Equation (3.1.63) looks complicated but can be interpreted as two impulses and a continuous section. The first impulse has a magnitude of $s(-10) = \frac{2}{25}$ at $y = -10$, which corresponds to the discontinuity in Fig. 3.1.16 at that point. The second term is an impulse of magnitude $1 - s(+10) = \frac{3}{25}$ at $y = +10$ and corresponds to the discontinuity in Fig. 3.1.16 at that point. The last term in Eq. (3.1.63) has a constant height of $\frac{1}{25}$ over the range $-10 < y < +10$ and corresponds to the straight-line portion in Fig. 3.1.16. The corresponding PDF is shown in Fig. 3.1.17.

The PDF in Fig. 3.1.17 is the derivative of the CDF in Fig. 3.1.16. The discontinuities in the CDF lead to the impulse functions in the PDF. Mixed random variables always lead to these features.

Summary. Section 3.1 presented the basic tools for describing continuous random variables. The major points follow:

- The distribution of probabilities of a continuous random variable may be described by a probability density function, PDF, or a cumulative distribution function, CDF. The PDF and the CDF contain the same information.
- The CDF is a probability and is useful for derivations and for analyzing problems. The PDF is a density of probability and is useful for calculating probabilities and expectations.
- Conditional CDFs and PDFs can be defined and provide useful tools for analysis and calculation.

In Sec. 3.2 we extend these concepts and definitions to the bivariate case, two random variables.

3.2 BIVARIATE RANDOM VARIABLES

3.2.1 Bivariate Probability Density Functions, PDFs

Mapping two random variables into the Cartesian plane. In this section we deal with two continuous random variables, call them X and Y. The two random variables map the sample space into the Cartesian plane, as shown in Fig. 3.2.1(a), that is, for each outcome of the chance experiment there is an associated point X, Y in the x, y plane.

Many physical problems introduce two or more such random variables. For examples, the location of an imperfection on a semiconductor wafer, the height and weight of an individual from a population, or the velocity of a molecule in a gas (three components = three random variables). The definitions and methodologies we introduce for two random variables in this section are routinely extended to three or more random variables.

Definition of bivariate PDF. Definitions are the same as for the discrete bivariate PMF, with appropriate accommodation for continuous space. For example, the bivariate PDF, $f_{XY}(x, y)$, also called the *joint PDF,* is defined as

$$f_{XY}(x, y)\, dx\, dy = P[(x < X \leq x + dx) \cap (y < Y \leq y + dy)] \tag{3.2.1}$$

and $f_{XY}(x, y)$ is thus a probability per unit area.[13] This definition applies to random variables that are continuous in both X and Y and is illustrated in Fig. 3.2.1(b). The definition requires that dx and dy be positive for the probability to be defined, from which it follows that the bivariate PDF must be positive.

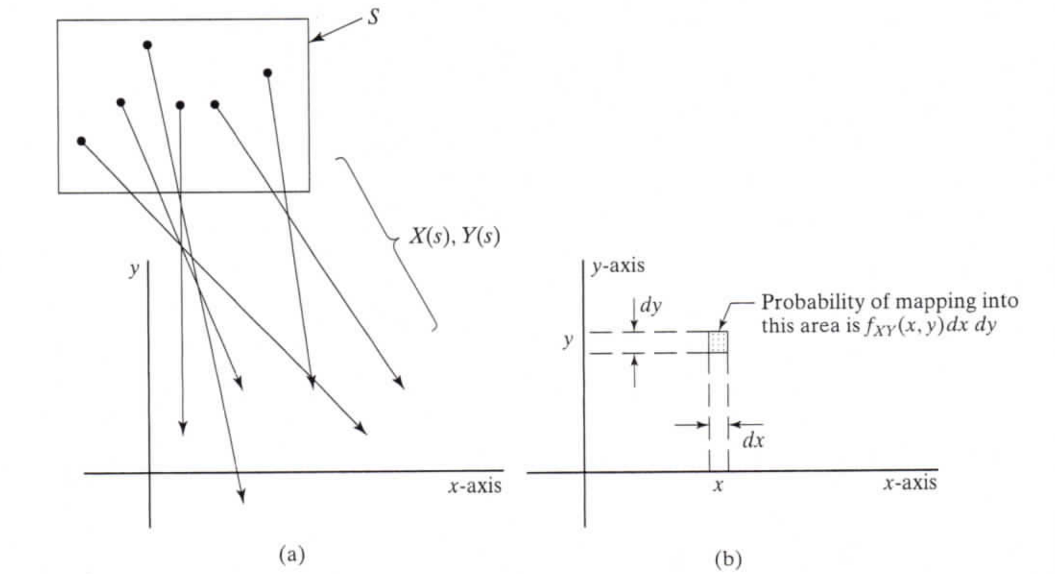

Figure 3.2.1 (a) Two random variables map the sample space into the Cartesian plane. Events are expressed as areas in this plane. (b) The bivariate PDF is expressed in terms of the probability that an *X*, *Y* point resulting from a single performance of the experiment falls in a small rectangle of area *dx dy*, located at *x*, *y*.

Independent random variables. When X and Y are independent random variables, the $\cap$ in Eq. (3.2.1) becomes $\cap \to \times$ and the result is

$$f_{XY}(x, y)\,dx\,dy = P[x < X \leq x + dx] \times P[y < Y \leq y + dy] = f_X(x)\,dx \times f_Y(y)\,dy \quad (3.2.2)$$

Because dx and dy are arbitrary in magnitude, it follows that when X and Y are independent the bivariate PDF is the product of the PDFs of the two random variables, as shown in Eq. (3.2.3). In this context, f_X and f_Y are called the *marginal PDFs*.

$$f_{XY}(x, y) = f_X(x) f_Y(y) \text{ for } x \text{ and } y \text{ independent} \quad (3.2.3)$$

If $f_{XY}(x, y)$ is nonzero over a region, then for independence the boundaries of that region must also be factorable into separate dependence on x and y.

Example 3.2.1: Bivariate PDF

Consider two independent random variables: X is uniformly distributed between 0 and 5, and Y is uniformly distributed between 5 and 5.5. Find the bivariate PDF, $f_{XY}(x, y)$.

Solution The PDFs of X and Y are $f_X(x) = 0.2, 0 < x \leq 5$, zow; $f_Y(y) = 2, 5 < y \leq 5.5$, zow. From Eq. (3.2.3), it follows that the joint PDF is

$$f_{XY}(x, y) = f_X(x) f_Y(y) = 0.2 \times 2 = 0.4, \;\; 0 < x \leq 5 \cap 5 < y \leq 5.5, \;\text{zow} \quad (3.2.4)$$

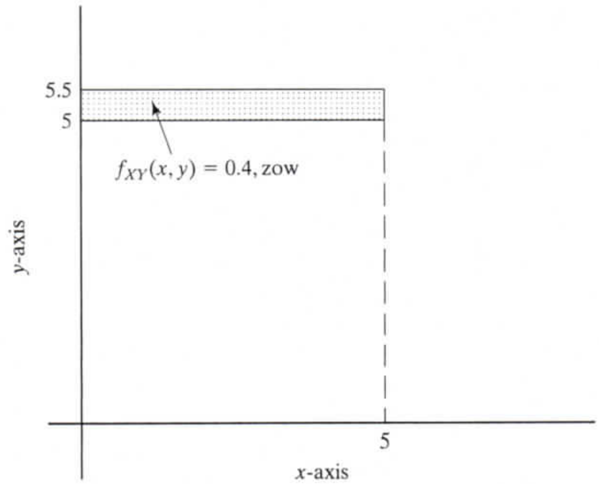

Figure 3.2.2 The joint PDF is a constant over a rectangular area. The area of the rectangle is 2.5, so the PDF has a "height" of $1/2.5 = 0.4$, such that the volume under the PDF surface is unity.

This PDF is shown in Fig. 3.2.2. Identification of the region where the PDF is nonzero is fully as important as the value of the PDF in that region. This is the significance of the "zow."

Calculating probabilities. The properties of the bivariate PDF are simple extensions of the single variable case. For example, instead of integrating in one dimension to find the probability that the random variable falls in a prescribed range, we integrate in two dimensions to find out the probability that the random variables fall into a prescribed region in the x, y plane, as indicated in Eq. (3.2.5) and in Fig. 3.2.3.

$$P[X, Y \in A] = \iint_A f_{XY}(x, y)\, dx\, dy \tag{3.2.5}$$

Equation (3.2.5) may be justified in the same way that we derived Eq. (3.1.4), except here we must divide the event represented by the area in Fig. 3.2.3 into little rectangles of dimensions $dx \times dy$, as shown in Fig. 3.2.1(b), and sum up, in an integral, the probabilities associated with each small rectangle. These rectangles constitute a partition of the event $\{X, Y \in A\}$.

Normalization. Bivariate PDFs are nonnegative functions that are normalized to unit volume when integrated over the x, y plane. This follows from Eq. (3.2.5) if we let A be the event that X, Y falls somewhere in the plane.

$$A = \{(-\infty < X < +\infty) \cap (-\infty < Y < +\infty)\} = S \tag{3.2.6}$$

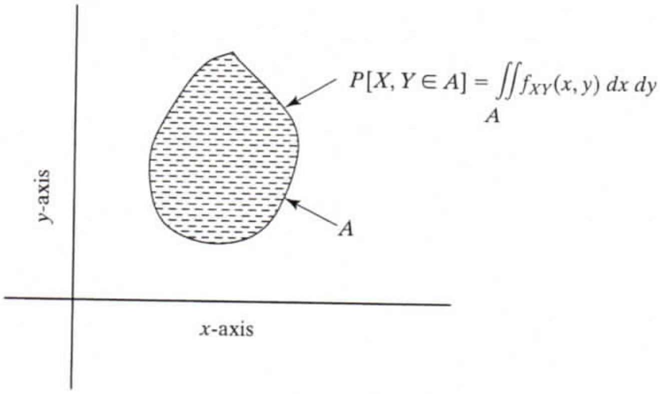

Figure 3.2.3 To calculate the probability of an event, we integrate the bivariate PDF over the area in x, y space representing the event.

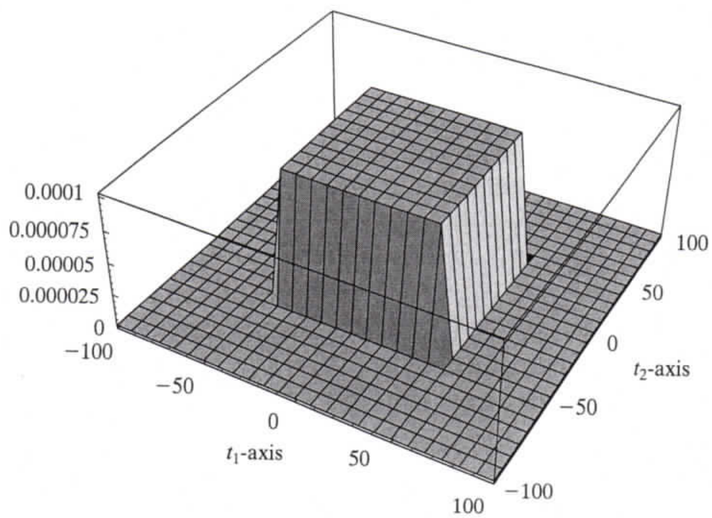

Figure 3.2.4 A bivariate PDF in three dimensions. This is the PDF for the "glitch" example later in this section.

where S is the certain event. Because $P[S] = 1$, Eq. (3.2.5) leads directly to

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dx\, dy = 1 \tag{3.2.7}$$

Normalized volume. When the PDF is plotted on an axis normal to the x, y plane, Eq. (3.2.7) can be interpreted as requiring unit volume under the PDF curve, as illustrated by Fig. 3.2.4 for the PDF in Example 3.2.4.

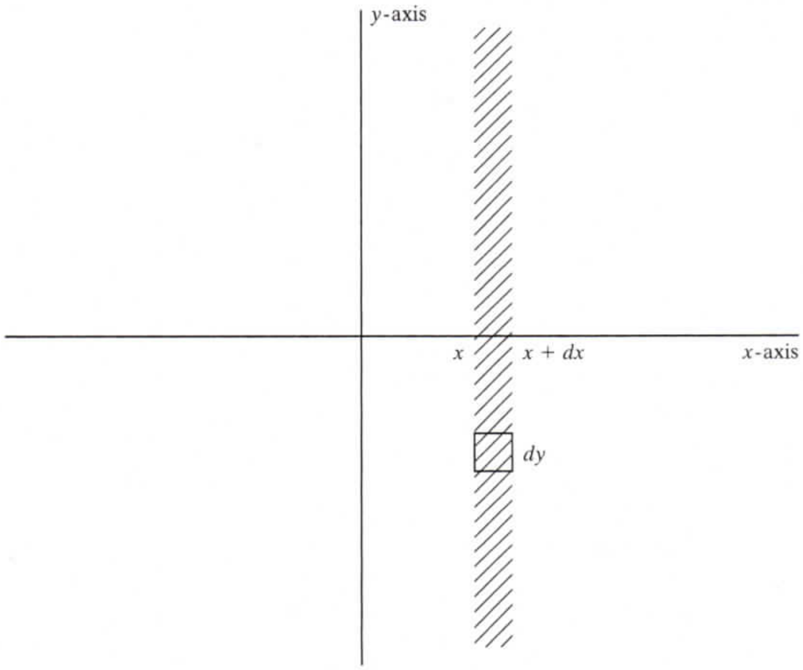

Figure 3.2.5 We can calculate the probability that X falls in a strip by integrating the bivariate PDF over the strip. This leads to the calculation of the marginal PDF on X shown in Eq. (3.2.9).

Marginal PDFs. Consider the following expression:

$$f_X(x)\,dx = P[(x < X \leq x + dx) \cap (-\infty < Y < +\infty)] = \left[\int_{-\infty}^{+\infty} f_{XY}(x, y)\,dy\right] dx \qquad (3.2.8)$$

The first term in Eq. (3.2.8) says that X falls in a small increment dx near x. The middle term says the same thing in terms of the definition of the PDF of X but puts in Y to change to the bivariate context, saying in effect that all values of Y are accepted. The last term calculates the probability in the middle term from the bivariate PDF, as shown in Fig. 3.2.5.

Writing Eq. (3.2.8) without the middle term and dropping the dx's on both sides gives

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\,dy \qquad (3.2.9)$$

In the bivariate context, $f_X(x)$ is called the *marginal* PDF of X. The idea is that we take all the probability in the strip and push it, as with a bulldozer, to the top margin of the plane. We might equally well call it the *axial* PDF if we think of all the probability in the plane heaped up on the x-axis. Often, Eq. (3.2.9) is called *integrating Y out of the distribution*, since in effect we are expressing that all values of Y are acceptable. In the same way we can integrate X out of the bivariate PDF to obtain the marginal distribution on Y:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x, y)\, dx \tag{3.2.10}$$

Example 3.2.2: Marginal PDFs

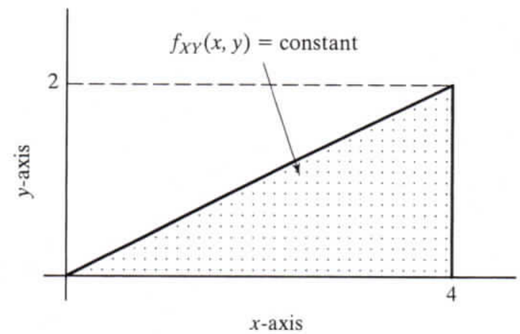

Figure 3.2.6 A point is chosen at random in the triangle shown, which means that the bivariate PDF is constant. We will calculate the marginal PDFs on X and Y.

Consider a point chosen at random in a triangle, as shown in Fig. 3.2.6.

The PDF. Our first task is to normalize the PDF. Because the bivariate PDF is constant over the triangle, its magnitude must be the reciprocal of the area of the triangle, which is 4. Hence,

$$f_{XY}(x, y) = \frac{1}{\text{area}} = \frac{1}{4} \text{ over the triangle, zow} \tag{3.2.11}$$

The marginal PDF for *X*. Using Eq. (3.2.9), we fix x and integrate from $-\infty$ to $+\infty$ in y. The only region in x where we will encounter any probability is in the region $0 < x \leq 4$. In that region the integrand is zero except between $y = 0$ and $y = x/2$, as shown in Fig. 3.2.7. Thus the result is

$$f_X(x) = \int_{y=0}^{y=x/2} \frac{1}{4}\, dy = \frac{1}{4} \times \frac{x}{2} = \frac{x}{8}, \quad 0 < x \leq 4, \text{ zow} \tag{3.2.12}$$

This PDF is shown in Fig. 3.2.8, and clearly is appropriately normalized.

Marginal PDF in *Y*. We can integrate out X and get the marginal PDF in Y, as shown in Eq. (3.2.10). Here the path is horizontal and yields a nonzero result only in the region $0 < y \leq 2$. The path of integration is shown in Fig. 3.2.9.

The required integration is

$$f_Y(y) = \int_{x=2y}^{x=4} \frac{1}{4}\, dx = \frac{1}{4}(4 - 2y), \quad 0 < y \leq 2, \text{ zow} \tag{3.2.13}$$

This PDF is shown in Fig. 3.2.10.

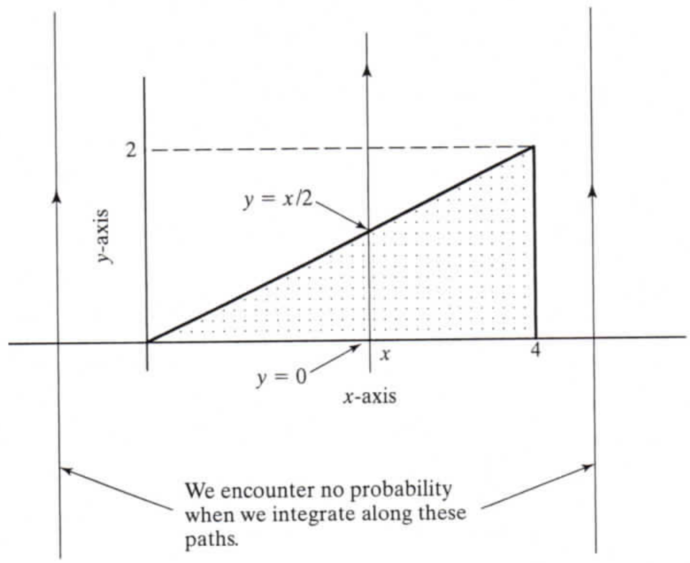

Figure 3.2.7 Integrating the PDF in the positive y direction. The only place in x where probability is encountered is $0 < x \leq 4$. All other paths give zero. When probability is encountered it is between $y = 0$ and $y = x/2$.

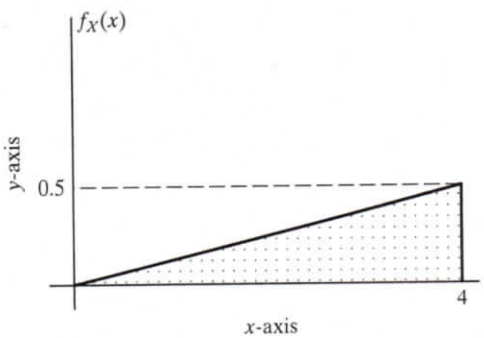

Figure 3.2.8 The marginal PDF for X. Notice the area under the curve is 1, as required for all PDFs.

Interpretation of the marginals. Look at Fig. 3.2.6 and think of the shaded triangle as so much sand. If you push the sand onto the x-axis, you get excess sand at $x = 4$, as shown by the marginal PDF on X in Fig. 3.2.8. If, however, you push the sand onto the y-axis, you get excess at $y = 0$, as shown for the marginal PDF on Y in Fig. 3.2.10. Of course, you would have the same situation if you pushed the sand to the "margins" at $x = \infty$ or $y = \infty$, but that is so far off that it is more convenient to (mentally) push the sand onto the axes.

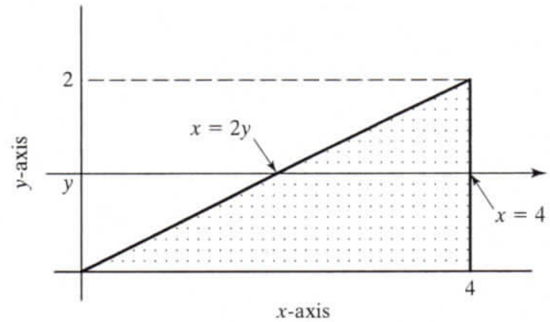

Figure 3.2.9 The integration in x encounters probability only between $x = 2y$ and $x = 4$. Only paths in the range $0 < y \le 2$ give nonzero results. The integral is evaluated in Eq. (3.2.13).

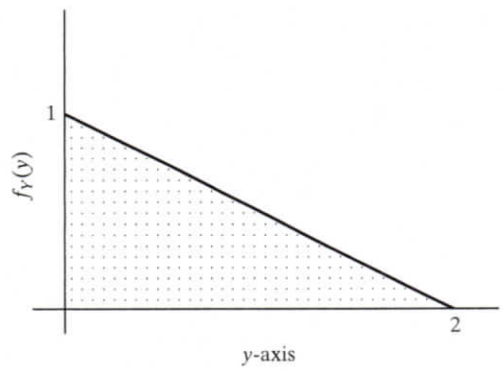

Figure 3.2.10 The marginal PDF in Y. Again, we note that the PDF is properly normalized.

You do it. Let X, Y be chosen at random in the quarter circle shown in Fig. 3.2.11. Find the marginal PDF in X. Evaluate your answer at $x = 0.5$, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 14.

Conditional PDFs. The bivariate case also allows the definition of conditional PDFs. These, like conditional probability in general, provide a useful analysis tool. We now define the conditional PDF, $f_{Y|X}(x, y)$. We begin with the definition of the bivariate PDF of Eq. (3.2.1), repeated in Eq. (3.2.14).

$$f_{XY}(x, y)\,dx\,dy = P[(x < X \le x + dx) \cap (y < Y \le y + dy)] \qquad (3.2.14)$$

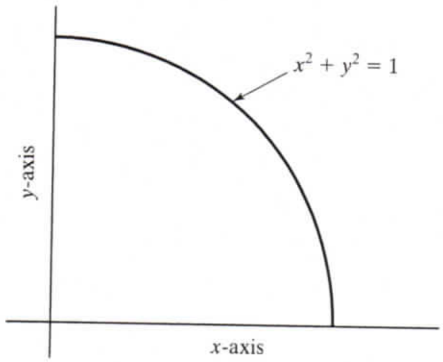

Figure 3.2.11 The PDF is uniform in the quarter circle. You are to find the marginal PDF in X.

The probability on the right side can be expressed as a conditional probability. By definition [Eq. (1.5.2)] of conditional probability,

$$f_{XY}(x, y)\,dx\,dy = \underbrace{P[(x < X \le x + dx)|(y < Y \le y + dy)]}_{f_{X|Y}(x,y)\,dx} \times \underbrace{P[(y < Y \le y + dy)]}_{f_Y(y)\,dy} \quad (3.2.15)$$

Equation (3.2.15) contains the definition of the conditional PDF, except that the conditioning term is usually abbreviated simply as $Y = y$, which is equivalent when the random variables are continuous and dy is infinitesimal. Thus we have

$$f_{X|Y}(x, y)\,dx = P[x < X \le x + dx|Y = y] \quad (3.2.16)$$

In consequence of the definition of the conditional PDF in Eq. (3.2.15), we have, after canceling the differentials

$$f_{XY}(x, y) = f_{X|Y}(x, y) \times f_Y(y) \quad (3.2.17)$$

and since Eq. (3.2.14) is symmetric in X and Y, we can equally derive

$$f_{XY}(x, y) = f_{Y|X}(x, y) \times f_X(x) \quad (3.2.18)$$

Example 3.2.3: Bivariate PDF

Consider the following description of two random variables. We choose X at random between 0 and 4; then we choose Y at random between 0 and $X/2$. We will derive the bivariate PDF. The space for this problem is identical with that shown in Fig. 3.2.6, but the bivariate PDF is no longer constant in the region because of the method used for choosing the values of X and Y. We show in Fig. 3.2.12 the region for these random variables.

We now derive the PDF. For X, we have

$$f_X(x) = \frac{1}{4}, \quad 0 < x \le 4, \quad \text{zow} \quad (3.2.19)$$

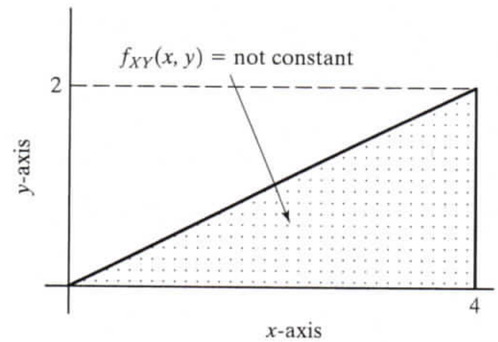

Figure 3.2.12 The region for the random variables is the same as in Fig. 3.2.6, but the joint PDF is no longer uniform because of the manner in which the random variables are defined.

and for Y we have

$$f_{Y|X}(x, y) = \frac{2}{x}, \ 0 < y \leq x/2, \text{ zow} \tag{3.2.20}$$

Note that the appropriate PDF for Y is a conditional PDF, since the PDF of Y depends on X. Using Eq. (3.2.18) we find the bivariate PDF as

$$f_{XY}(x, y) = f_{Y|X}(x, y) \times f_X(x) = \frac{2}{x} \times \frac{1}{4} = \frac{1}{2x}, \ 0 < y \leq x/2 \cap 0 < x \leq 4, \text{ zow} \tag{3.2.21}$$

As stated previously, the region of nonzero probability in the x, y plane is identical with the previous example, Fig. 3.2.6; however, the bivariate PDF is different: instead of a constant PDF we have the PDF given in Eq. (3.2.21). Both pick a point in the x, y plane "at random," but they use different methods and lead to different results.

You do it. Derive the marginal PDF of X for the PDF given in Eq. (3.2.21). Substitute $x = 2$, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

If you're having trouble, see endnote 15 for the answer.

Find the marginal PDF for *Y*. Find the marginal PDF for Y and substitute $y = 1$. Enter your answer in the cell box, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 16.

If you want to check the normalization and see a plot of this marginal PDF, see endnote 17.

Expectation. As in the case of a single continuous random variable, the summations used in the discrete case become integrals. For example, the expectation of a function of the two random variables would be computed as follows:

$$E[Z(X,Y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z(x,y) f_{XY}(x,y)\,dx\,dy \tag{3.2.22}$$

Conditional expectations. Conditional expectations can be defined and are extremely useful. For example, the expectation conditioned on event A would require the PDF conditioned on A,

$$f_{XY|A}(x,y)\,dx\,dy = \frac{P[(x < X \le x+dx) \cap (y < Y \le y+dy) \cap A]}{P[A]} \tag{3.2.23}$$

and the conditional expectation would be

$$E[Z(X,Y)|A] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z(x,y) f_{XY|A}(x,y)\,dx\,dy \tag{3.2.24}$$

Example 3.2.4: Two pulses

We now illustrate the use of the bivariate PDF to calculate probabilities and expectations. Two pulses in a digital system are supposed to arrive at a gate simultaneously but can vary by ± 50 ns owing to varying path delays. The pulses cause a glitch (false signal) whose width is the difference in arrival times. Consider that the delays are independent and uniformly distributed between -50 and $+50$ ns. Find the probability that the resulting glitch is longer than 60 ns, and also find the expected glitch width.

Solution Let T_1 and T_2 represent the amount of the two delays in nanoseconds. Because both are uniformly distributed between -50 and $+50$ the PDF of T_1 is $f_{T_1}(t_1) = 1/100, -50 < t_1 \le +50$, and similarly for T_2. Since the delays are independent, the joint PDF is

$$f_{T_1T_2}(t_1,t_2) = f_{T_1} \times f_{T_2} = 10^{-4}, -50 < t_1, t_2 \le +50, \text{ zow} \tag{3.2.25}$$

This PDF is nonzero in a square in the t_1, t_2 plane, 100 on a side and centered on the origin, as shown in Fig. 3.2.13.

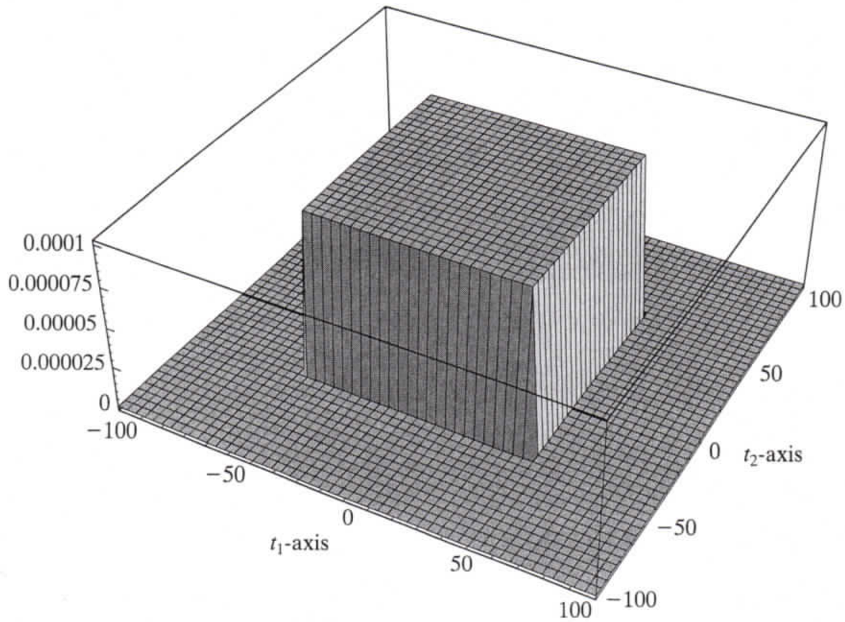

Figure 3.2.13 A three-dimensional plot of a PDF that is uniform over a square. The volume under the PDF curve must be 1. This is the PDF for Example 3.2.5.

Example 3.2.5: Calculating the required probability and expectation

The event. We are asked to determine the probability of the event $|T_1 - T_2| > 60$. The probability of this event can be determined by integrating the PDF over the regions of the t_1, t_2 plane where $|t_1 - t_2| > 60$. These regions are shown as the shaded triangles in Fig. 3.2.14.

The probability. To determine the probability of the event represented by the shaded triangles in the figure, we must integrate the joint PDF over these triangular regions. Fortunately, the PDF is a constant, and the regions are simple geometric figures, whose area we may determine from their triangular shape. The results are

$$P[|T_1 - T_2| \geq 60] = 10^{-4} \times \text{area of triangles} = 10^{-4} \times 2 \times \tfrac{1}{2}(40)(40) = 0.16 \qquad (3.2.26)$$

Had the shapes been irregular or the PDF a complicated function, we would have had more mathematical work to do, but the principle would have been the same.

You do it. What is the probability that the glitch width is less than 20 nanoseconds? Calculate your answer, and click Evaluate for a response.

```
myanswer = ;
```


Figure 3.2.14 The region
corners correspond to the
over these regions determi

```
Evaluate
```

For the answer, see endnote

The expectation. T
Eq. (3.2.22):

$E[|T_1$

Because of the symmetry o
$t_1 > t_2$ and double the resul
confusion. The region of int
We integrate first in t_2 a

$E[|T_1 - T_2|] = 2$

$= 2$

$= 2$

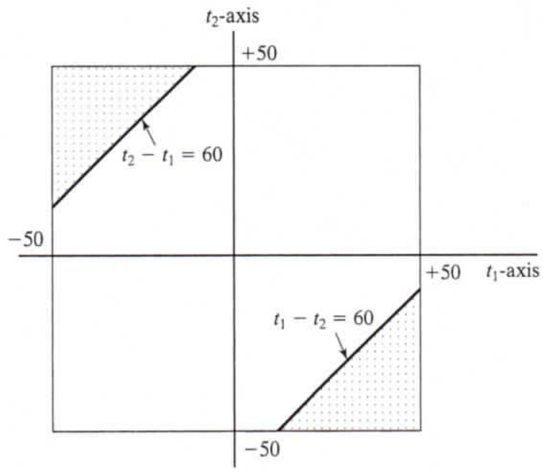

Figure 3.2.14 The region with nonzero probability is a square, 100 on a side. The shaded corners correspond to the event that $|T_1 - T_2|$ is greater than 60 ns. Integration of the PDF over these regions determines the probability.

```
Evaluate
```

For the answer, see endnote 18.

The expectation. The expected glitch width can be calculated from the definition in Eq. (3.2.22):

$$\begin{aligned} E[|T_1 - T_2|] &= \iint_{\text{square}} |t_1 - t_2| f_{T_1T_2}(t_1, t_2)\, dt_1\, dt_2 \\ &= 2 \times \iint_{t_1 > t_2} (t_1 - t_2) f_{T_1T_2}(t_1, t_2)\, dt_1\, dt_2 \end{aligned} \tag{3.2.27}$$

Because of the symmetry of the integrand we can evaluate the integral over the region where $t_1 > t_2$ and double the result. This gets rid of the absolute value bars and reduces the work and confusion. The region of integration is shown in Fig. 3.2.15.

We integrate first in t_2 and then in t_1. The details follow.

$$\begin{aligned} E[|T_1 - T_2|] &= 2 \int_{t_1=-50}^{t_1=+50} \left[\int_{t_2=-50}^{t_2=t_1} (t_1 - t_2) \times 10^{-4}\, dt_2 \right] dt_1 \\ &= 2 \times 10^{-4} \int_{-50}^{+50} \left[t_1 t_2 - \frac{t_2^2}{2} \right]_{-50}^{t_1} dt_1 \\ &= 2 \times 10^{-4} \int_{-50}^{+50} \left[\frac{t_1^2}{2} + 50\, t_1 + \frac{50^2}{2} \right] dt_1 = 33.3 \text{ ns} \end{aligned} \tag{3.2.28}$$

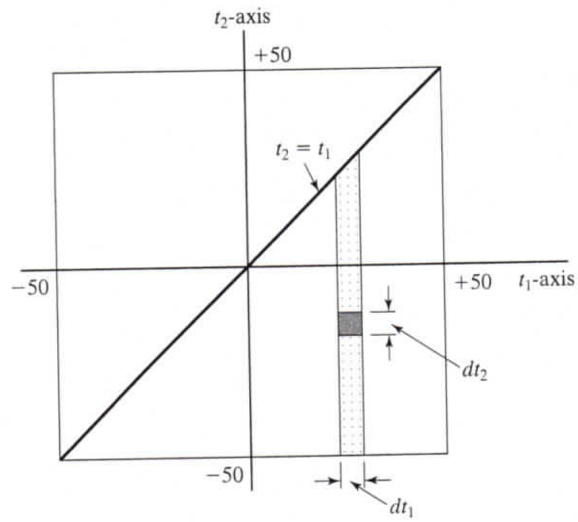

Figure 3.2.15 Integration of Eq. (3.2.27) for the expected glitch width. We integrate first in the vertical direction and then in the horizontal. We integrate only over the bottom triangle and double the result.

This appears to be a reasonable answer, since the maximum delay is 100 nanoseconds, but more delays are near zero (the region near the diagonal) than near the maximum (the corner regions).

You do it. Find the expectation of T_1, given that $T_1 > T_2$. Enter your answer in the cell box, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 19.

3.2.2 Bivariate Cumulative Distribution Functions, CDFs

Bivariate cumulative distribution functions, CDFs, find some use. For the bivariate case, the joint CDF is defined as

$$F_{XY}(x, y) = P[(X \leq x) \cap (Y \leq y)] \tag{3.2.29}$$

This corresponds to the probability that X, Y will fall to the left of and below x, y, as shown in Fig. 3.2.16.

From the definition, and the properties of the bivariate PDF, a number of consequences follow:

Figure 3.2.16 The CDF experiment X, Y will fa

- If either x or y example,
- If both argument

$F_{XY}(-$

- If one of the vari other variable. F

since the conditic *marginal CDF o*

- For independent CDFs:

$F_{XY}(x, y) =$

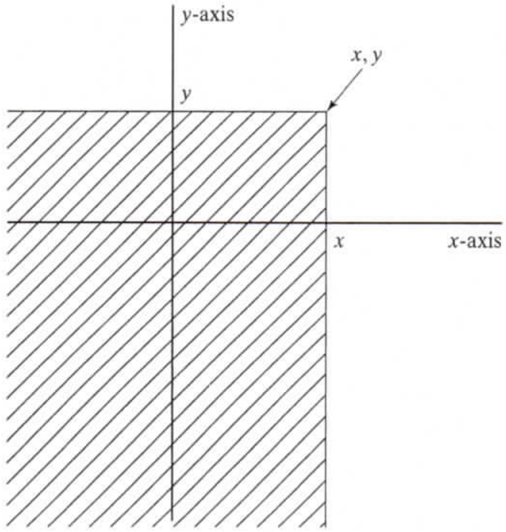

Figure 3.2.16 The CDF is defined as the probability that on one performance of the experiment *X*, *Y* will fall left of and below the point *x*, *y*, the crosshatched area shown.

- If either x or y is $-\infty$, the CDF must be zero. These represent impossible events, for example,

$$F_{XY}(-\infty, y) = P[(X \le -\infty) \cap (Y \le y)] = 0 \qquad (3.2.30)$$

- If both arguments are $+\infty$, these represent the certain event, which has a probability of 1:

$$F_{XY}(+\infty, +\infty) = P[(X \le +\infty) \cap (Y \le +\infty)] = P[S] = 1 \qquad (3.2.31)$$

- If one of the variables is $+\infty$, then the bivariate CDF reduces to the marginal CDF of the other variable. For example, if $x = +\infty$, then

$$\begin{aligned} F_{XY}(+\infty, y) &= P[(X \le +\infty) \cap (Y \le y)] = P[Y \le y] \\ &= F_Y(y), \text{ and likewise for } F_X(x) \end{aligned} \qquad (3.2.32)$$

since the condition $X \le +\infty$ is certain to be satisfied. For this reason $F_Y(y)$ is called the *marginal CDF on Y*, since $x \to +\infty$ takes us to the margin of the plane.

- For independent random variables, the joint CDF factors into the product of the marginal CDFs:

$$F_{XY}(x, y) = P[(X \le x) \cap (Y \le y)] = P[X \le x] \times P[Y \le y] = F_X(x) \times F_Y(y) \qquad (3.2.33)$$

- The joint PDF is the derivative of the joint CDF:

$$f_{XY}(x, y) = \frac{\partial^2}{\partial x\, \partial y} F_{XY}(x, y) \tag{3.2.34}$$

- Finally, from Eq. (3.2.29) and (3.2.5), we can derive the bivariate CDF from the bivariate PDF:

$$F_{XY}(x, y) = P[(X \leq x) \cap (Y \leq y)] = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(x', y')\, dx'\, dy' \tag{3.2.35}$$

Use of bivariate CDFs. Having defined the bivariate CDF and shown many of its properties, we now consider its value in solving problems. For one random variable, CDFs are very important in setting up problems. For two or more random variables, CDFs are of limited usefulness except in theoretical developments. The problem is that CDFs can be used to determine probabilities only if the region of interest is some form of rectangle, which seldom occurs in applications. Thus bivariate CDFs are useful conceptually and relate to the bivariate PDF in a straightforward way, but have limited use in setting up and solving problems, which is the focus of this book. The following example illustrates these points. We use the bivariate PDF in dealing with two random variables; the CDF is used only for a single random variable.

Example 3.2.6: Two delays

A message must be sent over a network through two computers. The delays in each computer are random variables uniformly distributed between 0 and 2 seconds and are independent. Find the expected delay, the standard deviation of the delay, and the PDF of the delay.

Solution The model for this problem is similar to that of the previous problem. We let T_1 and T_2 represent the two independent delay times and give their joint PDF as

$$f_{T_1 T_2}(t_1, t_2) = f_{T_1} \times f_{T_2} = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4},\ 0 < t_1, t_2 \leq 2,\ \text{zow} \tag{3.2.36}$$

Here we are interested in the combined delay $T = T_1 + T_2$. We thus are defining a third random variable that is a function of the original bivariate random variables, $T(T_1, T_2)$. We can clearly calculate the expected delay and the standard deviation of the delay using the algebra of expectation, but we will first calculate the PDF of the combined delay and, from that, calculate the mean and the standard deviation. Because this is a derivation, we will begin with the CDF of T and from that derive its PDF. This CDF is

$$F_T(t) = P[T \leq t] = P[T_1 + T_2 \leq t] = \iint_{\text{shaded region}} f_{T_1}(t_1) f_{T_2}(t_2)\, dt_1\, dt_2 \tag{3.2.37}$$

where the shaded area is that shown in Fig. 3.2.17.

Again the math is easy because the PDF is constant and the area is a triangle. The results are

$$F_T(t) = \frac{1}{2} t^2 \times 0.25 = \frac{t^2}{8},\ 0 \leq t \leq 2 \tag{3.2.38}$$

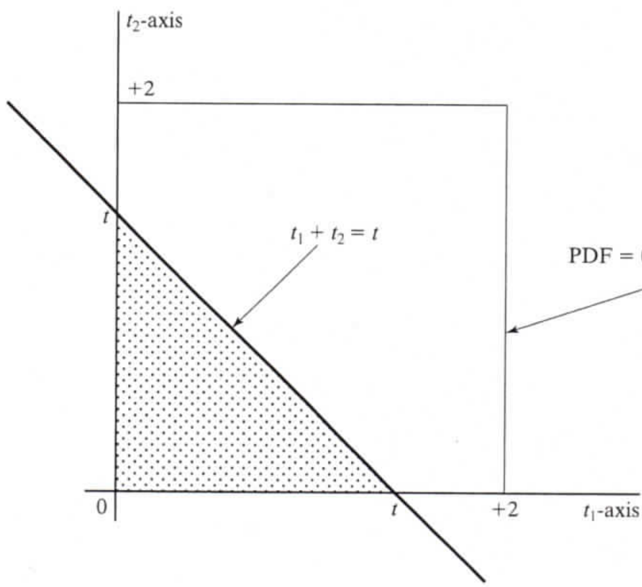

Figure 3.2.17 Calculation of the CDF of T requires integration of the bivariate PDF over the triangular region, which is simply the area times the PDF. For $t > 2$, the easiest calculation takes the area of the entire square and subtracts the area of a triangle.

We can use Eq. (3.2.38) only up to $t = 2$ because the geometric shape is different for $2 \le t \le 4$. Sparing you the details, we give the CDF in this region as

$$F_T(t) = \left(4 - \frac{(4-t)^2}{2}\right) \times 0.25, \; 2 \le t \le 4 \tag{3.2.39}$$

The CDF is 0 for $t < 0$, and 1 for $t > 4$, outside the two regions defined in Eqs. (3.2.38) and (3.2.39), and is plotted in Fig. 3.2.18.

The PDF is the derivative of the CDF:

$$\begin{aligned} f_T(t) &= \frac{d}{dt} F_T(t) = \frac{t}{4}, \; 0 \le t \le 2 \\ &= \frac{(4-t)}{4}, \; 2 \le t \le 4, \; \text{zow} \end{aligned} \tag{3.2.40}$$

which is shown in Fig. 3.2.19.

We now may complete the remaining parts of the example. We may use the PDF to determine the mean:

$$\mu_T = E[T] = \int_{-\infty}^{+\infty} t f_T(t)\, dt = \int_0^2 t \times \frac{t}{4}\, dt + \int_2^4 t \times \frac{(4-t)}{4}\, dt = 2 \tag{3.2.41}$$

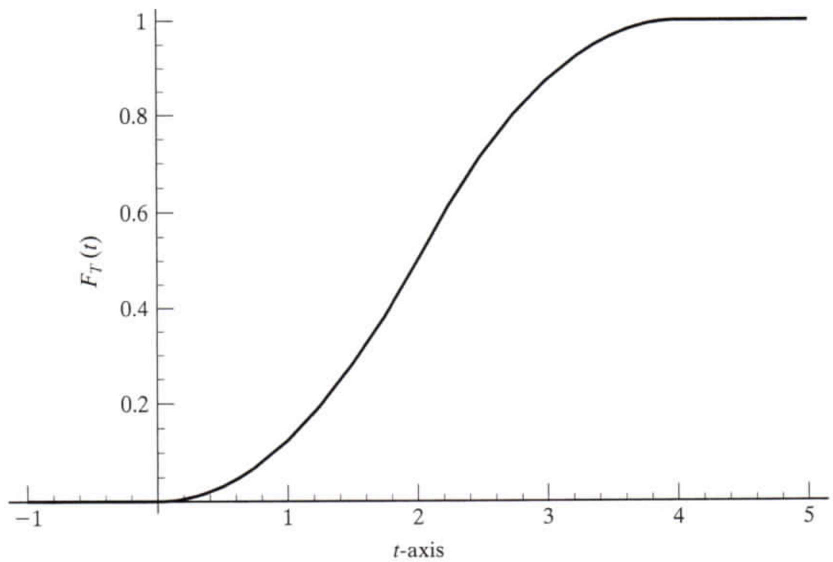

Figure 3.2.18 The CDF of the total delay.

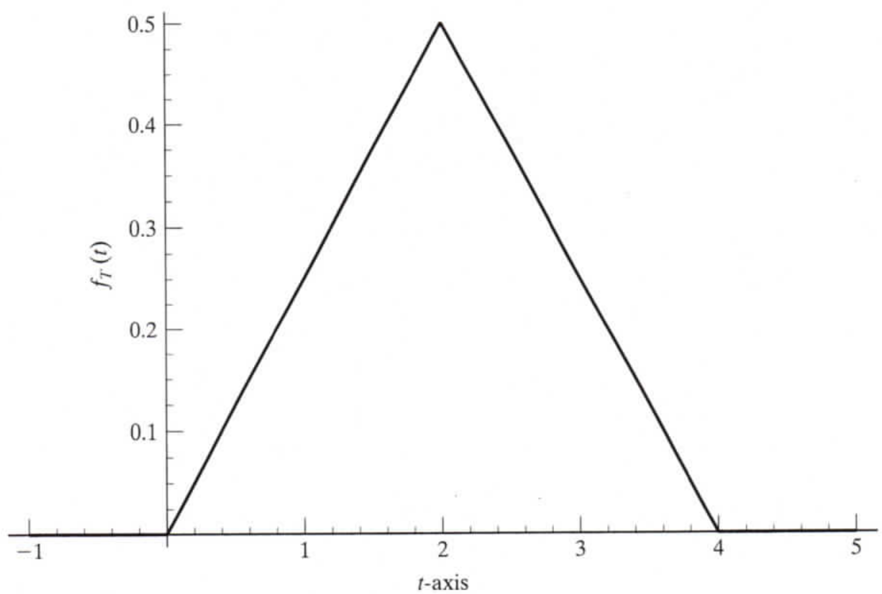

Figure 3.2.19 The PDF of the total delay. Note that the area is unity, as required of all PDFs.

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which clearly is the balance point of the distribution. The variance can be computed similarly:

$$Var[T] = E[(T-\mu_T)^2] = \int_0^2 (t-2)^2 \times \frac{t}{4}\,dt + \int_2^4 (t-2)^2 \times \frac{(4-t)}{4}\,dt = 0.667 \quad (3.2.42)$$

The standard deviation of the delay is therefore $\sigma_T = \sqrt{0.667} = 0.816$ second.

Using the algebra of expectation. We may confirm the mean and standard deviation calculations using the algebra of expectation. The mean is the sum of the means of the two random variables:

$$E[T] = E[T_1 + T_2] = E[T_1] + E[T_2] \quad (3.2.43)$$

Because T_1 and T_2 are uniformly distributed between 0 and 2, their individual means are

$$E[T_1] = E[T_2] = \frac{(0+2)}{2} = 1 \quad (3.2.44)$$

and therefore $E[T] = 2$, as in Eq. (3.2.41). Because T_1 and T_2 are independent, their variances add:

$$Var[T] = Var[T_1 + T_2] = Var[T_1] + Var[T_2] \quad (3.2.45)$$

For a random variable uniformly distributed between a and b, the variance is known to be $\frac{(b-a)^2}{12}$ [see Eq. (3.1.21)] and thus the variance of T is

$$Var[T] = \frac{2^2}{12} + \frac{2^2}{12} = \frac{2}{3} \quad (3.2.46)$$

which agrees with Eq. (3.2.42).

For a simulation of this example, see endnote 20.

You do it. Now, work out this example to test your understanding of the material in this section.

a. A bivariate CDF is $F_{XY}(x, y) = c(1 - e^{-2x})\tan^{-1}(y/6)$ for $0 < x, y < +\infty$, zow. First, you need to find c. Think about it (no calculation required) and enter your answer as "myanswer" in the cell box, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 21.

b. Now, find the marginal PDF for Y and give its value at $y = 2$.

```
myanswer = ;
```

```
Evaluate
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For the answer, see endnote 22.

Summary. This section presented the concept of bivariate random variables and showed how probability is described for two variables through PDFs and CDFs. We emphasized the use of the bivariate PDF to calculate probability and expectations. In the next section we extend this type of analysis to more than two random variables.

3.3 INDEPENDENT, IDENTICALLY DISTRIBUTED (IID) RANDOM VARIABLES AND THE CENTRAL LIMIT THEOREM

3.3.1 Independent, Identically Distributed (IID) Random Variables

Overview. In this section we present the concepts of multivariate random variables that are independent and have identical PDFs. These we call *IID* random variables. The first *I* stands for *independent,* the second *I* stands for *identical,* and the *D* stands for *distributed.* We will speak of the "first I" meaning independent, and the "second I" meaning identically distributed.

We will compute the mean, variance, and PDF of sums of such IID random variables. This leads naturally to the presentation of the central limit theorem, which expresses what happens when the number of IID random variables gets very large. Properly normalized, the sum of many IID random variables gives a random variable that has a Gaussian or normal PDF, which we introduce in the last subsection of this section. The full exploration of the Gaussian distribution comes in Sec. 3.4.

Why are we interested in IID random variables? One situation that yields IID random variables is the following. Take a basic experiment that yields a random variable. Repeat it to obtain random variables that are independent yet identical in their characteristics. Examples are repeating a measurement for a fluctuating quantity, measuring bolts out of a bin in a hardware store, or measuring antennae on fruit flies. For two measurements, this suggests the diagram in Fig. 3.3.1.

Note that we are using X_1 and X_2 for the random variables in Fig. 3.3.1. This makes more sense than using X and Y because we must also consider a large number of IID random variables. Besides, this notation suggests the identical characteristics of the random variables.

What is different between this situation and Bernoulli trials? Not much, really, except a Bernoulli trial has a binary result, and here the result is a random variable with its PDF.

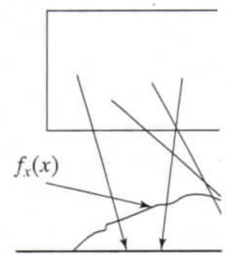

Figure 3.3.1 If we measur
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