3

Continuous Random Variables and the Gaussian Model

3.1 CONTINUOUS RANDOM VARIABLES

To this point we have described sample spaces and random variables that are discrete, involving either a finite or a countably infinite number of outcomes. In this chapter we introduce sample spaces and random variables of a continuous nature, involving an uncountable infinity of outcomes. In Sec. 3.1.1 we discuss differences between discrete and continuous spaces and define appropriate means to describe how probability is distributed to a continuous random variable.

3.1.1 Continuous Sample Spaces

Discrete and continuous number systems in engineering

Measured data. All real-world probability spaces are discrete. For example, all measured values have limited accuracy and hence are meaningful only out to, say, the fourth place. This means that, in effect, only discrete values can result from a measurement.

Computer output. Another example is any number produced by a computer or any other digital system. There exists a least significant bit somewhere in the system that limits the output to a discrete set of numbers.

A useful model. Often, a discrete sample space is well modeled as a continuum. If, for example, the accuracy of the data is good, one may have thousands of possible outcomes from a measurement. In such instances, using continuous numbers can offer real advantages. One advantage is that we do not have to worry about the details of the computer or measurement system that produced the numbers when we model with a continuous space.

Easy math. In addition, the math gets easier. Which calculation would you prefer to perform,

$$\sum_{i=0}^{100} i^4 = 0^4 + 1^4 + 2^4 + \cdots + 100^4 \quad \text{or} \quad \int_0^{100} x^4\,dx \qquad ? \tag{3.1.1}$$

These are essentially the same calculation, except the first is discrete and the second continuous. Clearly, you would rather do the integral rather than the sum. Of course, Mathematica has no trouble with either, so perhaps this is not a big advantage, but generally we like easy math. To see the Mathematica results, see endnote 1.

3.1.2 Describing Probabilities in Continuous Sample Spaces

The concept of a continuous sample space. Mathematicians can prove that there are an infinite number of rational numbers, numbers that can be expressed as a ratio of integers, in any finite interval of the real line, say, from 1.5 to 2.0. Furthermore, they tell us that between any two rational numbers there exists an infinity of irrational numbers. Hence, an uncountable infinity of points lies between 1.5 and 2.0 on the real axis.

Pick a number, any number. Consider now the sample space $S = \{1.5 < s \leq 2.0\}$, where all outcomes are equally likely. Thus the chance experiment is to pick a number at random between 1.5 and 2.0. Clearly we have an infinity of outcomes. It follows that $P[s = 1.75] = 0$, since 1.75 is only one of an infinity of outcomes. We illustrate this in the following way. Evaluating the following expression will generate $k = 25$ numbers from a population uniformly distributed between 1.5 and 2, printed with 10-place accuracy. Although it is possible that 1.750000000 is one of the numbers, this is rather unlikely. If you are running Mathematica, try it a few times and get a feel for how unlikely it is.

```
k=25;
Table[NumberForm[Random[Real, {1.5,2}],10],{k}]
```

```
{1.754301748, 1.830036463, 1.690279158,
 1.972605016, 1.718271984, 1.672246016,
 1.534970989, 1.955686084, 1.588103819,
 1.933640064, 1.788797722, 1.750123663, 1.826551125,
 1.787069398, 1.989428098, 1.972417095, 1.622214896,
 1.569336222, 1.811659055, 1.790547474, 1.85559485,
 1.851147333, 1.603731846, 1.585718837, 1.601293102}
```

A simple model for this experiment suggests that the probability of getting exactly 1.750000000 at least once in 25 tries is 5×10^{-8}. Clearly, if we had asked for 20-place accuracy, getting a prescribed answer would be even more unlikely.

Events of zero probability versus the impossible event. Thus the probability that s = any prescribed number in the space is exactly zero! This follows from common sense, and also from the classical definition of probability, that the probability of an event is the number of outcomes favorable to the event (one outcome) divided by the number of outcomes (an infinity of outcomes). But $\{s = 1.75\}$ is not the impossible event, since it can occur. Some number between 1.5 and 2.0 occurs each time the experiment is performed. We conclude that although the impossible event has zero probability, $P[\emptyset] = 0$, every event with zero probability is not an impossible event. The impossible event has zero probability by definition, whereas the outcome $\{s = 1.75\}$ has zero probability owing to numerical considerations.

Assigning probabilities to the outcomes for this space. We can assign probabilities to intervals of the sample space. For example, we expect that $P[1.5 < s \leq 2] = 1$, since this covers the entire range, which is the certain event and has a probability of 1. In general, we expect that the probability that the outcome will occur in a prescribed range is proportional to the width of that range, provided that the range lies within the bounds from 1.5 to and including 2. This means that $P[s_1 < s \leq s_2] = K(s_2 - s_1)$, where K is a constant, s_1 and s_2 are in the range between 1.5 and 2, and $s_1 < s_2$. Since with $s_1 = 1.5$ and $s_2 = 2$ we have the certain event, it follows that the constant, K, must be $\frac{1}{2-1.5} = 2$. Therefore we can calculate the probability that the outcome will fall in the region $s_1 < s \leq s_2$ to be $2(s_2 - s_1)$, provided s_1 and s_2 are in the range between 1.5 and 2, and $s_1 < s_2$. This probability is shown by the crosshatched area in Fig. 3.1.1.

You do it. (a) For the chance experiment just described, find $P[1.5 < s \leq 1.6]$. Enter your answer in the cell box, and click Evaluate for the response.

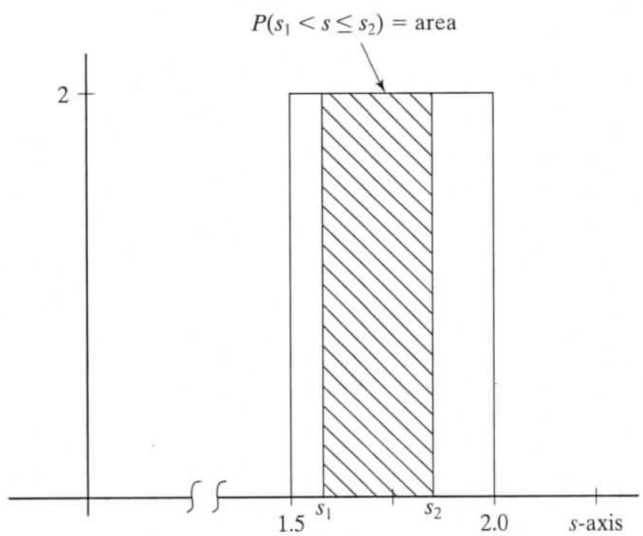


Figure 3.1.1 The crosshatched area gives the probability that s will fall in that range on any performance of the chance experiment.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 2.

(b) For the same situation, find $P[1.4 < s \leq 1.6]$.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 3.

Unequal outcomes. If all outcomes in the range are not equally likely, we can represent the distribution of probabilities by a general function rather than a uniform function, as shown in Fig. 3.1.2.

This way of defining probabilities is essentially the same as defining a random variable on the space and describing it by a probability *density* function. In other words, since continuous sample spaces are numerical, with every outcome of the experiment a number or set of numbers, we can simply consider these as random variables defined on the outcomes and discuss how the probability maps into the range of the random variable through a function that is like the probability mass function. We explore these ideas in the next section.

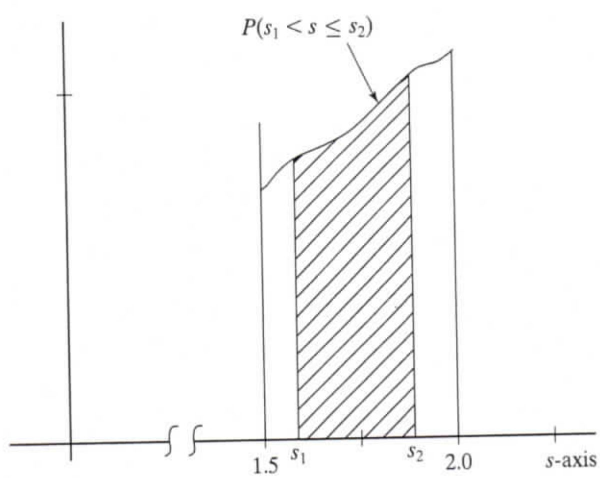


Figure 3.1.2 With outcomes that are not equally likely, we can still represent the probability as an area, which is to say, as an integral of some function.

Summary. The following are some major points about continuous sample spaces:

- Although all real-world numbers are discrete, the continuous number system offers many theoretical and practical advantages.
- With outcomes in a continuous space, probability is associated with regions of outcomes. A single outcome like $\{s = 1.75\}$ has zero probability.
- Because continuous outcomes are always numerical, the outcomes of a chance experiment and a random variable associated with those outcomes is essentially the same. Therefore, in continuous sample spaces we usually start with a random variable.

3.1.3 The Probability Density Function (PDF)

Probability density functions (PDFs). Figure 3.1.3 shows how the probability density function is defined.

In Fig. 3.1.3 we see that the sample space maps into the range of the random variable on the real axis, S_X, and the probabilities distribute according to some function, $f_X(x)$. This function is called the *probability density function*, PDF for short. Figure 3.1.3(b) and (c) show the relationship between a PMF, which gives the probability at a point when we have a discrete random variable, and the PDF, which gives the probability that the random variable will lie in an infinitesimal region. The probability that was concentrated at one point in the discrete case (PMF) is spread over a region in the continuous case (PDF).

Definition of probability density function, PDF. We define the PDF for a continuous random variable as follows:

$$P[x < X \leq x + dx] = f_X(x)\,dx, \quad \text{with } dx \geq 0 \tag{3.1.2}$$

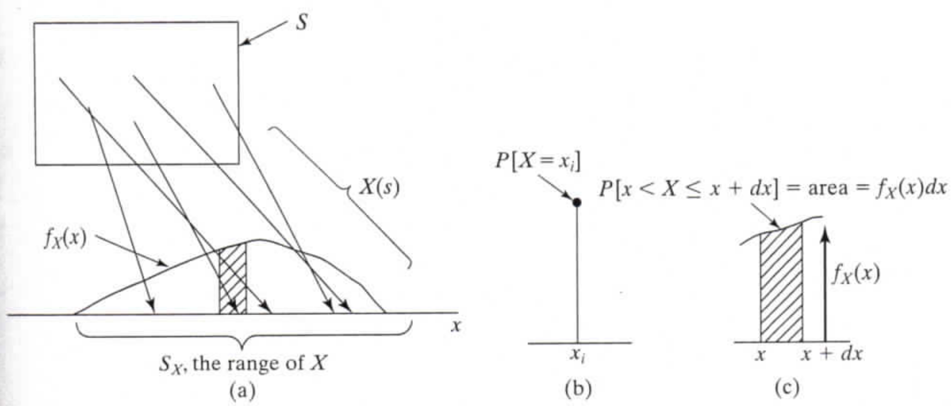


Figure 3.1.3 (a) The random variable is defined in the usual way. (b) With a discrete random variable, all the probability is lumped at one point. (c) With a continuous random variable, probability is distributed to all points. The crosshatched area shows the probability that the random variable, X, will fall in a region between x and $x + dx$.

Thus the PDF is defined in terms of the probability that on any given performance of the experiment the random variable falls in a narrow region in the vicinity of x. Formally, we let $dx \to 0$ according to the conventions of calculus. We may make Eq. (3.1.2) look more like a density by dividing by dx, as in Eq. (3.1.3):

$$f_X(x) = \frac{P[x < X \le x + dx]}{dx} \tag{3.1.3}$$

which we can interpret as a probability per unit distance in x.

Properties of the probability density function. The PDF has some properties in common with the PMF. Here we list some properties of the PDF.

1. The probability density function, PDF, is nonnegative $f_X(x) \ge 0$. This follows from Eq. (3.1.2) and the rule that probabilities are nonnegative.
2. The probability *mass* function, PMF, is a probability and must be less than or equal to 1, whereas the PDF is a *density* of probability and may be greater than 1. In the previous example $f_X(x) = 2$ in a certain region if $X(s) = s$.
3. The PDF may be used to determine the probability that the random variable, X, will fall in an interval:

$$P[x_1 < X \le x_2] = \int_{x_1}^{x_2} f_X(x)\, dx, \ \ x_1 \le x_2 \tag{3.1.4}$$

Discussion of Eq. (3.1.4). To show that Eq. (3.1.4) is true we consider a partition of the event $\{x_1 < X \le x_2\}$.

In Fig. 3.1.4, we break the region between x_1 and x_2 into many smaller regions of width Δx and label these regions $R_1, R_2, \ldots$. If X is to fall between x_1 and x_2, then it must fall in one and only one of these regions. Thus we may partition the event of interest as follows:

$$\{x_1 < X \le x_2\} = \{X \in R_1\} \cup \{X \in R_2\} \cup \{X \in R_3\} \cup \cdots \cup \{X \in R_n\} \tag{3.1.5}$$

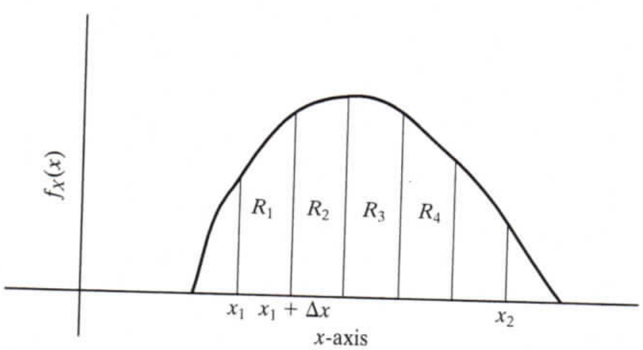


Figure 3.1.4 We break the region between x_1 and x_2 into many smaller regions of width Δx. These allow a partition of the event whose probability is calculated in Eq. (3.1.4).

where n is the number of regions. Because the events on the right of Eq. (3.1.5) are disjoint, we have a partition of the event in question and we may calculate its probability using Eq. (1.5.7) as follows:

$$P[x_1 < X \leq x_2] = P[X \in R_1] + P[X \in R_2] + P[X \in R_3] + \cdots + P[X \in R_n] \tag{3.1.6}$$

But the probabilities of each of the events on the right side can be expressed in terms of the PDF, Eq. (3.1.2). For example,

$$P[X \in R_1] = f_x(x_1)\Delta x \tag{3.1.7}$$

Thus in Eq. (3.1.6), as $\Delta x \to dx$ the sum becomes the integral in Eq. (3.1.4).

4. The PDF must be normalized. There are no restrictions on x_1 and x_2 in Eq. (3.1.4), so we can set $x_1 = -\infty$, and $x_2 = +\infty$. For these values the event on the left side of Eq. (3.1.4) becomes the certain event, and hence the integral of the PDF must be normalized to unity:

$$P[-\infty < X \leq +\infty] = \int_{-\infty}^{+\infty} f_X(x)\,dx = P[S] = 1 \tag{3.1.8}$$

Thus all PDFs are nonnegative functions with unit area. This is equivalent to the normalization condition for PMFs, Eq. (2.2.3). We might note that the equal sign at the upper limit of the region does not matter for continuous spaces; however, it does matter in discrete spaces and in spaces where continuous and discrete random variables are mixed. We use the less than or equal sign at the top of the interval to develop consistent habits.
5. The PDF has units. If, for example, x was a distance in meters, the PDF would have units of inverse meters. This follows because a probability is a pure number with no units.
6. The PDF is used to compute expectations. The mean is expressed in terms of the PDF as

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx \tag{3.1.9}$$

which follows from the reasoning that was used in defining the mean with the PMF. The integral replaces the sum that was used there, Eq. (2.3.5). For the expectation for functions of the random variable, X, we have

$$E[Y(X)] = \int_{-\infty}^{+\infty} y(x) f_X(x)\,dx \tag{3.1.10}$$

For example, the mean square value of a random variable and the variance of X would be

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx$$

and

$$Var[X] = \int_{-\infty}^{+\infty} (x - \mu_X)^2 f_X(x)\,dx = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx - \mu_X^2 = E[X^2] - \mu_X^2 \tag{3.1.11}$$

Summary. The PMF and the PDF play similar roles in the development of probability. In discrete spaces, we sum the PMF to normalize and compute probabilities and expectations. In continuous spaces, we integrate the PDF to normalize and compute probabilities and expectations. When we combine discrete and continuous random variables later in this chapter, we will use impulse functions to express the probability on a point in a continuous space.

3.1.4 The Uniform and Exponential PDFs

In Sec. 3.1.2 we used as an example (Fig. 3.1.1) of a continuous sample space a chance experiment in which the outcomes were numbers between 1.5 and 2.0, with all outcomes equally likely. We now have defined PDFs and can develop that example more fully. Our random variable is the outcome itself, $X(s) = s$, where s represents the equally likely outcomes, $\{1.5 < s \le 2.0\}$. In this example we will find the PDF of X and calculate the mean and variance of this random variable. Because all outcomes are equally likely, we require that the PDF be a constant, C:

$$f_X(x) = C, 1.5 < x \le 2, \text{ zow} \tag{3.1.12}$$

We can determine C through the normalization condition, Eq. (3.1.8):

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = \int_{1.5}^{2} C\,dx = 0.5\,C = 1, \text{ therefore } C = 2 \tag{3.1.13}$$

This is, of course, the same result that we determined earlier by similar reasoning, in different notation. The mean would be

$$\mu_X = E[X] = \int_{-\infty}^{+\infty} x f_X(x)\,dx = \int_{1.5}^{2} 2x\,dx = x^2|_{1.5}^{2} = 1.75 \tag{3.1.14}$$

which is the balance point of the PDF. To calculate the variance, we first calculate the mean square value:

$$E[X^2] = \int_{-\infty}^{+\infty} x^2 f_X(x)\,dx = \int_{1.5}^{2} 2x^2\,dx = \frac{2}{3}x^3|_{1.5}^{2} = 3.083 \tag{3.1.15}$$

and hence the variance is

$$Var[X] = E[X^2] - \mu_X^2 = 3.083 - (1.75)^2 = 0.02083 \tag{3.1.16}$$

and the standard deviation is

$$\sigma_X = \sqrt{0.02083} = 0.1443$$

Finally, we find the probability that the random variable, on a given performance of the experiment, falls within one standard deviation of the mean. This is a simple integral, as follows:

$$P[\mu_X - \sigma_X < X \le \mu_X + \sigma_X] = \int_{\mu_X - \sigma_X}^{\mu_X + \sigma_X} f_X(x)\,dx = \int_{1.606}^{1.894} 2\,dx = 0.577 \tag{3.1.17}$$

We note that this prob
symmetric PDFs.

The general prope
example to a uniform PD
the PDF

where C is a constant. Yo
something, we have built
click Evaluate. Use an orc

```
myanswer = "1/(?
```

```
Evaluate
```

For the answer, see endnot
As an example we plo
Now, calculate the exp
but use the general case, a
answer:

- Does the mean you

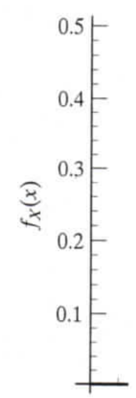


Figure 3.1.5 Plot of the PD
is 1 over the base, such that

We note that this probability is about $\frac{2}{3}$, slightly less in this case, which is typical for symmetric PDFs.

The general properties of a uniform PDF. Here we want you to generalize the previous example to a uniform PDF between two values, a and b, where $a < b$. Assume that you have the PDF

$$f_X(x) = C, \ a < x \leq b, \ \text{zow} \tag{3.1.18}$$

where C is a constant. Your first task is to find C. Because the height can be expressed as 1 over something, we have built that into the form for you to use. Enter your results in the cell box, and click Evaluate. Use an ordinary font (no italics).

```
myanswer = "1/(?)";
```

```
Evaluate
```

For the answer, see endnote 4.

As an example we plot the PDF for $a = 2$ and $b = 4$ in Fig. 3.1.5.

Now, calculate the expected value and the variance. You may follow the preceding example, but use the general case, a and b. Here are some questions you ought to ask yourself about your answer:

- Does the mean you derive look like the balance point of the distribution?

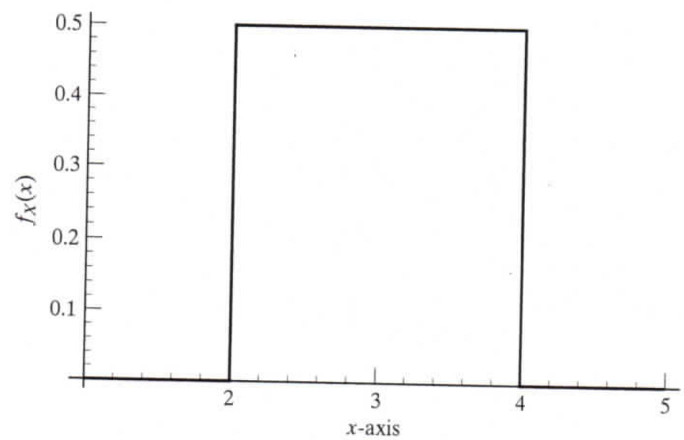


Figure 3.1.5 Plot of the PDF of a uniform distribution between 2 and 4. Note that the height is 1 over the base, such that the area is 1.

- Does the square root of the variance, the standard deviation, bear some relation to the width of the distribution? As a general rule for distributions that are somewhat symmetric, about two-thirds of the area under the PDF curve will lie within one standard deviation on both sides of the mean. A mathematical expression of the previous statement is

$$P[\mu - \sigma < X < \mu + \sigma] \approx \tfrac{2}{3} \tag{3.1.19}$$

Following is a test to see if your results are correct. Calculate the probability given in Eq. (3.1.19). Your answer should be a pure number near $\frac{2}{3}$, not dependent on a or b. Enter your answer in the cell box, and click Evaluate for a response.

```
myanswer = ? ;
```

```
Evaluate
```

For the answer, see endnote 5.

Summary of the properties of the uniform PDF. If the random variable is uniformly distributed between a and b, with $b > a$, it has a PDF of

$$f_X(x) = \frac{1}{b-a}, \quad a < x \leq b, \text{ zow} \tag{3.1.20}$$

and it has a mean and variance of

$$\mu_X = \frac{b+a}{2} \text{ and } \sigma_X^2 = \frac{(b-a)^2}{12} \tag{3.1.21}$$

The uniform PDF as a model. The uniform PDF can successfully model situations in which we know only the range of a random variable. For example, let us say we buy a new car, and the manufacturer guarantees the gasoline mileage to be between 24 and 27 miles per gallon (mpg). If we have no further knowledge, a reasonable model for the PDF of M, a random variable representing the unknown mileage, would be $f_M(m) = \frac{1}{3}$, $24 < m \leq 27$, zow.

Modeling ignorance. In one sense, use of the uniform PDF is demanded by our ignorance of the true PDF. As such, it is a conservative model. For the car mileage, our intuition might suggest that $M = 25.5$ mpg, being near the middle of the range, is more likely than $M = 24$ mpg. But one also could argue that $M = 24$ mpg is more likely; it all depends on how the manufacturer tests and tinkers with the cars to ensure that the specification is met. In the absence of such detailed knowledge, the uniform PDF is a good model.

The exponential PDF. Another useful model is the exponential PDF, defined as

$$f_T(t) = ae^{-at}, \quad t \geq 0, \quad a > 0, \text{ zow} \tag{3.1.22}$$

which is shown in Fig. 3.1.6 for $a = 1$.

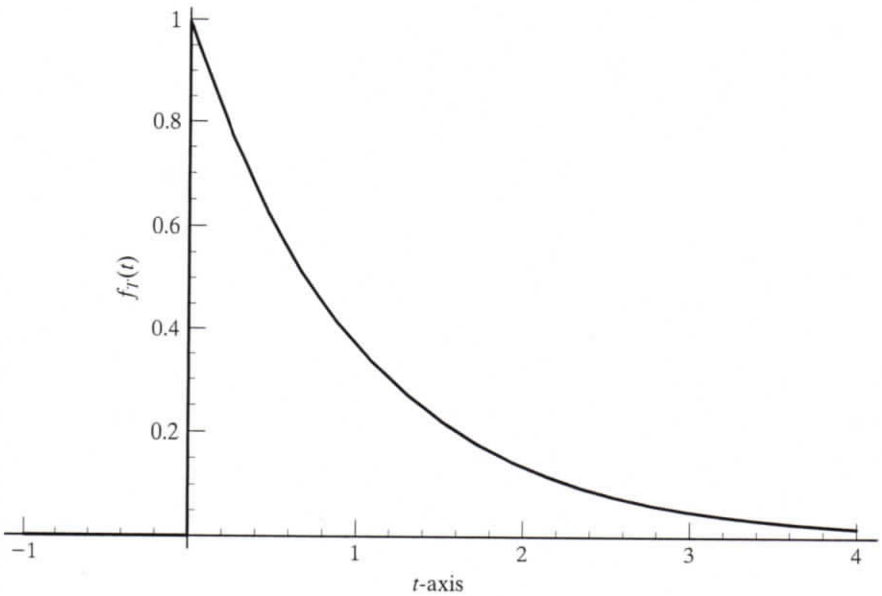


Figure 3.1.6 The exponential PDF is useful for modeling random events in time; hence we use T for the random variable. We will derive this distribution from basic assumptions in Chapter 5.

The exponential PDF is useful in describing random events in time; that is why we used T for the random variable in Eq. (3.1.22) and in Fig. 3.1.6. We will derive the exponential PDF from basic assumptions in Chapter 5; we introduce it here as a second example to illustrate the properties and uses of PDFs.

- The PDF is nonnegative and is normalized to 1:

$$\int_{-\infty}^{+\infty} f_T(t)\,dt = \int_0^{+\infty} ae^{-at}\,dt = 1 \tag{3.1.23}$$

- The mean is $\frac{1}{a}$:

$$\mu_T = E[T] = \int_{-\infty}^{+\infty} tf_T(t)\,dt = \int_0^{+\infty} tae^{-at}\,dt = \frac{1}{a}\int_0^{+\infty} xe^{-x}\,dx = \frac{1}{a} \tag{3.1.24}$$

- The mean square value is $\frac{2}{a^2}$:

$$E[T^2] = \int_{-\infty}^{+\infty} t^2 f_T(t)\,dt = \int_0^{+\infty} t^2 ae^{-at}\,dt = \frac{1}{a^2}\int_0^{+\infty} x^2 e^{-x}\,dx = \frac{2}{a^2} \tag{3.1.25}$$

- The variance is $\frac{1}{a^2}$:

$$\sigma_T^2 = E[(T - \mu_T)^2] = E[T^2] - \mu_T^2 = \frac{2}{a^2} - \left(\frac{1}{a}\right)^2 = \frac{1}{a^2} \tag{3.1.26}$$

- The probability that T falls within one standard deviation of the mean is

$$P[\mu_T - \sigma_T < T \le \mu_T + \sigma_T] = \int_0^{\frac{2}{a}} ae^{-at}\,dt = 1 - e^{-2} = 0.8647 \tag{3.1.27}$$

This is somewhat higher than the $\frac{2}{3}$ mentioned earlier, Eq. (3.1.19), for PDFs with symmetric shapes; the exponential PDF is highly skewed and departs significantly from this rule of thumb.

3.1.5 The Cumulative Distribution Function (CDF)

The cumulative distribution function, CDF. With continuous random variables we have an alternative way to describe how the probabilities distribute on the real line. This alternative, the cumulative distribution function, CDF, has the advantage that it is a probability.

Definition of the CDF. The CDF is defined as the probability that the random variable, X, is less than or equal to an independent variable, x: $F_X(x) = P[X \le x]$. This definition is illustrated in Fig. 3.1.7.

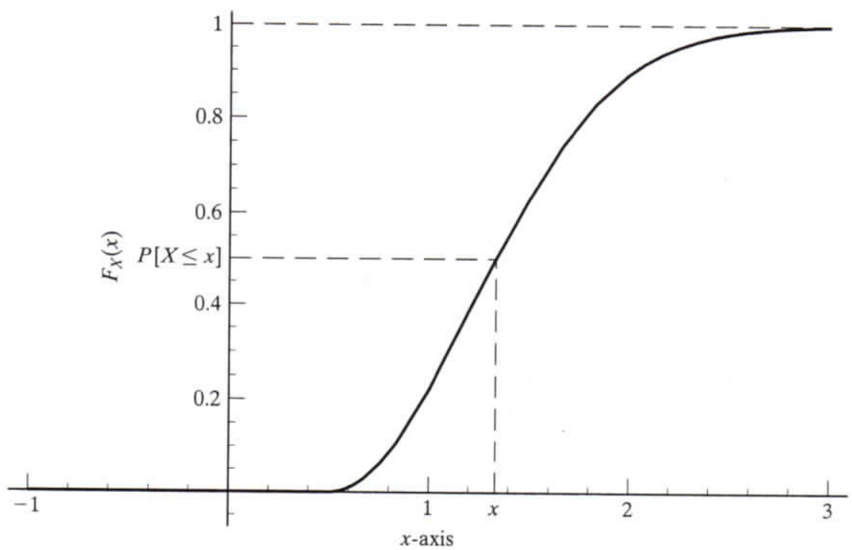


Figure 3.1.7 The height of the CDF is the probability that the random variable, X, is less than or equal to x. The CDF never decreases, because as x increases, more probability is included, and probability never goes away.

The CDF and the PDF. The CDF and the PDF are related through a derivative/integration operation. Comparing the definition of the CDF with Eq. (3.1.4) with $x_1 = -\infty$ and $x_2 = x$, you will realize that

$$F_X(x) = P[X \leq x] = \int_{-\infty}^{x} f_X(x')\, dx' \tag{3.1.28}$$

and

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{3.1.29}$$

follows immediately from differentiating Eq. (3.1.28). Note we must use the "dummy" variable, x', since x is the upper limit of the integral.

Comparing the CDF and the PDF. If we have the same information in the CDF as we have in the PDF, why do we need both? As we develop this subject you will see that generally we use the CDF in derivations and in setting up problems. The CDF is a probability, and we have many rules for manipulating, and much experience with, probabilities. The PDF, in contrast, is a probability density and is useful for calculating probabilities and expectations. The PDF is our calculation engine, whereas the CDF is our way out of the woods when we are lost. We need both even though they contain the same information.

In Fig. 3.1.8 we show the CDF for the uniform PDF with limits $a = 2$ and $b = 4$. This is the integral of the PDF in Fig. 3.1.5, provided you use the same a and b, of course.

The properties of the CDF. The properties of the CDF are as follows:

1. The CDF is a probability and hence must lie between 0 and 1: $0 \leq F_X(x) \leq 1$.

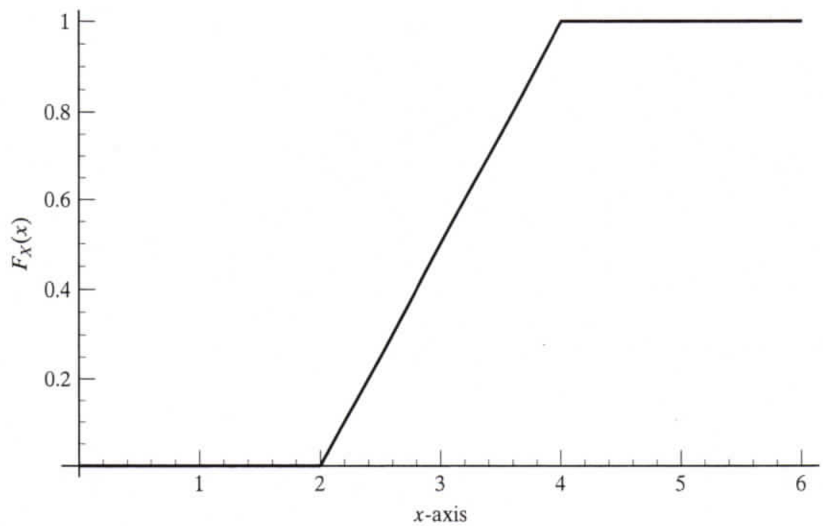


Figure 3.1.8 CDF for uniform PDF between the values of $a = 2$ and $b = 4$.

2. The CDF is nondecreasing, since $P[X \le x]$ can only gain, and never lose, probability as x increases.
3. The value at $-\infty$ is zero: $F_X(-\infty) = 0$. This represents the probability of an impossible event, $\emptyset = \{X \le -\infty\}$.
4. The value at $+\infty$ is 1: $F_X(+\infty) = 1$. This represents the probability of a certain event: $S = \{X \le +\infty\}$.
5. Because $\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$, and since these two events are disjoint, it follows that

$$P[X \le x_2] = P[X \le x_1] + P[x_1 < X \le x_2] \tag{3.1.30}$$

from which it follows immediately that

$$P[x_1 < X \le x_2] = P[X \le x_2] - P[X \le x_1] = F_X(x_2) - F_X(x_1) \tag{3.1.31}$$

Thus the CDF allows us to determine the probability that a random variable lies in a continuous range by taking the difference between the CDF at the limits of that range. If the CDF is known, this is easier than integrating the PDF.

Example 3.1.1

The CDF of a random variable is $F_X(x) = 1 - e^{-2x}$, $x \ge 0$, zow. Find the probability that on any performance of the experiment the random variable falls between 1 and 1.5.

Solution Using Eq. (3.1.31), we find

$$P[1 < X \le 1.5] = F_X(1.5) - F_X(1) = 1 - e^{-2\times 1.5} - (1 - e^{-2\times 1}) = 0.0855$$

By the way, notice that we always put the less than or equal sign at the top of the range. For a continuous random variable the *or equal* part contributes no probability, since the probability of the event $\{X = 1.5\}$ is zero; nevertheless, it is consistent to include the upper bound of the interval.

You do it. What is the PDF of the random variable described in the previous example? Work out your answer, substitute $x = 1$, enter your answer in the cell box, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 6.

Figure 3.1.9 The outcomes
the resistance of the resistor
the random variable with a

3.1.6 Use of PDFs: An Ext

We will illustrate these defini
concepts and definitions will

Example starting with a r

The chance experiment is to
the bin contains resistors betw
The outcome of the experime
outcome is the resistance of
experiment. The chance exper

Example 3.1.2: Value of th
Find the probability that the re

Solution We may calculate
105 Ω, using Eq. (3.1.4):

$P[100 < R$

This makes perfect sense beca

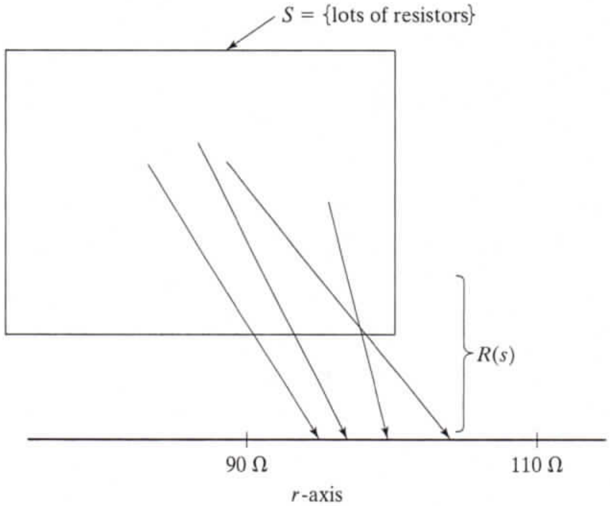


Figure 3.1.9 The outcomes of the chance experiments are resistors. The random variable is the resistance of the resistors. We are assuming sufficient number of resistors that we model the random variable with a uniform PDF between 90 and 110 Ω, as indicated in Eq. (3.1.32).

3.1.6 Use of PDFs: An Extended Example

We will illustrate these definitions and applications in the context of an extended example. New concepts and definitions will be introduced as required.

Example starting with a resistor chosen at random

The chance experiment is to choose a resistor from a bin containing "100-Ω" resistors. Actually, the bin contains resistors between 90 Ω and 110 Ω, with all values in this range equally likely. The outcome of the experiment is the resistor chosen. The random variable associated with the outcome is the resistance of the resistor, R. We will base a series of examples on this chance experiment. The chance experiment is illustrated in Fig. 3.1.9, and the PDF of the resistors is[7]

$$f_R(r) = 0.05\ \Omega^{-1},\ 90 < r \leq 110,\ \text{zow} \tag{3.1.32}$$

Example 3.1.2: Value of the resistance

Find the probability that the resistance of the chosen resistor is between 100 and 105 Ω.

Solution We may calculate the probability that the selected resistor falls between 100 and 105 Ω, using Eq. (3.1.4):

$$P[100 < R \leq 105] = \int_{100}^{105} f_R(r)\,dr = \int_{100}^{105} 0.05\,dr = 0.25 \tag{3.1.33}$$

This makes perfect sense because the 5-Ω range is one-fourth of the total range of 20 Ω.

Example 3.1.3: Power range

Assume the resistor is a 3-W (watt) resistor. What is the probability that the resistor is thermally stressed by having a power exceeding 3 W, assuming a voltage of 17 V is applied.

Solution We can address this question within the framework of our original experiment, or we can consider this a new chance experiment (choose a resistor and connect it to a 17-V battery) and a new random variable (the resulting power in the resistor). We will regard this as part of the original experiment but define a second random variable that is a function of the original random variable:

$$W(R) = \frac{(17)^2}{R} \tag{3.1.34}$$

This change can be considered a mapping of one random variable to another, from R to $W = (17)^2/R$. This mapping is shown in Fig. 3.1.10. We see that 90 Ω corresponds to 3.21 watts and 110 Ω to 2.63 watts. Hence the range for W will be $2.63 < W \leq 3.21$ watts.

Mapping events. The power 3 watts corresponds to a resistance of 96.3 Ω. The event $\{3 \leq W\}$ contains all values of the power that are greater than or equal to 3 watts. Each of these values corresponds to a resistor in the event $\{R \leq 96.3\}$. Thus we can map an event in W to an event in R.

Calculating the required probability. We may calculate the probability $P[3 \leq W]$ by calculating the probability of the event

$$P[R \leq 96.3] = \text{the area shaded in Fig. 3.1.10} = 0.05 \times (96.3 - 90) = 0.317 \tag{3.1.35}$$

This calculation is based on the mapping of an event in W to a corresponding event in R.

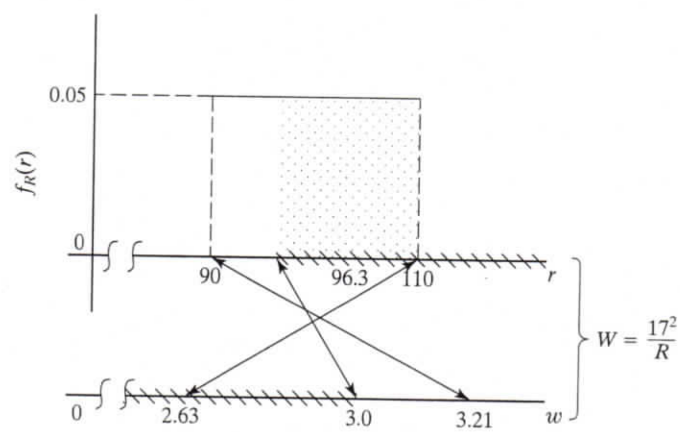


Figure 3.1.10 All the resistor values between 90 and 110 Ω map to powers between 2.63 and 3.21 watts. This can be considered a change in random variables from R to W.

You do it. What is the probability that the power falls in the range $2.7 < W \leq 2.9$ watts? Enter your answer in the cell box, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 8.

Example 3.1.4: Expected value of the power

Find the expected value of the power in the resistor: $E[W]$.

Solution Because the power, W, is a function of a random variable of known PDF, R, we can apply the definition in Eq. (3.1.10):

$$E[W(R)] = \int_{-\infty}^{+\infty} w(r) f_R(r)\, dr = \int_{90}^{110} \frac{17^2}{r} 0.05\, dr = 14.5 \ln\left(\frac{110}{90}\right) = 2.90 \text{ watts} \quad (3.1.36)$$

Thus the resistors connected to 17 V have an average power of 2.90 watts.

3.1.7 Conditional PDFs and Conditional Expectations

As we have stated repeatedly, conditional probabilities provide a powerful analysis tool. We therefore wish to explore the definition and uses of conditional probabilities in the analysis of probabilities in continuous spaces, using PDFs and CDFs. We continue the example with a conditional expectation. This in turn will require the definition and determination of a conditional PDF.

Example 3.1.5: Expected power of overstressed resistors

Find the expected value of the power in the thermally overstressed resistors: $E[W|W > 3]$.

Solution Resistors with more power than 3 watts are thermally overstressed. To calculate the average power of all such resistors we need the conditional PDF: $f_{R|W\geq 3}(r)$. The definition requires a slight modification of the definition of an (unconditional) PDF, Eq. (3.1.2).

$$f_{R|W\geq 3}(r)\, dr = \frac{P[(r < R \leq r + dr) \cap (W \geq 3)]}{P[W \geq 3]} \quad (3.1.37)$$

The denominator of Eq. (3.1.37) has been determined to be 0.317. The effect of the conditioning event in the numerator is to limit the domain of resistors to $90 < R \leq 96.3$. The condition thus has two effects on the PDF: (1) it limits the domain (the numerator), and (2) it renormalizes (the denominator). Thus the conditional PDF is

$$f_{R|W\geq 3}(r)\, dr = \frac{0.05}{0.317} dr \Rightarrow f_{R|W\geq 3}(r) = 0.158,\ 90 < r \leq 96.3,\ \text{zow} \quad (3.1.38)$$

The required conditional expectation can now be determined:

$$E[W(R)|W \geq 3] = \int_{-\infty}^{+\infty} w(r) f_{R|W\geq 3}(r)\,dr = \int_{90}^{96.3} \frac{17^2}{r} \times 0.158\,dr \qquad (3.1.39)$$

$$= 45.6 \ln\left(\frac{96.3}{90}\right) = 3.10 \text{ watts}$$

Thus the expected power in the thermally overstressed resistors is 3.10 watts.

You do it. Find the expected power of the resistors that yield a power in the range $2.7 < W \leq 2.9$ watts. Calculate the conditional PDF and then the conditional expectation. Enter your answer to at least four-place accuracy, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 9.

In this section we have defined and used conditional PDFs and conditional expectations in a specific context. In endnote 10 we show how to derive a conditional PDF in general.

3.1.8 Transforming Random Variables

Example 3.1.6: PDF of the power

Find the probability density function, PDF, of the power, $f_W(w)$.

Solution We are now transforming from one random variable to another, $W \to R$, and from one PDF to another, $f_R(r) \to f_W(w)$. The safest way to perform such a transformation is first to determine the cumulative distribution function of the power and then to determine the PDF from the CDF. By definition, the CDF of W is

$$F_W(w) = P[W \leq w] \qquad (3.1.40)$$

Critical regions. In general, the form of the CDF will be

$$\begin{aligned} F_W(w) &= (\text{something}),\ w < \text{some limit} \\ &= (\text{something else}),\ \text{lower} < w < \text{upper limit} \qquad (3.1.41) \\ &= (\text{something else}),\ w > \text{another limit, and so on} \end{aligned}$$

To determine the CDF we first have to identify the various critical regions. Toward that end, examine the mapping of resistance values to power values shown in Fig. 3.1.11.

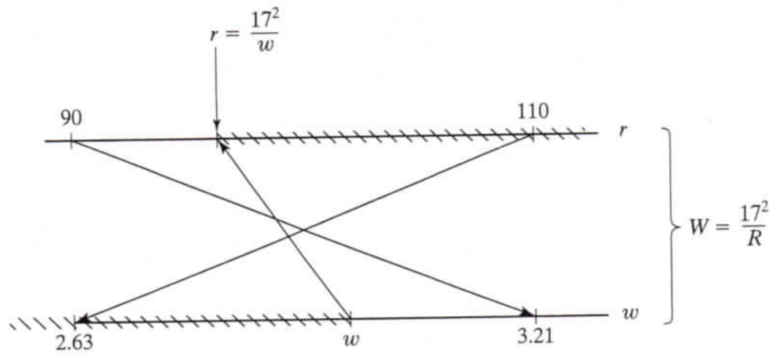


Figure 3.1.11 Here we show a general mapping of the event $\{W < w\}$ to the corresponding event in R, $\{R > 17^2/w\}$. This mapping allows us to calculate the CDF of W from the PDF of R.

In this case, the critical regions are seen to be $w \leq 2.63$, $2.63 < w \leq 3.21$, and $w > 3.21$. (Because the random variable is continuous we have assigned the powers $w = 2.63$ and $w = 3.21$ to the top of the ranges.) Thus our form is

$$\begin{aligned} F_W(w) &= \text{(something)},\ \ w \leq 2.63 \\ &= \text{(something else)},\ \ 2.63 < w \leq 3.21 \\ &= \text{(something else)},\ \ w > 3.21 \end{aligned} \tag{3.1.42}$$

Determining the CDF. We may determine the *somethings* by mapping the event $\{W \leq w\}$ to the identical event $\{R \geq \frac{17^2}{w}\}$, as shown in Fig. 3.1.11. The first *something* is 0 because no values of resistance lead to powers less than 2.63 watts. And the last *something else* is 1 because all values of resistance lead to powers below 3.21 watts. The middle *something else* comes from the mapping and inverse mapping relationship of the function $W(R)$.

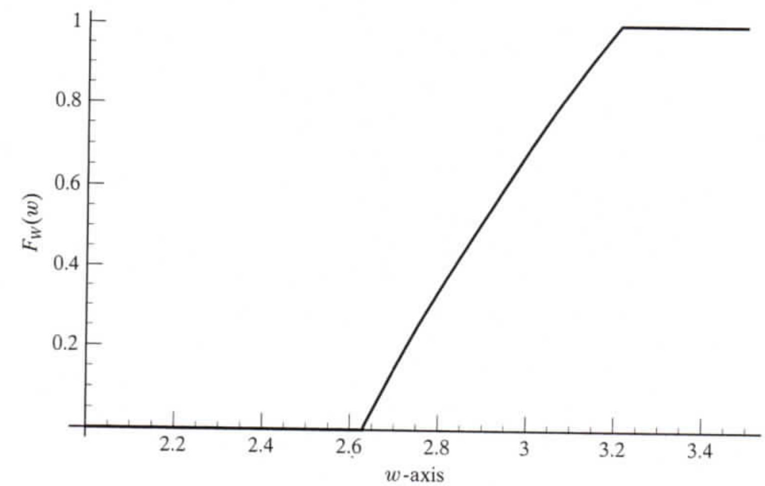
We can therefore determine the CDF in this middle region as

$$\begin{aligned} F_W(w) &= P[W \leq w] = P\left[R \geq \frac{17^2}{w}\right] \\ &= \int_{r=\frac{17^2}{w}}^{r=110} 0.05\,dr = 0.05\left(110 - \frac{17^2}{w}\right),\ 2.63 < w \leq 3.21 \end{aligned} \tag{3.1.43}$$

Thus the final result is

$$\begin{aligned} F_W(w) &= 0,\ \ w \leq 2.63 \\ &= 0.05\left(110 - \frac{17^2}{w}\right),\ \ 2.63 < w \leq 3.21 \\ &= 1,\ \ w > 3.21 \end{aligned} \tag{3.1.44}$$

The plot of this CDF follows.

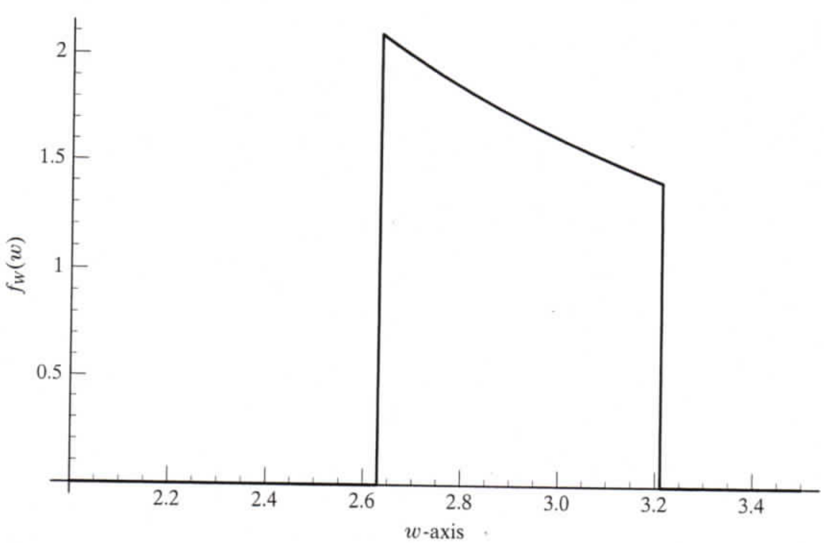


The CDF is 0 below and 1 above the middle range in Eq. (3.1.44), as shown.

Determining the PDF from the CDF. The PDF is the derivative of the CDF. The PDF is thus zero in the regions where the CDF is 0 or 1. In the middle region, the derivative is

$$f_W(w) = \frac{d}{dw}F_W(w) = \frac{d}{dw}0.05\left(110 - \frac{17^2}{w}\right) = \frac{14.5}{w^2},\ 2.63 < w \leq 3.21,\ \text{zow} \qquad (3.1.45)$$

The plot follows.



A check of a previo
calculated in Eq. (3.1.36):

$$E[W] = \int_{-\infty}^{+\infty} wf_w(w)$$

Example 3.1.7: PDF of
Let us say that someone ac
10% of the resistors are 50
same as before; namely, pi

Solution We will contin
we may treat the resistance
CDFs and PDFs. Let $A =$
assume that 50-Ω resistors
would be

We regard these conditional
These may be integrated to

F

and

F

These conditional CDFs loo
We now may formulate
as a partition of the space
we write

$F_R(r) = P[R \leq$

Thus we may express the C
$P[A] = 0.1$ and $P[B] = 0.9$
CDF is plotted in Fig. 3.1.12

A check of a previous calculation. We may use this PDF to check the expected power calculated in Eq. (3.1.36):

$$E[W] = \int_{-\infty}^{+\infty} w f_w(w)\, dw = \int_{2.63}^{3.21} w \times \frac{14.5}{w^2}\, dw = 14.5 \ln\left(\frac{3.21}{2.63}\right) = 2.90 \text{ watts} \qquad (3.1.46)$$

Example 3.1.7: PDF of new mixture of resistors

Let us say that someone accidentally dumped some "50-Ω resistors" into the 100-Ω bin, such that 10% of the resistors are 50 Ω, and 90% of the resistors are 100 Ω. The chance experiment is the same as before; namely, pick a resistor at random and note its resistance. Find the resulting PDF.

Solution We will continue to assume that many resistors of both types are involved and that we may treat the resistance as a continuous random variable. This problem calls for conditional CDFs and PDFs. Let $A = \{\text{a 50-}\Omega \text{ resistor is chosen}\}$ and $B = \{\text{a 100-}\Omega \text{ resistor is chosen}\}$. We assume that 50-Ω resistors are uniformly distributed between 45 and 55 Ω. The conditional PDFs would be

$$\begin{aligned} f_{R|A}(r) &= \frac{1}{55-45}, \quad 45 < r \leq 55, \text{ zow} \\ f_{R|B}(r) &= \frac{1}{110-90}, \quad 90 < r \leq 110, \text{ zow} \end{aligned} \qquad (3.1.47)$$

We regard these conditional PDFs as given information, based on our model for a 10% resistor. These may be integrated to yield the conditional CDFs.

$$F_{R|A}(r) = \int_{-\infty}^{r} f_{R|A}(r')\, dr' = P[R \leq r|A]$$

and

$$F_{R|B}(r) = \int_{-\infty}^{r} f_{R|B}(r')\, dr' = P[R \leq r|B] \qquad (3.1.48)$$

These conditional CDFs look similar to Fig. 3.1.8.

We now may formulate the (unconditional) CDF for the random variable R using A and B as a partition of the space of the experiment. Using the law of total probability [Eq. (1.5.7)], we write

$$\begin{aligned} F_R(r) = P[R \leq r] &= P[R \leq r|A] \times P[A] + P[R \leq r|B] \times P[B] \\ &= F_{R|A}(r) \times P[A] + F_{R|B}(r) \times P[B] \end{aligned} \qquad (3.1.49)$$

Thus we may express the CDF of R in terms of the conditional CDFs in Eq. (3.1.48). Using $P[A] = 0.1$ and $P[B] = 0.9$, in Eq. (3.1.49), we obtain the CDF of R. We skip the details. The CDF is plotted in Fig. 3.1.12.

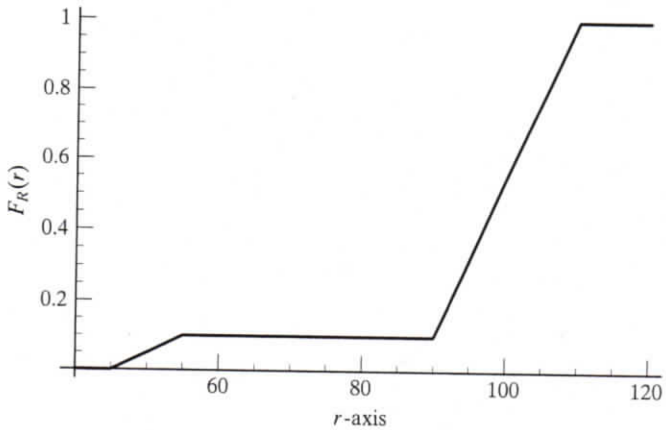


Figure 3.1.12 The CDF of the random variable that consists of 10% 50-Ω resistors and 90% 100-Ω resistors. In both cases we are assuming a uniform distribution over a ±10% range centered on the nominal value.

From the CDF in Eq. (3.1.49) we may derive the PDF of R by differentiation:

$$
\begin{aligned}
f_R(r) &= f_{R|A}(r) \times P[A] + f_{R|B}(r) \times P[B] \\
&= 0.1 \times 0.1,\ 45 < r \leq 55,\ \text{and} \\
&= 0.05 \times 0.9,\ 90 < r \leq 110,\ \text{zow}
\end{aligned} \tag{3.1.50}
$$

This PDF is shown in Fig. 3.1.13.

Example 3.1.8: Using conditional expectations

Find the expected value of R for this modified experiment.

Solution We apply the definition of expectation (mean), Eq. (3.1.9):

$$
\begin{aligned}
\mu_R = E[R] = \int_{-\infty}^{+\infty} r f_R(r)\, dr &= P[A] \times \int_{-\infty}^{+\infty} r f_{R|A}(r)\, dr + P[B] \times \int_{-\infty}^{+\infty} r f_{R|B}(r)\, dr \\
&= P[A] \times E[R|A] + P[B] \times E[R|B] \\
&= 0.1 \times 50 + 0.9 \times 100 = 95\ \Omega
\end{aligned} \tag{3.1.51}
$$

Generating random variables

Although we have shown how to transform random variables, we now add a very practical example that relates to calculator and computer simulations of random systems. The following is an instance of the problem we will address:

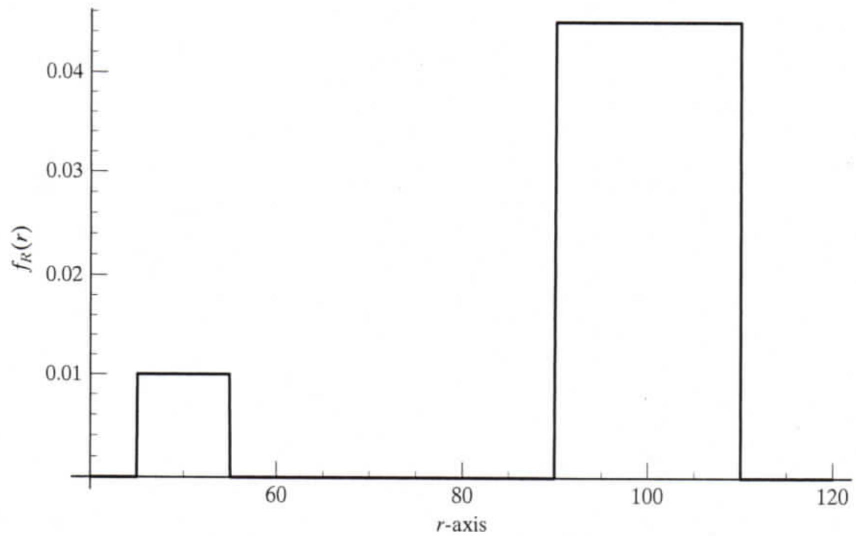


Figure 3.1.13 The PDF of the random variable that consists of 10% 50-Ω resistors and 90% 100-Ω resistors. In both cases we are assuming a uniform distribution over a ±10% range centered on the nominal value. This is the derivative of the CDF in Fig. 3.1.12.

In Eq. (3.1.45) we give the PDF of the power, W, as

$$f_W(w) = \frac{14.5}{w^2},\ 2.63 < w \le 3.21,\ \text{zow} \tag{3.1.52}$$

Let us say we wish to generate values of a random variable that has this distribution. Mathematica will generate random numbers of prescribed distributions, provided these are standard distributions. Your calculator will not even do that. Instead, it will generate random numbers that have a uniform distribution between the limits of 0 and 1. Mathematica does the same with the command "Random[ ]", as shown.

```
X = Table[Random[], {10}]
```

```
{0.926594, 0.336122, 0.390913, 0.544429, 0.57383,
 0.7714, 0.00138949, 0.846951, 0.81984, 0.769131}
```

Thus we need a method to transform uniformly distributed random numbers to numbers distributed according to some other PDF, such as shown in Fig. 3.1.14. Let us call the uniformly distributed random variable X, as before. We now need $W(X)$ for the transformation. We determine this transformation on the basis of the CDFs of X and W. Fig. 3.1.14 shows the areas corresponding to the CDFs of X and W and indicates an appropriate mapping to make the areas equal, which means the CDFs are equal, $F_X(x) = F_W(w(x))$.

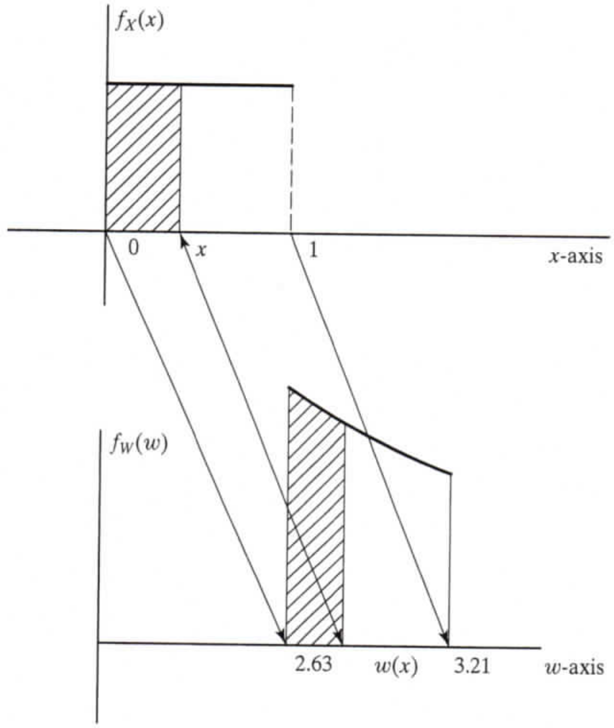


Figure 3.1.14 To transform the uniformly distributed random variable *X* into another random variable, in this case *W*, we need *W*(*X*) such that the CDFs are equal.

The crosshatched areas should be equal, such that

$$P[X \leq x] = P[W \leq w(x)] \tag{3.1.53}$$

which simply equates the CDFs of X and W. The critical parts of these CDFs are

$$F_X(x) = x, 0 < x \leq 1 \text{ and } F_W(w(x)) = 0.05\left(110 - \frac{17^2}{w(x)}\right), 2.63 < w(x) \leq 3.21 \tag{3.1.54}$$

Setting these equal, we have

$$x = 0.05\left(110 - \frac{17^2}{w(x)}\right) \tag{3.1.55}$$

To obtain the required transformation, we solve for $w(x)$, with the result

$$w(x) = \frac{17^2}{110 - 20x} \tag{3.1.56}$$

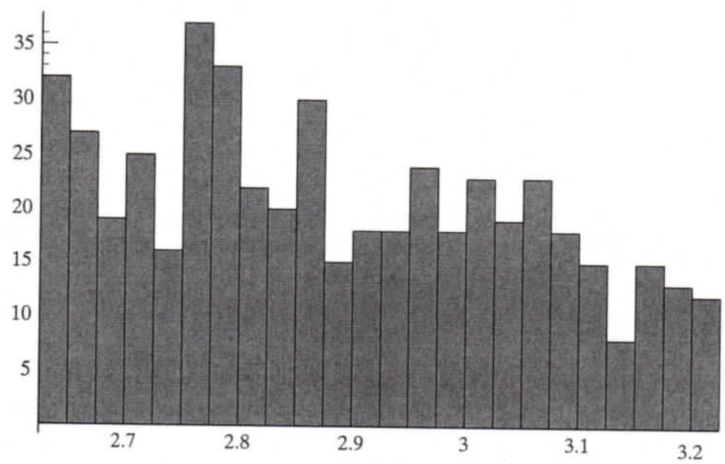
We now can generate the required numbers distributed according to the target PDF. The code is

```
X = Table[Random[], {10}];
        17²
W = ─────────── /. x → X
    110 - 20x
```

```
{2.81238, 2.89968, 3.18438, 2.89623, 3.06527, 2.72755, 3.19802,
 3.18 > 2.8656, 2.92151}
```

To demonstrate how the W values distribute, we execute the following cell for a histogram of 500 samples

```
X = Table[Random[], {500}];
        17²
W = ─────────── /. x → X;
    110 - 20x
Histogram[W];
```



You do it. Derive a transformation $t(x)$ that will transform a uniformly distributed random variable in the range $0 < X \leq 1$ to an exponential random variable with the PDF $f_T(t) = 3e^{-3t}, t \geq 0$, zow. Substitute $t = 0.4$ as myanswer, and click Evaluate for a response.

```
myanswer = ;
```

```
Evaluate
```

For the answer, see endnote 11.