4.8 Conditioning by an Event

An experiment produces two random variables, X and Y. We learn that the outcome (x, y) is an element of an event, B. We use the information $(x, y) \in B$ to construct a new probability model. If X and Y are discrete, the new model is a conditional joint PMF, the ratio of the joint PMF to $P[B]$. If X and Y are continuous, the new model is a conditional joint PDF, defined as the ratio of the joint PDF to $P[B]$. The definitions of these functions follow from the same intuition as Definition 1.6 for the conditional probability of an event. Section 4.9 considers the special case of an event that corresponds to an observation of one of the two random variables: either $B = \{X = x\}$, or $B = \{Y = y\}$.

efinition 4.9 **Conditional Joint PMF**

For discrete random variables X and Y and an event, B with $P[B] > 0$, the ***conditional joint PMF*** *of X and Y given B is*

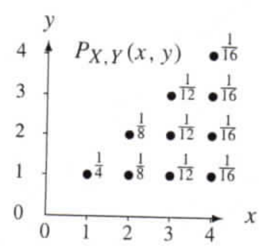
$$P_{X,Y|B}(x, y) = P[X = x, Y = y|B].$$

The following theorem is an immediate consequence of the definition.

ieorem 4.19 *For any event B, a region of the X, Y plane with $P[B] > 0$,*

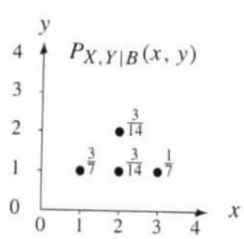
$$P_{X,Y|B}(x, y) = \begin{cases} \dfrac{P_{X,Y}(x, y)}{P[B]} & (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.13



Random variables X and Y have the joint PMF $P_{X,Y}(x, y)$ as shown. Let B denote the event $X + Y \le 4$. Find the conditional PMF of X and Y given B.

. . . .



Event $B = \{(1, 1), (2, 1), (2, 2), (3, 1)\}$ consists of all points (x, y) such that $x + y \le 4$. By adding up the probabilities of all outcomes in B, we find

$$P[B] = P_{X,Y}(1, 1) + P_{X,Y}(2, 1) + P_{X,Y}(2, 2) + P_{X,Y}(3, 1) = \frac{7}{12}.$$

The conditional PMF $P_{X,Y|B}(x, y)$ is shown on the left.

In the case of two continuous random variables, we have the following definition of the conditional probability model.

Definition 4.10 ***Conditional Joint PDF***
Given an event B with $P[B] > 0$, the conditional joint probability density function of X and Y is

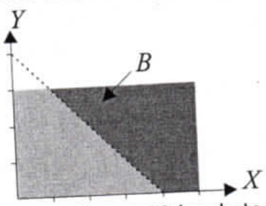
$$f_{X,Y|B}(x, y) = \begin{cases} \dfrac{f_{X,Y}(x, y)}{P[B]} & (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.14 X and Y are random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} 1/15 & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases} \tag{4.83}$$

Find the conditional PDF of X and Y given the event $B = \{X + Y \ge 4\}$.

We calculate $P[B]$ by integrating $f_{X,Y}(x, y)$ over the region B.



$$P[B] = \int_0^3 \int_{4-y}^5 \frac{1}{15}\, dx\, dy \tag{4.84}$$

$$= \frac{1}{15} \int_0^3 (1 + y)\, dy \tag{4.85}$$

$$= 1/2. \tag{4.86}$$

Definition 4.10 leads to the conditional joint PDF

$$f_{X,Y|B}(x, y) = \begin{cases} 2/15 & 0 \le x \le 5, 0 \le y \le 3, x + y \ge 4, \\ 0 & \text{otherwise.} \end{cases} \tag{4.87}$$

Corresponding to Theorem 4.12, we have

Theorem 4.20 ***Conditional Expected Value***
For random variables X and Y and an event B of nonzero probability, the conditional expected value of $W = g(X, Y)$ given B is

$$\textit{Discrete:} \quad E[W|B] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y|B}(x, y),$$

$$\textit{Continuous:} \quad E[W|B] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y|B}(x, y)\, dx\, dy.$$

Another notation for conditional expected value is $\mu_{W|B}$.

Definitio

Theorem

Exa

Exa

n 4.11 ***Conditional variance***

*The **conditional variance** of the random variable $W = g(X, Y)$ is*

$$\text{Var}[W|B] = E\left[\left(W - \mu_{W|B}\right)^2 |B\right].$$

Another notation for conditional variance is $\sigma^2_{W|B}$. The following formula is a convenient computational shortcut.

4.21

$$\text{Var}[W|B] = E\left[W^2|B\right] - (\mu_{W|B})^2.$$

nple 4.15 Continuing Example 4.13, find the conditional expected value and the conditional variance of $W = X + Y$ given the event $B = \{X + Y \leq 4\}$.

We recall from Example 4.13 that $P_{X,Y|B}(x, y)$ has four points with nonzero probability: $(1, 1)$, $(1, 2)$, $(1, 3)$, and $(2, 2)$. Their probabilities are $3/7$, $3/14$, $1/7$, and $3/14$, respectively. Therefore,

$$E[W|B] = \sum_{x,y} (x + y) P_{X,Y|B}(x, y) \tag{4.88}$$

$$= 2\frac{3}{7} + 3\frac{3}{14} + 4\frac{1}{7} + 4\frac{3}{14} = \frac{41}{14}. \tag{4.89}$$

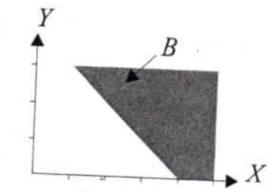
Similarly,

$$E\left[W^2|B\right] = \sum_{x,y} (x + y)^2 P_{X,Y|B}(x, y) \tag{4.90}$$

$$= 2^2\frac{3}{7} + 3^2\frac{3}{14} + 4^2\frac{1}{7} + 4^2\frac{3}{14} = \frac{131}{14}. \tag{4.91}$$

The conditional variance is $\text{Var}[W|B] = E[W^2|B] - (E[W|B])^2 = (131/14) - (41/14)^2 = 153/196$.

le 4.16 Continuing Example 4.14, find the conditional expected value of $W = XY$ given the event $B = \{X + Y \geq 4\}$.



For the event B shown in the adjacent graph, Example 4.14 showed that the conditional PDF of X, Y given B is

$$f_{X,Y|B}(x, y) = \begin{cases} 2/15 & 0 \leq x \leq 5, 0 \leq y \leq 3, (x, y) \in B, \\ 0 & \text{otherwise.} \end{cases}$$

d G in Quiz 4.2

$g = 2$	$g = 3$
0.24	0.12
0.16	0.08

pendent?

outed with prob-

(4.144)

$Z = \max(X_1, X_2)$.

with the property

neters $\mu_1, \sigma_1, \mu_2,$

$\rho < 1$.

$\sigma_1 = \sigma_2 = 1$, and
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hen $\rho = -0.9$ there
as $\rho \to \pm 1$.
PDF, we define

(4.145)

From Theorem 4.20,

$$E[XY|B] = \int_0^3 \int_{4-y}^5 \frac{2}{15} xy\,dx\,dy \tag{4.92}$$

$$= \frac{1}{15}\int_0^3 \left(x^2\Big|_{4-y}^5 \right) y\,dy \tag{4.93}$$

$$= \frac{1}{15}\int_0^3 \left(9y + 8y^2 - y^3\right) dy = \frac{123}{20}. \tag{4.94}$$

Quiz 4.8

(A) From Example 4.8, random variables L and T have joint PMF

$P_{L,T}(l,t)$	$t = 40$ *sec*	$t = 60$ *sec*
$l = 1$ *page*	0.15	0.1
$l = 2$ *pages*	0.3	0.2
$l = 3$ *pages*	0.15	0.1

(4.95)

For random variable $V = LT$, *we define the event* $A = \{V > 80\}$. *Find the conditional PMF* $P_{L,T|A}(l,t)$ *of L and T given A. What are* $E[V|A]$ *and* $\text{Var}[V|A]$?

(B) Random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} xy/4000 & 1 \le x \le 3, 40 \le y \le 60, \\ 0 & \text{otherwise.} \end{cases} \tag{4.96}$$

For random variable $W = XY$, *we define the event* $B = \{W > 80\}$. *Find the conditional joint PDF* $f_{X,Y|B}(l,t)$ *of X and Y given B. What are* $E[W|B]$ *and* $\text{Var}[W|B]$?

4.9 Conditioning by a Random Variable

In Section 4.8, we use the partial knowledge that the outcome of an experiment $(x, y) \in B$ in order to derive a new probability model for the experiment. Now we turn our attention to the special case in which the partial knowledge consists of the value of one of the random variables: either $B = \{X = x\}$ or $B = \{Y = y\}$. Learning $\{Y = y\}$ changes our knowledge of random variables X, Y. We now have complete knowledge of Y and modified knowledge of X. From this information, we derive a modified probability model for X. The new model is either a *conditional PMF of X given Y* or a *conditional PDF of X given Y*. When X and Y are discrete, the conditional PMF and associated expected values represent a specialized notation for their counterparts, $P_{X,Y|B}(x,y)$ and $E[g(X,Y)|B]$ in Section 4.8. By contrast, when X and Y are continuous, we cannot apply Section 4.8 directly because $P[B] = P[Y = y] = 0$ as discussed in Chapter 3. Instead, we define a conditional PDF as the ratio of the joint PDF to the marginal PDF.

Definition 4.12 ***Conditional PMF***

For any event $Y = y$ such that $P_Y(y) > 0$, the **conditional PMF** *of X given $Y = y$ is*

$$P_{X|Y}(x|y) = P[X = x|Y = y].$$

The following theorem contains the relationship between the joint PMF of X and Y and the two conditional PMFs, $P_{X|Y}(x|y)$ and $P_{Y|X}(y|x)$.

Theorem 4.22 *For random variables X and Y with joint PMF $P_{X,Y}(x, y)$, and x and y such that $P_X(x) > 0$ and $P_Y(y) > 0$,*

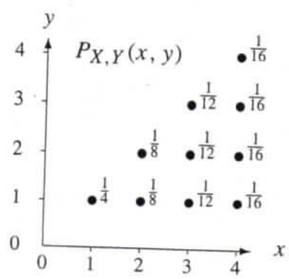
$$P_{X,Y}(x, y) = P_{X|Y}(x|y)\, P_Y(y) = P_{Y|X}(y|x)\, P_X(x).$$

Proof Referring to Definition 4.12, Definition 1.6, and Theorem 4.3, we observe that

$$P_{X|Y}(x|y) = P[X = x|Y = y] = \frac{P[X = x, Y = y]}{P[Y = y]} = \frac{P_{X,Y}(x, y)}{P_Y(y)}. \tag{4.97}$$

Hence, $P_{X,Y}(x, y) = P_{X|Y}(x|y)P_Y(y)$. The proof of the second part is the same with X and Y reversed.

Example 4.17



Random variables X and Y have the joint PMF $P_{X,Y}(x, y)$, as given in Example 4.13 and repeated in the accompanying graph. Find the conditional PMF of Y given $X = x$ for each $x \in S_X$.

....

To apply Theorem 4.22, we first find the marginal PMF $P_X(x)$. By Theorem 4.3, $P_X(x) = \sum_{y \in S_Y} P_{X,Y}(x, y)$. For a given $X = x$, we sum the nonzero probablities along the vertical line $X = x$. That is,

$$P_X(x) = \begin{cases} 1/4 & x = 1, \\ 1/8 + 1/8 & x = 2, \\ 1/12 + 1/12 + 1/12 & x = 3, \\ 1/16 + 1/16 + 1/16 + 1/16 & x = 4, \\ 0 & \text{otherwise}, \end{cases} = \begin{cases} 1/4 & x = 1, \\ 1/4 & x = 2, \\ 1/4 & x = 3, \\ 1/4 & x = 4, \\ 0 & \text{otherwise}. \end{cases}$$

Theorem 4.22 implies that for $x \in \{1, 2, 3, 4\}$,

$$P_{Y|X}(y|x) = \frac{P_{X,Y}(x, y)}{P_X(x)} = 4P_{X,Y}(x, y). \tag{4.98}$$

For each $x \in \{1,2,3,4\}$, $P_{Y|X}(y|x)$ is a different PMF.

$$P_{Y|X}(y|1) = \begin{cases} 1 & y = 1, \\ 0 & \text{otherwise.} \end{cases} \qquad P_{Y|X}(y|2) = \begin{cases} 1/2 & y \in \{1,2\}, \\ 0 & \text{otherwise.} \end{cases}$$

$$P_{Y|X}(y|3) = \begin{cases} 1/3 & y \in \{1,2,3\}, \\ 0 & \text{otherwise.} \end{cases} \qquad P_{Y|X}(y|4) = \begin{cases} 1/4 & y \in \{1,2,3,4\}, \\ 0 & \text{otherwise.} \end{cases}$$

Given $X = x$, the conditional PMF of Y is the discrete uniform $(1, x)$ random variable.

For each $y \in S_Y$, the conditional probability mass function of X, gives us a new probability model of X. We can use this model in any way that we use $P_X(x)$, the model we have in the absence of knowledge of Y. Most important, we can find expected values with respect to $P_{X|Y}(x|y)$ just as we do in Chapter 2 with respect to $P_X(x)$.

Theorem 4.23 ***Conditional Expected Value of a Function***

X and Y are discrete random variables. For any $y \in S_Y$, the conditional expected value of $g(X, Y)$ given $Y = y$ is

$$E[g(X,Y)|Y = y] = \sum_{x \in S_X} g(x,y) P_{X|Y}(x|y).$$

The conditional expected value of X given $Y = y$ is a special case of Theorem 4.23:

$$E[X|Y = y] = \sum_{x \in S_X} x P_{X|Y}(x|y). \tag{4.99}$$

Theorem 4.22 shows how to obtain the conditional PMF given the joint PMF, $P_{X,Y}(x, y)$. In many practical situations, including the next example, we first obtain information about marginal and conditional probabilities. We can then use that information to build the complete model.

Example 4.18 In Example 4.17, we derived the following conditional PMFs: $P_{Y|X}(y|1)$, $P_{Y|X}(y|2)$, $P_{Y|X}(y|3)$, and $P_{Y|X}(y|4)$. Find $E[Y|X = x]$ for $x = 1,2,3,4$.

Applying Theorem 4.23 with $g(x, y) = x$, we calculate

$$E[Y|X = 1] = 1, \qquad E[Y|X = 2] = 1.5, \tag{4.100}$$

$$E[Y|X = 3] = 2, \qquad E[Y|X = 4] = 2.5. \tag{4.101}$$

Now we consider the case in which X and Y are continuous random variables. We observe $\{Y = y\}$ and define the PDF of X given $\{Y = y\}$. We cannot use $B = \{Y = y\}$ in Definition 4.10 because $P[Y = y] = 0$. Instead, we define a *conditional probability density function*, denoted as $f_{X|Y}(x|y)$.

Definition 4.13 ***Conditional PDF***

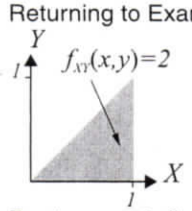
For y such that $f_Y(y) > 0$, the conditional PDF of X given $\{Y = y\}$ is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)}.$$

Definition 4.13 implies

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}. \tag{4.102}$$

Example 4.19 Returning to Example 4.5, random variables X and Y have joint PDF



$$f_{X,Y}(x, y) = \begin{cases} 2 & 0 \le y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.103}$$

For $0 \le x \le 1$, find the conditional PDF $f_{Y|X}(y|x)$. For $0 \le y \le 1$, find the conditional PDF $f_{X|Y}(x|y)$.

For $0 \le x \le 1$, Theorem 4.8 implies

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dy = \int_0^x 2\, dy = 2x. \tag{4.104}$$

The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \begin{cases} 1/x & 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.105}$$

Given $X = x$, we see that Y is the uniform $(0, x)$ random variable. For $0 \le y \le 1$, Theorem 4.8 implies

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x, y)\, dx = \int_y^1 2\, dx = 2(1 - y). \tag{4.106}$$

Furthermore, Equation (4.102) implies

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} 1/(1 - y) & y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.107}$$

Conditioned on $Y = y$, we see that X is the uniform $(y, 1)$ random variable.

We can include both expressions for conditional PDFs in the following formulas.

Theorem 4.24

$$f_{X,Y}(x, y) = f_{Y|X}(y|x)\, f_X(x) = f_{X|Y}(x|y)\, f_Y(y).$$

For each y with $f_Y(y) > 0$, the conditional PDF $f_{X|Y}(x|y)$ gives us a new probability model of X. We can use this model in any way that we use $f_X(x)$, the model we have in the absence of knowledge of Y. Most important, we can find expected values with respect to $f_{X|Y}(x|y)$ just as we do in Chapter 3 with respect to $f_X(x)$. More generally, we define the conditional expected value of a function of the random variable X.

Definition 4.14 ***Conditional Expected Value of a Function***

*For continuous random variables X and Y and any y such that $f_Y(y) > 0$, the **conditional expected value** of $g(X, Y)$ given $Y = y$ is*

$$E[g(X,Y)|Y=y] = \int_{-\infty}^{\infty} g(x,y) f_{X|Y}(x|y)\, dx.$$

The conditional expected value of X given $Y = y$ is a special case of Definition 4.14:

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx. \tag{4.108}$$

When we introduced the concept of expected value in Chapters 2 and 3, we observed that $E[X]$ is a number derived from the probability model of X. This is also true for $E[X|B]$. The conditional expected value given an event is a number derived from the conditional probability model. The situation is more complex when we consider $E[X|Y=y]$, the conditional expected value given a random variable. In this case, the conditional expected value is a different number for each possible observation $y \in S_Y$. Therefore, $E[X|Y=y]$ is a deterministic function of the observation y. This implies that when we perform an experiment and observe $Y = y$, $E[X|Y=y]$ is a function of the random variable Y. We use the notation $E[X|Y]$ to denote this function of the random variable Y. Since a function of a random variable is another random variable, we conclude that $E[X|Y]$ *is a random variable!* For some readers, the following definition may help to clarify this point.

Definition 4.15 ***Conditional Expected Value***

The conditional expected value $E[X|Y]$ is a function of random variable Y such that if $Y = y$ then $E[X|Y] = E[X|Y=y]$.

Example 4.20 For random variables X and Y in Example 4.5, we found in Example 4.19 that the conditional PDF of X given Y is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} 1/(1-y) & y \le x \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.109}$$

Find the conditional expected values $E[X|Y=y]$ and $E[X|Y]$.

Given the conditional PDF $f_{X|Y}(x|y)$, we perform the integration

$$E[X|Y=y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)\, dx \tag{4.110}$$

$$= \int_y^1 \frac{1}{1-y} x\, dx = \left.\frac{x^2}{2(1-y)}\right|_{x=y}^{x=1} = \frac{1+y}{2}. \tag{4.111}$$

Since $E[X|Y=y]=(1+y)/2$, $E[X|Y]=(1+Y)/2$.

An interesting property of the random variable $E[X|Y]$ is its expected value $E[E[X|Y]]$. We find $E[E[X|Y]]$ in two steps: first we calculate $g(y)=E[X|Y=y]$ and then we apply Theorem 3.4 to evaluate $E[g(Y)]$. This two-step process is known as *iterated expectation.*

Theorem 4.25 ***Iterated Expectation***

$$E\left[E\left[X|Y\right]\right]=E\left[X\right].$$

Proof We consider continuous random variables X and Y and apply Theorem 3.4:

$$E\left[E\left[X|Y\right]\right]=\int_{-\infty}^{\infty}E\left[X|Y=y\right]f_Y\left(y\right)dy. \tag{4.112}$$

To obtain this formula from Theorem 3.4, we have used $E[X|Y=y]$ in place of $g(x)$ and $f_Y(y)$ in place of $f_X(x)$. Next, we substitute the right side of Equation (4.108) for $E[X|Y=y]$:

$$E\left[E\left[X|Y\right]\right]=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}xf_{X|Y}\left(x|y\right)dx\right)f_Y\left(y\right)dy. \tag{4.113}$$

Rearranging terms in the double integral and reversing the order of integration, we obtain:

$$E\left[E\left[X|Y\right]\right]=\int_{-\infty}^{\infty}x\int_{-\infty}^{\infty}f_{X|Y}\left(x|y\right)f_Y\left(y\right)dy\,dx. \tag{4.114}$$

Next, we apply Theorem 4.24 and Theorem 4.8 to infer that the inner integral is simply $f_X(x)$. Therefore,

$$E\left[E\left[X|Y\right]\right]=\int_{-\infty}^{\infty}xf_X\left(x\right)dx. \tag{4.115}$$

The proof is complete because the right side of this formula is the definition of $E[X]$. A similar derivation (using sums instead of integrals) proves the theorem for discrete random variables.

The same derivation can be generalized to any function $g(X)$ of one of the two random variables:

Theorem 4.26

$$E\left[E\left[g(X)|Y\right]\right]=E\left[g(X)\right].$$

The following versions of Theorem 4.26 are instructive. If Y is continuous,

$$E\left[g(X)\right]=E\left[E\left[g(X)|Y\right]\right]=\int_{-\infty}^{\infty}E\left[g(X)|Y=y\right]f_Y\left(y\right)dy, \tag{4.116}$$

and if Y is discrete, we have a similar expression,

$$E\left[g(X)\right]=E\left[E\left[g(X)|Y\right]\right]=\sum_{y\in S_Y}E\left[g(X)|Y=y\right]P_Y\left(y\right). \tag{4.117}$$

Theorem 4.26 decomposes the calculation of $E[g(X)]$ into two steps: the calculation of $E[g(X)|Y = y]$, followed by the averaging of $E[g(X)|Y = y]$ over the distribution of Y. This is another example of iterated expectation. In Section 4.11, we will see that the iterated expectation can both facilitate understanding as well as simplify calculations.

Example 4.21 At noon on a weekday, we begin recording new call attempts at a telephone switch. Let X denote the arrival time of the first call, as measured by the number of seconds after noon. Let Y denote the arrival time of the second call. In the most common model used in the telephone industry, X and Y are continuous random variables with joint PDF

$$f_{X,Y}(x, y) = \begin{cases} \lambda^2 e^{-\lambda y} & 0 \le x < y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.118}$$

where $\lambda > 0$ calls/second is the average arrival rate of telephone calls. Find the marginal PDFs $f_X(x)$ and $f_Y(y)$ and the conditional PDFs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$.

For $x < 0$, $f_X(x) = 0$. For $x \ge 0$, Theorem 4.8 gives $f_X(x)$:

$$f_X(x) = \int_x^\infty \lambda^2 e^{-\lambda y}\, dy = \lambda e^{-\lambda x}. \tag{4.119}$$

Referring to Appendix A.2, we see that X is an exponential random variable with expected value $1/\lambda$. Given $X = x$, the conditional PDF of Y is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)} = \begin{cases} \lambda e^{-\lambda(y-x)} & y > x, \\ 0 & \text{otherwise.} \end{cases} \tag{4.120}$$

To interpret this result, let $U = Y - X$ denote the interarrival time, the time between the arrival of the first and second calls. Problem 4.10.15 asks the reader to show that given $X = x$, U has the same PDF as X. That is, U is an exponential (λ) random variable. Now we can find the marginal PDF of Y. For $y < 0$, $f_Y(y) = 0$. Theorem 4.8 implies

$$f_Y(y) = \begin{cases} \int_0^y \lambda^2 e^{-\lambda y}\, dx = \lambda^2 y e^{-\lambda y} & y \ge 0, \\ 0 & \text{otherwise.} \end{cases} \tag{4.121}$$

Y is the Erlang $(2, \lambda)$ random variable (Appendix A). Given $Y = y$, the conditional PDF of X is

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x, y)}{f_Y(y)} = \begin{cases} 1/y & 0 \le x < y, \\ 0 & \text{otherwise.} \end{cases} \tag{4.122}$$

Under the condition that the second call arrives at time y, the time of arrival of the first call is the uniform $(0, y)$ random variable.

In Example 4.21, we begin with a joint PDF and compute two conditional PDFs. Often in practical situations, we begin with a conditional PDF and a marginal PDF. Then we use this information to compute the joint PDF and the other conditional PDF.

Example 4.22 Let R be the uniform $(0, 1)$ random variable. Given $R = r$, X is the uniform $(0, r)$ random variable. Find the conditional PDF of R given X.

The problem definition states that

$$f_R(r) = \begin{cases} 1 & 0 \le r < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_{X|R}(x|r) = \begin{cases} 1/r & 0 \le x < r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.123}$$

It follows from Theorem 4.24 that the joint PDF of R and X is

$$f_{R,X}(r,x) = f_{X|R}(x|r)\, f_R(r) = \begin{cases} 1/r & 0 \le x < r < 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.124}$$

Now we can find the marginal PDF of X from Theorem 4.8. For $0 < x < 1$,

$$f_X(x) = \int_{-\infty}^{\infty} f_{R,X}(r,x)\, dr = \int_x^1 \frac{dr}{r} = -\ln x. \tag{4.125}$$

By the definition of the conditional PDF,

$$f_{R|X}(r|x) = \frac{f_{R,X}(r,x)}{f_X(x)} = \begin{cases} \frac{1}{-r\ln x} & x \le r \le 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.126}$$

4.9

(A) *The probability model for random variable A is*

$$P_A(a) = \begin{cases} 0.4 & a = 0, \\ 0.6 & a = 2, \\ 0 & \text{otherwise.} \end{cases} \tag{4.127}$$

The conditional probability model for random variable B given A is

$$P_{B|A}(b|0) = \begin{cases} 0.8 & b = 0, \\ 0.2 & b = 1, \\ 0 & \text{otherwise,} \end{cases} \qquad P_{B|A}(b|2) = \begin{cases} 0.5 & b = 0, \\ 0.5 & b = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{4.128}$$

(1) *What is the probability model for A and B? Write the joint PMF $P_{A,B}(a,b)$ as a table.*

(2) *If $A = 2$, what is the conditional expected value $E[B|A = 2]$?*

(3) *If $B = 0$, what is the conditional PMF $P_{A|B}(a|0)$?*

(4) *If $B = 0$, what is the conditional variance $\text{Var}[A|B = 0]$ of A?*

(B) *The PDF of random variable X and the conditional PDF of random variable Y given X are*

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1, \\ 0 & \text{otherwise,} \end{cases} \qquad f_{Y|X}(y|x) = \begin{cases} 2y/x^2 & 0 \le y \le x,\ 0 < x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(1) *What is the probability model for X and Y? Find $f_{X,Y}(x,y)$.*

(2) *If $X = 1/2$, find the conditional PDF $f_{Y|X}(y|1/2)$.*

(3) *If $Y = 1/2$, what is the conditional PDF $f_{X|Y}(x|1/2)$?*

(4) *If $Y = 1/2$, what is the conditional variance $\text{Var}[X|Y = 1/2]$?*

and for $1 \le z < 2$ (Fig. 6-12b)

$$f_z(z) = \int_{z-1}^{1} 1\,dx = 2 - z \tag{6-52}$$

Fig. 6-12c shows $f_z(z)$, which agrees with the convolution of two rectangular waveforms as well. ◀

EXAMPLE 6-9

z = x − y

▶ Let $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Determine $f_z(z)$.

From (6-37) and Fig. 6-13

$$F_z(z) = P\{\mathbf{x} - \mathbf{y} \le z\} = \int_{y=-\infty}^{\infty} \int_{x=-\infty}^{z+y} f_{xy}(x, y)\,dx\,dy$$

and hence

$$f_z(z) = \frac{dF_z(z)}{dz} = \int_{-\infty}^{\infty} f_{xy}(z + y, y)\,dy \tag{6-53}$$

If $\mathbf{x}$ and $\mathbf{y}$ are independent, then this formula reduces to

$$f_z(z) = \int_{-\infty}^{\infty} f_x(z + y) f_y(y)\,dy = f_x(-z) \otimes f_y(y) \tag{6-54}$$

which represents the convolution of $f_x(-z)$ with $f_y(z)$.

As a special case, suppose

$$f_x(x) = 0 \qquad x < 0, \qquad f_y(y) = 0 \qquad y < 0$$

In this case, $\mathbf{z}$ can be negative as well as positive, and that gives rise to two situations that should be analyzed separately, since the regions of integration for $z \ge 0$ and $z < 0$ are quite different.

For $z \ge 0$, from Fig. 6-14a

$$F_z(z) = \int_{y=0}^{\infty} \int_{x=0}^{z+y} f_{xy}(x, y)\,dx\,dy$$

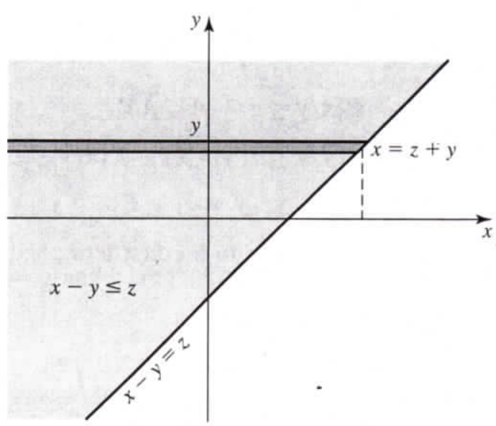


FIGURE 6-13

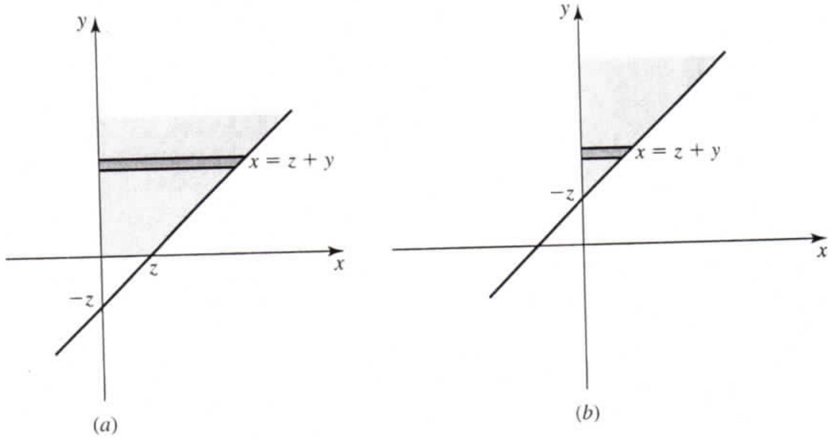


FIGURE 6-14

and for $z < 0$, from Fig. 6-14*b*

$$F_z(z) = \int_{y=-z}^{\infty} \int_{x=0}^{z+y} f_{xy}(x, y)\, dx\, dy$$

After differentiation, this gives

$$f_z(z) = \begin{cases} \int_0^{\infty} f_{xy}(z+y, y)\, dy & z \geq 0 \\ \int_{-z}^{\infty} f_{xy}(z+y, y)\, dy & z < 0 \end{cases} \tag{6-55}$$

◀

EXAMPLE 6-10

$\mathbf{z} = \mathbf{x}/\mathbf{y}$

▶ Let $\mathbf{z} = \mathbf{x}/\mathbf{y}$. Determine $f_z(z)$.

We have

$$F_z(z) = P\{\mathbf{x}/\mathbf{y} \leq z\} \tag{6-56}$$

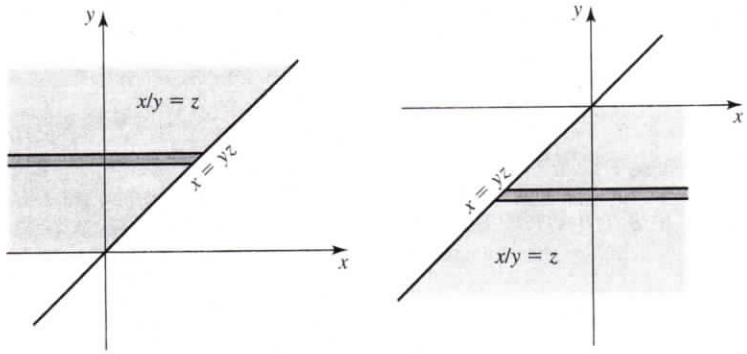
The inequality $x/y \leq z$ can be rewritten as $\mathbf{x} \leq \mathbf{y}z$ if $\mathbf{y} > 0$, and $\mathbf{x} \geq \mathbf{y}z$ if $\mathbf{y} < 0$. Hence the event $\{\mathbf{x}/\mathbf{y} \leq z\}$ in (6-56) needs to be conditioned by the event $A = \{\mathbf{y} > 0\}$ and its compliment $\overline{A}$. Since $A \cup \overline{A} = S$, by the partition theorem, we have

$$\begin{aligned} P\{\mathbf{x}/\mathbf{y} \leq z\} &= P\{\mathbf{x}/\mathbf{y} \leq z \cap (A \cup \overline{A})\} \\ &= P\{\mathbf{x}/\mathbf{y} \leq z, \mathbf{y} > 0\} + P\{\mathbf{x}/\mathbf{y} \leq z, \mathbf{y} < 0\} \\ &= P\{\mathbf{x} \leq \mathbf{y}z, \mathbf{y} > 0\} + P\{\mathbf{x} \geq \mathbf{y}z, \mathbf{y} < 0\} \end{aligned} \tag{6-57}$$

Fig. 6-15*a* shows the area corresponding to the first term, and Fig. 6-15*b* shows that corresponding to the second term in (6-57).

Integrating over these two regions, we get

$$F_z(z) = \int_{y=0}^{\infty} \int_{x=-\infty}^{yz} f_{xy}(x, y)\, dx\, dy + \int_{y=-\infty}^{0} \int_{x=yz}^{\infty} f_{xy}(x, y)\, dx\, dy \tag{6-58}$$



(a) $P(\mathbf{x} \le \mathbf{y}z, \mathbf{y} > 0)$ (b) $P(\mathbf{x} \ge \mathbf{y}z, \mathbf{y} < 0)$

FIGURE 6-15

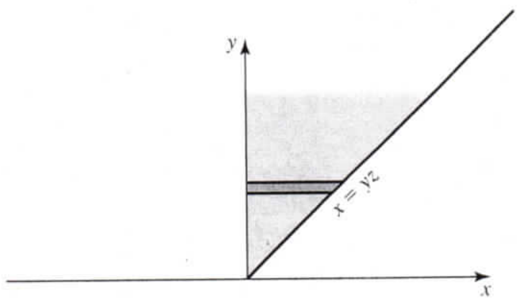


FIGURE 6-16

Differentiation gives

$$f_z(z) = \int_0^{\infty} y f_{xy}(yz, y)\, dy + \int_{-\infty}^{0} -y f_{xy}(yz, y)\, dy$$
$$= \int_{-\infty}^{\infty} |y| f_{xy}(yz, y)\, dy \tag{6-59}$$

Note that if $\mathbf{x}$ and $\mathbf{y}$ are non-negative random variables, then the area of integration reduces to that shown in Fig. 6-16.

This gives

$$F_z(z) = \int_{y=0}^{\infty} \int_{x=0}^{yz} f_{xy}(x, y)\, dx\, dy$$

or

$$f_z(z) = \int_{y=0}^{\infty} y f_{xy}(yz, y)\, dy \tag{6-60}$$

◀

EXAMPLE 6-11 ▶ $\mathbf{x}$ and $\mathbf{y}$ are jointly normal random variables with zero mean and

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1 - r^2}} e^{-\left[\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)\right]} \tag{6-61}$$

Show that the ratio $\mathbf{z} = \mathbf{x}/\mathbf{y}$ has a Cauchy density centered at $r\sigma_1/\sigma_2$.

SOLUTION
Inserting (6-61) into (6-59) and using the fact that $f_{xy}(-x, -y) = f_{xy}(x, y)$, we obtain

$$f_z(z) = \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \int_0^\infty y e^{-y^2/2\sigma_0^2}\, dy = \frac{\sigma_0^2}{\pi\sigma_1\sigma_2\sqrt{1-r^2}}$$

where

$$\sigma_0^2 = \frac{1-r^2}{\left(z^2/\sigma_1^2\right) - (2rz/\sigma_1\sigma_2) + \left(1/\sigma_2^2\right)}$$

Thus

$$f_z(z) = \frac{\sigma_1\sigma_2\sqrt{1-r^2}/\pi}{\sigma_2^2(z - r\sigma_1/\sigma_2)^2 + \sigma_1^2(1-r^2)} \tag{6-62}$$

which represents a Cauchy random variable centered at $r\sigma_1/\sigma_2$. Integrating (6-62) from $-\infty$ to z, we obtain the corresponding distribution function to be

$$F_z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\sigma_2 z - r\sigma_1}{\sigma_1\sqrt{1-r^2}} \tag{6-63}$$

◀

As an application, we can use (6-63) to determine the probability masses m_1, m_2, m_3, and m_4 in the four quadrants of the xy plane for (6-61). From the spherical symmetry of (6-61), we have

$$m_1 = m_3 \qquad m_2 = m_4$$

But the second and fourth quadrants represent the region of the plane where $x/y < 0$. The probability that the point (x, y) is in that region equals, therefore, the probability that the random variable $\mathbf{z} = \mathbf{x}/\mathbf{y}$ is negative. Thus

$$m_2 + m_4 = P(\mathbf{z} \le 0) = F_z(0) = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{r}{\sqrt{1-r^2}}$$

and

$$m_1 + m_3 = 1 - (m_2 + m_4) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{r}{\sqrt{1-r^2}}$$

If we define $\alpha = \arctan r/\sqrt{1-r^2}$, this gives

$$m_1 = m_3 = \frac{1}{4} + \frac{\alpha}{2\pi} \qquad m_2 = m_4 = \frac{1}{4} - \frac{\alpha}{2\pi} \tag{6-64}$$

Of course, we could have obtained this result by direct integration of (6-61) in each quadrant. However, this is simpler.

EXAMPLE 6-12

▶ Let $\mathbf{x}$ and $\mathbf{y}$ be independent gamma random variables with $\mathbf{x} \sim G(m, \alpha)$ and $\mathbf{y} \sim G(n, \alpha)$. Show that $\mathbf{z} = \mathbf{x}/(\mathbf{x}+\mathbf{y})$ has a beta distribution.

Proof.

$$f_{xy}(x, y) = f_x(x) f_y(y) = \frac{1}{\alpha^{m+n}\Gamma(m)\Gamma(n)} x^{m-1} y^{n-1} e^{-(x+y)/\alpha} \qquad x > 0 \qquad y > 0 \tag{6-65}$$

EXAMPLE 6-13

$\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$

Note that $0 < z < 1$, since $\mathbf{x}$ and $\mathbf{y}$ are non-negative random variables

$$F_z(z) = P\{\mathbf{z} \le z\} = P\left(\frac{\mathbf{x}}{\mathbf{x}+\mathbf{y}} \le z\right) = P\left(\mathbf{x} \le \mathbf{y}\frac{z}{1-z}\right)$$
$$= \int_0^\infty \int_0^{yz/(1-z)} f_{xy}(x, y)\, dx\, dy$$

where we have made use of Fig. 6-16. Differentiation with respect to z gives

$$f_z(z) = \int_0^\infty \frac{y}{(1-z)^2} f_{xy}(yz/(1-z), y)\, dy$$
$$= \int_0^\infty \frac{y}{(1-z)^2} \frac{1}{\alpha^{m+n}\Gamma(m)\Gamma(n)} \left(\frac{yz}{1-z}\right)^{m-1} y^{n-1} e^{-y/(1-z)\alpha}\, dy$$
$$= \frac{1}{\alpha^{m+n}\Gamma(m)\Gamma(n)} \frac{z^{m-1}}{(1-z)^{m+1}} \int_0^\infty y^{m+n-1} e^{-y/\alpha(1-z)}\, dy$$
$$= \frac{z^{m-1}(1-z)^{n-1}}{\Gamma(m)\Gamma(n)} \int_0^\infty u^{m+n-1} e^{-u}\, du = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)} z^{m-1}(1-z)^{n-1}$$
$$= \begin{cases} \dfrac{1}{\beta(m,n)} z^{m-1}(1-z)^{n-1} & 0 < z < 1 \\ 0 & \text{otherwise} \end{cases} \tag{6-66}$$

which represents a beta distribution. ◀

EXAMPLE 6-13

$\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$

▶ Let $\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$. Determine $f_z(z)$.

We have

$$F_z(z) = P\{\mathbf{x}^2 + \mathbf{y}^2 \le z\} = \iint_{x^2+y^2 \le z} f_{xy}(x, y)\, dx\, dy$$

But, $\mathbf{x}^2 + \mathbf{y}^2 \le z$ represents the area of a circle with radius $\sqrt{z}$, and hence (see Fig. 6-17)

$$F_z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{x=-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{xy}(x, y)\, dx\, dy$$

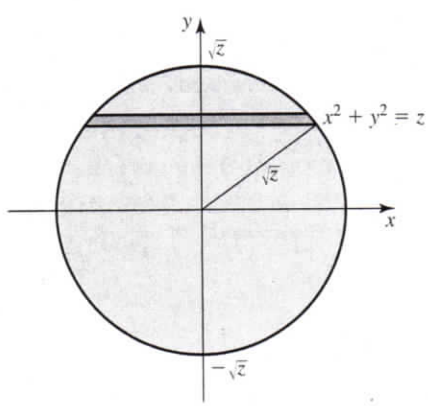


FIGURE 6-17

This gives

$$f_z(z) = \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \{f_{xy}(\sqrt{z-y^2}, y) + f_{xy}(-\sqrt{z-y^2}, y)\}\, dy \qquad (6\text{-}67)$$

◀

As an illustration, consider Example 6-14.

EXAMPLE 6-14

▶ **x** and **y** are independent normal random variables with zero mean and common variance σ^2. Determine $f_z(z)$ for $\mathbf{z} = \mathbf{x}^2 + \mathbf{y}^2$.

SOLUTION

Using (6-67), we get

$$\begin{aligned} f_z(z) &= \int_{-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(2 \cdot \frac{1}{2\pi\sigma^2} e^{(z-y^2+y^2)/2\sigma^2}\right) dy \\ &= \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z-y^2}}\, dy = \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\pi/2} \frac{\sqrt{z}\cos\theta}{\sqrt{z}\cos\theta}\, d\theta \\ &= \frac{1}{2\sigma^2} e^{-z/2\sigma^2} U(z) \end{aligned} \qquad (6\text{-}68)$$

where we have used the substitution $y = \sqrt{z}\sin\theta$. From (6-68), we have the following: If **x** and **y** are independent zero mean Gaussian random variables with common variance σ^2, then $\mathbf{x}^2 + \mathbf{y}^2$ is an exponential random variable with parameter $2\sigma^2$. ◀

EXAMPLE 6-15

$\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$

▶ Let $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$. Find $f_z(z)$.

SOLUTION

From Fig. 6-17, the present case corresponds to a circle with radius z^2. Thus

$$F_z(z) = \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{xy}(x, y)\, dx\, dy$$

and by differentiation,

$$f_z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2-y^2}} \{f_{xy}(\sqrt{z^2-y^2}, y) + f_{xy}(-\sqrt{z^2-y^2}, y)\}\, dy \qquad (6\text{-}69)$$

In particular, if **x** and **y** are zero mean independent Gaussian random variables as in the previous example, then

$$\begin{aligned} f_z(z) &= 2\int_0^{z} \frac{z}{\sqrt{z^2-y^2}} \frac{2}{2\pi\sigma^2} e^{-(z^2-y^2+y^2)/2\sigma^2}\, dy \\ &= \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^{z} \frac{1}{\sqrt{z^2-y^2}}\, dy = \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^{\pi/2} \frac{z\cos\theta}{z\cos\theta}\, d\theta \\ &= \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} U(z) \end{aligned} \qquad (6\text{-}70)$$

EXAMPLE 6-16

which represents a Rayleigh distribution. Thus, if $\mathbf{w} = \mathbf{x} + i\mathbf{y}$, where **x** and **y** are real independent normal random variables with zero mean and equal variance, then the random variable $|\mathbf{w}| = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ has a Rayleigh density. $\mathbf{w}$ is said to be a complex Gaussian random variable with zero mean, if its real and imaginary parts are independent. So far we have seen that the magnitude of a complex Gaussian random variable has Rayleigh distribution. What about its phase

$$\theta = \tan^{-1}\left(\frac{\mathbf{y}}{\mathbf{x}}\right) \tag{6-71}$$

Clearly, the principal value of $\boldsymbol{\theta}$ lies in the interval $(-\pi/2, \pi/2)$. If we let $\mathbf{u} = \tan\boldsymbol{\theta} = \mathbf{y}/\mathbf{x}$, then from Example 6-11, $\mathbf{u}$ has a Cauchy distribution (see (6-62) with $\sigma_1 = \sigma_2$, $r = 0$)

$$f_u(u) = \frac{1/\pi}{u^2+1} \qquad -\infty < u < \infty$$

As a result, the principal value of $\boldsymbol{\theta}$ has the density function

$$f_\theta(\theta) = \frac{1}{|d\theta/du|} f_u(\tan\theta) = \frac{1}{(1/\sec^2\theta)} \frac{1/\pi}{\tan^2\theta + 1}$$

$$= \begin{cases} 1/\pi & -\pi/2 < \theta < \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{6-72}$$

However, in the representation $\mathbf{x} + j\mathbf{y} = \mathbf{r}e^{j\boldsymbol{\theta}}$, the variable $\boldsymbol{\theta}$ lies in the interval $(-\pi, \pi)$, and taking into account this scaling by a factor of two, we obtain

$$f_\theta(\theta) = \begin{cases} 1/2\pi & -\pi < \theta < \pi \\ 0 & \text{otherwise} \end{cases} \tag{6-73}$$

To summarize, the magnitude and phase of a zero mean complex Gaussian random variable have Rayleigh and uniform distributions respectively. Interestingly, as we will show later (Example 6-22), these two derived random variables are also statistically *independent* of each other! ◀

Let us reconsider Example 6-15 where $\mathbf{x}$ and $\mathbf{y}$ are independent Gaussian random variables with nonzero means μ_x and μ_y respectively. Then $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ is said to be a Rician random variable. Such a scene arises in fading multipath situations where there is a dominant constant component (mean) in addition to a zero mean Gaussian random variable. The constant component may be the line of sight signal and the zero mean Gaussian random variable part could be due to random multipath components adding up incoherently. The envelope of such a signal is said to be Rician instead of Rayleigh.

EXAMPLE 6-16 ▶ Redo Example 6-15, where $\mathbf{x}$ and $\mathbf{y}$ are independent Gaussian random variables with nonzero means μ_x and μ_y respectively.

SOLUTION

Since

$$f_{xy}(x, y) = \frac{1}{2\pi\sigma^2} e^{-[(x-\mu_x)^2 + (y-\mu_y)^2]/2\sigma^2}$$

substituting this into (6-69) and letting $y = z\sin\theta$, $\mu = \sqrt{\mu_x^2 + \mu_y^2}$, $\mu_x = \mu\cos\phi$,

$\mu_y = \mu \sin\phi$, we get the Rician distribution to be

$$
\begin{aligned}
f_z(z) &= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} \int_{-\pi/2}^{\pi/2} \left(e^{z\mu\cos(\theta-\phi)/\sigma^2} + e^{-z\mu\cos(\theta+\phi)/\sigma^2} \right) d\theta \\
&= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} \left(\int_{-\pi/2}^{\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2}\, d\theta + \int_{\pi/2}^{3\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2}\, d\theta \right) \\
&= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{\sigma^2} I_0\left(\frac{z\mu}{\sigma^2}\right) \qquad (6\text{-}74)
\end{aligned}
$$

where

$$I_0(\eta) \triangleq \frac{1}{2\pi} \int_0^{2\pi} e^{\eta\cos(\theta-\phi)}\, d\theta = \frac{1}{\pi} \int_0^{\pi} e^{\eta\cos\theta}\, d\theta$$

is the modified Bessel function of the first kind and zeroth order. ◀

Order Statistics

In general, given any n-tuple $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, we can rearrange them in an increasing order of magnitude such that

$$\mathbf{x}_{(1)} \le \mathbf{x}_{(2)} \le \cdots \le \mathbf{x}_{(n)}$$

where $\mathbf{x}_{(1)} = \min(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$, and $\mathbf{x}_{(2)}$ is the second smallest value among $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$, and finally $\mathbf{x}_{(n)} = \max(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$. The functions *min* and *max* are nonlinear operators, and represent special cases of the more general order statistics. If $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ represent random variables, the function $\mathbf{x}_{(k)}$ that takes on the value $x_{(k)}$ in each possible sequence $(x_1, x_2, \ldots, x_n)$ is known as the kth-order statistic. $\{\mathbf{x}_{(1)}, \mathbf{x}_{(2)}, \ldots, \mathbf{x}_{(n)}\}$ represent the set of order statistics among n random variables. In this context

$$\mathbf{R} = \mathbf{x}_{(n)} - \mathbf{x}_{(1)} \qquad (6\text{-}75)$$

represents the range, and when $n = 2$, we have the *max* and *min* statistics.

Order statistics is useful when relative magnitude of observations is of importance. When worst case scenarios have to be accounted for, then the function $max(\cdot)$ is quite useful. For example, let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ represent the recorded flood levels over the past n years at some location. If the objective is to construct a dam to prevent any more flooding, then the height H of the proposed dam should satisfy the inequality

$$H > \max(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n) \qquad (6\text{-}76)$$

with some finite probability. In that case, the p.d.f. of the random variable on the right side of (6-76) can be used to compute the desired height. In another case, if a bulb manufacturer wants to determine the average time to failure (μ) of its bulbs based on a sample of size n, the sample mean $(\mathbf{x}_1 + \mathbf{x}_2 + \cdots + \mathbf{x}_n)/n$ can be used as an estimate for μ. On the other hand, an estimate based on the least time to failure has other attractive features. This estimate $\min(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n)$ may not be as good as the sample mean in terms of their respective variances, but the $min(\cdot)$ can be computed as soon as the first bulb fuses, whereas to compute the sample mean one needs to wait till the last of the lot extinguishes.

EXAMPLE 6-17

$\mathbf{z} = \max(\mathbf{x}, \mathbf{y})$
$\mathbf{w} = \min(\mathbf{x}, \mathbf{y})$