

Problem Solutions – Chapter 2

Problem 2.2.1 Solution

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Therefore, $\sum_{n=0}^2 P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \leq 1$ is

$$P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7 \quad (2)$$

Problem 2.2.2 Solution

From Example 2.5, we can write the PMF of X and the PMF of R as

$$P_X(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \\ 0 & \text{otherwise} \end{cases} \quad P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

From the PMFs $P_X(x)$ and $P_R(r)$, we can calculate the requested probabilities

(a) $P[X = 0] = P_X(0) = 1/8$.

(b) $P[X < 3] = P_X(0) + P_X(1) + P_X(2) = 7/8$.

(c) $P[R > 1] = P_R(2) = 3/4$.

Problem 2.2.3 Solution

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^4 P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1 \quad (1)$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \dots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30} \quad (2)$$

(c) The probability that V is even is

$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3} \quad (3)$$

(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6} \quad (4)$$

Problem 2.2.4 Solution

(a) We choose c so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1 \quad (1)$$

Thus $c = 8/7$.

(b)

$$P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7} \quad (2)$$

(c)

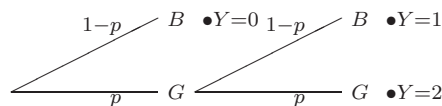
$$P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7} \quad (3)$$

(d)

$$P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7} \quad (4)$$

Problem 2.2.5 Solution

Using B (for Bad) to denote a miss and G (for Good) to denote a successful free throw, the sample tree for the number of points scored in the 1 and 1 is



From the tree, the PMF of Y is

$$P_Y(y) = \begin{cases} 1-p & y=0 \\ p(1-p) & y=1 \\ p^2 & y=2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 2.2.6 Solution

The probability that a caller fails to get through in three tries is $(1-p)^3$. To be sure that at least 95% of all callers get through, we need $(1-p)^3 \leq 0.05$. This implies $p = 0.6316$.

Problem 2.2.7 Solution

In Problem 2.2.6, each caller is willing to make 3 attempts to get through. An attempt is a failure if all n operators are busy, which occurs with probability $q = (0.8)^n$. Assuming call attempts are independent, a caller will suffer three failed attempts with probability $q^3 = (0.8)^{3n}$. The problem statement requires that $(0.8)^{3n} \leq 0.05$. This implies $n \geq 4.48$ and so we need 5 operators.

Problem 2.2.8 Solution

From the problem statement, a single is twice as likely as a double, which is twice as likely as a triple, which is twice as likely as a home-run. If p is the probability of a home run, then

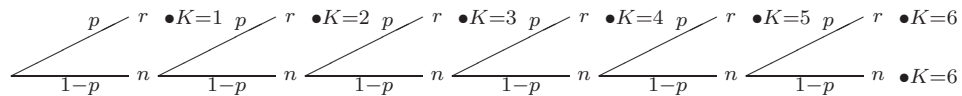
$$P_B(4) = p \quad P_B(3) = 2p \quad P_B(2) = 4p \quad P_B(1) = 8p \quad (1)$$

Since a hit of any kind occurs with probability of .300, $p + 2p + 4p + 8p = 0.300$ which implies $p = 0.02$. Hence, the PMF of B is

$$P_B(b) = \begin{cases} 0.70 & b = 0 \\ 0.16 & b = 1 \\ 0.08 & b = 2 \\ 0.04 & b = 3 \\ 0.02 & b = 4 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Problem 2.2.9 Solution

- (a) In the setup of a mobile call, the phone will send the “SETUP” message up to six times. Each time the setup message is sent, we have a Bernoulli trial with success probability p . Of course, the phone stops trying as soon as there is a success. Using r to denote a successful response, and n a non-response, the sample tree is



- (b) We can write the PMF of K , the number of “SETUP” messages sent as

$$P_K(k) = \begin{cases} (1-p)^{k-1}p & k = 1, 2, \dots, 5 \\ (1-p)^5p + (1-p)^6 = (1-p)^5 & k = 6 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Note that the expression for $P_K(6)$ is different because $K = 6$ if either there was a success or a failure on the sixth attempt. In fact, $K = 6$ whenever there were failures on the first five attempts which is why $P_K(6)$ simplifies to $(1-p)^5$.

- (c) Let B denote the event that a busy signal is given after six failed setup attempts. The probability of six consecutive failures is $P[B] = (1-p)^6$.
- (d) To be sure that $P[B] \leq 0.02$, we need $p \geq 1 - (0.02)^{1/6} = 0.479$.

Problem 2.3.1 Solution

- (a) If it is indeed true that Y , the number of yellow M&M's in a package, is uniformly distributed between 5 and 15, then the PMF of Y , is

$$P_Y(y) = \begin{cases} 1/11 & y = 5, 6, 7, \dots, 15 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b)

$$P[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = 5/11 \quad (2)$$

(c)

$$P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = 3/11 \quad (3)$$

(d)

$$P[8 \leq Y \leq 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = 5/11 \quad (4)$$

Problem 2.3.2 Solution

- (a) Each paging attempt is an independent Bernoulli trial with success probability p . The number of times K that the pager receives a message is the number of successes in n Bernoulli trials and has the binomial PMF

$$P_K(k) = \begin{cases} \binom{n}{k} p^k (1-p)^{n-k} & k = 0, 1, \dots, n \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) Let R denote the event that the paging message was received at least once. The event R has probability

$$P[R] = P[B > 0] = 1 - P[B = 0] = 1 - (1-p)^n \quad (2)$$

To ensure that $P[R] \geq 0.95$ requires that $n \geq \ln(0.05)/\ln(1-p)$. For $p = 0.8$, we must have $n \geq 1.86$. Thus, $n = 2$ pages would be necessary.

Problem 2.3.3 Solution

Whether a hook catches a fish is an independent trial with success probability h . The the number of fish hooked, K , has the binomial PMF

$$P_K(k) = \begin{cases} \binom{m}{k} h^k (1-h)^{m-k} & k = 0, 1, \dots, m \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 2.3.4 Solution

- (a) Let X be the number of times the frisbee is thrown until the dog catches it and runs away. Each throw of the frisbee can be viewed as a Bernoulli trial in which a success occurs if the dog catches the frisbee and runs away. Thus, the experiment ends on the first success and X has the geometric PMF

$$P_X(x) = \begin{cases} (1-p)^{x-1} p & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The child will throw the frisbee more than four times iff there are failures on the first 4 trials which has probability $(1-p)^4$. If $p = 0.2$, the probability of more than four throws is $(0.8)^4 = 0.4096$.

Problem 2.3.5 Solution

Each paging attempt is a Bernoulli trial with success probability p where a success occurs if the pager receives the paging message.

- (a) The paging message is sent again and again until a success occurs. Hence the number of paging messages is $N = n$ if there are $n - 1$ paging failures followed by a paging success. That is, N has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The probability that no more three paging attempts are required is

$$P[N \leq 3] = 1 - P[N > 3] = 1 - \sum_{n=4}^{\infty} P_N(n) = 1 - (1-p)^3 \quad (2)$$

This answer can be obtained without calculation since $N > 3$ if the first three paging attempts fail and that event occurs with probability $(1-p)^3$. Hence, we must choose p to satisfy $1 - (1-p)^3 \geq 0.95$ or $(1-p)^3 \leq 0.05$. This implies

$$p \geq 1 - (0.05)^{1/3} \approx 0.6316 \quad (3)$$

Problem 2.3.6 Solution

The probability of more than 500,000 bits is

$$P[B > 500,000] = 1 - \sum_{b=1}^{500,000} P_B(b) \quad (1)$$

$$= 1 - p \sum_{b=1}^{500,000} (1-p)^{b-1} \quad (2)$$

Math Fact B.4 implies that $(1-x) \sum_{b=1}^{500,000} x^{b-1} = 1 - x^{500,000}$. Substituting, $x = 1-p$, we obtain:

$$P[B > 500,000] = 1 - (1 - (1-p)^{500,000}) \quad (3)$$

$$= (1 - 0.25 \times 10^{-5})^{500,000} \approx \exp(-500,000/400,000) = 0.29. \quad (4)$$

Problem 2.3.7 Solution

Since an average of $T/5$ buses arrive in an interval of T minutes, buses arrive at the bus stop at a rate of $1/5$ buses per minute.

- (a) From the definition of the Poisson PMF, the PMF of B , the number of buses in T minutes, is

$$P_B(b) = \begin{cases} (T/5)^b e^{-T/5} / b! & b = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) Choosing $T = 2$ minutes, the probability that three buses arrive in a two minute interval is

$$P_B(3) = (2/5)^3 e^{-2/5} / 3! \approx 0.0072 \quad (2)$$

(c) By choosing $T = 10$ minutes, the probability of zero buses arriving in a ten minute interval is

$$P_B(0) = e^{-10/5}/0! = e^{-2} \approx 0.135 \quad (3)$$

(d) The probability that at least one bus arrives in T minutes is

$$P[B \geq 1] = 1 - P[B = 0] = 1 - e^{-T/5} \geq 0.99 \quad (4)$$

Rearranging yields $T \geq 5 \ln 100 \approx 23.0$ minutes.

Problem 2.3.8 Solution

(a) If each message is transmitted 8 times and the probability of a successful transmission is p , then the PMF of N , the number of successful transmissions has the binomial PMF

$$P_N(n) = \begin{cases} \binom{8}{n} p^n (1-p)^{8-n} & n = 0, 1, \dots, 8 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(b) The indicator random variable I equals zero if and only if $N = 8$. Hence,

$$P[I = 0] = P[N = 8] = 1 - P[I = 1] \quad (2)$$

Thus, the complete expression for the PMF of I is

$$P_I(i) = \begin{cases} (1-p)^8 & i = 0 \\ 1 - (1-p)^8 & i = 1 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Problem 2.3.9 Solution

The requirement that $\sum_{x=1}^n P_X(x) = 1$ implies

$$n = 1 : \quad c(1) \left[\frac{1}{1} \right] = 1 \quad c(1) = 1 \quad (1)$$

$$n = 2 : \quad c(2) \left[\frac{1}{1} + \frac{1}{2} \right] = 1 \quad c(2) = \frac{2}{3} \quad (2)$$

$$n = 3 : \quad c(3) \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} \right] = 1 \quad c(3) = \frac{6}{11} \quad (3)$$

$$n = 4 : \quad c(4) \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right] = 1 \quad c(4) = \frac{12}{25} \quad (4)$$

$$n = 5 : \quad c(5) \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \right] = 1 \quad c(5) = \frac{12}{25} \quad (5)$$

$$n = 6 : \quad c(6) \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} \right] = 1 \quad c(6) = \frac{20}{49} \quad (6)$$

As an aside, find $c(n)$ for large values of n is easy using the recursion

$$\frac{1}{c(n+1)} = \frac{1}{c(n)} + \frac{1}{n+1}. \quad (7)$$

Problem 2.3.10 Solution

- (a) We can view whether each caller knows the birthdate as a Bernoulli trial. As a result, L is the number of trials needed for 6 successes. That is, L has a Pascal PMF with parameters $p = 0.75$ and $k = 6$ as defined by Definition 2.8. In particular,

$$P_L(l) = \begin{cases} \binom{l-1}{5} (0.75)^6 (0.25)^{l-6} & l = 6, 7, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The probability of finding the winner on the tenth call is

$$P_L(10) = \binom{9}{5} (0.75)^6 (0.25)^4 \approx 0.0876 \quad (2)$$

- (c) The probability that the station will need nine or more calls to find a winner is

$$P[L \geq 9] = 1 - P[L < 9] \quad (3)$$

$$= 1 - P_L(6) - P_L(7) - P_L(8) \quad (4)$$

$$= 1 - (0.75)^6 [1 + 6(0.25) + 21(0.25)^2] \approx 0.321 \quad (5)$$

Problem 2.3.11 Solution

The packets are delay sensitive and can only be retransmitted d times. For $t < d$, a packet is transmitted t times if the first $t - 1$ attempts fail followed by a successful transmission on attempt t . Further, the packet is transmitted d times if there are failures on the first $d - 1$ transmissions, no matter what the outcome of attempt d . So the random variable T , the number of times that a packet is transmitted, can be represented by the following PMF.

$$P_T(t) = \begin{cases} p(1-p)^{t-1} & t = 1, 2, \dots, d-1 \\ (1-p)^{d-1} & t = d \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 2.3.12 Solution

- (a) Since each day is independent of any other day, $P[W_{33}]$ is just the probability that a winning lottery ticket was bought. Similarly for $P[L_{87}]$ and $P[N_{99}]$ become just the probability that a losing ticket was bought and that no ticket was bought on a single day, respectively. Therefore

$$P[W_{33}] = p/2 \quad P[L_{87}] = (1-p)/2 \quad P[N_{99}] = 1/2 \quad (1)$$

- (b) Suppose we say a success occurs on the k th trial if on day k we buy a ticket. Otherwise, a failure occurs. The probability of success is simply $1/2$. The random variable K is just the number of trials until the first success and has the geometric PMF

$$P_K(k) = \begin{cases} (1/2)(1/2)^{k-1} = (1/2)^k & k = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

- (c) The probability that you decide to buy a ticket and it is a losing ticket is $(1-p)/2$, independent of any other day. If we view buying a losing ticket as a Bernoulli success, R , the number of losing lottery tickets bought in m days, has the binomial PMF

$$P_R(r) = \begin{cases} \binom{m}{r} [(1-p)/2]^r [(1+p)/2]^{m-r} & r = 0, 1, \dots, m \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

- (d) Letting D be the day on which the j -th losing ticket is bought, we can find the probability that $D = d$ by noting that $j-1$ losing tickets must have been purchased in the $d-1$ previous days. Therefore D has the Pascal PMF

$$P_D(d) = \begin{cases} \binom{d-1}{j-1} [(1-p)/2]^j [(1+p)/2]^{d-j} & d = j, j+1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Problem 2.3.13 Solution

- (a) Let S_n denote the event that the Sixers win the series in n games. Similarly, C_n is the event that the Celtics win in n games. The Sixers win the series in 3 games if they win three straight, which occurs with probability

$$P[S_3] = (1/2)^3 = 1/8 \quad (1)$$

The Sixers win the series in 4 games if they win two out of the first three games and they win the fourth game so that

$$P[S_4] = \binom{3}{2} (1/2)^3 (1/2) = 3/16 \quad (2)$$

The Sixers win the series in five games if they win two out of the first four games and then win game five. Hence,

$$P[S_5] = \binom{4}{2} (1/2)^4 (1/2) = 3/16 \quad (3)$$

By symmetry, $P[C_n] = P[S_n]$. Further we observe that the series last n games if either the Sixers or the Celtics win the series in n games. Thus,

$$P[N = n] = P[S_n] + P[C_n] = 2P[S_n] \quad (4)$$

Consequently, the total number of games, N , played in a best of 5 series between the Celtics and the Sixers can be described by the PMF

$$P_N(n) = \begin{cases} 2(1/2)^3 = 1/4 & n = 3 \\ 2\binom{3}{1} (1/2)^4 = 3/8 & n = 4 \\ 2\binom{4}{2} (1/2)^5 = 3/8 & n = 5 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

- (b) For the total number of Celtic wins W , we note that if the Celtics get $w < 3$ wins, then the Sixers won the series in $3+w$ games. Also, the Celtics win 3 games if they win the series in 3, 4, or 5 games. Mathematically,

$$P[W = w] = \begin{cases} P[S_{3+w}] & w = 0, 1, 2 \\ P[C_3] + P[C_4] + P[C_5] & w = 3 \end{cases} \quad (6)$$

Thus, the number of wins by the Celtics, W , has the PMF shown below.

$$P_W(w) = \begin{cases} P[S_3] = 1/8 & w = 0 \\ P[S_4] = 3/16 & w = 1 \\ P[S_5] = 3/16 & w = 2 \\ 1/8 + 3/16 + 3/16 = 1/2 & w = 3 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

- (c) The number of Celtic losses L equals the number of Sixers' wins W_S . This implies $P_L(l) = P_{W_S}(l)$. Since either team is equally likely to win any game, by symmetry, $P_{W_S}(w) = P_W(w)$. This implies $P_L(l) = P_{W_S}(l) = P_W(l)$. The complete expression of for the PMF of L is

$$P_L(l) = P_W(l) = \begin{cases} 1/8 & l = 0 \\ 3/16 & l = 1 \\ 3/16 & l = 2 \\ 1/2 & l = 3 \\ 0 & \text{otherwise} \end{cases} \quad (8)$$

Problem 2.3.14 Solution

Since a and b are positive, let K be a binomial random variable for n trials and success probability $p = a/(a + b)$. First, we observe that the sum of over all possible values of the PMF of K is

$$\sum_{k=0}^n P_K(k) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} \quad (1)$$

$$= \sum_{k=0}^n \binom{n}{k} \left(\frac{a}{a+b}\right)^k \left(\frac{b}{a+b}\right)^{n-k} \quad (2)$$

$$= \frac{\sum_{k=0}^n \binom{n}{k} a^k b^{n-k}}{(a+b)^n} \quad (3)$$

Since $\sum_{k=0}^n P_K(k) = 1$, we see that

$$(a+b)^n = (a+b)^n \sum_{k=0}^n P_K(k) = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k} \quad (4)$$

Problem 2.4.1 Solution

Using the CDF given in the problem statement we find that

(a) $P[Y < 1] = 0$

(b) $P[Y \leq 1] = 1/4$

(c) $P[Y > 2] = 1 - P[Y \leq 2] = 1 - 1/2 = 1/2$

(d) $P[Y \geq 2] = 1 - P[Y < 2] = 1 - 1/4 = 3/4$

(e) $P[Y = 1] = 1/4$

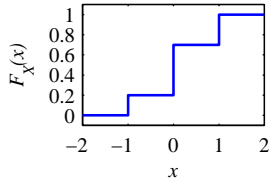
(f) $P[Y = 3] = 1/2$

(g) From the staircase CDF of Problem 2.4.1, we see that Y is a discrete random variable. The jumps in the CDF occur at the values that Y can take on. The height of each jump equals the probability of that value. The PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 2.4.2 Solution

(a) The given CDF is shown in the diagram below.



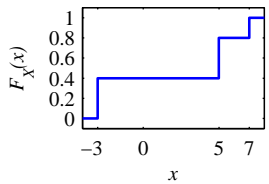
$$F_X(x) = \begin{cases} 0 & x < -1 \\ 0.2 & -1 \leq x < 0 \\ 0.7 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \quad (1)$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Problem 2.4.3 Solution

(a) Similar to the previous problem, the graph of the CDF is shown below.



$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1 & x \geq 7 \end{cases} \quad (1)$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Problem 2.4.4 Solution

Let $q = 1 - p$, so the PMF of the geometric (p) random variable K is

$$P_K(k) = \begin{cases} pq^{k-1} & k = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

For any integer $k \geq 1$, the CDF obeys

$$F_K(k) = \sum_{j=1}^k P_K(j) = \sum_{j=1}^k pq^{j-1} = 1 - q^k. \quad (2)$$

Since K is integer valued, $F_K(k) = F_K(\lfloor k \rfloor)$ for all integer and non-integer values of k . (If this point is not clear, you should review Example 2.24.) Thus, the complete expression for the CDF of K is

$$F_K(k) = \begin{cases} 0 & k < 1, \\ 1 - (1 - p)^{\lfloor k \rfloor} & k \geq 1. \end{cases} \quad (3)$$

Problem 2.4.5 Solution

Since mushrooms occur with probability $2/3$, the number of pizzas sold before the first mushroom pizza is $N = n < 100$ if the first n pizzas do not have mushrooms followed by mushrooms on pizza $n + 1$. Also, it is possible that $N = 100$ if all 100 pizzas are sold without mushrooms. the resulting PMF is

$$P_N(n) = \begin{cases} (1/3)^n(2/3) & n = 0, 1, \dots, 99 \\ (1/3)^{100} & n = 100 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For integers $n < 100$, the CDF of N obeys

$$F_N(n) = \sum_{i=0}^n P_N(i) = \sum_{i=0}^n (1/3)^i(2/3) = 1 - (1/3)^{n+1} \quad (2)$$

A complete expression for $F_N(n)$ must give a valid answer for every value of n , including non-integer values. We can write the CDF using the floor function $\lfloor x \rfloor$ which denote the largest integer less than or equal to X . The complete expression for the CDF is

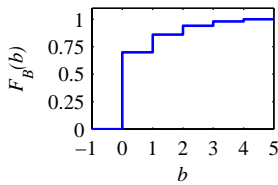
$$F_N(x) = \begin{cases} 0 & x < 0 \\ 1 - (1/3)^{\lfloor x \rfloor + 1} & 0 \leq x < 100 \\ 1 & x \geq 100 \end{cases} \quad (3)$$

Problem 2.4.6 Solution

From Problem 2.2.8, the PMF of B is

$$P_B(b) = \begin{cases} 0.70 & b = 0 \\ 0.16 & b = 1 \\ 0.08 & b = 2 \\ 0.04 & b = 3 \\ 0.02 & b = 4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The corresponding CDF is



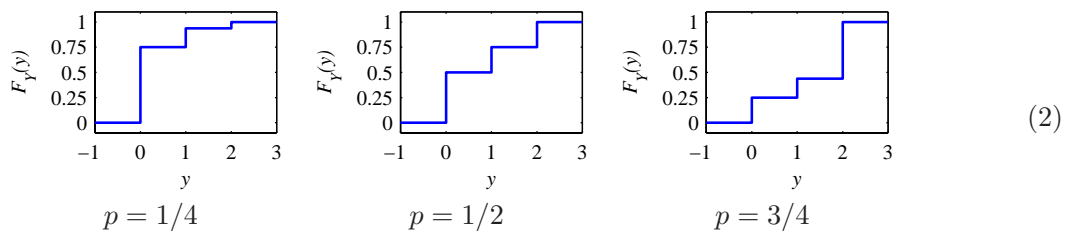
$$F_B(b) = \begin{cases} 0 & b < 0 \\ 0.70 & 0 \leq b < 1 \\ 0.86 & 1 \leq b < 2 \\ 0.94 & 2 \leq b < 3 \\ 0.98 & 3 \leq b < 4 \\ 1.0 & b \geq 4 \end{cases} \quad (2)$$

Problem 2.4.7 Solution

In Problem 2.2.5, we found the PMF of Y . This PMF, and its corresponding CDF are

$$P_Y(y) = \begin{cases} 1-p & y=0 \\ p(1-p) & y=1 \\ p^2 & y=2 \\ 0 & \text{otherwise} \end{cases} \quad F_Y(y) = \begin{cases} 0 & y < 0 \\ 1-p & 0 \leq y < 1 \\ 1-p^2 & 1 \leq y < 2 \\ 1 & y \geq 2 \end{cases} \quad (1)$$

For the three values of p , the CDF resembles



Problem 2.4.8 Solution

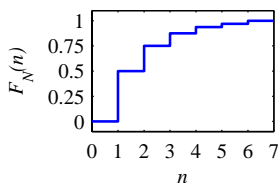
From Problem 2.2.9, the PMF of the number of call attempts is

$$P_N(n) = \begin{cases} (1-p)^{k-1}p & k=1, 2, \dots, 5 \\ (1-p)^5p + (1-p)^6 = (1-p)^5 & k=6 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For $p = 1/2$, the PMF can be simplified to

$$P_N(n) = \begin{cases} (1/2)^n & n=1, 2, \dots, 5 \\ (1/2)^5 & n=6 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The corresponding CDF of N is



$$F_N(n) = \begin{cases} 0 & n < 1 \\ 1/2 & 1 \leq n < 2 \\ 3/4 & 2 \leq n < 3 \\ 7/8 & 3 \leq n < 4 \\ 15/16 & 4 \leq n < 5 \\ 31/32 & 5 \leq n < 6 \\ 1 & n \geq 6 \end{cases} \quad (3)$$

Problem 2.5.1 Solution

For this problem, we just need to pay careful attention to the definitions of mode and median.

- (a) The mode must satisfy $P_X(x_{\text{mod}}) \geq P_X(x)$ for all x . In the case of the uniform PMF, any integer x' between 1 and 100 is a mode of the random variable X . Hence, the set of all modes is

$$X_{\text{mod}} = \{1, 2, \dots, 100\} \quad (1)$$

- (b) The median must satisfy $P[X < x_{\text{med}}] = P[X > x_{\text{med}}]$. Since

$$P[X \leq 50] = P[X \geq 51] = 1/2 \quad (2)$$

we observe that $x_{\text{med}} = 50.5$ is a median since it satisfies

$$P[X < x_{\text{med}}] = P[X > x_{\text{med}}] = 1/2 \quad (3)$$

In fact, for any x' satisfying $50 < x' < 51$, $P[X < x'] = P[X > x'] = 1/2$. Thus,

$$X_{\text{med}} = \{x | 50 < x < 51\} \quad (4)$$

Problem 2.5.2 Solution

Voice calls and data calls each cost 20 cents and 30 cents respectively. Furthermore the respective probabilities of each type of call are 0.6 and 0.4.

- (a) Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore the PMF of X is

$$P_X(x) = \begin{cases} 0.6 & x = 20 \\ 0.4 & x = 30 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The expected cost, $E[C]$, is simply the sum of the cost of each type of call multiplied by the probability of such a call occurring.

$$E[C] = 20(0.6) + 30(0.4) = 24 \text{ cents} \quad (2)$$

Problem 2.5.3 Solution

From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of Y is

$$E[Y] = \sum_y y P_Y(y) = 1(1/4) + 2(1/4) + 3(1/2) = 9/4 \quad (2)$$

Problem 2.5.4 Solution

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_x xP_X(x) = -1(0.2) + 0(0.5) + 1(0.3) = 0.1 \quad (2)$$

Problem 2.5.5 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_x xP_X(x) = -3(0.4) + 5(0.4) + 7(0.2) = 2.2 \quad (2)$$

Problem 2.5.6 Solution

From Definition 2.7, random variable X has PMF

$$P_X(x) = \begin{cases} \binom{4}{x}(1/2)^4 & x = 0, 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_{x=0}^4 xP_X(x) = 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \quad (2)$$

$$= [4 + 12 + 12 + 4]/2^4 = 2 \quad (3)$$

Problem 2.5.7 Solution

From Definition 2.7, random variable X has PMF

$$P_X(x) = \begin{cases} \binom{5}{x}(1/2)^5 & x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_{x=0}^5 xP_X(x) \quad (2)$$

$$= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} \quad (3)$$

$$= [5 + 20 + 30 + 20 + 5]/2^5 = 2.5 \quad (4)$$

Problem 2.5.8 Solution

The following experiments are based on a common model of packet transmissions in data networks. In these networks, each data packet contains a cyclic redundancy check (CRC) code that permits the receiver to determine whether the packet was decoded correctly. In the following, we assume that a packet is corrupted with probability $\epsilon = 0.001$, independent of whether any other packet is corrupted.

- (a) Let $X = 1$ if a data packet is decoded correctly; otherwise $X = 0$. Random variable X is a Bernoulli random variable with PMF

$$P_X(x) = \begin{cases} 0.001 & x = 0 \\ 0.999 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The parameter $\epsilon = 0.001$ is the probability a packet is corrupted. The expected value of X is

$$E[X] = 1 - \epsilon = 0.999 \quad (2)$$

- (b) Let Y denote the number of packets received in error out of 100 packets transmitted. Y has the binomial PMF

$$P_Y(y) = \begin{cases} \binom{100}{y}(0.001)^y(0.999)^{100-y} & y = 0, 1, \dots, 100 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The expected value of Y is

$$E[Y] = 100\epsilon = 0.1 \quad (4)$$

- (c) Let L equal the number of packets that must be received to decode 5 packets in error. L has the Pascal PMF

$$P_L(l) = \begin{cases} \binom{l-1}{4}(0.001)^5(0.999)^{l-5} & l = 5, 6, \dots \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

The expected value of L is

$$E[L] = \frac{5}{\epsilon} = \frac{5}{0.001} = 5000 \quad (6)$$

- (d) If packet arrivals obey a Poisson model with an average arrival rate of 1000 packets per second, then the number N of packets that arrive in 5 seconds has the Poisson PMF

$$P_N(n) = \begin{cases} 5000^n e^{-5000} / n! & n = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

The expected value of N is $E[N] = 5000$.

Problem 2.5.9 Solution

In this "double-or-nothing" type game, there are only two possible payoffs. The first is zero dollars, which happens when we lose 6 straight bets, and the second payoff is 64 dollars which happens unless we lose 6 straight bets. So the PMF of Y is

$$P_Y(y) = \begin{cases} (1/2)^6 = 1/64 & y = 0 \\ 1 - (1/2)^6 = 63/64 & y = 64 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected amount you take home is

$$E[Y] = 0(1/64) + 64(63/64) = 63 \quad (2)$$

So, on the average, we can expect to break even, which is not a very exciting proposition.

Problem 2.5.10 Solution

By the definition of the expected value,

$$E[X_n] = \sum_{x=1}^n x \binom{n}{x} p^x (1-p)^{n-x} \quad (1)$$

$$= np \sum_{x=1}^n \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} p^{x-1} (1-p)^{n-1-(x-1)} \quad (2)$$

With the substitution $x' = x - 1$, we have

$$E[X_n] = np \underbrace{\sum_{x'=0}^{n-1} \binom{n-1}{x'} p^{x'} (1-p)^{n-x'}}_1 = np \sum_{x'=0}^{n-1} P_{X_{n-1}}(x) = np \quad (3)$$

The above sum is 1 because it is the sum of a binomial random variable for $n - 1$ trials over all possible values.

Problem 2.5.11 Solution

We write the sum as a double sum in the following way:

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{i=0}^{\infty} \sum_{j=i+1}^{\infty} P_X(j) \quad (1)$$

At this point, the key step is to reverse the order of summation. You may need to make a sketch of the feasible values for i and j to see how this reversal occurs. In this case,

$$\sum_{i=0}^{\infty} P[X > i] = \sum_{j=1}^{\infty} \sum_{i=0}^{j-1} P_X(j) = \sum_{j=1}^{\infty} j P_X(j) = E[X] \quad (2)$$

Problem 2.6.1 Solution

From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (a) Since Y has range $S_Y = \{1, 2, 3\}$, the range of $U = Y^2$ is $S_U = \{1, 4, 9\}$. The PMF of U can be found by observing that

$$P[U = u] = P[Y^2 = u] = P[Y = \sqrt{u}] + P[Y = -\sqrt{u}] \quad (2)$$

Since Y is never negative, $P_U(u) = P_Y(\sqrt{u})$. Hence,

$$P_U(1) = P_Y(1) = 1/4 \quad P_U(4) = P_Y(2) = 1/4 \quad P_U(9) = P_Y(3) = 1/2 \quad (3)$$

For all other values of u , $P_U(u) = 0$. The complete expression for the PMF of U is

$$P_U(u) = \begin{cases} 1/4 & u = 1 \\ 1/4 & u = 4 \\ 1/2 & u = 9 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(b) From the PMF, it is straightforward to write down the CDF.

$$F_U(u) = \begin{cases} 0 & u < 1 \\ 1/4 & 1 \leq u < 4 \\ 1/2 & 4 \leq u < 9 \\ 1 & u \geq 9 \end{cases} \quad (5)$$

(c) From Definition 2.14, the expected value of U is

$$E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75 \quad (6)$$

From Theorem 2.10, we can calculate the expected value of U as

$$E[U] = E[Y^2] = \sum_y y^2 P_Y(y) = 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 5.75 \quad (7)$$

As we expect, both methods yield the same answer.

Problem 2.6.2 Solution

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The PMF of $V = |X|$ satisfies

$$P_V(v) = P[|X| = v] = P_X(v) + P_X(-v) \quad (2)$$

In particular,

$$P_V(0) = P_X(0) = 0.5 \quad P_V(1) = P_X(-1) + P_X(1) = 0.5 \quad (3)$$

The complete expression for the PMF of V is

$$P_V(v) = \begin{cases} 0.5 & v = 0 \\ 0.5 & v = 1 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(b) From the PMF, we can construct the staircase CDF of V .

$$F_V(v) = \begin{cases} 0 & v < 0 \\ 0.5 & 0 \leq v < 1 \\ 1 & v \geq 1 \end{cases} \quad (5)$$

(c) From the PMF $P_V(v)$, the expected value of V is

$$E[V] = \sum_v P_V(v) = 0(1/2) + 1(1/2) = 1/2 \quad (6)$$

You can also compute $E[V]$ directly by using Theorem 2.10.

Problem 2.6.3 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The PMF of $W = -X$ satisfies

$$P_W(w) = P[-X = w] = P_X(-w) \quad (2)$$

This implies

$$P_W(-7) = P_X(7) = 0.2 \quad P_W(-5) = P_X(5) = 0.4 \quad P_W(3) = P_X(-3) = 0.4 \quad (3)$$

The complete PMF for W is

$$P_W(w) = \begin{cases} 0.2 & w = -7 \\ 0.4 & w = -5 \\ 0.4 & w = 3 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(b) From the PMF, the CDF of W is

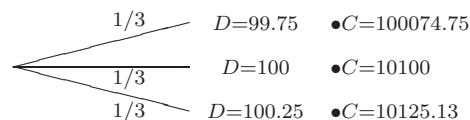
$$F_W(w) = \begin{cases} 0 & w < -7 \\ 0.2 & -7 \leq w < -5 \\ 0.6 & -5 \leq w < 3 \\ 1 & w \geq 3 \end{cases} \quad (5)$$

(c) From the PMF, W has expected value

$$E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2 \quad (6)$$

Problem 2.6.4 Solution

A tree for the experiment is



Thus C has three equally likely outcomes. The PMF of C is

$$P_C(c) = \begin{cases} 1/3 & c = 100,074.75, 10,100, 10,125.13 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Problem 2.6.5 Solution

- (a) The source continues to transmit packets until one is received correctly. Hence, the total number of times that a packet is transmitted is $X = x$ if the first $x - 1$ transmissions were in error. Therefore the PMF of X is

$$P_X(x) = \begin{cases} q^{x-1}(1-q) & x = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The time required to send a packet is a millisecond and the time required to send an acknowledgment back to the source takes another millisecond. Thus, if X transmissions of a packet are needed to send the packet correctly, then the packet is correctly received after $T = 2X - 1$ milliseconds. Therefore, for an odd integer $t > 0$, $T = t$ iff $X = (t + 1)/2$. Thus,

$$P_T(t) = P_X((t + 1)/2) = \begin{cases} q^{(t-1)/2}(1-q) & t = 1, 3, 5, \dots \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Problem 2.6.6 Solution

The cellular calling plan charges a flat rate of \$20 per month up to and including the 30th minute, and an additional 50 cents for each minute over 30 minutes. Knowing that the time you spend on the phone is a geometric random variable M with mean $1/p = 30$, the PMF of M is

$$P_M(m) = \begin{cases} (1-p)^{m-1}p & m = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The monthly cost, C obeys

$$P_C(20) = P[M \leq 30] = \sum_{m=1}^{30} (1-p)^{m-1}p = 1 - (1-p)^{30} \quad (2)$$

When $M \geq 30$, $C = 20 + (M - 30)/2$ or $M = 2C - 10$. Thus,

$$P_C(c) = P_M(2c - 10) \quad c = 20.5, 21, 21.5, \dots \quad (3)$$

The complete PMF of C is

$$P_C(c) = \begin{cases} 1 - (1-p)^{30} & c = 20 \\ (1-p)^{2c-10-1}p & c = 20.5, 21, 21.5, \dots \end{cases} \quad (4)$$

Problem 2.7.1 Solution

From the solution to Quiz 2.6, we found that $T = 120 - 15N$. By Theorem 2.10,

$$E[T] = \sum_{n \in S_N} (120 - 15n)P_N(n) \quad (1)$$

$$= 0.1(120) + 0.3(120 - 15) + 0.3(120 - 30) + 0.3(120 - 45) = 93 \quad (2)$$

Also from the solution to Quiz 2.6, we found that

$$P_T(t) = \begin{cases} 0.3 & t = 75, 90, 105 \\ 0.1 & t = 120 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

Using Definition 2.14,

$$E [T] = \sum_{t \in S_T} tP_T(t) = 0.3(75) + 0.3(90) + 0.3(105) + 0.1(120) = 93 \quad (4)$$

As expected, the two calculations give the same answer.

Problem 2.7.2 Solution

Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets T is a binomial random variable with mean

$$E [T] = 18250(0.001) = 18.25 \quad (1)$$

Since each winning ticket grosses \$1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E [R] = 1000E [T] = 18250 \quad (2)$$

But buying a lottery ticket everyday for 50 years, at \$2.00 a pop isn't cheap and will cost us a total of $18250 \cdot 2 = \$36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E [Q] = E [R] - 36500 = -18250 \quad (3)$$

Problem 2.7.3 Solution

Let X denote the number of points the shooter scores. If the shot is uncontested, the expected number of points scored is

$$E [X] = (0.6)2 = 1.2 \quad (1)$$

If we foul the shooter, then X is a binomial random variable with mean $E[X] = 2p$. If $2p > 1.2$, then we should not foul the shooter. Generally, p will exceed 0.6 since a free throw is usually even easier than an uncontested shot taken during the action of the game. Furthermore, fouling the shooter ultimately leads to the detriment of players possibly fouling out. This suggests that fouling a player is not a good idea. The only real exception occurs when facing a player like Shaquille O'Neal whose free throw probability p is lower than his field goal percentage during a game.

Problem 2.7.4 Solution

Given the distributions of D , the waiting time in days and the resulting cost, C , we can answer the following questions.

- (a) The expected waiting time is simply the expected value of D .

$$E [D] = \sum_{d=1}^4 d \cdot P_D(d) = 1(0.2) + 2(0.4) + 3(0.3) + 4(0.1) = 2.3 \quad (1)$$

- (b) The expected deviation from the waiting time is

$$E [D - \mu_D] = E [D] - E [\mu_d] = \mu_D - \mu_D = 0 \quad (2)$$

(c) C can be expressed as a function of D in the following manner.

$$C(D) = \begin{cases} 90 & D = 1 \\ 70 & D = 2 \\ 40 & D = 3 \\ 40 & D = 4 \end{cases} \quad (3)$$

(d) The expected service charge is

$$E[C] = 90(0.2) + 70(0.4) + 40(0.3) + 40(0.1) = 62 \text{ dollars} \quad (4)$$

Problem 2.7.5 Solution

As a function of the number of minutes used, M , the monthly cost is

$$C(M) = \begin{cases} 20 & M \leq 30 \\ 20 + (M - 30)/2 & M \geq 30 \end{cases} \quad (1)$$

The expected cost per month is

$$E[C] = \sum_{m=1}^{\infty} C(m)P_M(m) = \sum_{m=1}^{30} 20P_M(m) + \sum_{m=31}^{\infty} (20 + (m - 30)/2)P_M(m) \quad (2)$$

$$= 20 \sum_{m=1}^{\infty} P_M(m) + \frac{1}{2} \sum_{m=31}^{\infty} (m - 30)P_M(m) \quad (3)$$

Since $\sum_{m=1}^{\infty} P_M(m) = 1$ and since $P_M(m) = (1 - p)^{m-1}p$ for $m \geq 1$, we have

$$E[C] = 20 + \frac{(1 - p)^{30}}{2} \sum_{m=31}^{\infty} (m - 30)(1 - p)^{m-31}p \quad (4)$$

Making the substitution $j = m - 30$ yields

$$E[C] = 20 + \frac{(1 - p)^{30}}{2} \sum_{j=1}^{\infty} j(1 - p)^{j-1}p = 20 + \frac{(1 - p)^{30}}{2p} \quad (5)$$

Problem 2.7.6 Solution

Since our phone use is a geometric random variable M with mean value $1/p$,

$$P_M(m) = \begin{cases} (1 - p)^{m-1}p & m = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

For this cellular billing plan, we are given no free minutes, but are charged half the flat fee. That is, we are going to pay 15 dollars regardless and \$1 for each minute we use the phone. Hence $C = 15 + M$ and for $c \geq 16$, $P[C = c] = P[M = c - 15]$. Thus we can construct the PMF of the cost C

$$P_C(c) = \begin{cases} (1 - p)^{c-16}p & c = 16, 17, \dots \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Since $C = 15 + M$, the expected cost per month of the plan is

$$E[C] = E[15 + M] = 15 + E[M] = 15 + 1/p \quad (3)$$

In Problem 2.7.5, we found that the expected cost of the plan was

$$E[C] = 20 + [(1 - p)^{30}]/(2p) \quad (4)$$

In comparing the expected costs of the two plans, we see that the new plan is better (i.e. cheaper) if

$$15 + 1/p \leq 20 + [(1 - p)^{30}]/(2p) \quad (5)$$

A simple plot will show that the new plan is better if $p \leq p_0 \approx 0.2$.

Problem 2.7.7 Solution

We define random variable W such that $W = 1$ if the circuit works or $W = 0$ if the circuit is defective. (In the probability literature, W is called an indicator random variable.) Let R_s denote the profit on a circuit with standard devices. Let R_u denote the profit on a circuit with ultrareliable devices. We will compare $E[R_s]$ and $E[R_u]$ to decide which circuit implementation offers the highest expected profit.

The circuit with standard devices works with probability $(1 - q)^{10}$ and generates revenue of k dollars if all of its 10 constituent devices work. We observe that we can express R_s as a function $r_s(W)$ and that we can find the PMF $P_W(w)$:

$$R_s = r_s(W) = \begin{cases} -10 & W = 0, \\ k - 10 & W = 1, \end{cases} \quad P_W(w) = \begin{cases} 1 - (1 - q)^{10} & w = 0, \\ (1 - q)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Thus we can express the expected profit as

$$E[r_s(W)] = \sum_{w=0}^1 P_W(w) r_s(w) \quad (2)$$

$$= P_W(0)(-10) + P_W(1)(k - 10) \quad (3)$$

$$= (1 - (1 - q)^{10})(-10) + (1 - q)^{10}(k - 10) = (0.9)^{10}k - 10. \quad (4)$$

For the ultra-reliable case,

$$R_u = r_u(W) = \begin{cases} -30 & W = 0, \\ k - 30 & W = 1, \end{cases} \quad P_W(w) = \begin{cases} 1 - (1 - q/2)^{10} & w = 0, \\ (1 - q/2)^{10} & w = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Thus we can express the expected profit as

$$E[r_u(W)] = \sum_{w=0}^1 P_W(w) r_u(w) \quad (6)$$

$$= P_W(0)(-30) + P_W(1)(k - 30) \quad (7)$$

$$= (1 - (1 - q/2)^{10})(-30) + (1 - q/2)^{10}(k - 30) = (0.95)^{10}k - 30 \quad (8)$$

To determine which implementation generates the most profit, we solve $E[R_u] \geq E[R_s]$, yielding $k \geq 20/[(0.95)^{10} - (0.9)^{10}] = 80.21$. So for $k < \$80.21$ using all standard devices results in greater

revenue, while for $k > \$80.21$ more revenue will be generated by implementing all ultra-reliable devices. That is, when the price commanded for a working circuit is sufficiently high, we should build more-expensive higher-reliability circuits.

If you have read ahead to Section 2.9 and learned about conditional expected values, you might prefer the following solution. If not, you might want to come back and review this alternate approach after reading Section 2.9.

Let W denote the event that a circuit works. The circuit works and generates revenue of k dollars if all of its 10 constituent devices work. For each implementation, standard or ultra-reliable, let R denote the profit on a device. We can express the expected profit as

$$E[R] = P[W] E[R|W] + P[W^c] E[R|W^c] \quad (9)$$

Let's first consider the case when only standard devices are used. In this case, a circuit works with probability $P[W] = (1 - q)^{10}$. The profit made on a working device is $k - 10$ dollars while a nonworking circuit has a profit of -10 dollars. That is, $E[R|W] = k - 10$ and $E[R|W^c] = -10$. Of course, a negative profit is actually a loss. Using R_s to denote the profit using standard circuits, the expected profit is

$$E[R_s] = (1 - q)^{10}(k - 10) + (1 - (1 - q)^{10})(-10) = (0.9)^{10}k - 10 \quad (10)$$

And for the ultra-reliable case, the circuit works with probability $P[W] = (1 - q/2)^{10}$. The profit per working circuit is $E[R|W] = k - 30$ dollars while the profit for a nonworking circuit is $E[R|W^c] = -30$ dollars. The expected profit is

$$E[R_u] = (1 - q/2)^{10}(k - 30) + (1 - (1 - q/2)^{10})(-30) = (0.95)^{10}k - 30 \quad (11)$$

Not surprisingly, we get the same answers for $E[R_u]$ and $E[R_s]$ as in the first solution by performing essentially the same calculations. It should be apparent that indicator random variable W in the first solution indicates the occurrence of the conditioning event W in the second solution. That is, indicators are a way to track conditioning events.

Problem 2.7.8 Solution

- (a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9,366,819} \approx 1.07 \times 10^{-7} \quad (1)$$

- (b) Assuming each ticket is chosen randomly, each of the $2n - 1$ other tickets is independently a winner with probability q . The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1 - q)^{2n-1-k} & k = 0, 1, \dots, 2n - 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

- (c) Since there are $K_n + 1$ winning tickets in all, the value of your winning ticket is $W_n = n/(K_n + 1)$ which has mean

$$E[W_n] = nE\left[\frac{1}{K_n + 1}\right] \quad (3)$$

Calculating the expected value

$$E \left[\frac{1}{K_n + 1} \right] = \sum_{k=0}^{2n-1} \binom{2n-1}{k+1} P_{K_n}(k) \quad (4)$$

is fairly complicated. The trick is to express the sum in terms of the sum of a binomial PMF.

$$E \left[\frac{1}{K_n + 1} \right] = \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \quad (5)$$

$$= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \quad (6)$$

By factoring out $1/q$, we obtain

$$E \left[\frac{1}{K_n + 1} \right] = \frac{1}{2nq} \sum_{k=0}^{2n-1} \binom{2n}{k+1} q^{k+1} (1-q)^{2n-(k+1)} \quad (7)$$

$$= \frac{1}{2nq} \underbrace{\sum_{j=1}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j}}_A \quad (8)$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j = 0$. Thus

$$A = 1 - \binom{2n}{0} q^0 (1-q)^{2n-0} = 1 - (1-q)^{2n} \quad (9)$$

This implies

$$E \left[\frac{1}{K_n + 1} \right] = \frac{1 - (1-q)^{2n}}{2nq} \quad (10)$$

Our expected return on a winning ticket is

$$E[W_n] = nE \left[\frac{1}{K_n + 1} \right] = \frac{1 - (1-q)^{2n}}{2q} \quad (11)$$

Note that when $nq \ll 1$, we can use the approximation that $(1-q)^{2n} \approx 1 - 2nq$ to show that

$$E[W_n] \approx \frac{1 - (1 - 2nq)}{2q} = n \quad (nq \ll 1) \quad (12)$$

However, in the limit as the value of the prize n approaches infinity, we have

$$\lim_{n \rightarrow \infty} E[W_n] = \frac{1}{2q} \approx 4.683 \times 10^6 \quad (13)$$

That is, as the pot grows to infinity, the expected return on a winning ticket doesn't approach infinity because there is a corresponding increase in the number of other winning tickets. If it's not clear how large n must be for this effect to be seen, consider the following table:

n	10^6	10^7	10^8	(14)
$E[W_n]$	9.00×10^5	4.13×10^6	4.68×10^6	

When the pot is \$1 million, our expected return is \$900,000. However, we see that when the pot reaches \$100 million, our expected return is very close to $1/(2q)$, less than \$5 million!

Problem 2.7.9 Solution

(a) There are $\binom{46}{6}$ equally likely winning combinations so that

$$q = \frac{1}{\binom{46}{6}} = \frac{1}{9,366,819} \approx 1.07 \times 10^{-7} \quad (1)$$

(b) Assuming each ticket is chosen randomly, each of the $2n - 1$ other tickets is independently a winner with probability q . The number of other winning tickets K_n has the binomial PMF

$$P_{K_n}(k) = \begin{cases} \binom{2n-1}{k} q^k (1-q)^{2n-1-k} & k = 0, 1, \dots, 2n-1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Since the pot has $n + r$ dollars, the expected amount that you win on your ticket is

$$E[V] = 0(1-q) + qE\left[\frac{n+r}{K_n+1}\right] = q(n+r)E\left[\frac{1}{K_n+1}\right] \quad (3)$$

Note that $E[1/K_n + 1]$ was also evaluated in Problem 2.7.8. For completeness, we repeat those steps here.

$$E\left[\frac{1}{K_n+1}\right] = \sum_{k=0}^{2n-1} \frac{1}{k+1} \frac{(2n-1)!}{k!(2n-1-k)!} q^k (1-q)^{2n-1-k} \quad (4)$$

$$= \frac{1}{2n} \sum_{k=0}^{2n-1} \frac{(2n)!}{(k+1)!(2n-(k+1))!} q^k (1-q)^{2n-(k+1)} \quad (5)$$

By factoring out $1/q$, we obtain

$$E\left[\frac{1}{K_n+1}\right] = \frac{1}{2nq} \sum_{k=0}^{2n-1} \binom{2n}{k+1} q^{k+1} (1-q)^{2n-(k+1)} \quad (6)$$

$$= \frac{1}{2nq} \underbrace{\sum_{j=1}^{2n} \binom{2n}{j} q^j (1-q)^{2n-j}}_A \quad (7)$$

We observe that the above sum labeled A is the sum of a binomial PMF for $2n$ trials and success probability q over all possible values except $j = 0$. Thus $A = 1 - \binom{2n}{0} q^0 (1-q)^{2n-0}$, which implies

$$E\left[\frac{1}{K_n+1}\right] = \frac{A}{2nq} = \frac{1 - (1-q)^{2n}}{2nq} \quad (8)$$

The expected value of your ticket is

$$E[V] = \frac{q(n+r)[1 - (1-q)^{2n}]}{2nq} = \frac{1}{2} \left(1 + \frac{r}{n}\right) [1 - (1-q)^{2n}] \quad (9)$$

Each ticket tends to be more valuable when the carryover pot r is large and the number of new tickets sold, $2n$, is small. For any fixed number n , corresponding to $2n$ tickets sold,

a sufficiently large pot r will guarantee that $E[V] > 1$. For example if $n = 10^7$, (20 million tickets sold) then

$$E[V] = 0.44 \left(1 + \frac{r}{10^7}\right) \quad (10)$$

If the carryover pot r is 30 million dollars, then $E[V] = 1.76$. This suggests that buying a one dollar ticket is a good idea. This is an unusual situation because normally a carryover pot of 30 million dollars will result in far more than 20 million tickets being sold.

- (c) So that we can use the results of the previous part, suppose there were $2n - 1$ tickets sold before you must make your decision. If you buy one of each possible ticket, you are guaranteed to have one winning ticket. From the other $2n - 1$ tickets, there will be K_n winners. The total number of winning tickets will be $K_n + 1$. In the previous part we found that

$$E\left[\frac{1}{K_n + 1}\right] = \frac{1 - (1 - q)^{2n}}{2nq} \quad (11)$$

Let R denote the expected return from buying one of each possible ticket. The pot had r dollars beforehand. The $2n - 1$ other tickets are sold add $n - 1/2$ dollars to the pot. Furthermore, you must buy $1/q$ tickets, adding $1/(2q)$ dollars to the pot. Since the cost of the tickets is $1/q$ dollars, your expected profit

$$E[R] = E\left[\frac{r + n - 1/2 + 1/(2q)}{K_n + 1}\right] - \frac{1}{q} \quad (12)$$

$$= \frac{q(2r + 2n - 1) + 1}{2q} E\left[\frac{1}{K_n + 1}\right] - \frac{1}{q} \quad (13)$$

$$= \frac{[q(2r + 2n - 1) + 1](1 - (1 - q)^{2n})}{4nq^2} - \frac{1}{q} \quad (14)$$

For fixed n , sufficiently large r will make $E[R] > 0$. On the other hand, for fixed r , $\lim_{n \rightarrow \infty} E[R] = -1/(2q)$. That is, as n approaches infinity, your expected loss will be quite large.

Problem 2.8.1 Solution

Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0 \\ 0.7 & n = 1 \\ 0.1 & n = 2 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (a) $E[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$
 (b) $E[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$
 (c) $\text{Var}[N] = E[N^2] - E[N]^2 = 1.1 - (0.9)^2 = 0.29$
 (d) $\sigma_N = \sqrt{\text{Var}[N]} = \sqrt{0.29}$

Problem 2.8.2 Solution

From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of Y is

$$E[Y] = \sum_y y P_Y(y) = 1(1/4) + 2(1/4) + 3(1/2) = 9/4 \quad (2)$$

The expected value of Y^2 is

$$E[Y^2] = \sum_y y^2 P_Y(y) = 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 23/4 \quad (3)$$

The variance of Y is

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 23/4 - (9/4)^2 = 11/16 \quad (4)$$

Problem 2.8.3 Solution

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_x x P_X(x) = (-1)(0.2) + 0(0.5) + 1(0.3) = 0.1 \quad (2)$$

The expected value of X^2 is

$$E[X^2] = \sum_x x^2 P_X(x) = (-1)^2(0.2) + 0^2(0.5) + 1^2(0.3) = 0.5 \quad (3)$$

The variance of X is

$$\text{Var}[X] = E[X^2] - (E[X])^2 = 0.5 - (0.1)^2 = 0.49 \quad (4)$$

Problem 2.8.4 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of X is

$$E[X] = \sum_x xP_X(x) = -3(0.4) + 5(0.4) + 7(0.2) = 2.2 \quad (2)$$

The expected value of X^2 is

$$E[X^2] = \sum_x x^2P_X(x) = (-3)^2(0.4) + 5^2(0.4) + 7^2(0.2) = 23.4 \quad (3)$$

The variance of X is

$$\text{Var}[X] = E[X^2] - (E[X])^2 = 23.4 - (2.2)^2 = 18.56 \quad (4)$$

Problem 2.8.5 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^4 xP_X(x) = 0\binom{4}{0}\frac{1}{2^4} + 1\binom{4}{1}\frac{1}{2^4} + 2\binom{4}{2}\frac{1}{2^4} + 3\binom{4}{3}\frac{1}{2^4} + 4\binom{4}{4}\frac{1}{2^4} \quad (1)$$

$$= [4 + 12 + 12 + 4]/2^4 = 2 \quad (2)$$

The expected value of X^2 is

$$E[X^2] = \sum_{x=0}^4 x^2P_X(x) = 0^2\binom{4}{0}\frac{1}{2^4} + 1^2\binom{4}{1}\frac{1}{2^4} + 2^2\binom{4}{2}\frac{1}{2^4} + 3^2\binom{4}{3}\frac{1}{2^4} + 4^2\binom{4}{4}\frac{1}{2^4} \quad (3)$$

$$= [4 + 24 + 36 + 16]/2^4 = 5 \quad (4)$$

The variance of X is

$$\text{Var}[X] = E[X^2] - (E[X])^2 = 5 - 2^2 = 1 \quad (5)$$

Thus, X has standard deviation $\sigma_X = \sqrt{\text{Var}[X]} = 1$.

(b) The probability that X is within one standard deviation of its expected value is

$$P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] = P[2 - 1 \leq X \leq 2 + 1] = P[1 \leq X \leq 3] \quad (6)$$

This calculation is easy using the PMF of X .

$$P[1 \leq X \leq 3] = P_X(1) + P_X(2) + P_X(3) = 7/8 \quad (7)$$

Problem 2.8.6 Solution

(a) The expected value of X is

$$E[X] = \sum_{x=0}^5 xP_X(x) \quad (1)$$

$$= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} \quad (2)$$

$$= [5 + 20 + 30 + 20 + 5]/2^5 = 5/2 \quad (3)$$

The expected value of X^2 is

$$E[X^2] = \sum_{x=0}^5 x^2 P_X(x) \quad (4)$$

$$= 0^2 \binom{5}{0} \frac{1}{2^5} + 1^2 \binom{5}{1} \frac{1}{2^5} + 2^2 \binom{5}{2} \frac{1}{2^5} + 3^2 \binom{5}{3} \frac{1}{2^5} + 4^2 \binom{5}{4} \frac{1}{2^5} + 5^2 \binom{5}{5} \frac{1}{2^5} \quad (5)$$

$$= [5 + 40 + 90 + 80 + 25]/2^5 = 240/32 = 15/2 \quad (6)$$

The variance of X is

$$\text{Var}[X] = E[X^2] - (E[X])^2 = 15/2 - 25/4 = 5/4 \quad (7)$$

By taking the square root of the variance, the standard deviation of X is $\sigma_X = \sqrt{5/4} \approx 1.12$.

(b) The probability that X is within one standard deviation of its mean is

$$P[\mu_X - \sigma_X \leq X \leq \mu_X + \sigma_X] = P[2.5 - 1.12 \leq X \leq 2.5 + 1.12] \quad (8)$$

$$= P[1.38 \leq X \leq 3.62] \quad (9)$$

$$= P[2 \leq X \leq 3] \quad (10)$$

By summing the PMF over the desired range, we obtain

$$P[2 \leq X \leq 3] = P_X(2) + P_X(3) = 10/32 + 10/32 = 5/8 \quad (11)$$

Problem 2.8.7 Solution

For $Y = aX + b$, we wish to show that $\text{Var}[Y] = a^2 \text{Var}[X]$. We begin by noting that Theorem 2.12 says that $E[aX + b] = aE[X] + b$. Hence, by the definition of variance,

$$\text{Var}[Y] = E[(aX + b - (aE[X] + b))^2] \quad (1)$$

$$= E[a^2(X - E[X])^2] \quad (2)$$

$$= a^2 E[(X - E[X])^2] \quad (3)$$

Since $E[(X - E[X])^2] = \text{Var}[X]$, the assertion is proved.

Problem 2.8.8 Solution

Given the following description of the random variable Y ,

$$Y = \frac{1}{\sigma_X}(X - \mu_X) \quad (1)$$

we can use the linearity property of the expectation operator to find the mean value

$$E[Y] = \frac{E[X - \mu_X]}{\sigma_X} = \frac{E[X] - E[X]}{\sigma_X} = 0 \quad (2)$$

Using the fact that $\text{Var}[aX + b] = a^2 \text{Var}[X]$, the variance of Y is found to be

$$\text{Var}[Y] = \frac{1}{\sigma_X^2} \text{Var}[X] = 1 \quad (3)$$

Problem 2.8.9 Solution

With our measure of jitter being σ_T , and the fact that $T = 2X - 1$, we can express the jitter as a function of q by realizing that

$$\text{Var}[T] = 4 \text{Var}[X] = \frac{4q}{(1-q)^2} \quad (1)$$

Therefore, our maximum permitted jitter is

$$\sigma_T = \frac{2\sqrt{q}}{(1-q)} = 2 \text{ msec} \quad (2)$$

Solving for q yields $q^2 - 3q + 1 = 0$. By solving this quadratic equation, we obtain

$$q = \frac{3 \pm \sqrt{5}}{2} = 3/2 \pm \sqrt{5}/2 \quad (3)$$

Since q must be a value between 0 and 1, we know that a value of $q = 3/2 - \sqrt{5}/2 \approx 0.382$ will ensure a jitter of at most 2 milliseconds.

Problem 2.8.10 Solution

We wish to minimize the function

$$e(\hat{x}) = E[(X - \hat{x})^2] \quad (1)$$

with respect to \hat{x} . We can expand the square and take the expectation while treating \hat{x} as a constant. This yields

$$e(\hat{x}) = E[X^2 - 2\hat{x}X + \hat{x}^2] = E[X^2] - 2\hat{x}E[X] + \hat{x}^2 \quad (2)$$

Solving for the value of \hat{x} that makes the derivative $de(\hat{x})/d\hat{x}$ equal to zero results in the value of \hat{x} that minimizes $e(\hat{x})$. Note that when we take the derivative with respect to \hat{x} , both $E[X^2]$ and $E[X]$ are simply constants.

$$\frac{d}{d\hat{x}} (E[X^2] - 2\hat{x}E[X] + \hat{x}^2) = 2E[X] - 2\hat{x} = 0 \quad (3)$$

Hence we see that $\hat{x} = E[X]$. In the sense of mean squared error, the best guess for a random variable is the mean value. In Chapter 9 this idea is extended to develop minimum mean squared error estimation.

Problem 2.8.11 Solution

The PMF of K is the Poisson PMF

$$P_K(k) = \begin{cases} \lambda^k e^{-\lambda}/k! & k = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The mean of K is

$$E[K] = \sum_{k=0}^{\infty} k \frac{\lambda^k e^{-\lambda}}{k!} = \lambda \sum_{k=1}^{\infty} \frac{\lambda^{k-1} e^{-\lambda}}{(k-1)!} = \lambda \quad (2)$$

To find $E[K^2]$, we use the hint and first find

$$E[K(K-1)] = \sum_{k=0}^{\infty} k(k-1) \frac{\lambda^k e^{-\lambda}}{k!} = \sum_{k=2}^{\infty} \frac{\lambda^k e^{-\lambda}}{(k-2)!} \quad (3)$$

By factoring out λ^2 and substituting $j = k - 2$, we obtain

$$E[K(K-1)] = \lambda^2 \underbrace{\sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!}}_1 = \lambda^2 \quad (4)$$

The above sum equals 1 because it is the sum of a Poisson PMF over all possible values. Since $E[K] = \lambda$, the variance of K is

$$\text{Var}[K] = E[K^2] - (E[K])^2 \quad (5)$$

$$= E[K(K-1)] + E[K] - (E[K])^2 \quad (6)$$

$$= \lambda^2 + \lambda - \lambda^2 = \lambda \quad (7)$$

Problem 2.8.12 Solution

The standard deviation can be expressed as

$$\sigma_D = \sqrt{\text{Var}[D]} = \sqrt{E[D^2] - E[D]^2} \quad (1)$$

where

$$E[D^2] = \sum_{d=1}^4 d^2 P_D(d) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1 \quad (2)$$

So finally we have

$$\sigma_D = \sqrt{6.1 - 2.3^2} = \sqrt{0.81} = 0.9 \quad (3)$$

Problem 2.9.1 Solution

From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The probability of the event $B = \{Y < 3\}$ is $P[B] = 1 - P[Y = 3] = 1/2$. From Theorem 2.17, the conditional PMF of Y given B is

$$P_{Y|B}(y) = \begin{cases} \frac{P_Y(y)}{P[B]} & y \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & y = 1 \\ 1/2 & y = 2 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The conditional first and second moments of Y are

$$E[Y|B] = \sum_y y P_{Y|B}(y) = 1(1/2) + 2(1/2) = 3/2 \quad (3)$$

$$E[Y^2|B] = \sum_y y^2 P_{Y|B}(y) = 1^2(1/2) + 2^2(1/2) = 5/2 \quad (4)$$

The conditional variance of Y is

$$\text{Var}[Y|B] = E[Y^2|B] - (E[Y|B])^2 = 5/2 - 9/4 = 1/4 \quad (5)$$

Problem 2.9.2 Solution

From the solution to Problem 2.4.2, the PMF of X is

$$P_X(x) = \begin{cases} 0.2 & x = -1 \\ 0.5 & x = 0 \\ 0.3 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The event $B = \{|X| > 0\}$ has probability $P[B] = P[X \neq 0] = 0.5$. From Theorem 2.17, the conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.4 & x = -1 \\ 0.6 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = (-1)(0.4) + 1(0.6) = 0.2 \quad (3)$$

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = (-1)^2(0.4) + 1^2(0.6) = 1 \quad (4)$$

The conditional variance of X is

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = 1 - (0.2)^2 = 0.96 \quad (5)$$

Problem 2.9.3 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The event $B = \{X > 0\}$ has probability $P[B] = P_X(5) + P_X(7) = 0.6$. From Theorem 2.17, the conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = 5(2/3) + 7(1/3) = 17/3 \quad (3)$$

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = 5^2(2/3) + 7^2(1/3) = 33 \quad (4)$$

The conditional variance of X is

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \quad (5)$$

Problem 2.9.4 Solution

The event $B = \{X \neq 0\}$ has probability $P[B] = 1 - P[X = 0] = 15/16$. The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{4}{x} \frac{1}{15} & x = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The conditional first and second moments of X are

$$E[X|B] = \sum_{x=1}^4 x P_{X|B}(x) = 1 \binom{4}{1} \frac{1}{15} + 2 \binom{4}{2} \frac{1}{15} + 3 \binom{4}{3} \frac{1}{15} + 4 \binom{4}{4} \frac{1}{15} \quad (2)$$

$$= [4 + 12 + 12 + 4]/15 = 32/15 \quad (3)$$

$$E[X^2|B] = \sum_{x=1}^4 x^2 P_{X|B}(x) = 1^2 \binom{4}{1} \frac{1}{15} + 2^2 \binom{4}{2} \frac{1}{15} + 3^2 \binom{4}{3} \frac{1}{15} + 4^2 \binom{4}{4} \frac{1}{15} \quad (4)$$

$$= [4 + 24 + 36 + 16]/15 = 80/15 \quad (5)$$

The conditional variance of X is

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = 80/15 - (32/15)^2 = 176/225 \approx 0.782 \quad (6)$$

Problem 2.9.5 Solution

The probability of the event B is

$$P[B] = P[X \geq \mu_X] = P[X \geq 3] = P_X(3) + P_X(4) + P_X(5) \quad (1)$$

$$= \frac{\binom{5}{3} + \binom{5}{4} + \binom{5}{5}}{32} = 21/32 \quad (2)$$

The conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \binom{5}{x} \frac{1}{21} & x = 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The conditional first and second moments of X are

$$E[X|B] = \sum_{x=3}^5 x P_{X|B}(x) = 3 \binom{5}{3} \frac{1}{21} + 4 \binom{5}{4} \frac{1}{21} + 5 \binom{5}{5} \frac{1}{21} \quad (4)$$

$$= [30 + 20 + 5]/21 = 55/21 \quad (5)$$

$$E[X^2|B] = \sum_{x=3}^5 x^2 P_{X|B}(x) = 3^2 \binom{5}{3} \frac{1}{21} + 4^2 \binom{5}{4} \frac{1}{21} + 5^2 \binom{5}{5} \frac{1}{21} \quad (6)$$

$$= [90 + 80 + 25]/21 = 195/21 = 65/7 \quad (7)$$

The conditional variance of X is

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = 65/7 - (55/21)^2 = 1070/441 = 2.43 \quad (8)$$

Problem 2.9.6 Solution

- (a) Consider each circuit test as a Bernoulli trial such that a failed circuit is called a success. The number of trials until the first success (i.e. a failed circuit) has the geometric PMF

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (b) The probability there are at least 20 tests is

$$P[B] = P[N \geq 20] = \sum_{n=20}^{\infty} P_N(n) = (1-p)^{19} \quad (2)$$

Note that $(1-p)^{19}$ is just the probability that the first 19 circuits pass the test, which is what we would expect since there must be at least 20 tests if the first 19 circuits pass. The conditional PMF of N given B is

$$P_{N|B}(n) = \begin{cases} \frac{P_N(n)}{P[B]} & n \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} (1-p)^{n-20}p & n = 20, 21, \dots \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

- (c) Given the event B , the conditional expectation of N is

$$E[N|B] = \sum_n n P_{N|B}(n) = \sum_{n=20}^{\infty} n (1-p)^{n-20} p \quad (4)$$

Making the substitution $j = n - 19$ yields

$$E[N|B] = \sum_{j=1}^{\infty} (j+19)(1-p)^{j-1} p = 1/p + 19 \quad (5)$$

We see that in the above sum, we effectively have the expected value of $J + 19$ where J is geometric random variable with parameter p . This is not surprising since the $N \geq 20$ iff we observed 19 successful tests. After 19 successful tests, the number of additional tests needed to find the first failure is still a geometric random variable with mean $1/p$.

Problem 2.9.7 Solution

- (a) The PMF of M , the number of miles run on an arbitrary day is

$$P_M(m) = \begin{cases} q(1-q)^m & m = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

And we can see that the probability that $M > 0$, is

$$P[M > 0] = 1 - P[M = 0] = 1 - q \quad (2)$$

(b) The probability that we run a marathon on any particular day is the probability that $M \geq 26$.

$$r = P[M \geq 26] = \sum_{m=26}^{\infty} q(1-q)^m = (1-q)^{26} \quad (3)$$

(c) We run a marathon on each day with probability equal to r , and we do not run a marathon with probability $1-r$. Therefore in a year we have 365 tests of our jogging resolve, and thus 365 chances to run a marathon. So the PMF of the number of marathons run in a year, J , can be expressed as

$$P_J(j) = \begin{cases} \binom{365}{j} r^j (1-r)^{365-j} & j = 0, 1, \dots, 365 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

(d) The random variable K is defined as the number of miles we run above that required for a marathon, $K = M - 26$. Given the event, A , that we have run a marathon, we wish to know how many miles in excess of 26 we in fact ran. So we want to know the conditional PMF $P_{K|A}(k)$.

$$P_{K|A}(k) = \frac{P[K = k, A]}{P[A]} = \frac{P[M = 26 + k]}{P[A]} \quad (5)$$

Since $P[A] = r$, for $k = 0, 1, \dots$,

$$P_{K|A}(k) = \frac{(1-q)^{26+k} q}{(1-q)^{26}} = (1-q)^k q \quad (6)$$

The complete expression of for the conditional PMF of K is

$$P_{K|A}(k) = \begin{cases} (1-q)^k q & k = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

Problem 2.9.8 Solution

Recall that the PMF of the number of pages in a fax is

$$P_X(x) = \begin{cases} 0.15 & x = 1, 2, 3, 4 \\ 0.1 & x = 5, 6, 7, 8 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

(a) The event that a fax was sent to machine A can be expressed mathematically as the event that the number of pages X is an even number. Similarly, the event that a fax was sent to B is the event that X is an odd number. Since $S_X = \{1, 2, \dots, 8\}$, we define the set $A = \{2, 4, 6, 8\}$. Using this definition for A , we have that the event that a fax is sent to A is equivalent to the event $X \in A$. The event A has probability

$$P[A] = P_X(2) + P_X(4) + P_X(6) + P_X(8) = 0.5 \quad (2)$$

Given the event A , the conditional PMF of X is

$$P_{X|A}(x) = \begin{cases} \frac{P_X(x)}{P[A]} & x \in A \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 0.3 & x = 2, 4 \\ 0.2 & x = 6, 8 \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

The conditional first and second moments of X given A is

$$E[X|A] = \sum_x x P_{X|A}(x) = 2(0.3) + 4(0.3) + 6(0.2) + 8(0.2) = 4.6 \quad (4)$$

$$E[X^2|A] = \sum_x x^2 P_{X|A}(x) = 4(0.3) + 16(0.3) + 36(0.2) + 64(0.2) = 26 \quad (5)$$

The conditional variance and standard deviation are

$$\text{Var}[X|A] = E[X^2|A] - (E[X|A])^2 = 26 - (4.6)^2 = 4.84 \quad (6)$$

$$\sigma_{X|A} = \sqrt{\text{Var}[X|A]} = 2.2 \quad (7)$$

(b) Let the event B' denote the event that the fax was sent to B and that the fax had no more than 6 pages. Hence, the event $B' = \{1, 3, 5\}$ has probability

$$P[B'] = P_X(1) + P_X(3) + P_X(5) = 0.4 \quad (8)$$

The conditional PMF of X given B' is

$$P_{X|B'}(x) = \begin{cases} \frac{P_X(x)}{P[B']} & x \in B' \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 3/8 & x = 1, 3 \\ 1/4 & x = 5 \\ 0 & \text{otherwise} \end{cases} \quad (9)$$

Given the event B' , the conditional first and second moments are

$$E[X|B'] = \sum_x x P_{X|B'}(x) = 1(3/8) + 3(3/8) + 5(1/4) = 11/4 \quad (10)$$

$$E[X^2|B'] = \sum_x x^2 P_{X|B'}(x) = 1(3/8) + 9(3/8) + 25(1/4) = 10 \quad (11)$$

The conditional variance and standard deviation are

$$\text{Var}[X|B'] = E[X^2|B'] - (E[X|B'])^2 = 10 - (11/4)^2 = 39/16 \quad (12)$$

$$\sigma_{X|B'} = \sqrt{\text{Var}[X|B']} = \sqrt{39}/4 \approx 1.56 \quad (13)$$

Problem 2.10.1 Solution

For a binomial (n, p) random variable X , the solution in terms of math is

$$P[E_2] = \sum_{x=0}^{\lfloor \sqrt{n} \rfloor} P_X(x^2) \quad (1)$$

In terms of MATLAB, the efficient solution is to generate the vector of perfect squares $x = [0 \ 1 \ 4 \ 9 \ 16 \ \dots]$ and then to pass that vector to the `binomialpmf.m`. In this case, the values of the binomial PMF are calculated only once. Here is the code:

```
function q=perfectbinomial(n,p);
i=0:floor(sqrt(n));
x=i.^2;
q=sum(binomialpmf(n,p,x));
```

For a binomial $(100, 0.5)$ random variable X , the probability X is a perfect square is

```
>> perfectbinomial(100,0.5)
ans =
    0.0811
```

Problem 2.10.2 Solution

The random variable X given in Example 2.29 is just a finite random variable. We can generate random samples using the `finiterv` function. The code is

```
function x=faxlength8(m);
sx=1:8;
p=[0.15*ones(1,4) 0.1*ones(1,4)];
x=finiterv(sx,p,m);
```

Problem 2.10.3 Solution

First we use `faxlength8` from Problem 2.10.2 to generate m samples of the fax length X . Next we convert that to m samples of the fax cost Y . Summing these samples and dividing by m , we obtain the average cost of m samples. Here is the code:

```
function y=avgfax(m);
x=faxlength8(m);
yy=cumsum([10 9 8 7 6]);
yy=[yy 50 50 50];
y=sum(yy(x))/m;
```

Each time we perform the experiment of executing the function `avgfax`, we generate m random samples of X , and m corresponding samples of Y . The sum $\bar{Y} = \frac{1}{m} \sum_{i=1}^m Y_i$ is random. For $m = 10$, four samples of \bar{Y} are

```
>> [avgfax(10) avgfax(10) avgfax(10) avgfax(10)]
ans =
    31.9000    31.2000    29.6000    34.1000
>>
```

For $m = 100$, the results are arguably more consistent:

```
>> [avgfax(100) avgfax(100) avgfax(100) avgfax(100)]
ans =
    34.5300    33.3000    29.8100    33.6900
>>
```

Finally, for $m = 1000$, we obtain results reasonably close to $E[Y]$:

```
>> [avgfax(1000) avgfax(1000) avgfax(1000) avgfax(1000)]
ans =
    32.1740    31.8920    33.1890    32.8250
>>
```

In Chapter 7, we will develop techniques to show how \bar{Y} converges to $E[Y]$ as $m \rightarrow \infty$.

Problem 2.10.4 Solution

Suppose X_n is a Zipf ($n, \alpha = 1$) random variable and thus has PMF

$$P_X(x) = \begin{cases} c(n)/x & x = 1, 2, \dots, n \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The problem asks us to find the smallest value of k such that $P[X_n \leq k] \geq 0.75$. That is, if the server caches the k most popular files, then with $P[X_n \leq k]$ the request is for one of the k cached

files. First, we might as well solve this problem for any probability p rather than just $p = 0.75$. Thus, in math terms, we are looking for

$$k = \min \{k' | P [X_n \leq k'] \geq p\}. \quad (2)$$

What makes the Zipf distribution hard to analyze is that there is no closed form expression for

$$c(n) = \left(\sum_{x=1}^n \frac{1}{x} \right)^{-1}. \quad (3)$$

Thus, we use MATLAB to grind through the calculations. The following simple program generates the Zipf distributions and returns the correct value of k .

```
function k=zipfcache(n,p);
%Usage: k=zipfcache(n,p);
%for the Zipf (n,alpha=1) distribution, returns the smallest k
%such that the first k items have total probability p
pmf=1./(1:n);
pmf=pmf/sum(pmf); %normalize to sum to 1
cdf=cumsum(pmf);
k=1+sum(cdf<=p);
```

The program `zipfcache` generalizes 0.75 to be the probability p . Although this program is sufficient, the problem asks us to find k for all values of n from 1 to 10^3 !. One way to do this is to call `zipfcache` a thousand times to find k for each value of n . A better way is to use the properties of the Zipf PDF. In particular,

$$P [X_n \leq k'] = c(n) \sum_{x=1}^{k'} \frac{1}{x} = \frac{c(n)}{c(k')} \quad (4)$$

Thus we wish to find

$$k = \min \left\{ k' \mid \frac{c(n)}{c(k')} \geq p \right\} = \min \left\{ k' \mid \frac{1}{c(k')} \geq \frac{p}{c(n)} \right\}. \quad (5)$$

Note that the definition of k implies that

$$\frac{1}{c(k')} < \frac{p}{c(n)}, \quad k' = 1, \dots, k-1. \quad (6)$$

Using the notation $|A|$ to denote the number of elements in the set A , we can write

$$k = 1 + \left| \left\{ k' \mid \frac{1}{c(k')} < \frac{p}{c(n)} \right\} \right| \quad (7)$$

This is the basis for a very short MATLAB program:

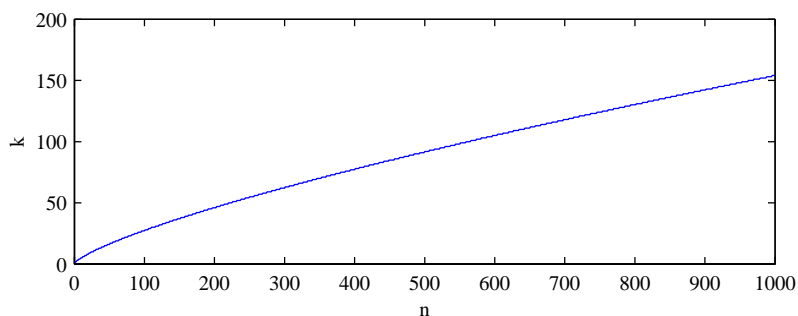
```
function k=zipfcacheall(n,p);
%Usage: k=zipfcacheall(n,p);
%returns vector k such that the first
%k(m) items have total probability >= p
%for the Zipf(m,1) distribution.
c=1./cumsum(1./(1:n));
k=1+countless(1./c,p./c);
```

Note that `zipfcacheall` uses a short MATLAB program `countless.m` that is almost the same as `count.m` introduced in Example 2.47. If `n=countless(x,y)`, then `n(i)` is the number of elements of `x` that are strictly less than `y(i)` while `count` returns the number of elements less than or equal to `y(i)`.

In any case, the commands

```
k=zipfcacheall(1000,0.75);
plot(1:1000,k);
```

is sufficient to produce this figure of k as a function of m :



We see in the figure that the number of files that must be cached grows slowly with the total number of files n .

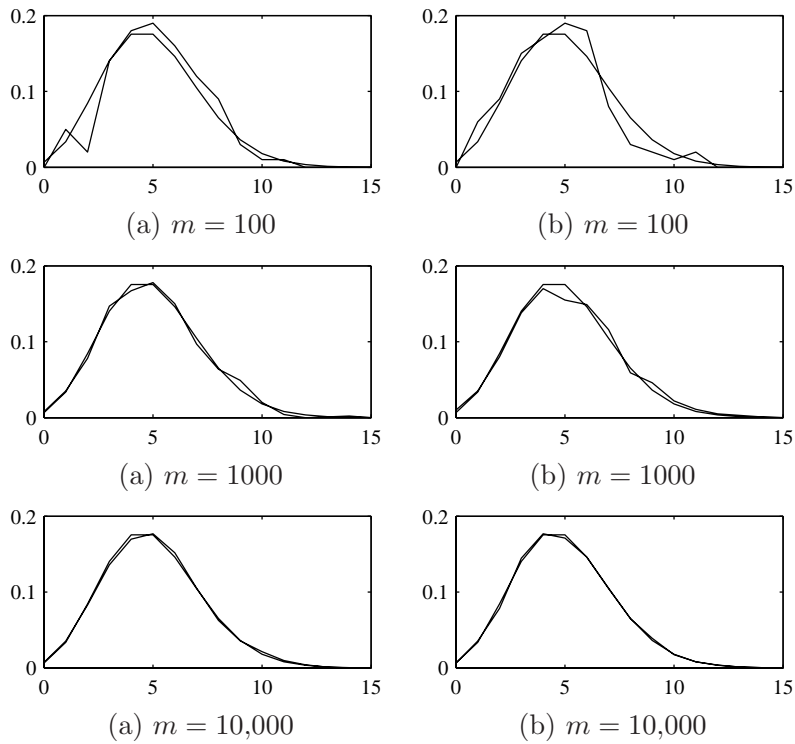
Finally, we make one last observation. It is generally desirable for MATLAB to execute operations in parallel. The program `zipfcacheall` generally will run faster than n calls to `zipfcache`. However, to do its counting all at once, `countless` generates and $n \times n$ array. When n is not too large, say $n \leq 1000$, the resulting array with $n^2 = 1,000,000$ elements fits in memory. For much large values of n , say $n = 10^6$ (as was proposed in the original printing of this edition of the text, `countless` will cause an “out of memory” error.

Problem 2.10.5 Solution

We use `poissonrv.m` to generate random samples of a Poisson ($\alpha = 5$) random variable. To compare the Poisson PMF against the output of `poissonrv`, relative frequencies are calculated using the `hist` function. The following code plots the relative frequency against the PMF.

```
function diff=poisontest(alpha,m)
x=poissonrv(alpha,m);
xr=0:ceil(3*alpha);
pxsample=hist(x,xr)/m;
pxsample=pxsample(:);
%pxsample=(countequal(x,xr)/m);
px=poissonpmf(alpha,xr);
plot(xr,pxsample,xr,px);
diff=sum((pxsample-px).^2);
```

For $m = 100, 1000, 10000$, here are sample plots comparing the PMF and the relative frequency. The plots show reasonable agreement for $m = 10000$ samples.

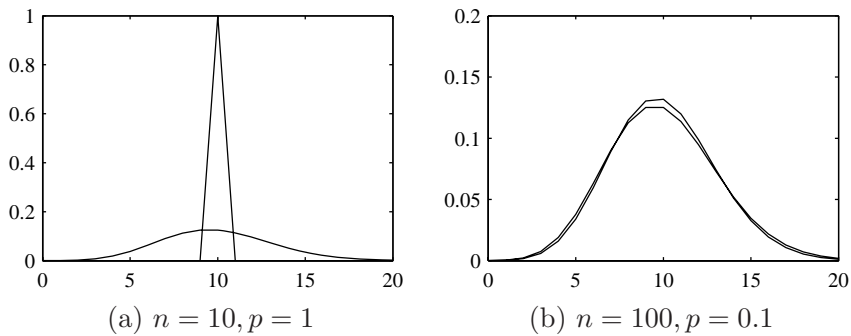


Problem 2.10.6 Solution

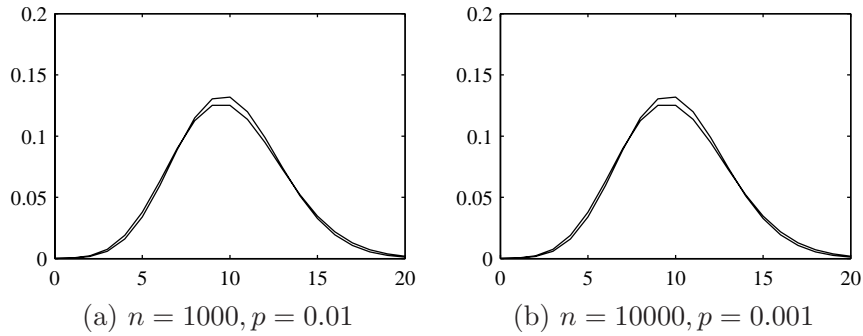
We can compare the binomial and Poisson PMFs for $(n, p) = (100, 0.1)$ using the following MATLAB code:

```
x=0:20;
p=poissonpmf(100,x);
b=binomialpmf(100,0.1,x);
plot(x,p,x,b);
```

For $(n, p) = (10, 1)$, the binomial PMF has no randomness. For $(n, p) = (100, 0.1)$, the approximation is reasonable:



Finally, for $(n, p) = (1000, 0.01)$, and $(n, p) = (10000, 0.001)$, the approximation is very good:



Problem 2.10.7 Solution

Following the Random Sample algorithm, we generate a sample value $R = \text{rand}(1)$ and then we find k^* such that

$$F_K(k^* - 1) < R < F_K(k^*). \quad (1)$$

From Problem 2.4.4, we know for integers $k \geq 1$ that geometric (p) random variable K has CDF $F_K(k) = 1 - (1 - p)^k$. Thus,

$$1 - (1 - p)^{k^* - 1} < R \leq 1 - (1 - p)^{k^*}. \quad (2)$$

Subtracting 1 from each side and then multiplying through by -1 (which reverses the inequalities), we obtain

$$(1 - p)^{k^* - 1} > 1 - R \geq (1 - p)^{k^*}. \quad (3)$$

Next we take the logarithm of each side. Since logarithms are monotonic functions, we have

$$(k^* - 1) \ln(1 - p) > \ln(1 - R) \geq k^* \ln(1 - p). \quad (4)$$

Since $0 < p < 1$, we have that $\ln(1 - p) < 0$. Thus dividing through by $\ln(1 - p)$ reverses the inequalities, yielding

$$k^* - 1 > \frac{\ln(1 - R)}{\ln(1 - p)} \geq k^*. \quad (5)$$

Since k^* is an integer, it must be the smallest integer greater than or equal to $\ln(1 - R)/\ln(1 - p)$. That is, following the last step of the random sample algorithm,

$$K = k^* = \left\lceil \frac{\ln(1 - R)}{\ln(1 - p)} \right\rceil \quad (6)$$

The MATLAB algorithm that implements this operation is quite simple:

```
function x=geometricrv(p,m)
%Usage: x=geometricrv(p,m)
% returns m samples of a geometric (p) rv
r=rand(m,1);
x=ceil(log(1-r)/log(1-p));
```

Problem 2.10.8 Solution

For the PC version of MATLAB employed for this test, `poissonpmf(n,n)` reported `Inf` for $n = n^* = 714$. The problem with the `poissonpmf` function in Example 2.44 is that the cumulative product that calculated $n^k/k!$ can have an overflow. Following the hint, we can write an alternate `poissonpmf` function as follows:

```
function pmf=poissonpmf(alpha,x)
%Poisson (alpha) rv X,
%out=vector pmf: pmf(i)=P[X=x(i)]
x=x(:);
if (alpha==0)
    pmf=1.0*(x==0);
else
    k=(1:ceil(max(x)))';
    logfacts =cumsum(log(k));
    pb=exp([-alpha; ...
            -alpha+ (k*log(alpha))-logfacts]);
    okx=(x>=0).*(x==floor(x));
    x=okx.*x;
    pmf=okx.*pb(x+1);
end
%pmf(i)=0 for zero-prob x(i)
```

By summing logarithms, the intermediate terms are much less likely to overflow.