two or more disjoint events, so that probabilities add [Eq. (1.3.10)]. Adding the probabilities we have

$$P[1 \text{ in } t + dt] = P[(1 \text{ in } t) \cap (0 \text{ in } dt)] + P[(0 \text{ in } t) \cap (1 \text{ in } dt)] \tag{5.1.8}$$

Because these are nonoverlapping regions in both probabilities and thus describe independent events, we can transform $\cap \to \times$. Translating Eq. (5.1.8) into the Poisson notation of Eq. (5.1.1) and using Eq. (5.1.2), we have

$$P(1, t + dt) = P(1, t) \times (1 - \lambda dt) + P(0, t) \times \lambda dt \tag{5.1.9}$$

Routine manipulation of Eq. (5.1.9) produces a differential equation for $P(1, t)$ as follows:

$$\frac{d}{dt} P(1, t) + \lambda P(1, t) = \lambda P(0, t) \tag{5.1.10}$$

where the term on the right side is known from Eq. (5.1.7). Equation (5.1.10) is of the form known as *exact* with an integrating factor of $e^{+\lambda t}$ and thus can be solved readily. The solution is

$$P(1, t) = \lambda t \, e^{-\lambda t} \tag{5.1.11}$$

For $\lambda = 4$, this solution looks like Fig. 5.1.7.

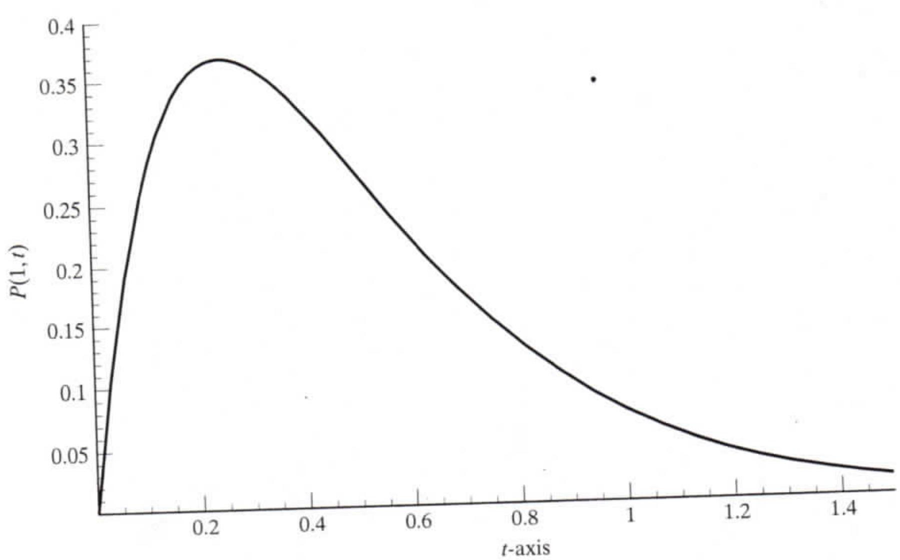

Figure 5.1.7 The probability of {1 event in t} starts at zero, rises to a maximum at $t = \frac{1}{\lambda}$, and then falls asymptotically to zero. Near $t = 0$ it is unlikely that an event has already occurred, and for large time more than one event is likely.

The higher-order Poisson probabilities. We may derive $P(k, t)$ for $k = 2, 3, \ldots$ using similar methods. The general result is

$$P(k,t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \ k = 0, 1, 2, \ldots, t \geq 0, \text{ zow} \tag{5.1.12}$$

Notice that Eq. (5.1.12), which is the Poisson distribution, is valid for $k = 0$ if we assume $0! = 1$.

The PMF of the Poisson random variable. If we use the language of random variables, we define the Poisson random variable, K, as the number of events occurring during the period of time of duration t. This Poisson random variable is discrete, having values of $K = 0, 1, 2, \ldots$. The probabilities of the various values of the random variable are described by the PMF

$$P_K(k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \ k = 0, 1, 2, \cdots, t \geq 0, \text{ zow} \tag{5.1.13}$$

The Poisson PMF is plotted for $\lambda t = 10.8$ in Fig. 5.1.8.

The most likely value of the Poisson random variable. This is a good place to look at some of the properties of the Poisson distribution. Let us take the ratio between $P(k-1, t)$

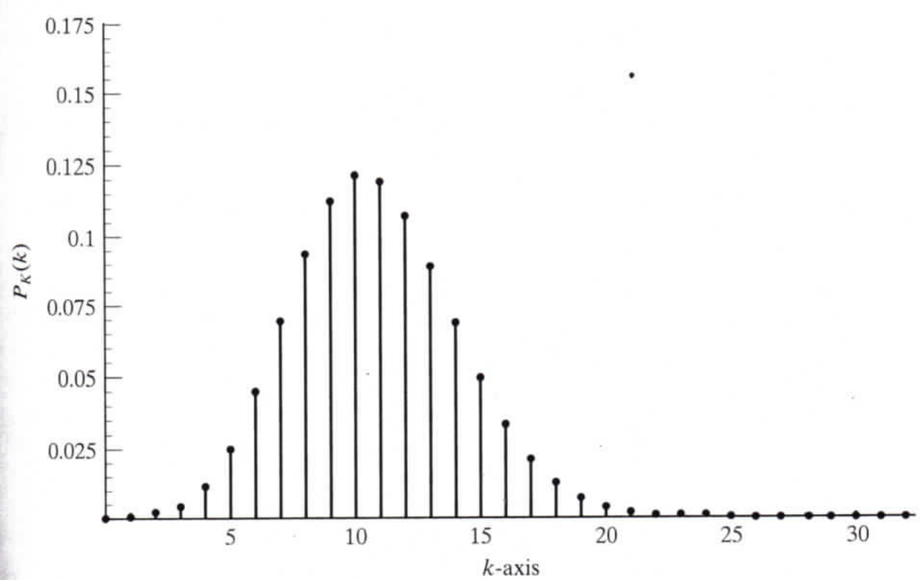

Figure 5.1.8 The Poisson distribution for $\lambda t = 10.8$. For large values of k the distribution approaches the Gaussian. If λt is an integer, there will always be two equally likely values of k, as explained in the text.

where we used Eq. (5.1.18) to make the final simplification. Two important conclusions follow from Eq. (5.1.19). The first follows from

$$\lambda = \frac{E[K]}{t} \tag{5.1.20}$$

which says that λ can be interpreted as the expected number of events in a period of time t divided by t. Thus λ is the average rate of events.

The second interpretation and conclusion is that λ and t always appear together and hence $\mu_K = \lambda t$ is a convenient argument for the Poisson distribution. We will therefore change the notation from "t" to "μ_K" in the Poisson distribution for K. Thus Eq. (5.1.12) can be rewritten in the form

$$P(k, \mu_K) = \frac{\mu_K^k}{k!} e^{-\mu_K} \tag{5.1.21}$$

This shows that the Poisson distribution is a one-parameter distribution: if one knows the expected number of events, then one knows the probability of 0, 1, 2, ... events. As you will see in the examples to follow, this feature of the Poisson distribution allows us to obtain many results from minimal information.

Equation (5.1.21) also shows the Poisson distribution without reference to λ or t or any underlying assumptions. In this guise, the Poisson random variable may be used as a model for any discrete random variable that takes on all integer values. To cite a far-out example, let us say we are interested in how many keys people carry. The chance experiment is to pick a person at random by some scheme, and the random variable is the number of keys that person is carrying. This random variable could well be approximately Poisson, but there is no obvious connection to the assumptions we made in deriving the Poisson.

The mean square value. The mean square value of the Poisson distribution can be readily calculated from the following identity that follows from the linearity of expectation:

$$E[K^2] = E[K(K-1)] + E[K] \tag{5.1.22}$$

The second term we determined in Eq. (5.1.19). The first term is similar to Eq. (5.1.19), except the first two terms of the sum vanish and the $k(k-1)$ in the numerator cancels the first two terms in the factorial. The result is

$$E[K^2] = \mu_K^2 + \mu_K \tag{5.1.23}$$

The variance of the Poisson random variable. The variance can be calculated as

$$Var[K] = E[K^2] - E^2[K] = \mu_K \tag{5.1.24}$$

where Eqs. (5.1.23) and (5.1.19) were used. Thus the mean and variance of the Poisson random variable are the same. This seems dimensionally wrong until one realizes that μ_K is dimensionless.

Summary. We derived a model for random events in time, such as raindrops hitting in a rain gauge. If we know the average rate, λ, and we have a period of time, t, then we may calculate the probability of k raindrops with Eq. (5.1.12). We may think of the number of raindrops as a random variable, K, described by a PMF, Eq. (5.1.13).

Conditions for using the Poisson distribution. In addition to the model explored for random events in time, we may think of the Poisson PMF as the distribution of a discrete random variable that satisfies certain criteria. The model is one-dimensional in that μ_K, the mean value, defines the entire distribution. The criteria for modeling with the Poisson distribution follow:

1. The outcomes must be integers in the range beginning with zero and having no upper limit. Thus the Poisson differs from the binomial distribution, for which an upper limit of events (successes) exists. In practice, an upper limit may exist but it should be much larger than the expected number of events; that is, $k_{max} \gg \mu_K$.
2. The random phenomena occur in a continuous medium. We used time as the medium for our derivations, but we might have used area, volume, or some other continuous medium.
3. The physical situation suggests events happening at random and independently in some continuous medium in which the rate or density of events is constant. We will illustrate this criterion in a series of examples that apply the Poisson distribution to situations where events are distributed in time, distance, area, and volume.

If these criteria are met, the Poisson should be a good model. The critical beginning is determining the expected number of events in the region, μ_K. From the mean we may calculate all the probabilities using Eq. (5.1.21). You will see how this approach works in the examples to follow.

5.1.3 Examples Using the Poisson Distribution

Example 5.1.1: Wrong numbers

As an example of random events in time, consider a family that gets, on average, three wrong-number calls per week. What is the probability they will receive exactly two wrong-number calls during a 4-day period?

Solution First, let us consider how the assumptions for the Poisson model are satisfied. The first two assumptions are clearly satisfied: a large number of calls could occur in a 2-day period, and time is a continuous medium. The third assumption is a bit shaky, since calls in the evening are more likely than in the early morning. Nevertheless, we will assume the third assumption is warranted and proceed.

We are given the expected number of calls per week, $E[K] = 3 = 7\lambda$, where time is measured in days; $\lambda = \frac{3}{7}$ calls/day. To calculate the probability of two wrong-number calls in a 4-day period, we need the expected number, which is

$$\mu_K = \frac{3 \text{ calls}}{7 \text{ days}} \times 4 \text{ days} = \frac{12}{7} = 1.714 \text{ calls} \tag{5.1.25}$$

Using Eq. (5.1.21), we calculate the probability of exactly two calls in 4 days to be

$$P(2, 1.714) = \frac{(1.714)^2}{2!} e^{-1.714} = 0.2646 \tag{5.1.26}$$

Example 5.1.2: Cars on a bridge

As an example of random events in distance, consider a highway that has, on average, 20 cars/mile. What is the probability that at any instant of time there will be 3 or more cars on a bridge that is 100 ft long.

Solution First, we must work out the expected number of cars on the bridge. This we can do by letting the units guide us:

$$\mu_K = \frac{20 \text{ cars}}{1 \text{ mile}} \times \frac{1 \text{ mile}}{5280 \text{ ft}} \times \frac{100 \text{ ft}}{1 \text{ bridge}} = 0.379 \frac{\text{car}}{\text{bridge}} \tag{5.1.27}$$

We assume cars are distributed in distance according to the Poisson model. This means, among other things, that the largest number of cars possible on the bridge would much exceed 0.379 car, which is reasonable. The probability of three or more cars on the bridge is best calculated by the following:

$$\begin{aligned} P(\geq 3, 0.379) &= 1 - P(< 3, 0.379) = 1 - [P(0, 0.379) + P(1, 0.379) + P(2, 0.379)] \\ &= 1 - e^{-0.379}\left(1 + 0.379 + \frac{(0.379)^2}{2!}\right) = 1 - 0.9932 = 0.006837 \end{aligned} \tag{5.1.28}$$

You do it. At a busy intersection, 22 accidents occur yearly on average. Assuming Poisson conditions are met, what is the probability of having 2 or more accidents in a given week?

```
myanswer = ?        ;
```

```
Evaluate
```

For the answer, see endnote 3.

Example 5.1.3: IC defects

This example involves random events in an area. In the manufacture of semiconductor integrated circuits (ICs), the process produces an average of 10 defects/wafer, and one wafer contains 20 ICs. Find the probability that an IC selected at random has no defects, and the expected number of good ICs/wafer.

Solution The expected number of defects is $\frac{10 \text{ defects}}{1 \text{ wafer}} \times \frac{1 \text{ wafer}}{20 \text{ ICs}} = 0.5 \frac{\text{defect}}{\text{IC}}$. From the Poisson distribution [Eq. (5.1.21)], the probability of zero defects on an IC is

$$P(0, 0.5) = e^{-0.5} = 0.607 \tag{5.1.29}$$

Thus an IC picked at random will have a probability of 0.607 of being without defect, or, alternatively, we expect 60.7% of the ICs to be good. That this is more than 50% follows because some of the ICs will have two or more defects if the distribution is truly random.

We now can consider the number of good ICs in the 20 that come from the wafer. For these, the conditions for binomial trials are met: independent results, same probability of success for each, and no concern for order. We may calculate the probability of 0, 1, . . . , 20 good ICs with the binomial distribution [Eq. (2.1.9)]:

$$P[k \text{ good ICs}] = B_{20}(k, 0.607) = \binom{20}{k}(0.607)^k(1-0.607)^{20-k} \tag{5.1.30}$$

Thus the expected number [Eq. (2.3.10)] of defect-free ICs on a wafer is

$$E[K] = np = 20 \times 0.607, \text{ or approximately 12/wafer} \tag{5.1.31}$$

Example 5.1.4: Raisins in a cake

This example involves random events in a volume. A cake recipe calls for 1 cup of raisins, which is approximately 360 raisins.[4] The cake is cut into 20 pieces. What is the probability that a piece will have at least 20 raisins?

Solution The average number of raisins/piece = 360/20 = 18 raisins/piece. The conditions for using the Poisson are reasonably met. We may therefore calculate the required probability as

$$P(\geq 20, 18) = \sum_{20}^{+\infty} P(k, 18) = \sum_{20}^{+\infty} \frac{(18)^k}{k!} e^{-18} \tag{5.1.32}$$

which we can use Mathematica to calculate:

```
Sum[PDF[PoissonDistribution[18], k], {k, 20, Infinity}];
N[%]
0.349084
```

If you are stuck on a desert island with only a set of math tables, you can perform the same calculation using the Gaussian as an approximation to the Poisson. We pointed out before that for large μ_K, the Poisson becomes bell-shaped. We can approximate the Poisson with a Gaussian distribution of the same mean and variance:

$$P(k, \mu_K) \approx N(\mu_K, \mu_K) \text{ for large } \mu_K \tag{5.1.33}$$

where we used [Eq. (5.1.24)], which gives $\sigma_K^2 = \mu_K$. The sum in Eq. (5.1.32) can be calculated as

$$\sum_{20}^{+\infty} P(k, 18) \approx \int_{19.5}^{+\infty} N(18, 18)\, dk = 1 - \Phi\left(\frac{19.5 - 18}{\sqrt{18}}\right) = 1 - \Phi(0.3536) = 0.362 \tag{5.1.34}$$

which compares favorably with the exact answer of 0.349, an error of 3.7%. Note that we had to use the continuity correction (see page 249), since half the probability associated with $k = 20$ lies between 19.5 and 20 in the Gaussian PDF.

You do it. A box of Raisin Bran contains 400 raisins. The box claims 10 servings per box. What is the probability of fewer than 36 raisins in a serving?

```
myanswer = ?        ;
```

```
Evaluate
```

For the answer, see endnote 5.

5.1.4 The Exponential PDF

Relationship between Poisson and binomial models. We now consider the exponential and the Erlang random variables, which are intimately related to the Poisson random variable. First, let us compare the relationship between the Poisson and binomial models. With the binomial, we have a discrete number of trials with independent events occurring or not occurring on each trial, with a constant probability of occurrence. If we let the number of trials go to infinity, and let the probability of success go to zero in such a way that the average number of successes remains the same, then we have the Poisson conditions. Thus there is a strong analogy between the Poisson and the binomial distributions, and under certain conditions we may approximate the computationally challenging binomial with the Poisson, which is relatively easy to calculate. This sort of approximation was attractive in precomputer times.[6]

Waiting-time distributions. Recall that the geometric distribution gives the probabilities of having to wait so many trials for the first success (event), and the Pascal distribution gives the probabilities of having to wait so many trials for the kth success (event). In the Poisson model, we have the probability of k events in a certain period of "time". The corresponding waiting-time distributions are the exponential, for the first event, and the *Erlang distribution,* for the kth event, as shown in Fig. 5.1.9.

Derivation of the exponential PDF. We now derive the PDF of the exponential random variable from the Poisson distribution. We begin with the CDF of the exponential random variable, which is defined as

$$F_T(t) = P[T \leq t] = 1 - P[T > t] \tag{5.1.35}$$

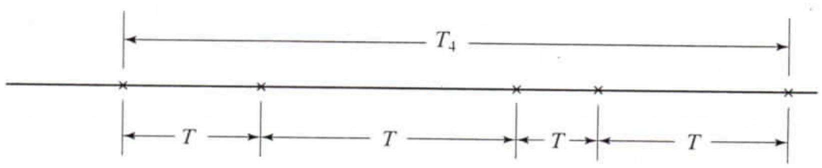

Figure 5.1.9 The random variable T_k is the waiting time for the kth event. Here we show the definitions for $k = 1$ and 4, the first events and the fourth event. We drop the subscript on $T_1 = T$ for notational simplicity. For Poisson conditions, this figure depicts the exponential random variable T and the Erlang random variable of order 4, T_4.

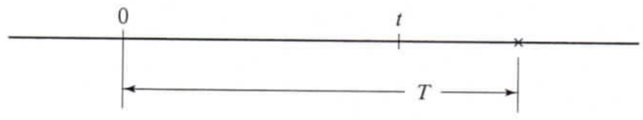

Figure 5.1.10 The event $\{T > t\}$ is equivalent to the event {zero events between 0 and t}, which is given by the Poisson distribution.

The second form is readily related to the Poisson distribution. The event $\{T > t\}$ is that pictured in Fig. 5.1.10, and is equal to the event $\{0 \text{ events in } 0 \to t\}$.

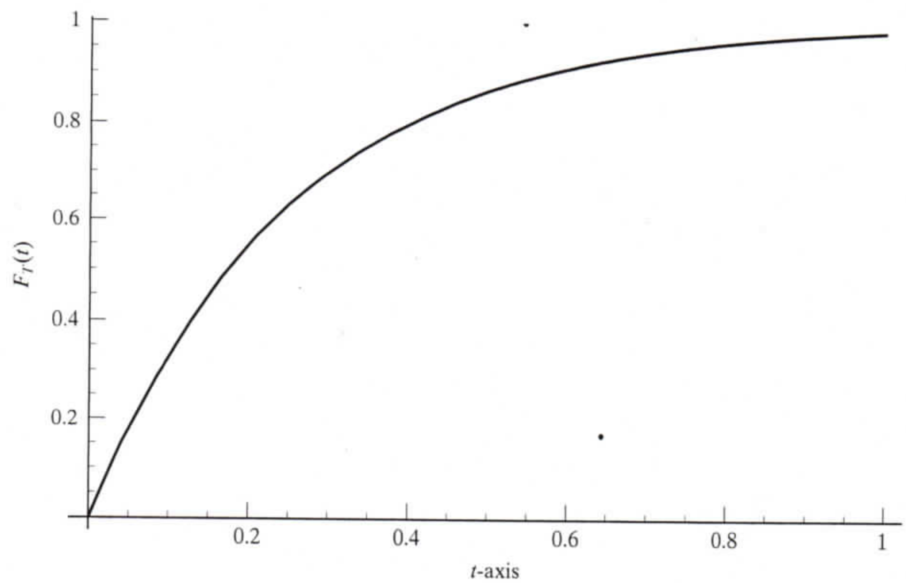

Figure 5.1.11 The CDF for T, the time to the first event in a Poisson process. $F_T(t)$ begins at zero because it takes some time for the first event to happen, and $F_T(t)$ approaches 1 asymptotically because the first event must eventually happen.

Thus we can rewrite Eq. (5.1.35) in the form

$$F_T(t) = 1 - P(0, \lambda t) = 1 - e^{-\lambda t}, t \geq 0, \text{ zow} \tag{5.1.36}$$

This CDF is pictured in Fig. 5.1.11 for $\lambda = 4$.

The PDF of T is given by the derivative of the CDF:

$$f_T(t) = \frac{d}{dt} F_T(t) = \lambda e^{-\lambda t},\ t \geq 0, \text{ zow} \tag{5.1.37}$$

where λ is the average rate of events. This PDF is shown in Fig. 5.1.12 for $\lambda = 4$.

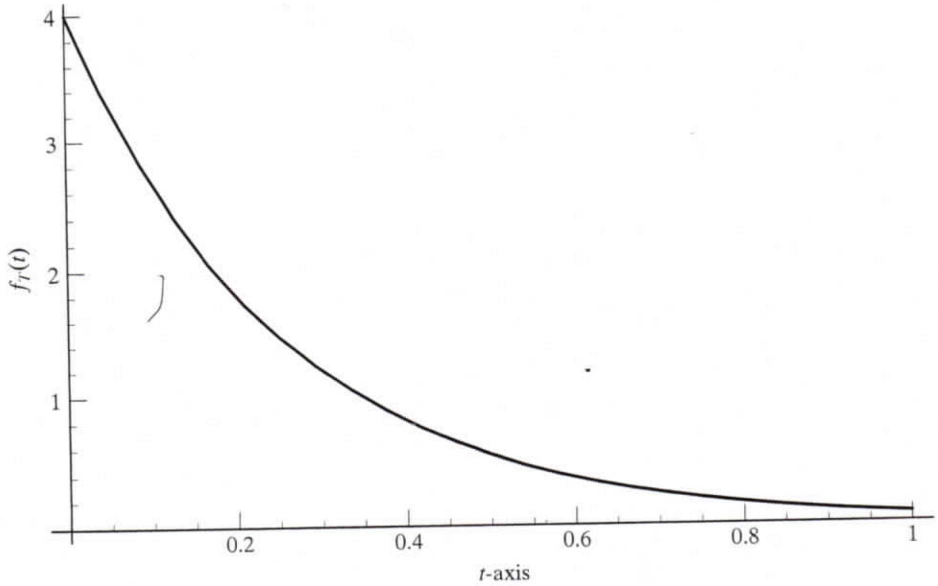

Figure 5.1.12 The PDF for T, the time to the first event in a Poisson process. Note that the most likely time for an event is "now." The time scale is determined by the average rate, λ, and the expectation (average) time to the first event is $\frac{1}{\lambda}$. For this plot, $\lambda = 4$.

Properties of the exponential random variable. The mean, mean square value, and variance were derived in Sec. 3.1, where we introduced [Eq. (3.1.22)] the exponential PDF as an example, without derivation. We repeat the values here:

$$\mu_T = E[T] = \frac{1}{\lambda}, \qquad E[T^2] = \frac{2}{\lambda^2}, \qquad \text{and} \qquad Var[T] = \frac{1}{\lambda^2} \tag{5.1.38}$$

Thus the mean and the standard deviation of the exponential random variable are equal.

No memory. The most striking property of the exponential random variable is that it, like the geometric distribution [Eq. (2.1.21)], has no memory, that it "begins again" at each instant of time. This is implied by the Poisson assumption of independent events, but it still is interesting to see how it works out in the mathematics. We can show this lack of memory by calculating the conditional PDF, with the condition that no event takes place between 0 and some time, call it t_0, as pictured in Fig. 5.1.13.

We begin with the conditional CDF:

$$F_{T\,|\,T>t_0}(t) = P[T \le t \,|\, T > t_0] = \frac{P[(T \le t) \cap (T > t_0)]}{P[T > t_0]} \tag{5.1.39}$$

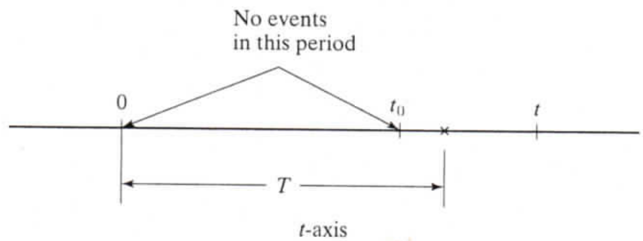

Figure 5.1.13 We are going to calculate the conditional probability of T, the time to the first event, with the condition that no event occurs in the period 0 to t_0, as shown. The result will show that the exponential random variable has no memory.

The numerator is equivalent to the probability of the event $\{t_0 < T \le t\}$, which we may calculate from the unconditional PDF of T:

$$P[t_0 < T \le t] = \int_{t_0}^{t} f_T(t)\,dt = \int_{t_0}^{t} \lambda e^{-\lambda t}\,dt = e^{-\lambda t_0} - e^{-\lambda t},\ t > t_0 \tag{5.1.40}$$

and the denominator of Eq. (5.1.39) we also can calculate from the unconditioned PDF of T:

$$P[T > t_0] = \int_{t_0}^{\infty} f_T(t)\,dt = \int_{t_0}^{\infty} \lambda e^{-\lambda t}\,dt = e^{-\lambda t_0} \tag{5.1.41}$$

Thus Eq. (5.1.39) reduces to

$$F_{T\,|\,T>t_0}(t) = \frac{e^{-\lambda t_0} - e^{-\lambda t}}{e^{-\lambda t_0}} = 1 - e^{-\lambda(t-t_0)},\ t > t_0,\ \text{zow} \tag{5.1.42}$$

which is shown in Fig. 5.1.14 for $\lambda = 4$.

We see that nothing is changed except the delay. The conditional PDF is the derivative of the conditional CDF, and will show no change except the delay.

$$f_{T\,|\,T>t_0}(t) = \frac{d}{dt}(1 - e^{-\lambda(t-t_0)}) = \lambda e^{-\lambda(t-t_0)},\ t > t_0,\ \text{zow} \tag{5.1.43}$$

which is shown in Fig. 5.1.15 for $\lambda = 4$.

The conditional PDF is merely delayed by t_0; there is no other difference between Figs. 5.1.12 and 5.1.15. This means that the past does not influence the future. This property of the exponential random variable is somewhat counterintuitive because we are accustomed to scheduled events. A city bus, for example, runs on a schedule, so the longer you wait for a bus, the more likely it becomes that the bus will appear. But many events are unscheduled, such as radioactive decay, supernova explosions, and auto accidents. For such unscheduled events, the exponential random variable may well model the times between events.

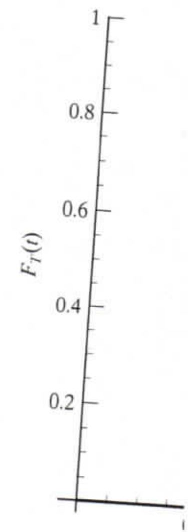

Figure 5.1.14 The conditi
that no event takes place
delay, which shows the lac

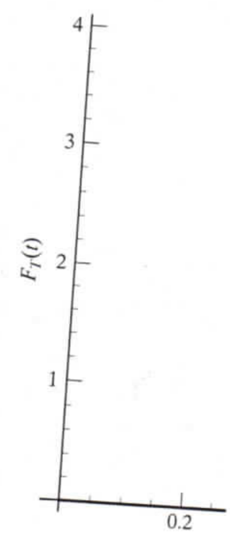

Figure 5.1.15 The conditiona
that no event occurs between
unconditioned PDF in Fig. 5.1.
random variable has no memo
when events have occurred in

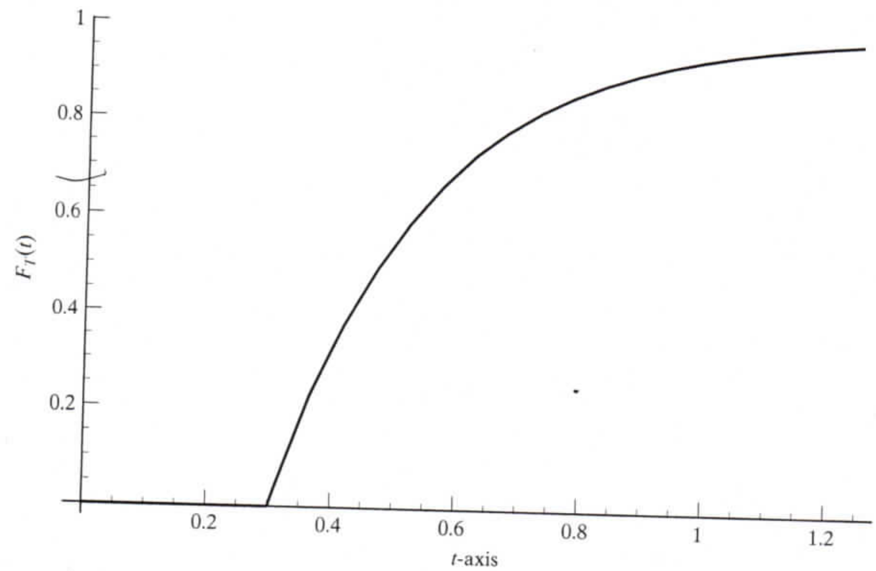

Figure 5.1.14 The conditional CDF of the exponential random variable, the condition being that no event takes place between 0 and $t_0 = 0.3$. This is identical to Fig. 5.1.11, except for the delay, which shows the lack of memory in the exponential random variable.

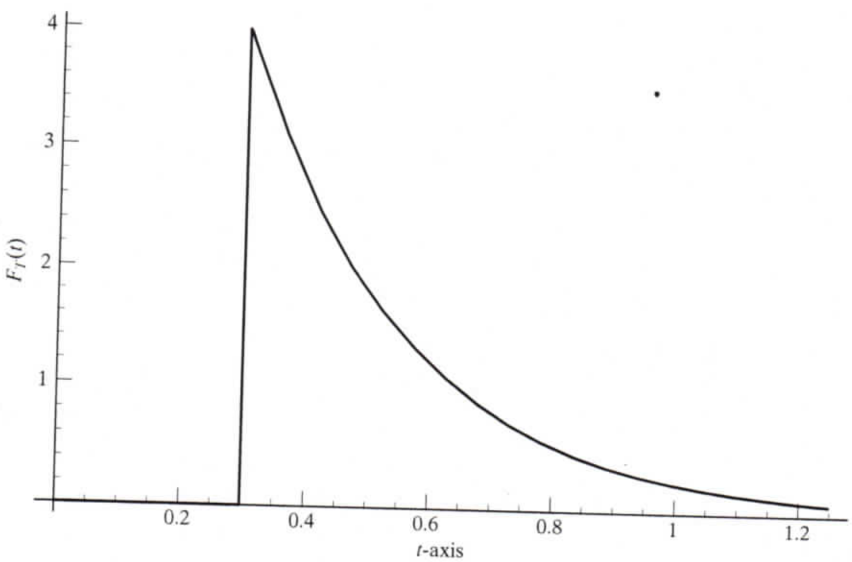

Figure 5.1.15 The conditional PDF of the exponential random variable, the condition being that no event occurs between 0 and $t_0 = 0.3$. Note that this graph is identical with the unconditioned PDF in Fig. 5.1.12, except for the delay. This tells us that the exponential random variable has no memory, that it begins afresh at each instant of time, regardless of when events have occurred in the past.

Example 5.1.5: Entering a busy street

Consider entering from a side street onto a busy thoroughfare. There are 40 cars/minute passing your side street, and it will take you 3 seconds to enter the stream of traffic safely. What is the probability that you will be able to enter without waiting for a break in the traffic?

Solution The rate of cars is

$$\lambda = 40\frac{\text{cars}}{\text{minute}} \times \frac{1\text{ minute}}{60\text{ seconds}} = \frac{2}{3}\frac{\text{cars}}{\text{second}} \tag{5.1.44}$$

We assume the cars are randomly distributed in time according to the Poisson assumptions, so the time to the first car is exponentially distributed with $\lambda = \frac{2}{3}$. The probability that the first car will reach the intersection 3 or more seconds after you arrive is

$$P[T > 3] = \int_3^\infty f_T(t)\,dt = \int_3^\infty \tfrac{2}{3}e^{-2t/3}\,dt = e^{-2} = 0.135 \tag{5.1.45}$$

You do it. At a busy intersection, 22 accidents occur yearly on the average. Assuming Poisson conditions are met, what is the probability that the next accident will occur in the next 6 days?

```
myanswer = ?        ;
```

```
Evaluate
```

For the answer, see endnote 7.

5.1.5 The Erlang Distribution

As the exponential random variable is the analog in continuous space to the geometric random variable for binomial trials, so the Erlang random variable is analogous to the Pascal. The Erlang random variable, T_k, is the waiting time from $t = 0$ (or any arbitrary time) to the kth event, as is shown for $k = 4$ in Fig. 5.1.16.

Thus the Erlang random variable of order k is the sum of k independent exponential random variables. We may derive the PDF of the Erlang random variable three ways. We will show you two of the three and merely mention the third.

Convolution. The method we only mention is convolution. Because the Erlang is the sum of independent exponential random variables, we may derive the PDF of the Erlang by multiple convolution of the exponential PDF. Although this method sounds horrible, it works out pretty well. But there is nothing new to be learned from this, so we will not do it here.

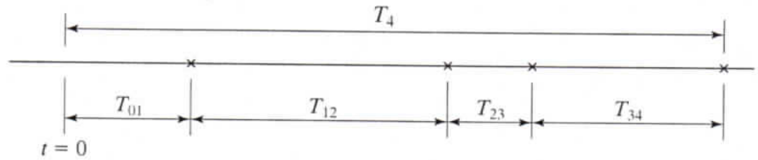

Figure 5.1.16 The time to the fourth event is the sum of the times between the origin and the first event, the first and the second, the second and the third, and the third and the fourth. Whether an event occurs at the origin does not matter, since the Poisson process has no memory.

Direct derivation. By definition [Eq. (3.1.2)] the PDF of T_k is

$$f_{T_k}(t)\,dt = P[t < T_k \le t + dt] \tag{5.1.46}$$

The event on the right can be expressed as the intersection of two independent events:

$$P[t < T_k \le t + dt] = P[(k - 1 \text{ events in } t) \cap (1 \text{ event in } dt)] \tag{5.1.47}$$

We now apply the Poisson assumptions (see page 456). As just stated, the two events on the right are independent because the two regions do not overlap (first assumption). Thus we can change $\cap \to \times$.

$$P[t < T_k \le t + dt] = \underbrace{P[k - 1 \text{ events in } t]}_{P(k-1,\,\lambda t)} \times \underbrace{P[1 \text{ event in } dt]}_{\lambda dt} \tag{5.1.48}$$

The first term is the Poisson probability of $k - 1$ events in time t, and the second term, by the third Poisson assumption, is λdt. Substituting Eq. (5.1.48) back into Eq. (5.1.46), canceling the dt's, and substituting Eq. (5.1.12), we have the kth-order Erlang PDF:

$$f_{T_k}(t) = \frac{(\lambda t)^{k-1}}{(k-1)!}e^{-\lambda t} \times \lambda = \frac{\lambda^k t^{k-1}}{(k-1)!}e^{-\lambda t},\ t \ge 0,\ \text{zow} \tag{5.1.49}$$

Figure 5.1.17 shows the Erlang PDF for $\lambda = 4$ and $k = 10$.

The Erlang CDF. The third way to derive the Erlang PDF is to start with the CDF. Usually, we start with the CDF in deriving the PDF, but you will see why we do not favor this approach here, although it does work. The CDF is by definition

$$F_{T_k}(t) = P[T_k \le t] = 1 - P[T_k > t] \tag{5.1.50}$$

But the event $\{T_k > t\}$ can be partitioned into the events $\{0 \text{ events in } t\} \cup \{1 \text{ event in } t\} \cup \cdots \cup \{k-1 \text{ events in } t\}$. These events are disjoint, so we may add their probabilities, with the result

$$P[T_k > t] = e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{k-1}}{(k-1)!}\right] \tag{5.1.51}$$

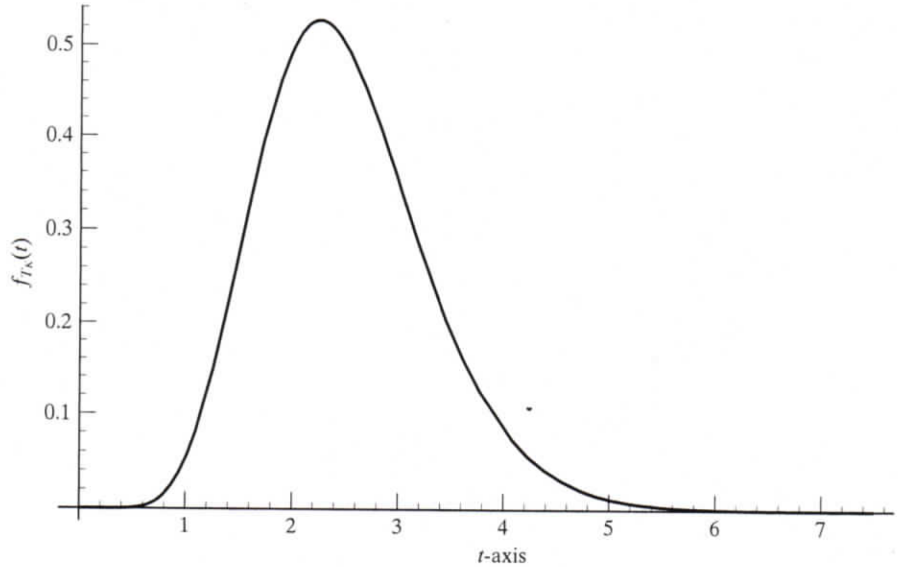

Figure 5.1.17 The PDF of the Erlang random variable of order $k = 10$. The expected value is $\frac{k}{\lambda}$, and the peak value of the PDF occurs at $\frac{k-1}{\lambda}$. The Erlang is the sum of k IID exponential random variables. For this plot, $\lambda = 4$.

and therefore the CDF of the Erlang random variable is

$$F_{T_k}(t) = 1 - P[T_k > t] = 1 - e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^{k-1}}{(k-1)!}\right],\ t \geq 0,\ \text{zow} \quad (5.1.52)$$

The derivative of the CDF in Eq. (5.1.52) is the PDF in Eq. (5.1.49), since all the terms in the derivative cancel out except the last. Likewise, the integral of Eq. (5.1.49) from 0 to t is Eq. (5.1.52), after you integrate by parts $k - 1$ times. We think you will agree that it is easier to derive the PDF and the CDF from basics rather than from each other.

Properties of the Erlang random variable. We may derive the mean and variance of the Erlang without doing any calculus. Because the Erlang is the sum of k IID exponential [Eq. (5.1.38)] random variables, it follows from the algebra of expectation that

$$E[T_k] = kE[T] = \frac{k}{\lambda} \quad (5.1.53)$$

Similarly, the variance is

$$Var[T_k] = k\ Var[T] = \frac{k}{\lambda^2} \quad (5.1.54)$$

Because the Erlang is the sum of k exponential random variables, for large k the central limit theorem suggests that the Erlang should approach the Gaussian. Using Eqs. (5.1.53) and (5.1.54),

we may express this as

which you may find use

Example 5.1.6: Waiti
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one being served, and it
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to drive away is the sum
variable of order six. Wh
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with $k = 6$ and $\lambda = \frac{1}{2}$. It i
distribution. We know tha
Poisson, exponential, and
(see page 456). From the (

$P[T_6 > 12] = 1 -$

This is readily interpreted a
minutes, but not 6 or more
and driven away.

You do it. The situat
Calculate the probability tha
places in the cell box, and c

```
myanswer = ?
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Evaluate
```

For the answer, see endnote

we may express this as

$$f_{T_k}(t) \approx N\left(\frac{k}{\lambda}, \frac{k}{\lambda^2}\right), \quad \text{for large } k \tag{5.1.55}$$

which you may find useful if stranded on a desert isle with some math tables.

Example 5.1.6: Waiting at the bank

You are waiting in line at the drive-in window at your bank. You are sixth in line, counting the one being served, and it takes, on average, 2 minutes to serve each customer. The service time for each customer is a random variable, which we will assume to be exponential.[8] Thus your time to drive away is the sum of the six exponential random variables, which is the Erlang random variable of order six. What is the probability that you will drive away more than 12 minutes from the present?

Solution From the expected (average) value of the wait time and Eq. (5.1.38), we derive that $\lambda = \frac{1}{2}$ car/minute. In terms of the Erlang PDF, the required probability is

$$P[T_6 > 12] = \int_{12}^{\infty} \frac{\lambda^k t^{k-1}}{(k-1)!} e^{-\lambda t}\, dt \tag{5.1.56}$$

with $k = 6$ and $\lambda = \frac{1}{2}$. It is better to use the CDF or to simply reason on the basis of the Poisson distribution. We know that the Poisson applies because of the intimate relationship between the Poisson, exponential, and Erlang distributions; that is, they all arise from the same assumptions (see page 456). From the CDF, Eq. (5.1.52), and $\lambda t = \frac{1}{2} \times 12 = 6$, we have

$$\begin{aligned} P[T_6 > 12] = 1 - F_{T_6}(12) &= e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \cdots + \frac{(\lambda t)^5}{5!}\right] \\ &= e^{-6}\left[1 + \frac{6}{1!} + \frac{(6)^2}{2!} + \cdots + \frac{(6)^5}{5!}\right] = 0.446 \end{aligned} \tag{5.1.57}$$

This is readily interpreted as the probability that 0, 1, 2, 3, 4, and 5 events take place in the 12 minutes, but not 6 or more, for then you, the sixth event, would have completed your business and driven away.

You do it. The situation is the same, but you are third and it takes 4 minutes per customer. Calculate the probability that you will be gone in 12 minutes, enter your answer to at least three places in the cell box, and click Evaluate for a response.

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myanswer = ?     ;
```

```
Evaluate
```

For the answer, see endnote 9.

we may express this as

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