(x)\, f_W(w)\,. \tag{8}$$

Since we have our desired factorization, W and X are independent.

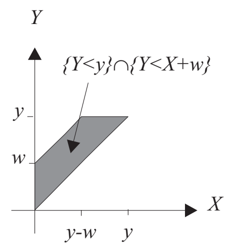
(b) Following the same procedure, we find the joint CDF of Y and W.

$$F_{W,Y}(w,y) = P[W \le w, Y \le y] = P[Y - X \le w, Y \le y] \tag{9}$$

$$= P[Y \le X + w, Y \le y]\,. \tag{10}$$

The region of integration corresponding to the event $\{Y \le x + w, Y \le y\}$ depends on whether $y < w$ or $y \ge w$. Keep in mind that although $W = Y - X \le Y$, the dummy arguments y and w of $f_{W,Y}(w,y)$ need not obey the same constraints. In any case, we must consider each case separately.

For $y > w$, the integration is



$$F_{W,Y}(w,y) = \int_0^{y-w}\int_u^{u+w} \lambda^2 e^{-\lambda v}\,dv\,du + \int_{y-w}^{y}\int_u^{y} \lambda^2 e^{-\lambda v}\,dv\,du \tag{11}$$

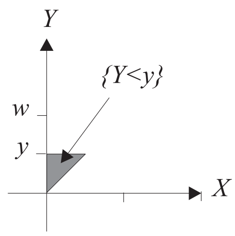
$$= \lambda\int_0^{y-w}\left[e^{-\lambda u} - e^{-\lambda(u+w)}\right]du + \lambda\int_{y-w}^{y}\left[e^{-\lambda u} - e^{-\lambda y}\right]du \tag{12}$$

It follows that

$$F_{W,Y}(w,y) = \left[-e^{-\lambda u} + e^{-\lambda(u+w)}\right]\Big|_0^{y-w} + \left[-e^{-\lambda u} - u\lambda e^{-\lambda y}\right]\Big|_{y-w}^{y} \tag{13}$$

$$= 1 - e^{-\lambda w} - \lambda w e^{-\lambda y}. \tag{14}$$

For $y \le w$,



$$F_{W,Y}(w,y) = \int_0^{y}\int_u^{y} \lambda^2 e^{-\lambda v}\,dv\,du \tag{15}$$

$$= \int_0^{y}\left[-\lambda e^{-\lambda y} + \lambda e^{-\lambda u}\right]du \tag{16}$$

$$= -\lambda u e^{-\lambda y} - e^{-\lambda u}\Big|_0^{y} \tag{17}$$

$$= 1 - (1 + \lambda y)e^{-\lambda y} \tag{18}$$

The complete expression for the joint CDF is

$$F_{W,Y}(w,y) = \begin{cases} 1 - e^{-\lambda w} - \lambda w e^{-\lambda y} & 0 \le w \le y \\ 1 - (1+\lambda y)e^{-\lambda y} & 0 \le y \le w \\ 0 & \text{otherwise} \end{cases} \tag{19}$$

Applying Theorem 4.4 yields

$$f_{W,Y}(w,y) = \frac{\partial^2 F_{W,Y}(w,y)}{\partial w\,\partial y} = \begin{cases} 2\lambda^2 e^{-\lambda y} & 0 \le w \le y \\ 0 & \text{otherwise} \end{cases} \tag{20}$$

The joint PDF $f_{W,Y}(w,y)$ doesn't factor and thus W and Y are dependent.

Problem 4.10.17 Solution

We need to define the events $A = \{U \leq u\}$ and $B = \{V \leq v\}$. In this case,

$$F_{U,V}(u,v) = P[AB] = P[B] - P[A^cB] = P[V \leq v] - P[U > u, V \leq v] \tag{1}$$

Note that $U = \min(X,Y) > u$ if and only if $X > u$ and $Y > u$. In the same way, since $V = \max(X,Y)$, $V \leq v$ if and only if $X \leq v$ and $Y \leq v$. Thus

$$P[U > u, V \leq v] = P[X > u, Y > u, X \leq v, Y \leq v] \tag{2}$$
$$= P[u < X \leq v, u < Y \leq v] \tag{3}$$

Thus, the joint CDF of U and V satisfies

$$F_{U,V}(u,v) = P[V \leq v] - P[U > u, V \leq v] \tag{4}$$
$$= P[X \leq v, Y \leq v] - P[u < X \leq v, u < X \leq v] \tag{5}$$

Since X and Y are independent random variables,

$$F_{U,V}(u,v) = P[X \leq v]\,P[Y \leq v] - P[u < X \leq v]\,P[u < X \leq v] \tag{6}$$
$$= F_X(v)\,F_Y(v) - (F_X(v) - F_X(u))\,(F_Y(v) - F_Y(u)) \tag{7}$$
$$= F_X(v)\,F_Y(u) + F_X(u)\,F_Y(v) - F_X(u)\,F_Y(u) \tag{8}$$

The joint PDF is

$$f_{U,V}(u,v) = \frac{\partial^2 F_{U,V}(u,v)}{\partial u \partial v} \tag{9}$$
$$= \frac{\partial}{\partial u}\left[f_X(v)\,F_Y(u) + F_X(u)\,f_Y(v)\right] \tag{10}$$
$$= f_X(u)\,f_Y(v) + f_X(v)\,f_Y(v) \tag{11}$$

Problem 4.11.1 Solution

$$f_{X,Y}(x,y) = ce^{-(x^2/8)-(y^2/18)} \tag{1}$$

The omission of any limits for the PDF indicates that it is defined over all x and y. We know that $f_{X,Y}(x,y)$ is in the form of the bivariate Gaussian distribution so we look to Definition 4.17 and attempt to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ. First, we know that the constant is

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \tag{2}$$

Because the exponent of $f_{X,Y}(x,y)$ doesn't contain any cross terms we know that ρ must be zero, and we are left to solve the following for $E[X]$, $E[Y]$, σ_X, and σ_Y:

$$\left(\frac{x - E[X]}{\sigma_X}\right)^2 = \frac{x^2}{8} \qquad \left(\frac{y - E[Y]}{\sigma_Y}\right)^2 = \frac{y^2}{18} \tag{3}$$

From which we can conclude that

$$E[X] = E[Y] = 0 \tag{4}$$
$$\sigma_X = \sqrt{8} \tag{5}$$
$$\sigma_Y = \sqrt{18} \tag{6}$$

Putting all the pieces together, we find that $c = \frac{1}{24\pi}$. Since $\rho = 0$, we also find that X and Y are independent.

Problem 4.11.2 Solution

For the joint PDF

$$f_{X,Y}(x,y) = ce^{-(2x^2-4xy+4y^2)}, \tag{1}$$

we proceed as in Problem 4.11.1 to find values for σ_Y, σ_X, $E[X]$, $E[Y]$ and ρ.

(a) First, we try to solve the following equations

$$\left(\frac{x - E[X]}{\sigma_X}\right)^2 = 4(1-\rho^2)x^2 \tag{2}$$

$$\left(\frac{y - E[Y]}{\sigma_Y}\right)^2 = 8(1-\rho^2)y^2 \tag{3}$$

$$\frac{2\rho}{\sigma_X\sigma_Y} = 8(1-\rho^2) \tag{4}$$

The first two equations yield $E[X] = E[Y] = 0$

(b) To find the correlation coefficient ρ, we observe that

$$\sigma_X = 1/\sqrt{4(1-\rho^2)} \qquad \sigma_Y = 1/\sqrt{8(1-\rho^2)} \tag{5}$$

Using σ_X and σ_Y in the third equation yields $\rho = 1/\sqrt{2}$.

(c) Since $\rho = 1/\sqrt{2}$, now we can solve for σ_X and σ_Y.

$$\sigma_X = 1/\sqrt{2} \qquad \sigma_Y = 1/2 \tag{6}$$

(d) From here we can solve for c.

$$c = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} = \frac{2}{\pi} \tag{7}$$

(e) X and Y are dependent because $\rho \neq 0$.

Problem 4.11.3 Solution

From the problem statement, we learn that

$$\mu_X = \mu_Y = 0 \qquad \sigma_X^2 = \sigma_Y^2 = 1 \tag{1}$$

From Theorem 4.30, the conditional expectation of Y given X is

$$E[Y|X] = \tilde{\mu}_Y(X) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(X - \mu_X) = \rho X \tag{2}$$

In the problem statement, we learn that $E[Y|X] = X/2$. Hence $\rho = 1/2$. From Definition 4.17, the joint PDF is

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{3\pi^2}}e^{-2(x^2-xy+y^2)/3} \tag{3}$$

Problem 4.11.4 Solution

The event B is the set of outcomes satisfying $X^2 + Y^2 \leq 2^2$. Of ocurse, the calculation of $P[B]$ depends on the probability model for X and Y.

(a) In this instance, X and Y have the same PDF

$$f_X(x) = f_Y(x) = \begin{cases} 0.01 & -50 \leq x \leq 50 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} 10^{-4} & -50 \leq x \leq 50, -50 \leq y \leq 50 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Because X and Y have a uniform PDF over the bullseye area, $P[B]$ is just the value of the joint PDF over the area times the area of the bullseye.

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = 10^{-4} \cdot \pi 2^2 = 4\pi \cdot 10^{-4} \approx 0.0013 \tag{3}$$

(b) In this case, the joint PDF of X and Y is inversely proportional to the area of the target.

$$f_{X,Y}(x,y) = \begin{cases} 1/[\pi 50^2] & x^2 + y^2 \leq 50^2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

The probability of a bullseye is

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = \frac{\pi 2^2}{\pi 50^2} = \left(\frac{1}{25}\right)^2 \approx 0.0016. \tag{5}$$

(c) In this instance, X and Y have the identical Gaussian $(0, \sigma)$ PDF with $\sigma^2 = 100$; i.e.,

$$f_X(x) = f_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-x^2/2\sigma^2} \tag{6}$$

Since X and Y are independent, their joint PDF is

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2} \tag{7}$$

To find $P[B]$, we write

$$P[B] = P\left[X^2 + Y^2 \leq 2^2\right] = \iint_{x^2+y^2 \leq 2^2} f_{X,Y}(x,y)\, dx\, dy \tag{8}$$

$$= \frac{1}{2\pi\sigma^2} \iint_{x^2+y^2 \leq 2^2} e^{-(x^2+y^2)/2\sigma^2}\, dx\, dy \tag{9}$$

This integral is easy using polar coordinates. With the substitutions $x^2 + y^2 = r^2$, and $dx\, dy = r\, dr\, d\theta$,

$$P[B] = \frac{1}{2\pi\sigma^2} \int_0^2 \int_0^{2\pi} e^{-r^2/2\sigma^2} r\, dr\, d\theta \tag{10}$$

$$= \frac{1}{\sigma^2} \int_0^2 r e^{-r^2/2\sigma^2}\, dr \tag{11}$$

$$= -e^{-r^2/2\sigma^2} \Big|_0^2 = 1 - e^{-4/200} \approx 0.0198. \tag{12}$$

Problem 4.11.5 Solution

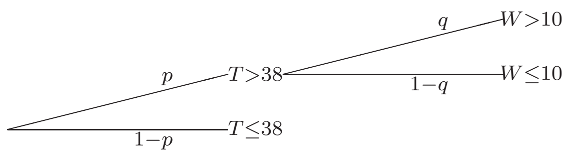
(a) The person's temperature is high with probability

$$p = P[T > 38] = P[T - 37 > 38 - 37] = 1 - \Phi(1) = 0.159. \tag{1}$$

Given that the temperature is high, then W is measured. Since $\rho = 0$, W and T are independent and

$$q = P[W > 10] = P\left[\frac{W-7}{2} > \frac{10-7}{2}\right] = 1 - \Phi(1.5) = 0.067. \tag{2}$$

The tree for this experiment is



The probability the person is ill is

$$P[I] = P[T > 38, W > 10] = P[T > 38]\, P[W > 10] = pq = 0.0107. \tag{3}$$

(b) The general form of the bivariate Gaussian PDF is

$$f_{W,T}(w,t) = \frac{\exp\left[-\frac{\left(\frac{w-\mu_1}{\sigma_1}\right)^2 - \frac{2\rho(w-\mu_1)(t-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{t-\mu_2}{\sigma_2}\right)^2}{2(1-\rho^2)}\right]}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \tag{4}$$

With $\mu_1 = E[W] = 7$, $\sigma_1 = \sigma_W = 2$, $\mu_2 = E[T] = 37$ and $\sigma_2 = \sigma_T = 1$ and $\rho = 1/\sqrt{2}$, we have

$$f_{W,T}(w,t) = \frac{1}{2\pi\sqrt{2}} \exp\left[-\frac{(w-7)^2}{4} - \frac{\sqrt{2}(w-7)(t-37)}{2} + (t-37)^2\right] \tag{5}$$

To find the conditional probability $P[I|T = t]$, we need to find the conditional PDF of W given $T = t$. The direct way is simply to use algebra to find

$$f_{W|T}(w|t) = \frac{f_{W,T}(w,t)}{f_T(t)} \tag{6}$$

The required algebra is essentially the same as that needed to prove Theorem 4.29. Its easier just to apply Theorem 4.29 which says that given $T = t$, the conditional distribution of W is Gaussian with

$$E[W|T = t] = E[W] + \rho\frac{\sigma_W}{\sigma_T}(t - E[T]) \tag{7}$$

$$\mathrm{Var}[W|T = t] = \sigma_W^2(1-\rho^2) \tag{8}$$

Plugging in the various parameters gives

$$E[W|T=t] = 7 + \sqrt{2}(t-37) \quad \text{and} \quad \text{Var}[W|T=t] = 2 \tag{9}$$

Using this conditional mean and variance, we obtain the conditional Gaussian PDF

$$f_{W|T}(w|t) = \frac{1}{\sqrt{4\pi}} e^{-\left(w-(7+\sqrt{2}(t-37))\right)^2/4}. \tag{10}$$

Given $T=t$, the conditional probability the person is declared ill is

$$\begin{aligned} P[I|T=t] &= P[W > 10|T=t] &(11)\\ &= P\left[\frac{W-(7+\sqrt{2}(t-37))}{\sqrt{2}} > \frac{10-(7+\sqrt{2}(t-37))}{\sqrt{2}}\right] &(12)\\ &= P\left[Z > \frac{3-\sqrt{2}(t-37)}{\sqrt{2}}\right] = Q\left(\frac{3\sqrt{2}}{2} - (t-37)\right). &(13) \end{aligned}$$

Problem 4.11.6 Solution

The given joint PDF is

$$f_{X,Y}(x,y) = de^{-(a^2x^2+bxy+c^2y^2)} \tag{1}$$

In order to be an example of the bivariate Gaussian PDF given in Definition 4.17, we must have

$$a^2 = \frac{1}{2\sigma_X^2(1-\rho^2)} \qquad c^2 = \frac{1}{2\sigma_Y^2(1-\rho^2)}$$

$$b = \frac{-\rho}{\sigma_X\sigma_Y(1-\rho^2)} \qquad d = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

We can solve for σ_X and σ_Y, yielding

$$\sigma_X = \frac{1}{a\sqrt{2(1-\rho^2)}} \qquad \sigma_Y = \frac{1}{c\sqrt{2(1-\rho^2)}} \tag{2}$$

Plugging these values into the equation for b, it follows that $b = -2ac\rho$, or, equivalently, $\rho = -b/2ac$. This implies

$$d^2 = \frac{1}{4\pi^2\sigma_X^2\sigma_Y^2(1-\rho^2)} = (1-\rho^2)a^2c^2 = a^2c^2 - b^2/4 \tag{3}$$

Since $|\rho| \le 1$, we see that $|b| \le 2ac$. Further, for any choice of a, b and c that meets this constraint, choosing $d = \sqrt{a^2c^2 - b^2/4}$ yields a valid PDF.

Problem 4.11.7 Solution

From Equation (4.146), we can write the bivariate Gaussian PDF as

$$f_{X,Y}(x,y) = \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2} \tag{1}$$

where $\tilde{\mu}_Y(x) = \mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X)$ and $\tilde{\sigma}_Y = \sigma_Y\sqrt{1-\rho^2}$. However, the definitions of $\tilde{\mu}_Y(x)$ and $\tilde{\sigma}_Y$ are not particularly important for this exercise. When we integrate the joint PDF over all x

and y, we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\tilde{\sigma}_Y\sqrt{2\pi}} e^{-(y-\tilde{\mu}_Y(x))^2/2\tilde{\sigma}_Y^2}\,dy}_{1}\,dx \quad (2)$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx \quad (3)$$

The marked integral equals 1 because for each value of x, it is the integral of a Gaussian PDF of one variable over all possible values. In fact, it is the integral of the conditional PDF $f_{Y|X}(y|x)$ over all possible y. To complete the proof, we see that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-(x-\mu_X)^2/2\sigma_X^2}\,dx = 1 \quad (4)$$

since the remaining integral is the integral of the marginal Gaussian PDF $f_X(x)$ over all possible x.

Problem 4.11.8 Solution

In this problem, X_1 and X_2 are jointly Gaussian random variables with $E[X_i] = \mu_i$, $\text{Var}[X_i] = \sigma_i^2$, and correlation coefficient $\rho_{12} = \rho$. The goal is to show that $Y = X_1X_2$ has variance

$$\text{Var}[Y] = (1+\rho^2)\sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + 2\rho\mu_1\mu_2\sigma_1\sigma_2. \quad (1)$$

Since $\text{Var}[Y] = E[Y^2] - (E[Y])^2$, we will find the moments of Y. The first moment is

$$E[Y] = E[X_1X_2] = \text{Cov}[X_1,X_2] + E[X_1]E[X_2] = \rho\sigma_1\sigma_2 + \mu_1\mu_2. \quad (2)$$

For the second moment of Y, we follow the problem hint and use the iterated expectation

$$E[Y^2] = E[X_1^2X_2^2] = E[E[X_1^2X_2^2|X_2]] = E[X_2^2E[X_1^2|X_2]]. \quad (3)$$

Given $X_2 = x_2$, we observe from Theorem 4.30 that X_1 is is Gaussian with

$$E[X_1|X_2=x_2] = \mu_1 + \rho\frac{\sigma_1}{\sigma_2}(x_2-\mu_2), \qquad \text{Var}[X_1|X_2=x_2] = \sigma_1^2(1-\rho^2). \quad (4)$$

Thus, the conditional second moment of X_1 is

$$E[X_1^2|X_2] = (E[X_1|X_2])^2 + \text{Var}[X_1|X_2] \quad (5)$$

$$= \left(\mu_1 + \rho\frac{\sigma_1}{\sigma_2}(X_2-\mu_2)\right)^2 + \sigma_1^2(1-\rho^2) \quad (6)$$

$$= [\mu_1^2 + \sigma_1^2(1-\rho^2)] + 2\rho\mu_1\frac{\sigma_1}{\sigma_2}(X_2-\mu_2) + \rho^2\frac{\sigma_1^2}{\sigma_2^2}(X_2-\mu_2)^2. \quad (7)$$

It follows that

$$E[X_1^2X_2^2] = E[X_2^2E[X_1^2|X_2^2]] \quad (8)$$

$$= E\left[[\mu_1^2 + \sigma_1^2(1-\rho^2)]X_2^2 + 2\rho\mu_1\frac{\sigma_1}{\sigma_2}(X_2-\mu_2)X_2^2 + \rho^2\frac{\sigma_1^2}{\sigma_2^2}(X_2-\mu_2)^2X_2^2\right]. \quad (9)$$

Since $E[X_2^2] = \sigma_2^2 + \mu_2^2$,

$$E\left[X_1^2 X_2^2\right] = \left(\mu_1^2 + \sigma_1^2(1-\rho^2)\right)(\sigma_2^2 + \mu_2^2) + 2\rho\mu_1 \frac{\sigma_1}{\sigma_2} E\left[(X_2-\mu_2)X_2^2\right] + \rho^2 \frac{\sigma_1^2}{\sigma_2^2} E\left[(X_2-\mu_2)^2 X_2^2\right]. \tag{10}$$

We observe that

$$\begin{aligned}
E\left[(X_2-\mu_2)X_2^2\right] &= E\left[(X_2-\mu_2)(X_2-\mu_2+\mu_2)^2\right] &(11)\\
&= E\left[(X_2-\mu_2)\left((X_2-\mu_2)^2 + 2\mu_2(X_2-\mu_2) + \mu_2^2\right)\right] &(12)\\
&= E\left[(X_2-\mu_2)^3\right] + 2\mu_2 E\left[(X_2-\mu_2)^2\right] + \mu_2 E\left[(X_2-\mu_2)\right] &(13)
\end{aligned}$$

We recall that $E[X_2-\mu_2] = 0$ and that $E[(X_2-\mu_2)^2] = \sigma_2^2$. We now look ahead to Problem 6.3.4 to learn that

$$E\left[(X_2-\mu_2)^3\right] = 0, \qquad E\left[(X_2-\mu_2)^4\right] = 3\sigma_2^4. \tag{14}$$

This implies

$$E\left[(X_2-\mu_2)X_2^2\right] = 2\mu_2\sigma_2^2. \tag{15}$$

Following this same approach, we write

$$\begin{aligned}
E\left[(X_2-\mu_2)^2 X_2^2\right] &= E\left[(X_2-\mu_2)^2(X_2-\mu_2+\mu_2)^2\right] &(16)\\
&= E\left[(X_2-\mu_2)^2\left((X_2-\mu_2)^2 + 2\mu_2(X_2-\mu_2) + \mu_2^2\right)\right] &(17)\\
&= E\left[(X_2-\mu_2)^2\left((X_2-\mu_2)^2 + 2\mu_2(X_2-\mu_2) + \mu_2^2\right)\right] &(18)\\
&= E\left[(X_2-\mu_2)^4\right] + 2\mu_2 E\left[X_2-\mu_2)^3\right] + \mu_2^2 E\left[(X_2-\mu_2)^2\right]. &(19)
\end{aligned}$$

It follows that

$$E\left[(X_2-\mu_2)^2 X_2^2\right] = 3\sigma_2^4 + \mu_2^2\sigma_2^2. \tag{20}$$

Combining the above results, we can conclude that

$$\begin{aligned}
E\left[X_1^2 X_2^2\right] &= \left(\mu_1^2 + \sigma_1^2(1-\rho^2)\right)(\sigma_2^2+\mu_2^2) + 2\rho\mu_1\frac{\sigma_1}{\sigma_2}(2\mu_2\sigma_2^2) + \rho^2\frac{\sigma_1^2}{\sigma_2^2}(3\sigma_2^4 + \mu_2^2\sigma_2^2) &(21)\\
&= (1+2\rho^2)\sigma_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + \mu_1^2\sigma_2^2 + \mu_1^2\mu_2^2 + 4\rho\mu_1\mu_2\sigma_1\sigma_2. &(22)
\end{aligned}$$

Finally, combining Equations (2) and (22) yields

$$\begin{aligned}
\text{Var}[Y] &= E\left[X_1^2X_2^2\right] - \left(E\left[X_1X_2\right]\right)^2 &(23)\\
&= (1+\rho^2)\sigma_1^2\sigma_2^2 + \mu_1^2\sigma_2^2 + \mu_2^2\sigma_1^2 + 2\rho\mu_1\mu_2\sigma_1\sigma_2. &(24)
\end{aligned}$$

Problem 4.12.1 Solution

The script `imagepmf` in Example 4.27 generates the grid variables `SX`, `SY`, and `PXY`. Recall that for each entry in the grid, `SX`. `SY` and `PXY` are the corresponding values of x, y and $P_{X,Y}(x,y)$. Displaying them as adjacent column vectors forms the list of all possible pairs x, y and the probabilities $P_{X,Y}(x,y)$. Since any MATLAB vector or matrix `x` is reduced to a column vector with the command `x(:)`, the following simple commands will generate the list:

```
>> format rat;
>> imagepmf;
>> [SX(:) SY(:) PXY(:)]
ans =
    800          400           1/5
   1200          400           1/20
   1600          400            0
    800          800           1/20
   1200          800           1/5
   1600          800           1/10
    800         1200           1/10
   1200         1200           1/10
   1600         1200           1/5
>>
```

Note that the command `format rat` wasn't necessary; it just formats the output as rational numbers, i.e., ratios of integers, which you may or may not find esthetically pleasing.

Problem 4.12.2 Solution

In this problem, we need to calculate $E[X]$, $E[Y]$, the correlation $E[XY]$, and the covariance $\text{Cov}[X,Y]$ for random variables X and Y in Example 4.27. In this case, we can use the script `imagepmf.m` in Example 4.27 to generate the grid variables `SX`, `SY` and `PXY` that describe the joint PMF $P_{X,Y}(x,y)$.

However, for the rest of the problem, a general solution is better than a specific solution. The general problem is that given a pair of finite random variables described by the grid variables `SX`, `SY` and `PXY`, we want MATLAB to calculate an expected value $E[g(X,Y)]$. This problem is solved in a few simple steps. First we write a function that calculates the expected value of any finite random variable.

```
function ex=finiteexp(sx,px);
%Usage: ex=finiteexp(sx,px)
%returns the expected value E[X]
%of finite random variable X described
%by samples sx and probabilities px
ex=sum((sx(:)).*(px(:)));
```

Note that `finiteexp` performs its calculations on the sample values `sx` and probabilities `px` using the column vectors `sx(:)` and `px(:)`. As a result, we can use the same `finiteexp` function when the random variable is represented by grid variables. For example, we can calculate the correlation $r = E[XY]$ as

```
r=finiteexp(SX.*SY,PXY)
```

It is also convenient to define a function that returns the covariance:

```
function covxy=finitecov(SX,SY,PXY);
%Usage: cxy=finitecov(SX,SY,PXY)
%returns the covariance of
%finite random variables X and Y
%given by grids SX, SY, and PXY
ex=finiteexp(SX,PXY);
ey=finiteexp(SY,PXY);
R=finiteexp(SX.*SY,PXY);
covxy=R-ex*ey;
```

The following script calculates the desired quantities:

```
%imageavg.m
%Solution for Problem 4.12.2
imagepmf; %defines SX, SY, PXY
ex=finiteexp(SX,PXY)
ey=finiteexp(SY,PXY)
rxy=finiteexp(SX.*SY,PXY)
cxy=finitecov(SX,SY,PXY)
```

```
>> imageavg
ex =
        1180
ey =
   860
rxy =
     1064000
cxy =
       49200
>>
```

The careful reader will observe that `imagepmf` is inefficiently coded in that the correlation $E[XY]$ is calculated twice, once directly and once inside of `finitecov`. For more complex problems, it would be worthwhile to avoid this duplication.

Problem 4.12.3 Solution

The script is just a MATLAB calculation of $F_{X,Y}(x, y)$ in Equation (4.29).

```
%trianglecdfplot.m
[X,Y]=meshgrid(0:0.05:1.5);
R=(0<=Y).*(Y<=X).*(X<=1).*(2*(X.*Y)-(Y.^2));
R=R+((0<=X).*(X<Y).*(X<=1).*(X.^2));
R=R+((0<=Y).*(Y<=1).*(1<X).*((2*Y)-(Y.^2)));
R=R+((X>1).*(Y>1));
mesh(X,Y,R);
xlabel('\it x');
ylabel('\it y');
```

For functions like $F_{X,Y}(x, y)$ that have multiple cases, we calculate the function for each case and multiply by the corresponding boolean condition so as to have a zero contribution when that case doesn't apply. Using this technique, its important to define the boundary conditions carefully to make sure that no point is included in two different boundary conditions.

Problem 4.12.4 Solution

By following the formulation of Problem 4.2.6, the code to set up the sample grid is reasonably straightforward:

```
function [SX,SY,PXY]=circuits(n,p);
%Usage: [SX,SY,PXY]=circuits(n,p);
%   (See Problem 4.12.4)
[SX,SY]=ndgrid(0:n,0:n);
PXY=0*SX;
PXY(find((SX==n) & (SY==n)))=p^n;
for y=0:(n-1),
    I=find((SY==y) &(SX>=SY) &(SX<n));
    PXY(I)=(p^y)*(1-p)* ...
        binomialpmf(n-y-1,p,SX(I)-y);
end;
```

The only catch is that for a given value of y, we need to calculate the binomial probability of $x - y$ successes in $(n - y - 1)$ trials. We can do this using the function call

```
binomialpmf(n-y-1,p,x-y)
```

However, this function expects the argument `n-y-1` to be a scalar. As a result, we must perform a separate call to `binomialpmf` for each value of `y`.

An alternate solution is direct calculation of the PMF $P_{X,Y}(x, y)$ in Problem 4.2.6. In this case, we calculate $m!$ using the MATLAB function `gamma(m+1)`. Because, `gamma(x)` function will calculate the gamma function for each element in a vector `x`, we can calculate the PMF without any loops:

```
function [SX,SY,PXY]=circuits2(n,p);
%Usage: [SX,SY,PXY]=circuits2(n,p);
%   (See Problem 4.12.4)
[SX,SY]=ndgrid(0:n,0:n);
PXY=0*SX;
PXY(find((SX==n) & (SY==n)))=p^n;
I=find((SY<=SX) &(SX<n));
PXY(I)=(gamma(n-SY(I))./(gamma(SX(I)-SY(I)+1)...
      .*gamma(n-SX(I)))).*(p.^SX(I)).*((1-p).^(n-SX(I)));
```

Some experimentation with `cputime` will show that `circuits2(n,p)` runs much faster than `circuits(n,p)`. As is typical, the `for` loop in `circuit` results in time wasted running the MATLAB interpretor and in regenerating the binomial PMF in each cycle.

To finish the problem, we need to calculate the correlation coefficient

$$\rho_{X,Y} = \frac{\text{Cov}[X, Y]}{\sigma_X \sigma_Y}. \tag{1}$$

In fact, this is one of those problems where a general solution is better than a specific solution. The general problem is that given a pair of finite random variables described by the grid variables `SX`, `SY` and PMF `PXY`, we wish to calculate the correlation coefficient

This problem is solved in a few simple steps. First we write a function that calculates the expected value of a finite random variable.

```
function ex=finiteexp(sx,px);
%Usage: ex=finiteexp(sx,px)
%returns the expected value E[X]
%of finite random variable X described
%by samples sx and probabilities px
ex=sum((sx(:)).*(px(:)));
```

Note that `finiteexp` performs its calculations on the sample values `sx` and probabilities `px` using the column vectors `sx(:)` and `px(:)`. As a result, we can use the same `finiteexp` function when the random variable is represented by grid variables. We can build on `finiteexp` to calculate the variance using `finitevar`:

```
function v=finitevar(sx,px);
%Usage: ex=finitevar(sx,px)
%   returns the variance Var[X]
%   of finite random variables X described by
%   samples sx and probabilities px
ex2=finiteexp(sx.^2,px);
ex=finiteexp(sx,px);
v=ex2-(ex^2);
```

Putting these pieces together, we can calculate the correlation coefficient.

```
function rho=finitecoeff(SX,SY,PXY);
%Usage:  rho=finitecoeff(SX,SY,PXY)
%Calculate the correlation coefficient rho of
%finite random variables X and Y
ex=finiteexp(SX,PXY); vx=finitevar(SX,PXY);
ey=finiteexp(SY,PXY); vy=finitevar(SY,PXY);
R=finiteexp(SX.*SY,PXY);
rho=(R-ex*ey)/sqrt(vx*vy);
```

Calculating the correlation coefficient of X and Y, is now a two line exercise..

```
>> [SX,SY,PXY]=circuits2(50,0.9);
>> rho=finitecoeff(SX,SY,PXY)
rho =
    0.4451
>>
```

Problem 4.12.5 Solution

In the first approach X is an exponential (λ) random variable, Y is an independent exponential (μ) random variable, and $W = Y/X$. we implement this approach in the function `wrv1.m` shown below.

In the second approach, we use Theorem 3.22 and generate samples of a uniform $(0,1)$ random variable U and calculate $W = F_W^{-1}(U)$. In this problem,

$$F_W(w) = 1 - \frac{\lambda/\mu}{\lambda/\mu + w}. \tag{1}$$

Setting $u = F_W(w)$ and solving for w yields

$$w = F_W^{-1}(u) = \frac{\lambda}{\mu}\left(\frac{u}{1-u}\right) \tag{2}$$

We implement this solution in the function `wrv2`. Here are the two solutions:

```
function w=wrv1(lambda,mu,m)
%Usage: w=wrv1(lambda,mu,m)
%Return m samples of W=Y/X
%X is exponential (lambda)
%Y is exponential (mu)

x=exponentialrv(lambda,m);
y=exponentialrv(mu,m);
w=y./x;
```

```
function w=wrv2(lambda,mu,m)
%Usage: w=wrv1(lambda,mu,m)
%Return m samples of W=Y/X
%X is exponential (lambda)
%Y is exponential (mu)
%Uses CDF of F_W(w)

u=rand(m,1);
w=(lambda/mu)*u./(1-u);
```

We would expect that `wrv2` would be faster simply because it does less work. In fact, its instructive to account for the work each program does.

- `wrv1` Each exponential random sample requires the generation of a uniform random variable, and the calculation of a logarithm. Thus, we generate $2m$ uniform random variables, calculate $2m$ logarithms, and perform m floating point divisions.

- `wrv2` Generate m uniform random variables and perform m floating points divisions.

This quickie analysis indicates that `wrv1` executes roughly $5m$ operations while `wrv2` executes about $2m$ operations. We might guess that `wrv2` would be faster by a factor of 2.5. Experimentally, we calculated the execution time associated with generating a million samples:

```
>> t2=cputime;w2=wrv2(1,1,1000000);t2=cputime-t2
t2 =
    0.2500
>> t1=cputime;w1=wrv1(1,1,1000000);t1=cputime-t1
t1 =
    0.7610
>>
```

We see in our simple experiments that `wrv2` is faster by a rough factor of 3. (Note that repeating such trials yielded qualitatively similar results.)